

The 3-state Potts model and Rogers–Ramanujan series

Research Article

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Abstract: We explain the appearance of Rogers–Ramanujan series inside the tensor product of two basic $A_2^{(2)}$ -modules, previously discovered by the first author in [Feingold A.J., Some applications of vertex operators to Kac–Moody algebras, In: Vertex Operators in Mathematics and Physics, Berkeley, November 10–17, 1983, Math. Sci. Res. Inst. Publ., 3, Springer, New York, 1985, 185–206]. The key new ingredients are (5, 6) Virasoro minimal models and twisted modules for the Zamolodchikov \mathcal{W}_3 -algebra.

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1. Introduction

Virasoro minimal models and their characters have a long (and fruitful) history in conformal field theory, string theory and of course in vertex algebra theory. It is by now well known that the character $\chi_{s,t}^{m,n}(q)$ of the minimal model $L(c_{s,t}, h_{s,t}^{m,n})$ is given by

$$\chi_{s,t}^{m,n}(q) = \frac{q^{h_{s,t}^{m,n} - c_{s,t}/24}}{(q)_\infty} \cdot \sum_{k \in \mathbb{Z}} q^{st k^2} (q^{k(mt-ns)} - q^{(mt+ns)k+mn}),$$

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where $2 \leq s, t \in \mathbb{Z}$ are coprime, $1 \leq m < s$, $1 \leq n < t$, $(q)_\infty = \phi(q) = \prod_{i=1}^{\infty} (1 - q^i)$ and

$$c_{s,t} = 1 - \frac{6(s-t)^2}{st}, \quad h_{s,t}^{m,n} = \frac{(mt-ns)^2 - (s-t)^2}{4st}. \quad (1)$$

In particular,

$$\chi_{2,5}^{1,1}(q) = q^{11/60} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+2})(1 - q^{5n+3})}, \quad \chi_{2,5}^{1,2}(q) = q^{-1/60} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})} \quad (2)$$

give the product sides of the famous Rogers–Ramanujan series which appear in many papers on representation theory of Virasoro and affine Lie algebras.

The results in this note arose as an attempt to find a representation theoretic explanation of the following theorem obtained in 1983 by the first author [11]:

Theorem 1.1.

Denote by $L(\Lambda_1)$ the basic (level one) highest weight module for the twisted affine Kac–Moody algebra $A_2^{(2)}$. Then

$$L(\Lambda_1) \otimes L(\Lambda_1) = L(2\Lambda_1) \otimes V_1 \oplus L(\Lambda_0) \otimes V_2, \quad (3)$$

such that suitably normalized and scaled characters of the multiplicity spaces V_1 and V_2 coincide with the product sides of Rogers–Ramanujan series (2).

This result was obtained in the principal mod 6 realization of $A_2^{(2)}$ also studied in [4, 7, 23], so it is perhaps not obvious what is its reformulation in the language of vertex algebras and twisted modules. The above result is somewhat unexpected because the coset spaces, which are clearly unitary modules for the Virasoro algebra, are essentially given by the characters of the $(s, t) = (2, 5)$ non-unitary minimal models! It is also interesting to notice that the principally specialized characters of $L(2\Lambda_1)$ and $L(\Lambda_0)$ are also given by the above Rogers–Ramanujan series (again, suitably normalized and scaled).

To explain the appearance of Rogers–Ramanujan q -series we recall first a pair of identities which can be traced back to Bytsko and Fring [6] (for further identities, see [6] and also [19, 21]):

$$\chi_{5,6}^{1,2}(q) + \chi_{5,6}^{1,4}(q) = \chi_{2,5}^{1,1}(q^{1/2}), \quad \chi_{5,6}^{2,2}(q) + \chi_{5,6}^{2,4}(q) = \chi_{2,5}^{1,2}(q^{1/2}). \quad (4)$$

We also mention the related identities

$$\chi_{5,6}^{2,1}(q) - \chi_{5,6}^{2,5}(q) = \chi_{2,5}^{1,1}(q^2), \quad \chi_{5,6}^{1,1}(q) - \chi_{5,6}^{1,5}(q) = \chi_{2,5}^{1,2}(q^2). \quad (5)$$

Although all these formulas can be checked directly (see [Appendix](#)) they are very interesting for several reasons. Their right hand sides are clearly (scaled) Rogers–Ramanujan series. Also, unlike the identities studied in [21] and [19], the above formulas are among characters of modules of *different* central charges (in our case, of central charges $-22/5$ and $4/5$). The key observation now is that the central charge of the coset spaces V_1 and V_2 equals $4/5$, the central charge of $(5, 6)$ minimal models. So we are immediately led to the following conclusion: relations (4) should be interpreted as decomposition formulas of the coset spaces V_1 and V_2 into irreducible Virasoro characters of central charge $4/5$. As we shall see this is indeed the case. In fact, there is something even deeper going on. It turns out that V_1 and V_2 are in fact (the only twisted) irreducible modules for a larger rational vertex algebra which we end up calling $\mathcal{W}_3(4/5)$, also known as the Zamolodchikov \mathcal{W}_3 -algebra. This algebra has already appeared in the physics literature under the name 3-State Potts model and more recently in the vertex algebra theory [16]. We first show that there are (at least) four different ways of thinking about $\mathcal{W}_3(4/5)$.

Theorem 1.2.

The following vertex operator algebras are isomorphic:

- (a) The parafermionic space $K_{\mathfrak{sl}_2}(3,0) \subset L_{\mathfrak{sl}_2}(3,0)$ [10].
- (b) A certain subalgebra M^0 of the lattice vertex algebra V_L , where $L = \sqrt{2}Q$ and Q is the root lattice of type A_2 , see [16, 20].
- (c) The coset vertex algebra $\mathcal{W}_{L_{\mathfrak{sl}_3}(\Lambda_0) \otimes L_{\mathfrak{sl}_3}(\Lambda_0)}^{L_{\mathfrak{sl}_3}(2\Lambda_0)}$ [15].
- (d) The simple affine \mathcal{W} -algebra $\mathbf{L}(4/5)$, obtained via Drinfeld–Sokolov reduction [2, 12].

In physics literature equivalence of (a), (c) and (d) is more or less known. Construction (b) is more recent (again, see [16]).

Having enough knowledge about $\mathcal{W}_3(4/5)$ we can now return to the cosets V_1 and V_2 . These are not ordinary modules for $\mathcal{W}_3(4/5)$, but rather τ -twisted $\mathcal{W}_3(4/5)$ -modules. Our main result is the following theorem about $\mathcal{W}_3(4/5)$.

Theorem 1.3.

For the rational vertex algebra $\mathcal{W}_3(4/5)$ the following holds:

- (a) $\text{Aut}(\mathcal{W}_3(4/5)) = \mathbb{Z}_2$.
- (b) If we denote by τ the nontrivial automorphism of $\mathcal{W}_3(4/5)$, then $\mathcal{W}_3(4/5)$ is τ -rational (i.e. $\mathcal{W}_3(4/5)$ has finitely many irreps and every τ -twisted module is completely reducible).
- (c) The algebra $\mathcal{W}_3(4/5)$ has precisely two inequivalent representations,

$$\mathcal{W}^\tau\left(\frac{1}{40}\right) = L\left(\frac{4}{5}, \frac{1}{40}\right) \oplus L\left(\frac{4}{5}, \frac{21}{40}\right) \quad \text{and} \quad \mathcal{W}^\tau\left(\frac{1}{8}\right) = L\left(\frac{4}{5}, \frac{1}{8}\right) \oplus L\left(\frac{4}{5}, \frac{13}{8}\right).$$

$$(d) \quad \chi(\mathcal{W}^\tau(1/40)) = \chi_{2,5}^{1,2}(q^{1/2}) \text{ and } \chi(\mathcal{W}^\tau(1/8)) = \chi_{2,5}^{1,1}(q^{1/2}).$$

Now we connect results from Theorem 1.1 with those in Theorem 1.3. Denote by σ the principal automorphism of order 6 of \mathfrak{sl}_3 , and let $L_{\mathfrak{sl}_3}(\Lambda_0)$ be the affine vertex algebra associated to $\widehat{\mathfrak{sl}_3}$ of level one. The basic $A_2^{(2)}$ -module $L_{\mathfrak{sl}_3}^\sigma(\Lambda_1)$ can be viewed as a σ -twisted $L_{\mathfrak{sl}_3}(\Lambda_0)$ -module, and its character is given by

$$\text{tr}_{L(\Lambda_1)} q^{L^\sigma(0)-c/24} = q^{-1/72} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{(6n+1)/6})(1 - q^{(6n+5)/6})},$$

where $L^\sigma(0)$ is the σ -twisted Virasoro operator.

