CONFORMAL FIELD THEORY, DILOGARITHMS, AND THREE DIMENSIONAL MANIFOLDS

WERNER NAHM
Physikalisches Institut der Universität Bonn
Nussalle 12 D-5300 Bonn 1
nahm@pib1.physik.uni-bonn.de

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The dilogarithm $Li_2(z)$ is defined by

$$\frac{d}{dz}Li_2(z) = -\frac{\log(1-z)}{z}, \quad \text{for non-real } z,$$

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{for} \quad |z| < 1 .$$

In particular $Li_2(1) = \pi^2/6$. Its analytic continuation is multivalued and will be considered below.

A q-deformed version of the dilogarithm is given by

$$-\sum_{n=1}^{\infty} \log(1-uq^n) , \quad |q| < 1.$$

When the absolute value of q is close to 1, one can approximate the sum by an integral, which yields $Li_2(u)/\log(q)$.

Dilogarithms have a long history both in physics and in mathematics, which would take to long to describe. In physics, they appear in the evaluation of Feynman graphs, which at present has no relations to the new applications considered below. The latter seem to have originated first in the investigations of Faddeev's Leningrad/ St.Petersburg group. On one hand, q-deformed dilogarithms describe the S-matrix of the sine-Gordon model [FK 1978, eq. 5.3], on the other hand the magnetization of the XXY model was linked to the central charge of the corresponding conformal theory and later calculated in dilogarithmic form, see e.g. [KR 1987]. Parts of the group dispersed, but the investigations were taken up elsewhere. In particular, much evidence was produced which links dilogarithms to conformal dimensions, see [FNO 1992]. We shall see that the corresponding dilogarithm identities correspond to elements of finite order of the so called Bloch group. The fact that the conformal dimensions are rational is closely linked to the finite order property.

On the other hand, infinite order elements of the Bloch group seem to be essential for the classification of three dimensional manifolds, one of the most interesting problems of present day mathematics. Some relation to the classification of two dimensional conformal field theories can be expected, since the latter yield topological field theories in three dimensions and those in turn yield invariants of manifolds of three (real) dimensions. These invariants are of the type of the well known Chern-Simons and η-invariants. Again, they are rational and related to the real part of dilogarithms. At least as important, however, is the volume invariant of Thurston's classification program for three-manifolds [Thurston 77,82], [DS 1982], [NZ 85]. This invariant yields imaginary values of the dilogarithm function, for Bloch group elements of infinite order.

For an elementary introduction to Thurston's program see [Meyerhofer 1992]. For convenience of the reader, I repeat some of the essential points. Three dimensional manifolds have canonical decompositions with respect to cuts along spheres and tori. For manifolds which are indecomposable with respects to such cuts, Thurston argued that they can be given a geometric structure. In other words, they can be written as the quotient of a homogeneous space with respect to the action of a discrete group. Thurston proved his conjecture under various conditions, but the general problem is still open.

In two dimensions it is easy to write any manifold in such a way, namely just as the quotient of its covering space by its fundamental group. In fact, the covering space of any compact two dimensional Riemannian manifold is either the sphere, the plane or the hyperbolic space, which all can be given a homogeneous metric. Apart from the torus, the curvature can be normalized to ±1, which gives the manifolds a canonical volume. Due to the Gauss-Bonnet theorem

$$\int R dV = 4\pi(1-g) ,$$

the classification by the genus g and the one by the normalized volume are equivalent.

For almost all values of the genus, indeed for $g \neq 0, 1$, the geometric structure is hyperbolic. In three dimensions, there is a sense in which the generic manifolds have a hyperbolic structure, too. Note first that every three dimensional manifolds can be constructed by Dehn twists around some link in the sphere S^3. To perform such twists, one cuts out a small tubular neighborhood around each knot component of the link, transforms the surface of each resulting solid torus by a diffeomorphism of the mapping class group $SL(2,Z)$, and glues it back in. More precisely, consider a solid torus $D \times S^1$, where D is the unit disk with polar coordinates r, ϕ. On S^1 we have an angle coordinate θ. The mapping class group of the torus surface is given by the transformations

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} \to \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix}, \qquad \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2,Z) .$$

Thus it can be identified with the modular group $SL(2,Z)/Z_2$. The transformations generated by $\binom{11}{01}$ can be continued to the whole solid torus, such that the corresponding Dehn twists do not change the topology of the manifold. Thus the resulting manifolds are given by the cosets of $\binom{pr}{qs}$ modulo this subgroup, in other words by the first matrix column (p,q). Except for special links or special small values of the (p,q), the manifold constructed by Dehn twists will be hyperbolic. In this sense, generic manifolds are hyperbolic.

