

SMALL-DISTANCE BEHAVIOUR IN FIELD THEORY

CONTENTS

1.	Renormalization.	22
1.1	Generating functional of Green's and vertex functions	22
1.2	Feynman rules, skeleton expansions, and renormalization conditions	23
1.3	Bethe-Salpeter equation	26
2.	Short-distance expansions.	28
2.1	Short-distance expansion in the narrow sense	28
2.2	Related expansion formulae	31
3.	Mass vertex insertion.	32
3.1	Derivation of PDEs	32
3.2	Integration of PDEs	35
4.	Asymptotic forms.	37
4.1	Existence of asymptotic forms	37
4.2	Properties of asymptotic forms	39
4.3	Relation to zero-mass theory	40
5.	Exceptional momenta.	42
5.1	Asymptotic forms at exceptional momenta	42
5.2	Asymptotic behaviour near exceptional momenta	49
5.3	Application to finite-mass corrections	50
6.	Question of true large-momenta behaviour	53
6.1	Assumptions relating to fixed points	53
6.2	Anomalous dimensions and conformal invariance	56
6.3	Consistency check of assumptions	57
7.	General renormalizable theory.	59
7.1	PDEs for the massive theory	59
7.2	PDEs for the massless theory	60
8.	Remarks on some renormalizable theories.	62
8.1	QED and gluon theory	62

8.2	Yukawa theories	63
8.3	Nonabelian gauge field theories	64
8.4	A note on asymptotically free theories	64
8.5	Uses of computable large-momenta behaviour	67
CONCLUSION		68
ACKNOWLEDGEMENT		69
REFERENCES		70

SMALL-DISTANCE BEHAVIOUR IN FIELD THEORY

These lectures give a review of techniques for large-momenta-behaviour analysis in renormalizable field theories, and of results thereby obtained. For the sake of brevity of formulae, the ϕ^4 theory will be used to explain the ideas involved. Only in the last two sections more general theories will be discussed and specific results for some particular theories mentioned.

1. Renormalization.

As already mentioned, only the ϕ^4 theory will be discussed here, and for our purpose it suffices to do so on a heuristic level. For a more rigorous treatment see e.g. [1] and [2]. Some more details to the material of this section can be looked up in [3], [4], and, in particular, [5].

1.1 Generating functionals of Green's and vertex functions.

Let L be the Lagrangian density of ϕ^4 theory, given in detail in (1.11) below. The generating functional of the Green's functions containing connected and disconnected parts, namely, the vacuum expectation values of time-ordered operator products,

$$G_{\text{disc}}\{j\} = \langle (e^{i \int J(x)\phi(x)dx})_+ \rangle \quad (1.1a)$$

is obtained [6] as the scalar product

$$G_{\text{disc}}\{J\} = \begin{matrix} J & J \\ \langle & | & \rangle \\ \text{out} & & \text{in} \end{matrix} \quad (1.1b)$$

of the in- and out-vacua of the theory with explicitly time dependent Lagrangian density

$$L^J = L + J\phi. \quad (1.1c)$$

The generating functional of the connected parts of the Green's functions $G_{\text{disc}}(x_1 \dots x_n)$ is

$$\begin{aligned} G\{j\} &= \ln G_{\text{disc}}\{J\} \\ &= \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{i^n}{n!} \int \dots \int dx_1 \dots dx_n J(x_1) \dots J(x_n) G(x_1 \dots x_n) \end{aligned} \quad (1.2)$$

where in the ϕ^4 model, in absence of spontaneous breaking of the $\phi \leftrightarrow -\phi$ symmetry, only the $G(x_1 \dots x_n)$ and $G_{\text{disc}}(x_1 \dots x_n)$ with n even are nonzero. For functional derivatives we use the notation

$$\frac{\delta^n G\{J\}}{\delta J(x_1) \dots \delta J(x_n)} = G_{x_1 \dots x_n}\{J\}$$

and similarly later for other functionals.

In renormalization theory more basic than the Green's functions are the amputated one-particle irreducible parts of the connected functions, $\Gamma(x_1 \dots x_n)$, here called vertex functions (VFs). They are introduced most conveniently as follows [6], [7]: Set

$$\frac{1}{i} G_x\{J\} = \int dy \, i\Delta_F^1(x-y)J(y) + O(J^3) \equiv A(x)\{J\}. \quad (1.3)$$

This can be solved to

$$J(x) = -i \int \Delta_F^{1-1}(x-y)A(y)dy + O(A^3). \quad (1.4)$$

The functional of A

$$\Gamma\{A\} = [G\{J\} - i \int dx A(x)J(x)] \Big|_{J=J\{A\}}. \quad (1.5)$$

satisfies

$$J(x) = i \frac{\delta \Gamma\{A\}}{\delta A(x)} \equiv i\Gamma_x\{A\} \quad (1.6)$$

and

$$\left. \delta \Gamma\{A\} \right|_{A \text{ fixed}} = \left. \delta G\{J\} \right|_{J \text{ fixed}}, \quad (1.7)$$

and the VFs are

$$\Gamma(x_1 \dots x_n) = \Gamma_{x_1 \dots x_n}\{0\}. \quad (1.8)$$

Their Fourier transforms are introduced by

$$\int dx_1 \dots dx_n \Gamma(x_1 \dots x_n) e^{i \sum p_i x_i} = (2\pi)^4 \delta(\sum p_i) \Gamma(p_1 \dots p_n)$$

and for $n=2$ we have [4]

$$\Gamma(p(-p)) = -G(p(-p))^{-1}. \quad (1.9)$$

1.2 Feynman rules, skeleton expansions, and renormalization conditions.

The Lagrangian L^J for ϕ^4 theory, written in unrenormalized fields ϕ_u (and with unrenormalized source J_u) is

$$L^J = \frac{1}{2} (\partial_\mu \phi_u \partial^\mu \phi_u - m_u^2 \phi_u^2) - \frac{g_u}{4!} \phi_u^4 + J_u \phi_u. \quad (1.10)$$