Similarly, the characters of the two standard level two $A_2^{(2)}$ -modules (if viewed as σ -twisted $L_{\mathfrak{sl}_3}(2\Lambda_0)$ -modules) are given by

$$\text{tr}_{L(2\Lambda_1)} q^{L^\sigma(0)-c/24} = q^{-1/72} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{(6n+1)/6})(1 - q^{(6n+5)/6})} \chi_{2,5}^{1,2}(q^{1/3})$$

and

$$\text{tr}_{L(\Lambda_0)} q^{L^\sigma(0)-c/24} = q^{1/6-1/72} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{(6n+1)/6})(1 - q^{(6n+5)/6})} \chi_{2,5}^{1,1}(q^{1/3}).$$

Now, formula (3) implies a q -series identity

$$q^{-1/72} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{(6n+1)/6})(1 - q^{(6n+5)/6})} = \chi_{2,5}^{1,2}(q^{1/3}) \chi_{2,5}^{1,2}(q^{1/2}) + \chi_{2,5}^{1,1}(q^{1/3}) \chi_{2,5}^{1,1}(q^{1/2}),$$

after we canceled $\text{tr}_{L(\Lambda_1)} q^{L^\sigma(0)-c/24}$ from both sides of (3). This formula can be traced back to Ramanujan. This gives the main result of our paper. □

Theorem 1.4.

As $(\sigma \times \tau)$ -twisted $L_{\text{sl}_3}(2\Lambda_0) \otimes \mathcal{W}_3(4/5)$ -module,

$$L_{\text{sl}_3}^\sigma(\Lambda_1) \otimes L_{\text{sl}_3}^\sigma(\Lambda_1) \cong L_{\text{sl}_3}^\sigma(2\Lambda_1) \otimes \mathcal{W}^\tau\left(\frac{1}{40}\right) \oplus L_{\text{sl}_3}^\sigma(\Lambda_0) \otimes \mathcal{W}^\tau\left(\frac{1}{8}\right).$$

We also have

$$\text{tr}_{V_1} q^{L(0)-c/24} = \chi_{2,5}^{1,2}(q^{1/2}), \quad \text{tr}_{V_2} q^{L(0)-c/24} = \chi_{2,5}^{1,1}(q^{1/2}).$$

where $L(0)$ is the coset Virasoro operator for the tensor product.

Finally, we mention that (5) can also be explained in terms of representations of $\mathcal{W}_3(4/5)$. This requires a modular invariance theorem for τ -twisted modules [9]. This was pursued in the last section.

Remark 1.5.

The algebra $\mathcal{W}_3(4/5)$ and some of its modules appear also in [18], where the coset Virasoro construction is applied to the investigation of the branching rule decomposition of level-1 irreducible $E_6^{(1)}$ -modules with respect to the affine subalgebra $F_4^{(1)}$.

2. The 3-state Potts model vertex algebra

Let us recall a few basic facts about the Virasoro algebra and its representation theory. We use $M(c, h)$ to denote the Virasoro Verma module of central charge c and lowest conformal weight h , and denote its lowest weight vector by $v_{c,h}$. We let $V(c, 0) = M(c, 0)/\langle L(-1)v_{c,0} \rangle$, the vacuum vertex algebra. We denote by $L(c, h)$ the unique irreducible quotient of $M(c, h)$.

Recall (1). We will focus on the central charge $c_{5,6} = 4/5$. It is well known that $L(4/5, 0)$, viewed as a vertex algebra, has (up to equivalence) precisely 10 irreducible modules [22]:

$$\begin{aligned} L\left(\frac{4}{5}, 0\right), \quad L\left(\frac{4}{5}, \frac{1}{8}\right), \quad L\left(\frac{4}{5}, \frac{2}{3}\right), \quad L\left(\frac{4}{5}, \frac{13}{8}\right), \quad L\left(\frac{4}{5}, 3\right), \\ L\left(\frac{4}{5}, \frac{2}{5}\right), \quad L\left(\frac{4}{5}, \frac{1}{40}\right), \quad L\left(\frac{4}{5}, \frac{1}{15}\right), \quad L\left(\frac{4}{5}, \frac{21}{40}\right), \quad L\left(\frac{4}{5}, \frac{7}{5}\right). \end{aligned} \tag{6}$$

It is also known that $L(4/5, 0) \oplus L(4/5, 3)$ can be equipped with a simple vertex operator algebra structure, cf. [16]. Of course, one can always define a vertex operator algebra structure on $V \oplus M$, where M is any module with integral grading by defining the action of M on M to be trivial. But such vertex algebra is not simple. As we shall see, there is a unique vertex operator algebra structure on $L(4/5, 0) \oplus L(4/5, 3)$.

Let us also recall that the space of irreducible characters of $L(4/5, 0)$ is 10-dimensional (i.e. the characters are linearly independent).

Proposition 2.1.

Let $(V, Y, 1)$ be a vertex algebra such that $V \cong L(4/5, 0) \oplus L(4/5, 3)$ as a module for the Virasoro algebra and such that $Y|_{L(4/5, 3) \otimes L(4/5, 3)} \neq 0$. If another vertex algebra $W \cong L(4/5, 0) \oplus L(4/5, 3)$ satisfies the same property, then $W \cong V$.

Proof. We certainly have an isomorphism $f = f_1|_{L(4/5, 0)} \oplus f_2|_{L(4/5, 3)}$ between V and W viewed as Virasoro modules (f is unique up to a choice of two nonzero scalars). Because of $Y(1, x) = \text{id}$, the map f_1 is uniquely determined sending the vacuum of V to the vacuum of W . Observe that in V , $\tilde{Y}_V = Y|_{L(4/5, 3) \otimes L(4/5, 3)}$ defines an intertwining operator of type

$$\begin{pmatrix} L(4/5, 0) \\ L(4/5, 3) & L(4/5, 3) \end{pmatrix},$$

otherwise this would contradict

$$I \begin{pmatrix} L(4/5, 3) & \\ L(4/5, 3) & L(4/5, 3) \end{pmatrix} = 0.$$

The same holds for \tilde{Y}_W . According to [13], this intertwining operator is unique up to a nonzero constant. Thus, after identification via f , we can find $v \neq 0$ such that $v\tilde{Y}_V = \tilde{Y}_W$. Therefore $(u, v) \mapsto (f_1(u), f_2(v)/\lambda)$, $\lambda^2 = v$, defines the wanted automorphism between V and W . \square

Existence of the vertex operator algebra satisfying the conditions in Proposition 2.1 has been established in [16]. We shall denote it by $\mathcal{W}_3(4/5)$. More precisely, we have [16]

Theorem 2.2.

