Mathematical Structures of Cohomological Field Theories

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Chapter 1

Introduction

A cohomological field theory (CohFT) is a Lagrangian field theory that possesses a scalar supersymmetry Q with $Q^2 = 0$ and a Q-exact energy-momentum tensor [Wit88; Wit91]. The physical operators of interest in such a theory are solutions to the following equations

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)} \tag{1.0.1}$$

with $Q\mathcal{O}^{(0)} = 0$ for $1 \leq p \leq n$, where d is the de Rham differential of our n-dimensional spacetime manifold M. The expectation value of $\int_{\gamma_p} \mathcal{O}^{(p)}$, where γ_p is a p-cycle in M, does not depend the metric on M. That is to say, it can be seen as a smooth invariant of M. Many famous invariants in mathematics, e.g., the Donaldson invariants, Gromov-Witten invariants and Seiberg-Witten invariants, can be obtained in this way.

There exist various mathematical approaches to cohomological field theories. The most known ones are Baulieu and Singer's approach [BS88] using the BRST cohomology, and Atiyah and Jeffrey's approach [AJ90] based on Mathai and Quillen's construction of the Thom class [MQ86]. In fact, these two different approaches can be related by an automorphism of the Weil model of the relevant equivariant cohomology, observed by Kalkman in [Kal93a; Kal93b]. Parallelly, there is also the AKSZ formalism [Ale+97], where everything is reformulated in the language of the so-called Q-manifolds and QP-manifolds. However, most of these approaches focus only on the construction of the Lagrangians, lacking a systematic treatment of the algebraic structures of the operators in a CohFT. It is the goal of this thesis to build a new mathematical framework for CohFTs unifying the previous ones, within which a complete classification of the solutions to (1.0.1) is available.

The first step toward such a framework is to generalize the theory of supermanifolds to a theory with richer grading structures. A supermanifold is an extension of a usual manifold by attaching Grassmann algebras locally to it [Kos77; Lei80; Man97]. The anticommutativity property of a fermionic field in physics can be then interpreted in terms of the anticommutativity of the Grassmann algebras. When multiplying two fermionic fields, one gets a bosonic field. This process can be tracked by assigning $0 \in \mathbb{Z}_2$ to bosonic fields and $1 \in \mathbb{Z}_2$ to fermionic fields. In fact, $\mathbb{Z}_2$ can be replaced by a commutative semiring $\mathcal{I}$ to yield the notion of an $\mathcal{I}$-graded manifold [Jia23], which includes supermanifolds, graded manifolds [KS21] and colored supermanifolds [CGP16] as special cases.

The scalar supersymmetry Q and the de Rham differential d appearing in (1.0.1) can be interpreted as vector fields of degrees $(0,1)$ and $(1,0)$ over a $\mathbb{Z}\times\mathbb{Z}$-graded (or bigraded) manifold [Jia22]. To be more precise, let E be a fiber bundle over M. The variational bicomplex of E is the double complex of local differential forms on the Fréchet manifold $M \times \Gamma(E)$. We call a differential form over $M \times \Gamma(E)$ local if it is the pullback of a differential form on the infinite jet bundle $J^\infty(E)$ under the infinite jet evaluation map $\mathrm{ev}^\infty : M \times \Gamma(E) \to J^\infty(E)$ which sends (x,ψ) to $j^\infty(\psi)(x)$, the infinite jet prolongation of ψ at x. $M \times \Gamma(E)$ together with the local differential forms over it can be seen as a bigraded manifold called the variational bigraded manifold of E. The vector fields Q and d can be obtained from the vertical and horizontal differentials on $J^\infty(E)$ by applying a change of coordinates. Moreover, the supersymmetry algebra can be extended to include a vector supersymmetry K, a vector field of degree $(1,-1)$ over such a bigraded manifold. K together with Q and d satisfies the following relations

$$Q^2 = 0, \quad QK + KQ = d, \quad Kd + dK = 0.$$

It can be used to produce particular solutions to (1.0.1) called K-sequences by setting $\mathcal{O}^{(p)} = \sum_{q=0}^{p} \frac{1}{(p-q)!} K^{p-q}\mathcal{W}^{(q)}$, where $\mathcal{W}^{(0)} = \mathcal{O}^{(0)}$ and $\mathcal{W}^{(q)}$ is any (non-exact) Q-closed function of degree $(q, n-q)$ for $1 \le q \le n$.

Let $\mathfrak{g}$ be a Lie algebra. There is a natural graded Lie superalgebra L associated to $\mathfrak{g}$. L is spanned by a "differential" Q_g of degree 1, a set of "contractions" ι_a of degree -1 and a set of "Lie derivatives" Lie_b of degree 0, $a, b = 1, \cdots, \dim(\mathfrak{g})$, satisfying the following relations

$$Q_g^2 = 0, \quad \iota_a\iota_b + \iota_b\iota_a = 0, \quad Q_g\iota_a + \iota_a Q_g = \mathrm{Lie}_a, \quad \mathrm{Lie}_a\iota_b - \iota_b\mathrm{Lie}_a = f_{ab}^c\iota_c,$$

where f_{ab}^c are the structure constants of $\mathfrak{g}$ (in terms of a given basis). An L-manifold is a graded manifold equipped with an L-action. A function over an L-manifold is said to be horizontal if it is annihilated by all ι_a. A horizontal function is said to be basic if it is annihilated by all Lie_a. Note that for a basic function f, $Q_g f$ is also basic. The cohomology associated to such functions and Q_g can be seen as a generalization of the equivariant cohomology.

When a cohomological field theory has gauge symmetries, i.e., when E is an associated bundle to some principal bundle and the Lagrangian is invariant under the corresponding gauge group, the variational bigraded manifold of E is canonically an L-manifold. In fact, the L-structure and QK-structure are compatible in the sense that

1. Q_g, ι_λ and Lie_λ are of degrees $(0,1)$, $(0,-1)$, $(0,0)$, respectively, where λ is an element in the Lie algebra of the gauge group;

2. Q_g coincides with Q, ι_λ anti-commutes with K;

3. $\mathrm{Lie}_\lambda K - K\mathrm{Lie}_\lambda = 0$ for all horizontal functions over the variational bigraded manifold.

An operator is called gauge invariant if it is basic with respect to the L-structure. Likewise, the vector symmetry K can be used to produce gauge invariant solutions to (1.0.1) when the QK-structure and the L-structure are compatible.

1.1 Main results

The main results of this thesis can be summarized as follows:

- We generalize the theory of supermanifolds to a theory of $\mathcal{I}$-graded manifolds. We present a detailed proof of Batchelor's theorem in this $\mathcal{I}$-graded setting.

- We build a new framework for CohFTs unifying the ones in [BS88; BS89; OSV89; AJ90; Bir+91; Kal93b; Bla93]. In this new framework, we revisit Witten's idea of topological twisting [Wit88] and show that the twisted $N = 2$ super Yang-Mills theory carries a family of QK-structures.

- Using the language of QK_g-manifolds, we generalize the notion of a Chern-Weil homomorphism and the construction of a universal Thom class to the infinite dimensional setting.

- We prove that every (gauge invariant) solution to (1.0.1) is cohomologically a K-sequence. That is, every solution to (1.0.1) is the sum of a K-sequence and an exact sequence. (A sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$ is called exact if $\mathcal{O}^{(p)} = Q\rho^{(p)} + d\rho^{(p-1)}$ for $p \geq 1$ and $\mathcal{O}^{(0)} = Q\rho^{(0)}$.)

1.2 Organization

This thesis is organized as follows:

In Chapter 2, we briefly describe three different approaches to equivariant cohomology, namely, the Weil model, the Cartan model and the Kalkman model. We use the last one to reformulate the Mathai-Quillen construction of a universal Thom class.

In Chapter 3, we review the standard mathematical construction of a BRST complex using a Chevalley-Eilenberg complex and a Koszul complex. We discuss its connection to the Kalkman model of equivariant cohomology.

In Chapter 4, we give a definition of an $\mathcal{I}$-graded manifold, where $\mathcal{I}$ is a countable cancellative commutative semiring. In this generalized setting, we prove the existence and uniqueness of an underlying manifold of an $\mathcal{I}$-graded manifold. We also prove Batchelor's theorem, namely that every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle.

Chapter 5 builds a new geometric framework for CohFTs upon the previous chapters. In this framework, we prove the main result of this thesis, namely that every (gauge invariant) solution to (1.0.1) is a K-sequence up to an exact sequence.

Chapter 6 provides a detailed treatment of supersymmetric field theories in physics. We investigate Witten's idea of topological twistings [Wit88] and its application to super Yang-Mills theories. We show that the twisting of a super Poincaré algebra gives naturally rise to a family of QK-algebras.

Chapter 7 generalizes the Mathai-Quillen formalism described in Chapter 2 to incorporate QK-structures and L-structures. In this new formalism, we discuss various examples such as topological Yang-Mills theory, topological quantum mechanics, and topological sigma model.

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Chapter 2

Equivariant cohomology

2.1 Three different models for equivariant cohomology

Naturally, one should expect that the equivariant cohomology of a G-space X tells us both the topological information of X and the information about the G-action on X. A naive choice is the cohomology of the quotient space X/G. This is not right, since X/G remembers nothing about the stabilizer of the group action at each point $x \in X$ unless the group action is free. The key idea here is to consider the Cartesian product of a "universal G-space" EG and X where

1. EG is contractible, hence does not provide any new topological information;

2. EG has a free G-action. (It follows that G acts also freely on $EG \times X$.)

Definition 2.1.1. A contractible space EG with a free G-action is called a universal G-space. Let X be a G-space. The quotient space $X_G = (EG \times X)/G$ is called a homotopy orbit space of X. In particular, the homotopy orbit space of a one-point space is called a classifying space for G, denoted by BG.

EG is universal in the sense of the following theorem [Hat02].

Theorem 2.1.1. *Let X be a topological space. Let EG be a universal G-space. There exists a bijective correspondence between the set of homotopy classes of maps $f : X \to BG$ and the set of isomorphism classes of principal G-bundles P over X, given by $f \mapsto f^{-1}EG$, where $f^{-1}EG$ is the pullback bundle of EG through f.*

The existence of EG for a topological Lie group G is a standard result due to Milnor [Mil56a; Mil56b]. His construction proceeds as follows. Recall that the join $X * Y$ of two topological spaces X and Y is defined as $X \times Y \times [0,1]/\sim$, where the equivalence relation $\sim$ is defined by $(x_1, y_1, t_1) \sim (x_2, y_2, t_2)$ if and only if $t_1 = t_2 = 0$ and $x_1 = x_2$, or, $t_1 = t_2 = 1$ and $y_1 = y_2$. Let's consider the n-fold join $G * \cdots * G$ of G. It is a $(n-1)$-connected space with a free G-action given by multiplying $g \in G$ simultaneously from the right to all of its factors. Together with the canonical inclusion

$$G * \cdots * G \overset{\iota_0}{\hookrightarrow} (G * \cdots * G) \times G \times [0,1] \to (G * \cdots * G) * G$$

where ι_0 identifies $G * \cdots * G$ with $G * \cdots * G \times \{\mathrm{Id}\} \times \{0\}$, the n-fold joins of G form a direct system. By construction, the direct limit of this system is a weakly contractible CW-complex and has a free G-action. It is, therefore, a universal G-space. As an example, one can consider $G = \mathrm{U}(1)$. The n-fold join of G is the sphere S^{2n+1}. $EG = \varinjlim S^{2n+1} = S^\infty$. This construction of EG is by no means unique (consider $G = \mathbb{Z}$ with $EG = \mathbb{R}$). However, it is unique up to homotopy as a result of Theorem 2.1.1.

Corollary 2.1.1. *Let EG_1 and EG_2 be two universal G-spaces. There exists G-equivariant maps $\phi : EG_1 \to EG_2$ and $\psi : EG_2 \to EG_1$ such that $\phi \circ \psi$ is homotopic to id_{EG_2}, and $\psi \circ \phi$ is homotopic to id_{EG_1}.*

Proof. By Theorem 2.1.1, the principal G-bundle $EG_1 \to BG_1$ corresponds to a map $f : BG_1 \to BG_2$ with $f^{-1}EG_2 \cong EG_1$. In other words, there exists a bundle morphism $\phi : EG_1 \to EG_2$ covering f which fiber-wisely is a homeomorphism. Similarly, we obtain a morphism of the other direction $\psi : EG_2 \to EG_1$. Since the underlying maps of both id_{EG_2} and $\phi \circ \psi$ is homotopic to each other by Theorem 2.1.1, we conclude that $\phi \circ \psi$ itself is homotopic to id_{EG_2}. Similarly, $\psi \circ \phi$ is homotopic to id_{EG_1} □

Definition 2.1.2. The equivariant cohomology of a G-space X, denoted by $H_G(X)$, is defined as the singular cohomology of the homotopy orbit space X_G.

Remark 2.1.1. Note that if the G-action on X is free, then $H_G(X) \cong H(X/G)$.

Let's switch to the smooth category and assume G to be compact. By the quotient manifold theorem, the homotopy orbit space X_G is also smooth and we can replace the singular cohomology of it in Definition 2.1.2 by its de Rham cohomology.

Remark 2.1.2. The universal G-space EG is usually obtained as a direct limit of some direct system of finite dimensional manifold, as is in the construction of Milnor. The de Rham complex of EG is then defined as the inverse limit of the induced inverse system of de Rham complexes over such finite dimensional manifolds.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. To each $\xi \in \mathfrak{g}$ we can associate a vector field v_ξ on a G-manifold, which again induces a contraction ι_ξ and a Lie derivative Lie_ξ on the de Rham complex of the G-manifold. Let d denote the de Rham differential. Fix a basis $\{\xi_a\}$ of $\mathfrak{g}$. Let ι_a and Lie_a denote the contraction and Lie derivative assoicated to ξ_a. d, ι_a, Lie_a satisfy the following relations

$$[\mathrm{Lie}_a, \mathrm{Lie}_b] = f^c_{ab}\mathrm{Lie}_c, \quad [\mathrm{Lie}_a, \iota_b] = f^c_{ab}\iota_c, \quad [\mathrm{Lie}_a, d] = 0, \tag{2.1.1}$$

$$\{d, d\} = 0, \quad \{\iota_a, \iota_b\} = 0, \quad \{d, \iota_a\} = \mathrm{Lie}_a. \tag{2.1.2}$$

Recall that a super Lie algebra $L = L_{even} \oplus L_{odd}$ is specified by [Lei80]

1. a Lie algebra L_{even};
2. an L_{even}-module L_{odd} and a bilinear pairing $[\cdot,\cdot]$ such that $[x,\cdot]$ is the action of $x \in L_{even}$ on L_{odd} and $[\cdot,x] := -[x,\cdot]$;
3. a symmetric bilinear paring $\{\cdot,\cdot\} : L_{odd} \times L_{odd} \to L_{even}$ that is a homomorphism of L_{even}-modules and satisfies the Jacobi identity

$$\{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} = 0$$

for $x, y, z \in L_{odd}$.

In this case, L_{even} is spanned by Lie_a. L_{odd} is spanned by d and ι_a. We have $L_{even} = \mathfrak{g}$ and $L_{odd} = \mathbb{R} \oplus \mathfrak{g}$. Elements of L_{even} act on $\mathbb{R}$ trivially and act on $\mathfrak{g}$ via the adjoint representation. The symmetric pairing is given by (2.1.2). Moreover, L is a graded Lie algebra by assigning degrees 0, 1 and -1 to Lie_a, d and ι_a, respectively. We write $L = L_{-1} \oplus L_0 \oplus L_1$ to emphasize this fact.

Remark 2.1.3. In fact, the Lie algebra actions of L_{even} on L integrates to a global Lie group action $\sigma : G \to \mathrm{Aut}(L)$ with $\sigma|_{L_{even}} = \mathrm{Ad}$, the adjoint action of G on $\mathfrak{g}$. (G, L) is called a super Harish-Chandra pair and is equivalent to a super Lie group [CCF11], which we denote by $G^\star$.

Definition 2.1.3. A $G^\star$-module is a graded vector space A together with two representations $\rho : G \to \mathrm{GL}(A)$ and $\tau : L \to \mathrm{gl}(A)$ which are consistent in the sense that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \rho(\exp(t(\cdot))) &= \tau|_{L_{even}}(\cdot), \\ \rho(g)\tau(\gamma)\rho(g^{-1}) &= \tau(\sigma(g)(\gamma)), \end{aligned}$$

for all $g \in G$ and $\gamma \in L$, where $\exp : \mathfrak{g} \to G$ is the exponential map, $\mathrm{gl}(A)$ is the set of linear maps $A \to A$, and $\mathrm{GL}(A)$ is the set of invertible linear maps $A \to A$ of degree 0. A morphism between $G^\star$-modules is a G-equivariant linear map of degree 0 which commutes with τ.

We often write γa directly to denote the action of $\gamma \in L$ on $a \in A$.

Definition 2.1.4. Let A be a $G^\star$-module. An element $\alpha \in A$ is horizontal if $\iota_a \alpha = 0$. A horizontal element α is basic if in addition $\mathrm{Lie}_a \alpha = 0$.

Let A_{hor} and A_{bas} denote the sub-module of horizontal and basic elements in A, respectively. It is easy to see that for $\alpha \in A_{bas}$, $d\alpha$ is also in A_{bas} because $\iota_a d\alpha = \mathrm{Lie}_a \alpha - d\iota_a \alpha = 0$ and $\mathrm{Lie}_a d\alpha = d\mathrm{Lie}_a \alpha = 0$. We use $H(A)$ to denote the cohomology of (A, d) and $H_{bas}(A)$ to denote the cohomology of (A_{bas}, d).

Definition 2.1.5. Let A and B be two $G^\star$-modules. A semi-homotopy is a linear map $K : A \to B$ of degree -1 which satisfies

$$\iota_a K + K\iota_a = 0, \tag{2.1.3}$$

and

$$B_{hor} \subset \ker(\mathrm{Lie}_a K - K\mathrm{Lie}_a).$$

A semi-homotopy K is said to be a homotopy if $(\mathrm{Lie}_a K - K\mathrm{Lie}_a) = 0$. Two morphisms τ_0 and $\tau_1 : A \to B$ are (semi-)chain homotopic if they are equal up to a (semi-)homotopy, i.e., if

$$\tau_1 - \tau_2 = dK + Kd.$$

Proposition 2.1.1. *Let τ_0 and $\tau_1 : A \to B$ be two morphisms between $G^\star$-modules. They induces the same morphism $H(A) \to H(B)$ if they are chain homotopic. They induces the same morphism $H_{bas}(A) \to H_{bas}(B)$ if they are chain semi-homotopic.*

Proof. Let $L = dK + Kd$ and $P = \mathrm{Lie}_a K - K\mathrm{Lie}_a$. It is not hard to show that $\iota_a L - L\iota_a = P$ and $\mathrm{Lie}_a L - L\mathrm{Lie}_a = dP - Pd$. If K is a homotopy, then $P = 0$ and L becomes a morphism of $G^\star$-modules, hence also a morphism $H(A) \to H(B)$. If K is a semi-homotopy, then L still commutes with ι_a and Lie_a when restricted to the basic parts of A and B, hence L becomes a morphism $H_{bas}(A) \to H_{bas}(B)$. The rest of the proof follows directly from the standard arguments of homological algebras. □

Recall that a graded algebra A is a graded vector space together with a multiplication satisfying $A_i A_j \subset A_{i+j}$ and an identity $1 \in A_0$. A is said to be commutative if $ab = (-1)^{d(a)d(b)} ba$, where $d(a)$ is the degree of $a \in A$. $D \in \mathrm{gl}(A)$ is called a derivation if $D(ab) = D(a)b + (-1)^{d(D)d(a)} aD(b)$. We use $\mathrm{Der}(A)$ to denote the set of derivations of A.

Definition 2.1.6. A $G^\star$-algebra is a $G^\star$-module A where A is a commutative graded algebra, ρ takes values in the automorphism group $\mathrm{Aut}(A)$ of A, and τ takes values in the derivation algebra $\mathrm{Der}(A)$ of A. A morphism between $G^\star$-algebras is a $G^\star$-module morphism which preserves the algebraic structure on A.

Definition 2.1.7. Let A and B be $G^\star$-algebras with a morphism $\phi : A \to B$. A semi-homotopy $K : A \to B$ is said to be a semi-homotopy relative to ϕ if

$$K(xy) = K(x)\phi(y) + (-1)^{d(x)}\phi(x)K(y)$$

for all $x, y \in A$.

Lemma 2.1.1. *If A is finitely generated, then a semi-homotopy K relative to ϕ is determined uniquely by its action on the generators of A.*

Proof. This follows directly from the fact that ι_a and Lie_a are derivations of the relevant $G^\star$-algebras, and the fact that ϕ commutes with both ι_a and Lie_a. □

The de Rham complex of a G-manifold is canonically a $G^\star$-algebra. In particular, $\Omega(EG)$ is a $G^\star$-algebra. The freeness of the G-action on EG is translated into the following definition [BGS13].

Definition 2.1.8. A $G^\star$-algebra E is said to be of type (C) if there exists a G-invariant free submodule C of the A_0-module A_1 such that the contractions

$$\iota_a : A_1 \to A_0$$

form a basis of C^*, the dual module of C over A_0.

Remark 2.1.4. C be can be seen as an algebraic analogue of the dual of the vertical bundle VP of a principal G-bundle P.

Example 2.1.1. The de Rham complex of a principal G-bundle is of type (C).

A $G^\star$-algebra E is of type (C) if and only if there are elements $\theta^a \in E_1$ such that

$$\iota_a\theta^b = \delta_a^b, \tag{2.1.4}$$

$$\mathrm{Lie}_a\theta^b = -f_{ac}^b\theta^c. \tag{2.1.5}$$

It follows from (2.1.4) and (2.1.5) that there exists elements $\phi^a \in E_2$ satisfying

$$d\theta^a = \phi^a - \frac{1}{2}f_{bc}^a\theta^b\theta^c. \tag{2.1.6}$$

The actions of d, ι_a and Lie_a on ϕ^b are uniquely determined by (2.1.4) to (2.1.6).

Definition 2.1.9. The connection of a $G^\star$-algebra E is of type (C) is defined as

$$\theta = \theta^a \otimes \xi_a \in E^1 \otimes \mathfrak{g}.$$

The curvature (of θ) of E is defined as

$$\phi = \phi^a \otimes \xi_a \in E^2 \otimes \mathfrak{g}.$$

Remark 2.1.5. It follows that $\phi = d\theta + \frac{1}{2}[\theta, \theta]$ and satisfies the second Bianchi identity $d\phi + [\theta, \phi] = 0$.

It is easy to show that the tensor product $A \otimes B$ of two $G^\star$-algebra is again a $G^\star$ algebra, and that $A \otimes B$ is of type (C) if B is of type (C).

Definition 2.1.10. The equivariant cohomology of a $G^\star$-algebra A, denoted by $H_G(A)$, is defined as $H_{bas}(A \otimes E)$, where E is an acyclic $G^\star$-algebra of type (C).

It is shown by Guillemin and Sternberg that Definition 2.1.10 does not depend on the choice of E (see Section 4.4 in [BGS13]), and that the two notions of equivariant cohomology coincide, i.e.,

Theorem 2.1.2 (Theorem 2.5.1 in [BGS13])**.** *Let X be a G-manifold. $H_G(X) \cong H_G(\Omega(X))$.*

There is a universal object in the category of (acyclic) $G^\star$-algebras of type (C).

Definition 2.1.11. The Weil Algebra of $\mathfrak{g}$ is a $G^\star$-algebra of type (C) with underlying commutative graded algebra

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes \mathrm{S}(\mathfrak{g}^*),$$

where Λ and S are the exterior power and the symmetric power, respectively. $W(\mathfrak{g})$ is graded by assigning degree 1 to elements of $\mathfrak{g}^* \subset \Lambda(\mathfrak{g}^*)$ and degree 2 to elements of $\mathfrak{g}^* \subset \mathrm{S}(\mathfrak{g}^*)$. The action ρ of G on $W(\mathfrak{g})$ is induced by its coadjoint action on $\mathfrak{g}$. The action τ of L on $W(\mathfrak{g})$ is specified by (2.1.4) to (2.1.6) and

$$\iota_a \phi^b = 0, \tag{2.1.7}$$

$$d\phi^a = f^a_{bc}\phi^b\theta^c, \tag{2.1.8}$$

$$\mathrm{Lie}_a \phi^b = -f^b_{ac}\phi^c, \tag{2.1.9}$$

where $\theta^a = \xi^a \otimes 1$, $\phi^a = 1 \otimes \xi^a$, $\{\xi^a\}$ is the dual basis of $\mathfrak{g}^*$. The connection (curvature) of $W(\mathfrak{g})$ is also referred to as the universal connection (curvature).

Proposition 2.1.2 (Theorem 3.2.1 in [BGS13])**.** *$(W(\mathfrak{g}), d)$ is acyclic.*

Note that $W(\mathfrak{g})_{bas} = \mathrm{S}(\mathfrak{g}^*)^G$, i.e., the space of G-invariant polynomials on $\mathfrak{g}$, and that d restricted to $W(\mathfrak{g})_{bas}$ is zero.

Proposition 2.1.3. *$H_{bas}(W(\mathfrak{g})) = \mathrm{S}(\mathfrak{g}^*)^G$.*

Let E be a $G^\star$-algebra of type (C). The connection and curvature on E determine maps

$$\mathfrak{g}^* \to E^1, \quad \mathfrak{g}^* \to E_2$$

which induce a homomorphism of $G^\star$-algebras

$$\phi_W : W(\mathfrak{g}) \to E,$$

which sends the universal connection and curvature of $W(\mathfrak{g})$ to the connection and curvature on E.

Theorem 2.1.3 (Theorem 3.3.1 in [BGS13])**.** *ϕ_W induces a morphism $\phi_{CW} : A_{bas} \to E_{bas}$, which again induces a morphism $H_{bas}(A) \to H_{bas}(E)$ which does not depend on the choice of connections and curvatures on E.*

Let P be a principal G-bundle over M equipped with a connection 1-form A and curvature 2-form F. The homomorphism ϕ_W induced by A and F is nothing but the Weil homomorphism which sends the universal connection θ and curvature ϕ to A and F, respectively. The homomorphism $\phi_{CW} : W(\mathfrak{g})_{bas} \cong \mathrm{S}(\mathfrak{g}^*)^G \to \Omega(M) \cong \Omega_{bas}(P)$ is the well-known Chern-Weil homomorphism.

2.1.1 Weil model

Let's consider the tensor product $W(\mathfrak{g}) \otimes \Omega(X)$. It has canonically a $G^\star$-algebra structure where the contractions, the Lie derivatives, and the differential are

$$\iota_a \otimes 1 + 1 \otimes \iota_a, \quad \mathrm{Lie}_a \otimes 1 + 1 \otimes \mathrm{Lie}_a, \quad d \otimes 1 + 1 \otimes d.$$

Definition 2.1.12. Let $\Omega_G(X)$ denote the basic part of $W(\mathfrak{g}) \otimes \Omega(X)$. Let d_W denote the differential on $\Omega_G(X)$ induced from the differential on $W(\mathfrak{g}) \otimes \Omega(X)$. $(\Omega_G(X), d_W)$ is called the Weil Model for the equivariant cohomology of a G-manifold X. d_W is called the Weil differential.

Note that $\Omega(P) \otimes \Omega(X) \cong \Omega(P \times X)$. The Weil homomorphism induces a homomorphism between $\Omega_G(X)$ and $\Omega(P \times_G X)$, where $P \times_G X$ is the associated bundle to P with fiber X, through the commutative diagram

$$\begin{array}{ccc} W(\mathfrak{g}) \otimes \Omega(X) & \xrightarrow{\phi_W \otimes \mathrm{id}} & \Omega(P \times X) \\ \uparrow & & \uparrow \\ \Omega_G(X) & \xrightarrow{\phi_{CW}} & \Omega(P \times_G X) \end{array}$$

With a slight abuse of notation, we denote this homomorphism again by ϕ_{CW}. ϕ_{CW} again induces a homomorphism of cohomologies

$$H_G(X) \to H(P \times_G X)$$

which does not depend on the choice of the connection on P.

2.1.2 Kalkman model

Definition 2.1.13. The automorphism map $j = \exp(-\theta^a \otimes \iota_a)$ of $W(\mathrm{g}) \otimes \Omega(X)$ is called the Mathai-Quillen map. The differential $d_K = j \circ d_W \circ j^{-1}$ is called the Kalkman differential. $(W(\mathrm{g}) \otimes \Omega(X), d_K)$ is called the Kalkman model of the equivariant cohomology of a G-manifold X.

Proposition 2.1.4. $d_K = d_W + \theta^a \otimes \mathrm{Lie}_a - \phi^a \otimes \iota_a$.

Proof. We need to show that $j \circ d_K = d_W \circ j$. Note that

$$\begin{aligned} d_W(\theta^a \otimes \iota_a) &= d_W\theta^a \otimes \iota_a - \theta^a \otimes d_W\iota_a \\ &= (d_W\theta^a) \otimes \iota_a - \theta^a d_W \otimes \iota_a - \theta^a \otimes L_a + \theta^a \otimes \iota_a d_W \\ &= -\frac{1}{2} f^a_{bc}\theta^b\theta^c \otimes \iota_a + \phi^a \otimes \iota_a - \theta^a \otimes L_a + (\theta^a \otimes \iota_a) d_W. \end{aligned}$$

We get $[d_W, (\theta^a \otimes \iota_a)] = -\frac{1}{2} f^a_{bc}\theta^b\theta^c \otimes \iota_a + \phi^a \otimes \iota_a - \theta^a \otimes L_a$. The next step is to compute $[[d_W, (\theta^a \otimes \iota_a)], (\theta^b \otimes \iota_b)]$. We have

$$[d_W, (\theta^a \otimes \iota_a)](\theta^d \otimes \iota_d) = \frac{1}{2} f^a_{bc}\theta^b\theta^c\theta^d \otimes \iota_a\iota_d - \phi^a\theta^d \otimes \iota_a\iota_d - \theta^a\theta^d \otimes L_a\iota_d$$

and

$$(\theta^d \otimes \iota_d)[d_W, (\theta^a \otimes \iota_a)] = -\theta^d \frac{1}{2} f^a_{bc}\theta^b\theta^c \otimes \iota_d\iota_a + \theta^d\phi^a \otimes \iota_d\iota_a + \theta^d\theta^a \otimes \iota_d L_a.$$

Hence,

$$[[d_W, (\theta^a \otimes \iota_a)], (\theta^d \otimes \iota_d)] = -\theta^a\theta^d \otimes [L_a, \iota_d] = -\theta^a\theta^d \otimes f^e_{ad}\iota_e.$$

It follows that

$$[[[d_W, (\theta^a \otimes \iota_a)], (\theta^d \otimes \iota_d)], (\theta^f \otimes \iota_f)] = [-\theta^a\theta^d \otimes f^e_{ad}\iota_e, (\theta^f \otimes \iota_f)] = 0.$$

Using $j = \exp(-\theta^a \otimes \iota_a) = \prod_a (1 - \theta^a \otimes \iota_a)$, we finally get

$$\begin{aligned} d_W j &= \sum_a (1 - \theta^1 \otimes \iota_1) \cdots (\frac{1}{2} f^a_{bc}\theta^b\theta^c \otimes \iota_a - \phi^a \otimes \iota_a + \theta^a \otimes L_a) \cdots (1 - \theta^d \otimes \iota_d) + j d_W \\ &= j(\sum_a (\frac{1}{2} f^a_{bc}\theta^b\theta^c \otimes \iota_a - \phi^a \otimes \iota_a + \theta^a \otimes L_a) - \sum_{a,b<c} f^a_{bc}\theta^b\theta^c \otimes \iota_a + d_W) \\ &= j(\sum_a (-\phi^a \otimes \iota_a + \theta^a \otimes L_a) + d_W) \\ &= j d_K. \end{aligned}$$

We use $f^c_{ab} = -f^c_{ba}$ in the second last step. □

2.1.3 Cartan model

Similarly, one can show that

$$\iota_a \otimes 1 = j \circ (\iota_a \otimes 1 + 1 \otimes \iota_a) \circ j^{-1}$$

and

$$(\mathrm{Lie}_a \otimes 1 + 1 \otimes \mathrm{Lie}_a) = j \circ (\mathrm{Lie}_a \otimes 1 + 1 \otimes \mathrm{Lie}_a) \circ j^{-1}.$$

Consequently, the basic part of $W(\mathrm{g}) \otimes \Omega(X)$ in the Kalkman model of X is $(S(\mathfrak{g}^*) \otimes \Omega(X))^G$, and the restriction of d_K to $(S(\mathfrak{g}^*) \otimes \Omega(X))^G$, denoted by d_C, takes the form $d_C = d \otimes 1 - \phi^a \otimes \iota_a$.

Definition 2.1.14. $((S(\mathfrak{g}^*) \otimes \Omega(X))^G, d_C)$ is called the Cartan Model of a G-manifold X. d_C is called the Cartan differential.

2.2 Mathai-Quillen formalism

Let $G = SO(n)$, $n = 2m$. Let $\mathfrak{g}$ be the Lie algebra of G. Let ρ be the standard representation of G on $V = \mathbb{R}^n$. We also use V to denote the n-dimensional translation group and its Lie algebra. Let θ and ϕ be the universal connection and curvature of the Weil algebra $W(\mathfrak{g})$. Let w^i and b_i denote the coordinate functions of V and V^* respectively. Let χ_i denote the odd coordinate functions of ΠV^*. In [MQ86], Mathai and Quillen defined the following element[1]

$$U = (2\pi)^{-n} \int d\chi db \exp\left(-b^t(b/2 + iw) + \frac{1}{2}\chi^t\phi\chi + i(dw + \theta w)^t\chi\right) \tag{2.2.1}$$

of degree n in $\Omega_G(V)$, where $\phi\chi = \phi^a \otimes \rho(\xi_a)\chi$, $\theta w = \theta^a \otimes \rho(\xi_a)w$, $\int d\chi db$ is the Berezin integral over the even variables b_i and the odd variables χ_i.

Proposition 2.2.1. *U is closed in $\Omega_G(V)$.*

To prove this proposition, let

$$L = b^t(b/2 + iw) - \frac{1}{2}\chi^t\phi\chi - i\chi^t(dw + \theta w). \tag{2.2.2}$$

L can be seen as an element in $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly} \otimes \Omega(V)$, where $\Omega(V^*)_{poly} = \mathrm{S}(V^*) \otimes \Lambda(V^*)$. Let $G \ltimes V$ be the semi-direct product of G and V induced by ρ. Let $G \ltimes V$ act on V through ρ only, i.e., we require that the translation part of the group acts trivially. Let $\mathfrak{g} \ltimes V$ denote its Lie algebra. It is easy to see that $W(\mathfrak{g} \ltimes V)$ is isomorphic to $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly}$ as graded commutative algebras. Moreover,

Lemma 2.2.1. *The Weil model $(W(\mathfrak{g} \ltimes V) \otimes \Omega(V), d_W)$ is isomorphic to the differential graded algebra $(W(\mathfrak{g}) \otimes \Omega(V^*)_{poly} \otimes \Omega(V), s)$. The differential s takes the form*

$$s = d_K \otimes 1 + 1 \otimes d,$$

where $d_K = d \otimes 1 + 1 \otimes \delta_K + \theta^a \otimes \mathrm{Lie}_a - \phi^a \otimes \iota_a$ is the Kalkman differential of $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly}$, with δ_K denoting the Koszul differential on $\Omega(V^*)_{poly}$.

[1] The imaginary unit i is introduced so that we will get an integrable Gaussian function of w after integrating out the "auxiliary" field b.

Proof. We need to show that the differential d on $W(\mathfrak{g} \ltimes V)$ is equivalent to the Kalkman differential d_K on $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly}$. Let ξ_a and t^i be bases of $\mathfrak{g}$ and V, respectively. We can write $\rho(\xi_a)t^i = \rho^i_{aj}t^j$. The Lie bracket of $\mathfrak{g} \ltimes V$ is given by

$$[(\xi_a, t^i), (\xi_b, t^j)] = ([\xi_a, \xi_b], \rho(\xi_a)t^j - \rho(\xi_b)t^i) = (f^c_{ab}\xi_c, \rho^j_{ak}t^k - \rho^i_{bk}t^k).$$

It then follows from (2.1.8) that

$$d\theta^a = \phi^a - \frac{1}{2}f^a_{bc}\theta^b\theta^c, \quad d\phi^a = -f^a_{bc}\theta^b\phi^c,$$
$$d\chi^i = b^i - \rho^i_{aj}\theta^a\chi^j, \quad db^i = \rho^i_{aj}\phi^a\chi^j - \rho^i_{aj}\theta^a b^j.$$

On the other hand, one easily can check that

$$\mathrm{Lie}_a b^i = -\rho^i_{aj}b^j, \quad \mathrm{Lie}_a\chi^i = -\rho^i_{aj}\chi^j,$$
$$\iota_a b^i = -\rho^i_{aj}\chi^j, \quad \iota_a\chi^i = 0.$$

The rest of the proof is straightforward. □

In fact, $W(\mathfrak{g} \ltimes V) \otimes \Omega(V)$, or equivalently $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly} \otimes \Omega(V)$, is also a $G^\star$-algebra with the standard G-action on V and its dual action on V^*, and

$$\mathrm{Lie}_a = (\mathrm{Lie}_a \otimes 1 + 1 \otimes \mathrm{Lie}_a) \otimes 1 + 1 \otimes 1 \otimes \mathrm{Lie}_a, \quad \iota_a = (\iota_a \otimes 1) \otimes 1 + 1 \otimes 1 \otimes \iota_a.$$

For simplicity, we omit the indices of the coordinate functions. The action of s on coordinate functions can then be written as

$$s\theta = \phi - \theta\theta, \quad s\phi = [\phi, \theta], \tag{2.2.3}$$
$$sw = dw, \quad sb = -\theta b + \phi\chi, \tag{2.2.4}$$
$$sdw = 0, \quad s\chi = b - \theta\chi. \tag{2.2.5}$$

Remark 2.2.1. It is always fun to check $s^2 = 0$ by direct computations. The non-trivial ones are $s^2 b = -(s\theta)b + \theta(sb) + (s\phi)\chi + \phi(s\chi) = -(\phi - \theta\theta)b + \theta(-\theta b + \phi\chi) + ([\phi, \theta])\chi + \phi(b - \theta\chi) = 0$, and $s^2\chi = sb - s(\theta)\chi + \theta(s\chi) = (-\theta b + \phi\chi) - (\phi - \theta\theta)\chi + \theta(b - \theta\chi) = 0$.

Lemma 2.2.2. *L is exact in $(W(\mathfrak{g} \ltimes V) \otimes \Omega(V))^0_{bas}$.*

Proof. The exactness of L follows from direct computations.

$$\begin{aligned} s\left(\chi^t(iw + b/2)\right) &= (b - \theta\chi)^t(iw + b/2) - \chi^t(idw + (-\theta b + \phi\chi)) \\ &= b^t(iw + b/2) - (-\chi^t\theta^t)(iw + b/2) - \chi^t(idw - \theta b/2) - \frac{1}{2}\chi^t\phi\chi \\ &= b^t(iw + b/2) - \frac{1}{2}\chi^t\phi\chi - \chi^t((i\theta w + \theta b/2) + (idw - \theta b/2)) \\ &= b^t(iw + b/2) - \frac{1}{2}\chi^t\phi\chi - \chi^t i(dw + \theta w) = L. \end{aligned}$$

We use the skew-symmetric property of θ in the third step. L is basic because $\chi^t(iw+b/2)$ is $\mathfrak{g}$-invariant and does not contain θ and dw. $\square$

Let $\mathcal{S}(V^*)$ denote the space of Schwartz functions over V^*. Let $\Omega(V^*)_S$ denote $\mathcal{S}(V^*)\otimes\Lambda(V^*)$. Note that $\Omega(V^*)_S$ is a $G^\star$-algebra as the subalgebra of $\Omega(V^*)$.

Definition 2.2.1. For every $\alpha\in\Omega(V^*)_S$, the super Fourier transform $\mathcal{F}$ of α is defined by

$$\mathcal{F}(\alpha)=\int_{V^*}\alpha\exp(-i(b^tw-\chi^t dw))\in\Omega(V).$$

Lemma 2.2.3. *$\mathcal{F}\circ\delta_K=d\circ\mathcal{F}$. Likewise, we also have $\mathcal{F}\circ\iota_a=\iota_a\circ\mathcal{F}$.*

Proof. It suffices to consider $f\otimes\beta$ where $f\in C^\infty(V^*)$ and $\beta\in\Lambda^p(V^*)$ for some p. We have

$$\begin{aligned}d\mathcal{F}(f\otimes\beta)&=d(\int_{V^*}f\exp(-i(b^tw))\otimes\beta\exp(i(\chi^t dw)))\\&=\int_{V^*}f\exp(-i(b^tw))(-ib_j)\otimes dw^j\beta\exp(-i(\chi^t dw))\\&=-(-1)^p\int_{V^*}f\exp(-i(b^tw))\otimes\beta\delta_K(\exp(-i(\chi^t dw)))\\&=\mathcal{F}(\delta_K(f\otimes\beta)).\end{aligned}$$

And

$$\begin{aligned}\iota_a\mathcal{F}(f\otimes\beta)&=\iota_a(\int_{V^*}f\exp(-i(b^tw))\otimes\beta\exp(i(\chi^t dw)))\\&=(-1)^p\int_{V^*}f\exp(-i(b^tw))(i\mathrm{Lie}_a(w^j))\otimes\beta\chi_j\exp(i(\chi^t dw))\\&=-(-1)^p\int_{V^*}f\exp(-i(b^tw))(iw^j)\otimes\beta\mathrm{Lie}_a(\chi_j)\exp(i(\chi^t dw))\\&=-\int_{V^*}f\iota_a(\exp(-i(b^tw)))\otimes\beta\exp(-i(\chi^t dw))\\&=\mathcal{F}(\iota_a(f\otimes\beta)).\end{aligned}$$

$\square$

Corollary 2.2.1. *$\mathcal{F}$ is morphism of $G^\star$-algebras.*

Proof. By Lemma 2.2.3, $\mathcal{F}$ commutes with the L-action. Note that $\exp(-i(b^tw-\chi^t dw))$ is invariant under the G-action. $\square$

Now, apply the Mathai-Quillen map $j = \exp(-\theta^a \otimes \iota_a)$ to $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly}$. This is equivalent to a change of coordinates which sends b to $b - \theta\chi$. In the new coordinates, we have $s = d \otimes 1 \otimes 1 + 1 \otimes \delta_K \otimes 1 + 1 \otimes 1 \otimes d$, and $L = s\alpha + i(b^t w - \chi^t dw)$, where $\alpha = \chi^t(b - \theta\chi)/2$. It is easy to show that α is a basic element in $W(\mathfrak{g}) \otimes \Omega(V^*)_{poly}$. Moreover, the element $\exp(-s\alpha)$ is a closed basic element in $W(\mathfrak{g}) \otimes \Omega(V^*)_S$ due to the Gaussian factor $\exp(-b^t b/2)$. Proposition 2.2.1 is then proved by observing that

$$\int d\chi db \exp(-L) = \mathcal{F}(\exp(-s\alpha)).$$

Note that integrating out b and χ will give us a factor $(2\pi)^m$. The component of U with top de Rham degree is $(2\pi)^{-m} \exp(-\frac{1}{2}w^2) dw^1 \dots dw^n$. It follows that $\int_V U = 1$. Let P be a principal G-bundle over a manifold Σ. Let A be a connection 1-form on P. Let E be an associated vector bundle to P of rank $2m$ equipped with a metric $(\cdot, \cdot)$ and a metric connection ∇ induced by A.