Introducing renormalized fields, mass squared, coupling constant, and source function, by

$$\begin{aligned} \phi &= Z_3^{-1/2} \phi_u & m^2 &= m_u^2 + \delta m^2 \\ g &= g_u Z_1^{-1} Z_3^2 & J &= Z_3^{1/2} J_u \end{aligned}$$

the Lagrangian takes the form

$$\begin{aligned} L^J &= \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{g}{4!} \phi^4 + J \phi \\ &+ (Z_3 - 1) \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + Z_3 \delta m^2 \frac{1}{2} \phi^2 - (Z_1 - 1) \frac{g}{4!} \phi^4 \end{aligned} \quad (1.11)$$

It is discussed e.g. in [3] and [5] how from (1.11) the Feynman rules follow, the role of the counter terms as subtraction terms for the superficially divergent VFs emerges, and skeleton expansions are obtained. The Feynman rules are: Use $\Delta_F = -i(m^2 - p^2 - i\epsilon)^{-1}$ for the propagators, $-ig$ for the vertices, and $(2\pi)^{-4} \int dk$ for each loop integration, and supply the symmetry factor S^{-1} , where S is the number of possible mappings of the graph on itself [2]. The superficial divergence degree D , i.e. divergence degree under overall power counting, is computed as follows: Let n_i be the number of ϕ legs and d_i the number of derivatives at the i^{th} vertex, L the number of loops, L the one of lines, and E the one of external ϕ arguments amputated upon, equal n in (1.8). Then

$$D = 4L - 2L + \sum_i d_i. \quad (1.12a)$$

Use of the familiar topological relation

$$\sum_i 1 + L = L + 1 \quad (1.12b)$$

and of the trivial one

$$E = \sum_i n_i - 2L$$

in (1.12a) yields

$$D = 4 - E - \sum_i (4 - n_i - d_i) \quad (1.13)$$

where $n_i + d_i$ is the ("operator") dimension of the i^{th} vertex: the (mass) dimension of ϕ is 1, and the one of a derivative also 1. We thus have the table

E =	2	4	≥ 6
D =	2	0	≤ -2
graphical notation	$- \frac{-1}{2}$		

The VFs for $E = 2k$, $k \geq 3$, are expandible in terms of the renormalization functions: the propagator (the negative inverse VF for $n=2$) and the $n=4$ VF.

Example: $E=6$: = $\sum_{15}$ + $\sum_{30}$ + ...

For $E=4$ one finds, written diagrammatically

$$\begin{aligned}
 \begin{matrix} E=4 \\ D=0 \end{matrix} \quad \text{Diagram} &= X + \frac{1}{6} \underbrace{\text{Diagram}}_{\substack{1,3 \text{ irred.} \\ \text{Diagram}}} + \frac{1}{2} \sum_3 \text{Diagram} \\
 &= \underbrace{\text{Diagram}}_{(i)} + \frac{1}{2} \text{Diagram}
 \end{aligned} \tag{1.14}$$

where is the Bethe-Salpeter (BS) kernel two-particle irreducible in the vertical channel, and itself expressed by

$$\text{Diagram} = (i) + \frac{1}{2} \text{Diagram} + \frac{1}{2} \text{Diagram} \tag{1.15}$$

For $E=2$, the negative inverse propagator, one finds

$$\begin{matrix} E=2: \\ D=2 \end{matrix} \quad - \frac{-1}{2} = i(p^2 - m^2) + X + \frac{1}{6} \text{Diagram} \cdot \tag{1.16}$$

Renormalization theory [8], [9] assures that to render integrals for superficially divergent VFs finite, one must give as many data for these functions as a polynomial in momenta (with invariances observed) of order D has; the fulfilment of these renormalization conditions is effected by the counter terms indicated by crosses, which are $-iZ_1 g$ in (1.14) and $i(Z_3 - 1)(p^2 - m^2) + iZ_3 \delta m^2$ in (1.16). In practice, it is simplest to subtract the Taylor expansion of order D at zero momenta from the sum of integrals for the function considered and to add that (invariant) polynomial of order D with finite coefficients that secures fulfilment of the renormalization conditions to the order in question. We choose

$$\Gamma(p(-p)) \Big|_{p^2=m^2} = 0 \quad (1.17a)$$

$$\frac{\partial}{\partial p^2} \Gamma(p(-p)) \Big|_{p^2=m^2} = i \quad (1.17b)$$

$$\Gamma(p_1 \dots p_4) \Big|_{\substack{\text{symm.pt. to } m^2 \\ p_i p_j = \frac{m^2}{3} (4\delta_{ij} - 1)}} = -ig \quad (1.17c)$$

which imply

$$\Gamma(p(-p)) = i(p^2 - m^2) + O(g^2) \quad (1.18a)$$

$$\Gamma(p_1 \dots p_4) = -ig + O(g^2) \quad (1.18b)$$

$$\Gamma(p_1 \dots p_{2n}) = O(g^n) \quad n \geq 3 \quad (1.18c)$$

For one procedure to convert (1.14) and (1.16) into manifestly finite form see [5], for a closely related one [3], for another [10]. Thereby, in intermediate steps meaningless expressions are manipulated; all calculations can be made meaningful, however, by regularizing the theory first e.g. as detailed in [3], and removing the regularization only in the final equations, in which all overlapping divergences are disentangled. The most elegant way to regularize, however, and of definite advantage in gauge field theories, is to perform all computations in $4-\epsilon$ dimensions and to let $\epsilon \rightarrow 0$ in the final expressions [11]: the Feynman rules lead upon integration to functions meromorphic in ϵ , and renormalization removes the poles at $\epsilon=0$.