The Dehn twist construction is very far from a classification, since it is highly non-unique and since there is no effective classification of knots. Nevertheless, the construction easily generates many hyperbolic manifolds of small volume. In particular, take the Dehn twists around the figure-of-eight knot. The volume is an increasing function of the partially ordered labels (p,q). Its minimum at (1,5) is conjectured to be the smallest value in the set $\mathcal{V}$ of all possible volumes of hyperbolic three-manifolds of curvature -1.

To calculate such a volume, one cuts the manifold into tetrahedral pieces. The volumes of hyperbolic tetrahedra first were calculated by Lobatchevsky. His formula is a bit complicated, but it simplifies a lot for ideal tetrahedra, for which the vertices lie at infinity. Disregarding lower dimensional submanifolds, any hyperbolic manifold can be cut up into such ideal tetrahedra. For compact manifolds, one just has to cut out a circular geodesic, which comes to lie at the infinite points of the hyperbolic space. The sides of the tetrahedra all wind around this geodesic and converge towards it.

In terms of the quaternions $1, i, j, k$, the points of hyperbolic space can be written in the form $X = x + iy + jz$, $z > 0$. Consider the matrices $\binom{ab}{cd} \in SL(2,C)$, where the complex numbers C are given by the linear combinations of the quaternions $1, i$. Their group acts on hyperbolic space by the transformations

$$X' = (aX + b)(cX + d)^{-1} .$$

To see this, note that

$$X' = (aX + b)(X^+ c^+ + d^+)|cX + d|^{-2} .$$

The j, k part of X' is proportional to

$$ajd^+ - bjc^+ = (ad - bc)j = j \ .$$

In particular,

$$z' = z|cX + d|^{-2} > 0 \ .$$

Since

$$dX' = -c^{-1}(cX + d)^{-1} c \, dX \, (cX + d)^{-1} \ ,$$

the metric

$$(dx^2 + dy^2 + dz^2)/z^2 = dX dX^+/z^2$$

is conserved under $SL(2, C)$. The boundary at infinity of hyperbolic space is given by the plane $z = 0$ plus a point at infinity, thus it is isomorphic to the Riemann sphere. Its transformations under $SL(2, C)$ are the usual rational linear ones.

The volume of an ideal tetrahedron depends on its four vertices $z_i \in C$, $i = 1, 2, 3, 4$. As we have seen it is invariant under rational linear transformations. Invariant functions of the z_i only depend on the double ratio

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \ .$$

Let the volume function be denoted by $V(z_1, z_2, z_3, z_4) = D(z)$. For real z, all vertices lie on one line, such that the volume vanishes. It is convenient to incorporate the tetrahedron orientation by the sign of the volume, such that $D(\bar{z}) = -D(z)$. Permuting the vertices yields the symmetry properties $D(z) = -D(1 - z) = -D(1/z)$. As long as no vertices coincide, D is a real analytic function of its argument.

The union of two tetrahedra joined along a face can be cut into three tetrahedra by using the body diagonal as a new edge. Similarly, each tetrahedron can be cut up into four tetrahedra by choosing another vertex in the interiour. This yields

$$\sum_{i=0}^{5} (-)^i V(\hat{z}_i) = 0 \ ,$$

where $\hat{z}_i$ denotes z_0, z_1, z_2, z_3, z_4 with z_i omitted. Equivalently one has

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1 - x}{1 - xy}\right) + D\left(\frac{1 - y}{1 - xy}\right) = 0 \ .$$

This is the five term identity, which has been discovered and rediscovered by several famous mathematicians. With the convention $D(\infty) = 0$, it implies the symmetry properties of D.