Theorem 2.2.1 (Theorem 4.10 in [MQ86])**.** *U is a universal Thom form in the sense that for any such E, the Chern-Weil homomorphism ϕ_{CW} sends U to a form representing the Thom class of E.*

Remark 2.2.2. We can also consider the Kalkman model of $W(\mathfrak{g} \ltimes V) \otimes \Omega(V)$. The differential s is locally given by

$$sw = dw - \theta w, \quad sb = -\theta b + \phi\chi, \tag{2.2.6}$$

$$sdw = -\theta dw + \phi w, \quad s\chi = b - \theta\chi. \tag{2.2.7}$$

This is not the right choice for constructing a universal Thom class, as s restricted to $\Omega(V)$ is not the de Rham differential. However, we will need this kind of differentials later to construct cohomological field theories with gauge symmetries.

Let v be a section of E, we can also obtain a representative for the Euler class $e_\nabla(E)$ of E by setting

$$e_\nabla(E) = v^*\phi_{CW}(U) = (2\pi)^{-n} \int d\chi db \exp\left(-L\right),$$

where $L = (b, b/2 + iv) - \frac{1}{2}(\chi, R\chi) - i\chi(\nabla v)$. Let x^μ denote the coordinate functions of Σ. Identifying dx^μ with η^μ, we have

$$L = (b, b/2 + iv) - i\chi(\nabla_\eta v) - \frac{1}{4}(\chi, R(\eta, \eta)\chi). \tag{2.2.8}$$

Let us take $E = T\Sigma$, $(\cdot, \cdot)$ to be a Riemannian metric g and ∇ to be the Levi-Civita connection determined by g. We have

$$sx^\mu = \eta^\mu, \quad sb_\mu = \Gamma^\nu_{\rho\mu}\eta^\rho b_\nu - \frac{1}{2}R^\nu_{\mu\rho\sigma}\eta^\rho\eta^\sigma\chi_\nu \tag{2.2.9}$$

$$s\eta^\mu = 0, \quad s\chi_\mu = b_\mu + \Gamma^\nu_{\rho\mu},\eta^\rho\chi_\nu, \tag{2.2.10}$$

where $\Gamma^{\nu}_{\rho\mu}$ is the Christoffel symbol and $R^{\nu}_{\mu\rho\sigma}$ is the Riemann curvature tensor. (2.2.8) together with the supersymmetry transformations (2.2.9) and (2.2.10) give us a 0-dimensional supersymmetric theory.

Remark 2.2.3. The Mathai-Quillen map j of $W(\mathfrak{g}) \otimes \Omega(V^*)$ induces a change of coordinates, namely,

$$b_\mu \to b_\mu + \Gamma^{\nu}_{\rho\mu}\eta^\rho\chi_\nu.$$

The differential s in the new coordinates takes the form

$$\begin{aligned} sx^\mu &= \eta^\mu, \quad sb_\mu = 0, \\ s\eta^\mu &= 0, \quad s\chi_\mu = b_\mu, \end{aligned}$$

which is exactly the BRST differential s appearing in [BS89]. The price one pays for this simplification is that L will no longer be covariant unless b is integrated out.

Chapter 3

BRST cohomology

3.1 Chevalley-Eilenberg complex

Since every compact simply connected Lie group G is uniquely determined by its Lie algebra $\mathfrak{g}$, it should be possible to obtain topological information of G from $\mathfrak{g}$. In fact, one can define a cohomology for $\mathfrak{g}$ (with coefficients in $\mathbb{R}$) that is isomorphic to the de Rham cohomology of G. Let X be a left invariant vector field over G. A differential form ω on G is said to be left invariant if $\mathrm{Lie}_X\omega = 0$. Let $\Omega_L(G)$ denote the set of left invariant differential forms on G. $\Omega_L(G)$ is stable under the de Rham differential d, hence a subcomplex of the de Rham complex $\Omega_{dR}(G)$.

Theorem 3.1.1. *Let G be a compact connected Lie group. The inclusion $\iota : \Omega_L(G) \hookrightarrow \Omega_{dR}(G)$ induces an isomorphism $\iota_* : H_L(G) \to H_{dR}(G)$.*

The proof of this theorem can be found in [CE48]. The key idea is to consider the following "averaging" operator

$$\begin{aligned} I : \Omega^p(G) &\to \Omega^p_L(G) \\ \omega &\mapsto \int_G d\mu L_g^*\omega, \end{aligned}$$

where $L_g : G \to G$ is the left multiplication map induced by $g \in G$ and $d\mu$ is the normalized Haar measure of G. One can show that I is well-defined and commutes with d. The injectivity of ι^* follows from that $\iota \circ I = \mathrm{id}$. The surjectivity of ι^* follows from that $\int_Z(\omega - I(\omega)) = 0$ for any p-cycle Z in G.

Since every left invariant vector field over G is uniquely determined by its value at the identity element of G, $\Lambda(\mathfrak{g}^*) \cong \Omega_L(G)$. Recall that for a differential k-form ω and vector fields $X_0, \cdots, X_k$, we have

$$(d\omega)(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega(X_0, \cdots, \hat{X}_i, \cdots, X_k) + \tag{3.1.1}$$

$$\sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k). \tag{3.1.2}$$

We arrive at the following definition.

Definition 3.1.1. The Chevalley-Eilenberg complex of a Lie algebra $\mathfrak{g}$ is the differential graded algebra $\Lambda(\mathfrak{g}^*) \otimes E$[1], where E is a left $\mathfrak{g}$-module, equipped with the differential d_{CE} defined by

$$(d_{CE}\omega)(\xi_0, \cdots, \xi_k) = \sum_{i=0}^{k} (-1)^i \xi_i \omega(\xi_0, \cdots, \hat{\xi}_i, \cdots, \xi_k) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_k).$$

The Lie algebra cohomology of $\mathfrak{g}$ with coefficients in E is of course defined as $H(\mathfrak{g}; E) = \frac{\ker d_{CE}}{\mathrm{im}\ d_{CE}}$.

Remark 3.1.1. It is easy to see that $H^0(\mathfrak{g}; E) = \{f \in E : \xi f = 0, \forall \xi \in \mathfrak{g}\}$, i.e., the subspace of $\mathfrak{g}$-invariant elements in E.

Remark 3.1.2. Fix a basis $\{\xi_a\}$ for $\mathfrak{g}$. Fix a dual basis $\{\theta^b\}$ of $\{\xi_a\}$ for $\mathfrak{g}^*$. Let f^a_{bc} denote the structure constants of $\mathfrak{g}$. Fix a basis $\{f_i\}$ for E. The left $\mathfrak{g}$-module structure of E is given by $\xi_a f_i = (\rho_a)^j_i f_j$. One can then express d_{CE} in a component fashion

$$d_{CE} f_i = (\rho_a)^j_i \theta^a f_j, \quad d_{CE}\theta^a = -\frac{1}{2} f^a_{bc} \theta^b \theta^c. \tag{3.1.3}$$

Corollary 3.1.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. $H_{dR}(G) \cong H(\mathfrak{g}; \mathbb{R})$.*

Example 3.1.1. Let $\mathfrak{g} = \mathfrak{su}(2)$. Let $E = \mathbb{R}$. One can choose a basis σ_a of $\mathfrak{g}$ such that

$$[\sigma_a, \sigma_b] = \sum_{c=1,2,3} \epsilon_{abc} \sigma_c, \tag{3.1.4}$$

where ϵ is the totally anti-symmetric tensor with $\epsilon_{123} = 1$.

1. $H^0(\mathfrak{g}, E) = \mathbb{R}$ by Remark 3.1.1.

[1]For our purpose, E is assumed to be a real vector space and the tensor product is taken over $\mathbb{R}$.

2. For degree 1, $\ker(d_{CE}) = \{\omega \in \mathfrak{g}^* : \omega([\xi_1, \xi_2]) = 0\}$. However, since $[\mathfrak{su}(2), \mathfrak{su}(2)] = \mathfrak{su}(2)$, $\ker(d_{CE}) = 0$. $H^1(\mathfrak{g}; E) = 0$.

3. For degree 2, $\ker(d_{CE}) = \{\omega \in \Lambda^2(\mathfrak{g}^*) : \omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_2, \xi_3], \xi_1) + \omega([\xi_3, \xi_1], \xi_2) = 0\}$. By (3.1.4), it is easy to see that $\ker(d_{CE}) = \Lambda^2(\mathfrak{g}^*)$. On the other hand, $\mathrm{im}(d_{CE}) = \{\omega \in \Lambda^2(\mathfrak{g}^*) : \exists \omega' \in \mathfrak{g}^*, \omega(\xi_1, \xi_2) = \omega'([\xi_1, \xi_2])\}$. Let θ^a be the dual basis of σ_a, one can write $\omega = \omega_{12}\theta^1 \wedge \theta^2 + \omega_{13}\theta^1 \wedge \theta^3 + \omega_{23}\theta^2 \wedge \theta^3$. One can then set $\omega' = \omega_{12}\theta^3 - \omega_{13}\theta^2 + \omega_{23}\theta^1$. Hence $\mathrm{im}(d_{CE}) = \Lambda^2(\mathfrak{g}^*)$. $H^2(\mathfrak{g}; E) = 0$.

4. For degree 3, we have $\mathrm{im}(d_{CE}) = 0$ by the last step. $H^3(\mathfrak{g}; E) = \mathbb{R}$.

These are precisely the de Rham cohomology groups of SU(2) (or SO(3)).

Example 3.1.2. Let M be a manifold. Let $\mathfrak{g}$ be $\mathfrak{X}(M)$, the space of vector fields over M. It is the Lie algebra of the diffeomorphism group of M. Let $E = C^\infty(M)$, the ring of smooth functions over M. $X \in \mathfrak{X}(M)$ acts on $f \in C^\infty(M)$ in the canonical way. By (3.1.1), the Lie algebra cohomology of $\mathfrak{g}$ is just the de Rham cohomology of M.

3.2 Koszul complex

For a commutative ring R, a regular sequence is a sequence $r_1, \cdots, r_d$ in R such that r_i is not a zero-divisor of $R/(r_1, \cdots, r_{i-1})$ for $i = 1, \ldots, d$ and $R/(r_1, \cdots, r_d)$ is not zero. We are interested in the case where $R = C^\infty(M)$ for some n-dimensional manifold M.

Lemma 3.2.1. *A function $f \in C^\infty(M)$ is a zero-divisor if and only if there exists a nonempty open subset U of M such that $U \subset f^{-1}(0)$.*

Proof. If f is a zero-divisor, then there exists a nonzero g such that $gf = 0$. This implies that $(g^{-1}(0))^c \subset f^{-1}(0)$. We then take $U = (g^{-1}(0))^c$. Note that U is not the empty set since g is not zero everywhere.

Suppose there exists a nonempty open subset U of M contained in $f^{-1}(0)$. Then one can find a bump function g such that g is everywhere zero outside of U. Hence $fg = 0$. □

Lemma 3.2.2. *If 0 is a regular value of f, then f is not a zero-divisor.*

Proof. By the preimage theorem, $f^{-1}(0)$ is an $(n-1)$-dimensional submanifold of M, hence cannot contain any nonempty open subset of M. □

Lemma 3.2.3. *If 0 is a regular value of f, then the ideal (f) generated by f is equal to the ideal $I = \{g \in C^\infty(M) : g|_{f^{-1}(0)} = 0\}$, i.e., the ideal of functions vanishing at the zero set of f.*

Proof. Obviously, $(f) \subset I$. We only need to prove the other inclusion.

By the implicit function theorem, one can find local coordinates $(y^1, \cdots, y^n)$ around $x \in M$ such that $y^n = f$ and $f(x) = 0 \Leftrightarrow y^n(x) = 0$. Let g be a smooth function vanishing on $f^{-1}(0)$. For $x \in f^{-1}(0)$, locally we have

$$\begin{aligned} g(y^1, \cdots, y^n) &= g(y^1, \cdots, y^n) - g(y^1, \cdots, 0) \\ &= \int_0^1 dt \frac{dg(y^1, \cdots, ty^n)}{dt} \\ &= f(\int_0^1 dt \frac{\partial g}{\partial y^n}) \\ &=: fh \end{aligned}$$

For $x \notin f^{-1}(0)$, locally we can define a function $h' = g/f$. Since both h and h' are unique, we can glue them together to get a global function h'' such that $fh'' = g$. □

Corollary 3.2.1. *Let f be a function with 0 as a regular value. Let $\Sigma = f^{-1}(0)$. Then $C^\infty(\Sigma) \cong C^\infty(M)/(f)$.*

Proof. This follows directly from the extension lemma of functions on closed subsets (See Lemma 2.26 in [Lee12].) and Lemma 3.2.3. □

Proposition 3.2.1. *Let $F = (f_1, \cdots, f_d)$ be a smooth map from M to $\mathbb{R}^d$. If 0 is a regular value of F, then $f_1, \cdots, f_d$ is a regular sequence of $C^\infty(M)$. Moreover,*

$$C^\infty(M)/(f_1, \cdots, f_d) \cong C^\infty(Z),$$

where $Z = F^{-1}(0)$.

Proof. We prove this proposition by induction. Let Z_k be the zero set of $f_1, \cdots, f_k$. Assume that $f_1, \cdots, f_k$ is a regular sequence and $C^\infty(Z_k) \cong C^\infty(M)/(f_1, \cdots, f_k)$. Since 0 is a regular value of F, the pullback of df_{k+1} to Z_k is everywhere nonzero over $Z_{k+1} \subset Z_k$. By Lemma 3.2.2, $f_{k+1}|_{Z_k}$ is not a zero-divisor of $C^\infty(Z_k)$, hence $f_1, \cdots, f_{k+1}$ is again a regular sequence of $C^\infty(M)$. Note that $(f_1, \cdots, f_{k+1}) = (f_1, \cdots, f_k) + (f_{k+1})$, we can apply Corollary 3.2.1 to obtain $C^\infty(Z_{k+1}) \cong C^\infty(Z_k)/(f_{k+1}|_{Z_k}) \cong C^\infty(M)/(f_1, \cdots, f_{k+1})$. □

Let's turn to the construction of the Koszul complex of the sequence $f_1, \cdots, f_d$.

Definition 3.2.1. Let $\Gamma = R^d$. Let F be an R-linear map from Γ to R. Note that F is equivalent to a sequence $f_1, \cdots, f_d$ in R. The Koszul complex of such sequence is the differential graded algebra $\Lambda(\Gamma)$ (the exterior algebra of is over R) equipped with the differential δ_K defined by

$$\delta_K(r_0 \wedge \cdots \wedge r_k) = \sum_{i=0}^{k} (-1)^i F(r_i) r_0 \wedge \cdots r_{i-1} \wedge r_{i+1} \wedge \cdots \wedge r_k.$$

The cohomology of $(\Lambda(\Gamma), \delta_K)$ is called the Koszul cohomology associated of the sequence $f_1, \cdots, f_d$.

Remark 3.2.1. We assign degree $-k$ to elements in $\Lambda^k(\Gamma)$ to ensure that the differential δ_K is of degree $+1$, which justifies the name "Koszul cohomology". It is also easy to see that the 0-th cohomology group is just the quotient $R/(f_1, \cdots, f_d)$.

Remark 3.2.2. F is called the gauge fixing function. It is said to be regular if $f_1, \cdots, f_d$ is a regular sequence.

Proposition 3.2.2. *The Koszul complex is acyclic for a regular F.*

Proof. See [Kos50]. □

In other words, for a regular sequence $f_1, \cdots, f_d$, we have the following free resolution

$$0 \to \Lambda^d(\Gamma) \xrightarrow{\delta_K} \dots \xrightarrow{\delta_K} \Lambda^1(\Gamma) = \Gamma \xrightarrow{f} R \to R/(f_1, \cdots, f_d) \to 0$$

known as the Koszul resolution of $R/(f_1, \cdots, f_d)$.

Remark 3.2.3. The converse of Proposition 3.2.2 is not true in general. For example, take $R = C^\infty(M)$ and $f_1, \cdots, f_d$ to be any everywhere nonzero functions. It is not hard to see that the Koszul cohomology groups are all trivial. But $f_1, \cdots, f_d$ is clearly not a regular sequence.

Example 3.2.1. Let $R = C^\infty(\mathbb{R}^d)$. Let $f_i = x_i$, where $x^1, \cdots, x^d$ are the Cartesian coordinates of $\mathbb{R}^d$. By Proposition 3.2.2, the corresponding Koszul complex is acyclic. By Proposition 3.2.1, the 0-th cohomology group is just $\mathbb{R}$.

A gauge fixing function F in the case of $R = C^\infty(M)$ is equivalent to a section of the trivial bundle $M \times \mathbb{R}^d$. As in the Mathai-Quillen construction of an Euler class, we also want F to be a section of a nontrivial vector bundle. For this purpose, we give the following generalization of Definition 3.2.1.

Definition 3.2.2. Let Γ be an R-module. Let F be an R-linear map from Γ to R. The Koszul complex of F is the differential graded algebra $\Lambda(\Gamma)$ equipped with the differential δ_K defined by

$$\delta_K(\gamma_0 \wedge \cdots \wedge \gamma_k) = \sum_{i=0}^{k} (-1)^i F(\gamma_i) \gamma_0 \wedge \cdots \gamma_{i-1} \wedge \gamma_{i+1} \wedge \cdots \wedge \gamma_k.$$

The cohomology of $(\Lambda(\Gamma), \delta_K)$ is called the Koszul cohomology of F.

In both the Hamiltonian formalism and the Lagrangian formalism of a physical theory, there exist natural choices for the gauge fixing function F in Definition 3.2.2.

Example 3.2.2. Let M be a symplectic manifold. Let G be a Lie group. Suppose there exists a Hamiltonian G-action on M. The distribution over M associated to such action is involutive, hence corresponds to a sub-bundle of TM, denoted by H. Fix a basis ξ_a for the Lie algebra $\mathfrak{g}$ of G, we have $\Gamma(H) = \mathrm{span}_{C^\infty}\{X_a\}$, where X_a is the fundamental vector field generated by ξ_a. One can then define a $C^\infty(M)$-linear map from $\Gamma(H)$ to $C^\infty(M)$ by sending X_a to f_a, the Hamiltonian function of X_a. Note that F commutes with the $\mathfrak{g}$-actions on $\Gamma(H)$ and $C^\infty(M)$, that is, $F(\xi(X_a)) = F([X_\xi, X_a]) = \{f_\xi, f_a\} = \xi(F(X_a))$, where $\{\cdot,\cdot\}$ is the Poisson bracket on M. If the action is free, then H is the trivial bundle $M \times \mathfrak{g}$. F is equivalent to a moment map of the action. If 0 is, moreover, a regular value of the moment map, then by Proposition 3.2.2, the Koszul complex of F is acyclic.

Example 3.2.3. Let M be a manifold. Let S be a function on M. The 1-form dS induces a $C^\infty(M)$-linear map

$$\begin{aligned} F : \mathfrak{X}(M) &\to C^\infty(M) \\ X &\mapsto dS(X). \end{aligned}$$

Let G be the subgroup of the diffeomorphism group of M which keeps S invariant. Let $\mathfrak{g}$ be the Lie algebra of G. Note that F commutes with the $\mathfrak{g}$-actions on $\mathfrak{X}(M)$ and $C^\infty(M)$, since $F(\xi(X)) = F([X_\xi, X]) = [X_\xi, X]S = X_\xi(XS) = \xi(F(X))$.

Definition 3.2.2 can be again generalized to the following one.

Definition 3.2.3. Let Γ be an R-module. Let A be a commutative R-algebra. Let F be an R-linear map from Γ to A. F is referred to as a gauge fixing function. The Koszul complex of F is the differential graded algebra $\Lambda(\Gamma) \otimes A$ equipped with the differential δ_K defined by

$$\delta_K(\gamma_0 \wedge \cdots \wedge \gamma_k \otimes a) = \sum_{i=0}^{k} (-1)^i \gamma_0 \wedge \cdots \gamma_{i-1} \wedge \gamma_{i+1} \wedge \cdots \wedge \gamma_k \otimes F(\gamma_i) a.$$

The cohomology of $(\Lambda(\Gamma) \otimes A, \delta_K)$ is called the (generalized) Koszul cohomology of F.

Remark 3.2.4. Let's consider the A-module $A \otimes \Gamma$ and its exterior algebra $\Lambda_A(A \otimes \Gamma)$ over A. It is not hard to show that $\Lambda_A(A \otimes \Gamma) \cong \Lambda(\Gamma) \otimes A$. Moreover, the R-linear map $F : \Gamma \to A$ induces an A-linear map

$$\begin{aligned} \tilde{F} : A \otimes \Gamma &\to A \\ a \otimes X &\mapsto aF(X). \end{aligned}$$

The Koszul complex of F coincides with the Koszul complex of $\tilde{F}$ in the sense of Definition 3.2.2.

Example 3.2.4. Let $A = \mathrm{S}(\Gamma)$, the symmetric algebra of Γ over R. Let F be the canonical embedding of Γ into $\mathrm{S}(\Gamma)$. We call the resultant Koszul complex the universal Koszul complex. It is acyclic when Γ is a flat R-module [Bou07].

Example 3.2.5. Let L be an R-module. Let $F : L \to \Gamma$ be an R-linear map. F induces a dual map from Γ^* to L^*, which again induces an R-linear map from Γ^* to $\mathrm{S}(L^*)$. The Koszul differential takes the form

$$(\delta_K\omega)(r_1, \cdots, r_k)(a_0, \cdots, a_l) = \sum_{i=0}^{k} \omega(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_k)(r_i, a_0, \cdots, a_l).$$

For a specific application of Example 3.2.5, let's consider the following geometric construction. Let H be a Lie group with Lie algebra $\mathfrak{h}$. Let H act transitively on a manifold M. We have the following short exact sequence of vector bundles

$$0 \to K_{\mathfrak{h}} \to V_{\mathfrak{h}} \to TM \to 0,$$

where $V_{\mathfrak{h}} = M \times \mathfrak{h}$, $V_{\mathfrak{h}} \to TM$ is obtained by combing the infinitesimal action $\mathfrak{h} \to \Gamma(TM)$ with the evaluation map $M \times \Gamma(TM) \to TM$, and $K_{\mathfrak{h}}$ is the kernel of $V_{\mathfrak{h}} \to TM$. We now take Γ to be $\Gamma(V_{\mathfrak{h}})$, L to be $\Gamma(K_h)$ and F to be the canonical inclusion. This is exactly the Koszul complex considered by Kalkman in [Kal93a]. The corresponding differential graded algebra is $(\Lambda(\mathfrak{h}^*) \otimes_{\mathbb{R}} \Gamma(\mathrm{S}(K_{\mathfrak{h}}^*)), \delta_K)$, where δ_K is defined in Example 3.2.5. The degree assigned to elements in $\Gamma(\mathrm{S}^1(K_{\mathfrak{h}}^*))$ is 2 instead of 0, and the degree assigned to elements in $\Lambda^1(\mathfrak{h}^*)$ is 1 instead of -1. It was shown by Kalkman that the corresponding Koszul cohomology is isomorphic to the de Rham cohomology of M via the cochain map

$$\begin{aligned}\Phi : \Omega^k(M) = \Gamma(\Lambda^k T^*M) \to \Lambda^k(\mathfrak{h}^*) \otimes_{\mathbb{R}} \Gamma(\mathrm{S}^0(K_{\mathfrak{h}}^*)) \subset \Lambda(\mathfrak{h}^*) \otimes_{\mathbb{R}} \Gamma(\mathrm{S}(K_{\mathfrak{h}}^*))\\ \omega \mapsto \Phi(\omega),\end{aligned}$$

where $\Phi(\omega)(\xi_1, \cdots, \xi_k) := \omega(X_{\xi_1}, \cdots, X_{\xi_k})$ (see Theorem 2.3.2 in [Kal93a]).

Remark 3.2.5. As is seen in the above examples, sometimes, the gauge fixing function F is not good enough and we do not get an acyclic Koszul complex. In the case where Γ is a free R-module, however, one can always obtain a new complex with trivial m-th cohomology group by attaching generators of degree $m+1$ to the old one, $m \geq 1$. Iterating and taking the direct limit, one will obtain an acyclic complex called the Koszul-Tate complex [Tat57; PP17]. I do not know if a similar generalization exists in the case of a general R-module Γ. For this reason, I will restrict the discussion to Koszul-complexes when it comes to the construction of a BRST complex.

3.3 BRST complex

Roughly speaking, a BRST complex is the combination of a Chevalley-Eilenberg complex and an acyclic Koszul complex (or Koszul-Tate complex). The acyclic property is needed so that the computation of the BRST cohomology can be greatly simplified. That being said, we will also discuss an example where a non-acyclic Koszul complex is involved at the end of this section.

3.3.1 The first way of defining a BRST complex

There exist many different approaches to the definition of a BRST complex. Most of them can be described in two different ways depending on the type of the Koszul complex involved. The first way uses the following data.

1. A Lie group G with Lie algebra $\mathfrak{g}$.

2. A G-manifold M with the unital commutative ring $R = C^\infty(M)$ of smooth functions over M. Note that there is a canonical $\mathfrak{g}$-action on R.

3. A vector bundle E over M, or equivalently, the finitely generated projective R-module $\Gamma = \Gamma(E)$. Choosing a partition of unity over M, it is not hard to construct an isomorphism $\Lambda_R(\Gamma) \cong \Gamma(\Lambda E)$, where $\Gamma(\Lambda E)$ is the space of sections of the exterior algebra of the vector bundle E.

4. A $\mathfrak{g}$-action on $\Gamma(E)$ satisfying Leibniz's rule

$$\xi(fX) = \xi(f)X + f\xi(X),$$

   where $\xi \in \mathfrak{g}$, $f \in C^\infty(M)$ and $X \in \Gamma(E)$. This equips the exterior algebra $\Lambda(\Gamma)$ of Γ over R with a canonical $\mathfrak{g}$-module structure. In fact, let $\mathrm{T}(\Gamma)$ be the tensor product of Γ over R. It has a canonical $\mathfrak{g}$-action by setting

$$\xi(X_1 \otimes \cdots \otimes X_m) = \xi(X_1) \otimes \cdots \otimes X_m + \cdots + X_1 \otimes \cdots \otimes \xi(X_m).$$

   This action is well-defined because

$$\begin{aligned} \xi(fX \otimes Y) &= \xi(fX) \otimes Y + fX \otimes Y \\ &= \xi(f)X \otimes Y + f\xi(X) \otimes Y + fX \otimes Y \\ &= \xi(X) \otimes fY + X \otimes \xi(fY) \\ &= \xi(X \otimes fY). \end{aligned}$$

   The $\mathfrak{g}$-module structure on $\Lambda(\Gamma)$ is then induced from the one on $\mathrm{T}(\Gamma)$.

5. A R-linear map $F : \Gamma \to R$ called the gauge fixing function which commutes with the $\mathfrak{g}$-actions on Γ and R.

Consider the bicomplex $\Lambda(\mathfrak{g}^*)\otimes\Lambda(\Gamma)$ equipped with the Chevalley-Eilenberg differential with values in the $\mathfrak{g}$-module $\Lambda(\Gamma)$

$$d_{CE}:\Lambda^p(\mathfrak{g}^*)\otimes\Lambda^q(\Gamma)\to\Lambda^{p+1}(\mathfrak{g}^*)\otimes\Lambda^q(\Gamma),$$

and the Koszul differential associated to the gauge fixing function $F:\Gamma\to R$

$$\delta_K:\Lambda^p(\mathfrak{g}^*)\otimes\Lambda^q(\Gamma)\to\Lambda^p(\mathfrak{g}^*)\otimes\Lambda^{q-1}(\Gamma).$$

$\Lambda(\mathfrak{g}^*)\otimes\Lambda(\Gamma)$ is bigraded by assigning degree 1 to elements in $\mathfrak{g}^*$ and degree -1 to elements in Γ. Let $\omega\otimes f\in\Lambda^1(\mathfrak{g}^*)\otimes R$ and $1\otimes r\in\Lambda^0(\mathfrak{g}^*)\otimes\Gamma$. We have

$$\delta_K(d_{CE}(\omega\otimes f))=0=d_{CE}(\delta_K(\omega\otimes f)),$$

and

$$\delta_K(d_{CE}(1\otimes r))=\delta_K(\theta^a\otimes\xi_a r)=\theta^a\otimes F(\xi_a r)=\theta^a\otimes\xi_a F(r)=d_{CE}(\delta_K(1\otimes r)).$$

Therefore $d_{CE}\delta_K-\delta_K d_{CE}=0$ for all elements in the $\Lambda(\mathfrak{g}^*)\otimes\Lambda(\Gamma)$ by Leibniz's rule. The bicomplex structure is indeed well-defined. One can then define the BRST complex to be the total complex of this bicomplex with the BRST differential defined to be the total differential $s=d_{CE}+(-1)^{\bullet}\delta_K$. When the Koszul complex is acyclic, it is not hard to show that $H^{\bullet}_s=H^{\bullet}_{d_{CE}}(H^0_{\delta_K})$ using standard methods of spectral sequences.

For example, we can take M to be a symplectic manifold with a Hamiltonian G-action, E to be the vector bundle H defined in Example 3.2.2 and F to be the gauge fixing function induced by the moment map of the G-action. Note that for $\xi\in\mathfrak{g}$, $X\in\Gamma(H)\subset\Gamma(TM)$ and $f\in C^\infty(M)$, we have

$$\xi(fX)=[X_\xi,fX]=X_\xi(f)X+f[X_\xi,X]=\xi(f)X+f\xi(X).$$

We recover the BRST complex defined in [KS87].

As another example, we can take E to be the tangent bundle over a G-manifold M, and F to be the gauge fixing function induced by a G-invariant function S on M as in Example 3.2.3. Likewise, for $\xi\in\mathfrak{g}$, $X\in\Gamma(TM)$ and $f\in C^\infty(M)$, we have $\xi(fX)=\xi(f)X+f\xi(X)$. The corresponding BRST complex is indeed well-defined.

3.3.2 The second way of defining a BRST complex

The second way of constructing a BRST complex uses almost the same data as the first one, except that it involves the universal Koszul complex of Γ instead. The bicomplex is then $\Lambda(\mathfrak{g}^*)\otimes\Lambda(\Gamma)\otimes\mathrm{S}(\Gamma)$ [2] equipped with the Chevalley-Eilenberg differential with values in the trivial $\mathfrak{g}$-module $\Lambda(\Gamma)\otimes\mathrm{S}(\Gamma)$

$$d_{CE}:\Lambda^r(\mathfrak{g}^*)\otimes\Lambda^p(\Gamma)\otimes\mathrm{S}^q(\Gamma)\to\Lambda^{r+1}(\mathfrak{g}^*)\otimes\Lambda^p(\Gamma)\otimes\mathrm{S}^q(\Gamma),$$

[2] The first $\otimes$ is over $\mathbb{R}$, while the second is over R. Elements in $\Lambda^r(\mathfrak{g}^*)\otimes\Lambda^p(\Gamma)\otimes\mathrm{S}^q(\Gamma)$ are of degree $(r,-p)$.

and the Koszul differential of the universal Koszul complex

$$\delta_K : \Lambda^r(\mathfrak{g}^*) \otimes \Lambda^p(\Gamma) \otimes \mathrm{S}^q(\Gamma) \to \Lambda^r(\mathfrak{g}^*) \otimes \Lambda^{p-1}(\Gamma) \otimes \mathrm{S}^{q+1}(\Gamma).$$

Since Γ is projective, the universal Koszul complex is acyclic. Hence $H_s^\bullet = H^\bullet_{d_{CE}}(H^0_{\delta_K})$.

For example, let's consider the Faddeev-Popov method of Yang-Mills theory [FP67]. Let P be a (trivial) principal H-bundle over a 4-dimensional Minkowski space (N, g). Recall that the Yang-Mills functional is defined as $S = \frac{1}{4}\int_N d\mathrm{vol}_g \mathrm{tr}(F^{\mu\nu}F_{\mu\nu})$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the stress-energy tensor of the vector potential A_μ, $F^{\mu\nu} = g^{\mu\sigma}g^{\nu\rho}F_{\sigma\rho}$. S is invariant under the action of the gauge symmetry group $\mathcal{H} = C^\infty(M, H)$. $h \in \mathcal{H}$ acts on A_μ as $h(A_\mu) = h\partial_\mu h^{-1} + hA_\mu h^{-1}$. The Lie algebra of $\mathcal{H}$ can be identified with $C^\infty(M, \mathfrak{h})$. Let D_μ denote the covariant derivative associated to A_μ. $\xi \in C^\infty(M, \mathfrak{h})$ acts on A_μ as $\xi(A_\mu) = -D_\mu\xi$. The BRST complex is defined by setting $\mathfrak{g}$ to be $C^\infty(M, \mathfrak{h})$, M to be $\mathcal{A}$, the infinite dimensional space of all vector potentials, and Γ to be the trivial vector bundle over $\mathcal{A}$ with fiber $\mathfrak{g}$. An element $\xi \in \mathfrak{g}^*$ induces a generator $c := \xi \otimes 1 \otimes 1$ of degree 1, a generator $\bar{c} := 1 \otimes \xi \otimes 1$ of degree -1 and a generator $b := 1 \otimes 1 \otimes \xi$ of degree 0. The BRST differential takes the following form

$$sA_\mu = -D_\mu c, \quad sc = -\frac{1}{2}[c, c]$$
$$s\bar{c} = b, \quad sb = 0.$$

Recall that $sc = -\frac{1}{2}[c, c]$ should be understood as $sc^a = -\frac{1}{2}f^a_{bc}c^bc^c$, where $c^a = \xi^a \otimes 1 \otimes 1$.

3.3.3 BRST model for equivariant cohomology

Let's consider the Koszul complex defined at the end of Section 3.2. The corresponding differential graded algebra is $(\Lambda(V^*_\mathfrak{h}) \otimes_{C^\infty(M)} \mathrm{S}(\Gamma(K^*_\mathfrak{h})) \cong \Lambda(\mathfrak{h}^*) \otimes_\mathbb{R} \Gamma(\mathrm{S}(K^*_\mathfrak{h})), \delta_K)$. By construction, $\Gamma(K_\mathfrak{h})$, and hence also $\Gamma(\mathrm{S}(K^*_\mathfrak{h}))$ are $\mathfrak{h}$-modules. One can then equip $\Lambda(\mathfrak{h}^*) \otimes_\mathbb{R} \Gamma(\mathrm{S}(K^*_\mathfrak{h}))$ with the Chevalley-Eilenberg differential d_{CE} with values in $\Gamma(\mathrm{S}(K^*_\mathfrak{h}))$. Fix a basis $\{\xi^a\}$ for $\mathfrak{h}^*$. ξ^a induces a generator $\theta^a := \xi^a \otimes 1$ of degree 1 and a generator $\phi^a := 1 \otimes \xi^a$ of degree 2 (through the inclusion $K_\mathfrak{h} \hookrightarrow V_h$). θ^a and ϕ^a generate $\Lambda(\mathfrak{h}^*) \otimes_\mathbb{R} \Gamma(\mathrm{S}(K^*_\mathfrak{h}))$ over $C^\infty(M)$. We have

$$d_{CE}\theta^a = -\frac{1}{2}f^a_{bc}\theta^b\theta^c, \quad d_{CE}\phi^a = -f^a_{bc}\theta^b\phi^c$$
$$\delta_K\theta^a = \phi^a, \quad \delta_K\phi^a = 0.$$

It is not hard to verify that $d_{CE}\delta_K = -\delta_K d_{CE}$. The BRST differential is then defined to be $s = d_{CE} + \delta_K$. One may immediately realize that s is just the differential of the Weil algebra $\Lambda(\mathfrak{h}^*) \otimes_\mathbb{R} \Gamma(\mathrm{S}(K^*_\mathfrak{h})) \cong W(\mathfrak{h})$ when the H-manifold M is a point.

Let $\mathfrak{g}$ be an ideal of $\mathfrak{h}$. Let G be a normal subgroup of H with Lie algebra $\mathfrak{g}$. The transitive H-action induces another transitive $G \rtimes H$-action, which again induces the following exact sequences of vector bundles

$$0 \to V_{\mathfrak{g}} \oplus K_{\mathfrak{h}} \to V_{\mathfrak{g}} \oplus V_{\mathfrak{h}} \to TM \to 0,$$

where the second map is defined by sending (ξ, ν) to $(\xi + \nu, -\xi)$ for $\xi \in V_{\mathfrak{g}}(x)$, $\nu \in K_{\mathfrak{h}}(x)$, $x \in M$. The BRST complex in this case is isomorphic to $W(\mathfrak{g}) \otimes \Lambda(\mathfrak{h}^*) \otimes \Gamma(\mathrm{S}(K^*_{\mathfrak{h}}))$ with the BRST differential $s = s' + \delta^{\mathfrak{h}}_K$, where $s' = d_{CE} + \delta^{\mathfrak{g}}_K$. The cohomology $H_{\delta^{\mathfrak{h}}_K}(W(\mathfrak{g}) \otimes \Lambda(\mathfrak{h}^*) \otimes \Gamma(\mathrm{S}(K^*_{\mathfrak{h}})))$ equipped with the differential s' coincides with the Kalkman model of the equivariant cohomology of the G-manifold M. We refer the reader to the thesis [Kal93a] by Kalkman for more detail about this construction.

Chapter 4

Monoidally graded geometry

4.1 Commutative monoids and parity functions

Let $(\mathcal{I}, 0, +)$ be a commutative monoid. Let $\mathbb{Z}_q$ denote the cyclic group of order q.

Definition 4.1.1. A parity function is a (non-trivial) monoid homomorphism $p : \mathcal{I} \to \mathbb{Z}_2$.

Not every $\mathcal{I}$ has a non-trivial parity function. For example, there is no non-trivial homomorphism from $\mathbb{Z}_q$ to $\mathbb{Z}_2$ when q is odd. Let $\mathcal{I}_a$ denote $p^{-1}(a)$ for $a \in \mathbb{Z}_2$. We have $\mathcal{I}_a + \mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. Recall that an element x in $\mathcal{I}$ is called cancellative if $x + y = x + z$ implies $y = z$ for all y and z in $\mathcal{I}$. Suppose that there is a cancellative element in $\mathcal{I}_1$. It is easy to see that such an element induces an injective map from $\mathcal{I}_a$ to $\mathcal{I}_{a+1}$. It follows from the Cantor-Bernstein theorem that there exists a bijection between $\mathcal{I}_0$ and $\mathcal{I}_1$. A monoid is called cancellative if every element in it is cancellative. We have shown that

Proposition 4.1.1. *Let $\mathcal{I}$ be a commutative cancellative monoid. If $\mathcal{I}$ has a non-trivial parity function p, then the submonoid $\mathcal{I}_0$ and its complement $\mathcal{I}_1$ have the same cardinality.*

Remark 4.1.1. In the finite case, Proposition 4.1.1 is no longer true if we drop the cancellative condition. For example, we can consider the commutative monoid defined by the following table. A non-trivial p is defined by setting $p(0) = p(b) = 0$ and $p(a) = 1$.

	0	a	b
0	0	a	b
a	a	b	a
b	b	a	b

Table 4.1.1: A commutative non-cancellative monoid of order 3.

The question now is, given an appropriate commutative cancellative monoid $\mathcal{I}$, how can one construct a parity function for it? If $\mathcal{I}$ is finite, it is not hard to show that $\mathcal{I}$ is actually an abelian group. The fundamental theorem of finite abelian groups then tells us that $\mathcal{I}$ is isomorphic to a direct product of cyclic groups of prime-power order. By Proposition 4.1.1, one of these cyclic groups must be $\mathbb{Z}_{2^k}$, $k \geq 1$. We can write $\mathcal{I} = \mathbb{Z}_{2^k} \times \cdots$ and define p by sending $(x, \cdots) \in \mathcal{I}$ to $a - 1 \pmod 2$, where a is the order of $x \in \mathbb{Z}_{2^k}$. If $\mathcal{I}$ is infinite, the construction of p is hard, perhaps not possible in general. However, one can easily work out the case when $\mathcal{I}$ is free. ($\mathcal{I}$ is then cancellative, but not a group.) Let $\mathcal{I}_0$ be the submonoid of elements generated by an even number of generators. Let $\mathcal{I}_1$ be the subset of elements generated by an odd number of generators. Note that $\mathcal{I}_a + \mathcal{I}_b \subseteq \mathcal{I}_{a+b}$. We obtain a parity function which sends elements in $\mathcal{I}_a$ to a. As an example, let $\mathcal{I}$ be $\mathbb{N}$, the monoid of natural numbers under addition. p is then defined by sending even numbers to 0 and odd numbers to 1.