1.3 Bethe-Salpeter equation.

For the BS equation that is part of (1.14) we use the matrix notation

$$I = B + IGB = B + BGI \quad (1.19a)$$

or equivalently

$$(1-GB)(1+GI) = (1+GI)(1-GB) = (1-BG)(1+IG) = (1+IG)(1-BG) = 1. \quad (1.19b)$$

Here I is the four-point VF, B the BS kernel, and G stands for the pair of propagators. Momenta are indicated as

$$\Gamma((- \frac{1}{2}q-p) (- \frac{1}{2}q+p)(\frac{1}{2}q+p')(\frac{1}{2}q-p')) \leftrightarrow p_I^q p'^q$$

$$(2\pi)^4 [\delta(p-p') + \delta(p+p')] G(-\frac{1}{2}q-p) G(-\frac{1}{2}q+p) \leftrightarrow G^q$$

$$(2\pi)^4 [\delta(p-p') + \delta(P+p')] \leftrightarrow 1$$

and are suppressed if internal (and then integrated over with a factor $\frac{1}{2} (2\pi)^{-4}$ supplied), or if external (left relative, right relative, and through going or total) and kept general. In (1.19) it is understood that internal relative momenta fit unless a momentum associated with the factor to the left or right is indicated to be fixed, and thus constant in the internal-momentum integration, indicated for clarity additionally by a bracket $>$ or $<$. For a G-link, this amounts to let arguments coalesce in coordinate space. B contains an ill-defined constant since the $-iZ_1 g$ term in $\textcircled{1}$ in (1.14) is not appropriate to make B finite when the regularization is removed since it is chosen to render I finite instead. Thus, only B subtracted once in an arbitrary way is finite after regularization removal. We briefly indicate how I is obtained (e.g., recursively in perturbation theory) from the subtracted B. In the following, for simplicity of notation we write formulae as if the momenta set at which I is specified (as $\overset{\circ}{I}^\circ$) were all momenta vanishing. From (1.19a) follows

$$\overset{\circ}{I} - \overset{\circ}{I}^\circ = \overset{\circ}{B} - \overset{\circ}{B}^\circ + \overset{\circ}{I} \overset{\circ}{G} (\overset{\circ}{B} - \overset{\circ}{B}^\circ) \quad (1.20)$$

which yields

$$\overset{\circ}{I} = (\overset{\circ}{I}^\circ + \overset{\circ}{B} - \overset{\circ}{B}^\circ) [1 - \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^\circ)]^{-1} \quad (1.21)$$

The inverse defined e.g. by expansion (sufficient for perturbation theory). Similarly, as (1.21)

$$\overset{\circ}{I} = [1 - (\overset{\circ}{B} - \overset{\circ}{B}^\circ) \overset{\circ}{G}]^{-1} (\overset{\circ}{I}^\circ + \overset{\circ}{B} - \overset{\circ}{B}^\circ) \quad (1.22)$$

is derived, where $\overset{\circ}{I}$ from (1.21) is to be inserted. Finally, (2.2) below yields

$$I = [1 - (1 + \overset{\circ}{I} \overset{\circ}{G})(\overset{\circ}{B} - \overset{\circ}{B}^\circ) \overset{\circ}{G} - \overset{\circ}{I}(\overset{\circ}{G} - \overset{\circ}{G}^\circ)]^{-1} [\overset{\circ}{I} + (1 + \overset{\circ}{I} \overset{\circ}{G})(\overset{\circ}{B} - \overset{\circ}{B}^\circ)] \quad (1.23)$$

which is, with $\overset{\circ}{I}$ from (1.22) inserted, the desired result. B itself can be recovered from

$$\overset{\circ}{B}^\circ = \frac{\overset{\circ}{I}^\circ - \overset{\circ}{I} \overset{\circ}{G} (\overset{\circ}{B}^\circ - \overset{\circ}{B}^\circ)}{1 + \overset{\circ}{I} \overset{\circ}{G} >}$$

which is divergent in perturbation theory from order g^2 on when the regularization is removed. We shall also use the BS equation for the absorptive part of the four-point vertex. We introduce

$$P \overset{q}{I} P' + P \overline{\overset{q}{I}} P' \leftrightarrow P \overset{q}{I} P'$$

for $(p-p')^2 \geq 4m^2$, $p_0 - p'_0 \geq 2m$, $(-\frac{1}{2}q-p)^2 < 9m^2 \dots$

$\dots (\frac{1}{2}q-p')^2 < 9m^2$, and

$$(2\pi)^4 \delta(p-p') \bar{G}(-\frac{1}{2}q+p) G(-\frac{1}{2}q-p) \leftrightarrow \overset{q}{\tilde{G}},$$

$$(2\pi)^4 \delta(p-p') \leftrightarrow \overset{\sim}{1},$$

and

$$p \overset{q}{B} p' + p \overset{q}{B} p' + \dots \leftrightarrow p \overset{q}{V} p'$$

where $\tilde{V}$ in the sum over all contributions to $\tilde{I}$ not $\tilde{G}$ - reducible in the horizontal channel. The equation

$$\tilde{I} = \tilde{V} + \tilde{I} \tilde{G} \tilde{V} = \tilde{V} + \tilde{V} \tilde{G} \tilde{I}, \quad (1.24a)$$

where in intermediate integrations the factor $\frac{1}{2}$ is to be dropped, was derived in [12] and can be recast into the forms

$$(\overset{\sim}{1} - \tilde{G} \tilde{V})(\overset{\sim}{1} + \tilde{G} \tilde{I}) = \dots = 1 \quad (1.24b)$$

analogous to (1.19b). The ill-defined (imaginary) constant in B drops out in $\tilde{V}$.