Under very weak regularity conditions, which are obviously true for the tetrahedron volumes, the five term identity and the relatity properties yield

$$D(z) = \Im(Li_2(z)) + arg(1-z)\log|z| \, .$$

The dilogarithm Li_2 has a multivalued analytic continuation, but $D(z)$ becomes a continuous function on the whole plane. Away from the singularities at $z=0$ and $z=1$, it is real analytic.

Splitting the five term identity for $D(z)$ into its holomorphic and antiholomorphic parts, one obtains a five term identity for the Rogers dilogarithm

$$L(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x) \, ,$$

namely

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \pi^2/2 \, .$$

The constant $3L(1) = \pi^2/2$ on the right hand side is determined by putting $x=y=0$. The identity is valid for $x, y \in (0,1)$.

Thus volumes of hyperbolic three-manifolds have the form

$$V = \sum_k D(z_k) \, .$$

The fact that the tetrahedra fit together to form a manifold without boundary yields the closure condition

$$\sum_k [z_k] \wedge [1-z_k] = 0 \, ,$$

where the symbol $[z]$ fulfils the single relation $[xy] = [x] + [y]$ and the wedge product is defined by bilinearity and antisymmetry. Replacing adjoining tetrahedra as in the five term relation conserves this condition, since

$$\begin{gathered}[x] \wedge [1-x] + [y] \wedge [1-y] = \\ [xy] \wedge [1-xy] + \left[\tfrac{x(1-y)}{1-xy}\right] \wedge \left[\tfrac{1-x}{1-xy}\right] + \left[\tfrac{y(1-x)}{1-xy}\right] \wedge \left[\tfrac{1-y}{1-xy}\right] \, ,\end{gathered}$$

as can be checked easily.

The Bloch group [Bloch 78] is defined by the formal sums $\sum_k n_k(z_k)$, $z_k \in C$, $n_k \in Z$, which satisfy the closure condition

$$\sum_k n_k [z_k] \wedge [1-z_k] = 0 \, ,$$

modulo the formal sums coming from the five term identity, for which this condition always is satisfied. Moreover, one uses the convention $(\infty) = 0$, which implies $(0) = (1) = 0$. This is no significant restriction, since $15(\infty)$

vanishes by the five term identity. The map $D : \sum_k n_k(z_k) \to \sum_k n_k D(z_k)$ is a well defined map from the Bloch group elements to the real numbers. The volume set $\mathcal{V}$ belongs to the image of this map.

Since the closure equation is essentially algebraic, this implies that $\mathcal{V}$ is a countable set. Admitting disjoint unions and manifolds with boundaries, this set becomes closed and additive. Moreover, one can show that it is well ordered. In other words, for every volume there is a unique next larger volume. Accumulation points only arise by convergence towards upper limits, not towards lower ones. Let $\mathcal{V}'$ be the subset of accumulation points of $\mathcal{V}$ and iterate this procedure to obtain the sets $\mathcal{V}^{(n)}$ of n-fold accumulation points. One finds that all of these sets are non-empty, though they have empty intersection. In the language of ordinal numbers, this is expressed by the fact that the ordinal number of $\mathcal{V}$ is ω^ω.

Many elements of the Bloch group can be produced by the equations

$$\log(1-z_i) = \sum_j B_{ij} \log(z_i) \ ,$$

$i = 1, \dots, r$. If one makes Dehn twist around the figure-eight knot one finds $r = 2$ and

$$B = -\frac{1}{p+q}\begin{pmatrix} p & q \\ q & p \end{pmatrix} \ .$$

So far, we have considered some standard manifold mathematics. Now let us cosider the partition functions of some conformal field theories, which yield new, but apparently related features. On the Hilbert space of such a theory one has the action of left and right Virasoro algebras with generators L_n, L'_n and central extension c. The Hamiltonian is given by $H = L_0 + L'_0$ is the momentum by $P = L_0 - L'_0$. For our purposes it is convenient to shift these generators, such that $\tilde{L}_0 = L_0 - c/24$ and analogously for L'_0. Consider the partition function

$$Z(\tau, \bar{\tau}) = tr(\exp(2\pi i(\tilde{L}_0\tau - \tilde{L}'_0\bar{\tau})))$$

of such a theory, such that the imaginary part of τ can be identified with the inverse temperature.