Let $K(\mathcal{I})$ denote the Grothendieck group of $\mathcal{I}$. Recall that it can be constructed as follows. Let $\sim$ be the equivalence relation on $\mathcal{I} \times \mathcal{I}$ defined by $(a_1, a_2) \sim (b_1, b_2)$ if there exists a $c \in \mathcal{I}$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. The quotient $K(\mathcal{I}) = \mathcal{I} \times \mathcal{I} / \sim$ has a group structure by $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)]$.

Proposition 4.1.2. *Let p be a parity function for $\mathcal{I}$. The map*

$$\begin{aligned} p' : K(\mathcal{I}) &\to \mathbb{Z}_2 \\ [(a_1, a_2)] &\mapsto p(a_1) + p(a_2) \end{aligned}$$

is well-defined and gives a parity function for $K(\mathcal{I})$.

Remark 4.1.2. When $\mathcal{I}$ is cancellative, it can be seen as a submonoid of $K(\mathcal{I})$ by the embedding

$$\begin{aligned} \iota : \mathcal{I} &\to K(\mathcal{I}) \\ a &\mapsto [(a, 0)]. \end{aligned}$$

For this reason, we sometimes simply write $a - b$ to denote $[(a, b)] \in K(\mathcal{I})$. The cancellative property is not necessary for the proof of Proposition 4.1.2. But it guarantees the non-triviality of p', since p' restricted to $\mathcal{I}$ must coincide with p.

Proof. Let (a_1, a_2) and (b_1, b_2) represent the same element of $K(\mathcal{I})$, i.e., there exist some c such that $a_1 + b_2 + c = a_2 + b_1 + c$. One then concludes that $a_1 + b_2$ and $a_2 + b_1$ must have the same parity. Note that, for $a, b \in \mathbb{Z}_2$, $a = b$ if and only if $a + b = 0$. We have

$$p'([(a_1, a_2)]) + p'(([b_1, b_2)]) = p(a_1 + b_2) + p(a_2 + b_1) = 0.$$

Hence $p'([a_1, a_2]) = p'([b_1, b_2])$. □

As an example, consider $K(\mathbb{N}) = \mathbb{Z}$, the monoid of integers under addition. The parity function p' induced from the parity function p for $\mathbb{N}$ again sends even numbers to 0 and odd numbers to 1.

4.2 Monoidally graded ringed spaces

Let R be a commutative ring. Let $\mathcal{I}$ be a countable commutative cancellative monoid equipped with a parity function p.

Definition 4.2.1. An $\mathcal{I}$-graded R-module is an R-module V with a family of sub-modules $\{V_i\}_{i\in\mathcal{I}}$ indexed by $\mathcal{I}$ such that $V = \bigoplus_{i\in\mathcal{I}} V_i$. $v \in V$ is said to be homogeneous if $v \in V_i$ for some $i \in \mathcal{I}$. We use $d(v)$ to denote the degree of v, $d(v) = i$.

Given two $\mathcal{I}$-graded R-modules V and W, we make the direct sum $V \oplus W$ and the tensor product $V \otimes W$ into $\mathcal{I}$-graded R-modules by setting

$$V \oplus W = \bigoplus_{i\in\mathcal{I}} (V_i \oplus W_i), \quad V \otimes W = \bigoplus_{k\in\mathcal{I}} \left(\bigoplus_{i+j=k} V_i \otimes W_j \right).$$

We can also make the space $\mathrm{Hom}(V, W)$ of R-linear maps from V to W into a $K(\mathcal{I})$-graded R-module $\mathrm{Hom}(V, W) = \bigoplus_{\alpha\in K(\mathcal{I})} \mathrm{Hom}(V, W)_\alpha$ by setting

$$\mathrm{Hom}(V, W)_\alpha = \{f \in \mathrm{Hom}(V, W) | f(V_i) \subset W_j, [(j, i)] = \alpha\}.$$

A morphism from V to W is just an element of $\mathrm{Hom}(V, W)_0$.

Remark 4.2.1. $\mathrm{Hom}(V, W)$ is in general not $\mathcal{I}$-graded. This is because we should assign degree "$j - i$" to a map f which maps elements in V_i to elements in $w \in W_j$. But the minus operation does make sense for a general monoid $\mathcal{I}$. So we have to work with $K(\mathcal{I})$, the group completion of $\mathcal{I}$. Note that $V^* = \mathrm{Hom}(V, R)$, the dual of V, is in particular $K(\mathcal{I})$-graded. (The degree of elements in V_i^* is $-i$.) Hence $V^* \otimes W$, which is isomorphic to $\mathrm{Hom}(V, W)$, is $K(\mathcal{I})$-graded by assigning degree $j - i$ to elements in $V_i^* \otimes W_j$. Everything is consistent.

Now, suppose that $\mathcal{I}$ also has a commutative multiplicative structure which is compatible with the additive structure. That is, it is a commutative cancellative semiring. We write ab as the multiplication of a and b in $\mathcal{I}$.

Definition 4.2.2. An $\mathcal{I}$-graded R-module A is called an $\mathcal{I}$-graded R-algebra if A is a unital associative R-algebra and if the multiplication $\mu : A \otimes A \to A$ is a morphism of $\mathcal{I}$-graded R-modules. We write $xy = \mu(x \otimes y)$ as the shorthand notation for multiplications of A. A is said to be commutative if

$$xy - (-1)^{p(x)p(y)} yx = 0 \tag{4.2.1}$$

for all homogeneous $x, y \in A$, where $(-1)^{(\cdot)}$ is the sign function of $\mathbb{Z}_2$ which sends $0 \in \mathbb{Z}_2$ to 1 and $1 \in \mathbb{Z}_2$ to -1.

Remark 4.2.2. Here we have to be careful about the sign factor appearing on the right hand side of (4.2.1). Although both of $\mathcal{I}$ and $\mathbb{Z}_2$ are semirings[1], p is not necessarily a semi-ring homomorphism and we do not have $p(d(x)d(y)) = p(d(x))p(d(y))$ in general. (As a counterexample, consider the case of $\mathcal{I} = \mathbb{Z} \times \mathbb{Z}$.) For simplicity, we write $p(x)p(y)$ to bypass this ambiguity. We will specify the convention we use when it comes to explicit computations. Note that the notion of commutativity in the $\mathcal{I}$-graded setting is well-defined because

$$(-1)^{p(x)p(y)} = (-1)^{p(y)p(x)}$$

and

$$(-1)^{p(x)p(yz)} = (-1)^{p(x)p(y)+p(x)p(z)}$$

for both conventions. In other words, the monoidal category of $\mathcal{I}$-graded R-modules together with the braiding

$$\begin{aligned} s_{V,W} : V \otimes W &\to W \otimes V \\ x \otimes y &\mapsto (-1)^{p(x)p(y)} y \otimes x \end{aligned}$$

is a symmetric monoidal category.

Morphisms of $\mathcal{I}$-graded algebras are simply linear maps of degree 0 which preserves the algebraic structures. We use Comm-Alg$_{\mathcal{I}}$ to denote the category of commutative $\mathcal{I}$-graded algebras.

Definition 4.2.3. The tensor algebra $\mathrm{T}(V)$ is the $\mathcal{I}$-graded R-module $\mathrm{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes^n}$, together with the tensor product $\otimes$ as the canonical multiplication. The symmetric algebra $\mathrm{S}(V)$ is the quotient algebra of $\mathrm{T}(V)$ by the $\mathcal{I}$-graded two-sided ideal generated by

$$v \otimes w - (-1)^{p(v)p(w)} w \otimes v,$$

where $v, w \in V \subset \mathrm{T}(V)$ are homogeneous.

Remark 4.2.3. $\mathrm{S}(V)$ has a canonical $\mathbb{N}$-grading inherited from $\mathrm{T}(V)$ which should not be confused with its $\mathcal{I}$-grading. We write $\mathrm{S}(V) = \bigoplus_{n\in\mathbb{N}} \mathrm{S}^n(V)$ to indicate that fact. Note that $\mathrm{S}^0(V) = R$, but $\mathrm{S}(V)_0$, the sub-space of homogeneous elements of degree 0, is in general larger than R.

$\mathrm{S}(V)$ is universal in the sense that, given a commutative $\mathcal{I}$-graded R-algebra A and a morphism $f : V \to A$. There exists a unique algebraic homomorphism $\tilde{f} : \mathrm{S}(V) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xhookrightarrow{\iota} & \mathrm{S}(V) \\ & \searrow^{f} & \downarrow{\tilde{f}} \\ & & A \end{array}$$

[1] The multiplicative structure on $\mathbb{Z}_2$ is inherited from the one on $\mathbb{Z}$.

where $\iota : V \to \mathrm{S}(V)$ is the canonical embedding. Note that $\tilde{f}$ preserves the $\mathcal{I}$-grading, i.e., it is a morphism in Comm-Alg$_{\mathcal{I}}$. Choosing A to be R (viewed as an $\mathcal{I}$-graded R-algebra whose components of non-zero degree are 0.) and f to be the zero map, we obtain an R-algebra homomorphism from $\mathrm{S}(V)$ to R. We denote this map by ϵ. Note that $\ker\epsilon = \bigoplus_{n>0} \mathrm{S}^n(V)$.

Let k be a field and R be a commutative k-algebra. Let A be a commutative $\mathcal{I}$-graded k-algebra.

Definition 4.2.4. A k-algebra epimorphism $\epsilon : A \to R$ is called a body map of A if $\ker\epsilon \supset I$, where I is the ideal in A generated by homogeneous elements of non-zero degree.

By definition, ϵ preserves the $\mathcal{I}$-grading of A.

Definition 4.2.5. Let ϵ be a body map of A. A is said to be projected if the short exact sequence

$$0 \longrightarrow \ker\epsilon \longrightarrow A \xrightarrow{\epsilon} R \longrightarrow 0$$

splits.

The splitting gives A an R-module structure depending on ϵ, with respect to which ϵ becomes an R-algebra homomorphism. Conversely, A is projected if A has an R-module structure and ϵ preserves that structure.

Lemma 4.2.1. *Let V be an $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be an R-linear body map of $\mathrm{S}(V)$. Then ϵ is unique.*

Proof. In this case, $\mathrm{S}(V) = R \oplus I$ where $I = \bigoplus_{n>0} \mathrm{S}^n(V)$. Since $I \subset \ker\epsilon$ and ϵ is R-linear, the only possible choice of ϵ is the canonical one. $\square$

Remark 4.2.4. Let V be as in Lemma 4.2.1. Suppose $A \cong \mathrm{S}(V)$ as $\mathcal{I}$-graded k-algebras. In particular, this implies that A admits a decomposition $A = A' \oplus I$ where $A' \cong R$ and I is the ideal generated by homogeneous elements of non-zero degree. Let ϵ be a body map of A. Since $I \subset \ker\epsilon$, ϵ is determined by $\epsilon|_{A'}$. In other words, ϵ is determined by a k-algebra endomorphism of R.

More can be said if V is free.

Lemma 4.2.2. *Let V be a free $\mathcal{I}$-graded R-module with $V_0 = 0$. Let ϵ be an R-linear body map of $\mathrm{S}(V)$. (By Lemma 4.2.1, ϵ is the canonical one.) Let I denote the kernel of ϵ. Then there exists an R-algebra isomorphism*

$$\mathrm{S}(V) \cong \mathrm{S}(I/I^2),$$

where I^2 is the square of the ideal I.

Proof. Let $\iota : V \hookrightarrow \mathrm{S}(V)$ be the canonical embedding. Since $I = \bigoplus_{n>0} \mathrm{S}^n(V)$, we have $\iota(V) \subset I$, which yields another embedding $V \hookrightarrow I/I^2 \hookrightarrow \mathrm{S}(I/I^2)$, which induces the desired isomorphic map between $\mathrm{S}(V)$ and $\mathrm{S}(I/I^2)$. □

Definition 4.2.6. The $\mathcal{I}$-graded algebra of formal power series on V is the R-module

$$\overline{\mathrm{S}(V)} = \prod_{n\in\mathbb{N}} \mathrm{S}^n(V)$$

equipped with the canonical algebraic multiplication.

Remark 4.2.5. As is in the case of $\mathcal{I} = \mathbb{Z}$ [Fai17], it is actually crucial to work with $\overline{\mathrm{S}(V)}$ instead of $\mathrm{S}(V)$ when the even part of V is non-trivial. The former allows us to have a coordinate description of morphisms between "$\mathcal{I}$-graded domains", a notion of partition of unity for "$\mathcal{I}$-graded manifolds", and more.

Let I be the kernel of the canonical body map of $\mathrm{S}(V)$. One can equip $\mathrm{S}(V)$ with the so-called I-adic topology.[2] Moreover, one can consider the I-adic completion of $\mathrm{S}(V)$ which is defined as the inverse limit

$$\widehat{\mathrm{S}(V)}_I := \varprojlim \mathrm{S}(V)/I^n$$

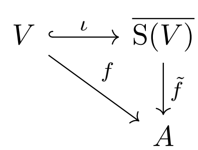
of the inverse system $((\mathrm{S}(V)/I^n)_{n\in\mathbb{N}}, (\pi_{m,n})_{n\leq m\in\mathbb{N}})$, where $\pi_{m,n} : \mathrm{S}(V)/I^m \to \mathrm{S}(V)/I^n$ is the canonical projection. Note that there is also a canonical projection $\mathrm{S}(V) \to \mathrm{S}(V)/I^n$ for each $n \in \mathbb{N}$. By the universal property of the inverse limit, one obtains a morphism

$$\iota_I : \mathrm{S}(V) \to \widehat{\mathrm{S}(V)}_I$$

with kernel being $\bigcap_{n\geq 0} I^n = \{0\}$. On the other hand, it is easy to see that $\mathrm{S}(V)/I^n \cong \bigoplus_{i=0}^{n-1} \mathrm{S}^i(V)$ for $n \geq 1$. It follows that there is a canonical isomorphism $\widehat{\mathrm{S}(V)}_I \cong \overline{\mathrm{S}(V)}$ under which ι_I coincides with the canonical inclusion $\mathrm{S}(V) \hookrightarrow \overline{\mathrm{S}(V)}$.

In fact, $\mathrm{S}(V)$ can be made into a metric space such that $\overline{\mathrm{S}(V)}$ is the completion of $\mathrm{S}(V)$ with respect to the metric structure [Sin11]. The metric-induced topology on $\overline{\mathrm{S}(V)}$, with a slight abuse of notation, coincides with the I-adic topology on $\overline{\mathrm{S}(V)}$, where $I = \prod_{n>0} \mathrm{S}^n(V)$.

Lemma 4.2.3. *Let A be a commutative $\mathcal{I}$-graded R-algebra. Let J be an ideal of A such that A is J-adic complete. $\overline{\mathrm{S}(V)}$ is universal in the sense that, given a morphism $f : V \to A$ such that $f(V) \subset J$, there exists a unique (continuous) algebraic homomorphism $\tilde{f} : \overline{\mathrm{S}(V)} \to A$ such that the following diagram commutes*



[2]To each point x of $\mathrm{S}(V)$ one assigns a collection of subsets $\mathcal{B}(x) = \{x + I^n\}_{x\in A, n>0}$. The I-adic topology is then the unique topology on $\mathrm{S}(V)$ such that $\mathcal{B}(x)$ forms a neighborhood base of x for all x.

Proof. We already know that f induces a unique morphism $f' : \mathrm{S}(V) \to A$ such that $f' \circ \iota = f$. By assumption, f' extends naturally to a morphism $\tilde{f} : \overline{\mathrm{S}(V)} \to \widehat{A}_J \cong A$.
Claim: $\tilde{f}$ is continuous.
Proof: It suffices to show that $\tilde{f}^{-1}(J^m)$ is a neighborhood of 0 for any $m \in \mathbb{N}$. By assumption, $I \subset \tilde{f}^{-1}(J)$. It follows that $I^m \subset \tilde{f}^{-1}(J)^m \subset \tilde{f}^{-1}(J^m)$. ■

Since $\mathrm{S}(V)$ is dense in $\overline{\mathrm{S}(V)}$ and $\tilde{f}|_{\mathrm{S}(V)} = f'$, $\tilde{f}$ is also unique. □

Remark 4.2.6. Likewise, we have a canonical body map of $\overline{\mathrm{S}(V)}$ induced from the zero map $V \to R$. Similar results like Lemma 4.2.1 and Lemma 4.2.2 also hold. For example, we have

$$\overline{\mathrm{S}(V)} \cong \overline{\mathrm{S}(I/I^2)},$$

where V and I are as in Lemma 4.2.2.

Lemma 4.2.4. *Let ϵ be the canonical body map of $\overline{\mathrm{S}(V)}$. Then for $f \in \overline{\mathrm{S}(V)}$, f is invertible if and only if $\epsilon(f)$ is invertible.*

Proof. "⇒": Trivial.
"⇐": Suppose $\epsilon(f) = c$ where $c \in R$ is invertible. We can write $f = c + f'$ where $f' \in \prod_{n\geq 1} \mathrm{S}^n(V)$. Note that $(f')^k \in \prod_{n\geq k} \mathrm{S}^n(V)$ for all $k > 0$. We can then set the inverse of f to be the formal sum $f^{-1} := c^{-1} \sum_{k\in\mathbb{N}} (-1)^k (c^{-1} f')^k$. ($f^{-1}$ is well-defined because the formal sum restricted to each $\mathrm{S}^n(V)$ is a finite sum.) □

Corollary 4.2.1. *$\overline{\mathrm{S}(V)}$ is local if R is local.*

Proof. Choose a non-unit $f \in \overline{\mathrm{S}(V)}$. Let $c = \epsilon(f)$. By Lemma 4.2.4, c is a non-unit. Since R is local, $1 - c$ is invertible. $1 - f$ is then a unit by Lemma 4.2.4. □

Recall that a ringed space $(X, \mathcal{O})$ is a topological space X with a sheaf of rings $\mathcal{O}$ on X.

Definition 4.2.7. An $\mathcal{I}$-graded ringed space is a ringed space $(X, \mathcal{O})$ such that

1. $\mathcal{O}(U)$ is an $\mathcal{I}$-graded algebra for any open subset U of X;
2. the restriction morphism $\rho_{V,U} : \mathcal{O}(U) \to \mathcal{O}(V)$ is a morphism of $\mathcal{I}$-graded algebras.

A morphism between two $\mathcal{I}$-graded ringed spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is just a morphism $\varphi = (\tilde{\varphi}, \varphi^*)$ between ringed spaces such that $\varphi^*_U : \mathcal{O}_2(U) \to \mathcal{O}_1(\tilde{\varphi}^{-1}(U))$ preserves the $\mathcal{I}$-grading for any open subset U of X_2.

Let (X, C) be a ringed space where $C(U)$ are commutative rings. One can define $\mathcal{I}$-graded C-modules and commutative $\mathcal{I}$-graded C-algebras in a similar way. In particular, the structure sheaf $\mathcal{O}$ of an $\mathcal{I}$-graded ringed space can be viewed as an $\mathcal{I}$-graded C-algebra if C is a sub-sheaf of $\mathcal{O}$ such that $C(U)$ are homogeneous sub-algebras of degree 0 of $\mathcal{O}(U)$.

Definition 4.2.8. Let $\mathcal{F}$ be an $\mathcal{I}$-graded C-module. The formal symmetric power $\overline{\mathrm{S}(\mathcal{F})}$ of $\mathcal{F}$ is the sheafification of the presheaf

$$U \to \overline{\mathrm{S}(\mathcal{F}(U))},$$

where $\overline{\mathrm{S}(\mathcal{F}(U))}$ is the $\mathcal{I}$-graded algebra of formal power series on the $C(U)$-module $\mathcal{F}(U)$.

By definition, $\overline{\mathrm{S}(\mathcal{F})}$ is a commutative $\mathcal{I}$-graded C-algebra.

Lemma 4.2.5. *Let $\mathcal{A}$ be a commutative $\mathcal{I}$-graded C-algebra. Let $\mathcal{B}$ be a sub-sheaf of $\mathcal{A}$ such that $\mathcal{A}(U)$ is $\mathcal{B}(U)$-adic complete for all open subsets U. $\overline{\mathrm{S}(\mathcal{F})}$ is universal in the sense that, given a morphism of $\mathcal{I}$-graded C-modules $F : \mathcal{F} \to \mathcal{A}$ such that $F(\mathcal{F}(U)) \subset \mathcal{B}(U)$ for all open subsets U, there exists a unique morphism of $\mathcal{I}$-graded C-algebras $\tilde{F} : \overline{\mathrm{S}(\mathcal{F})} \to \mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F} & \overset{\iota}{\hookrightarrow} & \overline{\mathrm{S}(\mathcal{F})} \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle \tilde{F}} \\ & & \mathcal{A} \end{array}$$

where $\iota : \mathcal{F} \to \overline{\mathrm{S}(\mathcal{F})}$ is the canonical monomorphism.

Proof. This follows directly from the universal property of sheafification[3] and the universal property of $\overline{\mathrm{S}(\mathcal{F}(U))}$ stated in Lemma 4.2.3. □

To end this section, we state the following lemma taken from [Man97].

Lemma 4.2.6. *Let*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{4.2.2}$$

be a short exact sequence of C-modules where $\mathcal{F}$ and $\mathcal{G}$ are locally free C-modules. Then the obstruction of the existence of a splitting of (4.2.2) can be represented as an element in the first sheaf cohomology group $H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))$ of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

4.2.1 Monoidally graded domains

Throughout this subsection, V is a real $\mathcal{I}$-graded vector space with $V_0 = 0$. The dimension of the homogeneous sub-space V_i of V is m_i. (We also assume that only finitely many of m_i are non-zero.)

[3]That is, given a presheaf $\mathcal{F}$, a sheaf $\mathcal{G}$, and a presheaf morphism $F : \mathcal{F} \to \mathcal{G}$, there exists a unique sheaf morphism $\tilde{F} : \mathcal{F}^\sharp \to \mathcal{G}$ such that $\tilde{F} \circ \iota = F$, where $\mathcal{F}^\sharp$ is the sheafification of $\mathcal{F}$ and $\iota : \mathcal{F} \to \mathcal{F}^\sharp$ is the canonical morphism.

Definition 4.2.9. Let U be a domain of $\mathbb{R}^n$. An $\mathcal{I}$-graded domain $\mathcal{U}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(U, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of $\overline{\mathrm{S}(V)}$-valued smooth functions.

Remark 4.2.7. $\mathcal{U}$ is a locally ringed space by Corollary 4.2.1.

For example, a domain U with the sheaf C^∞ of smooth functions on U is an $\mathcal{I}$-graded domain of dimension $n|(0, \cdots)$, which is denoted again by U for simplicity.

Lemma 4.2.7. *Let $F : C^\infty \to C^\infty$ be an endomorphism of sheaves of commutative rings on U. Then F must be the identity.*

Proof. First, we show that F is actually an endomorphism of sheaves of unital $\mathbb{R}$-algebras on U. It suffices to show that F restricted to any open subset of U sends a constant function to itself. We know this is true for $\mathbb{Q}$-valued constant functions. Now, if F sends a constant function f to a non-constant function g, then one can find two rational numbers b_1 and b_2 such that $g - b_1$ and $g - b_2$ are non-invertible. But then the pre-images $f - b_1$ and $f - b_2$ are non-invertible, which implies that f is non-constant: a contradiction. To show that g actually equals f, use the fact that the only field endomorphism of $\mathbb{R}$ is the identity.

Let $p \in U$. F induces a unital ring endomorphism F_p on the stalk C^∞_p. On the other hand, for any open neighborhood $U_p \subset U$ of p, the evaluation map

$$\begin{aligned} \mathrm{ev} : C^\infty(U_p) &\to \mathbb{R} \\ f &\mapsto f(p) \end{aligned}$$

induces a map $\mathrm{ev}_p : C^\infty_p \to \mathbb{R}$. For $f_p \in C^\infty_p$, it is easy to see that f_p is invertible if and only if $\mathrm{ev}_p(f_p) \neq 0$. Let $c = \mathrm{ev}_p(F_p(f_p))$. $f_p - c$ is non-invertible. Hence $\mathrm{ev}_p(f_p) = c$. In other words, for any open subset U' of U, we have $F_{U'}(f)(p) = f(p)$ for all $f \in C^\infty(U')$ and all $p \in U'$. This implies $F = \mathrm{id}$. □

A morphism between $\mathcal{I}$-graded domains is just a morphism of $\mathcal{I}$-graded locally ringed spaces. Recall that we have the canonical body map $\epsilon : C^\infty(U) \otimes \overline{\mathrm{S}(V)} \to C^\infty(U)$.

Proposition 4.2.1. *There exists a unique monomorphism $\varphi : U \to \mathcal{U}$ with $\tilde{\varphi} = \mathrm{id}$.*

Proof. Existence is guaranteed by ϵ. Uniqueness follows from Remark 4.2.4 and Lemma 4.2.7. □

We also have a canonical morphism for the other direction $\mathcal{U} \to U$ induced by the canonical embedding $\iota : C^\infty(U) \to C^\infty(U) \otimes \overline{\mathrm{S}(V)}$.[4] Note that $\epsilon \circ \iota = \mathrm{id}$ on $C^\infty(U)$.

[4] There will be no longer such a canonical morphism if we go the category of $\mathcal{I}$-graded manifolds.

Proposition 4.2.2. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\varphi} & \mathcal{U}_2 \\ \uparrow & & \uparrow \\ U_1 & \xrightarrow{\tilde{\varphi}} & U_2 \end{array}$$

Proof. Let U be an open subset of U_2. Let $f \in \mathcal{O}_2(U)$. We need to show that

$$\epsilon(\varphi^*(f)) = \epsilon(f) \circ \tilde{\varphi}.$$

Suppose this does not hold. One can find a $p \in \tilde{\varphi}^{-1}(U)$ such that $\epsilon(\varphi^*(f))(p) = c \neq \epsilon(f)(\tilde{\varphi}(p))$. Then there exists an open neighborhood $U' \subset U$ of $\tilde{\varphi}(p)$ such that $\epsilon(f) - c$ is invertible. By Lemma 4.2.4, $f - c$ is also invertible on U', which implies that $\varphi^*(f - c)$ is invertible on $\tilde{\varphi}^{-1}(U') \subset \tilde{\varphi}^{-1}(U)$, which contradicts the fact that $\epsilon(\varphi^*(f - c))$ is non-invertible on $\tilde{\varphi}^{-1}(U')$. □

Definition 4.2.10. A coordinate system of $\mathcal{U}$ is a collection of functions $(x^\mu, \theta^{i,a})$ such that

1. x^μ are elements of $\mathcal{O}(U)_0$ such that $\epsilon(x^\mu)$ form a coordinate system of U;
2. $\theta^{i,a}$ are homogeneous elements of $\mathcal{O}(U)$ of degree $d(\theta^{i,a}) = i$, $i \neq 0$ and $a = 1, \cdots, m_i$, which generate $\mathcal{O}(U)$ as a $C^\infty(U)$-algebra.

Suppose that $\mathcal{I}$ can be given a total order $<$. It follows that any function $f \in \mathcal{O}(U)$ can be written uniquely in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(x^\mu) \prod_{j \in \mathcal{J}} (\theta^j)^{\beta^j}, \tag{4.2.3}$$

where

- $\mathcal{J} \in \mathrm{Pow}(\mathcal{I})$, $\beta = (\beta^j)_{j \in \mathcal{J}}$, $\beta^j = (\beta^j_1, \ldots, \beta^j_{m_j})$, $\beta^j_k \in \{0, 1\}$ if $p(j) = 1$, $\beta^j_k \in \mathbb{N}$ if $p(j) = 0$;
- $(\theta^j)^{\beta^j} = (\theta^{j,1})^{\beta^j_1} \cdots (\theta^{j,m_j})^{\beta^j_{m_j}}$, the product $\prod_{j \in \mathcal{J}} (\theta^j)^{\beta^j}$ is arranged in a proper order such that $(\theta^j)^{\beta^j}$ is on the left of $(\theta^{j'})^{\beta^{j'}}$ whenever $j < j'$;
- For a smooth function $g \in C^\infty(U)$, the notation $g(x^\mu)$ should be understood as

$$g(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(\epsilon(x^\mu))(x^1 - \epsilon(x^1))^{i_1} \cdots (x^n - \epsilon(x^n))^{i_n}. \tag{4.2.4}$$

  Hence, $g(x^\mu)$ is an element in $\mathcal{O}(U)_0$ instead of $C^\infty(U)$.

The sum in (4.2.3) is well-defined because, by assumption, only finitely many of m_j are non-zero.

Remark 4.2.8. One may wonder how we obtain (4.2.3). In fact, by definition, every function f can be expressed in the form

$$f = \sum_{\mathcal{J}} \sum_{\beta} f_{\mathcal{J},\beta}(\epsilon(x^\mu)) \prod_{j \in \mathcal{J}} (\theta^j)^{\beta^j}.$$

One can then define a map from $\mathcal{O}(U)$ to itself by sending $g(\epsilon(x^\mu))$ to $g(x^\mu)$. Now consider another map which sends $g(\epsilon(x^\mu))$ to $g^-(x^\mu)$, where

$$g^-(x^\mu) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{1}{i_1! \cdots i_n!} \partial_1^{i_1} \cdots \partial_n^{i_n} g(\epsilon(x^\mu))(\epsilon(x^1) - x^1)^{i_1} \cdots (\epsilon(x^n) - x^n)^{i_n}.$$

Using the binomial theorem, it is easy to see that the second map is the inverse of the first. In fact, the reader may notice that the map $g(\epsilon(x^\mu)) \mapsto g(x^\mu)$ is exactly the "Grassmann analytic continuation map" defined in [Rog07].

Corollary 4.2.2. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be as in Proposition 4.2.2. $\tilde{\varphi}$ is uniquely determined by φ^*.*

Proof. Let $(x^\mu, \theta^{i,a})$ be a coordinate system of $\mathcal{U}_2$. By Proposition 4.2.2, one has $\tilde{\varphi}^\mu = \epsilon(\varphi^* x^\mu)$, where $(\tilde{\varphi}^\mu)$ is $\tilde{\varphi}$ expressed in the coordinate system $(\epsilon(x^\mu))$ of U_2. □

Let ev be the evaluation map of $C^\infty(U)$ at $p \in U$. Let s_p denote ev $\circ\, \epsilon$. Let I_p denote the kernel of s_p. We follow [Lei80] to prove the following lemmas.

Lemma 4.2.8. *For any functions $f \in \mathcal{O}(U)$ and any integer $k \geq 0$, there is a polynomial P_k in the coordinates $(x^\mu, \theta^{i,a})$ such that $f - P_k \in I_p^{k+1}$.*

Proof. Use the classical Hadamard lemma and the decomposition (4.2.3). □

Lemma 4.2.9. *Let f and g be functions of $\mathcal{O}(U)$, then $f = g$ if and only if $f - g \in I_p^k$ for all $k \in \mathbb{N}$ and $p \in U$. In other words, $\bigcap_{p \in U} \bigcap_{k \in \mathbb{N}} I_p^k = \{0\}$.*

Proof. Let $h = f - g$. Apply the decomposition (4.2.3) to h, then by Lemma 4.2.8, $h_{\mathcal{J},\beta} = 0$ for all $\mathcal{J}$ and β. Hence $h = 0$. □

Lemma 4.2.10. *Any morphism of $\mathcal{I}$-graded $\mathbb{R}$-algebras $s : \mathcal{O}(U) \to \mathbb{R}$ must take the form $s = s_p$.*

Proof. Since we assume $V_0 = 0$, s can be reduced to a morphism $C^\infty(U) \to \mathbb{R}$. Let x^μ be a coordinate system of U. Let $f^\mu = x^\mu - s(x^\mu)$ and $h = \sum_\mu (f^\mu)^2$. Then $s(h) = 0$, which implies that h is non-invertible. In other words, there exists $p \in U$ such that $x^\mu(p) = s(x^\mu)$ for all μ. Now suppose there exists an $f \in C^\infty(U)$ such that $s(f) \neq s_p(f) = f(p)$. Consider the function $h' = h + (f - s(f))^2$. Since $h > 0$ for all points of $U/\{p\}$. We know $h' > 0$ on U. But this contradicts the fact that $s(h') = 0$. Hence s must equal s_p. □

Theorem 4.2.1. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{U}_1 = (U_1, \mathcal{O}_1)$ to $\mathcal{U}_2 = (U_2, \mathcal{O}_2)$. Let $(x^\mu, \theta^{i,a})$ be a coordinate system of $\mathcal{U}_2$. Then φ^* is uniquely determined by the equations*

$$\varphi^* x^\mu = y^\mu, \quad \varphi^* \theta^{i,a} = \eta^{i,a},$$

where $y^\mu \in \mathcal{O}(U_1)_0$, $\eta^{i,a} \in \mathcal{O}(U_1)_i$ and $(\epsilon(y^\mu))(p) \in U_2$ for all $p \in U_1$.

Proof. Let $f \in \mathcal{O}_2(U_2)$. By (4.2.3), to construct $\varphi^* f$, we only need to define $\varphi^* f_{\mathcal{J},\beta}$. But this is straightforward: one just replaces x^μ with y^μ and $\theta^{i,a}$ with $\eta^{i,a}$ in (4.2.4). By construction, we have $\varphi^* 1 = 1$, $\varphi^*(f+g) = \varphi^* f + \varphi^* g$, and $\varphi^*(fg) = \varphi^* f \varphi^* g$, hence φ^* is well-defined.

Now suppose there exists another φ'^* which equals φ^* on coordinates. Then they also equal on all polynomials of $(x^\mu, \theta^{i,a})$. By Lemma 4.2.8 and Lemma 4.2.9, $\varphi'^* = \varphi^*$. □

Remark 4.2.9. Theorem 4.2.1 can be seen as a generalization of the Global Chart Theorem in the $\mathbb{Z}_2$-graded setting (see Theorem 4.2.5 in [CCF11]).

Corollary 4.2.3. *Let $\varphi^* : \mathcal{O}_2(U_2) \to \mathcal{O}_1(U_1)$ be a ring homomorphism which preserves the $\mathcal{I}$-grading. Then there exists a unique morphism $\varphi' : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\varphi'^* = \varphi^*$.*

Proof. First, one can easily show that φ^* is actually an $\mathbb{R}$-algebra homomorphism using arguments similar to those in Lemma 4.2.7. Choose a point $p \in U_1$, by Lemma 4.2.10, the morphism $s_p \circ \varphi^*$ must take the form $s_{p'}$ for some $p' \in U_2$. It follows that $\varphi^*(I_{p'}) \subset I_p$. Let $(x^\mu, \theta^{i,a})$ be a coordinate system of $\mathcal{U}_2$, we then have $\varphi^* x^\mu - \epsilon(x^\mu)(p') \in I_p$. Hence $(\epsilon(\varphi^* x^\mu))(p) \in U_2$ for all $p \in U_1$. Next, observe that a coordinate system of U_2 restricted to any open subset of it gives a coordinate system of that open subset. Now apply Theorem 4.2.1 and Corollary 4.2.2. □

4.2.2 Monoidally graded manifolds

Definition 4.2.11. Let M be a n-dimensional manifold. An $\mathcal{I}$-graded manifold $\mathcal{M}$ of dimension $n|(m_i)_{i\in\mathcal{I}}$ is an $\mathcal{I}$-graded ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain of dimension $n|(m_i)_{i\in\mathcal{I}}$. That is, for each $x \in M$, there exist an open neighborhood U_x of x, an $\mathcal{I}$-graded domain $\mathcal{U}$, and an isomorphism of locally ringed spaces

$$\varphi = (\tilde{\varphi}, \varphi^*) : (U_x, \mathcal{O}_M|_{U_x}) \to \mathcal{U}.$$

φ is called a chart of $\mathcal{M}$ on U_x.[5]

M with the sheaf C^∞ of smooth functions on M is an $\mathcal{I}$-graded manifold of dimension $n|(0, \cdots)$, which is denoted again by M for simplicity. We call M together with a morphism $\mathcal{O} \to C^\infty$ an underlying manifold of $\mathcal{M}$. Equivalently, an underlying manifold of $\mathcal{M}$ is a morphism $\varphi : M \to \mathcal{M}$ with $\tilde{\varphi} = \text{id}$.

[5] We often refer to U_x as a chart too.

Let $x \in M$. An open neighborhood U of x on which $\mathcal{O}(U) \cong C^\infty(U) \otimes \overline{\mathrm{S}(V)}$ is called a splitting neighborhood. Clearly, every chart is a splitting neighborhood, but not vice versa. The set of splitting neighborhoods form a base of the topology of M. For a splitting U, there exists sub-algebras $C(U)$ and $D(U)$ of $\mathcal{O}(U)$ such that $C(U) \cong C^\infty(U)$, $D(U) \cong \overline{\mathrm{S}(V)}$ and $\mathcal{O}(U) = C(U) \otimes D(U)$. This induces an epimorphism

$$\epsilon : \mathcal{O}(U) \to C^\infty(U)$$

of graded commutative $\mathbb{R}$-algebras, which is a body map of $\mathcal{O}(U)$.

Definition 4.2.12. A local coordinate system of $\mathcal{M}$ is the data $(U, x^\mu, \theta^{i,a})$ where

1. U is a splitting neighborhood of $\mathcal{M}$;
2. $x^1, \ldots, x^n$ are elements of $C(U)$ such that $\epsilon(x^1), \ldots, \epsilon(x^n)$ are local coordinate functions of M on U;
3. $\theta^{i,a}$ are homogeneous elements of $\mathcal{D}(U)$ of degree $d(\theta^{i,a}) = i$, $i \neq 0$ and $a = 1, \cdots, m_i$, which generate $\mathcal{O}(U)$ as a $C(U)$-algebra.

Remark 4.2.10. By Theorem 4.2.1, every local coordinate system determines a chart (non-canonically).

Now, let U be an arbitrary open subset of M. We can choose a collection of charts $\{U_\alpha\}$ such that $U = \bigcup_\alpha U_\alpha$. For $f \in \mathcal{O}(U)$, one can apply the restriction morphisms to f to get a sequence of sections f_α in $\mathcal{O}(U_\alpha)$. Now, apply ϵ to each of them to get a sequence of smooth functions $\tilde{f}_\alpha$ in $C^\infty(U_\alpha)$. By Proposition 4.2.1, $\tilde{f}_\alpha$ must be compatible with each other, hence can be glued together to get a smooth function $\tilde{f}$ over U. In this way, we construct a body map for every open subset of M, which are compatible with restrictions. In other words, ϵ can be seen as a sheaf morphism from $\mathcal{O}$ to C^∞.

Proposition 4.2.3. *There exists a unique monomorphism $\varphi : M \to \mathcal{M}$ with $\tilde{\varphi} = \mathrm{id}$.*

Proof. Existence is guaranteed by ϵ. Uniqueness follows from Proposition 4.2.1. □

Proposition 4.2.4. *Let $\varphi = (\tilde{\varphi}, \varphi^*)$ be a morphism from $\mathcal{M} = (M, \mathcal{O}_M)$ to $\mathcal{N} = (N, \mathcal{O}_N)$. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow{\tilde{\varphi}} & N \end{array}$$

Proof. The proof is essentially the same as the one of Proposition 4.2.2. □

Lemma 4.2.11. *Let $\mathcal{O}^1$ be the kernel of ϵ. $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. That is, for any open subset U, $\mathcal{O}(U)$ is $\mathcal{O}^1(U)$-adic complete.*

Proof. Let $\widehat{\mathcal{O}}$ be the $\mathcal{O}^1$-adic completion of $\mathcal{O}$.[6] There exists a canonical morphism $\iota : \mathcal{O} \to$

[6]For each open subset U, one has $\widehat{\mathcal{O}}(U) = \varprojlim \mathcal{O}(U)/\mathcal{O}^n(U)$, where $\mathcal{O}^n(U)$ is the n-th power of $\mathcal{O}^1(U)$.

$\widehat{\mathcal{O}}$. Since $\mathcal{O}$ is locally $\mathcal{O}^1$-adic complete, the induced stalk morphism $\iota_p : \mathcal{O}_p \to \widehat{\mathcal{O}}_p$ is an isomorphism for each $p \in M$. It follows that $\mathcal{O}$ is $\mathcal{O}^1$-adic complete. □

Definition 4.2.13. An $\mathcal{I}$-graded manifold $\mathcal{M}$ is called projected if there exists a splitting of the short exact sequence of sheaves of rings

$$0 \longrightarrow \mathcal{O}^1 \longrightarrow \mathcal{O} \xrightarrow{\epsilon} C^\infty \longrightarrow 0, \tag{4.2.5}$$

where $\mathcal{O}^1$ is the kernel of ϵ.

The structure sheaf $\mathcal{O}$ of a projected manifold is a C^∞-module.

Definition 4.2.14. A projected $\mathcal{I}$-graded manifold $\mathcal{M}$ is called split if there exists a splitting of the short exact sequence of C^∞-modules

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}^1 \xrightarrow{\pi} \mathcal{O}^1/\mathcal{O}^2 \longrightarrow 0, \tag{4.2.6}$$

where $\mathcal{O}^2$ is the square of $\mathcal{O}^1$, π is the canonical quotient map.

Let $\mathcal{O}$ be the structure sheaf of a projected $\mathcal{I}$-graded manifold. Let $\mathcal{F}$ denote the sheaf $\mathcal{O}^1/\mathcal{O}^2$. $\mathcal{F}$ is an $\mathcal{I}$-graded C^∞-module and we can define its formal symmetric power $\overline{\mathrm{S}(\mathcal{F})}$. By construction, the ringed space $\mathcal{M}_S = (M, \overline{\mathrm{S}(\mathcal{F})})$ is also a projected $\mathcal{I}$-graded manifold.