2. Short-distance expansions.

Short distance (SD) expansions were proposed by Wilson and proven to all orders of perturbation theory by Zimmermann [2]. Here we give an elementary derivation of the simplest formulae that we will need later on. More details to this section can be found in the appendices A of [13] and [14].

2.1 Short-distance expansion in the narrow sense.

From (1.19a) follows

$$I - \overset{\circ}{I} = B - \overset{\circ}{B} + (I - \overset{\circ}{I})GB + \overset{\circ}{I}(G - \overset{\circ}{G})B + \overset{\circ}{I}\overset{\circ}{G}(B - \overset{\circ}{B}) \quad (2.1)$$

and therefrom with (1.19b)

$$I - \overset{\circ}{I} = (1 + \overset{\circ}{I}\overset{\circ}{G})(B - \overset{\circ}{B})(1 + GI) + \overset{\circ}{I}(G - \overset{\circ}{G})I. \quad (2.2)$$

Inserting this into the identity

$$GI = \overset{\circ}{G}\overset{\circ}{I} + (G - \overset{\circ}{G})I + \overset{\circ}{G}(I - \overset{\circ}{I})$$

yields

$$\begin{aligned} 1 + GI &= 1 + \overset{\circ}{G}\overset{\circ}{I} + (G - \overset{\circ}{G})I \\ &+ (1 + \overset{\circ}{G}\overset{\circ}{I})\overset{\circ}{G}(B - \overset{\circ}{B})(1 + GI) + \overset{\circ}{G}\overset{\circ}{I}(G - \overset{\circ}{G})I \end{aligned}$$

$$= (1 + \overset{\circ}{G}\overset{\circ}{I}) [1 + \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^{\circ})(1 + \overset{\circ}{G}I) + (\overset{\circ}{G} - \overset{\circ}{G})\overset{\circ}{I}] . \quad (2.3)$$

The transposed form of (1.22)

$$\overset{\circ}{I} = (\overset{\circ}{I}^{\circ} + \overset{\circ}{B} - \overset{\circ}{B}^{\circ}) [1 - \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^{\circ})]^{-1} \quad (2.4)$$

is equivalent to

$$1 + \overset{\circ}{G}\overset{\circ}{I} = (1 + \overset{\circ}{G}\overset{\circ}{I}^{\circ}) [1 - \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^{\circ})]^{-1} . \quad (2.5)$$

Inserting (2.5) in (2.3) gives

$$1 + \overset{\circ}{G}I = (1 + \overset{\circ}{G}\overset{\circ}{I}^{\circ})(1 + W) \quad (2.6)$$

where we used the abbreviation

$$W = -1 + [1 - \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^{\circ})]^{-1} [1 + \overset{\circ}{G}(\overset{\circ}{B} - \overset{\circ}{B}^{\circ})(1 + \overset{\circ}{G}I) + (\overset{\circ}{G} - \overset{\circ}{G})\overset{\circ}{I}] . \quad (2.7)$$

Acting with (2.6) on the suitably normalized bare vertex (a, upon regularization, finite constant in momentum space) gives

$$\gamma(1 + \overset{\circ}{G}I) = \Gamma = \gamma(1 + \overset{\circ}{G}\overset{\circ}{I}^{\circ})(1 + W) = \overset{\circ}{\Gamma}^{\circ} < (1 + W) \quad (2.8)$$

where we use the notation

$$\Gamma((\frac{1}{2} q+p) (\frac{1}{2} q-p), (-q)) \leftrightarrow \overset{q}{\Gamma}^p .$$

Hereby, $-q$ is the momentum acting on the (suitably normalized, see below) finite composite operator $1/2 N_2(\phi^2)$ in the sense of Zimmermann [2]. In the present context, matrix elements of this operator are defined in terms of skeleton expansions in which the Γ just introduced appears, and for the normalization of which we choose Zimmermann's

$$\Gamma(00,0; m^2, g) = 1. \quad (2.9)$$

Normalization independent, (2.6) takes the form

$$1 + \overset{\circ}{G}I = \overset{\circ}{G}\overset{\circ}{I}^{\circ} \frac{\Gamma}{\overset{\circ}{\Gamma}^{\circ}} + 1 + W . \quad (2.10)$$

Using this in the skeleton expansion of a general matrix element, (2.10) is equivalent to

$$N_O(\phi(x+\xi)\phi(x-\xi)) = E(\xi) \frac{1}{2} N_2(\phi^2(x)) + N_2(\phi(x+\xi)(x-\xi)) \quad (2.11)$$

in Zimmermann's notation [2]. In (2.10) and (2.11), the parts most singular as the left relative momentum goes to infinity, or $\xi \rightarrow 0$, respectively, is isolated and seen to be factorizable; the remainder term admits to be integrated over that relative momentum, or to let $\xi \rightarrow 0$, respectively, whereby the cofactor of the first term is again obtained. That that term, $<(1+W)$ in (2.8), is finite follows from examining all pos-

sibly divergent integrations involved in (2.7) and (2.8) and observing that they have a subtraction ("from outside") yielding convergence. The general matrix element of (2.11) corresponds to attaching to (2.10) from the right the diagrams

$$\begin{array}{c} \text{---} \square \text{---} \\ \text{---} \square \text{---} \end{array} \quad + \quad \left\{ \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \square \text{---} \end{array} \right\} \geq 4 \text{ arguments}$$

2 irred

where the square functions are the general connected (amputated) functions with only the two-particle-irreducibility imposed. It is this condition that prevents a divergent integral to appear in the attaching-process.

