Universal Scaling Relations in Electron-Phonon Superconductors

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We study linear scaling relations in electron-phonon superconductors. By combining numerical and analytical techniques, we find linear Homes scaling relations between the zero-temperature superfluid density and the normal-state dc conductivity. This phenomenon arises via either a large impurity scattering rate or inelastic scattering of electrons and Einstein phonons at large electron-phonon coupling. Thus, our Letter shows that Homes scaling is more universal than either cuprate or BCS-like physics and is, instead, a fundamental result in a wide class of superconductors.

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Introduction—Unconventional superconductors are often characterized by exotic normal phases above a high critical temperature T_c [1,2]. One notable example is the cuprates [3–8], which exhibit a non-Fermi-liquid normal state at optimal doping known as the “strange metal” [9–11]. The unique electronic transport observed in the strange-metal phase [12–15] has been attributed to a linear scaling relation between observables in the $T = 0$ ground state and the $T = T_c$ normal state [16,17]. The near-universal linear scaling behavior seen in clean high-T_c superconductors was formally believed to be a hallmark of these quantum critical compounds. Nevertheless, linear scaling relationships have been generally considered within a wide array of physical phenomena, such as in quantum Hall physics [18], weak localization [19], and dirty BCS superconductors [20].

This Letter concerns universal scaling relations beyond both high-T_c and BCS-like superconductors. In regard to the latter, such scaling relationships are confined to the dirty limit [20–26], where there is a linear relationship between the $T = 0$ superfluid density and the normal-state electrical conductivity just above $T = T_c$. As articulated by de Gennes [21], this linear relationship is a fundamental result for superconducting matter in the BCS limit, provided (i) there is a diffuse scattering mechanism, and (ii) the theory is gauge invariant. Our Letter builds upon de Gennes’ criteria by softening condition (i) to the less restrictive constraints of Galilean noninvariance and momentum relaxation.

Within the context of high-T_c superconductors, similar linear scaling laws have been studied in the hope of identifying a universal fingerprint for these materials [27]. The first attempt to formulate such a relation was provided by Pimenov *et al.* [28], who suggested linear scaling between the zero-temperature normalized superfluid density $n_s(\tau)/n \equiv n_s(\tau, T = 0)/n$ and $\sigma(\tau) \cdot \tau^{-1}$, where $\sigma(\tau) \equiv \sigma(\tau, T = T_c{}^{+})$ is the normal-state dc conductivity at $T = T_c{}^{+} \equiv T_c + 0^{+}$ and τ is the scattering time. This “Pimenov scaling” relation (which was partially motivated by the earlier “Uemura scaling” relation between $n_s(\tau)/n$ and T_c [29–32]) failed to serve as a universal hallmark for high-T_c physics, since heavily doped samples of certain YBaCuO species violated the proposed scaling law [33,34]. The work of both Uemura *et al.* and Pimenov *et al.* led to the landmark result of Homes *et al.* [17], who identified that the so-called “Homes scaling” relation between $n_s(\tau)/n$ and $\sigma(\tau) \cdot T_c$ was a more universal feature of high-T_c superconductors.

Unlike Uemura and Pimenov scaling, Homes scaling is obeyed in a wide class of compounds regardless of doping and other sample details [17,28,35–44]. While it was suggested by Zaanen [16] that Planckian dissipation in the normal state of the cuprates (and, thus, strange-metal physics itself) is fundamentally tied to Homes scaling, both Zaanen and Homes pointed out that Homes scaling is present in low-T_c compounds such as Pb and Nb [16,17,45]. Therefore, the wide range of applicability of such linear scaling relations may very well suggest certain universal physics underlying a broad class of superconductors. Nevertheless, theoretical works which demonstrate linear scaling beyond the BCS limit are quite scarce [26,46–49].