Lemma 4.2.12. *Let $\mathcal{M} = (M, \mathcal{O})$ be a projected $\mathcal{I}$-graded manifold. $\mathcal{M}$ is split if and only if $\mathcal{M} \cong \mathcal{M}_S$.*

Proof. Let $\iota : \mathcal{F} \to \overline{\mathrm{S}(\mathcal{F})}$ be the canonical monomorphism. ι splits the short exact sequence (4.2.6). Now if $\mathcal{M}$ is split, one can find a monomorphism $F : \mathcal{F} \to \mathcal{O}$ of C^∞-modules such that $F(\mathcal{F}(U)) \subset \mathcal{O}^1(U)$ for any open subset U. By Lemma 4.2.5 and Lemma 4.2.11, there exists a unique C^∞-algebra morphism $\tilde{F} : \overline{\mathrm{S}(\mathcal{F})} \to \mathcal{O}$ such that $\tilde{F} \circ \iota = F$. By Remark 4.2.6, $\tilde{F}$ induces an isomorphism for each stalk. Hence $\mathcal{M} \cong \mathcal{M}_S$. □

Lemma 4.2.13. *Every projected $\mathcal{I}$-graded manifold is split.*

Proof. Due to the existence of a smooth partition of unity on M, $H^q(M, \mathrm{Hom}(\mathcal{O}^1/\mathcal{O}^2, \mathcal{O}^2))$ vanishes for $q \geq 1$. By Lemma 4.2.6, there is no obstruction of the existence of a splitting of (4.2.6). □

Lemma 4.2.14. *Every $\mathcal{I}$-graded manifold is projected.*

Proof. Let $\mathcal{O}_{(i)} = \mathcal{O}/\mathcal{O}^{i+1}$. Let $\phi_{(0)} : C^\infty \to \mathcal{O}_{(0)}$ be the identity. (By Proposition 4.2.3, there is a unique identification $\mathcal{O}_{(0)} \cong C^\infty$.) One can construct by induction on i mappings $\phi_{(i+1)} : C^\infty \to \mathcal{O}_{(i+1)}$ such that $\pi_{i+1,i} \circ \phi_{(i+1)} = \phi_{(i)}$, where $\pi_{i+1,i} : \mathcal{O}_{i+1} \to \mathcal{O}_i$ is the canonical epimorphism. As is shown in [Man97], one can construct an element

$$\omega(\phi_{(i)}) \in H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0)$$

as the obstruction to the existence of $\phi_{(i+1)}$, where $\mathcal{T}$ is the tangent sheaf of M. Due to the existence of a smooth partition of unity on M, $H^1(M, (\mathcal{T} \otimes \mathrm{S}^{i+1}(\mathcal{F}))_0) = 0$ and $\omega(\phi_{(i)}) = 0$. It follows that there exists a unique morphism $\phi : C^\infty \to \varprojlim \mathcal{O}_{(i)}$ such that $\pi_i \circ \phi = \phi_{(i)}$, where $\pi_i : \varprojlim \mathcal{O}_{(i)} \to \mathcal{O}_i$ is the canonical epimorphism. By Lemma 4.2.11, ϕ can be seen as a morphism from C^∞ to $\mathcal{O}$. Note that $\pi_0 = \epsilon$ and $\pi_0 \circ \phi = \phi_{(0)} = \mathrm{id}$. ϕ splits (4.2.5). □

Corollary 4.2.4. *Every $\mathcal{I}$-graded manifold is split.*

Let V be a (finite dimensional) $\mathcal{I}$-graded vector space. An $\mathcal{I}$-graded vector bundle $\pi : E \to M$ is a vector bundle such that the local trivialization map $\varphi_U : \pi^{-1}(U) \to U \times V$ is a morphism of $\mathcal{I}$-graded vector spaces when restricted to $\pi^{-1}(x)$, $x \in U \subset M$. In other words, $E = \bigoplus_{k \in \mathcal{I}} E_k$ where E_k are vector bundles whose fibers consist of elements of degree k. Let $l \in \mathcal{I}$. To any $\mathcal{I}$-graded vector bundle E we can associate an $\mathcal{I}$-graded ringed space $E[l]$ with the underlying topological space being M and the structure sheaf being the sheaf of sections of $\overline{\mathrm{S}(\bigoplus_{k \in \mathcal{I}} (E_{k+l})^*)}$. This is an $\mathcal{I}$-graded manifold in our sense if the fiber of E does not contain elements of degree k' such that $k' + l = 0$. Corollary 4.2.4 can then be rephrased as

Theorem 4.2.2. *Every $\mathcal{I}$-graded manifold can be obtained from an $\mathcal{I}$-graded vector bundle.*

4.2.3 Monoidally graded manifolds with auxiliary parts

Throughout this subsection, V is a real $\mathcal{I}$-graded vector space with $\dim V_i = m_i$, $m_0 \neq 0$.

Definition 4.2.15. An $\mathcal{I}$-graded domain with auxiliary parts is an $\mathcal{I}$-graded ringed space $\mathcal{U} = (U, \mathcal{O})$, where U is a domain of $\mathbb{R}^n$ and $\mathcal{O}$ is the sheaf of $\overline{\mathrm{S}(V)}$-valued smooth functions over U. An $\mathcal{I}$-graded manifold with auxiliary parts is an $\mathcal{I}$-graded ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain with auxiliary parts.

Lemma 4.2.15. *There exists a unique R-linear body map $\epsilon : \overline{\mathrm{S}(V \otimes_{\mathbb{R}} R)} \to R$, where $R = C^\infty(U)$.*

Proof. ϵ is determined by its restriction to $\overline{\mathrm{S}(V_0 \otimes_{\mathbb{R}} R)}$, which is again determined by the morphism $V_0 \otimes_{\mathbb{R}} R \to R$ by the universal property of the algebra of formal power series. Let ξ be a non-zero element in V_0. Consider the element $f = \sum_{n=0}^{\infty} f_n \otimes \xi^n \in \overline{\mathrm{S}(V_0 \otimes_{\mathbb{R}} R)}$, where $f_n = n^2$ are constant functions over U. Since the radius of convergence of f is zero at each point of U, ϵ must send $1 \otimes \xi$ to 0. In other words, the morphism $V_0 \otimes_{\mathbb{R}} R \to R$ has to be the zero morphism. □

Proposition 4.2.5. *There exists a unique monomorphism $\varphi : U \to \mathcal{U}$ with $\tilde{\varphi} = \mathrm{id}$*

Proof. This follows directly from Lemmas 4.2.15 and 4.2.7. □

It follows from Proposition 4.2.5 that there exists a unique underlying manifold of an $\mathcal{I}$-graded manifold with auxiliary parts. The proof of Batchelor's theorem is essentially the same as the case where $V_0 = 0$.

Note that $\overline{\mathrm{S}(V)} \cong \overline{\mathrm{S}(V_0)} \otimes \overline{\mathrm{S}(V_{\neq 0})}$, where $V_{\neq 0} = \bigoplus_{i \neq 0} V_i$. We denote the sub-algebra $\mathrm{S}(V_0) \otimes \overline{\mathrm{S}(V_{\neq 0})}$ of $\overline{\mathrm{S}(V)}$ by $\overline{\mathrm{S}(V)}_{poly}$.

Definition 4.2.16. An $\mathcal{I}$-graded domain with polynomial auxiliary parts is an $\mathcal{I}$-graded ringed space $\mathcal{U} = (U, \mathcal{O})$, where U is a domain of $\mathbb{R}^n$ and $\mathcal{O}$ is the sheaf of $\overline{\mathrm{S}(V)}_{poly}$-valued smooth functions over U. An $\mathcal{I}$-graded manifold with polynomial auxiliary parts is an $\mathcal{I}$-graded ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ which is locally isomorphic to an $\mathcal{I}$-graded domain with polynomial auxiliary parts.

It is not hard to show that Batchelor's theorem still holds for $\mathcal{I}$-graded manifolds with polynomial auxiliary parts. However, the uniqueness of an underlying manifold is lost in this case, because there exist nontrivial morphisms from the ring of polynomials to the ring of smooth functions.

Definition 4.2.17. Let $\mathcal{M}$ be an $\mathcal{I}$-graded manifold with polynomial auxiliary parts. Fix a splitting for $\mathcal{M}$, that is, identify $\mathcal{M}$ with an $\mathcal{I}$-graded vector bundle E. Let $\mathcal{M}_{red}$ be the $\mathcal{I}$-graded manifold obtained from $E_{\neq 0} := \bigoplus_{i \neq 0} E_i$. $\mathcal{M}_{red}$ is called the reduced $\mathcal{I}$-graded manifold of $\mathcal{M}$. We say that a morphism $\varphi : \mathcal{M}_{red} \to \mathcal{M}$ integrates out the auxiliary parts of $\mathcal{M}$ if it is induced by the canonical inclusion $E_{\neq 0} \to E$ together with a morphism $\Gamma(\mathrm{S}(E_0^*)) \to C^\infty(M)$.

4.3 Vector fields and differential forms

Throughout this section, every algebra is assumed to be real. To define vector fields and differential forms over an $\mathcal{I}$-graded manifold, we need to generalize the notion of an $\mathcal{I}$-graded R-algebra such that R can be taken to be a commutative $\mathcal{I}$-graded algebra instead of just a usual commutative ring.

Definition 4.3.1. Let R be an associative commutative $\mathcal{I}$-graded algebra. An $\mathcal{I}$-graded left R-module V is a left R-module with an $\mathcal{I}$-grading $V = \bigoplus_{i \in \mathcal{I}} V_i$ which is preserved by the scalar multiplications, i.e.,

$$R_i A_j \subset A_{i+j},$$

for all $i, j \in \mathcal{I}$.

Remark 4.3.1. An $\mathcal{I}$-graded left R-module V is automatically a right R-module by setting $vr = (-1)^{p(r)p(v)} rv$ for $r \in R$ and $v \in V$. Clearly, the right multiplications also preserve the $\mathcal{I}$-grading on V. We therefore drop the prefixes "left-" and "right-" from now on.

Let V and W be two $\mathcal{I}$-graded R-modules. It is straightforward to define $V \oplus W$, $V \otimes W$ (viewed as the tensor product of a right R-module and a left R-module) and $\mathrm{Hom}(V, W)$. Like before, the first two are naturally $\mathcal{I}$-graded, while the last one is $K(\mathcal{I})$-graded.

Definition 4.3.2. An $\mathcal{I}$-graded algebra A over R is an $\mathcal{I}$-graded R-module A equipped with an algebraic multiplication which preserves the $\mathcal{I}$-grading on A such that

$$r(ab) = (ra)b = (-1)^{p(r)p(a)} a(rb)$$

for $r \in R$ and $a, b \in A$. If R is unital, then $1a = a$ should also hold for all $a \in A$, where $1 \in R$ is the identity element. A is said to be commutative if $ab = (-1)^{p(a)p(b)} ba$ for all $a, b \in A$.

Again, one can easily define the tensor algebra $\mathrm{T}(V)$, the symmetric algebra of $\mathrm{S}(V)$, and the algebra of formal power series $\overline{\mathrm{S}(V)}$ of an $\mathcal{I}$-graded R-module V. They are all $\mathcal{I}$-graded algebras over R.

Definition 4.3.3. An $\mathcal{I}$-graded Lie algebra (over R) is an $\mathcal{I}$-graded algebra L (over R) whose multiplications (denoted by $[\cdot, \cdot]$) satisfy

$$[a, b] = -(-1)^{p(a)p(b)} [b, a], \tag{4.3.1}$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)} [b, [a, c]], \tag{4.3.2}$$

for all $a, b, c \in L$.

The space of endomorphisms $\mathrm{Hom}(A, A)$ (or $\mathrm{gl}(A)$) of an $\mathcal{I}$-graded real vector space A is an associative $K(\mathcal{I})$-graded algebra. It can be also viewed as a $K(\mathcal{I})$-graded Lie algebra by setting

$$[f, g] = f \circ g - (-1)^{p(f)p(g)} g \circ f$$

for all $f, g \in \mathrm{Hom}(A, A)$. In the case of A being an $\mathcal{I}$-graded algebra, an endomorphism D is said to be a derivation if

$$D(ab) = D(a)b + (-1)^{p(D)p(a)} aD(b). \tag{4.3.3}$$

It is easy to check that derivations of A form a $\mathcal{K}(I)$-graded Lie subalgebra of $\mathrm{gl}(A)$, denoted by $\mathrm{Der}(A)$.

Definition 4.3.4. Let $\mathcal{M} = (M, \mathcal{O})$ be an $\mathcal{I}$-graded manifold. Let U be an open subset of M. A (local) vector field over $\mathcal{M}$ is a derivation of $\mathcal{O}(U)$.

Vector fields over $\mathcal{M}$ actually form a sheaf called the "tangent sheaf" of $\mathcal{M}$. To prove this, we need the partition of unity Lemma in the $\mathcal{I}$-graded setting.

Lemma 4.3.1. *Let $f \in \mathcal{O}(M)$ such that $\epsilon(f)(x) \neq 0$ for all $x \in M$. f is invertible.*

Proof. Choose an open cover $\{U_\alpha\}$ of $\mathcal{I}$-graded charts of M. Let f_α denote $\rho_{U_\alpha,M}(f)$. Each f_α is invertible by Lemma 4.2.4. Let f_α^{-1} denote the inverse of f_α. By uniqueness of the inverse, f_α^{-1} are compatible with each other, hence can be glued to give a section $f^{-1} \in \mathcal{O}(M)$, which is the inverse of f. □

Lemma 4.3.2. *Let $\{U_\alpha\}$ be an open cover of M. There exists a locally finite refinement $\{V_\beta\}$ of $\{U_\alpha\}$ and a family of functions $\{l_\beta \in \mathcal{O}(M)_0\}$ such that*

1. $\operatorname{supp} l_\beta \subset V_\beta$ *is compact and* $\epsilon(l_\beta) \geq 0$ *for all* β*;*
2. $\sum_\beta l_\beta = 1$.

Proof. First, find a partition of unity $\{\tilde{l}_\beta\}$ of M subordinate to $\{V_\beta\}$. Choose $l'_\beta \in \mathcal{O}(V_\beta)$ such that $\epsilon(l'_\beta) = \tilde{l}_\beta$. Since $\tilde{l}_\beta$ are invertible, we then set l_β to be $(\sum_\beta l'_\beta)^{-1} l'_\beta$. □

Lemma 4.3.3. *Let U be open in M. Let W be closed in M and $W \subset U$. Then for any $f \in \mathcal{O}(U)$, there exists a $g \in \mathcal{O}(M)$ and an open neighborhood V of W in U such that $\rho_{V,U}(f) = \rho_{V,M}(g)$ and* $\operatorname{supp} g \subset \operatorname{supp} f$.

Lemma 4.3.4. *Let U and V be open in M such that $V \subset U$. Let D be a derivation of $\mathcal{O}(U)$. Then there exists a unique derivation D' of $\mathcal{O}(V)$ such that $D'(\rho_{V,U}(f)) = \rho_{V,U}(D(f))$ for all $f \in \mathcal{O}(U)$.*

We skip the proofs of Lemmas 4.3.3 and 4.3.4 since they are essentially the same as the ones in the $\mathbb{Z}_2$-graded case [Lei80]. Lemma 4.3.4 actually implies that the vector fields over $\mathcal{M}$ form a presheaf $\mathfrak{X}$ on M.

Proposition 4.3.1. *$\mathfrak{X}$ is a sheaf.*

Proof. Let U be an open subset of M with an open cover $\{U_\alpha\}$. Let $D_\alpha \in \mathfrak{X}(U_\alpha)$ be compatible with each other. We define a $D \in \mathfrak{X}(U)$ by setting $D(f)$ to be unique function obtained by gluing $D_\alpha(f_\alpha)$, where $f \in \mathcal{O}(U)$ and $f_\alpha = \rho_{U_\alpha,U}(f)$. $D(f)$ is well defined because $\rho_{U_\alpha\cap U_\beta,U_\alpha}(D_\alpha(f_\alpha)) = \rho_{U_\alpha\cap U_\beta,U_\alpha}(D_\alpha)(\rho_{U_\alpha\cap U_\beta,U_\alpha}(f_\alpha)) = \rho_{U_\alpha\cap U_\beta,U_\beta}(D_\beta)(\rho_{U_\alpha\cap U_\beta,U_\beta}(f_\beta)) = \rho_{U_\alpha\cap U_\beta,U_\beta}(D_\beta(f_\beta))$. □

Definition 4.3.5. $\mathcal{X}$ is called the tangent sheaf of $\mathcal{M}$.

Remark 4.3.2. We also use $\mathcal{T}$ to denote the tangent sheaf of $\mathcal{M}$.

Let $\mathcal{U} = (U, \mathcal{O}|_U)$ be a chart of $\mathcal{M}$ with coordinates $(x^\mu, \theta^{i,a})$.

Proposition 4.3.2. *$\mathfrak{X}$ is a locally free $K(\mathcal{I})$-graded $\mathcal{O}$-module.*

Proof. Consider the derivations $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial \theta^{i,a}}$ of $\mathcal{O}(U)$ defined by setting $\frac{\partial}{\partial x^\mu}$ to be the usual partial derivatives on U, and $\frac{\partial}{\partial \theta^{i,a}}\theta_{j,b} = \delta_{ij}\delta_{ab}$. Let $D \in \mathfrak{X}(U)$. Let $D' = D^\mu \frac{\partial}{\partial x^\mu} + D^{i,a}\frac{\partial}{\partial \theta^{i,a}}$, where $D^\mu = D(x^\mu)$ and $D^{i,a} = D(\theta^{i,a})$. By construction, $(D - D')f = (D - D')(f - P_k) \in I_p^{k+1}$ for all $p \in U$, $f \in \mathcal{O}(U)$ and polynomials P_k such that $f - P_k \in I_p^{k+1}$. It follows from Lemma 4.2.9 that $(D - D')f = 0$ for all $f \in \mathcal{O}(U)$.Hence D is uniquely determined by its action on x^μ and $\theta^{i,a}$. $\mathfrak{X}$ is a locally free $\mathcal{O}$-module locally spanned by $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial \theta^{i,a}}$, where $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial \theta^{i,a}}$ are of degrees 0 and $[(0,i)] \in K(\mathcal{I})$, respectively. □

Definition 4.3.6. The sheaf $\mathcal{T}^* := \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}, \mathcal{O})$ is called the cotangent sheaf of $\mathcal{M}$.

Proposition 4.3.3. *$\mathcal{T}^*$ is a locally free $\mathcal{I}$-graded $\mathcal{O}$-module.*

Proof. $\mathcal{T}^*$ is locally spanned by dx^μ and $d\theta^{i,a}$, where dx^μ is the dual of $\frac{\partial}{\partial x^\mu}$ and $d\theta^{i,a}$ is the dual of $\frac{\partial}{\partial \theta^{i,a}}$. Note that dx^μ is of degree 0 and $d\theta^{i,a}$ is of degree i. □

We want to define a notion of differential forms on $\mathcal{M}$. Let V be an $\mathcal{I}$-graded R-module. Let $V[odd]$ denote the $\mathbb{N}\times\mathcal{I}$-graded R-module defined by setting $V[odd]_{i,j} = 0$ for all $i \neq 1$ and $V[odd]_{1,j} = V_j$. The parity function on $\mathbb{N}\times\mathcal{I}$ is defined by setting $p(i,j) = p(i) + p(j)$. The exterior algebra $\Lambda(V)$ of V is then defined to be the symmetric algebra $\mathrm{S}(V[odd])$. We write $\Lambda(V) = \bigoplus_{k\in\mathbb{N}} \Lambda^k(V)$ where $\Lambda^k(V) = \mathrm{S}^k(V[odd])$. An element ω of $\Lambda^k(V)$ is said to be of form degree k. We set $\overline{\Lambda(V)} = \prod_{k\in\mathbb{N}} \Lambda^k(V)$ to be the "completion" of $\Lambda(V)$.

Definition 4.3.7. The sheaf Ω of differential forms on $\mathcal{M}$ is defined to be the sheafification of the presheaf

$$U \mapsto \overline{\Lambda(\mathcal{T}^*(U))}.$$

A k-form ω on $\mathcal{M}$ is a global section of Ω of form degree k. ω is said to even (odd) if it is even (odd) with respect to its $\mathcal{I}$-grading. In particular, M together with Ω can be viewed as an $\mathbb{N}\times\mathcal{I}$-graded manifold $\mathcal{M}_\Omega$. The de Rham differential on Ω is a vector field d of degree $(1,0)$ over $\mathcal{M}_\Omega$ locally of the form

$$d = dx^\mu \frac{\partial}{\partial x^\mu} + d\theta^{i,a}\frac{\partial}{\partial \theta^{i,a}}.$$

Let X be a vector field of degree $l \in K(\mathcal{I})$ over $\mathcal{M}$ locally of the form $X = X^\mu \frac{\partial}{\partial x^\mu} + X^{i,a}\frac{\partial}{\partial \theta^{i,a}}$. The contraction of X on Ω is a vector field ι_X of degree $(-1,-l)$[7] over $\mathcal{M}_\Omega$ locally of the form

$$\iota_X = X^\mu \frac{\partial}{\partial dx^\mu} + X^{i,a}\frac{\partial}{\partial d\theta^{i,a}}.$$

[7] -1 is the inverse of $1 \in K(\mathbb{N}) = \mathbb{Z}$, $-l$ is the inverse of l in $K(\mathcal{I})$.

The Lie derivative with respect to X on Ω is a vector field

$$\mathrm{Lie}_X = [d, \iota_X]$$

of degree $(0, -l)$, where $[\cdot, \cdot]$ is the bracket of the $K(\mathcal{I})$-graded Lie algebra of vector fields over $\mathcal{M}_\Omega$.

Remark 4.3.3. Note that there exists a nontrivial parity-preserving monoid morphism $\mathbb{N} \to \mathcal{I}$ which sends $1 \in \mathbb{N}$ to an odd element l in $\mathcal{I}$. $V[odd]$ can be then viewed as an $\mathcal{I}$-graded vector space $V[l]$ shifted by degree l. It follows that M together with Ω can be viewed (non-canonically) as an $\mathcal{I}$-graded manifold (with auxiliary parts).

4.4 Monoidally graded manifolds with symmetries

Let $\mathcal{I}$ be a commutative ring. Let $\mathfrak{g} = \bigoplus_{i \in \mathcal{I}} \mathfrak{g}_i$ be an $\mathcal{I}$-graded Lie algebra. Let $\mathcal{M} = (M, \mathcal{O})$ be an $\mathcal{I}$-graded manifold.

Definition 4.4.1. An (infinitesimal) $\mathfrak{g}$-action on $\mathcal{M}$ is a $\mathcal{I}$-graded Lie algebra homomorphism

$$\tau : \mathfrak{g} \to \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the $\mathcal{I}$-graded Lie algebra of vector fields over $\mathcal{M}$.

Let G_0 be a Lie group with Lie algebra $\mathfrak{g}_0$.

Definition 4.4.2. An $\mathcal{I}$-graded Lie group $\mathcal{G}$ is a pair $(G_0, \mathfrak{g})$ together with a group homomorphism

$$\sigma : G_0 \to \mathrm{Aut}(\mathfrak{g}),$$

where $\mathrm{Aut}(\mathfrak{g})$ is the automorphism group of $\mathfrak{g}$, such that $\sigma|_{\mathfrak{g}_0} = \mathrm{Ad}$, the adjoint action of G_0 on $\mathfrak{g}_0$.

Definition 4.4.3. A (global) $\mathcal{G}$-action on $\mathcal{M}$ is a pair $\rho = (\rho_0, \tau)$ where

1. ρ_0 is a group homomorphism

   $$\rho_0 : G_0 \to \mathrm{Diff}(\mathcal{M}),$$

   where $\mathrm{Diff}(\mathcal{M})$ is the diffeomorphism group of $\mathcal{M}$, i.e., the group of invertible morphisms $\varphi : \mathcal{M} \to \mathcal{M}$

2. τ is an action of $\mathfrak{g}$ on $\mathcal{M}$,

such that

$$d\rho_0|_{\mathrm{id}} = \tau|_{\mathfrak{g}_0},$$
$$\rho_0(g)^* \tau(\xi) \rho_0(g^{-1})^* = \tau(\sigma(g)(\xi)),$$

for $g \in G_0$ and $\xi \in \mathfrak{g}$. ρ is called the globalization of the infinitesimal $\mathfrak{g}$-action τ.

4.5 (Bi)graded manifolds with (infinitesimal) symmetries

4.5.1 L-manifolds

Let $\mathcal{M} = (M, \mathcal{O})$ be a graded manifold. Let $\mathfrak{X}$ be the tangent sheaf of $\mathcal{M}$.

Definition 4.5.1. A cohomological vector field Q on a graded manifold $\mathcal{M}$ is a vector field of degree 1 satisfying $Q^2 = 0$. A Q-manifold is then a graded manifold equipped with a cohomological vector field Q. A Q-manifold is also called as a differential graded manifold.

The cohomology of a Q-manifold can be defined in a straightforward way. By Theorem 4.2.2, it actually corresponds to a cohomology theory of a graded vector bundle E.

Example 4.5.1. Let M be a manifold. Let $\mathcal{M} = T[1]M$. Let (x^μ, η^μ) be a local coordinate system, we define

$$Q = \eta^\mu \frac{\partial}{\partial x^\mu}.$$

The cohomology of $\mathcal{M}$ is the de Rham cohomology of M.

Example 4.5.2. Let V be a vector space, viewed as a vector bundle over a point. Let $\mathcal{M} = V[-1] \oplus V[0]$. Let (x^μ, η^μ) be a local coordinate system, we define

$$Q = x^\mu \frac{\partial}{\partial \eta^\mu}.$$

The cohomology of $\mathcal{M}$ is nothing but the Koszul cohomology defined in Example 3.2.1.

Definition 4.5.2. A Lie algebroid is a vector bundle $E \to M$ with a Lie bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(E)$, and a vector bundle morphism $a : E \to TM$, called as an anchor, such that the following Leibniz rule holds:

$$[X, fY] = f[X, Y] + a(X)(f)Y, \tag{4.5.1}$$

where $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$.

Remark 4.5.1. The Leibniz rule (4.5.1) guarantees that a is also a Lie algebra homomorphism. In fact, we have

$$\begin{aligned}
a([X,Y])(f)Z &= [[X,Y], fZ] - f[[X,Y], Z] \\
&= [[X, fZ], Y] + [X, [Y, fZ]] - f[[X,Y], Z] \\
&= [f[X,Z] + a(X)(f)Z, Y] + [X, f[Y,Z] + a(Y)(f)(Z)] - f[[X,Y], Z] \\
&= -(f[Y,[X,Z]] + a(Y)(f)[X,Z] + a(X)(f)[Y,Z] + a(Y)(a(X)(f))Z) \\
&\quad + (f[X,[Y,Z]] + a(X)(f)[Y,Z] + a(Y)(f)[X,Z] + a(X)(a(Y)(f))Z) \\
&\quad - f[[X,Y], Z] \\
&= [a(X), a(Y)](f)Z.
\end{aligned}$$

for any $f \in C^\infty(M)$ and $Z \in \Gamma(E)$. Hence $a([X,Y]) = [a(X), a(Y)]$.

Fix a local basis θ^a of E^* and a local coordinate system x^μ on M. This yields a local coordinate system of the graded manifold $E[1]$. Any vector field Q of degree 1 on $E[1]$ can be written as

$$Q = -\frac{1}{2} f^a_{bc}(x)\theta^b\theta^c \frac{\partial}{\partial\theta^a} + \rho^\mu_a(x)\theta^a \frac{\partial}{\partial x^\mu}.$$

Let ε_a be the dual of θ^a. Let $f = f^a(x)\varepsilon_a$ and $g = g^a(x)\varepsilon_a$ be two sections of E. Q induces a vector bundle morphism via

$$a(X) = f^a(x)\rho^\mu_a(x)\frac{\partial}{\partial x^\mu},$$

and a bracket on $\Gamma(E)$ via

$$[X,Y] = f^a(x)g^b(x)f^c_{ab}(x)\varepsilon_c + a(X)(g^a)\varepsilon_a - a(Y)(f^a)\varepsilon_a.$$

Proposition 4.5.1 ([Vai97])**.** *$E \to M$ is a Lie algebroid if and only if Q is cohomological.*

Proof. The induced bracket satisfies (4.5.1) automatically. It remains to show that it is a Lie bracket if and only if $Q^2 = 0$. Note that a section of E can be naturally identified as a vector field of degree -1 on $E[1]$ by replacing (locally) ε_a with $\frac{\partial}{\partial\theta^a}$. Under this identification, it is not hard to see that

$$[X,Y]_E = [[Q,X],Y]. \tag{4.5.2}$$

Here we use $[\cdot,\cdot]_E$ to denote the bracket on $\Gamma(E)$. The brackets on the right hand side of (4.5.2) are brackets of vector fields on $E[1]$. It follows that

$$\begin{aligned} [[X,Y]_E,Z]_E &= [[Q,[[Q,X],Y]],Z] \\ &= [[[Q,[Q,X]],Y] + [[Q,X],[Q,Y]],Z] \\ &= \frac{1}{2}[[[[Q,Q],X],Y],Z] + [[[Q,X],Z],[Q,Y]] + [[Q,X],[[Q,Y],Z]] \\ &= \frac{1}{2}[[[[Q,Q],X],Y],Z] + [[X,Z]_E,Y]_E + [X,[Y,Z]_E]_E. \end{aligned}$$

Hence, $[\cdot,\cdot]_E$ is a Lie bracket if and only if $\frac{1}{2}[Q,Q] = Q^2 = 0$. □

In the case where $E = TM$, $f^a_{bc}(x) = 0$ and $\rho^\mu_a(x) = \delta^\mu_a$, we recover Example 4.5.1.

As another special case, we can take M to be a G-manifold, E to be the trivial bundle $M \times \mathfrak{g}$ over M, $f^a_{bc}(x)$ to be the structure constants of $\mathfrak{g}$, and $\rho^\mu_a(x)$ to be induced from the infinitesimal action of $\mathfrak{g}$ on M. The Q-cohomology of $\mathcal{M}$ is nothing but the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $C^\infty(M)$. The Q-closed condition of a function S of degree 0 is equivalent to the G-invariance condition of S.

Definition 4.5.3. A P-manifold is a graded manifold $\mathcal{M}$ equipped with a non-degenerate closed 2-form ω of odd degree n. n is also referred to as the degree of $\mathcal{M}$.

Since ω is non-degenerate, it associates to any $f \in \mathcal{O}(M)$ a Hamiltonian vector field X_f just as in the bosonic case, X_f is defined by setting

$$\iota_{X_f}\omega = df.$$

We can define the graded Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{O}(M)$ by setting

$$\{f, g\} = -(-1)^{p(f)}\iota_{X_f}\iota_{X_g}\omega.$$

Definition 4.5.4. A QP-manifold is a P-manifold equipped with a Hamiltonian cohomological vector field Q.

By definition, the Hamiltonian function S associated to Q is Q-closed, hence provides an action functional of a cohomological field theory.

QP-manifolds were first defined in [Ale+97]. They provide a powerful mechanism to construct the action functional S of a CohFT by setting S to be the Hamiltonian function associated to Q.

Remark 4.5.2. The cohomological vector field Q can be seen as the generator of a $(0|1)$-dimensional Lie superalgebra $\mathfrak{q}$. From this point of view, a Q-manifold is just a graded manifold equipped with an action of $\mathfrak{q}$. As is shown above, the bosonic symmetries of an action functional S can sometimes be fully captured by the $\mathfrak{q}$-invariance condition of S. If S also has fermionic symmetries, $\mathfrak{q}$ should be replaced by the graded Lie algebra L defined in Section 2.1. In fact, the fermionic symmetries of S can be captured by the condition $\iota_a S = 0$. It follows that $\mathrm{Lie}_a S = 0$ via Cartan's magic formula $Q\iota_a + \iota_a Q = \mathrm{Lie}_a$. (With a slight abuse of notation, we use Q to denote the differential of L.)

Definition 4.5.5. An L-manifold is a graded manifold equipped with an L-action.

By definition, every L-manifold is a Q-manifold. The commutative graded algebra $\mathcal{O}(M)$ of functions over $\mathcal{M}$ is an L-module. In particular, the L-module $W(\mathfrak{g}) \otimes \Omega(X)$ associated to a G-manifold X can now be viewed as the commutative graded algebra of functions over the L-manifold $T[1]X \oplus \underline{\mathfrak{g}}[1] \oplus \underline{\mathfrak{g}}[2]$, where $\underline{\mathfrak{g}}$ is the trivial bundle $X \times \mathfrak{g}$. For a general L-manifold $\mathcal{M}$, the cohomology of $\mathcal{O}(M)_{bas}$ can be then viewed as a generalization of the equivariant cohomology.

We want to construct a basic Q-closed action functional S.

Definition 4.5.6. An LP-manifold is a P-manifold equipped with a Hamiltonian L-action. Namely, the fundamental vector fields generated by Lie_a, Q, ι_a in L are Hamiltonian vector fields.

We denote the Hamiltonian functions associated to Lie_a and ι_a by L_a and I_a, respectively.

Example 4.5.3. Let $\mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[2]$, where $\mathfrak{g}$ is viewed as a bundle over a point. Let (θ^a, ϕ_b) be a (local) coordinate system. The canonical symplectic form on $\mathcal{M}$ takes the form $\omega = d\theta^a \wedge d\phi_a$. In this case, the Hamiltonian functions I_a, L_a and S are of degrees 2, 3 and 4, respectively. We can write

$$\begin{aligned} I_a &= f_{1a}^b \phi_b + f_{2abc}\theta^b\theta^c, \\ L_a &= h_{2ac}^b \phi_b \theta^c + h_{3abcd}\theta^b\theta^c\theta^d, \\ S &= g_2^{ab}\phi_a\phi_b + g_{3bc}^a \phi_a\theta^b\theta^c + g_{4abcd}\theta^a\theta^b\theta^c\theta^d, \end{aligned}$$

generally. They need to satisfy

$$\{L_a, I_b\} = f_{ab}^c I_c, \quad \{S, S\} = 0, \quad \{I_a, I_b\} = 0, \quad \{S, I_a\} = L_a.$$

When $\mathfrak{g}$ is compact and semi-simple,[8] one can find a set of solutions

$$I_a = \phi_a, \quad L_a = -f_{ac}^b \phi_b \theta^c, \quad S = \frac{1}{2} g^{ab}\phi_a\phi_b - \frac{1}{2} f_{bc}^a \phi_a \theta^b\theta^c,$$

where g^{ab} is the Killing metric on $\mathfrak{g}$. Using g^{ab} to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, these solutions recover the L-module structure of the Weil algebra $W(\mathfrak{g})$.

The Hamiltonian function S associated to Q can not be basic in general, because

$$\iota_a(S) = -\{I_a, S\} = -L_a.$$

and L_a cannot be 0 when the even part of the L-action is non-trivial. In other words, LP-manifolds cannot help us to find basic Q-closed S like the way QP-manifolds help one to find Q-closed S.

4.5.2 QK-manifolds

We are particularly interested in the class of L-manifolds where $\mathfrak{g}$ is abelian. The L-structure of such graded manifold is given by $2n+1$ vector fields satisfying

$$[\mathrm{Lie}_\mu, \mathrm{Lie}_\nu] = 0, \quad [\mathrm{Lie}_\mu, Q] = 0, \quad [\mathrm{Lie}_\mu, \iota_\nu] = 0, \tag{4.5.3}$$

$$[Q, Q] = 0, \quad [Q, \iota_\mu] = \mathrm{Lie}_\mu, \quad [\iota_\mu, \iota_\nu] = 0, \tag{4.5.4}$$

for $\mu, \nu = 1, \ldots, n$, where $n = \dim \mathfrak{g}$.

[8] This property guarantees that $f_{ab}^c = f_{bc}^a = f_{ca}^b$ in orthonormal coordinates.

Example 4.5.4. Let $\mathcal{M} = T[1]\mathbb{R}^n$. Let (x^μ, θ^ν) be a (local) coordinate system. The L-structure is given by

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad \iota_\mu = \frac{\partial}{\partial \theta^\mu}, \quad \mathrm{Lie}_\mu = \frac{\partial}{\partial x^\mu}. \tag{4.5.5}$$

It is not hard to check that (4.5.5) satisfy (4.5.3) and (4.5.4). There is no interesting basic Q-closed function S on $\mathcal{M}$. This is because $I_\mu(S) = 0$ and $L_\mu(S) = 0$ force S to be independent of x^μ and θ^μ, hence the only possible candidates for such S are constant functions.

Example 4.5.4 can be easily generalized to a non-flat case. To achieve that, we need to work in a bigraded setting instead of the graded setting. The parity function is defined by

$$\begin{aligned} p : \mathbb{Z} \times \mathbb{Z} &\to \mathbb{Z}_2 \\ (i, j) &\mapsto i + j \pmod 2. \end{aligned}$$

Note that

$$p(d(x)d(y)), \quad p(d(x))p(d(y))$$

are not the same in this case. As mentioned before, we use the first one as our sign convention for the bigraded setting. We say a commutative bigraded algebra A is of the first (or the second) kind if $p(a)p(b)$ is set to be $p(d(a)d(b))$ (or $p(d(x))p(d(y))$), for $a, b \in A$. These two conventions can be connected by the following lemma.

Lemma 4.5.1. *Let A be a commutative bigraded algebra of the first kind. Let A' be a bigraded algebra with the same underlying bigraded vector space as A and a new algebraic product $\cdot_{A'}$ defined by*

$$a \cdot_{A'} b = (-1)^{j_a i_b} ab$$

where a is of degree (i_a, j_a) and b is of degree (i_b, j_b). A' is then a commutative bigraded algebra of the second kind. If D is a derivation of A of degree (i, j), then D' defined by

$$D'(a) = (-1)^{j i_a} D(a)$$

is a derivation of A'.

Proof. By definition, $a \cdot_{A'} b = (-1)^{j_a i_b} ab = (-1)^{j_a i_b + i_a i_b + j_a j_b} ba = (-1)^{(i_a + j_a)(i_b + j_b)} b \cdot_{A'} a$. We need to verify that D' satisfies Leibniz's rule.

$$\begin{aligned} D'(a \cdot_{A'} b) &= (-1)^{j(i_a + i_b)} (-1)^{j_a i_b} D(ab) \\ &= (-1)^{j(i_a + i_b) + j_a i_b} (D(a) b + (-1)^{i i_a + j j_a} a D(b)) \\ &= (-1)^{j(i_a + i_b) + j_a i_b} ((-1)^{(j + j_a) i_b} D(a) \cdot_{A'} b + (-1)^{i i_a + j j_a + j_a (i + i_b)} a \cdot_{A'} D(b)) \\ &= D'(a) \cdot_{A'} b + (-1)^{(i+j)(i_a + j_a)} a \cdot_{A'} D'(b). \end{aligned}$$

□

Example 4.5.5. Consider the bigraded manifold $\mathcal{M} = (T \oplus T)[(1,1)]M$ where M is an n-dimensional manifold. Let $(x^\mu, \eta^\nu, \theta^\sigma)$ be a local coordinate system where η^ν are of degree $(1,0)$ while θ^σ are of degree $(0,1)$. Vector fields analogous to those in Example 4.5.4 are given by

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad K = \eta^\mu \frac{\partial}{\partial \theta^\mu}, \quad L = \eta^\mu \frac{\partial}{\partial x^\mu}. \tag{4.5.6}$$

It is easy to check that instead of (4.5.3) and (4.5.4), they satisfy

$$Q^2 = 0, \quad L^2 = 0, \tag{4.5.7}$$

$$QL - LQ = 0, \quad KL + LK = 0, \quad QK + KQ = L. \tag{4.5.8}$$

Note that K^2 does not vanish, and that the relations $QL - LQ = 0$ and L^2 are not independent from the rest of (4.5.7) and (4.5.8). In fact, we have

$$QL - LQ = Q(QK + KQ) - (QK + KQ)Q = Q^2K - KQ^2 = 0$$

by using $Q^2 = 0$ and $L = QK + KQ$, and

$$L^2 = \frac{1}{2}(L(QK + KQ) + (QK + KQ)L) = \frac{1}{2}(QLK - KLQ - QLK + KLQ) = 0$$

by using $KL + LK = 0$ and $L = QK + KQ$.

Remark 4.5.3. If we use the second convention for $p(x)p(y)$, we will have

$$QL + LQ = 0, \quad KL - LK = 0, \quad QK - KQ = L. \tag{4.5.9}$$

instead of (4.5.8).

Definition 4.5.7. The QK Lie bigraded algebra is the Lie bigraded algebra $\mathcal{K}$ spanned by Q of degree $(0,1)$, K of degree $(1,-1)$ and L of $(1,0)$ with brackets

$$[Q,Q] = [K,K] = [L,L] = [Q,L] = [K,L] = 0, \quad [Q,K] = L.$$

Definition 4.5.8. A QK-manifold is a bigraded manifold $\mathcal{M} = (M, \mathcal{O})$ equipped with a $\mathcal{K}$-action. Equivalently, $\mathcal{M}$ is a QK-manifold if it has three vector fields Q of degree $(0,1)$, K of degree $(1,-1)$ and L of degree $(1,0)$ satisfying

$$Q^2 = 0, \quad QK + KQ = L, \quad LK + KL = 0.$$

Definition 4.5.9. A QK-algebra of length n is the unital associative bigraded algebra $\mathcal{K}_n$ generated by the generators Q of degree $(0,1)$, K of degree $(1,-1)$ and L of degree $(1,0)$ subject to the relations

$$Q^2 = 0, \quad QK + KQ = L, \quad LK + KL = 0, \tag{4.5.10}$$

and

$$K^{n+1} = 0 \tag{4.5.11}$$

for some $n \in \mathbb{N}$.

There is a canonical inclusion $i_{n,m} : \mathcal{K}_n \hookrightarrow \mathcal{K}_m$ for $m \geq n$. We refer to the direct limit of the system $((\mathcal{K}_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\leq m\in\mathbb{N}})$ as the QK-algebra of length ∞ and denote it by $\mathcal{K}_\infty$. Clearly, $\mathcal{K}_\infty$ is the unital associative bigraded algebra generated by the generators Q, K and L subject only to the relations (4.5.10).

Remark 4.5.4. Note that the commutative bigraded algebra of functions over a QK-manifold is canonically a $\mathcal{K}_\infty$-module (and a $\mathcal{K}_n$-module if $K^{n+1} = 0$). In particular, it can be viewed as a bicomplex with horizontal differential L and vertical differential Q.