The vertex algebra $\mathcal{W}_3(4/5)$ is rational with the following irreducible modules (we also write their decompositions viewed as Vir -modules):

$$\begin{aligned} \mathcal{W}(0) &= L\left(\frac{4}{5}, 0\right) \oplus L\left(\frac{4}{5}, 3\right), & \mathcal{W}\left(\frac{2}{5}\right) &= L\left(\frac{4}{5}, \frac{2}{5}\right) \oplus L\left(\frac{4}{5}, \frac{7}{5}\right), \\ \mathcal{W}\left(\frac{2}{5}, +\right) &= L\left(\frac{4}{5}, \frac{2}{3}\right), & \mathcal{W}\left(\frac{2}{5}, -\right) &= L\left(\frac{4}{5}, \frac{2}{3}\right), \\ \mathcal{W}\left(\frac{1}{15}, +\right) &= L\left(\frac{4}{5}, \frac{1}{15}\right), & \mathcal{W}\left(\frac{1}{15}, -\right) &= L\left(\frac{4}{5}, \frac{1}{15}\right). \end{aligned}$$

Now we prove the equivalence of (b) and (d) in Theorem 1.2.

Theorem 2.3.

Vertex algebras $\mathcal{W}_3(4/5)$ and $L(4/5)$ are isomorphic.

Proof. Recall first that $L(4/5)$ is the unique irreducible quotient of the universal affine \mathcal{W} -algebra $M_{\text{sl}_3}(4/5)$, modulo the maximal ideal. We can think of $M_{\text{sl}_3}(4/5)$ as the vertex algebra obtained via Drinfeld–Sokolov reduction from a universal affine vertex algebra associated to sl_3 , or constructed by using free fields via Miura transformation. Either way, it is known that $M_{\text{sl}_3}(4/5)$ is freely generated by the conformal vector ω and another *primary vector* of degree three, $w_{-1}\mathbf{1}$, that is $L(0)w_{-1}\mathbf{1} = 3w_{-1}\mathbf{1}$ and $L(n)w_{-1}\mathbf{1} = 0$ for $n \geq 1$. It is also known that $Y(w_{-1}\mathbf{1}, x)w_{-1}\mathbf{1}$ is nonzero both in $M_{\text{sl}_3}(4/5)$ and in the quotient $L_{\text{vac}}(4/5)$. Let us record

$$\text{tr}_{M_{\text{sl}_3}(4/5)} q^{L(0)} = \frac{1}{(q^2; q)_\infty (q^3; q)_\infty},$$

where $(a; q)_\infty = \prod_{i=0}^\infty (1 - aq^i)$. More importantly, by using Frenkel–Kac–Wakimoto’s character formula [12] (proven by Arakawa [2]), we compute the character of $L(4/5)$ as

$$\text{tr}_{L(4/5)} q^{L(0)-1/12} = q^{-1/12} (q)_\infty^{-2} \sum_{m,n \in \mathbb{Z}} \sum_{w \in S_3} \epsilon_w q^{|5w(\rho) + 20n\alpha_1 + 20m\alpha_2 - 4\rho|^2/40},$$

where S_3 is the Weyl group of sl_3 , α_1 and α_2 are simple roots, ρ is the half-sum of positive roots and $|\lambda|^2 = \langle \lambda, \lambda \rangle$ is normalized such that $|\alpha|^2 = 2$ for each simple root α . An easy computation shows that

$$\text{tr}_{L(4/5)} q^{L(0)-1/12} = q^{-1/30} (1 + q^2 + 2q^4 + 3q^6 + \dots).$$

In fact we can show (by expanding both sides in q -series) that the following identity holds for $m \leq 50$:

$$\text{tr}_{L(4/5)} q^{L(0)-1/24} \equiv \chi_{5,6}^{1,1}(q) + \chi_{5,6}^{1,5}(q) \pmod{q^m}. \quad (7)$$

Thus, the character of $L(4/5)$ equals the character of $\mathcal{W}_3(4/5)$ up to degree 50. Let us consider the Virasoro submodule $M \subset L(4/5)$ generated by ω and $w_{-1}\mathbf{1}$. By Proposition 2.1, it is sufficient to show $U(\text{Vir}) \cdot \mathbf{1} = L(4/5, 0) \subset L(4/5)$ and $U(\text{Vir}) \cdot w_{-1}\mathbf{1} = L(4/5, 3)$. Indeed, if that is the case, then $L(4/5, 0)$ is a rational subalgebra of $L(4/5)$, which decomposes as a sum of $L(4/5, 0)$ -modules with integral conformal weights. But classification of $L(4/5, 0)$ -modules and (7) imply $L(4/5) \cong L(4/5, 0) \oplus L(4/5, 3)$, and the rest of the proof now follows from Proposition 2.1.

Denote by M_1 the cyclic Virasoro module $U(\text{Vir}) \cdot \mathbf{1}$ and by M_2 the cyclic module $U(\text{Vir}) \cdot w_{-1}\mathbf{1}$. By the universal property of Verma modules, these cyclic modules are quotients of $M(4/5, 0)$ (even $V(4/5, 0)$) and $M(4/5, 3)$, respectively. We claim that $M_1 \cap M_2 = \{0\}$. If not, $M_1 \cap M_2$ is a nontrivial submodule of a quotient of $M(4/5, 0)$ and of $M(4/5, 3)$. Embedding structure for Verma modules among the minimal series shows that this is impossible (the two modules belong to different blocks). If we let $\chi_W(q) = \text{tr}_W q^{L(0)-c/24}$, we get

$$\chi_{L(4/5)}(q) \geq \chi_{M_1}(q) + \chi_{M_2}(q) \geq \chi_{5,6}^{1,1}(q) + \chi_{5,6}^{1,5}(q),$$

where \geq has an obvious meaning for two q -series with non-negative integer coefficients. Now, relation (7) implies

$$\chi_{M_1}(q) + \chi_{M_2}(q) \equiv \chi_{5,6}^{1,1}(q) + \chi_{5,6}^{1,5}(q) \pmod{q^{50}}.$$

Thus $\chi_{M_1}(q) = \chi_{5,6}^{1,1}(q)$ and $\chi_{M_2}(q) = \chi_{5,6}^{1,5}(q)$, and hence $U(\text{Vir})\mathbf{1} = L(4/5, 0)$ and $U(\text{Vir})w_{-1}\mathbf{1} = L(4/5, 3)$. \square

Remark 2.4.

The previous theorem gives a representation theoretic proof of the q -series identity

$$q^{-1/12}(q)_\infty^{-2} \cdot \sum_{m,n \in \mathbb{Z}} \sum_{w \in S_3} \epsilon_w q^{|5w(\rho) + 20n\alpha_1 + 20m\alpha_2 - 4\rho|^2/40} = \chi_{5,6}^{1,1}(q) + \chi_{5,6}^{1,5}(q),$$

which presumably can be checked directly by applying methods similar to those used in [Appendix](#).

Theorem 2.5.

Denote by $\mathcal{W}_{\text{sl}_3(2)}^{\text{sl}_3(1) \otimes \text{sl}_3(1)}$ the coset vertex algebra obtained via the embedding $L_{\text{sl}_3}(2, 0) \hookrightarrow L_{\text{sl}_3}(1, 0) \otimes L_{\text{sl}_3}(1, 0)$. Then, as vertex operator algebras, $\mathcal{W}_3(4/5) \cong \mathcal{W}_{\text{sl}_3(2)}^{\text{sl}_3(1) \otimes \text{sl}_3(1)}$.