In our Letter, we provide numerical evidence and theoretical justification for Homes scaling in a strongly correlated model of superconductivity distinct from both BCS-type and high-T_c-like physics. Specifically, we study a general family of scaling relations given by

$$\frac{n_s(\tau, \lambda)}{n} = \eta(\tau, \lambda) \frac{\sigma(\tau, \lambda) \cdot \psi}{\omega_p^2/(8\pi^2)}, \tag{1}$$

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where λ quantifies the interaction strength, $\eta(\tau, \lambda)$ is a proportionality factor, ω_p is the plasma frequency of the free-electron gas, and $\psi = T_c$ (τ^{-1}) for Homes (Pimenov) scaling. (See Sec. IV A of the Supplemental Material [50] for a distinction between linear proportionality and linear scaling). We argue that BCS physics cannot explain such scaling relations outside of a dirty, weak-coupling scenario. In this Letter, we go beyond BCS theory and consider scaling relations of the form given in Eq. (1) using the framework of Eliashberg theory [51–71]. Electron-phonon interactions provide an additional parameter (besides τ) with which to "tune" the normal-state conductivity and superfluid density. Likewise, strong electron-phonon coupling results in a violation of the Planckian bound [72], making Eliashberg theory an ideal setting to investigate the universality of Homes scaling.

We find linear scaling behavior in the electron-phonon system in both the clean and dirty limits [73–76], with a fundamental ingredient for such scaling behavior being Galilean noninvariance and momentum relaxation, either via elastic impurity scattering or inelastic scattering between electrons and Einstein phonons in an isotropic system [77–79].

Numerical calculations on the imaginary frequency axis—We consider the isotropic, single-band Eliashberg equations on the imaginary Matsubara frequency axis [70,80]. Eliashberg theory goes beyond BCS theory by incorporating a dynamical electron-phonon interaction [81], and thus, the gap function $\Delta(i\omega_n)$ and renormalization function $Z(i\omega_n)$ depend upon the fermionic Matsubara frequencies $\omega_n = (2n+1)\pi T$. For simplicity, we assume an Einstein (or Holstein) phonon model [65,70,82] with a dimensionless electron-phonon coupling λ and an Einstein phonon frequency ω_E.

We iteratively solve the Eliashberg equations for a fixed $\lambda \in [0.3, 100]$, with convergence criteria of the Matsubara summation determined by an algorithm discussed in the Supplemental Material [50]. The gap and the renormalization functions follow a Lorentzian structure for all values of λ, as already noted for $\lambda \lessapprox 0.5$ [83]. While Eliashberg theory remains valid for large coupling strengths as long as ω_E is much smaller than the Fermi energy ϵ_F [61,69,84], λ is usually no more than 3.5–4 in most present-day materials [60,85]. The motivation for considering the large-λ limit follows from the formulation of asymptotically strong Eliashberg theory [61,64,86,87], in which the Eliashberg equations reduce to a universal theory characterized by an Einstein phonon spectrum [64]. Therefore, our results for an Einstein phonon model with $\lambda \gg 1$ should remain appropriate for other strongly coupled models of

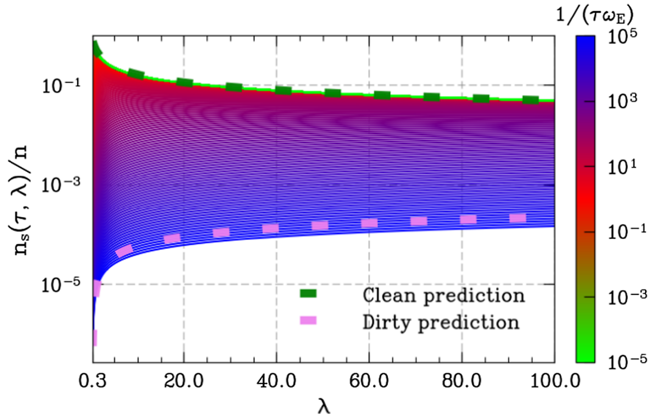


FIG. 1. Superfluid density $n_s(\tau, \lambda)/n$ versus λ for various scattering rates. In the clean limit, $n_s(\tau, \lambda)/n \sim 1/Z_0$ (green dashed curve), while $n_s(\tau, \lambda)/n \sim \pi\tau\Delta_0$ (violet dashed curve) in the dirty limit.

Eliashberg superconductivity. Although several arguments have been presented for a theoretical upper bound λ_c on the electron-phonon coupling strength in realistic materials [88–91], the motivation for considering arbitrary λ (and, in particular, the asymptotically strong limit) is to explore the universal behavior of the Homes slope, regardless of the precise numerical value for λ_c.