Let A be a unital associative bigraded algebra. It can be viewed as a Lie bigraded algebra in a natural way. Let $\phi : \mathcal{K} \to A$ be a morphism of Lie bigraded algebras.

Proposition 4.5.2. *$\mathcal{K}_\infty$ is the universal enveloping algebra of $\mathcal{K}$. That is, there exists a unique morphism $\tilde{\phi} : \mathcal{K}_\infty \to A$ of unital associative bigraded algebras such that the following diagram commutes,*

$$\begin{array}{ccc} \mathcal{K} & \overset{\iota}{\hookrightarrow} & \mathcal{K}_\infty \\ & \searrow^{\phi} & \downarrow{\tilde{\phi}} \\ & & A \end{array}$$

where $\iota : \mathcal{K} \hookrightarrow \mathcal{K}_\infty$ is the canonical inclusion.

Proof. This follows directly from the construction of $\mathcal{K}_\infty$. □

Lemma 4.5.2. *Every element α in $\mathcal{K}_n$ (or $\mathcal{K}_\infty$) can be uniquely written in the form*

$$\alpha = p_0(K) + p_1(K)Q + p_2(K)L + p_3(K)QL, \tag{4.5.12}$$

where $p_i(K)$ are polynomials in K.

Proof. Consider a word w where K appears to the right of Q, i.e.,

$$w = \cdots QK \cdots .$$

One can replace QK by $L - QK$ and move L from the middle to the rightmost using the properties $LQ = QL$ and $LK = -KL$ to get

$$w = \cdots KQ \cdots + \cdots L.$$

Repeating this procedure until there is no K on the right of Q, one has

$$w = \sum w_K w_{QL}$$

where w_K is expressed purely in K, w_{QL} is expressed purely in Q and L. The proof is completed by using $Q^2 = L^2 = 0$ and $QL = LQ$. □

Corollary 4.5.1. *$\mathcal{K}_\infty \cong \mathrm{S}[\mathcal{K}]$ as bigraded vector spaces.*

Remark 4.5.5. Corollary 4.5.1 can be seen as a special case of the Poincaré–Birkhoff–Witt theorem in the bigraded setting.

We prove the following lemmas for later use.

Lemma 4.5.3. *Let $p(K)$ be a polynomial in K, Then*

$$[Q, p(K)] = Lp'(K),$$

where $[\cdot,\cdot]$ is the canonical Lie bigraded bracket on $\mathcal{K}_n$ (or $\mathcal{K}_\infty$), p' is the derivative of p.

Proof. Note that $[Q,\cdot]$ is a derivation of degree $(0,1)$, we have

$$\begin{aligned}[Q, K^p] &= [Q,K]K^{p-1} - K[Q,K]K^{p-2} + \cdots \\ &= LK^{p-1} - KLK^{p-2} + \cdots \\ &= pLK^{p-1},\end{aligned}$$

where we use $KL = -LK$. □

Note that $\exp(K) := \sum_{p=0}^{\infty} \frac{1}{p!}K^p$ is a well-defined element in $\mathcal{K}_n$ since K is nilpotent.

Lemma 4.5.4. *Let Ω be a $\mathcal{K}_n$-module. We have*

$$\exp(K)(\alpha\beta) = (\exp(K)\alpha)(\exp(K)\beta)$$

for $\alpha, \beta \in \Omega$ with $p(d(\alpha)) = 0$.

Proof. Since $p(d(\alpha)) = 0$, we have $K(\alpha\beta) = (K\alpha)\beta + \alpha(K\beta)$. More generally, we have

$$\begin{aligned}\frac{1}{p!}K^p(\alpha\beta) &= \frac{1}{p!}\sum_{j=0}^{p}\binom{p}{j}(K^j\alpha)(K^{p-j}\beta) \\ &= \sum_{j=0}^{p}\frac{1}{j!(p-j)!}(K^j\alpha)(K^{p-j}\beta).\end{aligned}$$

It follows that

$$\begin{aligned}\exp(K)(\alpha\beta) &= \sum_{p=0}^{\infty}\sum_{j=0}^{p}\frac{1}{j!(p-j)!}(K^j\alpha)(K^{p-j}\beta) \\ &= \sum_{j=0}^{\infty}\sum_{p=j}^{\infty}\frac{1}{j!(p-j)!}(K^j\alpha)(K^{p-j}\beta) \\ &= (\sum_{j=0}^{\infty}\frac{1}{j!}K^j\alpha)(\sum_{l=0}^{\infty}\frac{1}{l!}K^l\beta) \\ &= (\exp(K)\alpha)(\exp(K)\beta).\end{aligned}$$

□

Chapter 5

Cohomological Lagrangian field theories

5.1 Geometry preliminaries

5.1.1 Whitney topologies

Let X, Y be smooth manifolds. Let $f, g : X \to Y$ be smooth maps. Let $p \in X$.

1. f is said to have 0-th order contact with g at p if $f(p) = g(p) = q \in Y$.

2. Let k be a positive integer. f is said to have k-th order contact with g at p if df has $(k-1)$-th order contact with dg at (p, v) for all $v \in T_p(X)$. We write $f \sim_k g$ at p to denote this equivalence relation.

Lemma 5.1.1. *$f \sim_k g$ at p if and only if their partial derivatives up to order k agree at p in some chart around p.*

Proof. See Lemma 2.2. in [GG12]. □

Let $J^k(X,Y)_{p,q}$ denote the set of equivalence classes under "$\sim_k$ at p" of mappings $f : X \to Y$. Let $J^k(X,Y) = \bigsqcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q}$. An element in $J^k(X,Y)$ is called a k-jet (of mappings from X to Y). To be more precise, for $f \in C^\infty(X,Y)$, there is a canonical mapping

$$j^k(f) : X \to J^k(X,Y)$$

called the k-jet of f defined by sending p to $j^k(f)(p) = [f]_{\sim_k \,\mathrm{at}\, p} \in J^k(X,Y)_{p,f(p)}$, the equivalence class of f. Note that $J^0(X,Y)$ is just $X \times Y$, a smooth manifold, and that $j^0(f) : X \to X \times Y$ is a smooth map whose image is the graph of f. Note also that there is a canonical projection

$$\pi_{k,l} : J^l(X,Y) \to J^k(X,Y)$$

sending $j^l(f)(p)$ to $j^k(f)(p)$ for $l \geq k$.

Let A_n^k be the vector space of polynomials in n variables of degrees $\leq k$ with zero constant terms. Let $B_{n,m}^k = \bigoplus_{i=1}^m A_n^k$. $B_{n,m}^k$ is isomorphic to a Euclidean space, and is, in particular, a smooth manifold. Let U and V be two open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For a smooth $f : U \to V$, $x_0 \in U$ and $y_0 = f(x_0) \in V$, define $T_k f : U \to A_n^k$ by setting $T_k f(x_0)$ to be the polynomial in x given by the first k terms of the Taylor series of f at x_0 after the constant term. There is a canonical bijection

$$\begin{aligned} T_{U,V} : J^k(U,V) &\to U \times V \times B_{n,m}^k \\ j^k(f)(x_0) &\mapsto (x_0, y_0, T_k f^1(x_0), \cdots, T_k f^m(x_0)), \end{aligned}$$

where $(f^1, \cdots, f^m)$ is the coordinate expression of f. By Lemma 5.1.1, $T_{U,V}$ is well-defined.

Proposition 5.1.1. *Let X and Y be smooth manifolds with $n = \dim X$ and $m = \dim Y$. $J^k(X,Y)$ is a smooth manifold of dimension $n + m + \dim B_{n,m}^k$. Moreover, let $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be atlases of X and Y, respectively. Let $U'_\alpha = \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ and $V'_\beta = \psi_\beta(V_\beta) \subset \mathbb{R}^m$. For $f \in C^\infty(U_\alpha, V_\beta)$, we write $f_{\alpha\beta}$ to denote the smooth map $\varphi_\alpha^{-1} \circ f \circ \psi_\beta \in C^\infty(U'_\alpha, V_\beta)$. Let $\Phi_{\alpha\beta} : C^\infty(U_\alpha, V_\beta) \to C^\infty(U'_\alpha, V'_\beta)$ be the map that sends f to $f_{\alpha\beta}$. Then $(C^\infty(U_\alpha, V_\beta), T_{U'_\alpha, V'_\beta} \circ \Phi_{\alpha\beta})$ form an atlas for $J^k(X,Y)$.*

Proof. See Theorem 2.7. in [GG12]. □

Using the charts defined in Proposition 5.1.1, it is not hard to show that

1. The k-jet $j^k(f) : X \to J^k(X,Y)$ of a smooth map f is smooth.
2. Let $g : Y \to Z$ be smooth. Then the map $g_* : J^k(X,Y) \to J^k(X,Z)$ defined by sending $j^k(f)$ to $j^k(g \circ f)$ is smooth.
3. The canonical projections $\pi_{k,l} : J^l(X,Y) \to J^k(X,Y)$ for $l \geq k \geq 0$ are all smooth.

The projection $\pi = \pi_{0,k} : J^k(X,Y) \to X \times Y$ makes $J^k(X,Y)$ into a smooth fiber bundle over $X \times Y$ with fiber isomorphic to $B_{n,m}^k$.[1] It follows, $J^k(X,Y)$ is also a smooth fiber bundle over X with fiber isomorphic to $Y \times B_{n,m}^k$.

The topologies on $J^k(X,Y)$, $k \geq 0$, induced by the smooth structures give rise to a topology on $C^\infty(X,Y)$ known as the Whitney C^∞ topology.

Definition 5.1.1. Let X, Y be smooth manifolds.

1. The Whitney C^k topology $\mathcal{W}_k$ on $C^\infty(X,Y)$ is the topology generated by the base

   $$\{M^k(U) : U \subset J^k(X,Y) \text{ open}\},$$

   where $M^k(U) = \{f \in C^\infty(X,Y) : j^k(f)(X) \subset U\}$.

[1] Note that $J^k(X,Y)$ is, however, only an affine bundle. It is a vector bundle if $Y = \mathbb{R}^m$.

2. The Whitney C^∞ topology on $C^\infty(X,Y)$ is the topology generated by the base $\mathcal{W} = \bigcup_{k=0}^\infty \mathcal{W}_k$.

Remark 5.1.1. It is worth mentioning that there is another topology on $C^\infty(X,Y)$, the compact-open topology, which is generated by the subbase

$$\{CO(K,U) : K \subset X \text{ compact}, U \subset Y \text{ open}\},$$

where $CO(K,U) = \{f \in C^\infty(X,Y) : f(K) \subset U\}$. If X is compact, the compact-open topology and the Whitney C^∞ topology on $C^\infty(X,Y)$ coincide. If X is non-compact, however, the Whitney C^∞ topology is strictly finer than the compact-open topology [KM97].

The Whitney C^∞ topology has the following nice properties.

Proposition 5.1.2. *Let X, Y be smooth manifolds. The mapping*

$$\begin{aligned} j^k : C^\infty(X,Y) &\to C^\infty(X, J^k(X,Y)) \\ f &\mapsto j^k(f) \end{aligned}$$

is continuous in the Whitney C^∞ topology.

Proof. See Proposition 3.4. in [GG12]. □

Proposition 5.1.3. *Let X, Y and Z be smooth manifolds. Let $g : Y \to Z$ be smooth. Then the mapping*

$$\begin{aligned} g_* : C^\infty(X,Y) &\to C^\infty(X,Z) \\ f &\mapsto g \circ f \end{aligned}$$

is continuous in the Whitney C^∞ topology.

Proof. See Proposition 3.5. in [GG12]. □

Proposition 5.1.4. *Let X, Y and Z be smooth manifolds. Then $C^\infty(X,Y) \times C^\infty(X,Z)$ is homeomorphic (in the Whitney C^∞ topology) with $C^\infty(X, Y \times Z)$ via the standard identification $(f,g) \mapsto f \times g$, where $(f \times g)(x) = (f(x), g(x))$.*

Proof. See Proposition 3.6. in [GG12]. □

Proposition 5.1.5. *Let X, Y and Z be smooth manifolds with X compact. Then the mapping $C^\infty(X,Y) \times C^\infty(Y,Z) \to C^\infty(X,Z)$ defined by $(f,g) \mapsto g \circ f$ is continuous in the Whitney C^∞ topology.*

Proof. See Proposition 3.9. in [GG12]. □

Corollary 5.1.1. *The evaluation map*

$$\begin{aligned} \mathrm{ev} : X \times C^\infty(X,Y) &\to Y \\ (p,f) &\mapsto f(p) \end{aligned}$$

is continuous in the Whitney C^∞ topology.

Proof. Take X in Proposition 5.1.5 to be the one-point space. □

It is possible to construct a Fréchet manifold structure on $C^\infty(X,Y)$. For a compact X, the smooth structure is compatible with the Whitney C^∞ topology. For a non-compact X, the topology induced by the smooth structure is strictly finer than the Whitney C^∞ topology. We refer the reader to the book [KM97] by Kriegl and Michor for more detail.

5.1.2 Infinite jet bundles

Let $\pi : Y \to X$ be a smooth fiber bundle over an n-dimensional manifold X with fiber Z being an m-dimensional manifold, where Z is the fiber of Y. Let s be a section of Y, that is, a smooth map $s : X \to Y$ such that $\pi \circ s = \mathrm{id}_X$. We use $\Gamma(Y)$ to denote the set of sections of Y.

Lemma 5.1.2. $\Gamma(Y)$ *is a closed subset of* $C^\infty(X,Y)$ *(in the Whitney C^∞ topology).*

Proof. Consider the map $\pi_* : C^\infty(X,Y) \to C^\infty(X,X)$ defined by sending f to $\pi \circ f$, where $\pi : Y \to X$ is the canonical projection. It is a continuous map by Proposition 5.1.3. $\Gamma(Y)$ is closed in $C^\infty(X,Y)$ because it can identified as $\pi_*^{-1}(\mathrm{id}_X)$, i.e., the inverse image of the closed subset $\{\mathrm{id}_X\}$ of $C^\infty(X,X)$. □

Let $J^k(Y)_{p,q}$ denote the set of equivalence classes under $\sim_k$ at p of sections s with $s(p) = q$. Let $J^k(Y) = \bigsqcup_{p\in X, q\in Y_p} J^k(Y)_{p,q}$. $J^k(Y)$ is a subset of $J^k(X,Y)$. Likewise, it is easy to show that $J^k(Y)$ is closed in $J^k(X,Y)$. It follows from Proposition 5.1.1 that $J^k(Y)$ is a smooth manifold of dimension $n + m + \dim B^k_{n,m}$. Moreover, it is a fiber bundle over X with fiber isomorphic to $Z \times B^k_{n,m}$.

Definition 5.1.2. $J^k(Y)$ is called the k-th jet bundle of Y. Let $s \in \Gamma(Y)$. $j^k(s) : X \to J^k(Y)$ is a section of $J^k(Y)$ called the k-th jet prolongation of s.

Note that $J^0(Y)$ is just Y itself.

Proposition 5.1.6. *The mapping*

$$\begin{aligned} j^k : \Gamma(Y) &\to \Gamma(J^k(Y)) \\ s &\mapsto j^k(s) \end{aligned}$$

is continuous in the Whitney C^∞ topology.

Proof. This follows directly from Proposition 5.1.2. □

Proposition 5.1.7. *The mapping*

$$\begin{aligned} \mathrm{ev}^k : X \times \Gamma(Y) &\to J^k(Y) \\ (p, f) &\mapsto j^k(f)(p) \end{aligned}$$

is continuous with the Whitney C^∞ topology on the left.

Remark 5.1.2. We call ev^k the k-th jet evaluation map.

Proof. This follows directly from Corollary 5.1.5 and Proposition 5.1.6. □

Proposition 5.1.8. *The k-th jet evaluation map is an open map.*

Proof. Let U be open in X. Let V be open in $J^k(Y)$. Recall that $M^k(V) = \{s \in \Gamma(Y) : j^k(s)(X) \subset V\}$. The open subsets of the form $U \times M^k(V)$ form a base for $X \times \Gamma(Y)$. It is easy to see that $\mathrm{ev}^k(U \times M^k(V)) = \pi^{-1}(U) \cap V$, where $\pi : J^k(Y) \to X$ is the canonical projection. Since taking unions commutes with taking images, ev^k maps every open subset of $X \times \Gamma(Y)$ to an open subset of $J^k(Y)$. □

Proposition 5.1.9. *Let Y' be another smooth fiber bundle over X. Let $Y \times_X Y'$ denote the fiber product of Y and Y' over X. Then $\Gamma(Y) \times \Gamma(Y')$ is homeomorphic (in the Whitney C^∞ topology) with $\Gamma(Y \times_X Y')$ via the standard identification $(s, t) \mapsto s \times t$, where $(s \times t)(x) = (s(x), t(x))$.*

Proof. This follows directly from Proposition 5.1.4. □

The inverse system $((J^k(Y))_{k \geq 0}, (\pi_{k,l})_{0 \leq k \leq l})$ does not have an inverse limit in the category of finite dimensional smooth manifolds. However, it does has an inverse limit $J^\infty(Y)$ in the category of topological spaces. To be more precise, $J^\infty(Y)$ is the set of sequences $(z_0, z_1, \cdots)$ with $z_k \in J^k(Y)$ satisfying $\pi_{k,l}(z_l) = z_k$ for $l \geq k$. The topology on $J^\infty(Y)$ is defined as the coarsest topology such that the canonical projections $\pi_{\infty,k} : J^\infty(Y) \to J^k(Y)$ are continuous. In fact, $J^\infty(Y)$ is a closed subset of the product space $\prod_{k \geq 0} J^k(Y)$ since each $J^k(Y)$ is a Hausdorff space (see Theorem 1 in [Mau97]). It is not hard to show that the canonical projection $\pi_\infty : J^\infty(Y) \to X$ is (open and) continuous. $J^\infty(Y)$ is then a topological fiber bundle over X. Note that the jet prolongations $j^k(s)$ of a section $s \in \Gamma(Y)$ are compatible with this inverse system. We therefore obtain a map $s \mapsto j^\infty(s) = (j^0(s), j^1(s), \cdots)$.

Definition 5.1.3. $J^\infty(Y)$ is called the infinite jet bundle of Y. $j^\infty(s)$ is called the infinite jet prolongation of $s \in \Gamma(Y)$.

Note that the jet evaluation maps ev^k are also compatible with the inverse system. We obtain an infinite jet evaluation map $\mathrm{ev}^\infty : X \times \Gamma(Y) \to J^\infty(Y)$.

Proposition 5.1.10. *j^∞ is continuous;* ev^∞ *is continuous and open.*

Proof. Since $J^\infty(Y)$ is a subset of $\prod_{k\geq 0} J^k(Y)$, j^∞ and ev^∞ can be identified as the maps $(j^0, j^1, \cdots) : \Gamma(Y) \to J^\infty(Y) \subset \prod_{k\geq 0} J^k(Y)$ and $(\mathrm{ev}^0, \mathrm{ev}^1, \cdots) : X \times \Gamma(Y) \to J^\infty(Y) \subset \prod_{k\geq 0} J^k(Y)$, respectively. Now apply Lemmas 5.1.7 and 5.1.8, and the fact that the product of continuous and open maps is again continuous and open, respectively. □

It is possible to construct a Fréchet manifold structure for $J^\infty(Y)$ which is compatible with the inverse limit topology, such that the maps j^∞ and ev^∞ are smooth. More precisely, $J^\infty(Y)$ is a Fréchet manifold modelled on the Fréchet space $\mathbb{R}^\infty$, the space of infinite sequences of real numbers. Let U be an open subset of X such that $\pi^{-1}(U) \cong U \times Z$. Let V be a chart of Z. Y can then be covered by charts of the form $U \times V$ with coordinate functions x^μ, u^j. A coordinate chart of $J^\infty(Y)$ is of the following form

$$\begin{aligned} J^\infty(Y) \supset \pi^{-1}_{\infty,0}(U \times V) &\to \mathbb{R}^\infty \\ j^\infty(s)(p) &\mapsto (x^\mu(p), u^j(s(p)), \cdots, \partial_I(u^j(s(p))), \cdots) \end{aligned}$$

where ∂_I is the partial derivative in x^μ with respect to the multi-index $I = (\mu_1, \ldots, \mu_n)$.

A function over $J^\infty(Y)$ is smooth if and only if it is locally a pullback of a smooth function over $J^k(Y)$ for some k, that is, for $f \in C^\infty(J^\infty(Y))$, there exists an open neighborhood of the form $\pi^{-1}_{\infty,k}(U)$ for each point in $J^\infty(Y)$, where U is an open subset of $J^k(Y)$, such that $f|_{\pi^{-1}_{\infty,k}(U)}$ is the pullback of a smooth function $U \to \mathbb{R}$. Starting from $C^\infty(J^\infty(Y))$, it is straightforward to define the set $\mathfrak{X}(J^\infty(Y))$ of vector fields over $J^\infty(Y)$ (as the set of linear derivations of $C^\infty(J^\infty(Y))$) and the set $\Omega(J^\infty(Y))$ of differential forms over $J^\infty(Y)$ (as the set of multilinear alternating maps from $\mathfrak{X}(J^\infty(Y))$ to $C^\infty(J^\infty(Y))$). Both $\mathfrak{X}(J^\infty(Y))$ and $\Omega^1(J^\infty(Y))$, the set of 1-forms, can be viewed as the spaces of sections of vector bundles over $J^\infty(Y)$, the tangent bundle $TJ^\infty(Y)$ and the cotangent bundle $T^*J^\infty(Y)$, respectively. Note that the fiber of $TJ^\infty(Y)$ is $\mathbb{R}^\infty$, while the fiber of $T^*J^\infty(Y)$ is $\mathbb{R}^\infty_0$, the subspace of $\mathbb{R}^\infty$ consisting of those sequences with only finitely many non-zero components. In other words, a vector field over $J^\infty(Y)$ is locally an infinite sum of coordinate derivatives on all $J^k(Y)$, while a differential form on $J^\infty(Y)$ is locally a pullback of a differential form on $J^k(Y)$ for some k, just like the case of a smooth function. We refer the reader to the paper by Takens [Tak79], the book [Sau89] by Saunders, and the thesis by Delgado [Del18] for more detail. From now on, we assume that such smooth structures are given for both $\Gamma(Y)$ and $J^\infty(Y)$.

Remark 5.1.3. There is another notion for smooth functions over $J^\infty(Y)$, which requires $f \in C^\infty(J^\infty(Y))$ to be globally a pullback of a smooth function over $J^k(Y)$ for some k [And92]. We prefer the locally-pullback version because the presheaves of smooth functions and differential forms over $J^\infty(Y)$ are actually sheaves. Moreover, the sheaf of smooth functions is nice enough to allow a partition of unity [Tak79].

Let's consider the short exact sequence of vector bundles

$$0 \to VJ^\infty(Y) \to TJ^\infty(Y) \to \pi_\infty^* TX \to 0, \tag{5.1.1}$$

where $VJ^\infty(Y)$ is the vertical bundle of $J^\infty(Y)$ viewed as a fiber bundle over X, i.e., the kernel of π_∞. This short exact sequence admits a canonical splitting $C : \pi_\infty^* TX \to TJ^\infty(Y)$ defined as follows. Let $p \in X$ and $j^\infty(s)(p) \in J^\infty(Y)$. Let $\xi \in T_pX$, we then set $C(j^\infty(s)(p), \xi) = dj^\infty(s)(p)\xi$, where $dj^\infty(s)$ is the tangent map of $j^\infty(s) : X \to J^\infty(Y)$. C is well-defined, i.e., it does not depend on the choice of s. It induces a horizontal bundle $HJ^\infty(Y)$ and an integrable distribution $\mathcal{C} = \Gamma(HJ^\infty(Y)) \subset \Gamma(TJ^\infty(Y))$ over $J^\infty(Y)$.

Definition 5.1.4. C is called the Cartan connection. $\mathcal{C}$ is called the Cartan distribution.

Due to the splitting of (5.1.1), we can write $\Omega(J^\infty(Y)) = \bigoplus_{p,q} \Omega^{p,q}(J^\infty(Y))$, where $\Omega^{p,q}(J^\infty(Y)) = \Gamma(\Lambda^p H^* J^\infty(Y) \otimes \Lambda^q V^* J^\infty(Y))$. An element of $\Omega^{p,q}(J^\infty(Y))$ is said to be of horizontal degree p and vertical degree q. The de Rham differential d on $\Omega(J^\infty(Y)) = \Gamma(\Lambda T^* J^\infty(Y))$ can be constructed as follows. Let α be a differential form on $J^\infty(Y)$. It is locally represented by forms on finite dimensional jet bundles $J^k(Y)$. $d\alpha$ is then defined by applying the finite dimensional de Rham differentials to these forms and then gluing. d splits correspondingly as $d = d_h + d_v$, where $d_h : \Omega^{p,q}(J^\infty(Y)) \to \Omega^{p+1,q}(J^\infty(Y))$ and $d_v : \Omega^{p,q}(J^\infty(Y)) \to \Omega^{p,q+1}(J^\infty(Y))$.

5.2 The geometric setting of CohLFTs

Let $\pi : Y \to M$ be a fiber bundle over an n-dimensional compact manifold M with fiber Z being an m-dimensional manifold. Let $\Gamma(Y)$ be the space of sections of Y. Let's consider the de Rham complex $\Omega(M \times \Gamma(Y))$ of differential forms on $M \times \Gamma(Y)$. It is a bicomplex bigraded according to the product structure of $M \times \Gamma(Y)$, i.e.,

$$\Omega(M \times \Gamma(Y)) = \bigoplus_{p,q} \Omega^{p,q}(M \times \Gamma(Y)).$$

Correspondingly, the de Rham differential d_{tot} on $M \times \Gamma(Y)$ breaks into two parts $d_{tot} = d + \delta$, where d is the de Rham differential on M and δ is the de Rham differential on $\Gamma(Y)$.

There is a canonical sub-bicomplex $\Omega_{loc}(M \times \Gamma(Y))$ [Zuc87]. Let $J^\infty(Y)$ be the infinite jet bundle of Y. Let ev^∞ be the infinite jet evaluation map from $M \times \Gamma(Y)$ to $J^\infty(Y)$. The pullback $(\mathrm{ev}^\infty)^*\Omega(J^\infty(Y))$ is stable under both d and δ, hence a sub-bicomplex, which is called the variational bicomplex of Y and denoted by $\Omega_{loc}(M \times \Gamma(Y))$. Elements in $\Omega_{loc}(M \times \Gamma(Y))$ are called local forms. d and δ restricted to $\Omega_{loc}(M \times \Gamma(Y))$ are called the horizontal differential, denoted by d_h, and the vertical differential, denoted by d_v, respectively.

Remark 5.2.1. Since ev^∞ is an open map, the pullbacks of differential forms on $J^\infty(Y)$ actually form a sheaf over $M \times \Gamma(Y)$.

Let U be an open neighborhood of M such that $\pi^{-1}U \cong U \times Z$. Let V be a coordinate chart of Z. Y can then be covered by coordinate charts of the form $U \times V$ with coordinate functions $x^1, \ldots, x^n, u^1, \ldots, u^m$. Let $\mathcal{W}(U,V)$ be the set of pairs (x, ψ) such that $\psi(x)$ is in V, it is then an open neighborhood of $M \times \Gamma(Y)$. We can explicitly define functions x^μ and u^j_I on $\mathcal{W}(U,V)$ by setting

$$x^\mu(x,\psi) = x^\mu(x), \quad u^j_I(x,\psi) = \partial_I(u^j(\psi(x))),$$

where ∂_I is the partial derivative in x^μ with respect to the multi-index $I = (\mu_1, \ldots, \mu_n)$. By definition, a local function on $\mathcal{W}(U,V)$ depends only on finitely many of x^μ and u^j_I. In particular, x^μ and u^j_I themselves are local functions. Their derivatives dx^μ and δu^j_I can be viewed as local forms of degree $(1,0)$ and $(0,1)$ respectively. We can write any local (k,l)-form ω as a finite sum

$$\omega = f^{I_1,\ldots,I_l}_{\mu_l,\ldots,\mu_k,j_1,\ldots,j_l} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \wedge \delta u^{j_1}_{I_1} \wedge \cdots \wedge \delta u^{j_l}_{I_l},$$

where each $f^{(\ldots)}_{(\ldots\ldots)}$ is a local function.

In fact, the above discussion can be easily generalized to the case of a "graded fiber bundle" Y. That is, we require Y to be a graded vector bundle over Y_0 with fiber consisting of elements of nonzero degree, where Y_0 is a fiber bundle over M with fiber consisting of elements of degree 0. The only subtleties of this generalization are

1. we should assign degree $(0, d(u^j))$ to the local function u^j_I and degree $(0, d(u^j)+1)$ to the local form δu^j_I where $d(u^j)$ is the degree of u^j induced from the grading of Y;

2. a local function should be polynomial in u^j_I when $d(u^j) \neq 0$.

Local forms obtained in this way can be seen as a sheaf of commutative bigraded algebras of the second kind over $M \times \Gamma(Y_0)$. We can turn this sheaf into a sheaf of commutative bigraded algebras of the first kind by applying Lemma 4.5.1. In other words, we obtain an infinite dimensional bigraded manifold $\mathcal{M}_Y$ from Y. We call $\mathcal{M}_Y$ the variational bigraded manifold of Y. Local forms can be viewed simply as functions over $\mathcal{M}_Y$.

Remark 5.2.2. In the physics literature, the vertical degree of a local form ω viewed as a function over $\mathcal{M}_Y$ is called as the ghost number of ω. One should not confuse it with the vertical degree of ω viewed as a differential form.

The differentials d_h and d_v can be viewed as vector fields over $\mathcal{M}_Y$ of degree $(1,0)$ and $(0,1)$, respectively. (Note that by Lemma 4.5.1, $d_h d_v - d_v d_h = 0$.) They act on x^μ, u^j_I, dx^μ and δu^j_I as

$$d_h(x^\mu) = dx^\mu, \quad d_h(u^j_I) = u^j_{I\cup\{\mu\}} dx^\mu, \quad d_h(dx^\mu) = 0, \quad d_h(\delta u^j_I) = \delta u^j_{I\cup\{\mu\}} dx^\mu,$$
$$d_v(x^\mu) = 0, \quad d_v(u^j_I) = \delta u^j_I, \quad d_v(dx^\mu) = 0, \quad d_v(\delta u^j_I) = 0.$$

We write $d_h = d_{h1} + d_{h2}$ where d_{h1} is defined by

$$d_{h1}(x^\mu) = dx^\mu, \quad d_{h1}(u^j_I) = 0, \quad d_{h1}(dx^\mu) = 0, \quad d_{h1}(\delta u^j_I) = 0,$$

and $d_{h2} = d_h - d_{h1}$.

Proposition 5.2.1. *There is a canonical QK-structure on a variational bigraded manifold where $Q = d_v$, $L = d_{h2}$ and K is defined as follows*

$$K(x^\mu) = 0, \quad K(u^j_I) = 0, \quad K(dx^\mu) = 0, \quad K(\delta u^j_I) = u^j_{I\cup\{\mu\}} dx^\mu.$$

Proof. K is a globally well-defined vector field of degree $(1, -1)$. One can easily check that $QK + KQ = L$ and $KL + LK = 0$. □

Definition 5.2.1. A QK_v-manifold is a variational bigraded manifold equipped with a $\mathcal{K}$-action such that the fundamental vector field generated by $L \in \mathcal{K}$ coincides with d_{h2}.

A QK_v-structure on a variational bigraded manifold $\mathcal{M}_Y$ is hence a generalization of the canonical QK-structure on $\mathcal{M}_Y$.

Definition 5.2.2. A cohomological (Lagrangian) field theory is a pair $(\mathcal{M}_Y, \mathcal{L})$ where $\mathcal{M}_Y$ is a QK_v-manifold and $\mathcal{L}$ is a Q-exact function on $\mathcal{M}_Y$ of degree $(n, 0)$.

Remark 5.2.3. More generally, one should consider a Lagrangian $\mathcal{L}$ such that

1. $\mathcal{L}$ is Q-closed up to an L-exact term;
2. The energy-momentum tensor of $\mathcal{L}$ is Q-exact.

Let Γ_Y denote the graded manifold $(\Gamma(Y), \Omega)$ where Ω is the sheaf of differential forms on $\Gamma(Y)$. (We call Γ the (graded) configuration space of the corresponding theory.) The action functional of $\mathcal{L}$ is a function S of degree 0 on Γ_Y defined by

$$S = \int_M \mathcal{L}.$$

In most cases, the cohomological vector field Q does not depend on coordinates x^μ and dx^μ. Hence it can be viewed as a cohomological vector field on Γ_Y. We have $QS = Q(\int_M \mathcal{L}) = \int_M Q\mathcal{L} = 0$. In other words, Γ_Y is a Q-manifold equipped with a Q-closed function S.

Remark 5.2.4. From now on, every function over a variational bigraded manifold considered by us will be assumed implicitly to be independent of x^μ. For this reason, we will often not distinguish $L = d_{h2}$ and the de Rham differential d.

Definition 5.2.3. A pre-observable $\mathcal{O}$ is a function on $\mathcal{M}_Y$ such that $Q\mathcal{O}$ is d-exact and $d\mathcal{O}$ is Q-exact. An observable is a Q-closed function on Γ_Y.

Let γ be a submanifold representative of a p-cycle in M. Let $\mathcal{O}[\gamma]$ be a function on Γ_Y defined by $\mathcal{O}[\gamma] = \int_\gamma \mathcal{O}$. $\mathcal{O}[\gamma]$ is Q-closed, hence an observable on Γ_Y. Note that the Q-cohomology class of $\mathcal{O}[\gamma]$ is independent of the choice of representatives of γ. In other words, we have a well-defined map $H_\bullet(M) \to H^\bullet(\Gamma_Y)$ defined by sending γ to $\mathcal{O}[\gamma]$.

Definition 5.2.4. Let $\mathcal{O}^{(0)}$ be a Q-closed function of degree $(0, n)$. The descendant sequence of $\mathcal{O}^{(0)}$ is a sequence of pre-observables $\{\mathcal{O}^{(p)}\}_{p=0}^n$ satisfying

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)} \tag{5.2.1}$$

for $p = 1, \dots, n$. (5.2.1) is called the (topological) descent equations.

Definition 5.2.5. Let $\mathcal{O}^{(0)}$ be a Q-closed function of degree $(0, n)$. The standard K-sequence of $\mathcal{O}^{(0)}$ is a sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$, where

$$\mathcal{O}^{(p)} := \frac{1}{p!} K^p \mathcal{O}^{(0)}$$

for $p = 1, \dots, n$.

Proposition 5.2.2. *The standard K-sequence is a descendant sequence.*

Proof. We have $Q\mathcal{O}^{(p)} = \frac{1}{p!} Q K^p \mathcal{O}^{(0)} = \frac{1}{p!}[Q, K^p]\mathcal{O}^{(0)} = \frac{1}{(p-1)!} L K^{p-1} \mathcal{O}^{(0)} = d\mathcal{O}^{(p-1)}$ for $p \geq 1$, where we use $Q\mathcal{O}^{(0)} = 0$ and Lemma 4.5.3. □

Definition 5.2.6. Let $\mathcal{W}^{(q)}$ be a Q-closed function of degree $(q, n-q)$, $1 \leq q \leq n$. A (general) K-sequence of $\mathcal{O}^{(0)}$ is a sequence $\{\mathcal{O}^{(p)}\}_{p=0}^n$, where

$$\mathcal{O}^{(p)} := \frac{1}{p!} K^p \mathcal{O}^{(0)} + \sum_{q=1}^{p} \frac{1}{(p-q)!} K^{p-q} \mathcal{W}^{(q)}$$

for $p = 1, \dots, n$.

Likewise, one can show that

Proposition 5.2.3. *Every (general) K-sequence is a descendant sequence.*

Remark 5.2.5. In physics literature [Sor+98; BBT05], the vector field K is known as the vector supersymmetry. A similar result as Proposition 5.2.2 is also proved in [PS08].(See Proposition 5.14 there.) What we will show later is that the converse of Proposition 5.2.3 is also true in a cohomological sense.

Lemma 5.2.1. *Let $\{\mathcal{O}^{(i)}\}_{i=0}^n$ be such that $\mathcal{O}^{(i)} = Q\rho^{(i)} + d\rho^{(i-1)}$ for $i > 0$ and $\mathcal{O}^{(0)} = Q\rho^{(0)}$, where $\rho^{(i)}$ is an arbitrary function of degree $(i, n-i-1)$. Then, $\{\mathcal{O}^i\}_{i=0}^n$ is a solution to (5.2.1).*

Proof. $Q\mathcal{O}^{(p)} = Q(Q\rho^{(p)} + d\rho^{(p-1)}) = d(Q\rho^{(p-1)}) = d\mathcal{O}^{(p-1)}$. □

Definition 5.2.7. A sequence of the form in Lemma 5.2.1 is called an exact sequence.

Recall that the functions over a bigraded manifold form a bicomplex

$$\Omega = \bigoplus_{(p,q)\in[0,\ldots,n]\times\mathbb{Z}} \Omega^{p,q}$$

with commuting differentials Q and L. Moreover, L is homotopic to 0 since $L = QK + KQ$, where K is interpreted as a homotopy operator.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{Q} & \Omega^{p,q-1} & \xrightarrow{Q} & \Omega^{p,q} & \xrightarrow{Q} & \Omega^{p,q+1} & \xrightarrow{Q} & \cdots \\
 & \swarrow K & 0\big\downarrow\big\downarrow L & \swarrow K & 0\big\downarrow\big\downarrow L & \swarrow K & 0\big\downarrow\big\downarrow L & \swarrow K & \\
\cdots & \xrightarrow{Q} & \Omega^{p+1,q-1} & \xrightarrow{Q} & \Omega^{p+1,q} & \xrightarrow{Q} & \Omega^{p+1,q+1} & \xrightarrow{Q} & \cdots
\end{array}$$

Let H_L and H_Q denote the horizontal and vertical cohomology of Ω, respectively. Note that both of them are naturally bigraded. We have obtained the following result.

Proposition 5.2.4. *$H_L^{p,q}(H_Q) \cong H_Q^{p,q}$ for all $0 \le p \le n$ and $q \in \mathbb{Z}$.*

Let Ω_{tot} denote the total complex of Ω. Let H_{tot} denote the cohomology of Ω_{tot}. Let's consider the filtration $\Omega^r_{tot,0} \subset \Omega^r_{tot,1} \subset \Omega^r_{tot,2} \subset \cdots \subset \Omega^r_{tot}$ on Ω^r_{tot}, where $\Omega^r_{tot,i} = \bigoplus_{\substack{p+q=r\\ p\ge n-i}} \Omega^{p,q}$. This filtration is preserved by the total differential, hence induces a filtration $H^r_{tot,0} \subset H^r_{tot,1} \subset H^r_{tot,2} \subset \cdots \subset H^r_{tot}$ on H^r_{tot}. Let $GH^r_{tot,i}$ denote $H^r_{tot,i}/H^r_{tot,i-1}$. We have $H^r_{tot} \cong \bigoplus_{i=0}^n GH^r_{tot,i}$.

Theorem 5.2.1. *For each $r \in \mathbb{Z}$ and $0 \le i \le n$, there exists a surjective map*

$$f : H^r_{tot,i} \to H_L^{l,r-l}(H_Q), \tag{5.2.2}$$

where $l = n - i$. Moreover, (5.2.2) induces an isomorphism

$$GH^r_{tot,i} \cong H_L^{l,r-l}(H_Q). \tag{5.2.3}$$

Proof. Let $\mathcal{O} = \sum_{l\le p\le n} \mathcal{O}^{p,r-p}$ be a closed element in $\Omega^r_{tot,i}$. We have $Q\mathcal{O}^{l,r-l} = 0$ and $d\mathcal{O}^{l,r-l} = Q\mathcal{O}^{l+1,r-l-1}$. Note that $\mathcal{O}^{l,r-l}$ is Q-exact if $\mathcal{O}$ is exact. We define f to be the map induced by

$$\begin{aligned} \tilde{f} : \Omega^r_{tot} &\to \Omega^{l,r-l} \\ \mathcal{O} &\mapsto \mathcal{O}^{l,r-l} \end{aligned}$$

f is a well-defined map between cohomologies. To prove the surjectivity of f, note that for each $\mathcal{O}^{l,r-l} \in \Omega^{l,r-l}$, the element $\sum_{p=0}^i \frac{1}{p!} K^p \mathcal{O}^{l,r-l} \in \tilde{f}^{-1}(\mathcal{O}^{l,r-l})$ is closed in $\Omega^r_{tot,i}$. The isomorphism (5.2.3) then follows directly from the observation that $\mathrm{Ker}(\tilde{f}) = \Omega^r_{tot,i-1}$. □

Corollary 5.2.1. *There is an isomorphism* $H^r_{tot} \cong \bigoplus_{i=0}^n H_Q^{i,r-i}$ *of graded modules.*

Proof. This follows directly from Proposition 5.2.4 and Theorem 5.2.1. □

Remark 5.2.6. There is a simpler proof for Corollary 5.2.1 if we adopt the second sign convention for the bigraded setting. By Remark 4.5.3, Q anticommutes with L and commutes with K. The total differential is just $Q - L$. Consider the "Mathai-Quillen map" $j = \exp(K)$ of Ω. Note that the expression $\exp(K)$ is well-defined because K is nilpotent in this setting. By Lemma 4.5.3, we have

$$j \circ Q \circ j^{-1} = \exp(K)([Q, \exp(-K)] + \exp(-K)Q) = Q - L.$$

In other words, the total cohomology of Ω is equal to the Q-cohomology of Ω.

The reason we adopt the first sign convention is just that we want to make the algebraic meaning of K more transparent.

One can easily see from the proof of Theorem 5.2.1 that Corollary 5.2.1 is equivalent to the following statement.