The partition functions of conformal theories are invariant under modular transformations

$$\tau \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z) \ .$$

Using $\tau \to -1/\tau$, one reads off the high temperature behaviour

$$Z \sim \exp\left(\frac{\pi i}{12} c_{\text{eff}}(1/\tau - 1/\bar{\tau})\right) \ .$$

The quantity c_{eff} is called left and right effective central charge of the theory. Since the partition function diverges at large temperature, the effective central charges have to be non-negative. For a unitary theory, they coincide with the central charge c of the Virasoro algebra. More generally, c_{eff} is given by the maximum of $c - 24h$, where h runs over eigenvalues of L_0.

The set $\mathcal{C}$ of values of c_{eff} for all possible conformal theories is conjectured to share many properties of $\mathcal{V}$. It is additive, since the tensor product of theories yields the sum of the effective central charges. At least for rational theories, the effective central charges are rational, and this may be generally true. In fact, $\mathcal{C}$ is conjectured to be well ordered, with the same ordinal number ω^{ω} as $\mathcal{V}$.

For rational theories one has

$$Z(\tau,\bar{\tau}) = \sum_i Z_i(\tau) Z_i'(\bar{\tau}) \ ,$$

where the sum is finite and runs over the superselection sectors of the theory. Asymptotically, all the Z_i are proportional. The proportionality constants are called conformal dimensions. With $q = \exp(2\pi i \tau)$ the asymptotic behaviour is obtained for $|q|$ close to 1. One finds

$$Z_i(\tau) \sim \exp(-\frac{\pi^2}{6} c_{\text{eff}} / \log(q)).$$

The normalization is chosen such that $\delta_0 = 1$ for the vacuum character Z_0.

For $c_{\text{eff}} < 1$, the rational conformal theories are classified by pairs (p,q) of natural numbers > 1 without common prime divisors. One has $c_{\text{eff}} = 1 - 6/pq$, such that the possible values in $\mathcal{C}$ are given by $c_{\text{eff}} = 1 - 6/n$, where n runs over those natural numbers which are not prime powers. For $c_{\text{eff}} > 1$ not much is known.

Let us calculate partition functions and effective central charges for a few simple theories. The partition function of a free boson is proportional to

$$P(\tau) = \prod_k (1 - q^k)^{-1} \ ,$$

One has

$$P(\tau) = \sum_k p(k) q^k \ ,$$

where $p(k)$ counts the additive partitions of k into natural numbers. If we denote the number of partitions into exactly n natural numbers by $p_n(k)$, we find

$$Z(\tau) = \sum_n \sum_k p_n(k) q^k \ .$$

Any partition can be written in the form $k = m_1 + m_2 + \ldots + m_n$, where $m_i \geq m_{i+1}$. Writing $l_i = m_i - m_{i+1}$, the l_i are independent variables. Since $k = \sum_i il_i$, one finds

$$\sum_k p_n(k)q^k = (q)_n^{-1} ,$$

where

$$(q)_n = (1-q)(1-q^2)\ldots(1-q^n) .$$

For a free fermion, we have a very similar partition function, except that the m_i have to be different due to the Pauli principle. To get independent variables, we have to write $l_i = m_i - m_{i+1} - 1$. Moreover, we can use integral or half-integral k. In the latter case we obtain

$$Z_0(\tau) = q^{\tilde{h}_0} \sum_n q^{n^2/2}/(q)_n ,$$

in the former

$$Z_1(\tau) = q^{\tilde{h}_1} \sum_n q^{n(n+1)/2}/(q)_n .$$

The values of the $\tilde{h}_i$ are given by the lowest eigenvalues of $\tilde{L}_0$ in the coresponding superselection sector. Their minimal value is $-c_{\text{eff}}/24$.