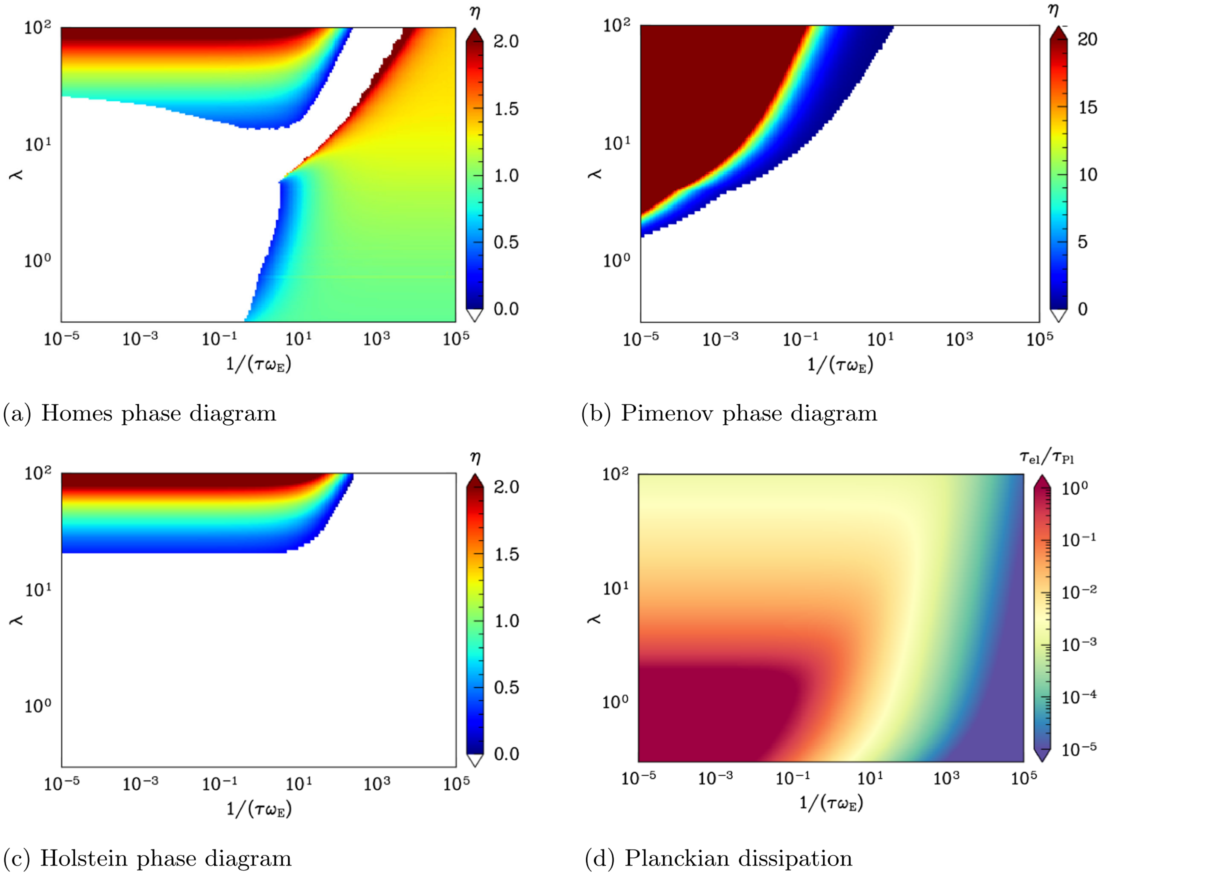
After numerically obtaining the gap and renormalization functions, we calculate the superfluid density for arbitrary τ, λ, and T [26,62,92–98]

$$\frac{n_s(\tau, \lambda, T)}{n} = \pi T \sum_{n=-\infty}^{\infty} \frac{\Delta^2(i\omega_n)}{\omega_n^2 + \Delta^2(i\omega_n)} \times \frac{1}{Z(i\omega_n)\sqrt{\omega_n^2 + \Delta^2(i\omega_n)} + 1/(2\tau)}. \quad (2)$$

In Fig. 1, we plot the $T = 0$ superfluid density $n_s(\tau, \lambda)/n$ versus λ. In the dirty limit $1/(\tau\omega_E) \gg 1$, severe suppression of $n_s(\tau, \lambda)/n$ occurs regardless of the interaction strength [20,23,98,99]. In the clean limit, we find that $n_s(\tau, \lambda)/n$ goes as $1/Z_0$, where $Z_0 \equiv \lim_{T\to 0} Z(i\omega_0)$ is the $T = 0$ limit of the renormalization function. This is in stark contrast to the clean BCS limit, where $n_s/n \to 1$ as $1/(\tau\omega_E) \to 0$ [20,26,98,100].

