Inhomogeneous charged black branes
— A novel stable phase —

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Abstract

The final fate of the Gregory-Laflamme (GL) instability is one of the most interesting problems in the black hole physics. To know the effects of charge in this context, we construct non-uniform black strings with a magnetic charge by higher-order perturbations. At the linear order of the perturbation, we see that the GL mode vanishes at the point where the background solution becomes thermodynamically stable. This vanishing resembles a second-order phase transition which is characterized by a universal power-law behavior. Solving the higher-order perturbations, the thermodynamical stability of the non-uniform strings in microcanonical, canonical and grandcanonical ensembles is investigated. We find that for fixed spacetime dimensions there exist critical charges at which the stability of non-uniform states changes. The charge can serve as a parameter that controls not only the stability of uniform black strings but also that of non-uniform states. Possible three dimensional (mass-tension-charge) phase diagram is proposed.

1 Introduction

It is known that black objects with translational symmetries, such as black branes and black strings, suffer from the Gregory-Laflamme (GL) instability, breaking the translational symmetries [1]. To know the endpoint of the instability, there are extensive studies to construct black objects in Kaluza-Klein (KK) spacetimes [2]. See [3] for reviews. One of the interesting features is that the phase structure, therefore the endpoint of the GL instability, would depend on the spacetime dimensions [4, 5].

Owing to the extensive studies, the phase structure of KK black holes in vacuum has been clarified gradually. However, the gravity in fundamental theories *inevitably* couples to other fields, such as gauge ones. Although we know that the linear perturbation of black branes strongly depends on the extremality of background solutions, the roles of extremality in non-linear regimes have not been known so far. Our understanding of the phase structure of charged black strings/branes is restricted to very special cases [6, 5], in which the systems can be translated to a vacuum system. The aim of this work is to see effects of a charge in non-linear regimes. We perform the higher-order static perturbations of magnetic black strings, and then investigate the thermodynamical properties of constructed solutions in detail. We see that the charge (extremality) controls the stability of black strings in non-linear regimes as well as in the linear regime [3].

2 Method: higher-order static perturbation

We consider $D = (d+1)$-dimensional spacetime in which the gravity couples to a $(d-2)$-form field $\mathcal{F}_{d-2}$. The governing equations are

$$R_{\mu\nu} = \frac{1}{2(d-3)!}\mathcal{F}_{\mu}{}^{\mu_2\cdots\mu_{d-2}}\mathcal{F}_{\nu\mu_2\cdots\mu_{d-2}} - \frac{d-3}{2(d-1)!}g_{\mu\nu}\mathcal{F}^2, \qquad \nabla_\mu \mathcal{F}^{\mu\mu_2\cdots\mu_{d-2}} = 0, \qquad \mathrm{d}\mathcal{F}_{d-2} = 0. \tag{1}$$

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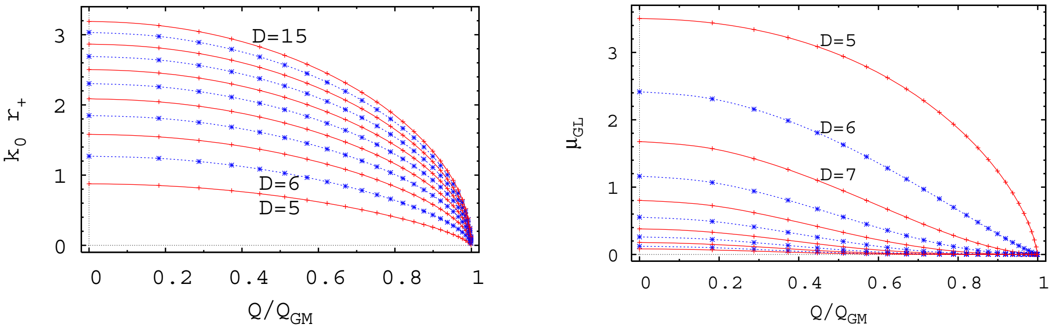
[3] This article corresponds to a short version of paper [7] with some new results and figures added.



(a) (b)

Figure 1: The charge dependance of the GL critical wavenumber, (a), and that of critical mass, (b).

We will find non-uniform black string solutions perturbatively with the metric ansatz given by

$$ds^2 = -e^{2a(r,z)} f_+ dt^2 + e^{2b(r,z)} f_- \left(\frac{dr^2}{f_+ f_-^2} + dz^2 \right) + e^{2c(r,z)} r^2 d\Omega_{d-2}^2,$$

$$f_\pm(r) = 1 - \left(\frac{r_\pm}{r} \right)^{d-3}, \qquad \mathcal{F}_{d-2} = Q_m \, \mathrm{Vol}_{\Omega_{d-2}}, \tag{2}$$