Using the function Γ constructed in (2.8) we can define Green's functions

$$\begin{aligned} & \langle (\phi(x_1) \dots \phi(x_{2n}) \frac{1}{2} N_2(\phi^2(y_1)) \dots \frac{1}{2} N_2(\phi^2(y_\ell)))_+ \rangle_{\text{conn.}} \\ & = G(x_1 \dots x_{2n}, y_1 \dots y_\ell; m^2, g) \end{aligned}$$

and corresponding vertex functions

$$\Gamma(x_1 \dots x_{2n}, y_1 \dots y_\ell; m^2, g)$$

and Fourier transforms (with momentum-conserving delta function taken out, as before)

$$\Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g)$$

by skeleton expansion, with superficial divergence, from (1.13),

$$D = \dim \Gamma = 4 - 2n - 2\ell. \quad (2.12)$$

In view of (2.12) and (2.9), only for $n = 0$, $\ell = 2$ must a further renormalization condition be imposed, and we choose

$$\Gamma(, 00; m^2, g) = 0. \quad (2.13)$$

The final, subtractive renormalization that ensures (2.13) is best imposed on the subtracted form

$$\Pi - \overset{\circ}{\Pi} = \Gamma(G - \overset{\circ}{G})\overset{\circ}{\Gamma}^T + \Gamma G(B - \overset{\circ}{B})\overset{\circ}{G}\overset{\circ}{\Gamma}^T \quad (2.14)$$

of the skeleton expansion for

$$\Gamma(, q(-q); m^2, g) \leftrightarrow \overset{q}{\Pi}$$

which is easily derived from (cp. (2.8))

$$\begin{aligned}\Pi &= \gamma(G + GIG)\gamma^T + \text{const} \\ &= \Gamma G \gamma^T + \text{const} = \gamma \Gamma^T + \text{const}.\end{aligned}$$

From (2.12) and (2.13) follow

$$\Gamma(p_1 p_2, q_1; m^2, g) = 1 + O(g), \quad (2.15a)$$

$$\Gamma(q, (-q); m^2, g) = O(1). \quad (2.15b)$$

2.2 Related expansion formulae.

(2.2) can be written

$$I = \overset{\circ}{I} \left[1 + G(B - \overset{\circ}{B})(1 + GI) + (G - \overset{\circ}{G})I \right] + (B - \overset{\circ}{B})(1 + GI). \quad (2.16)$$

Inserting here (2.4) yields, with (2.7) and (2.8)

$$I = \overset{\circ}{I} \circ (\overset{\circ}{\Gamma} \circ)^{-1} \Gamma + (\overset{\circ}{B} - \overset{\circ}{B} \circ)(1 + W) + (B - \overset{\circ}{B})(1 + GI) \quad (2.17)$$

which is, for large left relative momentum, a SD expansion for an amputated function in contrast to (2.10).

From (2.8) and (1.19) follows the subtracted BS equation for Γ

$$\begin{aligned}\Gamma - \overset{\circ}{\Gamma} &= \Gamma G B - \overset{\circ}{\Gamma} \overset{\circ}{G} \overset{\circ}{B} = (\Gamma - \overset{\circ}{\Gamma}) \overset{\circ}{G} \overset{\circ}{B} + \Gamma (G - \overset{\circ}{G}) \overset{\circ}{B} + \Gamma G (B - \overset{\circ}{B}) \\ &= \Gamma (G - \overset{\circ}{G}) \overset{\circ}{I} + \Gamma G (B - \overset{\circ}{B})(1 + GI)\end{aligned}$$

and herefrom, using (2.17) at zero total momentum in transposed form,

$$\Gamma = \overset{\circ}{\Gamma} + (\Pi - \overset{\circ}{\Pi})(\overset{\circ}{\Gamma} \circ)^{-1} \overset{\circ}{I} + \Gamma R \quad (2.18)$$

where we have used (2.14) and the abbreviation

$$R = G(B - \overset{\circ}{B}) + \left[G - \overset{\circ}{G} + G(B - \overset{\circ}{B}) \overset{\circ}{G} \right] (1 + \overset{\circ}{W}^T) (\overset{\circ}{B} \circ \overset{\circ}{B}).$$

(2.18) is an expansion formula for large (right) relative momentum.

Inserting (2.18) into (2.17) subtracted from itself at zero momentum yields

$$\begin{aligned}I &= \overset{\circ}{I} + \overset{\circ}{I} \circ (\overset{\circ}{\Gamma} \circ)^{-1} (\Pi - \overset{\circ}{\Pi})(\overset{\circ}{\Gamma} \circ)^{-1} \overset{\circ}{I} \\ &+ (\overset{\circ}{B} - \overset{\circ}{B} \circ)(W - \overset{\circ}{W}) + (B - \overset{\circ}{B})(1 + GI) + \overset{\circ}{I} \circ (\overset{\circ}{\Gamma} \circ)^{-1} \Gamma R,\end{aligned} \quad (2.19)$$

an expansion formula for both the left and right relative momentum

large compared to the total momentum. (2.17-19) will be used in sect. 5.