Proof. Observe first that the central charge of the coset vertex algebra is

$$4 - \frac{16}{5} = \frac{4}{5},$$

with the Virasoro generator

$$\omega_1 \otimes \mathbf{1} + \mathbf{1} \otimes \omega_1 - \omega_2,$$

where ω_k stands for the Sugawara generator of level $k \neq -3$. This coset is unitary so (as a Virasoro module, or $L(4/5, 0)$ -module) it decomposes as a direct sum of irreducible modules of central charge $4/5$. The graded dimension can be now easily computed by using the rationality of $L(4/5, 0)$. Alternatively, we can recall a result from [15], where the character of $\mathcal{W}_{\text{sl}_3(2)}^{\text{sl}_3(1) \otimes \text{sl}_3(1)}$ is computed by using modular invariance. Either way,

$$\mathcal{W}_{\text{sl}_3(2)}^{\text{sl}_3(1) \otimes \text{sl}_3(1)} = L\left(\frac{4}{5}, 0\right) \oplus L\left(\frac{4}{5}, 3\right)$$

as Virasoro modules. Finally to finish the proof we need

$$Y_{L(4/5, 3) \otimes L(4/5, 3)} \neq 0$$

in the coset algebra. This was proven by Bowknegt et al. in [3, 5], where it was shown that $\mathcal{W}_{\text{sl}_3(2)}^{\text{sl}_3(1) \otimes \text{sl}_3(1)}$ has a degree 3 generator $w_{-1}\mathbf{1}$ such that the brackets $[w_n, w_m]$ satisfy the relations as in $L(4/5)$. \square

3. τ -twisted $\mathcal{W}_3(4/5)$ -modules and the structure of $A_\tau(\mathcal{W}_3(4/5))$

In this part V is a vertex operator algebra and $\tau \in \text{Aut } V$ has order two. Recall the notion of a weak τ -twisted V -module M . By definition we require a decomposition $V = V^0 \oplus V^1$, and the twisted vertex operator map $Y^\tau(\cdot, x)$

$$Y^\tau(u, x) \in (\text{End } M)[[x^{1/2}, x^{-1/2}]]$$

such that

$$\begin{aligned} Y^\tau(u, x) &= \sum_{n \in \mathbb{Z} + r/2} u(n) x^{-n-1}, & u \in V^r, \quad u(n)v = 0, \quad n \gg 0, \\ Y^\tau(1, x) &= \text{Id}_M, \end{aligned}$$

and the Jacobi identity holds.

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y^\tau(u, x_1) Y^\tau(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y^\tau(v, x_2) Y^\tau(u, x_1) \\ = x_2^{-1} \left(\frac{x_1 - x_0}{x_2} \right)^{r/2} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y^\tau(Y(u, x_0)v, x_2), \end{aligned} \tag{8}$$

where $u \in V^r$ and $v \in V^s$. Clearly, from the Jacobi identity it follows that M is an untwisted (weak) V^0 -module.

- (1) If we also require M to be graded $M = \bigoplus_v M_v$, where the grading is induced by the spectrum of $L(0)$ and is bounded from below, and has finite dimensional graded components, we say that M is a τ -twisted V -module.
- (2) If we instead require M to be $(\mathbb{N}/2)$ -gradable, such that $M = \bigoplus_{n \in \mathbb{N}/2} M(n)$, $u(m)M(n) \subset M(n+m-\deg u-1)$, then M is called *admissible* [8].

Let M be an irreducible τ -twisted V -module. Then there is λ such that we have the following decomposition with respect to the spectrum of $L(0)$:

$$M = \bigoplus_{n=0}^{\infty} (M_{\lambda+n} \oplus M_{\lambda+n+1/2}), \tag{9}$$

so that if we let $M^i = M_{\lambda+n+i/2}$, $i \in \{0, 1\}$, the vertex operator map $Y^\tau(\cdot, x)$ is compatible with this \mathbb{Z}_2 -decomposition, a consequence of

$$u(m)M_v \subset M_{v+m-\deg u-1},$$

for homogeneous u . Again, M^i is a V^0 -module for $i \in \{0, 1\}$. A vertex operator algebra V is said to be τ -rational if every admissible τ -twisted V -module is completely reducible [8].

We also discuss intertwining operators among irreducible (ordinary) V -modules [13]. Without giving the full definition, let us record that an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1 \end{pmatrix} \begin{pmatrix} W_3 \\ W_2 \end{pmatrix}$ is a linear map $\mathcal{Y}(u, x) \in \text{Hom}(W_2, W_3)\{x\}$, $u \in W_1$, such that (among other things) the following Jacobi identity holds:

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) \mathcal{Y}(v, x_2) w - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(v, x_2) Y(u, x_1) w = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y(u, x_0)v, x_2) w,$$

where $u \in V$, $v \in W_1$ and $w \in W_2$.

Let us now focus on the twisted Jacobi identity (8) when $u \in V^1$ and $v \in V^0$. We shall see that the corresponding identity is an essentially intertwining operator map between V^0 -modules. Because $u \in V^1$, then $Y^\tau(u, x)$ restricted on M^i is mapped to M^{i+1} where we use the mod 2 exponent notation. Denote by $I(u, x)$ this restriction. Since $v \in V^0$,

its action on M^i is the usual V^0 -module action, so we write $Y_0(u, x)$ instead of $Y^\tau(u, x)$. The twisted Jacobi identity now reads (after we apply it to a vector w):

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) I(u, x_1) Y_0(v, x_2) w - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_0(v, x_2) I(u, x_1) w \\ = x_2^{-1} \left(\frac{x_1 - x_0}{x_2} \right)^{1/2} \delta \left(\frac{x_1 - x_0}{x_2} \right) I(Y(u, x_0)v, x_2) w. \end{aligned} \quad (10)$$

Now, apply the substitution $x_1 \leftrightarrow x_2$ and $x_0 \mapsto -x_0$. The identity then is

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_0(v, x_1) I(v, x_2) w - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) I(u, x_2) Y_0(v, x_1) w \\ = x_1^{-1} \left(\frac{x_2 + x_0}{x_1} \right)^{1/2} \delta \left(\frac{x_2 + x_0}{x_1} \right) I(Y_0(u, -x_0)v, x_1) w. \end{aligned} \quad (11)$$

Because of $Y(u, x)v = e^{xL(-1)}Y_0(v, -x)u$ (the skew-symmetry), the right hand-side can be rewritten as

$$x_1^{-1} \left(\frac{x_2 + x_0}{x_1} \right)^{1/2} \delta \left(\frac{x_2 + x_0}{x_1} \right) I(e^{-x_0L(-1)}Y_0(v, x_0)u, x_1) w = x_1^{-1} \left(\frac{x_2 + x_0}{x_1} \right)^{1/2} \delta \left(\frac{x_2 + x_0}{x_1} \right) I(Y_0(v, x_0)u, x_1 - x_0) w.$$

To finish the proof observe first that, by twisted weak associativity, we can always choose positive $k \in \mathbb{N}$ such that

$$x_2^{k+1/2} (x_2 + x_0)^k I(Y_0(v, x_0)u, x_2) w$$

involves only positive (integral!) powers of the variable x_2 . Consider

$$x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_1^k x_2^{k+1/2} I(Y_0(v, x_0)u, x_2) w), \quad (12)$$

which also equals

$$x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) ((x_2 + x_0)^k x_2^{k+1/2} I(Y_0(v, x_0)u, x_2) w).$$

Now, we are allowed to replace in (12) the x_2 variable with $x_1 - x_0$, so we get

$$x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (x_1^k x_2^k (x_1 - x_0)^{1/2} I(Y_0(v, x_0)u, x_1 - x_0) w).$$

Finally, we multiply the last expression by $x_2^{-k-1/2} x_1^{-k}$ and obtain

$$x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) I(Y_0(v, x_0)u, x_2) w = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \left(\frac{x_1 - x_0}{x_2} \right)^{1/2} I(Y_0(v, x_0)u, x_1 - x_0) w.$$

Consequently, (11) and the last formula imply the following result.

Proposition 3.1.

The map $I(\cdot, x)$ defines an intertwining operator among V^0 -modules of type $\left(\begin{smallmatrix} M^{i+1} \\ V^1 & M^i \end{smallmatrix}\right)$. The identity (10) is equivalent to the Jacobi identity for $I(\cdot, x)$.

We also have the following useful result.

Lemma 3.2.

Let M be an irreducible τ -twisted V -module for a simple vertex algebra V . Then for every nonzero $u \in V$, $m \in M$,

$$Y^\tau(u, x)m \neq 0.$$

The proof is clear (otherwise $Y^\tau(a, x)m = 0$ for every $a \in V$, which is impossible because M is cyclic). This statement, in particular, yields $M^1 \neq 0$ and $M^0 \neq 0$.