The localized Einstein phonon acts as an impurity [69]. As a consequence, the dc conductivity for the $T > T_c$ normal state is modified from the Drude result [41,101,102]. Extending previous work done on the Einstein-phonon model [103], we obtain $\sigma(\tau, \lambda, T) \equiv [\omega_p^2/(4\pi)] \cdot \zeta(\tau, \lambda, T)$, where we define [41,61,65,79,101–103]

$$\zeta(\tau, \lambda, T) \equiv \frac{1}{2\pi\lambda T} \int_0^\infty \frac{\mathrm{sech}^2\left(\frac{\omega_E}{2T}x\right)}{\coth\left(\frac{\omega_E}{2T}\right) - \frac{1}{2}\left\{\tanh\left[\frac{\omega_E}{2T}(1-x)\right] + \tanh\left[\frac{\omega_E}{2T}(1+x)\right]\right\} + \frac{1}{\pi\lambda\tau\omega_E}} dx. \quad (3)$$



(a) Homes phase diagram

(b) Pimenov phase diagram

(c) Holstein phase diagram

(d) Planckian dissipation

FIG. 2. (a)–(c) Phase diagrams for scaling relations of the form Eq. (1). The color denotes the value of the Homes, Pimenov, and Holstein slopes. The white region is where the respective linear scaling relation breaks down, as determined by our numerical algorithm (see Sec. VI B of the Supplemental Material [50]). In all instances, universal scaling exists in some regime of the strong-coupling limit. (d) The ratio of the normal-state scattering time τ_{el} to the Planckian lifetime $\tau_{\mathrm{Pl}} \equiv \hbar/(k_B T_c)$ on the λ versus $1/(\tau\omega_E)$ grid. We identify τ_{el} as $\zeta(\tau,\lambda)$ given in Eq. (3).

In the dirty limit, the above expression reduces to $\zeta(\tau,\lambda,T) = \tau$, reproducing the Drude result [104]. In the clean limit, Eq. (3) reduces to $\zeta(\lambda,T) = [1/(2\pi\lambda\omega_E)] \cdot \sinh(\omega_E/T)$, yielding a finite dc conductivity independent of τ. At $T = T_c$, we can simplify Eq. (3) further by recalling the semianalytical formula for the Eliashberg critical temperature derived by Combescot [105] for arbitrary λ assuming an Einstein phonon model, given by $T_c = a\omega_E(e^{2/\lambda} - 1)^{-1/2}$ where $a \approx 0.256$. As such, the dc conductivity $\sigma(\tau,\lambda)$ for the $T = T_c$ normal state can be cast as a function purely of τ and λ.

The previous result motivates us to consider scaling relations between $n_s(\tau,\lambda)/n$ and $\sigma(\tau,\lambda)$ for a wide range of τ and λ. We consider scaling relations of the form Eq. (1) with $\psi = T_c$ (Homes), $\psi = \tau^{-1}$ (Pimenov), and $\psi = \omega_E$ (Holstein). Results for these scaling relations are shown in Figs. 2(a), 2(b), and 2(c), respectively. The slope of the superfluid density versus $\sigma(\tau,\lambda) \cdot \psi/[\omega_p^2/(8\pi^2)]$ is plotted on a grid of λ versus $1/(\tau\omega_E)$, with the lack of a color denoting a breakdown of scaling between $n_s(\tau,\lambda)/n$ and $\sigma(\tau,\lambda)$. In Fig. 2(a), we see that Homes scaling is obeyed in the dirty weak-coupling limit, as predicted by BCS theory [20,23]. In the clean strong-coupling limit, we see that Homes scaling is obeyed for $\lambda \gtrsim 2 \times 10^1$, with Holstein scaling also emerging in a similar regime of the "phase diagram." Pimenov scaling appears to remain valid in the strong-coupling regime, although strong λ dependence emerges in the clean limit.

Our numerical results indicate that universal scaling relations of the form given in Eq. (1) can be explained within the framework of Eliashberg theory; namely, by virtue of strong interactions between electrons and Einstein phonons. Note that Homes scaling fails only in the clean weak-coupling limit and for certain intermediate values of λ and $1/(\tau\omega_E)$. The former violation occurs due to the superfluid density "flattening" to unity as both interactions and the scattering rate are decreased. The latter violation of Homes scaling is more nontrivial and is the result of nonlinear "back-bending" phenomena [50]. Finally, the work of Zaanen [16] argued that Homes scaling is a consequence of Planckian dissipation in the normal state. In Fig. 2(d), we plot the ratio of the total scattering time over the Planckian time and find no correlation between scaling behavior and the onset of Planckian dissipation in the normal state. Violation of the Planckian bound (and onset of a "super-Planckian" timescale) in Fig. 2(d) agrees with Ref. [72].