We finally derive an expansion formula for absorptive parts used recently by Mueller [15]. From (1.24) follows, by the same algebraic steps, the analog of (2.17), for zero total momentum

$$\begin{aligned}\overset{\circ}{I} &= \overset{\circ}{I} \circ (\overset{\circ}{\Gamma} \circ)^{-1} \overset{\circ}{\Gamma} + (\overset{\circ}{V} - \overset{\circ}{V} \circ)(1 + \overset{\circ}{W}) \\ &\equiv \overset{\circ}{I} \circ \overset{\circ}{\Gamma}_M + \overset{\circ}{I} \text{ reg}\end{aligned} \quad (2.20a)$$

with

$$\overset{\circ}{\Gamma}_M = \langle (1 + \overset{\circ}{W}) \rangle \quad (2.20b)$$

and

$$1 + \overset{\circ}{W} = [1 - \overset{\circ}{G} (\overset{\circ}{V} - \overset{\circ}{V}^0)]^{-1}. \quad (2.20c)$$

$\overset{\circ}{I}^{\text{reg}}$ vanishes, like the analogous term in (2.17), for large left relative momentum squared. Actually, Mueller [15] first separates (1.24) for zero total momentum with respect to a conserved angular quantum number, whereupon (1.24a) becomes a one-dimensional integral equation, with integration over the square of the intermediate relative momentum. (2.20) is then to be read for fixed angular quantum number with, where indicated, the right relative momentum squared zero rather than zero as a vector. The vanishing of $\overset{\circ}{I}^{\text{reg}}$ hereby for large left relative momentum squared is nontrivial.

3. Mass vertex insertion.

The references for this section are [16], [13], and, in particular, [14].

3.1 Derivation of PDEs.

Consider in the computation of all Green's and vertex functions from the Lagrangian (1.11) (amended also by the source term $\frac{1}{2} KN_2(\phi^2)$ to generate $\frac{1}{2} N_2(\phi^2)$ - arguments) the replacement

$$L \rightarrow L - s \frac{1}{2} m^2 N_2(\phi^2) \phi(g) \equiv L^s \quad (3.1)$$

where

$$\phi(g) = 1 + O(g^2) \quad (3.2)$$

will be specified later. The generating functional of the Green's functions

$$G_{\text{disc}} \{J, K\} = \langle (e^{i \int J \phi + \frac{i}{2} \int KN_2(\phi^2)})_+ \rangle$$

then changes accordingly:

$$\begin{aligned} G_{\text{disc}}^s \{J, K\}(m^2, g) &= \text{const. } G_{\text{disc}} \{J, K - sm^2 \phi(g)\}(m^2, g) \\ &= G_{\text{disc}} \{(1 + \Delta z(s))J, (1 + \Delta u(s))K\}(m^2 + \Delta m^2(s), g + \Delta g(s)) \\ &\quad \cdot \exp \left[- \frac{i}{2} \Delta v(s) \int K^2(x) dx \right] \end{aligned} \quad (3.3)$$

where the first equality is a consequence of the integrated form of the Schwinger action principle [6], and the second equality expresses that the change (3.1) is one within the family of ϕ^4 theories which

are completely parametrized by the renormalization parameters m^2 and g , that (1.17b) and (2.12), to be upheld, require further multiplicative factors to be allowed, and that (2.13), to be upheld, may require to change the constant in (2.14) and thus to provide for an extra subtraction term for the $n=0$, $\ell=2$ VF.

Since $N_2(\phi^2)$ is a finite operator, all $\Delta z(S)$, $\Delta u(S)$ etc. in (3.3) are finite and, in fact, computable in renormalized perturbation theory, see appendix C of [13] where, however, the formula following (C.7) should read

$$s=m^{-2} \phi(g)^{-1} \int_{m^2}^{m^2(s)} dm^2(s') Z(s')^{-1} \phi(g(s')) \\ \cdot \exp \left[- \int_2^{m^2(s')} m^2(s'')^{-1} dm^2(s'') \eta(g(s'')) \right] .$$

Differentiating (3.3) with respect to s at $s=0$ gives with (1.2)

$$-im^2 \phi(g) \frac{1}{i} \int dx \frac{\delta}{\delta K(x)} G\{J, K\}(m^2, g) \\ = \left[\gamma(g) \int dx J(x) \frac{\delta}{\delta J(x)} + (2\gamma(g) + \eta(g)) \int dx K(x) \frac{\delta}{\delta K(x)} \right. \\ \left. + m^2 \frac{\partial}{\partial m^2} + \beta(g) \frac{\partial}{\partial g} \right] G\{J, K\}(m^2, g) - i\kappa(g) \frac{1}{2} \int K(x)^2 dx \quad (3.4)$$

where we have introduced the functions

$$\frac{\partial \Delta m^2(s)}{\partial s} \Big|_{s=0} = m^2 \quad \text{by normalization } (\phi(g) \text{ choice}) \quad (3.5a)$$

$$\frac{\partial \Delta g}{\partial s} \Big|_{s=0} = \beta(g) \quad (3.5b)$$

$$\frac{\partial \Delta z(s)}{\partial s} \Big|_{s=0} = \gamma(g) \quad (3.5c)$$

$$\frac{\partial \Delta u(s)}{\partial s} \Big|_{s=0} = 2\gamma(g) + \eta(g) \quad (3.5d)$$

$$\frac{\partial \Delta v(s)}{\partial s} \Big|_{s=0} = \kappa(g). \quad (3.5e)$$