Finally let us consider the conformal theory given by the Lee-Yang edge singularity. The theory is minimal, such that all holomorphic fields are generated by the energy momentum density $T(z)$. The normal ordered product $: TT :$ is proportional to the second derivative of T and does not yield an independent state. Taking Fourier coefficients, one sees that products of the form L_nL_n and L_nL_{n+1} can be reduced to simpler ones, which looks like an extended Pauli principle. For the partition function this means $m_i \geq m_{i+1} + 2$. With the independent variables $l_i = m_i - m_{i+1} - 2$ one finds

$$Z_0(\tau) = q^{\tilde{h}_0} \sum_n q^{n(n+1)}/(q)_n$$

and

$$Z_1(\tau) = q^{\tilde{h}_1} \sum_n q^{n^2}/(q)_n .$$

in the two superselection sectors.

When we tensorize r theories of this kind, the characters are multiplied. To write the product characters in analogous form, we consider n as a vector with r components and define $(q)_n = (q)_{n_1} \ldots (q)_{n_r}$. The exponent in the numerator becomes a quadratic form

$$Q(n) = \frac{1}{2}nBn + bn + \tilde{h} ,$$

with a diagonal $r \times r$ matrix B and a vector b which depends on the representation.

Other conformal theories have characters of analogous form, but with more general symmetric matrices B. Thus we consider characters of the form

$$\sum_{n_1,\dots,n_r} q^{Q(n)}/(q)_n \ .$$

To calculate c_{eff}, one has to evaluate all these sums for q close to 1. This can be done by a saddle point calculation. First one interpolates the summands by a continuous function of n, using

$$(q)_m^{-1} = (q)_\infty^{-1} \prod_k^\infty (1 - q^m q^k) \ .$$

The logarithm of the product is essentially the q-deformed dilogarithm of q^m. In leading order it can be replaced by $-Li_2(q^m)/\log(q)$. Now $(q)_\infty$ is essentially the Dedekind η-function, whose modular behaviour is well known. In particular, we have

$$\log(q)_\infty \sim Li_2(1)/\log(q)$$

in leading order.

Varying the n_i yields a stationary point for $x_i = q^{n_i}$ with

$$\log(1 - x_i) = \sum_j B_{ij} \log(x_i) \ .$$

For the effective central charge one obtains

$$c_{\text{eff}} = \sum_i L(1 - x_i)/L(1) \ ,$$

where we used the Rogers dilogarithm and its properties $L(1) = \pi^2/6$, $L(1) - L(x) = L(1 - x)$.

For the free boson, $B = 0$ and $c_{\text{eff}} = 1$. For the free fermion $B = 1/2$, which yields $c_{\text{eff}} = 1/2$. For the Lee-Yang edge singularity $B = 2$, such that $1 - x = x^2$, which yields the golden ratio. The five term identity immediately yields $5L(x) = 3L(1)$ and $c_{\text{eff}} = 2/5$.

The characters of conformal field theories are modular, which essentially means that the saddle point approximation gets no perturbative correction, in the sense of a power series expansion in τ. To find these corrections, one uses the expansion

$$\begin{aligned}\textstyle\sum_{k=1}^{\infty} \log(1 - uq^k) &= Li_2(u)/2\pi i\tau - \tfrac{1}{2}\log(1 - u) \\ &\quad + \textstyle\sum_m \sum_k \frac{u^m}{m}(2\pi i m\tau)^{2k+1} \frac{B_{2k+2}}{(k+2)!}\end{aligned}$$

with $u = q^n$. Here the B_k are the Bernoulli numbers, which apart from B_1 vanish for odd k. Finally one uses the fact that $q^{1/24}(q)_\infty$ is a modular form and has no perturbative corrections [NRT 93].

An alternative expression for these corrections can be obtained by the following method. First one writes the character in the form

$$\oint \sum_n u^{-n} q^{Q(n)} \sum_m u^m/(q)_m du/2\pi i u \ .$$

The first sum can be transformed by Poisson summation. For the second sum one uses

$$\sum_m u^m/(q)_m = \prod_{k=0}^{\infty}(1-uq^k)^{-1}$$

and its expansion given in the previous paragraph.