where $r = r_+$ and $r = r_-$ correspond to an outer and an inner horizons. Q_m is a constant proportional to a magnetic charge Q. By setting $a = b = c = 0$, we have a uniform black string solution. Note that the form field Eq. (2) is a *general* solution even for $a, b, c \neq 0$. This fact makes our analysis simple since we do not have to perturb the form field independently. The physical property of the background solution to be noted is that the specific heat, which specifies the thermodynamical stability, is negative for small charge $0 \leq Q < Q_{\rm GM}$ and positive for $Q_{\rm GM} < Q < M$, where $Q_{\rm GM}$ and M are a critical charge, characterized only by dimensions, and the mass, respectively. The Gubser-Mitra (or correlated stability) conjecture asserts that the GL instability exists iff the string is locally thermodynamically unstable [8]. Indeed, we will see that the GL instability does not exist for $Q > Q_{\rm GM}$.

We expand the metric function $X(r,z)$ $(X = a, b, c)$ around the uniform solution as

$$X(r,z) = \sum_{n=0}^{\infty} \epsilon^n X_n(r) \cos(nKz), \quad X_n(r) = \sum_{p=0}^{\infty} \epsilon^{2p} X_{n,p}(r), \quad K = \sum_{q=0}^{\infty} \epsilon^{2q} k_q, \tag{3}$$

where $X_{0,0}(r) = 0$ is imposed. Here, K is the GL critical wavenumber and ϵ is an expansion parameter. Substituting these expansions into the Einstein equations, we obtain ODEs for $X_{n,p}(r)$ at $\mathcal{O}(\epsilon^{n+2p})$.

3 Results: properties of non-uniform charged strings

Solving the first order perturbations, the GL critical mode for each given charge Q is obtained [10]. The charge dependence of the critical wavenumber is shown in Fig. 1(a). One can see that the critical wavenumber vanishes at $Q = Q_{\rm GM}$. In addition, we find that the vanishing of the wavenumber obeys a power law near the GM point, $k_0 \propto |Q - Q_{\rm GM}|^\beta$, irrespective of dimensions. The universal "critical exponent" β is nearly 1/2, which resembles a second-order phase transition [9]. In Fig. 1(b), we show the charge dependence of the conventionally-used dimensionless mass parameter for the critical string. The mass parameter in the present case is defined by

$$\mu \equiv \frac{16\pi G_D M}{L^{D-3}}, \tag{4}$$

where G_D is a D-dimensional gravitational constant and L is the compactification length of the z-direction. The critical mass $\mu_{\rm GL}$ is estimated by setting $L = 2\pi/k_0$. The uniform string can exist stably for a parameter region of $\mu > \mu_{\rm GL}$ and $0 \leq Q < M$ in the (μ, Q) plane.

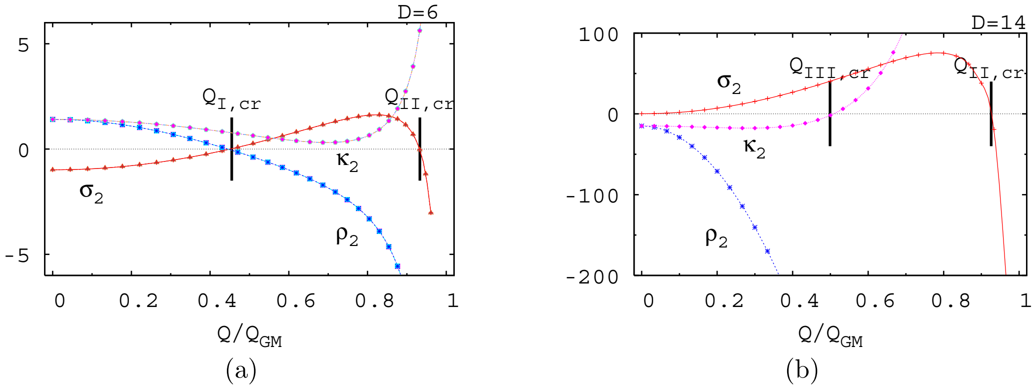


Figure 2: The difference of entropy and free energies between the uniform and non-uniform black strings in suitable ensembles. One can find three critical charges, at which the stability of non-uniform solution changes.

Solving the higher-order perturbations (up to third order), we can compare the entropy S, Helmholtz free energy F $(= M - TS)$ and Gibbs free energy G $(= M - TS - \nu Q)$ between Uniform and Non-Uniform black strings in suitable ensembles. We denote the differences of these thermodynamical functions by

$$S_{\rm NU}/S_{\rm U} \simeq 1 + \sigma_2 \epsilon^4, \qquad F_{\rm NU}/F_{\rm U} \simeq 1 + \rho_2 \epsilon^4, \qquad G_{\rm NU}/G_{\rm U} \simeq 1 + \kappa_2 \epsilon^4. \tag{5}$$

For example, the fourth-order coefficient of the Gibbs free energy is given by

$$\kappa_2 = -\frac{d-3}{2}\left[\delta S_1 + (d-2)\delta T_1\right]\delta T_1 - \frac{(d-1)q^{d-3}}{2(1-q^{d-3})}\left[\delta Q_1 + 2(d-3)\delta T_1 - \frac{1-(d-2)q^{d-3}}{1-q^{d-3}}\delta\nu_1\right]\delta\nu_1, \tag{6}$$

where $q \equiv r_-/r_+$ and $(\delta S_1, \delta T_1, \delta Q_1, \delta\nu_1)$ are the second-order changes of entropy, temperature, charge, and chemical potential, respectively. See [7] for the similar expressions for σ_2 and ρ_2. In Fig. 2, we show the charge dependence of these quantities for $D = 6$, (a), and $D = 14$, (b).