Using now (1.3), (1.6) and (1.7) gives

$$-im^2 \phi(g) \frac{1}{i} \int dx \frac{\delta}{\delta K(x)} \Gamma\{A, K\}(m^2, g) \\ = \left[-\gamma(g) \int dx A(x) \frac{\delta}{\delta A(x)} + (2\gamma(g) + \eta(g)) \int dx K(x) \frac{\delta}{\delta K(x)} \right. \\ \left. + m^2 \frac{\partial}{\partial m^2} + \beta(g) \frac{\partial}{\partial g} \right] \Gamma\{A, K\}(m^2, g) -$$

$$-\frac{i}{2} \kappa(g) \int dx K(x)^2. \quad (3.6)$$

Or, going from the generating functional to the VFs themselves,

$$O_{P_{2n}, \ell} \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) = -im^2 \phi(g) \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2, g) - i \kappa(g) \delta_{no} \delta_{\ell 2} \quad (3.7a)$$

$$O_{P_{2n}, \ell} \equiv m^2 \frac{\partial}{\partial m^2} + \beta(g) \frac{\partial}{\partial g} - 2n\gamma(g) + \ell(2\gamma(g) + \eta(g)). \quad (3.7b)$$

Requiring consistency of (3.7) with the five renormalization conditions (1.17a-c), (2.9), (2.13) allows to compute the five functions in (3.5), e.g. as power series in g :

$$\beta(g) = b_0 g^2 + b_1 g^3 + \dots \quad b_0 = 3(32\pi^2)^{-1} > 0 \quad (3.8a)$$

$$b_1 = -17(2^9 3\pi^4)^{-1}$$

$$\gamma(g) = c_0 g^2 + c_1 g^3 + \dots \quad c_0 = (2^{11} 3\pi^4)^{-1} \quad (3.8b)$$

$$\eta(g) = h_0 g + h_1 g^2 + \dots \quad h_0 = \frac{1}{3} b_0 \quad (3.8c)$$

$$\kappa(g) = \kappa_0 + \kappa_1 g + \dots \quad \kappa_0 = \frac{1}{3} b_0 \quad (3.8d)$$

$$\phi(g) = 1 + f_2 g^2 + \dots \quad (3.8e)$$

The value of b_1 was computed in [17].

Consider replacing the renormalization conditions (1.17b-c), (2.9), (2.13), by others such that (1.18a-b) and (2.15a-b) remain valid for the new functions $\bar{\Gamma}$ with $\bar{g}$ replacing g , e.g. replacing (1.17b-c) by

$$\begin{aligned} \bar{\Gamma}(00, ; m^2, \bar{g}) &= -im^2 \\ \bar{\Gamma}(0000, ; m^2, \bar{g}) &= -i\bar{g} \text{ etc.} \end{aligned} \quad (3.9)$$

Then renormalization theory [9] assures that

$$\begin{aligned} \bar{\Gamma}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, \bar{g}) \\ = \bar{Z}_1(\bar{g})^n \bar{Z}_2(\bar{g})^\ell \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g(\bar{g})) \\ + i \delta_{no} \delta_{\ell 2} \bar{F}(\bar{g}), \end{aligned} \quad (3.10)$$

must hold, with, according to (1.18a-b) and (2.15a-b), power series

$$\bar{Z}_1(\bar{g}) = 1 + O(\bar{g}^2) \quad (3.11a)$$

$$\bar{Z}_2(\bar{g}) = 1 + O(\bar{g}) \quad (3.11b)$$

$$g(\bar{g}) = \bar{g} + O(\bar{g}^2) \quad (3.11c)$$

$$\bar{F}(\bar{g}) = O(1). \quad (3.11d)$$

Inserting (3.10) into the PDEs for $\bar{T}$, analogous to (3.7), gives relations between the new functions $\bar{\beta}(\bar{g})$ etc. and the ones in (3.7), e.g.

$$\beta(g(\bar{g}))^{-1} dg = \bar{\beta}(\bar{g})^{-1} d\bar{g}. \quad (3.12)$$

This relation is consistent with (3.11c) and the analog of (3.8a),

$$\bar{\beta}(\bar{g}) = \bar{b}_0 \bar{g}^2 + \bar{b}_1 \bar{g}^3 + \dots,$$

only if

$$\bar{b}_0 = b_0, \bar{b}_1 = b_1. \quad (3.13a)$$

Similarly, one derives

$$\bar{c}_0 = c_0, \bar{h}_0 = h_0, \bar{\kappa}_0 = \kappa_0. \quad (3.13b)$$

Use of conventions (3.9) and of (3.13a) simplifies somewhat the computation of b_0 and b_1 . If only $\beta(g)$ and $\bar{\beta}(\bar{g})$ are given, $g(\bar{g})$ in (3.11c) is obtainable from (3.12) only if in addition an integration constant, easily obtained from (3.10), is known.