We shall need a few results about τ -twisted Zhu's algebra $A_\tau(V) = V/O(V)$ following [8], where $O(V)$ is the span of vectors of the form

$$a \circ b = \text{Res}_x \frac{(1+x)^{\deg a - 1 + \delta_r + r/2}}{x^{1+\delta_r}} Y(a, x)b,$$

where $a \in V^r$ and where $\delta_0 = 1$ and $\delta_1 = 0$, and the multiplication $*$ on $A_\tau(V)$ is induced via

$$a * b = \text{Res}_x \frac{(1+x)^{\deg a}}{x} Y(a, x)b.$$

For every $a \in V^0$ and $b \in V^1$ we have $a * b = 0 \bmod O(V)$, so $b = 0$ as an element in $A_\tau(V)$.

Now we specialize $V = \mathcal{W}_3(4/5) = L(4/5, 0) \oplus L(4/5, 3)$.

Lemma 3.3.

The τ -twisted Zhu algebra $A_\tau(\mathcal{W}_3(4/5))$ is a quotient of the polynomial algebra $k[x]$.

Proof. Denote by π the natural projection from $\mathcal{W}_3(4/5)$ to $A_\tau(\mathcal{W}_3(4/5))$. Then $\pi(v) = 0$ for $v \in A(\mathcal{W}_3(4/5))_1$ [8]. If we denote $A_0(\mathcal{W}_3(4/5)) = A(\mathcal{W}_3(4/5))$, where $A(\mathcal{W}_3(4/5))$ is the usual Zhu's algebra of $\mathcal{W}_3(4/5)$, we clearly have an isomorphism $A_\tau(\mathcal{W}_3(4/5)) \cong A(\mathcal{W}_3(4/5))_0/I$ where I is a certain ideal. \square

We are primarily interested in irreducible τ -twisted $\mathcal{W}_3(4/5)$ -modules. Every such module is an ordinary module for $L(4/5, 0)$, so it decomposes as a direct sum of (ordinary) modules given on the list in (6). Because of (9), any irreducible τ -twisted $\mathcal{W}_3(4/5)$ -module is heavily constrained with respect to the spectrum of $L(0)$. More precisely,

Proposition 3.4.

Let M be an irreducible τ -twisted $\mathcal{W}_3(4/5)$ -module as in (9). Then M , viewed as an $L(4/5, 0)$ -module, is isomorphic to either

$$\bigoplus_{i \in I} L\left(\frac{4}{5}, \frac{1}{40}\right) \oplus \bigoplus_{j \in J} L\left(\frac{4}{5}, \frac{21}{40}\right) \quad \text{or} \quad \bigoplus_{k \in K} L\left(\frac{4}{5}, \frac{1}{8}\right) \oplus \bigoplus_{l \in L} L\left(\frac{4}{5}, \frac{13}{8}\right),$$

where I, J, K, L are finite sets.

Proof. The spectrum of $L(0)$ on $M = M^0 \oplus M^1$ must be contained inside the set $\lambda + \mathbb{N}/2$, where λ is the lowest weight of irreducible $L(4/5, 0)$ -modules (6). Also, in addition $M^i \neq 0$. This implies that the absolute value of the difference between the lowest conformal weights of M^0 and of M^1 must lie within the set $\mathbb{N} + 1/2$. An easy inspection of allowed weights gives two possibilities: $\lambda = 3/8$ or $\lambda = 1/40$. \square

The next result follows from the fusion rules for $L(4/5, 0)$ [16, 22] (see also [14]).

Lemma 3.5.

The module $L(4/5, 3)$ is a simple current (i.e. it permutes equivalence classes of irreducible modules under the fusion product \times). In particular,

$$\begin{aligned} L\left(\frac{4}{5}, 3\right) \times L\left(\frac{4}{5}, \frac{13}{8}\right) &= L\left(\frac{4}{5}, \frac{1}{8}\right), & L\left(\frac{4}{5}, 3\right) \times L\left(\frac{4}{5}, \frac{1}{8}\right) &= L\left(\frac{4}{5}, \frac{13}{8}\right), \\ L\left(\frac{4}{5}, 3\right) \times L\left(\frac{4}{5}, \frac{1}{40}\right) &= L\left(\frac{4}{5}, \frac{21}{40}\right), & L\left(\frac{4}{5}, 3\right) \times L\left(\frac{4}{5}, \frac{21}{40}\right) &= L\left(\frac{4}{5}, \frac{1}{40}\right). \end{aligned}$$

Now let us examine the number of irreducible summands in decompositions in Proposition 3.4.

Proposition 3.6.

Let M be as in Proposition 3.4. Then, viewed as a Vir -module,

$$M \cong L\left(\frac{4}{5}, \frac{1}{40}\right) \oplus L\left(\frac{4}{5}, \frac{21}{40}\right) \quad \text{or} \quad M \cong L\left(\frac{4}{5}, \frac{1}{8}\right) \oplus L\left(\frac{4}{5}, \frac{13}{8}\right).$$

Proof. We prove the first assertion. Because the twisted module in question is irreducible and $A_\tau(\mathcal{W}_3(4/5))$ is commutative, the top level must be an (irreducible) one-dimensional module. Therefore the set J must be a singleton. If $|J| \geq 2$, the direct sum $\bigoplus_{j \in J} L(4/5, 21/40)$ decomposes into at least two irreducible $L(4/5, 0)$ -modules. To rule out this case we apply the same argument as in the proof of [16, Lemma 5.3]. This argument and Lemma 3.5 yield a $\mathcal{W}_3(4/5)$ -module on $L(4/5, 21/40)$, such that $L(4/5, 0)$ acts trivially on it. But this is clearly a contradiction, so $|J| = 1$. \square

Theorem 3.7.

Let M be an irreducible τ -twisted $\mathcal{W}_3(4/5)$ -module such that

$$M \cong L\left(\frac{4}{5}, \frac{1}{40}\right) \oplus L\left(\frac{4}{5}, \frac{21}{40}\right) \quad \text{or} \quad M \cong L\left(\frac{4}{5}, \frac{1}{8}\right) \oplus L\left(\frac{4}{5}, \frac{13}{8}\right).$$

Then such M is unique up to isomorphism.

Proof. Recall that irreducible $\mathcal{W}_3(4/5)$ -modules are in one-to-one correspondence with the modules for the Zhu algebra $A(\mathcal{W}_3(4/5))$, which is isomorphic to a quotient of the polynomial algebra $k[x]$, where $x = [\omega]$. Every such module is one-dimensional so the top level of an irreducible $\mathcal{W}_3(4/5)$ -module must be one-dimensional. The rest follows from Proposition 3.6. \square

In the next section (cf. Proposition 4.5) we construct two (irreducible) τ -twisted $\mathcal{W}_3(4/5)$ -modules $\mathcal{W}^\tau(1/40)$ and $\mathcal{W}^\tau(1/8)$, which decompose as modules in Theorem 3.7. Consequently, combined with Lemma 3.3 we immediately obtain

Corollary 3.8.

The vertex algebra $\mathcal{W}_3(4/5)$ has precisely two τ -twisted irreducible modules. Moreover, the twisted Zhu algebra $A_\tau(\mathcal{W}_3(4/5))$ is isomorphic to $\mathbb{C}[x]/\langle (x - 1/40)(x - 1/8) \rangle$.

Theorem 3.9.

The vertex algebra $\mathcal{W}_3(4/5)$ is C_2 -cofinite and τ -rational.

Proof. The vertex algebra $\mathcal{W}_3(4/5)$ contains a C_2 -cofinite subalgebra (i.e. $L(4/5, 0)$, with the same conformal vector) thus it is C_2 -cofinite itself.