The origin of Homes scaling—To understand the physical origin of Homes scaling observed in our numerical calculations, we perform a semianalytical calculation of the $T = 0$ superfluid density on the imaginary frequency axis. Taking $\Delta(i\omega_n) \approx \lim_{T\to 0}\Delta(i\omega_0) \equiv \Delta_0$ and $Z(i\omega_n) \approx \lim_{T\to 0} Z(i\omega_0) \equiv Z_0$, we find [50]

$$\frac{n_s(\tau,\lambda)}{n} = \frac{\pi}{2Z_0\gamma_0}\left[1 + \frac{4}{\pi\sqrt{1-\gamma_0^2}}\arctan\left(\frac{\gamma_0 - 1}{\sqrt{1-\gamma_0^2}}\right)\right], \quad (4)$$

where $\gamma_0 \equiv 1/(2\tau\Delta_0 Z_0)$. A similar result may be derived on the real frequency axis assuming a constant complex gap [50,81,106–108]. In the dirty limit of Eq. (4), $\gamma_0 \gg 1$, and thus, the above expression simplifies to $\sim\pi\tau\Delta_0$, in analogy to Nam's result for the dirty BCS superfluid density [23,24]. We identify the Homes slope in this scenario with the ratio $\Delta_0/(2T_c)$. Taking the clean limit ($\gamma_0 \ll 1$), Eq. (4) reduces to $\sim 1/Z_0$. In the BCS limit, $Z_0 = 1$, leading to a breakdown of Homes scaling in the clean limit due to a vanishing Homes slope. However, for finite λ, the superfluid density is suppressed below unity and scales as the inverse of $Z(i\omega_0)$. As such, our semianalytical estimate for the clean superfluid density agrees with the results given in Fig. 1.

We emphasize that the renormalization Z_0 is a crucial ingredient for the realization of Homes scaling in the clean strong-coupling limit. This can be seen by recalling Eq. (3), from which a rough prediction of the clean Homes proportionality factor may be calculated (up to a constant) to be $\eta_H(\lambda) \sim I^{-1}(\lambda)\cdot(\lambda/Z_0)$, where $I(\lambda)$ is the dimensionless integral introduced in Eq. (3) with $T = T_c$. We note that $Z_0 \sim \sqrt{\lambda}$ and $I(\lambda) \sim 1$ as $\lambda \to \infty$ [64,109]. As such, the Homes proportionality factor $\eta_H(\tau,\lambda)$ is found to be a slowly varying function of λ in this limit. Such weak dependence on λ is a hallmark of Homes scaling induced by strong electron-phonon coupling, and sets it apart from Homes scaling in the weakly coupled dirty system, where the Homes slope is a constant set at $\mathbf{\Delta}_0/(2T_c) \sim 0.8825$ [23] and where $\mathbf{\Delta}_0$ is the $T = 0$ BCS gap. Weak λ dependence in the Homes slope is similarly observed at large λ and small $1/(\tau\omega_E)$ in Fig. 2(a).

If Z_0 is set to unity, then $\eta_H(\tau,\lambda) \sim \lambda$ in the clean strong-coupling limit, and thus, Homes scaling breaks down. Homes scaling also breaks down in the case of clean superconductors where superconductivity is mediated by bosons with a finite momentum-dependent dispersion [110] in which electromagnetic vertex corrections in the superfluid density cancel any dependence on the mass renormalization. However, in the case of a dynamical gap mediated by dispersionless Einstein bosons, this cancellation does not occur [110]. This motivates us to propose that Homes scaling, while a poor signature of high T_c and normal-state Planckian dissipation, is instead a universal hallmark of some general Galilean noninvariance and momentum relaxation. The former ensures a nonunity superfluid density [111], while the latter ensures a finite dc conductivity [112]. In the present Letter, both Galilean noninvariance and momentum relaxation are achieved by virtue of elastic scattering of electrons by impurities or inelastic scattering of electrons by Einstein phonons. Note that the opposite is not universally true; i.e., Galilean noninvariance and momentum relaxation does not always result in a linear Homes slope, as evident from Fig. 2(a) [113].