The perturbation expansion has exactly the same form as before, except for the substitution of τ by $-\tau$ and of $Q(n) = \frac{1}{2}nBn + bn + \tilde{h}$, by

$$Q'(n) = \frac{1}{2}nB^{-1}n + nB^{-1}b + \tilde{h}' \ ,$$

where

$$\tilde{h}' = \tilde{h} - \frac{r}{24} + \frac{1}{2}bB^{-1}b \ .$$

In other words, when a pair B, b yields partition functions without power law correction, then $B^{-1}, B^{-1}b$ does the same. The duality between B and B^{-1} generalizes the level rank duality known from the characters of Kac-Moody algebras, as we shall see.

First, however, let us study non-perturbative corrections. Using the Jacobi triple product identity

$$(1-u)\prod_n(1-uq^n)(1-q^n)(1-u^{-1}q^n) = \sum_n(-)^n u^n q^{n(n-1)/2}$$

and Poisson summation of the right hand side plus reverse application of the Jacobi triple product identity one finds

$$q^{\frac{1}{12}}\prod_n(1-uq^n)(1-u^{-1}q^n) = \frac{iu^{1/2}}{u-1}\tilde{u}^{-1/2}\tilde{q}^{\frac{1}{12}}e^{-\pi i z^2/\tau}F(\tilde{u},\tilde{q}) \ ,$$

where $u = exp(2\pi i z)$, $\tilde{u} = exp(-2\pi i z/\tau)$, $q = exp(2\pi i \tau)$, $\tilde{q} = exp(-2\pi i/\tau)$,

$$F(\tilde{u},\tilde{q}) = (1-\tilde{u})\prod_n(1-\tilde{u}\tilde{q}^n)(1-\tilde{u}^{-1}\tilde{q}^n) \ .$$

We split the right hand side into a factor regular at zero and one regular at ∞. In particular, the dilogarithm symmetry yields

$$\begin{aligned}&\tfrac{iu^{1/2}}{u-1}\tilde{u}^{-1/2}\tilde{q}^{\frac{1}{12}}e^{-\pi i z^2/\tau}\\ &= \exp((Li_2(u) + Li_2(u^{-1}))/2\pi i\tau - \tfrac{1}{2}\log(1-u) - \tfrac{1}{2}\log(1-u^{-1}) \ .\end{aligned}$$

By the residue theorem we have

$$\log F = -\frac{1}{2}\int_{-i\infty}^{i\infty} \log(1-e^{-2\pi i s/\tau})\left(\frac{e^{2\pi i s}u+1}{e^{2\pi i s}u-1} + \frac{e^{2\pi i s}+u}{e^{2\pi i s}-u}\right) ds \ .$$

This yields the unique splitting

$$\begin{aligned}&\textstyle\sum_{k=1}^{\infty}\log(1-uq^k) = Li_2(u)/2\pi i\tau - \tfrac{1}{2}\log(1-u) -\\ &\quad -\tfrac{1}{2}\textstyle\int_0^\infty \log(1-e^{-2\pi i s/\tau})\left(\tfrac{e^{2\pi i s}u+1}{e^{2\pi i s}u-1} + \tfrac{e^{2\pi i s}+u}{e^{2\pi i s}-u}\right) ds \ ,\end{aligned}$$

which is exact for $|u| < 1$ and takes care of non-perturbative terms.

Contour integration yield the contributions of characters with non-minimal $\tilde{h}$. One must be careful with its analytic continuation, since the dilogarithm has cuts at arguments $0, 1$. To handle these difficulties, it is convenient to define the Rogers dilogarithm $L(x)$ only modulo $4\pi^2$ and as a function of U, V, such that

$$e^U = 1 - e^V = x.$$

Starting from $0 < x < 1$ and real U, V, one obtains by analytic continuation a function $L(U, V)$ which is one valued modulo $4\pi^2$. The five term identity now takes the form

$$\sum_{i=1}^{5} L(U_i, V_i) = \pi^2/2 \qquad \mathrm{mod}\ 4\pi^2 \ ,$$

if $U_{i-1} + U_{i+1} = V_i$ and $U_{i+5} = U_i$, $V_{i+5} = V_i$. Note that U, V are well defined on the Riemann surface of the dilogarithm.They are found by deforming the integration contour into the Riemann surface of the dilogarithm and using the variables U, V instead of x. One finds stationary points for