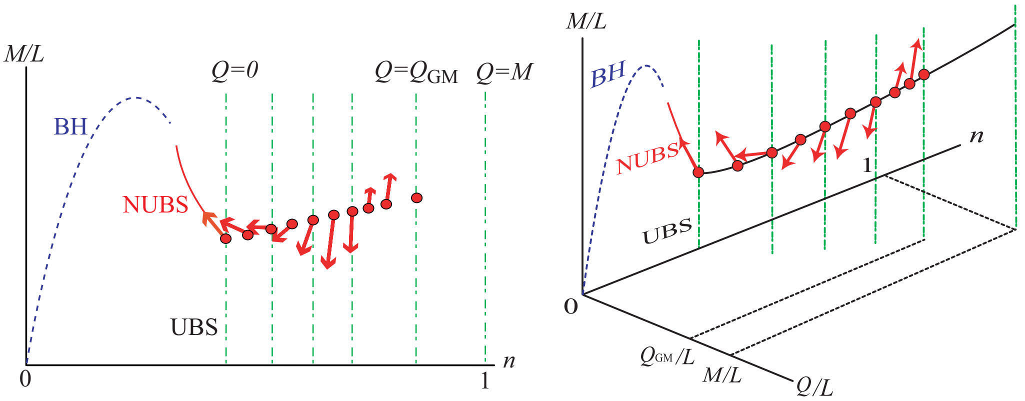
First, let us focus on the behavior of entropy difference σ_2 in $D = 6$. σ_2, being negative at $Q = 0$, increases as the background charge Q increases and become positive in an intermediate charge region, $Q_{\rm I,cr} < Q < Q_{\rm II,cr}$. For $Q_{\rm II,cr} < Q < Q_{\rm GM}$, σ_2 is negative again. The non-uniform black strings are entropically favored over the uniform one in the intermediate region. This also suggests that the phase transition from uniform to non-uniform phases is second (or higher) order. This behavior is common for $5 \leq D \leq 13$. For $D \geq 14$, the increase/decrease-behavior of σ_2 is similar but there is a different feature that σ_2 is positive at $Q = 0$ [4]. Therefore the first critical charge $Q_{\rm I,cr}$ does not exist for $D \geq 14$.

Next, let us see the the behavior of ρ_2. ρ_2 monotonically decreases as the charge increases for all dimensions. For vacuum cases it has been known that ρ_2 is positive for $5 \leq D \leq 12$ and negative for $D \geq 13$ [5, 11]. Thus, there exists a critical charge for $5 \leq D \leq 13$, which almost coincides with the critical charge $Q_{\rm I,cr}$.

Last, let us see the stability in a grandcanonical ensemble, which was not discussed in [7]. The Gibbs free energy is compared between the uniform and non-uniform strings with the temperature and chemical potential kept fixed. For $D = 6$, κ_2 is always positive to imply the non-uniform configuration is unstable for all non-extremal black strings. While, for $D = 14$ (in fact for $D \geq 13$) κ_2 is negative around the neutral point and increases as the background charge increases. Thus, a “third”critical charge $Q_{\rm III,cr}$, below which the non-uniform string is favored in the grandcanonical ensembles, appears.

4 Future prospects

The non-uniform charged black strings are constructed by the static perturbations for various dimensions. We have seen that the charge controls the stability of non-uniform strings as well as that of uniform strings.



(a) (b)

Figure 3: A possible (M, n) diagram, (a), and a (M, n, Q) diagram, (b) of Black Hole (BH), Uniform Black String (UBS) and Non-Uniform Black String (NUBS). Projecting (b) onto the $Q = 0$ plane, we obtain (a). Each arrow indicates the direction of the non-uniform string branch emanates.

We show possible phase diagrams in Fig. 3 suggested by the perturbation analysis in this paper. It is necessary to give some physical/intuitive understanding to the sign changes of $(\sigma_2, \rho_2, \kappa_2)$ and those phase diagrams. The application of the Landau-Ginzburg theory of phase transition would provide us good understanding of these [11]. The roles of the critical charges in the time evolution of GL instability and the gauge theories, predicted by the gauge/gravity dual, will be interesting [12].

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