We emphasize that our main focus is not to explain Homes scaling in high-T_c superconductors. Rather, we demonstrate linear Homes scaling in a strongly interacting model by virtue of a mechanism that is not solely due to impurity scattering. Similarly, we note that the Coulomb interaction should not have an appreciable effect on the Homes slope in electron-phonon superconductors, as this interaction does not significantly affect observables in the weak-coupling limit [70,86] and is negligible compared to the divergent electron-phonon coupling in the asymptotically strong limit [64]. In this way, Homes scaling can be seen as a robust consequence of Cooper pairing and some general disorder or dissipation.

The asymptotically strong limit—The consideration of an Einstein model is important for the strong-coupling analysis, as the $\lambda \to \infty$ limit is universally described by an Einstein spectrum satisfying $\lambda\omega_E^2 = 2$ [64]. For $\lambda \to \infty$, the Homes slope in the dirty limit reduces to a universal constant given by $\sim 1/(3a) \neq \Delta_0/(2T_c)$ [50], where $a \approx 0.256$ [105]. However, in the clean limit, then, $\sigma(\lambda)T_c/[\omega_p^2/(8\pi^2)] = \lambda^{-1}$ for all τ, while the superfluid density scales as $\lambda^{-1/2}$. This results in a diverging Homes proportionality factor proportional to $\sqrt{\lambda}$, and the

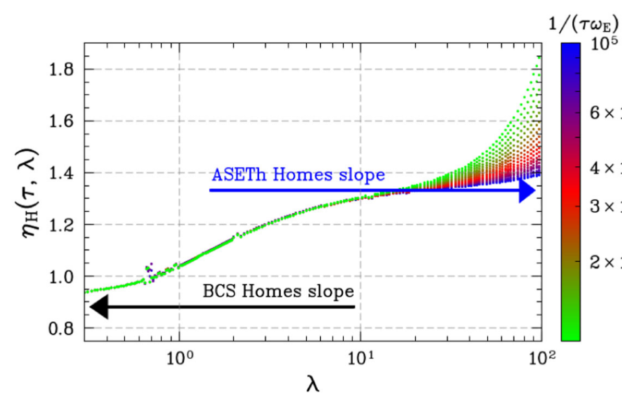


FIG. 3. Homes proportionality factor $\eta_H(\tau,\lambda)$ in the dirty limit plotted versus the electron-phonon coupling λ. As the system becomes dirtier in the large-λ limit, the Homes proportionality factor agrees with the asymptotic prediction.

breakdown of Homes scaling as $\lambda \to \infty$. Similar analysis suggests that Pimenov and Holstein scaling break down in the dirty and clean limits as $\lambda \to \infty$.

In Fig. 3, we show the dirty limit of the Homes factor plotted versus λ, from $\lambda = 0.3$ to $\lambda = 100$. Extrapolation of the small-λ data to $\lambda \to 0$ gives a Homes factor of ~0.88, in agreement with our theoretical BCS prediction. Extrapolation of the large-λ data to $\tau\omega_E \to 0$ yields a dirty Homes slope of ~1.35 for $\lambda = 100$ which is in agreement with our prediction for $\lambda \to \infty$ via the asymptotic Eliashberg equations [50]. Once again, we note that our main conclusions do not depend upon the precise value of some upper cutoff λ_c. Rather, our value of $\eta_H \sim 1/(3a)$ should be interpreted as a universal upper limit to the dirty Homes slope in Eliashberg theory, regardless of λ_c.

Conclusions—By combining numerical and analytical techniques for electron-phonon superconductors at weak and strong coupling and arbitrary scattering rates, we find that Homes-like scaling relations are not solely correlated with large T_c [17], normal-state Planckian dissipation [16], or some large scattering rate [20]. Instead, we find that Homes scaling is closely connected to Galilean noninvariance and momentum relaxation and, thus, remains valid in the clean limit for large electron-phonon coupling λ assuming an Einstein phonon model [115]. Pimenov and Holstein scaling are shown to emerge for strong enough electron-phonon coupling for certain values of the scattering rate, while the Homes slope approaches a universal constant for $\lambda \to \infty$ in the dirty limit. Our numerical values of the Homes slope for $\lambda \sim 0.3$ and $\lambda \sim 100$ are in agreement with the theoretical predictions of BCS theory and asymptotically strong Eliashberg theory, respectively.

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