To prove the rationality we follow the standard arguments as from [16, Theorem 5.6], which we essentially repeat here. We only have to prove complete reducibility. So let M be an arbitrary admissible $\mathcal{W}_3(4/5)$ -module. We may split $M = M_{(1/8)} \oplus M_{(1/40)}$, where the weights of $M_{(1/8)}$ are contained inside $1/8 + \mathbb{N}_{\geq 0}/2$, and the weights of $M_{(1/40)}$ are in $1/40 + \mathbb{N}_{\geq 0}/2$. Indeed, this follows from complete reducibility with respect to $L(4/5, 0)$, Lemma 3.5, Proposition 3.1 and the fact that allowed lowest weights are $1/8$ and $1/40$. Our proof of complete reducibility of $M_{(1/8)}$ is essentially the same as the proof for $M_{(1/40)}$, so let us assume $M = M_{(1/8)}$ for simplicity. Consider the (top) weight $1/8$ subspace $M(1/8)$ of $M_{(1/8)}$. This is also an $A(\mathcal{W}_3(4/5))$ -module. An easy analysis shows that, as $L(4/5, 0)$ -module, $M \cong L(4/5, 1/8)^{\oplus m} \oplus L(4/5, 13/8)^{\oplus n}$, with some multiplicities m and n . The multiplicity m is precisely the dimension of the weight $1/8$ subspace. To finish the proof we have to argue that $m = n$ and $M \cong (L(4/5, 1/8) \oplus L(4/5, 13/8))^{\oplus m}$, as $\mathcal{W}_3(4/5)$ -modules. Choose $0 \neq v \in M(1/8)$ and consider $\mathcal{W}_3(4/5) \cdot v$. We claim that $\mathcal{W}_3(4/5) \cdot v \cong L(4/5, 1/8) \oplus L(4/5, 13/8)$. Clearly there could be only one copy of $L(4/5, 1/8)$ inside $\mathcal{W}_3(4/5) \cdot v$. Also, from the fusion rules, Lemma 3.5, restriction of the module map $Y|_{L(4/5, 3) \otimes L(4/5, 1/8)}$, where $L(4/5, 1/8) \subset \mathcal{W}_3(4/5) \cdot v$, must land inside W , where W is isomorphic to $L(4/5, 13/8)$, or is plainly zero. If the image

is zero then $L(4/5, 1/8)$ becomes an irreducible module for $\mathcal{W}_3(4/5)$, contradicting our classification of $\mathcal{W}_3(4/5)$ -modules in Proposition 3.6. We conclude $\mathcal{W}_3(4/5) \cdot v \cong L(4/5, 1/8) \oplus L(4/5, 13/8)$.

Now, take a nonzero vector $v' \in M(1/8)$ in the complement of $\mathbb{C}v$ and repeat the procedure. Then $\mathcal{W}_3(4/5) \cdot v'$ must be trivial or $L(4/5, 13/8)$. The latter case cannot occur because of Proposition 3.6. This way we obtain a decomposition of submodule of M isomorphic to $(L(4/5, 1/8) \oplus L(4/5, 13/8))^{\oplus m}$. The condition $m = n$ must be satisfied, otherwise we could quotient M with the submodule $(L(4/5, 1/8) \oplus L(4/5, 13/8))^{\oplus m}$ and obtain a module of lowest weight $13/8$, again a contradiction. \square

4. Standard $A_2^{(2)}$ -modules and construction of τ -twisted $\mathcal{W}_3(4/5)$ -modules

In this section, our focus is the Kac–Moody Lie algebra of type $A_2^{(2)}$ and its standard modules. We will denote by σ a principal (order 6) automorphism of $\widehat{\mathfrak{sl}_3}$ such that

$$\mathfrak{sl}_3 = \bigoplus_{j=0}^5 \mathfrak{sl}_3[j], \quad \mathfrak{sl}_3[j] = \{a \in \mathfrak{sl}_3 : \sigma(a) = \xi^j a\},$$

where ξ is a primitive 6th root of unity. Let Λ be a dominant weight of level l for $A_2^{(2)}$. We denote by $L_{\mathfrak{sl}_3}^\sigma(\Lambda)$ the σ -twisted $A_2^{(2)}$ -module, which is also an $L_{\mathfrak{sl}_3}(l\Lambda_0)$ -module [17]. In particular, $L_{\mathfrak{sl}_3}^\sigma(\Lambda_2)$ (which is of level 2) and $L_{\mathfrak{sl}_3}^\sigma(2\Lambda_1)$ are σ -twisted $L_{\mathfrak{sl}_3}(2\Lambda_0)$ -modules. Theorem 1.1 gives construction of both modules inside the tensor product of the basic module $L_{\mathfrak{sl}_3}^\sigma(\Lambda_1) \otimes L_{\mathfrak{sl}_3}^\sigma(\Lambda_1)$. Indeed, it is easy to see that $v_{\Lambda_1} \otimes v_{\Lambda_1}$ is a highest weight module for $L_{\mathfrak{sl}_3}^\sigma(2\Lambda_1)$ and the vector $(f_1 \cdot v_{\Lambda_1}) \otimes v_{\Lambda_1} - v_{\Lambda_1} \otimes (f_1 \cdot v_{\Lambda_1})$ generates the module $L_{\mathfrak{sl}_3}^\sigma(\Lambda_0)$. Here $\{e_0, f_0, h_0, e_1, f_1, h_1\}$ is the canonical set of generators of $A_2^{(2)}$. We denote the twisted module map with $Y^\sigma(\cdot, x)$. In particular, for any $x \in \mathfrak{sl}_3[j]$, we have

$$Y^\sigma(x(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} x \left(n + \frac{j}{6} \right) x^{-n-j/6-1}.$$

Consider

$$\omega_{k=l} = \frac{1}{2(l+3)} \sum_{i=1}^8 u_i(-1) \bar{u}_i(-1) \mathbf{1} \in L_{\mathfrak{sl}_3}(l\Lambda_0),$$

the Sugawara conformal vector of central charge $8l/(l+3)$, where $\{u_i\}$ and $\{\bar{u}_i\}$ are conveniently chosen orthogonal bases such that $\sigma(u_i) = \sigma^{-1}(\bar{u}_i)$, so that $u_i(-1) \bar{u}_i(-1) \mathbf{1}$ is fixed under the automorphism (we can actually choose $\{\bar{u}_i\}$ to be a permutation of $\{u_i\}$ [4]). Thus

$$Y(\omega_l, x) = \sum_{n \in \mathbb{Z}} L_{k=l}^\sigma(0) x^{-n-2}.$$

Denote by

$$\omega = \omega_{k=1} \otimes \mathbf{1} - \mathbf{1} \otimes \omega_{k=1} - \omega_{k=2} \in L_{\mathfrak{sl}_3}(\Lambda_0) \otimes L_{\mathfrak{sl}_3}(\Lambda_0)$$

the coset Virasoro generator of central charge $2 + 2 - 16/5 = 4/5$, and let

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}.$$

For related coset constructions see [1]. The following lemma is a crucial technical fact.

Lemma 4.1.

We have

$$L_{k=1}^\sigma(0) v_{\Lambda_1} = \frac{5}{72} v_{\Lambda_1}, \quad L_{k=2}^\sigma(0) v_{2\Lambda_1} = \frac{41}{360} v_{2\Lambda_1}, \quad L_{k=2}^\sigma(0) v_{\Lambda_0} = \frac{13}{180} v_{\Lambda_0}.$$

Consequently,

$$L(0) v_{2\Lambda_1} = \frac{1}{40} v_{2\Lambda_1}, \quad L(0) v_{\Lambda_0} = \frac{1}{8} v_{\Lambda_0}.$$

Proof. We prove only the first formula $L(0)v_{2\Lambda_1} = v_{2\Lambda_1}/40$, the other formula is proven along the same lines. Recall that for the Virasoro algebra operator $L(0)$ we picked generators $u_i, \bar{u}_i, i \in \{1, \dots, 8\}$, such that

$$\sigma(u_i) = \xi^i u_i, \quad \sigma(\bar{u}_i) = \xi^{6-i} \bar{u}_i.$$

Thus we have to compute expressions

$$\text{Coeff}_{x^{-2}} Y^\sigma(u_i(-1)\bar{u}_i(-1)\mathbf{1}, x),$$

contributing to $L(0)$, acting on the highest weight vectors v_{Λ_1} and on $v_{2\Lambda_1} = v_{\Lambda_1} \otimes v_{\Lambda_1}$. For that we use a version of the Jacobi identity (8) with $u = u_i(-1)\mathbf{1}$ and $v = \bar{u}_i(-1)\mathbf{1}$, where the automorphism τ is now σ . In this setup

$$\text{Res}_{x_2} \text{CT}_{x_0} \text{Res}_{x_1} (\text{RHS of (8)}) = \text{Res}_{x_2} \text{CT}_{x_0} \text{Res}_{x_1} x_2^{-1} \left(\frac{x_1 - x_0}{x_2} \right)^{-j/6} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y^\sigma(Y(u_i, x_0)\bar{u}_i, x_2),$$

which equals (by [4, 17])

$$\text{Res}_{x_2} x_2 Y^\sigma(u_i(-1)\bar{u}_i(-1)\mathbf{1}, x_2) + \frac{j}{6} [u_i, \bar{u}_i](0) + \binom{j/6}{2} l, \quad (13)$$

where l is the level.