Theorem 5.2.2. *Every descendant sequence is a K-sequence up to an exact sequence.*

Let's assume that there is a well-defined notion of integration $\int$ on Γ_Y. The partition function Z of S is defined as

$$Z = \int \exp(-S).$$

The expectation value of an observable $\mathcal{O}$ is the integration

$$\langle \mathcal{O} \rangle = \int \exp(-S)\mathcal{O}.$$

In [Wit88], assuming that $\mathcal{O}$ does not depend on the Riemannian metric g of M, Witten observed that the expectation value $\langle \mathcal{O} \rangle$ is also independent of the choice of g if the energy momentum tensor T of S is Q-exact, i.e., $T = Q(G)$ for some G.[2] More precisely, he computed

$$\delta_g \langle \mathcal{O} \rangle = -\int \exp(-S)T\mathcal{O} = -\int \exp(-S)Q(G)\mathcal{O} = -\int Q(\exp(-S)G\mathcal{O}) = 0,$$

where he used that S and $\mathcal{O}$ are Q-closed, and that

the integration of a Q-exact function vanishes. (5.2.4)

[2] CohFTs are therefore a special class of topological quantum field theories (TQFTs), though the latter do not have a constructive definition in mathematics.

(5.2.4) is a bold assumption. It can be seen as an infinite dimensional analogue of Stokes' theorem. In fact, one can argue that this is indeed the case for a QK_v manifold $\mathcal{M}_Y$ equipped with the canonical QK-structure. For such $\mathcal{M}_Y$, Q is just the vertical differential d_v, i.e., the de Rham differential on Γ_Y. In Section 6, we will show that, in most CohFTs, Q are just d_v expressed in different coordinates.

Fix a sequence $\{\gamma_i\}_{i=0}^n$ of cycles of degrees $0, 1, \ldots, n$ in M and a descent sequence $\{\mathcal{O}^{(i)}\}_{i=0}^n$. We can get a sequence $\{\mathcal{O}^{(i)}[\gamma_i]\}_{i=0}^n$ of Q-closed observables. Obviously, $\mathcal{O}^{(i)}[\gamma_i]$ is Q-exact if $\{\mathcal{O}^{(i)}\}_{i=0}^n$ is an exact sequence. Using assumption (5.2.4), it is easy to see that Q-exact observables have vanishing expectation values. In other words, K-sequences are the only physically interesting solutions to the descent equations (5.2.1).

5.3 CohLFTs with gauge symmetries

Let P be a principal G-bundle over M, where G is a compact Lie group. The gauge symmetries are described by the automorphism group

$$\mathcal{G} := \{f : P \to P | \pi \circ f = \pi, f(pg) = f(p)g, \forall p \in P, g \in G\}.$$

Let $\mathrm{Ad}P = P \times_G G$, where G acts on itself by conjugation. We have a natural identification $\mathcal{G} \cong \Gamma(\mathrm{Ad}P)$. The Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$ can then be identified as $\Gamma(\mathrm{ad}P)$, i.e., the space of sections of the adjoint bundle of P.

Recall that a connection 1-form on a principal G-bundle P is a G-equivariant 1-form A with values in the Lie algebra $\mathfrak{g}$ such that $A(K_\xi) = \xi,\ \xi \in \mathfrak{g}$, where K_ξ is the fundamental vector field generated by ξ on P. The curvature 2-form of A is defined to be $F = dA + \frac{1}{2}[A, A]$. F is a basic form, and satisfies the second Bianchi identity $d_A F = 0$, where $d_A = d + [A, \cdot]$ is the covariant derivative associated to A.

Proposition 5.3.1. *For any principal bundle, the space of all connections $\mathcal{A}$ is an affine space modeled on $\Omega^1(\mathrm{ad}P)$. $\mathcal{A}$ has a natural $\mathcal{G}$-action, with its infinitesimal action given by*

$$\begin{aligned} \mathcal{A} \times \mathrm{Lie}(\mathcal{G}) &\to T\mathcal{A} \\ A \times \lambda &\mapsto (A, -d_A\lambda) \end{aligned}$$

where we use identifications $\mathrm{Lie}(\mathcal{G}) \cong \Omega^0(\mathrm{ad}P)$ and $T_A(\mathcal{A}) \cong \Omega^1(\mathrm{ad}P)$.

For our purpose, we need to identify $\mathcal{A}$ with the space of sections of some fiber bundle over M. Let P be a fiber bundle over M. Let J^1P be the first jet bundle of P. J^1P is an affine bundle modeled on the vector bundle $T^*M \otimes_M VP$, where VP is the vertical bundle over P and the tensor product is taken over M. Let $j^1\phi : J^1P \to J^1P$ denote the jet prolongation of a bundle automorphism $\phi : P \to P$. Such operations satisfy the chain rules

$$j^1(\phi_1 \circ \phi_2) = j^1(\phi_1) \circ j^1(\phi_2),$$
$$j^1(\mathrm{id}_P) = \mathrm{Id}_{J^1P}.$$

Thus, J^1P also has a principal G-action. The quotient space $C = J^1P/G$ is then an affine bundle modeled on the vector bundle $(T^*M \otimes_M VP)/G \cong T^*M \otimes \mathrm{ad}P$ over M.

Proposition 5.3.2 ([Sar93])**.** *There exists a bijection between $\mathcal{A}$, the affine space of connection 1-forms on P, and the set $\Gamma(C)$, the affine space of global sections of C.*

From now on, we will consider graded fiber bundles Y of the form

$$Y = C \times_M E,$$

where E is an associated bundle to P. (We assign degree 0 to elements of the fiber of C.) Let L_g denote the graded Lie algebra associated to the Lie algebra $\mathrm{Lie}(\mathcal{G})$. L_g is spanned by elements $\delta_\lambda, \iota_\lambda$ and Q_g for each $\lambda \in \mathrm{Lie}(\mathcal{G})$. They are of degrees 0, -1, 1, respectively, and satisfy

$$[\delta_{\lambda_1}, \delta_{\lambda_2}] = \delta_{[\lambda_1,\lambda_2]}, \quad [\delta_{\lambda_1}, \iota_{\lambda_2}] = \iota_{[\lambda_1,\lambda_2]}, \quad [\delta_\lambda, Q_g] = 0,$$
$$[Q_g, Q_g] = 0, \quad [\iota_{\lambda_1}, \iota_{\lambda_2}] = 0, \quad [Q_g, \iota_\lambda] = \delta_\lambda.$$

Note that we use the new notation δ_λ to denote the Lie derivatives.

Definition 5.3.1. Let $\mathcal{M}_Y$ be a QK_v-manifold. An L_g-structure on $\mathcal{M}_Y$ is said to be compatible with the QK_v-structure on $\mathcal{M}_Y$ if

1. Q_g, ι_λ and δ_λ are of degrees $(0,1)$, $(0,-1)$, $(0,0)$, respectively;

2. Q_g coincides with Q, ι_λ anticommutes with K.

$\mathcal{M}_Y$ together with the compatible QK_v-structure and L_g-structure is called a QK_{vg}-manifold.

Definition 5.3.2. Let $\mathcal{M}_Y$ be a QK_{vg}-manifold. $\mathcal{M}_Y$ is said to be simple if

$$[\delta_\lambda, K] = 0. \tag{5.3.1}$$

It is said to be h-simple if (5.3.1) only hold true for horizontal functions.

Lemma 5.3.1. $[\iota_\lambda, L] = [\delta_\lambda, K]$.

Proof. This follows from direct computations.

$$\begin{aligned}[\iota_\lambda, L] &= [\iota_\lambda, [Q, K]] \\ &= [[\iota_\lambda, Q], K] - [Q, [\iota_\lambda, K]] \\ &= [\delta_\lambda, K],\end{aligned}$$

where we use $[\iota_\lambda, Q] = \delta_\lambda$ and $[\iota_\lambda, K] = 0$. ∎

Definition 5.3.3. A cohomological (Lagrangian) gauge field theory (CohGFT) is a CohFT $(\mathcal{M}_Y, \mathcal{L})$, where $\mathcal{M}_Y$ is a QK_{vg}-manifold and $\mathcal{L}$ is basic with respect to the L_g-structure on $\mathcal{M}_Y$. The CohGFT is said to be simple (or h-simple) if $\mathcal{M}_Y$ is simple (or h-simple).

$\mathcal{L}$ being basic can be seen as a generalization of the notion of gauge invariance in bosonic theories. In most cases, δ_λ, ι_λ and Q do not depend on coordinates x^μ and dx^μ. They then gives Γ_Y an L_g-structure. The action functional S is a gauge invariant Q-closed function on Γ_Y.

Definition 5.3.4. A gauge invariant pre-observable $\mathcal{O}$ is a basic pre-observable on $\mathcal{M}_Y$. A gauge invariant observable is a basic observable on Γ_Y.

By definition, the observable $\mathcal{O}[\gamma]$ associated to a gauge invariant pre-observable $\mathcal{M}_Y$ and a cycle γ in M is a gauge invariant observable. Let $\mathcal{O}^{(0)}$ be a gauge invariant pre-observable of degree $(0, n)$. A natural question to ask is: Can we find a descendant sequence of $\mathcal{O}^{(0)}$ that is also gauge invariant?

Proposition 5.3.3. *The basic functions over an h-simple QK_{vg}-manifold $\mathcal{M}_Y$ are preserved by the $\mathcal{K}$-action.*

Proof. Let f be a basic function over $\mathcal{M}_Y$. We have

$$\iota_\lambda(Qf) = [\iota_\lambda, Q]f = \delta_\lambda f = 0, \quad \iota_\lambda(Kf) = [\iota_\lambda, K]f = 0.$$

We also have

$$\delta_\lambda(Qf) = [\delta_\lambda, Q]f = 0, \quad \delta_\lambda(Kf) = [\delta_\lambda, K]f = 0,$$

where we use the h-simple property, i.e., that $[\delta_\lambda, K]$ vanishes for horizontal functions. Since $L = QK + KQ$, the basic functions are also preserved by L, hence the $\mathcal{K}$-action. □

Recall that a K-sequence of $\mathcal{O}^{(0)}$ is specified by Q-closed functions $\mathcal{W}^{(1)}, \cdots, \mathcal{W}^{(n)}$.

Corollary 5.3.1. *In an h-simple CohGFT, a K-sequence of a gauge invariant pre-observable $\mathcal{O}^{(0)}$ is gauge invariant if $\mathcal{W}^{(1)}, \cdots, \mathcal{W}^{(n)}$ are gauge invariant.*

Now, let's turn back to the world of homological algebras. The QK_{vg} manifold $\mathcal{M}_Y$ give us a bicomplex Ω just like before. But this time we have a canonical sub-bicomplex, namely the sub-bicomplex Ω_{bas} which consists of gauge invariant elements. For an h-simple CohGFT, Ω_{bas} is stable under both Q, K and L. Let H_{tot} and H_Q denote the total cohomology and vertical cohomology of Ω_{bas}, respectively. Likewise, we have the following isomorphism

$$H^r_{tot}(\Omega_{bas}) \cong \bigoplus_{i=0}^{n} H_Q^{i,r-i}(\Omega_{bas}),$$

which particularly implies that

Theorem 5.3.1. *In an h-simple CohGFT, every gauge invariant descendant sequence is a K-sequence up to an exact sequence.*

In fact, it is necessary to consider Ω_{bas} instead of Ω. This is because there is a large class of CohGFTs with trivial Q-cohomologies. (For example, Ω equipped with the canonical QK_v-structure is Q-acyclic if the associated bundle E to P in the construction of Y is a vector bundle.) Therefore, one has to restrict to Ω_{bas} to obtain nontrivial observables. Geometrically, Ω_{bas} determines a QK_v-manifold $\mathcal{M}_{bas}$ as a submanifold of the QK_{vg}-manifold $\mathcal{M}_Y$. The Lagrangian $\mathcal{L}$ restricted to $\mathcal{M}_{bas}$ is also a Q-closed function of degree $(n, 0)$. Thus, $(\mathcal{M}_{bas}, \mathcal{L})$ is a CohFT. It makes more sense to consider the partition function and expectation values of observables in $(\mathcal{M}_{bas}, \mathcal{L})$, since the path integrals in $(\mathcal{M}_Y, \mathcal{L})$ always carry a redundant factor due the presence of gauge symmetries.

Chapter 6

Supersymmetric Lagrangian field theories

6.1 Algebra preliminaries

6.1.1 Spinors and vectors

Let V be a vector space over a field $\mathbb{K}$ with a quadratic form Q. The Clifford algebra $\mathrm{Cl}(V,Q)$ is the quotient algebra $\mathrm{T}(V)/\mathcal{I}_Q(V)$ of the tensor algebra $\mathrm{T}(V)$ of V, where $\mathcal{I}_Q$ is the two-sided ideal generated by all elements of the form

$$v \otimes v - Q(v)1.$$

We will always assume Q to be nondegenerate and $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$. In the real case, we write $\mathrm{Cl}(r,s)$ to denote the real Clifford algebra when $V = \mathbb{R}^{r+s}$ and $Q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$.[1] In the complex case, we write $\mathbb{C}\mathrm{l}(n)$ to denote the complex Clifford algebra when $V = \mathbb{C}^n$ and $Q(z) = z_1^2 + \cdots + z_n^2$.

The Clifford algebra $\mathrm{Cl}(V,Q)$ is filtered in a natural way by construction. Moreover, there is a filtration-preserving canonical vector space isomorphism λ between it and the exterior algebra ΛV

$$\begin{aligned} \lambda : \Lambda V &\to \mathrm{Cl}(V,Q) \\ v_1 \wedge \cdots \wedge v_r &\longmapsto \frac{1}{r!}\sum_{\sigma} \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(r)} \end{aligned}$$

which the sum is taken over the symmetric group on $\{1, \cdots, r\}$.

[1] We often write $\mathbb{R}^{r,s}$ to denote this quadratic space.

Lemma 6.1.1 ([LM16]). *With respect to the canonical isomorphism λ, Clifford multiplication between $v \in V$ and $\varphi \in \Lambda^r V$ can be written as*

$$\begin{aligned} v \cdot \varphi &= v \wedge \varphi + \iota_{v^\vee} \varphi, \\ \varphi \cdot v &= (-1)^r \left(v \wedge \varphi - \iota_{v^\vee} \varphi \right), \end{aligned}$$

where $v^\vee = Q(v, \cdot) \in V^\vee$.

There are two fundamental involutions on $\mathrm{Cl}(V,Q)$: the principal automorphism α which extends the antipodal map $v \mapsto -v$ on V, and the principal anti-automorphism β which descends from the involution on $\mathrm{T}(V)$ which sends $v_1 \otimes \cdots v_r$ to $v_r \otimes \cdots \otimes v_1$. Associated to α is the decomposition $\mathrm{Cl}(V,Q) = \mathrm{Cl}_0(V,Q) \oplus \mathrm{Cl}_1(V,Q)$ where $\mathrm{Cl}_i(V,Q) = \{\varphi \in \mathrm{Cl}(V,Q) : \alpha(\varphi) = (-1)^i \varphi\}$. Since $\alpha(\varphi_1\varphi_2) = \alpha(\varphi_1)\alpha(\varphi_2)$, this decomposition makes $\mathrm{Cl}(V,Q)$ into a $\mathbb{Z}_2$-graded algebra. The decomposition of $\mathrm{Cl}(V,Q)$ associated to β is less addressed in literature. For later use, we remark that for a p-vector $w \in \Lambda^p V$,

$$\beta(\lambda(w)) = (-1)^{\frac{p(p-1)}{2}} \lambda(w). \tag{6.1.1}$$

Let $e_1, \ldots, e_{r+s}$ be a positively oriented Q-orthonormal basis. We define the chirality operator to be $\omega = e_1 \cdots e_{r+s}$. In the complex case, one can consider $\omega = i^{\lfloor \frac{n}{2} \rfloor} e_1 \cdots e_n$ where $e_1, \ldots, e_n$ is a positively oriented Q-orthonormal basis of $\mathbb{C}^n$. The chirality operator lies in the center of the real (or complex) Clifford algebra when $r+s$ (or n) is odd. Moreover, one can check that $\omega^2 = 1$ in the complex case, and that $\omega^2 = (-1)^{\frac{l(l+1)}{2}}$ in the real case, where $l = s - r$. Let $\pi_\pm = \frac{1}{2}(1 \pm \omega)$, we have the following decompositions.

1. For n odd, $\mathbb{C}\mathrm{l}(n) = \mathbb{C}\mathrm{l}_+(n) \oplus \mathbb{C}\mathrm{l}_-(n)$, where $\mathbb{C}\mathrm{l}_\pm(n) \equiv \pi_\pm \mathbb{C}\mathrm{l}(n)$. Since ω is in the center of $\mathbb{C}\mathrm{l}(n)$, $\mathbb{C}\mathrm{l}_\pm(n)$ are two-sided ideals. Moreover, they are isomorphic to each other thourgh α because $\alpha\pi_\pm = \pi_\mp\alpha$.

2. Similarly, for $s - r = 3 \pmod 4$, $\mathrm{Cl}(r,s) = \mathrm{Cl}_+(r,s) \oplus \mathrm{Cl}_-(r,s)$, where $\mathrm{Cl}_\pm(r,s) \equiv \pi_\pm \mathrm{Cl}(r,s)$ are isomorphic two-sided ideals.

Let $\mathrm{Cl}^\times(V,Q)$ denote the group of units in $\mathrm{Cl}(V,Q)$. The pin group $\mathrm{Pin}(V,Q)$ is a subgroup of $\mathrm{Cl}^\times(V,Q)$ generated by $v \in V$ with $|Q(v)| = 1$, where $|a|$ is the absolute value of $a \in \mathbb{K}$. The spin group $\mathrm{Spin}(V,Q)$ is defined as the subgroup $\mathrm{Pin}(V,Q) \bigcap \mathrm{Cl}_0^\times(V,Q)$ of $\mathrm{Pin}(V,Q)$, where $\mathrm{Cl}_0^\times(V,Q)$ is the group of units in $\mathrm{Cl}_0(V,Q)$. (The notations $\mathrm{Spin}(r,s)$ and $\mathrm{Spin}^c(n)$ should be self-evident.) Note that $\mathrm{Spin}(V,Q)$ has a subgroup $\mathbb{Z}_2 = \{1, -1\}$, where 1 is the identity element of $\mathrm{Cl}(V,Q)$. It is not hard to show that $\mathrm{Spin}^c(n) \cong \mathrm{Spin}(r,s) \times_{\mathbb{Z}_2} \mathrm{U}(1)$ when $n = r + s$ [Jos17].

Definition 6.1.1. Let $\rho : \mathrm{Spin}(V,Q) \to \mathrm{End}_{\mathbb{K}}(W)$ be a $\mathbb{K}$-representation of the spin group. We say that ρ is a vector representation if $\mathbb{Z}_2 \subset \ker\rho$, and a spinor representation if $-1 \notin \ker\rho$. An element of W is called a vector (or spinor) if ρ is a vector (or spinor) representation.

The group $\mathrm{Cl}^\times(V,Q)$ acts on $\mathrm{Cl}(V,Q)$ naturally through the adjoint representation Ad, which is defined by $\mathrm{Ad}_\varphi(x) = \varphi^{-1}x\varphi$. For $v \in V$ with $Q(v) \neq 0$ and $w \in V$, we have

$$\mathrm{Ad}_v(w) = v^{-1}wv = Q(v)^{-1}vwv = Q(v)^{-1}v(2\langle v,w\rangle - vw) = 2\frac{\langle v,w\rangle}{\langle v,v\rangle}v - w,$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form on V associated to Q. In other words, the action of Ad_v preserves the quadratic form on V. It can be seen as the composition of the reflection map across the hyperplane orthogonal to v and the antipodal map. Using the Cartan-Dieudonné theorem, one can show that there is the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V,Q) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V,Q) \longrightarrow 1$$

where $\mathrm{SO}(V,Q)$ is the special Q-orthogonal group of V. In other words, $\mathrm{Spin}(V,Q)/\mathbb{Z}_2 \cong \mathrm{SO}(V,Q)$. A vector representation of $\mathrm{Spin}(V,Q)$ then descends naturally to a representation of $\mathrm{SO}(V,Q)$. On the other hand, any non-trivial representation of $\mathrm{Cl}^0(V,Q)$ restricts to a representation of $\mathrm{Spin}(V,Q)$ which sends -1 to a nontrivial involution, i.e., a spinor representation. Fix a Q-orthonormal basis for V. Let $\mathrm{F}(V,Q)$ denote the finite group generated by elements of this basis. $\mathrm{F}(V,Q)$ is a subgroup of $\mathrm{Pin}(V,Q)$. Let $\mathrm{F}_0(V,Q)$ denote its intersection with $\mathrm{Spin}(V,Q)$. It is easy to see that a representation of $\mathrm{Spin}(V,Q)$ is uniquely determined by its restriction to $\mathrm{F}_0(V,Q)$. Now observe that

$$\mathrm{Cl}_0(V,Q) \cong \mathbb{K}F_0(V,Q)/\mathbb{K}\cdot\{(-1)+1\},$$

where $\mathbb{K}F_0(V,Q)$ is the group algebra of $F_0(V,Q)$. It follows that every spinor representation of $\mathrm{Spin}(V,Q)$ comes from a (nontrivial) representation of $\mathrm{Cl}_0(V,Q)$.

Clifford algebras and their even parts can be fully characterized as matrix algebras [LM16].

$s-r \pmod 8$	$\mathrm{Cl}(r,s)$	$\mathrm{Cl}_0(r,s)$	$\mathbb{C}\mathrm{l}(n)$	$\mathbb{C}\mathrm{l}_0(n)$
0	$\mathbb{R}(2^l)$	$\mathbb{R}(2^{l-1})\oplus\mathbb{R}(2^{l-1})$	$\mathbb{C}(2^l)$	$\mathbb{C}(2^{l-1})\oplus\mathbb{C}(2^{l-1})$
1	$\mathbb{C}(2^l)$	$\mathbb{R}(2^l)$	$\mathbb{C}(2^l)\oplus\mathbb{C}(2^l)$	$\mathbb{C}(2^l)$
2	$\mathbb{H}(2^{l-1})$	$\mathbb{C}(2^{l-1})$	$\mathbb{C}(2^l)$	$\mathbb{C}(2^{l-1})\oplus\mathbb{C}(2^{l-1})$
3	$\mathbb{H}(2^{l-1})\oplus\mathbb{H}(2^{l-1})$	$\mathbb{H}(2^{l-1})$	$\mathbb{C}(2^l)\oplus\mathbb{C}(2^l)$	$\mathbb{C}(2^l)$
4	$\mathbb{H}(2^{l-1})$	$\mathbb{H}(2^{l-2})\oplus\mathbb{H}(2^{l-2})$	$\mathbb{C}(2^l)$	$\mathbb{C}(2^{l-1})\oplus\mathbb{C}(2^{l-1})$
5	$\mathbb{C}(2^l)$	$\mathbb{H}(2^{l-1})$	$\mathbb{C}(2^l)\oplus\mathbb{C}(2^l)$	$\mathbb{C}(2^l)$
6	$\mathbb{R}(2^l)$	$\mathbb{C}(2^{l-1})$	$\mathbb{C}(2^l)$	$\mathbb{C}(2^{l-1})\oplus\mathbb{C}(2^{l-1})$
7	$\mathbb{R}(2^l)\oplus\mathbb{R}(2^l)$	$\mathbb{R}(2^l)$	$\mathbb{C}(2^l)\oplus\mathbb{C}(2^l)$	$\mathbb{C}(2^l)$

Table 6.1.1: Clifford algebras and their even parts. We set $l = s-r \geq 0$ in the real case and $n = s+r$, $l = \lfloor \frac{n}{2} \rfloor$ in the complex case. We use $\mathbb{K}(N)$ to denote $N\times N$ matrices with values in $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ is the algebra of quaternions.

Let's explain a few physics terminology based on Table 6.1.1 and the previous discussion.

1. Consider the complex spinor representation of $\mathrm{Spin}(r,s)$ obtained by restricting an irreducible complex representation $\mathbb{C}\mathrm{l}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ to $\mathrm{Spin}(r,s) \subset \mathbb{C}\mathrm{l}(n)$. We call such representation the Dirac representation and an element in S a Dirac spinor.

2. For n even, the Dirac representation is the direct sum of two inequivalent irreducible complex spinor representations obtained from $\mathbb{C}\mathrm{l}_0(n) \to \mathrm{End}_{\mathbb{C}}(S_\pm)$. We have $\omega S_\pm = \pm S_\pm$, where ω is the chirality operator. We call such representations the chiral (or Weyl) representations. An element in $S_\pm$ is called a right-handed (or left-handed) Weyl spinor.

3. For $s - r = 0, 6, 7 \pmod 8$, a Dirac representation carries a real structure. It is the complexification of a real spinor representation obtained from $\mathrm{Cl}(r,s) \to \mathrm{End}_{\mathbb{R}}(S)$, which is called the Majorana representation. An element in S is called a Majorana spinor. For $s - r = 0 \pmod 8$, the Majorana representation is the direct sum of two irreducible real representations obtained from $\mathrm{Cl}_0(r,s) \to \mathrm{End}_{\mathbb{R}}(S_\pm)$. An element in $S_\pm$ is called a right-handed (or left-handed) Majorana-Weyl spinor.

4. For $s - r = 2, 3, 4 \pmod 8$, a Dirac representation carries a quaternionic structure. It is obtained from $\mathrm{Cl}(r,s) \to \mathrm{End}_{\mathbb{H}}(S)$ and is called the symplectic Majorana representation. An element in S is called a symplectic Majorana spinor. For $s - r = 4 \pmod 8$, the Majorana representation is a direct sum of two irreducible quaternionic representations obtained from $\mathrm{Cl}_0(r,s) \to \mathrm{End}_{\mathbb{H}}(S_\pm)$. An element in $S_\pm$ is called a right-handed (or left-handed) symplectic-Majorana-Weyl spinor.

We make the following remark for later use.

Remark 6.1.1. Let S be a real irreducible representation of $\mathrm{Cl}(r,s)$. S is then also a spinor representation. It is clear from Table 6.1.1 that, for $s - r = 0, 1, 2, 4 \pmod 8$, S is a direct sum of two irreducible spinor representations.

6.1.2 Super Poincaré algebras and Dirac operators

An n-dimensional Minkowski space is an affine space M modeled on $\mathbb{R}^n$, together with a metric g making the tangent space T_xM isomorphic to $\mathbb{R}^{1,n-1}$ for every $x \in M$. Let $\mathrm{O}(1, n-1)$ denote the orthogonal group of $\mathbb{R}^{1,n-1}$. The full symmetry group of T_xM is the semi-direct product $\mathbb{R}^{1,n-1} \rtimes \mathrm{O}(1, n-1)$. The Lie algebra $\mathfrak{p}$ of this group is usually referred to as the Poincaré algebra. The Poincaré group is then defined to be the simply connected group uniquely determined by this Lie algebra. More precisely, it is the semi-direct product $\mathbb{R}^{1,n-1} \rtimes \mathrm{Spin}^0(1, n-1)$, where $\mathrm{Spin}^0(1, n-1)$ is the identity component of the spin group $\mathrm{Spin}(1, n-1)$. By the spin-statistics theorem in physics, the Lorentz invariance of a quantum field theory requires that vectors must be commutative and spinors must be anti-commutative. This is a hint that there exists a class of theories whose symmetry groups are super generalizations of the Poincaré groups. To define such super Lie groups and their super Lie algebras, we need the following proposition.

Proposition 6.1.1 ([Del99]). *Let* $\gamma : \mathrm{Spin}^0(1, n-1) \to \mathrm{End}_{\mathbb{R}}(S)$ *be an irreducible real spinor representation of* $\mathrm{Spin}^0(1, n-1)$. *The commutant* Z *of* S *is* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *Up to a real factor, there exists a unique symmetric morphism* $\Gamma : S \otimes S \to \mathbb{R}^{1,n-1}$, *and it is* Z_1*-invariant, where* Z_1 *is the group of unit elements in* Z. *Moreover, the bilinear form* $\langle v, \Gamma(\cdot, \cdot)\rangle$ *on* S *is either positive or negative definite for* $v \in \mathbb{R}^{1,n-1}$ *with* $Q(v) > 0$.

Remark 6.1.2. For a reducible spinor module S that is a direct sum of irreducible $S^{(\alpha)}$ with pairings Γ^α, one can set $\Gamma \equiv \sum c_\alpha \Gamma^\alpha$, where $c_\alpha \in \mathbb{R}/\{0\}$. One can always choose c_α properly such that $\langle v, \Gamma(\cdot, \cdot)\rangle$ is positive definite for $v \in \mathbb{R}^{1,n-1}$ with $Q(v) > 0$. We say such Γ satisfies the positive condition.

Definition 6.1.2. Let S be a real spinor representation of $\mathrm{Spin}^0(1, n-1)$.Let $S^\vee$ be its dual representation. A super Poincaré algebra is a super Lie algebra $\mathfrak{p}_s$ with the even part being the Poincaré algebra $\mathfrak{p}$ and the odd part being $S^\vee$. The bracket $[\cdot, \cdot]$ on the odd part of $\mathfrak{p}_s$ is given by a symmetric morphism $S^\vee \otimes S^\vee \to \mathbb{R}^{1,n-1}$. The existence of such paring is guaranteed by Proposition 6.1.1. $\mathfrak{p}_s$ is said to be an $N = i$ super Poincaré algebra if S is the direct sum of i irreducible spinor representations, $n - 2 \neq 0 \pmod 4$. It is said to be an $N = (i, j)$ super Poincaré algebra if S is the direct sum of i and j copies of the two inequivalent spinor representations, $n - 2 = 0 \pmod 4$.

Remark 6.1.3. Note that there is a sub-algebra $\mathfrak{l} := \mathbb{R}^{1,n-1} \oplus S^\vee$ of $\mathfrak{p}_s$. In fact, $\mathfrak{l}$ can be viewed as the super Lie algebra of "super translations". One can also consider bilinear pairings of spinors which take values in $\Lambda^2\mathbb{R}^{1,n-1} \cong \mathfrak{so}(1, n-1)$. Spinors in such pairings (if they exist) can be then seen as a "super rotation". However, since $\mathfrak{so}(1, n-1)$ is non-abelian, one has to check carefully if the Jacobi identity still holds for such pairings.

The following proposition gives an explicit relation between γ and Γ.

Proposition 6.1.2 ([Del99]). *Let* S *be a real spinor representation of* $\mathrm{Spin}^0(1, n-1)$, *not necessarily irreducible. Let* $\Gamma : S \otimes S \to \mathbb{R}^{1,n-1}$ *be a symmetric pairing of* S *satisfying the positive condition. Let* $S^\vee$ *denote the dual representation of* S. *There exists a unique* $\Gamma^\vee : S^\vee \otimes S^\vee \to \mathbb{R}^{1,n-1}$ *satisfying the positive condition such that, if* Γ *and* $\Gamma^\vee$ *are reinterpreted as morphisms* $\tilde{\gamma} : \mathbb{R}^{1,n-1} \to \mathrm{Hom}_{\mathbb{R}}(S, S^\vee)$ *and* $\tilde{\gamma}^\vee : \mathbb{R}^{1,n-1} \to \mathrm{Hom}_{\mathbb{R}}(S^\vee, S)$[2], *then* $\tilde{\gamma}(v)\tilde{\gamma}^\vee(v) = \tilde{\gamma}^\vee(v)\tilde{\gamma}(v) = Q(v)$. *Moreover, the spinor module structure induced by the* $\mathrm{Cl}(1, n-1)$*-module structure* $(\tilde{\gamma}, \tilde{\gamma}^\vee)$ *on* $S \oplus S^\vee$ *coincides with* $(\gamma, \gamma^\vee)$.

Let S, Γ and $\Gamma^\vee$ be as in Proposition 6.1.2. Let $S_0 \equiv S \oplus S^\vee$. Let γ_0 denote the Clifford action induced by Γ and $\Gamma^\vee$ on S_0.

[2]More explicitly, $\tilde{\gamma}$ is given by

$$(\tilde{\gamma}(v)(s))(t) = \langle \Gamma(s, t), v\rangle,$$

where $s, t \in S$ and $v \in \mathbb{R}^{1,n-1}$. $\tilde{\gamma}^\vee$ can be determined in a similar way.

Remark 6.1.4. Note that, by Remark 6.1.1, S_0 is an irreducible representation of $\mathrm{Cl}(1, n-1)$ if S is an irreducible spinor representation and $n-2 = 0, 1, 2, 4 \pmod 8$. Moreover, S and $S^\vee$ are isomorphic to the two symplectic Majorana representations $S_\pm$ in the case of $n - 2 = 0 \pmod 4$.

On S_0 we can define a symmetric invariant form $(\cdot, \cdot)$ by setting

$$\left(s + s^\vee, t + t^\vee\right) \equiv s^\vee(t) + t^\vee(s),$$

and a equivariant pairing $[\cdot, \cdot]$ by setting

$$\left[s + s^\vee, t + t^\vee\right] \equiv \Gamma(s,t) + \Gamma^\vee(s^\vee, t^\vee).$$

The symmetry of $[\cdot, \cdot]$ implies that

$$\left(\gamma_0(v)(s + s^\vee), t + t^\vee\right) = \left(s + s^\vee, \gamma_0(v)(t + t^\vee)\right). \tag{6.1.2}$$

Definition 6.1.3. Let $\psi : M \to S^\vee$ be a spinor field on M. Let $\{e_\alpha\}_{\alpha=1}^n$ be a (local) orthonormal vector field over M. Let e^α be the dual of e_α. The Dirac term of ψ is defined as

$$\psi \not{D} \psi(x) \equiv \sum_\alpha \langle e^\alpha, \Gamma^\vee(\psi, \nabla_{e_\alpha}\psi)\rangle(x), \quad x \in M, \tag{6.1.3}$$

where ∇ is Levi-Civita connection on the spinor bundle. In this case, ∇_{e_α} is just the ordinary derivative.

We can rewrite (6.1.3) in a more common form using $(\cdot, \cdot)$ and $[\cdot, \cdot]$ on S_0.

$$\psi \not{D} \psi = \sum_\alpha \langle e^\alpha, [\psi, \nabla_{e_\alpha}\psi]\rangle = \sum_\alpha \left(e^\alpha \psi, \nabla_{e_\alpha}\psi\right) = \sum_\alpha \left(\psi, e^\alpha \nabla_{e_\alpha}\psi\right) = \left(\psi, \not{D}\psi\right),$$

where $\not{D} \equiv \sum_\alpha e^\alpha \nabla_{e_\alpha}$ is the so-called Dirac operator.

Proposition 6.1.3. *The Dirac operator is anti-self-adjoint with respect to the nondegenerate bilinear form $\int_M \mathrm{dvol}_g(\cdot, \cdot)$ on the space of spinor fields.*

Proof. Let ψ_1 and ψ_2 be two spinor fields. Using (6.1.2), it is not hard to show that

$$\operatorname{div}[\psi_1, \psi_2] = \left(\psi_1, \not{D}\psi_2\right) + \left(\not{D}\psi_1, \psi_2\right),$$

where div denote the divergence operator. □

For later applications, we also want to study "super Poincaré algebras" and Dirac terms in Euclidean signatures. We need the following proposition from [LM16].

Proposition 6.1.4. *Let S_0 be a real representation of* $\mathrm{Cl}(0,n)$. *There exists a (positive definite) symmetric invariant form $(\cdot,\cdot)$ on S_0 such that*

$$(es, et) = (s,t)$$

for all $s,t \in S$ and for all $e \in \mathbb{R}^n$ with $|e| = 1$.

Corollary 6.1.1. *Let $(\cdot,\cdot)$ and S_0 be as above. Then for any $v \in \mathbb{R}^n$,*

$$(vs, t) = -(s, vt)$$

for all $s,t \in S_0$.

Proof. Assume $|v| \neq 0$. Then $(vs,t) = ((v/|v|)vs, (v/|v|)t) = \frac{1}{|v|^2}\left(v^2 s, vt\right) = -(s,vt)$. □

Proposition 6.1.4 and Corollary 6.1.1 imply that Dirac operators in the Euclidean cases are self-adjoint with respect to $(\cdot,\cdot)$. A nontrivial Dirac term, therefore, requires the spinor fields to be commutative. To remedy this, we define $(\cdot,\cdot)_\omega \equiv (\omega(\cdot),\cdot)$, where ω is the chirality operator of $\mathrm{Cl}(0,n)$. It is easy to see that $(\cdot,\cdot)_\omega$ is symmetric when $n = 0, 3 \pmod 4$.

Corollary 6.1.2. *Let $(\cdot,\cdot)_\omega$ be the invariant form as above. Then for any $v \in \mathbb{R}^n$ and $n = 0, 1 \pmod 4$,*

$$(vs,t)_\omega = (vt,s)_\omega$$

for all $s,t \in S_0$.

Proof. $(vs,t)_\omega = (-1)^n(\omega s, vt) = (s, \beta(\omega)vt) = (-1)^{\frac{n(n-1)}{2}}(\omega vt, s) = (-1)^{\frac{n(n-1)}{2}}(vt,s)_\omega$. □

It follows that the Dirac operator is anti-self-adjoint with respect to $(\cdot,\cdot)_\omega$ if and only if $n = 0 \pmod 4$. Moreover, we can define a pairing $[\cdot,\cdot]_\omega : S_0 \otimes S_0 \to \mathbb{R}^n$ associated to $(\cdot,\cdot)_\omega$ by setting

$$\langle v, [s,t]_\omega\rangle = (vs,t)_\omega \tag{6.1.4}$$

for all $s,t \in S_0$ and $v \in \mathbb{R}^n$. By construction, it is a symmetric pairing when $n = 0 \pmod 4$. Thus, we can use it to define a super extension of the Lie algebra $\mathfrak{so}(0,n) \ltimes \mathbb{R}^{0,n}$. With a slight abuse of notation, we still call this algebra the super Poincaré algebra.

6.2 Super Yang-Mills theory

6.2.1 N=1 super Yang-Mills theory

Let (M, g) be a n-dimensional Minkowski spacetime. Let S_0 be a $\mathrm{Cl}(1, n-1)$ module with a symmetric invariant form $(\cdot,\cdot)$ and a symmetric equivariant pairing $[\cdot,\cdot]$ as defined in the previous section. We also need the following extra geometric data to define our Lagrangian.

1. A trivial principal G-bundle P over M and a connection 1-form A on it. Since P is trivial, we can also think of A as a section of $\Omega^1(\mathrm{ad}P)$, where $\mathrm{ad}P$ is the adjoint bundle associated to P. The corresponding covariant derivative d_A can then be written as $d_A = d + A$.

2. An irreducible real spinor field ψ that is in the adjoint representation of G, i.e., a section of $\Pi S^\vee \otimes \mathrm{ad}P$. Since all bundles are trivial, ψ is simply a map $M \to \Pi S^\vee \otimes \mathfrak{g} \subset \Pi S_0 \otimes \mathfrak{g}$.[3]

3. A twisted Dirac operator $\not{D}_A : \Gamma(\Pi S_0 \otimes \mathrm{ad}P) \to \Gamma(\Pi S_0 \otimes \mathrm{ad}P)$. We can write $\not{D}_A = \not{D} + A$, where $\not{D}$ is the usual Dirac operator. $A = \sum_i A_i dx^i$ acts on $\psi = \sum_a \psi_a \otimes g_a$ as $A\psi = \sum_{i,a} (\lambda(dx^i)\psi_a) \otimes [A_i, g_a]$. Note that $\not{D}_A$ interchanges elements of ΠS and $\Pi S^\vee$.

4. An invariant form on $\Pi S_0 \otimes \mathfrak{g}$ obtained by taking the tensor product of $(\cdot,\cdot)$ on S_0 and the Ad-invariant inner product Tr on $\mathfrak{g}$. For simplicity, we denote this invariant form again by $(\cdot,\cdot)$.

We consider the minimal extension of the bosonic Yang-Mills Lagrangian.

$$\mathcal{L} = -\frac{1}{4}\langle F, F\rangle + \frac{1}{2}\left(\psi, \not{D}_A \psi\right), \tag{6.2.1}$$

where F is the curvature 2-form of A, the inner product $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$-valued differential forms is induced by the Minkowski metric on M and the G-invariant inner product on $\mathfrak{g}$. The supersymmetry transformations are

$$\delta_\epsilon A = [\epsilon, \psi], \tag{6.2.2}$$

$$\delta_\epsilon \psi = \frac{1}{2} F\epsilon, \tag{6.2.3}$$

where $\epsilon : M \to \Pi S^\vee \subset \Pi S_0$ is a parallel spinor field, i.e., $\nabla\epsilon = 0$. The pairing $[\cdot,\cdot]$ in (6.2.2) is justified by the identification $T_x^* M \cong \mathbb{R}^{1,n-1}$, $x \in M$. Similarly, the curvature form F acts on ϵ via the identification $\Lambda T_x^* M \cong \mathrm{Cl}(T_x^* M)$, $x \in M$. Note that these transformations are equivariant with respect to the gauge transformations of A and ψ.

[3] With a slight abuse of notation, we use the same notation to denote a spinor module and a spinor bundle.

To show that $\mathcal{L}$ is supersymmetric, we need to prove that $\delta_\epsilon \mathcal{L}$ is a total divergence, and that (6.2.2) and (6.2.3) indeed closed to a supersymmetry algebra.

For the bosonic term, we have

$$\begin{aligned}\delta_\epsilon \langle F, F\rangle &= 2\langle F, \delta_\epsilon F\rangle \\ &= 2\langle F, d_A \delta_\epsilon A\rangle \\ &= 2\langle d_A^\star F, [\epsilon, \psi]\rangle + \textit{divergence} \\ &= 2\left((d_A^\star F)\epsilon, \psi\right) + \textit{divergence}.\end{aligned}$$

For the fermionic term, we have

$$\begin{aligned}\delta_\epsilon \left(\psi, \not{D}_A \psi\right) &= \left(\delta_\epsilon \psi, \not{D}_A \psi\right) + \left(\psi, (\delta_\epsilon \not{D}_A)\psi\right) + \left(\psi, \not{D}_A \delta_\epsilon \psi\right) \\ &= \left(\delta_\epsilon \psi, \not{D}_A \psi\right) + \text{tri } \psi + \left(\psi, \not{D}_A \delta_\epsilon \psi\right) \\ &= -2\left(\not{D}_A \delta_\epsilon \psi, \psi\right) + \text{tri } \psi + \textit{divergence} \\ &= -\left(\not{D}_A (F\epsilon), \psi\right) + \text{tri } \psi + \textit{divergence},\end{aligned}$$

where tri $\psi = (\psi, [\epsilon, \psi]\psi)$.