$$V_i = \sum_j B_{ij} U_i \ .$$

For the $\tilde{h}$ we obtain

$$-\tilde{h} = \sum_i L(U_i, V_i)/(4\pi^2) \qquad \mathrm{mod}\,1 \ .$$

For any given stationary point we can find others by adding $2\pi i n_i$ to V_i and $2\pi i m_i$ to U_i, as long as the m_i, n_i are integral and $n = Bm$. This changes the value of $\tilde{h}$ by $nm/2$. For bosonic theories, the eigenvalues of L_0 are integrally space. Thus B must have a form such that nm only takes even values. Such matrices may be called even. As long as the matrix elements are integral, this terminology coincides with the usual condition that even matrices have even diagonal entries. For fermionic theories, half-integral spacing is allowed in the Neveu-Schwarz sector.

The sum $\sum_i [x_i]$ belongs to the Bloch group, since one checks easily $\sum_i [x_i] \wedge [1 - x_i] = 0$. Moreover, $\sum_i D(x_i) = 0$, since the eigenvalues $\tilde{h}$ of L_0 are real. It has been proven that this property implies that $\sum_i [x_i]$ is a torsion element of the Bloch group. In other words, for some N, $\sum_i N[x_i]$ vanishes by the five term relation.

Already at the present stage, we obtain a simple interpretation of level rank dualities. In the simplest case, this duality relates the $SU(N)$ Kac-Moody algebra at level M and the $SU(M)$ Kac-Moody algebra at level N. In particular, the central charges of the two theories add up to the integer $NM - 1$. In our formalism, every torsion element $\sum_{i=1}^{r} [x_i]$ of the Bloch group has a dual element $\sum_{i=1}^{r} [1 - x_i]$. The corresponding matrix B' is just the inverse of B, as expected. The effective central charges add up to the matrix rank r.

In terms of $L(U, V)$, the Bloch group is given by equivalence classes of sums $\sum_i n_i(U_i, V_i)$,

$$\exp(U_i) + \exp(V_i) = 1 \ ,$$

with the closure condition $\sum_i U_i \wedge V_i = 0$, using the ordinary wedge product over the vector space of complex numbers. If $N \sum_i U_i \wedge V_i$ is a sum of five term relations in U, V, the five term identity for $L(U, V)$ implies that the denominators of $\tilde{h}$ have to divide $8N$. This is obviously true for the examples considered above. Since the free fermion yields $N = 2$ and in particular $h = 1/16$, the result is the best possible of this form.

There is some hope to classify the matrices B which yield torsion elements. For $r = 1$, the only relevant elements are $[1/2]$, $[1 - \tau]$ and the golden ratio defined by $\tau + \tau^2 = 1$. The corresponding matrices for the first two cases are $B = 1$ and $B = 2$. The theories are the ones described above and the effective central charges are $1/2$, $2/5$. The first case is self-dual, the second one has a dual theory with $B = 1/2$ and effective central charge $3/5$. The free fermion is given by the (3,4) minimal model, the two others by the (2,5) and (3,5) models.

More generally, the vanishing of perturbative corrections to the saddle point approximation yields strong and calculable constraints on $B, b, \tilde{h}$.

For $r = 2$, M. Terhoeven made a classification under the plausible assumption that every allowed matrix B admits $b = 0$. In this case, one always finds $\tilde{h} = -c_{\mathrm{eff}}/24$, such that the level rank duality between the conformal dimensions just yields $c_{\mathrm{eff}} + c'_{eff} = r$.

For $r = 2$, this assumption yields three exceptional cases and one series. The exceptional cases are given by the (2,7), (3,7) and (3,8) minimal models and correspond to $B = 2\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, $B = \frac{1}{2}\left(\begin{smallmatrix}3&2\\2&4\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$. The series is of the form

$$B = \frac{1}{p+q}\begin{pmatrix} p & q \\ q & p \end{pmatrix}.$$

The corresponding characters are theta functions and the central extension is 1.

A relation of this series to the Dehn twists of the figure-eight knot seems evident, but so far the close formal analogies cannot yet be explained. In particular, it would be very interesting to relate the renormalization flow of the conformal models to the continuous interpolation between Dehn twist is considered in [NZ 85] and in [Yoshida 85].

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