Taking the same residues, now of the left hand side of the Jacobi identity, gives

$$\text{Res}_{x_2} \text{CT}_{x_0} \text{Res}_{x_1} (\text{LHS of (8)}) = \text{CT}_{x_2} : Y^\sigma(u_i, x_2) Y^\sigma(\bar{u}_i, x_2) :, \quad (14)$$

the constant term of a twisted normally ordered product. Comparing the formulas (13) and (14), and summing over i , gives an expression for $L_{k=1}^\sigma(0)$. It is now easy to get $L_{k=1}^\sigma(0)v_{\Lambda_1} = 5v_{\Lambda_1}/72$. Next, we use the Sugawara operator $L_{k=2}^\sigma(0)$ and act on $v_{2\Lambda_1}$. The only nonzero contributions when calculating these operators come from $(j/6)[u_i, \bar{u}_i](0)$ and $\binom{j/6}{2} l$. After we sum over i we get $L_{k=2}^\sigma(0)v_{2\Lambda_1} = 41v_{2\Lambda_1}/360$ and finally $L(0)v_{2\Lambda_1} = v_{2\Lambda_1}/40$. \square

Remark 4.2.

Presumably the computation in Lemma 4.1 can be carried out via an explicit realization of $A_2^{(2)}$ on the twisted Fock space obtained in [11].

The next result comes immediately from Theorem 2.5 and [15]:

Proposition 4.3.

We have the following decomposition of $L_{\text{sl}_3}(2, 0) \otimes \mathcal{W}_3(4/5)$ -modules:

$$L_{\text{sl}_3}(\Lambda_0) \otimes L_{\text{sl}_3}(\Lambda_0) = L_{\text{sl}_3}(2\Lambda_0) \otimes \mathcal{W}(0) \oplus L_{\text{sl}_3}(\Lambda_1) \otimes \mathcal{W}(2/5).$$

The next goal is to find a twisted version of Proposition 4.3. The automorphism σ acts diagonally on $L_{\text{sl}_3}(\Lambda_0) \otimes L_{\text{sl}_3}(\Lambda_0)$, and is denoted by $\sigma \times \sigma$. We have to see how it behaves when restricted to the subalgebra $L_{\text{sl}_3}(2\Lambda_0) \otimes \mathcal{W}_3(4/5)$.

Lemma 4.4.

The automorphism σ preserves $\mathcal{W}_3(4/5)$. More precisely, we have $\sigma|_{\mathcal{W}_3(4/5)} = \tau$. Thus, $\sigma \times \sigma|_{L_{\text{sl}_3}(2\Lambda_0) \otimes \mathcal{W}_3(4/5)} = \sigma \times \tau$.

Proof. The automorphism σ acts (diagonally) on the tensor product $L_{\text{sl}_3}(\Lambda_0) \otimes L_{\text{sl}_3}(\Lambda_0)$. Therefore, σ also preserves $L_{\text{sl}_3}(2\Lambda_0)$. Since $\mathcal{W}_3(4/5)$ is the commutant of $L_{\text{sl}_3}(2\Lambda_0)$, by definition $a_m b = 0$ for all $a \in L_{\text{sl}_3}(2\Lambda_0)$ and $b \in \mathcal{W}_3(4/5)$, $m \geq 0$. But then $\sigma(a_m b) = \sigma(a)_m \sigma(b) = 0$, and hence $\sigma(b) \in \mathcal{W}_3(4/5)$. Recall $\text{Aut } \mathcal{W}_3(4/5) = \mathbb{Z}_2$. If $\sigma|_{\mathcal{W}_3(4/5)} = 1$, then V_1 would be an ordinary $\mathcal{W}_3(4/5)$ -module. An easy inspection of modules in Theorem 2.2 implies that this is impossible. The proof follows. \square

Consequently, we reached the desired decomposition analogous to the one in Proposition 4.3.

Proposition 4.5.

As $(\sigma \times \tau)$ -twisted $L_{\text{sl}_3}(2, 0) \otimes \mathcal{W}_3(4/5)$ -modules,

$$L_{\text{sl}_3}^\sigma(\Lambda_1) \otimes L_{\text{sl}_3}^\sigma(\Lambda_1) = L_{\text{sl}_3}^\sigma(2\Lambda_1) \otimes \mathcal{W}^\tau\left(\frac{1}{40}\right) \oplus L_{\text{sl}_3}^\sigma(\Lambda_0) \otimes \mathcal{W}^\tau\left(\frac{1}{8}\right),$$

where $\mathcal{W}^\tau(1/40)$ and $\mathcal{W}^\tau(1/8)$ have lowest conformal weights $1/40$ and $1/8$, respectively.

Proof. The vertex algebra $L_{\text{sl}_3}(2\Lambda_0)$ is σ -rational, thus $L_{\text{sl}_3}^\sigma(\Lambda_1) \otimes L_{\text{sl}_3}^\sigma(\Lambda_1)$ decomposes as a direct sum of σ -twisted $L_{\text{sl}_3}(2\Lambda_0)$ -modules. This decomposition is described in Theorem 1.1. As $\mathcal{W}_3(4/5)$ is the commutant of $L_{\text{sl}_3}(2, 0) \subset L_{\text{sl}_3}(\Lambda_0) \otimes L_{\text{sl}_3}(\Lambda_0)$, and $L_{\text{sl}_3}^\sigma(\Lambda_1) \otimes L_{\text{sl}_3}^\sigma(\Lambda_1)$ is a twisted $(\sigma \times \tau)$ -module by Lemma 4.4, the multiplicity spaces in Theorem 1.1 are naturally τ -twisted $\mathcal{W}_3(4/5)$ -modules. The proof now follows from Lemma 4.1. \square

5. Modular invariance

Now we are ready to “explain” relations (5), and in particular the negative sign appearing in both identities. First we recall a result from [9], where a version of Zhu’s of modular invariance theorem was extended to general C_2 -cofinite rational τ -twisted vertex algebras (here τ is at first an automorphism of finite order).

The setup is as follows. Pick a pair of commuting automorphisms (g, h) (of finite order) of V , where V is C_2 -cofinite, and satisfies all the rationality and finiteness conditions as in [9]. Then we let

$$T_M(g, h, v, q) = \text{tr}_M \phi(h) o(v) q^{L(0) - c/24},$$

where M is h -stable g -twisted sector. This holomorphic function in $|q| < 1$, $q = e^{2\pi iy}$, satisfies the modular transformation property under

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad ad - bc = 1, \quad \text{i.e.,} \quad T_M(g, h, v, q) \upharpoonright_{\gamma \cdot q} = \sum_W \sigma_W T_W((g, h)\gamma, q),$$

where the summation goes over all $g^a h^c$ -twisted sectors W which are $g^b h^d$ -stable, and $(g, h)\gamma = (g^a h^c, g^b h^d)$. Observe that the modular invariance mixes several twisted sectors.