Lemma 6.2.1.

$$\not{D}_A(F\epsilon) = -(d_A^\star F)\epsilon.$$

Proof. Recall that the untwisted Dirac operator on a Clifford bundle takes the form

$$\not{D} = d - d^\star.$$

Since ϵ is constant, we have

$$\not{D}_A(F\epsilon) = (\not{D}_A F)\epsilon = (d_A F - d_A^\star F)\epsilon = -(d_A^\star F)\epsilon.$$

In the last step we use the Bianchi identity. □

Putting everything together, we have proved

Proposition 6.2.1.

$$\delta_\epsilon \mathcal{L} = \frac{1}{2}\text{tri } \psi + \textit{divergence}. \tag{6.2.4}$$

It remains to find out the dimensions in which tri ψ vanishes. We have

$$\begin{aligned}\text{tri } \psi &= \sum \left(\psi^a, \left[\epsilon, \psi^b\right] \psi^c\right) \langle g_a, [g_b, g_c]\rangle \\ &= \sum \left(\epsilon, [\psi^a, \psi^c]\, \psi^b\right) \langle g_a, [g_b, g_c]\rangle.\end{aligned}$$

The last step follows from the definition of $[\cdot,\cdot]$. Since $\langle g_a, [g_b, g_c]\rangle$ is totally antisymmetric, tri $\psi = 0$ if and only if the totally antisymmetric part of $\left[\psi^a, \psi^b\right]\psi^c$ vanishes. If ψ is even, this is automatically true because $[\cdot,\cdot]$ is symmetric. In our case, we need to show that the totally symmetric part of $\left[\psi^a, \psi^b\right]\psi^c$ vanishes for even spinors ψ^a, ψ^b, ψ^c, i.e.,

$$\left[\psi^a, \psi^b\right]\psi^c + \left[\psi^b, \psi^c\right]\psi^a + \left[\psi^c, \psi^a\right]\psi^b = 0. \tag{6.2.5}$$

In fact, (6.2.5) is satisfied in dimensions $n = 3, 4, 6, 10$. For this we provide the argument of Deligne [Del99]. One can also consult [BH09] for an alternative proof using the division algebra techniques.

Lemma 6.2.2. *The following statements are equivalent:*

1. $[\psi, \psi]\,\psi = 0$ *for all even spinors* ψ.
2. $[\psi, \phi]\,\chi + [\phi, \chi]\,\psi + [\chi, \psi]\,\phi = 0$ *for all even spinors* ψ, ϕ, χ.
3. $|\,[\psi, \psi]\,|^2 = 0$ *for all even spinors* ψ.

Proof. Let V and W be two $\mathbb{K}$-vector spaces. Let $k \in \mathbb{N}$. Recall that a totally symmetric multilinear map $f : \mathcal{T}^k V \to W$ vanishes if and only if its associated map f_k of degree k on V vanishes, i.e.,

$$f(\underbrace{v, v, \ldots, v}_{k}) = 0, \quad v \in V.$$

Now observe that $[\psi, \psi]\,\psi$ is the associated cubic map of $[\psi, \phi]\,\chi$ and $|\,[\psi, \psi]\,|^2 = \langle \psi, [\psi, \psi]\,\psi\rangle$ is the associated quartic map of $\langle \theta, [\psi, \phi]\,\chi\rangle$. □

Now let $v \neq 0$ be an isotropic vector in $\mathbb{R}^{1,n-1}$. $\gamma(v) : S \to S^\vee$ has a non-trivial kernel Ker v. We have $\langle v, [\psi, \psi]\rangle = (\gamma(v)\psi, \psi) = 0$ for all $\psi \in$ Ker v. Recall that two vectors in a Minkowski spacetime are orthogonal only if

1. At least one of them is spacelike.
2. They are parallel lightlike vectors.

By Proposition 6.1.1, $[\psi, \psi]$ must be parallel to v. Hence $|\,[\psi, \psi]\,|^2 = 0$.

Now observe that for $n = 3, 4, 6, 10$, the spin group is $SL(2, \mathbb{K})$ and acts transitively on $S^\vee = \mathbb{K}^2$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Hence $|\,[\psi, \psi]\,|^2$ vanishes identically on S.

Theorem 6.2.1. *The super Yang-Mills Lagrangian (6.2.1) is supersymmetric in* $n = 3, 4, 6, 10$.

Let's derive the equations of motions of $\mathcal{L}$. The variation of the bosonic term is

$$\delta\langle F,F\rangle = 2\langle d_A^\star F,\delta A\rangle + divergence.$$

For the fermionic part, we have

$$\delta\left(\psi, \not{D}_A\psi\right) = -2\left(\not{D}_A\psi,\delta\psi\right) - \langle\delta A,[\psi,\psi]\rangle + divergence,$$

where $[\psi,\psi]=\sum_{i,j}[\psi_i,\psi_j]\otimes[g_i,g_j]$. Altogether we have

$$\delta\mathcal{L} = -\frac{1}{2}\langle d_A^\star F,\delta A\rangle - \frac{1}{2}\langle[\psi,\psi]\,,\delta A\rangle - \left(\not{D}_A\psi,\delta\psi\right) + divergence. \tag{6.2.6}$$

Proposition 6.2.2. *The equations of motion of $\mathcal{L}$ are*

$$d_A^\star F = -\left[\psi,\psi\right] \tag{6.2.7}$$

$$\not{D}_A\psi = 0. \tag{6.2.8}$$

Now let's investigate the algebra generated by (6.2.2) and (6.2.3). One needs to check

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}] = \delta_{[\epsilon_1,\epsilon_2]^\vee},$$

where $[\epsilon_1,\epsilon_2]^\vee(x)$ is the dual of $[\epsilon_1,\epsilon_2](x)$, $x\in M$.

Lemma 6.2.3. *$[\epsilon_1,\epsilon_2]^\vee$ is a constant vector field if ϵ_1 and ϵ_2 are parallel spinor fields.*

Proof. Let $v,w\in\Gamma(TM)$, we have

$$\begin{aligned} w\langle v,[\epsilon_1,\epsilon_2]^\vee\rangle &= w(v\epsilon_1,\epsilon_2)\\ &= (\nabla_w(v\epsilon_1),\epsilon_2) + (v\epsilon_1,\nabla_w\epsilon_2)\\ &= ((\nabla_w v)\epsilon_1,\epsilon_2) + (v\nabla_w\epsilon_1,\epsilon_2) + \langle v,[\epsilon_1,\nabla_w\epsilon_2]^\vee\rangle\\ &= \langle\nabla_w v,[\epsilon_1,\epsilon_2]^\vee\rangle + \langle v,[\nabla_w\epsilon_1,\epsilon_2]^\vee\rangle + \langle v,[\epsilon_1,\nabla_w\epsilon_2]^\vee\rangle. \end{aligned}$$

On the other hand,

$$w\langle v,[\epsilon_1,\epsilon_2]^\vee\rangle = \langle\nabla_w v,[\epsilon_1,\epsilon_2]^\vee\rangle + \langle v,\nabla_w[\epsilon_1,\epsilon_2]^\vee\rangle.$$

Thus,

$$\nabla_w[\epsilon_1,\epsilon_2]^\vee = [\nabla_w\epsilon_1,\epsilon_2]^\vee + [\epsilon_1,\nabla_w\epsilon_2]^\vee,$$

for all $w\in\Gamma(TM)$. □

Proposition 6.2.3. *$[\delta_{\epsilon_1},\delta_{\epsilon_2}]A = \delta_{[\epsilon_1,\epsilon_2]^\vee}A$ up to a gauge transformation generated by $-\iota_{[\epsilon_1,\epsilon_2]^\vee}A$.*

Proof. For a vector field η on M, we have

$$\delta_\eta A = \mathrm{Lie}_\eta A = \iota_\eta dA + d\iota_\eta A = \iota_\eta F - \iota_\eta(A \wedge A) + d\iota_\eta A = \iota_\eta F + d_A(\iota_\eta A).$$

On the other hand,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, A = \delta_{\epsilon_1}([\epsilon_2, \psi]) - \delta_{\epsilon_2}([\epsilon_1, \psi]) = \frac{1}{2}\left([\epsilon_2, F\epsilon_1] - [\epsilon_1, F\epsilon_2]\right).$$

Now take an arbitrary 1-form μ on M, we have

$$\begin{aligned}
\langle \mu, [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, A\rangle &= \frac{1}{2}((\mu\epsilon_2, F\epsilon_1) - (\mu\epsilon_1, F\epsilon_2)) \\
&= \frac{1}{2}((\epsilon_2, \mu F\epsilon_1) - (\beta(F)\mu\epsilon_1, \epsilon_2)) \\
&= \left(\frac{1}{2}(F\mu - \mu F)\epsilon_1, \epsilon_2\right) \\
&= -\left((\iota_{\mu^\vee} F)\epsilon_1, \epsilon_2\right) \\
&= -\iota_{[\epsilon_1,\epsilon_2]^\vee}\iota_{\mu^\vee} F \\
&= \langle \mu, \iota_{[\epsilon_1,\epsilon_2]^\vee} F\rangle.
\end{aligned}$$

In the fourth line we used Lemma 6.1.1. It follows that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, A = \delta_\eta A - d_A(\iota_\eta A).$$

□

To verify the commutation relation for the spinor field ψ, we have to impose the equation of motion $\not{D}_A\psi = 0$. Physically, this means that the supersymmetry only holds for an on-shell ψ.

Proposition 6.2.4. *For an on-shell ψ, $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, \psi = \delta_{[\epsilon_1,\epsilon_2]^\vee}\psi$ up to a gauge transformation generated by $-\iota_{[\epsilon_1,\epsilon_2]^\vee} A$ if and only if* $\mathrm{tri}\ \psi = 0$.

Proof. Let v denote the vector field $[\epsilon_1, \epsilon_2]^\vee$ on M. The Lie derivative of a spinor field ψ in direction v is given by

$$\mathrm{Lie}_v\psi = \iota_v \nabla\psi - \frac{1}{4}(\nabla v^\vee)\psi$$

In our case, the second term above vanishes, we have

$$\delta_v\psi = \mathrm{Lie}_v\psi = \iota_v\nabla\psi.$$

On the other hand,

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\psi &= \delta_{\epsilon_1}(\frac{1}{2}F\epsilon_2) - \delta_{\epsilon_2}(\frac{1}{2}F\epsilon_1) \\ &= \frac{1}{2}\left((d_A\,[\epsilon_1,\psi])\epsilon_2 - (d_A\,[\epsilon_2,\psi])\epsilon_1\right) \\ &= \frac{1}{2}\not{D}_A\left([\epsilon_1,\psi]\,\epsilon_2 - [\epsilon_2,\psi]\,\epsilon_1\right) \\ &= \frac{1}{2}\not{D}_A([\epsilon_1,\epsilon_2]\,\psi).\end{aligned}$$

To pass to the last line, we use

$$[\epsilon_2,\psi]\,\epsilon_1 + [\psi,\epsilon_1]\,\epsilon_2 + [\epsilon_1,\epsilon_2]\,\psi = 0,$$

which is equivalent to the condition tri $\psi = 0$. Note that

$$\begin{aligned}\not{D}_A(v\psi) &= \sum_\alpha e^\alpha \nabla_{A,e_\alpha} v\psi \\ &= \sum_\alpha e^\alpha(\nabla_{e_\alpha} v)\psi + \sum_\alpha e^\alpha v \nabla_{e_\alpha}\psi + A(v\psi) \\ &= 0 + \sum_\alpha (2\langle e^\alpha, v\rangle - ve^\alpha)\nabla_{e_\alpha}\psi + 2[\iota_{v^\vee}A,\psi] - v(A\psi) \\ &= 2\sum_\alpha \langle v, e^\alpha\rangle \nabla_{e_\alpha}\psi + 2[\iota_{v^\vee}A,\psi] - v\not{D}_A\psi \\ &= 2\iota_{v^\vee}\nabla\psi + 2[\iota_{v^\vee}A,\psi],\end{aligned}$$

where $v = [\epsilon_1,\epsilon_2]$. We have $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\psi = \iota_{[\epsilon_1,\epsilon_2]^\vee}\nabla\psi + [\iota_{[\epsilon_1,\epsilon_2]^\vee}A,\psi]$. □

6.2.2 Dimensional reduction and N=2 super Yang-Mills theory

In this subsection, we give a derivation of 4-dimensional $N = 2$ Euclidean super Yang-Mills theory from applying a spacetime reduction to the 6-dimensional $N = 1$ super Yang-Mills theory.

Let A' be a connection 1-form on the $(n+2)$-dimensional Minkowski spacetime M' that is invariant under the translations generated by $\frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^{n+1}}$, where $\{x^0, \ldots, x^{n+1}\}$ are the standard coordinates on M'. We can write

$$A' = A + \phi_1 dx^0 + \phi_2 dx^{n+1},$$

where A only involves dx^i, $1 \le i \le n$. Let M be the quotient of M' by the above translations. A can be viewed as a connection 1-form on M and ϕ_i, $i = 1, 2$ can be viewed as scalar fields with values in the adjoint bundle ad$P$ on M. The curvature form of A' takes the form

$$F' = F + d_A\phi_1 \wedge dx^0 + d_A\phi_2 \wedge dx^{n+1} + [\phi_1, \phi_2]dx^0 \wedge dx^{n+1},$$

which is a sum of orthogonal terms. It follows that

$$\begin{aligned}\langle F', F'\rangle_{M'} &= \langle F, F\rangle_{M'} + \langle d_A\phi_1, d_A\phi_1\rangle_{M'} - \langle d_A\phi_2, d_A\phi_2\rangle_{M'} - |[\phi_1, \phi_2]|^2 \\ &= \langle F, F\rangle_M - \langle d_A\phi_1, d_A\phi_1\rangle_M + \langle d_A\phi_2, d_A\phi_2\rangle_M - |[\phi_1, \phi_2]|^2.\end{aligned}$$

Remark 6.2.1 (Caveat)**.** By our sign convention[4], the dimensional reduction of the Minkowski metric $\langle\cdot,\cdot\rangle_{M'}$ to its Euclidean part produces a negative Euclidean metric $-\langle\cdot,\cdot\rangle_M$. Thus, after the reduction, any inner product of two differential forms of odd degrees will gain an extra minus sign in front of it. With this in mind, we will omit the lower-script of our metric.

The behaviours of the terms involving fermionic fields under the dimensional reduction are more complicated than the pure bosonic ones. To proceed, we need the following lemmas.

Lemma 6.2.4. $\mathrm{Cl}(r, s) \otimes \mathrm{Cl}(1, 1) \cong \mathrm{Cl}(r + 1, s + 1)$ *for all* $r, s \geq 0$.

Proof. Let $e_1, \dots, e_{r+1}, \epsilon_1, \dots, \epsilon_{s+1}$ be a Q-orthonormal basis for $\mathbb{R}^{r+1,s+1}$. Let $e'_1, \dots, e'_r$, $\epsilon'_1, \dots, \epsilon'_s$ and e''_1, ϵ''_1 be the bases for $\mathbb{R}^{r,s}$ and $\mathbb{R}^{1,1}$, respectively. Consider the map $f : \mathbb{R}^{r+1,s+1} \to \mathrm{Cl}(r, s) \otimes \mathrm{Cl}(1, 1)$ by setting

$$f(e_i) = \begin{cases} e'_i \otimes e''_1\epsilon''_1 & 1 \leq i \leq r \\ 1 \otimes e''_1 & i = r + 1 \end{cases}$$

and

$$f(\epsilon_j) = \begin{cases} \epsilon'_j \otimes e''_1\epsilon''_1 & 1 \leq j \leq s \\ 1 \otimes \epsilon''_1 & i = s + 1 \end{cases}$$

and then extending linearly. □

Corollary 6.2.1. *Let* $S_{1,n+1}$ *be a real irreducible spinor representation of* $\mathrm{Spin}(1, n + 1)$, $n = 0 \pmod 4$*. Let* $S_{0,n}$ *and* $S_{1,1}$ *be real irreducible spinor representations of* $\mathrm{Spin}^0(0, n)$ *and* $\mathrm{Spin}(1, 1)$*, respectively. Then*

$$S_{1,n+1} \cong (S_{0,n} \otimes S_{1,1}) \oplus (S^\vee_{0,n} \otimes S^\vee_{1,1}) \text{ or } (S_{0,n} \otimes S^\vee_{1,1}) \oplus (S^\vee_{0,n} \otimes S_{1,1})$$

as spinor representations of $\mathrm{Spin}(1, n + 1)$.

[4] $\langle dx^0, dx^0\rangle_{M'} = -\langle dx^j, dx^j\rangle_{M'} = 1$, $1 \leq j \leq n + 1$.

Proof. Recall that $S_{1,n+1} \oplus S_{1,n+1}^\vee$ forms a Clifford module such that the Clifford multiplication of $v \in \mathbb{R}^{1,n+1}$ maps $S_{1,n+1}$ to $S_{1,n+1}^\vee$ and vice versa. and that $S_{1,n+1} \oplus S_{1,n+1}^\vee$ and $S_{0,n} \oplus S_{0,n}'$ are irreducible Clifford modules when $n = 0,1,2,4 \pmod 8$. By Lemma 6.2.4, we have $S_{1,n+1} \oplus S_{1,n+1}^\vee \cong (S_{1,n} \oplus S_{1,n+1}^\vee) \otimes (S_{1,1} \oplus S_{1,1}^\vee)$, and that $\mathfrak{spin}(1,n+1) \cong (\mathfrak{spin}(0,n) \otimes 1) \oplus (1 \otimes \mathfrak{spin}(1,1)) \oplus \mathrm{span}\{e_i' \otimes e_1'', e_i' \otimes \epsilon_1'', \epsilon_j' \otimes e_1'', \epsilon_j' \otimes \epsilon_1''\}_{1 \le i \le r,\ 1 \le j \le s}$ as vector sub-spaces of $\mathrm{Cl}(1,n+1)$, where $\mathfrak{spin}(r,s)$ is the Lie algebra of the spin group $\mathrm{Spin}(r,s)$. It is not hard to check that both $(S_{0,n} \otimes S_{1,1}) \oplus (S_{0,n}^\vee \otimes S_{1,1}^\vee)$ and $(S_{0,n} \otimes S_{1,1}^\vee) \oplus (S_{0,n}^\vee \otimes S_{1,1})$ are invariant under the action of $\mathfrak{spin}(1,n+1)$. To finish the proof, note that the dimensions of $(S_{0,n} \otimes S_{1,1}) \oplus (S_{0,n}^\vee \otimes S_{1,1}^\vee)$, $(S_{0,n} \otimes S_{1,1}^\vee) \oplus (S_{0,n}^\vee \otimes S_{1,1})$ and $S_{1,n+1}$ are the same. □

Remark 6.2.2. By Remark 6.1.4, we can choose $S_{1,n+1}$, $S_{0,n}$, $S_{1,1}$ to be the representations with positive chirality and $S_{1,n+1}^\vee$, $S_{0,n}^\vee$, $S_{1,1}^\vee$ to be the representations with negative chirality. We then have $S_{1,n+1}^+ \cong (S_{0,n}^+ \otimes S_{1,1}^+) \oplus (S_{0,n}^- \otimes S_{1,1}^-)$ and $S_{1,n+1}^- \cong (S_{0,n}^+ \otimes S_{1,1}^-) \oplus (S_{0,n}^- \otimes S_{1,1}^+)$. It follows that a Weyl (Weyl-Majorana) spinor in dimension $(1,n+1)$ reduces to a Dirac (Majorana) spinor in dimension $(0,n)$.

Lemma 6.2.5. *Let S be the real irreducible representation of* $\mathrm{Cl}(1,1)$*. Then there exists a symmetric invariant form $(\cdot,\cdot)$ on S such that*

$$(vs,t) = (vt,s)$$

for all $s,t \in S$ and for all $v \in \mathbb{R}^{1,1}$. Moreover, one can show that $(\cdot,\cdot)_\omega$ is anti-symmetric and $(vs,t)_\omega = (vt,s)_\omega$.

Proof. Without loss of generality, we work with the Weyl-Majonara representation and set $e_1'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\epsilon_1'' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows that $\omega_{1,1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We then define the invariant forms $(\cdot,\cdot)$ and $(\cdot,\cdot)_\omega$ by setting

$$(s,t) \equiv s^T e_1'' t, \quad (s,t)_\omega \equiv -s^T \epsilon_1'' t,$$

for all $s,t \in S \cong \mathbb{R}^2$. □

We are now ready to prove the following proposition, which plays an important role in the dimensional reduction of the fermionic part.

Proposition 6.2.5. *Let $(\cdot,\cdot)_{0,n}$ and $(\cdot,\cdot)_{1,1}$ be the symmetric invariant forms as defined in Proposition 6.1.4 and Lemma 6.2.5, respectively. Then the invariant form $(\cdot,\cdot)_{1,n+1} \equiv (\cdot,\cdot)_{0,n} \otimes (\cdot,\cdot)_{1,1}$ and its associated equivariant pairing $[\cdot,\cdot]_{1,n+1}$ are symmetric.*

Proof. By Lemmas 6.2.4 and 6.2.5, we have

$$\begin{aligned}
(f(\epsilon_i)(\cdot),\cdot)_{1,n+1} &= (\epsilon_i'(\cdot),\cdot)_{0,n} \otimes (\omega_{1,1}(\cdot),\cdot)_{1,1} \\
&= (-1)^2(\cdot,\epsilon_i'(\cdot))_{0,n} \otimes (\cdot,\omega_{1,1}(\cdot))_{1,1} \\
&= (\cdot, f(\epsilon_i)(\cdot))_{1,n+1},
\end{aligned}$$

for $1 \leq i \leq n$, and

$$\begin{aligned}(f(e_1)(\cdot), \cdot)_{1,n+1} &= (\cdot, \cdot)_{0,n} \otimes (e_1''(\cdot), \cdot)_{1,1} \\ &= (\cdot, \cdot)_{0,n} \otimes (\cdot, e_1''(\cdot))_{1,1} \\ &= (\cdot, f(e_1)(\cdot))_{1,n+1},\end{aligned}$$

and

$$\begin{aligned}(f(\epsilon_{n+1})(\cdot), \cdot)_{1,n+1} &= (\cdot, \cdot)_{0,n} \otimes (\epsilon_1''(\cdot), \cdot)_{1,1} \\ &= (\cdot, \cdot)_{0,n} \otimes (\cdot, \epsilon_1''(\cdot))_{1,1} \\ &= (\cdot, f(\epsilon_{n+1})(\cdot))_{1,n+1}.\end{aligned}$$

□

For brevity, we will omit the lower-scripts of $(\cdot, \cdot)_{r,s}$ and $[\cdot, \cdot]_{r,s}$ from now on.

Let $s^{\pm}_{1,n+1}, t^{\pm}_{1,n+1} \in S^{\pm}_{1,n+1}$ for $n = 0, 4 \pmod 8$, we can write $s^{+}_{1,n+1} = s^{+}_{0,n} \otimes s^{+}_{1,1} + s^{-}_{0,n} \otimes s^{-}_{1,1}$ and $s^{-}_{1,n+1} = s^{+}_{0,n} \otimes s^{-}_{1,1} + s^{-}_{0,n} \otimes s^{+}_{1,1}$ and similarly for $t^{\pm}_{1,n+1}$. We choose the normalization conditions for $s^{\pm}_{1,1}$ and $t^{\pm}_{1,1}$ such that

$$s^{+}_{1,1} = t^{+}_{1,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad s^{-}_{1,1} = t^{-}_{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It follows that

$$(s^{\pm}_{1,n+1}, t^{\mp}_{1,n+1}) = (s^{+}_{0,n}, t^{+}_{0,n}) + (s^{-}_{0,n}, t^{-}_{0,n}) = (s_{0,n}, t_{0,n}).$$

It is not hard to show that

$$\langle f(v), [s^{+}_{1,n+1}, t^{+}_{1,n+1}] \rangle = \langle v, -[s_{0,n}, t_{0,n}]_{\omega} \rangle,$$

for all $v \in \mathbb{R}^{0,n} \subset \mathbb{R}^{1,n+1}$, and that

$$\langle f(e_1), [s^{+}_{1,n+1}, t^{+}_{1,n+1}] \rangle = (s_{0,n}, t_{0,n}), \quad \langle f(\epsilon_{n+1}), [s^{+}_{1,n+1}, t^{+}_{1,n+1}] \rangle = (s_{0,n}, t_{0,n})_{\omega}.$$

By Remark 6.2.1, we have

$$[\epsilon^{+}_{1,n+1}, \psi^{+}_{1,n+1}] = [\epsilon_{0,n}, \psi_{0,n}]_{\omega} + (\epsilon_{0,n}, \psi_{0,n}) dx^0 - (\epsilon_{0,n}, \psi_{0,n})_{\omega} dx^{n+1}. \tag{6.2.9}$$

It is also not hard to show that

$$F\epsilon^{+}_{1,n+1} = F\epsilon_{0,n} - (d_A\phi_1)\omega\epsilon_{0,n} - (d_A\phi_2)\epsilon_{0,n} + [\phi_1, \phi_2]\omega\epsilon_{0,n}. \tag{6.2.10}$$

(6.2.9) together with (6.2.10) indicate the supersymmetry transformation behaviours of the field contents of the $N = 2$ theory. For the dimensional reduction of the $N = 1$ Dirac term, we have

$$\not{D}_A \psi^+_{1,n+1} = -\omega \not{D}_A \psi_{0,n} + [\phi_1, \psi_{0,n}] + \omega[\phi_2, \psi_{0,n}].$$

It follows that

$$(\psi^+_{1,n+1}, \not{D}_A \psi^+_{1,n+1}) = -(\psi_{0,n}, \not{D}_A \psi_{0,n})_\omega + (\psi_{0,n}, [\phi_1, \psi_{0,n}]) + (\psi_{0,n}, [\phi_2, \psi_{0,n}])_\omega.$$

To conclude, we obtain the following $N = (1, 1)$ Euclidean super Yang-Mills Lagrangian in dimension $n = 0 \pmod 4$ if there exists a $N = 1$ theory on $(n+2)$-dimensional Minkowski spacetime.

$$\begin{aligned}\mathcal{L} = \frac{1}{4}\langle F, F\rangle + \frac{1}{2}(\psi, \not{D}_A \psi)_\omega - \frac{1}{4}\langle d_A\phi_1, d_A\phi_1\rangle + \frac{1}{4}\langle d_A\phi_2, d_A\phi_2\rangle \\ - \frac{1}{2}(\psi, [\phi_1, \psi]) - \frac{1}{2}(\psi, [\phi_2, \psi])_\omega - \frac{1}{4}|[\phi_1, \phi_2]|^2,\end{aligned} \tag{6.2.11}$$

where ψ is a map $\psi : M \to \Pi S_0 \otimes \mathfrak{g}$, $S_0 = S^+ \oplus S^-$ is the irreducible real representation of the Clifford algebra. The corresponding supersymmetry transformations are

$$\delta_\epsilon A = [\epsilon, \psi]_\omega, \tag{6.2.12}$$

$$\delta_\epsilon \phi_1 = (\epsilon, \psi), \tag{6.2.13}$$

$$\delta_\epsilon \phi_2 = -(\epsilon, \psi)_\omega, \tag{6.2.14}$$

$$\delta_\epsilon \psi = \frac{1}{2}\left(F\epsilon - (d_A\phi_1)\omega\epsilon - (d_A\phi_2)\epsilon + [\phi_1, \phi_2]\omega\epsilon\right), \tag{6.2.15}$$

where $\epsilon : M \to \Pi S_0$ is a parallel spinor field.

Remark 6.2.3. The Lagrangian and supersymmetry transformations are exactly those in [Zum77] (up to some scalings of parameters). They can also be obtained by performing the so-called Wick rotation to the $N = 2$ super Yang-Mills theory on $\mathbb{R}^{1,3}$ [BT97; VW96].

Let $\phi = \frac{1}{2}(\phi_2 + \phi_1)$ and $\phi^* = \frac{1}{2}(\phi_2 - \phi_1)$. For later use, we want to express the Lagrangian and the supersymmetry transformations in terms of $\psi^\pm$. We have

$$\mathcal{L} = \frac{1}{4}\langle F, F\rangle + (\psi^+, \not{D}_A \psi^-) + \langle d_A\phi, d_A\phi^*\rangle - (\psi^+, [\phi, \psi^+]) + (\psi^-, [\phi^*, \psi^-]) - |[\phi, \phi^*]|^2, \tag{6.2.16}$$

and

$$\delta_{\epsilon^+} A = -[\epsilon^+, \psi^-], \quad \delta_{\epsilon^+}\phi = 0, \quad \delta_{\epsilon^+}\phi^* = -(\epsilon^+, \psi^+), \tag{6.2.17}$$

$$\delta_{\epsilon^+}\psi^+ = \frac{1}{2}F_-\epsilon^+ + [\phi, \phi^*]\epsilon^+, \quad \delta_{\epsilon^+}\psi^- = -(d_A\phi)\epsilon^+, \tag{6.2.18}$$

$$\delta_{\epsilon^-} A = [\epsilon^-, \psi^+], \quad \delta_{\epsilon^-}\phi = (\epsilon^-, \psi^-), \quad \delta_{\epsilon^-}\phi^* = 0, \tag{6.2.19}$$

$$\delta_{\epsilon^-}\psi_+ = -(d_A\phi^*)\epsilon^-, \quad \delta_{\epsilon^-}\psi_- = \frac{1}{2}F_+\epsilon^- - [\phi, \phi^*]\epsilon^-. \tag{6.2.20}$$

Remark 6.2.4. Using the isomorphism between $\mathrm{Cl}(0,n)\otimes\mathrm{Cl}(0,4)\cong\mathrm{Cl}(0,n+4)$, one can obtain supersymmetric theories on $\mathbb{R}^n$ from those on $\mathbb{R}^{n+4}$ in a similar fashion. Combining this observation with what we have shown above gives the $N=4$ super Yang-Mills theory on $\mathbb{R}^4$.

6.2.3 Twisted N=2 super Yang Mills theory

The largest sub-group of the automorphism group of a super Poincaré algebra which fixes its underlying Poincaré algebra is called its R-symmetry group. Let S be a spinor representation of $\mathrm{Spin}^0(1,n-1)$. By Remark 6.1.4, we know that S can be assumed as a direct sum of N irreducible spinor representations for $n-2\neq 0 \pmod 4$, and as a direct sum of N^+ and N^- copies of two inequivalent irreducible spinor representations for $n-2=0 \pmod 4$. Using Proposition 6.1.1, one can obtain a nice classification result for R-symmetry groups [Var04]. See Table 6.2.1.

$n-2 \pmod 8$	1, 7	0	3, 5	4	2, 6
$\mathcal{I}_R$	$\mathrm{SO}(N)$	$\mathrm{SO}(N^+)\times\mathrm{SO}(N^-)$	$\mathrm{Sp}(N)$	$\mathrm{Sp}(N^+)\times\mathrm{Sp}(N^-)$	$\mathrm{U}(N)$

Table 6.2.1: R-symmetry groups $\mathcal{I}_R$ for n-dimensional Minkowski spaces.

Note that the dimensional reduction procedure introduced in the previous subsection preserves the corresponding R-symmetry groups. Therefore, one can easily find R-symmetry groups (or at least their subgroups) for Euclidean spaces. Let's consider an n-dimensional Euclidean supersymmetric theory. Suppose that the R-symmetry group $\mathcal{I}_R$ is also a symmetry of our theory. That is to say, the configuration space should carry an action of the semi-direct product of the R-symmetry group and the super Poincaré group; the action functional should be invariant under this new action. Since R-symmetry groups fix the underlying Poincaré algebras, we have a sub- symmetry group of the form

$$\mathrm{Spin}(0,n)\times\mathcal{I}_R,$$

which is of course also a symmetry of our theory. We also assume there exists a non-trivial group homomorphism h_R from $\mathrm{Spin}(0,n)$ to $\mathcal{I}_R$. The term topological twisting refers to the change of the ways of embedding $\mathrm{Spin}(0,n)$ into $\mathrm{Spin}(0,n)\times\mathcal{I}_R$. More precisely, we change the canonical embedding $(\mathrm{id},0)$ to the "twisted" embedding (id,h_R).

Let Φ denote the field contents of our original theory, which transform under some representation ρ of $\mathrm{Spin}(0,n)\times\mathcal{I}_R$. Note that Φ transforms under the new representation $\rho\circ(\mathrm{id},h_R)|_{\mathrm{Spin}(0,n)}$ of $\mathrm{Spin}(0,n)$ after the topological twisting.

Example 6.2.1 (Twisting of the $N=(1,1)$ super Poincaré algebra of $\mathbb{R}^4$.)**.** In this case, we have $\mathrm{Spin}(0,4)\cong\mathrm{Sp}_+(1)\times\mathrm{Sp}_-(1)$, and a R symmetry group $\mathcal{I}_R$ inherited from the the R symmetry group of $N=1$ super Poincaré algebra of $\mathbb{R}^{1,5}$. By checking Table 6.2.1, we find that $\mathcal{I}_R \cong \mathrm{Sp}(1)$. Thus, we can define the twisting homomorphism h_R by setting

$$\begin{aligned} h_R : \mathrm{Sp}_+(1) \times \mathrm{Sp}_-(1) &\to \mathcal{I}_R \\ g = (g_+, g_-) &\mapsto g_+ \end{aligned}$$

The irreducible spinor representation on $\mathbb{R}^{1,5}$ gives us two inequivalent irreducible spinor representations S^+ and S^- after applying the dimensional reduction. We have $S^+ \cong S^- \cong \mathbb{H}$ as vector spaces. The spin group acts on $S^\pm$ via

$$\begin{aligned} \mathrm{Spin}(0,4) \times S^\pm &\to S^\pm \\ g = (g_+, g_-) \times s^\pm &\mapsto g_\pm s^\pm \end{aligned}$$

and the R symmetry group acts on $S^\pm$ via

$$\begin{aligned} \mathcal{I}_R \times S^\pm &\to S^\pm \\ g \times s^\pm &\mapsto s^\pm g^* \end{aligned}$$

where all the multiplications are given by quaternionic multiplications, g^* denotes the conjugate of $g \in \mathbb{H}$. The two actions commute because $\mathbb{H}$ is an associative algebra. The action of $\mathrm{Spin}(0,4) \times \mathcal{I}_R$ on $S^\pm$ is indeed well defined.

The new actions of $\mathrm{Spin}(0,4)$ on $S^\pm$ after twisting is given by

$$\begin{aligned} \mathrm{Spin}(0,4) \times S^+ &\to S^+ \\ g \times s^+ &\mapsto g_+ s^+ g_+^* \end{aligned}$$

and

$$\begin{aligned} \mathrm{Spin}(0,4) \times S^- &\to S^- \\ g \times s^- &\mapsto g_- s^- g_+^* \end{aligned}$$

One can show that after twisting, S^+ becomes $\mathbb{R} \oplus \Lambda^2_- \mathbb{R}^4$ and S^- becomes $\mathbb{R}^4$ as representations of $\mathrm{Spin}(0,4)$, where $\Lambda^2_- \mathbb{R}^4$ is the vector space of anti-self-dual 2-forms. In other words, since all bundles over $\mathbb{R}^4$ are trivial, the twisting procedure does nothing but reorganizes the field components of the original theory in a different way by turning spinor fields into differential forms. The twisted supersymmetric theories can be put on a more general spacetime, because the existence of covariant differential forms puts far fewer restrictions on the geometry of the spacetime manifold than the existence of covariant spinors.

Now, let's examine the twisted super Poincaré algebra closely. Before the twisting, the paring $[\cdot,\cdot]$ on $S^\pm$ is given by [BH09]

$$\begin{aligned} [\cdot,\cdot] : (S^+ \oplus S^-) \times (S^+ \oplus S^-) &\to \mathbb{R}^4 \\ (s^+, s^-) \times (t^+, t^-) &\mapsto t^-(s^+)^* + s^-(t^+)^* \end{aligned}$$

Identifying $\mathbb{R}$, $\Lambda^2_-\mathbb{R}^4$ with the real part and imaginary part of $\mathbb{H}$, respectively. We have

$$[\cdot,\cdot] : (\mathbb{R} \oplus \Lambda^2_-\mathbb{R}^4 \oplus \mathbb{R}^4) \times (\mathbb{R} \oplus \Lambda^2_-\mathbb{R}^4 \oplus \mathbb{R}^4) \to \mathbb{R}^4$$
$$(\eta_1, \chi_1, \upsilon_1) \times (\eta_2, \chi_2, \upsilon_2) \mapsto \upsilon_2(\eta_1 - \chi_1) + \upsilon_1(\eta_2 - \chi_2)$$

There is a sub-pairing

$$[\cdot,\cdot] : (\mathbb{R} \oplus \mathbb{R}^4) \times (\mathbb{R} \oplus \mathbb{R}^4) \to \mathbb{R}^4$$
$$(\eta_1, \upsilon_1) \times (\eta_2, \upsilon_2) \mapsto \upsilon_2\eta_1 + \upsilon_1\eta_2$$

which gives us a subalgebra $\mathfrak{l}_t = \mathbb{R}^4 \oplus (\mathbb{R} \oplus \mathbb{R}^4)$ of the twisted super Poincaré algebra. Let w_i, η, υ_i, $i = 1, \dots, 4$, be a basis of $\mathfrak{l}_t$. We have

$$[w_i, w_j] = 0, \quad [w_i, \eta] = 0, \quad [w_i, \upsilon_j] = 0, \tag{6.2.21}$$
$$[\eta, \eta] = 0, \quad [\eta, \upsilon_i] = w_i, \quad [\upsilon_i, \upsilon_j] = 0. \tag{6.2.22}$$

One immediately recognizes that (6.2.21) and (6.2.22) are just (4.5.3) and (4.5.4) in the disguise. In other words, we have reproduced the graded Lie algebra L associated to an abelian Lie algebra by twisting a super Poincaré algebra. The twisted supersymmetric theory can be then given naturally a QK-structure. Since the Lagrangian is invariant under the $\mathfrak{l}_t$-action, it is also Q-closed. We then obtain a 4-dimensional CohFT.

Remark 6.2.5. One can also work with the twisted superalgebra defined by (6.2.21) and (6.2.22) directly, and then use the standard superfield method in physics literature to construct action functionals on flat spacetimes. This idea was studied in [BBM08].

As another example, one can twist the $N = (2, 2)$ super Poincaré algebra of $\mathbb{R}^4$ obtained by applying dimensional reduction to the $N = (1, 1)$ super Poincaré algebra of $\mathbb{R}^{1,5}$. The R-symmetry group in this case is $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, which is isomorphic to the spin group. Therefore, there exist three different homomorphisms h_R (up to automorphisms). The twisting associated to the identity one is called the geometric Langlands twisting [KW07].

Example 6.2.2 (Geometric Langlands twisting of the $N = (2, 2)$ super Poincaré algebra of $\mathbb{R}^4$)**.** In this case, the odd part of the super Poincaré algebra is $S_l \oplus S_r$, where $S_l \cong S_r \cong S^+ \oplus S^-$, $S^\pm \cong \mathbb{H}$. The R-symmetry group acts on $S_l \oplus S_r$ via

$$(\mathcal{I}_R \cong \mathrm{Sp}_l(1) \times \mathrm{Sp}_r(1)) \times S_l \oplus S_r \to S_l \oplus S_r$$
$$(g_l, g_r) \times (s_l, s_r) \mapsto (s_l g_l^*, s_r g_r^*).$$

It is not hard to see that, after the twisting, S_l becomes $\mathbb{R} \oplus \mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$ and S_r becomes $\mathbb{R} \oplus \mathbb{R}^4 \oplus \Lambda^2_+\mathbb{R}^4$. Again, we are only interested in the $\mathbb{R} \oplus \mathbb{R}^4$ parts of S_l and S_r. The pairings $[\cdot,\cdot]$ on each of them are identical to the one defined in Example 6.2.1. We then obtain an abelian Lie superalgebra $\mathfrak{l}_t = \mathbb{R}^4 \oplus (\mathbb{R} \oplus \mathbb{R}^4)_l \oplus (\mathbb{R} \oplus \mathbb{R}^4)_r$. The Lie bigraded algebra $\mathcal{K}_{GL}$ associated to $\mathfrak{l}_t$ is spanned by Q_l of degree $(0,1)$, Q_r of degree $(0,1)$, K_l of degree $(1,-1)$, K_r of degree $(1,-1)$, and L of degree $(1,0)$. The only non-trivial brackets between these basis elements are

$$[Q_l, K_l] = [Q_r, K_r] = L.$$

The universal enveloping algebra $\mathcal{K}_{GL\infty}$ of $\mathcal{K}_{GL}$ is generated by Q_l, Q_r, K_l, K_r, and L, which are subject to the relations

$$\begin{aligned}
&Q_l^2 = 0, \quad Q_lK_l + K_lQ_l = L, \quad K_lL + LK_l = 0,\\
&Q_r^2 = 0, \quad Q_rK_r + K_rQ_r = L, \quad K_rL + LK_r = 0,\\
&Q_lQ_r + Q_rQ_l = 0, \quad K_lK_r - K_rK_l = 0, \quad Q_lK_r + K_rQ_l = 0, \quad Q_rK_l + K_lQ_r = 0.
\end{aligned}$$

There exists a $\overline{T}\mathbb{RP}^1$-family of QK-algebras as subalgebras of $\mathcal{K}_{GL\infty}$, where $\overline{T}\mathbb{RP}^1$ is an affine bundle modelled on the tangent bundle of the 1-dimensional projective space $\mathbb{RP}^1$. Let $(u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ be such that $u_1v_1 + u_2v_2 = 1$. We define

$$Q_{\vec{u}} = u_1Q_l + u_2Q_r, \quad K_{\vec{v}} = v_1K_l + v_2K_r.$$

It is straightforward to verify that

$$Q_{\vec{u}}^2 = 0, \quad Q_{\vec{u}}K_{\vec{v}} + K_{\vec{v}}Q_{\vec{u}} = L, \quad K_{\vec{v}}L + LK_{\vec{v}} = 0,$$

where $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Let (u_1', u_2', v_1', v_2') be another element in $\mathbb{R}^4$ such that $u_1'v_1' + u_2'v_2' = 1$. Obviously, (u_1, u_2, v_1, v_2) and (u_1', u_2', v_1', v_2') determine the same QK-algebra (up to a scaling factor) if there exists an $a \in \mathbb{R}/\{0\}$ such that $u_1' = au_1$, $u_2' = au_2$, $v_1' = v_1/a$, $v_2' = v_2/a$. On the other hand, if we fix (u_1, u_2, v_1, v_2) and let $\Delta K_{\vec{u}} = -u_2K_l + u_1K_r$, then $Q_{\vec{u}}$, $K_{\vec{v}} + s\Delta K_{\vec{u}}$, and L form a QK-algebra for any $s \in \mathbb{R}$.