Now specialize $g = h = \tau$, where τ is of order two.

Claim: $\tau \circ M \cong M$ for every irreducible τ -twisted module M .

The τ -twisted module $\tau \circ M$ is defined via

$$\tilde{Y}(u, x) = Y^\tau(\tau u, x).$$

Because $\tau \circ M$ is $M = M^0 \oplus M^1$ as a vector space, we let

$$\sigma: M \rightarrow \tau \circ M, \quad \sigma|_{M^0} = 1, \quad \sigma|_{M^1} = -1.$$

We claim that σ is the wanted isomorphism. If $u \in V^0$, $\sigma(Y^\tau(u, x)m) = Y^\tau(u, x)\tau m = Y^\tau(u, x)\sigma m$, and if $u \in V^1$, $\sigma(Y^\tau(u, x)m) = (-1)Y^\tau(u, x)\sigma m = \tilde{Y}(u, x)\sigma m$. This proves the claim.

Notice that not every ordinary V -module is τ -stable. For instance, untwisted $\mathcal{W}_3(4/5)$ -module $\mathcal{W}(2/5, -)$ is not isomorphic to $\mathcal{W}(2/5, +)$, although the former is obtained as a τ -twist from the later (cf. Theorem 2.2). Thus when $g^b h^d \neq 1$

we can omit the stability condition (always satisfied). From now on we are interested only in vacuum twisted characters so we let $u = 1$. Consider the standard generators S and T of the modular group, corresponding to $y \mapsto -1/y$ and $y \mapsto y + 1$, respectively. For the S matrix $a = 0, b = 1, c = -1$ and $d = 0$, and for the T -matrix $a = 1, b = 1, c = 0$ and $d = 1$.

Under the S transformation

$$T_M(\tau, 1, 1, q)|_{q \cdot S} = \sum_W c_W T_W(1, \tau, 1, q),$$

where the summation is over untwisted modules which are τ -stable. Similarly,

$$T_M(\tau, 1, 1, q)|_{q \cdot T} = \sum_W d_W T_W(\tau, \tau, 1, q),$$

where the summation is over τ -twisted modules. We also have

$$T_M(\tau, \tau, 1, q)|_{q \cdot S} = \sum_W e_W T_W(\tau, \tau, 1, q),$$

where the summation is over τ -twisted modules (which are τ -fixed), and

$$T_M(\tau, \tau, 1, q)|_{q \cdot T} = \sum_W f_W T_W(\tau, 1, 1, q),$$

where the summation is over τ -twisted modules. Moreover,

$$T_M(1, \tau, 1, q)|_{q \cdot S} = \sum_W g_W T_W(\tau, 1, 1, q),$$

where the summation is over τ -twisted modules, and

$$T_M(1, \tau, 1, q)|_{q \cdot T} = \sum_W h_W T_W(1, \tau, 1, q),$$

where the summation is over τ -stable $\mathcal{W}_3(4/5)$ -modules. In the above formulas c_W, d_W, \dots, h_W are some constants. We summarize all these relations as

Corollary 5.1.

The vector space spanned by

$$\{T_M(1, \tau, 1, q) : M \text{ is untwisted and } \tau\text{-stable}\} \quad \text{and} \quad \{T_M(\tau, \tau^\epsilon, 1, q) : M \text{ is } \tau\text{-twisted and } \epsilon = 0, 1\}$$

is modular invariant.

Going back to $\mathcal{W}_3(4/5)$. There are two τ -stable irreducible untwisted $\mathcal{W}_3(4/5)$ -modules, namely $\mathcal{W}(0)$ and $\mathcal{W}(2/5)$. So the relevant modular invariant space has a basis

$$T_{\mathcal{W}^{\tau}(1/40)}(\tau, \tau^\epsilon, 1, q) = \chi_{5,6}^{1,2}(q) + (-1)^\epsilon \chi_{5,6}^{1,4}(q), \quad T_{\mathcal{W}^{\tau}(1/8)}(\tau, \tau^\epsilon, 1, q) = \chi_{5,6}^{2,2}(q) + (-1)^\epsilon \chi_{5,6}^{2,4}(q),$$

where $\epsilon \in \{0, 1\}$, and

$$T_{\mathcal{W}(0)}(1, \tau, 1, q) = \chi_{5,6}^{2,1}(q) - \chi_{5,6}^{2,5}(q), \quad T_{\mathcal{W}(2/5)}(1, \tau, 1, q) = \chi_{5,6}^{1,1}(q) - \chi_{5,6}^{1,5}(q).$$

Thus, the left hand-sides in (5) are simply the twisted characters of irreducible (untwisted) $\mathcal{W}_3(4/5)$ -modules. Combined with the expected identities given on the right hand-sides in (5), or simply by using modular invariance arguments, we easily get another (more natural) basis.

Proposition 5.2.

For $V = \mathcal{W}_3(4/5)$ and τ as above, the vector space spanned by expressions in Corollary 5.1 is 6-dimensional with a basis

$$\{\chi_{2,5}^{1,1}(q^2), \chi_{2,5}^{1,2}(q^2), \chi_{2,5}^{1,1}(q^{1/2}), \chi_{2,5}^{1,2}(q^{1/2}), \chi_{2,5}^{1,1}(-q^{1/2}), \chi_{2,5}^{1,2}(-q^{1/2})\}.$$

Appendix

In this section we discuss q -series identities underlying (4)–(5). The idea is relatively simple so we prove only the first of identities in (4) here, and leave the rest to the reader. Similar methods can be used to prove other identities.

Proposition 5.3.

We have

$$\chi_{5,6}^{1,2}(q) + \chi_{5,6}^{1,4}(q) = \chi_{2,5}^{1,1}(q^{1/2}). \quad (15)$$

Proof. The left hand side of (15) is equal to

$$q^{11/120} \sum_{m \in \mathbb{Z}} \left(q^{30m^2-4m} - q^{30m^2+16m+2} + q^{30m^2-14m+3/2} - q^{30m^2+26m+4+3/2} \right) / (q)_{\infty}.$$

Thus it is sufficient to prove

$$\sum_{m \in \mathbb{Z}} \left(q^{30m^2-4m} - q^{30m^2+16m+2} + q^{30m^2-14m+3/2} - q^{30m^2+26m+4+3/2} \right) = \frac{(q)_{\infty}}{\prod_{n=0}^{\infty} (1 - q^{(5n+2)/2})(1 - q^{(5n+3)/2})}. \quad (16)$$

Recall the quintuple product identity

$$\begin{aligned} \sum_{m \in \mathbb{Z}} (-1)^m q^{3m^2+m} z^{3m+1} + \sum_{m \in \mathbb{Z}} (-1)^m q^{3m^2-m} z^{3m} \\ = (1+z) \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{4n-2}z^2)(1 - q^{4n-2}z^{-2})(1 + q^{2n}z)(1 + q^{2n}z^{-1}). \end{aligned} \quad (17)$$

We rewrite the left hand-side as

$$\sum_{m \in \mathbb{Z}} q^{12m^2+2m} z^{6m+1} - \sum_{m \in \mathbb{Z}} q^{12m^2+14m+4} z^{6m+4} + \sum_{m \in \mathbb{Z}} q^{12m^2-2m} z^{6m} - \sum_{m \in \mathbb{Z}} q^{12m^2+10m+2} z^{6m+3}.$$

Substitute now in (17) q for $q^{5/2}$ and z for $q^{-3/2}$, and multiply the resulting expression with $q^{3/2}$. Then (17) turns into the left hand-side of (16). The proof now follows from an easy identity

$$(1 + q^{3/2}) \prod_{n=1}^{\infty} (1 - q^{5n})(1 - q^{10n-8})(1 - q^{10n-2})(1 + q^{5n-3/2})(1 + q^{5n+3/2}) = \frac{(q)_{\infty}}{\prod_{n=0}^{\infty} (1 - q^{(5n+2)/2})(1 - q^{(5n+3)/2})}. \quad \square$$

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