Note that there exists a natural $\mathrm{SL}(2,\mathbb{R})$-action on theses QK-algebras. More precisely, for $g \in \mathrm{SL}(2,\mathbb{R})$, we set

$$gQ_{\vec{u}} = Q_{g\vec{u}}, \quad gK_{\vec{v}} = K_{(g^{-1})^t\vec{v}}, \quad gL = L,$$

In other words, the Geometric Langlands twisting of the $N = (2,2)$ supersymmetric theory gives us a $\mathbb{RP}^1$-family of CohFTs which can be related to each other via a natural $\mathrm{SL}(2,\mathbb{R})$-action.

Remark 6.2.6. One can generalize the above examples by considering bigraded algebra generated by $2k+1$ generators: Q_i of degree $(1,0)$, K_j of degree $(1,-1)$, L of degree $(1,0)$, $i,j = 1,\cdots,k$, with non-trivial brackets being

$$[Q_i, K_i] = L, \quad i = 1, \cdots, k.$$

Such a bigraded algebra should be obtained from the $N = (k, k)$ super Poincaré algebra of $\mathbb{R}^4$. It follows that there exists a $\mathbb{RP}^{k-1}$-family of CohFTs obtained by twisting $N = (k, k)$ supersymmetric theories, which can be related to each other via a natural $\mathrm{SL}(k, \mathbb{R})$-action. For $k = 1, 2$, we recover Examples 6.2.1 and 6.2.2. However, I do not know if it makes sense to talk about $N = (k, k)$ supersymmetric theories in physics for $k \geq 3$.

Let's work out the details of Example 6.2.1 for the supersymmetric Yang-Mills theory. We can reorganize components of the right-handed spinor field ψ^+ as (η, χ) and reinterpret the twisted left-handed spinor ψ^- as υ, where (η, υ, χ) is a section of the parity reversed vector bundle $\Pi\left(\Lambda^0 T^*M \oplus \Lambda^1 T^*M \oplus \Lambda^2_- T^*M\right)$. After twisting, the Lagrangian (6.2.16) takes the form

$$\begin{aligned}\mathcal{L} = \frac{1}{4}\langle F, F\rangle - \langle \upsilon, d_A\eta\rangle + 2\langle \chi, d_A\upsilon\rangle + \langle d_A\phi, d_A\phi^*\rangle \\ - \langle \eta, [\phi, \eta]\rangle - 2\langle \chi, [\phi, \chi]\rangle + \langle \upsilon, [\phi^*, \upsilon]\rangle - |[\phi, \phi^*]|^2. \qquad (6.2.23)\end{aligned}$$

The factor 2 in front of the χ-relevant terms appears because of our normalization conventions for $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$.

It remains to determine the twisted supersymmetry transformations. The scalar supersymmetry transformation Q can be easily read off from (6.2.17) and (6.2.18), we have

$$\begin{aligned}&QA = -\upsilon, \quad Q\phi = 0, \quad Q\phi^* = -\eta, \\ &Q\eta = [\phi, \phi^*], \quad Q\chi = \frac{1}{2}F_-, \quad Q\upsilon = -d_A\phi.\end{aligned} \qquad (6.2.24)$$

Proposition 6.2.6. *For on-shell χ, $Q^2 = 0$ up to a gauge transformation generated by $-\phi$.*

Proof. This is a result of direct computations.

$$\begin{aligned}&Q^2(A) = -Q(\upsilon) = -d_A(-\phi), \\ &Q^2(\phi) = 0 = -[\phi, \phi], \\ &Q^2(\phi^*) = -Q(\eta) = -[\phi, \phi^*], \\ &Q^2(\eta) = Q([\phi, \phi^*]) = -[\phi, \eta], \\ &Q^2(\chi) = \frac{1}{2}Q(F_-) = -\frac{1}{2}(d_A\upsilon)_- = -[\phi, \chi], \\ &Q^2(\upsilon) = -Q(d_A\phi) = -[\phi, \upsilon].\end{aligned}$$

In the fourth line we use the equation of motion of χ. □

Proposition 6.2.7. *For on-shell χ, the Lagrangian (6.2.23) is Q-exact up to a topological term. More precisely, it can be rewritten as*

$$\mathcal{L} = Q(\mathcal{V}) + \frac{1}{4}\mathrm{tr}\left(F \wedge F\right), \qquad (6.2.25)$$

where $\mathcal{V} = \langle \chi, F_-\rangle - \langle \upsilon, d_A\phi^*\rangle - \langle \eta, [\phi, \phi^*]\rangle$.

Proof. We have

$$
\begin{aligned}
Q(\mathcal{V}) &= Q(\langle \chi, F_- \rangle - \langle \upsilon, d_A\phi^* \rangle - \langle \eta, [\phi, \phi^*] \rangle) \\
&= \frac{1}{2}\langle F_-, F_- \rangle - \langle \chi, d_A(-\upsilon) \rangle - \langle -d_A\phi, d_A\phi^* \rangle + \langle \upsilon, [-\upsilon, \phi^*] \rangle + \langle \upsilon, d_A(-\eta) \rangle \\
&\quad - |[\phi, \phi^*]|^2 + \langle \eta, [\phi, -\eta] \rangle \\
&= \frac{1}{2}\langle F^-, F^- \rangle + \langle \chi, d_A\upsilon \rangle + \langle \chi, d_A\upsilon - 2[\phi, \chi] \rangle + \langle d_A\phi, d_A\phi^* \rangle + \langle \upsilon, [\phi^*, \upsilon] \rangle - \langle \upsilon, d_A\eta \rangle \\
&\quad - |[\phi, \phi^*]|^2 - \langle \eta, [\phi, \eta] \rangle \\
&= \mathcal{L} - \frac{1}{4}(\langle F^+, F^+ \rangle - \langle F^-, F^- \rangle) \\
&= \mathcal{L} - \frac{1}{4}\mathrm{tr}\,(F \wedge F)\,,
\end{aligned}
$$

In the third line we use the equation of motion of χ. □

It is a lot harder to write down the explicit expressions for the vector supersymmetry transformation K due to the Clifford multiplications involved in the expression of δ_{ϵ^-}. However, it is easy to guess from (6.2.19) and (6.2.20) that

$$
\begin{aligned}
&KA \propto \chi, \quad K\phi \propto \upsilon, \quad K\phi^* = 0, \\
&K\eta \propto d_A\phi^*, \quad K\chi \propto \star d_A\phi^*, \quad K\upsilon \propto F_+.
\end{aligned}
\tag{6.2.26}
$$

We will not bother ourselves to determine the coefficients of K. (We also do not encourage the reader to do so.) Instead, we remark that with the right coefficients, one should have $QK + KQ = d$ up to a term of the form $dx^\mu \delta_{A_\mu}$ for on-shell η, υ, and χ, where δ_{A_μ} is the gauge transformation generated by A_μ.

Chapter 7

Mathai-Quillen formalism revisited: a generalization

7.1 Mathai-Quillen formalism with gauge symmetries

Let P be a principal G-bundle over an n-dimensional manifold M. Let $\mathrm{ad}P$ denote the adjoint bundle of P. Let $\mathcal{A}$ denote the affine space of connection 1-forms on P. Recall that $\mathcal{A}$ can be identified with $\Gamma(C)$ where C is an affine bundle over M. Let V and W be two associated vector bundles to P. We consider the variational bigraded manifold $\mathcal{M}_Y$ associated to the graded fiber bundle

$$Y = \mathrm{ad}P \times_M C \times_M V \times_M W,$$

where the grading is defined by assigning elements of the fibers of $\mathrm{ad}P$, C, V and W degrees 1, 0, -2 and -1, respectively.

A bundle chart of P induces a local coordinate system

$$(x^\mu,\ \theta^a_I,\ A^a_{\mu;I},\ w^i_I,\ \chi^i_I,\ dx^\mu,\ \delta\theta^a_I,\ \delta A^a_{\mu;I},\ \delta w^i_I,\ \delta\chi^i_I) \tag{7.1.1}$$

for $\mathcal{M}_Y$. The degrees of the above coordinate functions are

$$(0,0),\ (0,1),\ (0,0),\ (0,-2),\ (0,-1),\ (1,0),\ (0,2),\ (0,1),\ (0,-1),\ (0,0).$$

For simplicity, we omit the indices of the coordinate functions and use ϕ, υ, ψ, b to denote the odd coordinates $\delta\theta$, δA, δw, $\delta\chi$, respectively. There exist a family of QK_v-structures parameterized by $t \in \mathbb{R}$ on $\mathcal{M}_Y$. The cohomological vector field Q is defined by setting

$$Q\theta = \phi,\ QA = \upsilon,\ Qw = \psi,\ Q\chi = b,$$

and the action of Q on the other coordinates to be 0. The homotopy operator K is defined by setting

$$K\theta = tA,\ K\phi = d\theta - t\upsilon,\ K\upsilon = dA,\ K\psi = dw,\ Kb = d\chi,$$

and the action of K on the other coordinates to be 0. Q and K are of degrees $(0, 1)$ and $(1, -1)$, respectively. From now on, we set $t = 1$.

Remark 7.1.1. The notations we adopt here need more explanation. For example, when we write $Qw = \psi$, we actually mean a family of equations $Qw_I^j = \psi_I^j$. Likewise, when we write $K\psi = dw$, we mean $K\psi_I^j = w^j_{I\cup\{\mu\}}dx^\mu$. The reader may question that the equations $K\theta = A$ and $K\phi = d\theta - \upsilon$ are illegal because K need to be of degree $(1, -1)$. However, what we really mean by writing A is $A^a_{\mu;I}dx^\mu$ instead of $A^a_{\mu;I}$. Likewise, we write υ to mean $\upsilon^a_{\mu;I}dx^\mu$.

By construction, $\mathcal{M}_Y$ can be equipped with an L_g-action. Note that the Lie($\mathcal{G}$)-action on $\mathcal{A}$ is not linear. This will cause problems when we apply changes of coordinates later. Hence, we require that Lie($\mathcal{G}$) acts on $\mathcal{A}$ through the adjoint action instead. The contractions ι_λ are then defined by setting

$$\iota_\lambda\theta = \lambda,\ \iota_\lambda\phi = -[\lambda, \theta],\ \iota_\lambda\upsilon = -[\lambda, A],\ \iota_\lambda\psi = -\lambda w,\ \iota_\lambda b = -\lambda\chi,$$

and its action on the other coordinates to be 0. However, $\iota_\lambda K + K\iota_\lambda \neq 0$,[1] i.e., the L_g-structure is not compatible with the QK_υ-structure. This issue will be solved later.

We apply the Mathai-Quillen map to express the QK-structure in new coordinates. We have

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], \quad Q\phi = -[\theta, \phi],\\
&QA = \upsilon - [\theta, A], \quad Q\upsilon = -[\theta, \upsilon] + [\phi, A]\\
&Qw = \psi - \theta w, \quad Q\psi = -\theta\psi + \phi w,\\
&Q\chi = b - \theta\chi, \quad Qb = -\theta b + \phi\chi,
\end{aligned}$$

as a generalization of the Kalkman differential in Remark 2.2.2. We also have

$$\begin{aligned}
&K\theta = A, \quad K\phi = d\theta - \upsilon,\\
&KA = 0, \quad K\upsilon = dA + [A, A],\\
&Kw = 0, \quad K\psi = dw + Aw,\\
&K\chi = 0, \quad Kb = d\chi + A\chi.
\end{aligned}$$

One can verify $Q^2 = 0$ and $QK + KQ = L$ by direct computations. The advantage of the new coordinates is that we have a simpler expression for ι_λ, namely, we have

$$\iota_\lambda\theta = \lambda,$$

and 0 for ι_λ acting on the other coordinates.

[1] One can easily check that $(\iota_\lambda K + K\iota_\lambda)\phi = \iota_\lambda(d\theta - \upsilon) - K[\lambda, \theta] = d\lambda + [\lambda, A] - [\lambda, A] = d\lambda$.

In order to fix the incompatibility between K and ι_λ, and to change the Lie($\mathcal{G}$)-action on $\mathcal{A}$ back to the correct one, we apply the following change of coordinates

$$\upsilon \to \upsilon - d\theta.$$

We have now

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], \quad Q\phi = -[\theta, \phi], \\
&QA = \upsilon + d_A\theta, \quad Q\upsilon = -[\theta, \upsilon] - d_A\phi \\
&Qw = \eta - \theta w, \quad Q\psi = -\theta\psi + \phi w, \\
&Q\chi = b - \theta\chi, \quad Qb = -\theta b + \phi\chi,
\end{aligned}$$

and

$$\begin{aligned}
&K\theta = A, \quad K\phi = -\upsilon, \\
&KA = 0, \quad K\upsilon = 2F, \\
&Kw = 0, \quad K\psi = d_A w, \\
&K\chi = 0, \quad Kb = d_A\chi,
\end{aligned}$$

where $d_A = d + A$ and $F = dA + \frac{1}{2}[A, A]$ can be interpreted as the covariant derivative and the curvature of A. We set ι_λ to be of the same form as before. It is then not hard to see that $\delta_\lambda A$ give us the correct gauge transformation of a connection 1-form, and that $\iota_\lambda K + K\iota_\lambda = 0$.[2]

Theorem 7.1.1. *$\mathcal{M}_Y$ is an h-simple QK_{vg} manifold.*

Proof. It remains to check the h-simple property. Apply the vector fields $K\delta_\lambda - \delta_\lambda K$ to coordinate functions. The only non-vanishing one is

$$(K\delta_\lambda - \delta_\lambda K)\theta = K(-[\lambda, \theta]) - \delta_\lambda A = -[\lambda, A] - d_A\lambda = -d\lambda.$$

However, functions dependent on θ are not in the kernel of ι_λ. □

Remark 7.1.2. Cohomological vector fields Q of the above form were first given in [OSV89]. It was noticed there that Q recovers the saclar supersymmetry in [Wit88] by setting θ to be 0, i.e., by restricting to the horizontal functions over $\mathcal{M}_Y$.

It remains to specify the Lagrangians and observables. For the Lagrangian, we set

$$\mathcal{L} = Q(\langle(\theta, \upsilon, \psi, \chi), (0, f_1, f_2, f_3 + b)\rangle)d\text{vol}, \tag{7.1.2}$$

[2] In fact, we have $\iota_\lambda K = K\iota_\lambda = 0$.

where f_1, f_2, f_3 are the coefficient functions of a $\mathcal{G}$-equivariant vector field of degree $(0,-1)$ over $\mathcal{M}_Y$, $\langle\cdot,\cdot\rangle$ is a $\mathcal{G}$-invariant inner product on the tangent space, and $d$vol is the volume form on M, which can be regarded as a function over $\mathcal{M}_Y$ of degree $(n,0)$. By construction, $\mathcal{L}$ is a Q-exact basic function over $\mathcal{M}_Y$. It is easy to see that $\mathcal{L}$ is an infinite dimensional generalization of (2.2.2) in the Mathai-Quillen construction of an Euler class. Since one cannot have gauge symmetries in the finite dimensional case, this generalization is essentially non-trivial.

Remark 7.1.3. $\theta, \upsilon, \psi, \chi$ should be viewed as the coefficient functions of the (odd) Euler vector field (which is of degree $(0,0)$) over $\mathcal{M}_Y$.

Remark 7.1.4. f_1, f_2 and f_3 are referred to as the gauge fixing functions in the physics literature. In our case, they should be carefully chosen to have vertical degrees $-2, 0$ and 0, respectively. In this way, $\mathcal{L}$ is homogeneous of degree $(n,0)$.

Let $\mathcal{O}^{(0)}$ be a gauge invariant pre-observable of degree 0. It can't be Q-exact, otherwise the expectation values of the corresponding observables will vanish. For simplicity, let's assume that n is even. A reasonable choice is then $\mathcal{O}^{(0)} = \mathrm{Tr}(\phi^m)$, where $m = n/2$. Using Lemma 4.5.4, the standard K-sequence of $\mathcal{O}^{(0)}$ can be found as

$$\sum_{p=0}^{n} \mathcal{O}^{(p)} = \exp(K)\mathcal{O}^{(0)} = \mathrm{Tr}(\phi_K^m), \tag{7.1.3}$$

where

$$\phi_K = \exp(K)\phi = \phi - \upsilon - F$$

can be interpreted as the curvature 2-form on the principal G-bundle $\mathcal{P} = (P \times \mathcal{A})/\mathcal{G} \to M \times \mathcal{A}/\mathcal{G}$ [BS88]. In this sense, (7.1.3) is nothing but a Chern class of $\mathcal{P}$. With a slight abuse of notation, we call $\mathcal{P}$ the universal G-bundle, and ϕ_K the universal curvature 2-form on $\mathcal{P}$, though both of them depend apparently on the choice of P. Likewise, we set

$$\theta_K = \exp(K)\theta = \theta + A.$$

θ is called the universal connection 1-form on $\mathcal{P}$. In fact, the de Rham complex of $\mathcal{P}$ is a commutative bigraded algebra of the second kind. By Lemma 4.5.1, we should reset Q and K to be $(-1)^p Q$ and $(-1)^q K$, where p and q are the horizontal degrees of the functions acted by Q and K, respectively. The universal curvature 2-form ϕ_K takes the form $\phi - \upsilon + F$ instead. Let d_{tot} denote the total differential associated to Q and L. It is not hard to show that

$$\phi_K = d_{tot}\theta_K + \frac{1}{2}[\theta_K, \theta_K], \quad d_{tot}\phi_K + [\theta_K, \phi_K] = 0.$$

In the next subsection, we will consider the case where P is the trivial G-bundle over M. We will give a notion of Weil homomorphism in the infinite dimensional setting which sends θ_K and ϕ_K to connection and curvature forms on a mapping space.

7.1.1 Topological Yang-Mills theory

The geometric setting is specified by the following data.

1. P is a principal SU(2)-bundle;
2. $V = \mathrm{ad}P$, $W = \mathrm{ad}P \otimes \Lambda^2_-(T^*M)$.

The Lagrangian is specified by the gauge fixing functions

$$f_1 = d_A w, \quad f_2 = [\phi, w], \quad f_3 = F_-,$$

where F_- is the anti-self-dual part of the curvature F. Note that f_1 is of degree -2, f_2 is of degree 0, and f_3 is of degree 0. The pre-observables are determined by $\mathcal{O}^{(0)} = \mathrm{Tr}(\phi^2)$. We have

$$\begin{aligned}
\mathcal{O}^{(0)} &= \mathrm{Tr}(\phi^2),\\
\mathcal{O}^{(1)} &= -2\mathrm{Tr}(\phi\upsilon),\\
\mathcal{O}^{(2)} &= \mathrm{Tr}(\upsilon \wedge \upsilon - 2\phi F),\\
\mathcal{O}^{(3)} &= 2\mathrm{Tr}(\upsilon \wedge F),\\
\mathcal{O}^{(4)} &= \mathrm{Tr}(F \wedge F).
\end{aligned}$$

Remark 7.1.5. The topological Yang-Mills theory can also be obtained by twisting the $N = (1, 1)$ supersymmetric Yang-Mills theory. The QK-structure obtained from the twisting is more complicated than the QK-structure above. More precisely, since χ and b are $\mathfrak{su}(2)$-valued (anti-self-dual) 2-forms, there exists a family of QK-structures parameterized by $(r, s, t, u) \in \mathbb{R}^4$ by setting

$$\begin{aligned}
&Q\theta = \phi,\ QA = \upsilon,\ Qw = \psi,\ Q\chi = b + rF_-,\ Qb = -r(d_A\upsilon)_-,\\
&KA = s\chi,\ K\theta = tA,\ K\phi = d\theta - t\upsilon,\ K\upsilon = dA - s(b + rF_-),\ K\psi = dw,\\
&K\chi = u \star d_A w,\ Kb = d\chi + rs d_A\chi - u \star ([\upsilon, w] + d_A\psi),
\end{aligned}$$

and the action of Q and K on the other coordinates to be 0. It is easy to verify that $Q^2 = 0$ and $QK + KQ = d$.

Likewise, we set $t = 1$. After applying the change of coordinates induced by the Mathai-Quillen map, we get

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], \quad Q\phi = -[\theta, \phi],\\
&QA = \upsilon + d_A\theta, \quad Q\upsilon = -[\theta, \upsilon] - d_A\phi,\\
&Qw = \psi - [\theta, w], \quad Q\psi = -[\theta, \psi] + [\phi, w],\\
&Q\chi = b + rF_- - [\theta, \chi], \quad Qb = -r(d_A\upsilon)_- - [\theta, (b + rF_-)] + [\phi, \chi],
\end{aligned}$$

and

$$\begin{aligned}
&K\theta = A, \quad K\phi = -\upsilon, \\
&KA = s\chi, \quad K\upsilon = 2F - s(b + rF_-), \\
&Kw = 0, \quad K\psi = d_A w, \\
&K\chi = u \star d_A w, \quad Kb = (1+rs)d_A\chi - u\star([\upsilon, w] + d_A\psi - [\theta, d_A w]).
\end{aligned}$$

Note that the expression of Kb involves θ. To make the QK-structure compatible with the L_g-structure, we have to set $u = 0$.

Let's also set $r = 0$ for simplicity. We then have

$$\begin{aligned}
\theta_K &= \theta + A + \frac{s}{2}\chi, \\
\phi_K &= \phi - \upsilon - F + \frac{s}{2}(b + rF_-) + \frac{s}{2}d_A\chi + \frac{s^2}{8}[\chi, \chi].
\end{aligned}$$

The standard K-sequence of $\mathcal{O}^{(0)} = \mathrm{Tr}(\phi^2)$ takes the form

$$\begin{aligned}
\mathcal{O}^{(0)} &= \mathrm{Tr}(\phi^2), \\
\mathcal{O}^{(1)} &= -2\mathrm{Tr}(\phi\upsilon), \\
\mathcal{O}^{(2)} &= \mathrm{Tr}(\upsilon \wedge \upsilon + \phi(sb - 2F)), \\
\mathcal{O}^{(3)} &= \mathrm{Tr}(s\phi d_A\chi - \upsilon \wedge (sb - 2F), \\
\mathcal{O}^{(4)} &= \mathrm{Tr}((sb/2 - F) \wedge (sb/2 - F) - s\upsilon \wedge d_A\chi + s^2\phi[\chi, \chi]/4).
\end{aligned}$$

Let K_0 denote the vector symmetry in the special case of $s = 0$, i.e., the vector symmetry defined in Section 6. One can easily check the above standard K-sequence is nothing but the general K_0-sequence specified by

$$\mathcal{W}^{(1)} = 0, \quad \mathcal{W}^{(2)} = s\mathrm{Tr}(\phi b), \quad \mathcal{W}^{(3)} = 0, \quad \mathcal{W}^{(4)} = -\frac{s^2}{4}\mathrm{Tr}(b \wedge b + \phi[\chi, \chi]).$$

Note that both $\mathcal{W}^{(2)}$ and $\mathcal{W}^{(4)}$ are gauge invariant and Q-exact. In fact, we have

$$\mathcal{W}^{(2)} = sQ(\mathrm{Tr}(\phi\chi)), \quad \mathcal{W}^{(4)} = -\frac{s^2}{4}Q(\mathrm{Tr}(b \wedge \chi)).$$

The standard K-sequence is equivalent to the standard K_0-sequence up to an exact sequence.

Further examples like Kapustin-Witten theory can also be incorporated into this framework with ease.

7.2 Chern-Weil homomorphism for mapping spaces

Every physical theory with non-trivial dynamics involves the notion of connections. Theories with gauge symmetries are characterized by the interpretation of connections as variables, while theories with no gauge symmetry usually have their connections fixed. One approach to construct Lagrangians for rigid CohFTs, therefore, is to start with a "universal" CohGFT and apply a generalization of the Chern-Weil homomorphism. In this subsection, we explain this idea in detail.

Let the graded fiber bundle Y be as in the previous subsection, i.e., $Y = \mathrm{ad}P \times_M C \times_M V \times_M W$. Let's consider the trivial principal G-bundle P over M with $G = \mathrm{SO}(2m)$. $\mathcal{G}$, $\mathrm{Lie}(\mathcal{G})$, $\mathcal{A}$ can then be identified with $C^\infty(M, G)$, $C^\infty(M, \mathfrak{g})$ and $\Gamma(T^*M) \otimes \mathfrak{g}$, respectively. Let V be a real rank $2m$ vector bundle associated to P by the fundamental representation of $\mathrm{SO}(2m)$. Let W be the dual bundle of V. $\mathcal{G}$ acts on $\Gamma(V)$ and $\Gamma(W)$ fiber-wisely. By Remark 2.2.2, we need to change the QK-structure on $\mathcal{M}_Y$ by resetting

$$Qw = \psi, \quad Q\psi = 0, \quad Kw = 0, \quad K\psi = dw.$$

The L_g-structure also changes correspondingly. Namely, we should redefine ι_λ by setting

$$\iota_\lambda \psi = -\lambda w.$$

Moreover, we reassign degree $(0, 0)$ to w and degree $(0, 1)$ to ψ. We then consider $\mathcal{L}$ of the form

$$\mathcal{L} = Q(\chi(w) + \langle \chi, b \rangle) d\mathrm{vol}. \tag{7.2.1}$$

It is easy to verify that $\mathcal{L}$ is homogeneous of degree $(n, 0)$.

On the other hand, let Σ be a $2m$-dimensional Riemannian manifold, $2m \geq n$. Let P_Σ be a principal G-bundle over Σ equipped with a connection 1-form A. Let Y_0' denote the trivial fiber bundle $M \times P_\Sigma$ over M. Note that $\Gamma(Y_0') = C^\infty(M, P_\Sigma)$, which can be viewed as a principal $\mathcal{G}$-bundle over the mapping space $C^\infty(M, \Sigma)$. Therefore, the variational bigraded manifold $\mathcal{M}_{Y_0'}$ associated to Y_0' carries a canonical L_g-action.

Lemma 7.2.1. *$\mathcal{M}_{Y_0'}$ equipped with the canonical QK_v-structure and L_g-structure is a simple QK_{vg}-manifold, hence particularly an h-simple QK_{vg}-manifold.*

Proof. We need to show that $[K, \iota_\lambda] = 0$ and $[K, \delta_\lambda] = 0$. Let $(x^\mu, u_I^j, dx^\mu, \delta u_I^j)$ be a local coordinate system. It is not hard to see that we only need to check both properties for coordinates δu_I^j. For the first one, we have

$$[K, \iota_\lambda]\delta u_I^j = K(\delta_\lambda u_I^j) + \iota_\lambda(u_{I\cup\{\mu\}}^j dx^\mu) = 0,$$

where we use $[Q, \iota_\lambda] = \delta_\lambda$. For the second one, it is equivalent to show that $[L, \iota_\lambda] = 0$. We have

$$[L, \iota_\lambda]\delta u_I^j = L(\delta_\lambda u_I^j) - (\delta_\lambda u_{I\cup\{\mu\}}^j)dx^\mu = \partial_\mu(\delta_\lambda u_I^j)dx^\mu - (\delta_\lambda u_{I\cup\{\mu\}}^j)dx^\mu = 0,$$

where we use $\delta_\lambda u_I^j = \partial_I(\delta_\lambda u^j)$. □

The evaluation map

$$\mathrm{ev} : M \times C^\infty(M, P_\Sigma) \to P_\Sigma$$

pulls back A to a $\mathfrak{g}$-valued 1-form on $M \times C^\infty(M, P_\Sigma)$. It decomposes into two parts: the horizontal part A_h along M, and the vertical part A_v along $C^\infty(M, P_\Sigma)$. Let ∇_h and ∇_v denote the covariant derivatives associated to A_h and A_v, respectively. We can write

$$\nabla_h = d + A_h, \quad \nabla_v = \delta + A_v,$$

where d is the horizontal differential and δ is the vertical differential. We have three types of curvatures:

$$R_h = \nabla_h^2, \quad R_v = \nabla_v^2, \quad R_m = \nabla_v \nabla_h + \nabla_h \nabla_v,$$

where the subscript m of R_m stands for the word "mixed". We have

$$R_h = dA_h + \frac{1}{2}[A_h, A_h], \quad R_v = \delta A_v + \frac{1}{2}[A_v, A_v], \quad R_m = dA_v + \delta A_h + [A_h, A_v].$$

By a simple analysis of degrees, we also have the following four types of Bianchi identities.

$$\nabla_h R_h = 0, \quad \nabla_v R_v = 0, \quad \nabla_v R_m + \nabla_h R_v = 0, \quad \nabla_h R_m + \nabla_v R_h = 0.$$

Remark 7.2.1. Recall that the sign convention we choose for a variational bigraded manifold is of the first kind. In particular, we should have $d\delta = \delta d$. By Lemma 4.5.1, this can be achieved by redefining δ to be $(-1)^p \delta$, where p is the horizontal degree of the function acted by δ. Correspondingly, the expressions for the curvature R_m and the third one of the above Bianchi identities change. We have

$$R_m = dA_v - \delta A_h + [A_h, A_v], \quad -\nabla_v R_m + \nabla_h R_v = 0.$$

G can be viewed as a subgroup of $\mathcal{G}$ by identifying its element with the corresponding constant functions in $C^\infty(M, G)$. Thus, A_v, R_v can be viewed as $\mathrm{Lie}(\mathcal{G})$-valued 1-form and 2-form of $\Omega_{loc}(M \times C^\infty(M, P_\Sigma))$, respectively. They determine maps

$$\mathrm{Lie}(\mathcal{G})^* \to \Omega_{loc}^{\bullet,1}(M \times C^\infty(M, P_\Sigma)), \quad \mathrm{Lie}(\mathcal{G})^* \to \Omega_{loc}^{\bullet,2}(M \times C^\infty(M, P_\Sigma)).$$

which induce a map

$$\phi_1 : \Omega_{loc}(M \times \Gamma(\mathrm{ad}P)) \to \Omega_{loc}(M \times C^\infty(M, P_\Sigma))$$

sending θ and ϕ to A_v and R_v, respectively. This is the usual Weil homomorphism in the infinite dimensional setting. Let $(x^\mu, dx^\mu, A^a_{\mu;I}, \upsilon^a_{\mu;I})$ be a coordinate system of $\Omega_{loc}(M \times \mathcal{A})$. We can also define a connection fixing map

$$\phi_2 : \Omega_{loc}(M \times \mathcal{A}) \to \Omega_{loc}(M \times C^\infty(M, P_\Sigma))$$

which sends $A \in \mathcal{A}$ to A_h and the vertical differential υ of A to $-R_m$. Combining ϕ_1 and ϕ_2, we obtain a map

$$\phi_W : \Omega_{loc}(M \times \Gamma(\mathrm{ad}P) \times \mathcal{A}) \to \Omega_{loc}(M \times C^\infty(M, P_\Sigma)),$$

which we refer to as the Weil homomorphism for mapping spaces. By definition, it sends the universal connection θ_K and curvature ϕ_K defined in the previous subsection to the connection and curvature on $M \times C^\infty(M, P_\Sigma)$, respectively.

Theorem 7.2.1. *ϕ_W preserves the h-simple QK_{vg}-structure.*

Proof. This follows from direct computations. Let's check first that $K\phi_W = \phi_W K$. We have

$$K(\phi_W(A)) = KA_h = 0 = \phi_W(KA), \quad K(\phi_W(\theta)) = KA_v = A_h = \phi_W(K\theta).$$

Since Q is just the vertical differential δ for $\mathcal{M}_{Y_0'}$, we have

$$KR_v = K(QA_v + \frac{1}{2}[A_v, A_v]) = LA_v - QKA_v + [A_h, A_v] = dA_v - \delta A_h + [A_h, A_v] = R_m,$$
$$KR_m = K(LA_v - QA_h + [A_h, A_v]) = -LKA_v - LA_h - [A_h, KA_v] = -2dA_h - [A_h, A_h] = -2R_h.$$

Thus,

$$K(\phi_W(\upsilon)) = -KR_m = 2R_h = \phi_W(2F) = \phi_W(K\upsilon), \quad K(\phi_W(\phi)) = KR_v = R_m = \phi_W(K\phi).$$

The next step is to check $Q\phi_W = \phi_W Q$. We have

$$\phi_W(Q\theta) = \phi_W(\phi) - \frac{1}{2}\phi_W([\theta, \theta]) = R_v - [A_v, A_v] = \delta A_v = Q(\phi_W(\theta)),$$
$$\phi_W(Q\phi) = \phi_W(-[\theta, \phi]) = -[A_v, R_v] = \delta R_v = Q(\phi_W(\phi)),$$

where we use the Bianchi identity $\nabla_v R_v = 0$. We also have

$$\phi_W(QA) = \phi_W(\upsilon) + \phi_W(d_A\theta) = -R_m + dA_v + [A_h, A_v] = -\delta A_h = Q(\phi_W(A)),$$
$$\phi_W(Q\upsilon) = \phi_W(-[\theta, \upsilon]) - \phi_W(d_A\phi) = [A_v, R_m] - dR_v - [A_h, R_v] = -\delta R_m = Q(\phi_W(\upsilon)),$$

where we use the Bianchi identity $-\nabla_v R_m + \nabla_h R_v = 0$. It follows that

$$L\phi_W = \phi_W L.$$

To prove that ϕ_W preserves the L_g-structure, it suffices to check $\phi_W \iota_\lambda = \iota_\lambda \phi_W$. By construction, we have

$$\iota_\lambda \phi_W(\theta) = \iota_\lambda A_v = \lambda, \; \iota_\lambda \phi_W(\phi) = \iota_\lambda R_v = 0, \; \iota_\lambda \phi_W(A) = \iota_\lambda A_h = 0.$$

It remains to show that $\iota_\lambda \phi_W(\upsilon) = 0$. In fact, we can check that

$$\begin{aligned}
\iota_\lambda R_m &= \iota_\lambda(LA_v - QA_h + [A_h, A_v]) \\
&= [\iota_\lambda, L]A_v + L\lambda - [\iota_\lambda, Q]A_h + Q\iota_\lambda A_h + [A_h, \lambda] \\
&= [\delta_\lambda, K]A_v - \delta_\lambda A_h + [A_h, \lambda] \\
&= -K(\delta_\lambda A_v) + [A_h, \lambda] \\
&= K([\lambda, A_v]) + [A_h, \lambda] \\
&= 0,
\end{aligned}$$

where we use $[\iota_\lambda, L] = [\delta_\lambda, K]$. □

Let $Y' = Y_0' \times_M V \times_M W$. We can extend ϕ_W naturally to a map

$$\Omega_{loc}(M \times \Gamma(Y)) \to \Omega_{loc}(M \times \Gamma(Y')),$$

which is denoted again by ϕ_W with a slight abuse of notation. There exists a natural map

$$\begin{aligned}
\Gamma(Y_0') \cong C^\infty(M, P_\Sigma) &\to \Gamma(T^*M) \otimes \mathfrak{g} \cong \Gamma(Y_0) \\
f &\mapsto A(Tf),
\end{aligned}$$

where we identify the tangent map Tf of f as a section of $T^*M \otimes f^*TP_\Sigma$, and identify A as a map $\Gamma(TP_\Sigma) \to \mathfrak{g}$. This map together with ϕ_W determines a morphism $\mathcal{M}_{Y'} \to \mathcal{M}_Y$ of h-simple QK_{vg}-manifolds.

From now on, we choose P_Σ to be the frame bundle of Σ and A to be the Levi-Civita connection. $(\Omega_{loc}(M \times \Gamma(Y'))_{bas}$ can be identified with $\Omega_{loc}(M \times \Gamma(Y_\Sigma))$, where Y_Σ is the trivial graded fiber bundle $M \times (T\Sigma \times_\Sigma T^*\Sigma)$ over M, with $(Y_\Sigma)_0$ being the trivial bundle $M \times \Sigma$ over M. ϕ_W induces a homomorphism

$$\phi_{CW} : \Omega_{loc}(M \times \Gamma(Y))_{bas} \to \Omega_{loc}(M \times \Gamma(Y_\Sigma)),$$

which we refer to as the Chern-Weil homomorphism for mapping spaces.

Let $(x, u, w, \chi, dx, \delta u, \psi, b)$ be a local coordinate system of $\mathcal{M}_{Y_\Sigma}$. The QK-structure on $\mathcal{M}_{Y_\Sigma}$ is given by

$$\begin{aligned}
&Qu = \delta u, \quad Q\delta u = 0, \quad Qw = \psi, \quad Q\psi = 0, \quad Q\chi = b - A_v\chi, \quad Qb = -A_v b + R_v\chi, \\
&Ku = 0, \quad K\delta u = du, \quad Kw = 0, \quad K\psi = dw, \quad K\chi = 0, \quad Kb = d\chi + A_h\chi.
\end{aligned}$$

To define a CohFT, we simply set the Lagrangian to be

$$\mathcal{L} = Q(\langle \chi, f + b \rangle) d\text{vol}, \tag{7.2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the tangent spaces of $\mathcal{M}_{Y_\Sigma}$ induced by the Riemannian metric on Σ, f is a vector field over $\mathcal{M}_{Y_\Sigma}$ induced by a vector field over $C^\infty(M, \Sigma)$, and $d$vol is the volume form on M. (7.2.2) can be seen as the pullback through f of the image of (7.2.1) under ϕ_{CW}.

Remark 7.2.2. More generally, one can take W in the construction of $\mathcal{M}_Y$ and $\mathcal{M}_{Y'}$ to be the tensor product of the dual bundle of V and $\Lambda^p T^*M$, and f to be induced by a section of the bundle $\Lambda^p T^*M \times C^\infty(M, T\Sigma)$ over $M \times C^\infty(M, \Sigma)$.

Remark 7.2.3. Lagrangians of the form (7.2.2) can be found in [BS89; Bla93]

For pre-observables of the CohFT, consider the map

$$\begin{aligned} \mathrm{e} : M \times \Gamma(Y_\Sigma) \cong M \times C^\infty(M, T\Sigma) \times C^\infty(M, T^*\Sigma) \to \Sigma \\ (x, f_1, f_2) \mapsto \pi(f_1(x)), \end{aligned}$$

where $\pi : T\Sigma \to \Sigma$ is the canonical projection. Let α be a closed n-form on Σ. Let $\mathcal{O} = \mathrm{e}^*\alpha$. $\mathcal{O}$ can be decomposed as $\mathcal{O} = \sum_{p=0}^n \mathcal{O}^{(p)}$, where $\mathcal{O}^{(p)}$ is of horizontal degree p. Locally, α can be written as $\alpha_{i_1,\cdots,i_n} du^{i_1} \wedge \cdots \wedge du^{i_n}$. We then have

$$\mathcal{O}^{(p)} = \binom{n}{p} \alpha_{i_1,\cdots,i_n} d_h u^{i_1} \wedge \cdots \wedge d_h u^{i_p} \wedge \delta u^{i_{p+1}} \wedge \cdots \wedge \delta u^{i_n}.$$

One can check that $\mathcal{O}^{(p)} = \frac{1}{p} K \mathcal{O}^{(p-1)}$. In other words, $\{\mathcal{O}^{(p)}\}_{p=0}^n$ is the standard K-sequence of $\mathcal{O}^{(0)}$.

7.2.1 Topological quantum mechanics

The geometric setting is specified by the following data.

1. M is the real line $\mathbb{R}$;

2. Σ is a Riemannian manifold equipped with a Morse function h.

The Lagrangian is specified by the gauge fixing function

$$f = \frac{du}{dt} + \mathrm{grad} h,$$

where t is the parameter of $\mathbb{R}$. The pre-observables are specified by the 1-form $\alpha = dh$ on Σ. We have

$$\begin{aligned} \mathcal{O}^{(0)} &= \partial_i h \delta u^i, \\ \mathcal{O}^{(1)} &= \partial_i h u^i_t dt. \end{aligned}$$

Remark 7.2.4. In dimension 1, there is no spinor or R-symmetry, hence no topological twisting. The topological quantum mechanics is just the $N = 2$ supersymmetric quantum mechanics.

7.2.2 Topological sigma model

The geometric setting is specified by the following data.

1. (M, j) is a Riemann surface;

2. (Σ, ω, J) is a Kähler manifold.

The Lagrangian is specified by the gauge fixing function

$$f = \bar{\partial}_J u,$$

where $\bar{\partial}_J u = \frac{1}{2}(Du + J \circ Du \circ j)$, Du is the total differential of u. Note that f is a section of the bundle $T^*M \times C^\infty(M, T\Sigma)$ over $M \times C^\infty(M, \Sigma)$. The pre-observables are specified by $\alpha = \omega$, the symplectic form on Σ. We have

$$\begin{aligned}
\mathcal{O}^{(0)} &= \omega_{i_1 i_2} \delta u^{i_1} \delta u^{i_2}, \\
\mathcal{O}^{(1)} &= 2\omega_{i_1 i_2} u^{i_1}_\mu dx^\mu \delta u^{i_2}, \\
\mathcal{O}^{(2)} &= \omega_{i_1 i_2} u^{i_1}_\mu u^{i_2}_\nu dx^\mu dx^\nu.
\end{aligned}$$

Remark 7.2.5. The topological sigma model can also be obtained by twisting the $N = 2$ supersymmetric non-linear sigma model.

Further examples like topological M theory can also be incorporated into this framework with ease.

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Selbstständigkeitserklärung

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