3.2 Integration of PDEs.

The ordinary DE of the characteristic of the PDEs (3.7) is

$$\frac{dm^2}{m^2} = \frac{dg}{\beta(g)}.$$

The solution we parametrize by λ :

$$m^2, g \rightarrow m^2 \lambda^2, g(\lambda)$$

where $g(\lambda)$ is defined by

$$\ln \lambda^2 = \int_g^{g(\lambda)} dg' \beta(g')^{-1}. \quad (3.14)$$

or, equivalently, by

$$g(\lambda) = \rho^{-1} (\ln \lambda^2 + \rho(g)) \quad (3.15a)$$

where

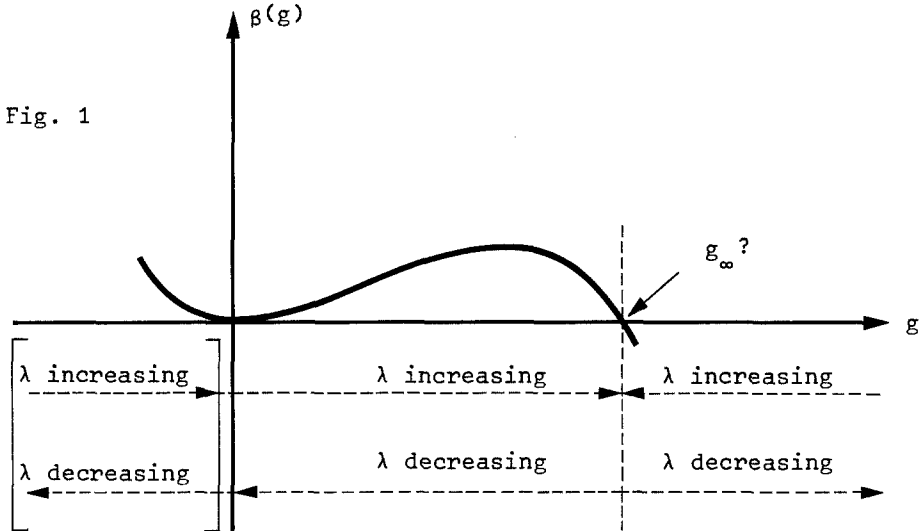
$$\int_g^g \frac{dg'}{\beta(g')} = \rho(g). \quad (3.15b)$$

From (3.15) follows

$$\begin{aligned} F(g(\lambda)) &= F(\rho^{-1}(\ln \lambda^2 + \rho(g))) = \sum_{r=0}^{\infty} \frac{(\ln \lambda^2)^r}{r!} \left[\frac{\partial}{\partial \rho(g)} \right]^r F(\rho^{-1}(\rho(g))) \\ &= \sum_{r=0}^{\infty} \frac{(\ln \lambda^2)^r}{r!} \left[\beta(g) \frac{\partial}{\partial g} \right]^r F(g) \end{aligned} \quad (3.16)$$

which yields the perturbation expansion of e.g. $g(\lambda)$ upon setting $F(g) \equiv g$ and using (3.8a). From (3.14) follows that $g(\lambda)$ increases

or decreases as a function of increasing λ if $\beta(g(\lambda))$ is positive or negative, respectively. In view of (3.8a) we obtain Fig. 1 where it is assumed that $\beta(g)$ has for positive g a first (and first-order) zero



at g_∞ . The existence of such a zero is uncertain, but the fact [17] that $b_1 < 0$ suggests it. For $g < 0$, the ϕ^4 theory can be argued to be unstable [18]. For $0 < g < g_\infty$, (3.15) gives

$$g(\lambda) = \frac{1}{b_0 \ln(\lambda^{-2})} - \frac{b_1 \ln \ln \lambda^{-2}}{b_0^3 (\ln \lambda^{-2})^2} + O\left(\frac{1}{(\ln \lambda)^2}\right), \quad \lambda \rightarrow 0. \quad (3.17)$$

We now introduce the functions

$$a(g) = \exp \left[2 \int_0^g \frac{dg'}{\beta(g')} \gamma(g') \right] = 1 + 2 \frac{c_0}{b_0} g + \dots \quad (3.18a)$$

$$h(g) = \exp \left[\int_0^g \frac{dg'}{\beta(g')} \eta(g') \right] = g^{1/3} [1 + \dots g + \dots] \quad (3.18b)$$

$$k(g) = \int_0^g \frac{dg'}{\beta(g')} a(g')^2 h(g')^2 \kappa(g') = -g^{-1/3} [1 + \dots g + \dots] \quad (3.18c)$$

where in (3.18b) and (3.18c) the integration constants have been chosen conveniently. The integral of (3.7) is now

$$\begin{aligned} & \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) \\ &= a(g)^{n-\ell} a(g(\lambda))^{-n+\ell} h(g)^{-\ell} h(g(\lambda))^\ell \\ & \quad \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2 \lambda^2, g(\lambda)) - \end{aligned}$$

$$\begin{aligned}
& -i\delta_{no} \delta_{\ell 2} a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))] \\
& -im^2 \int_{\lambda^2}^1 d\lambda'^2 a(g)^{n-\ell} a(g(\lambda'))^{-n+\ell} h(g)^{-\ell} h(g(\lambda'))^{\ell} \phi(g(\lambda')) \\
& \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_{\ell} 0; m^2 \lambda'^2, g(\lambda'))
\end{aligned} \tag{3.19}$$

where the sum of the first two terms on the r.h.s. can also be written as $\Gamma^{s(g,\lambda)}(p_1 \dots p_{2n}, q_1 \dots q_{\ell}; m^2, g)$, the superscript meant in the sense of (3.1) and (3.3), (see Appendix C of [13], with, however, the correction of formulae as indicated above) for

$$s(g, \lambda) = \phi(g)^{-1} a(g) h(g) \int_1^{\lambda^2} d\lambda'^2 \phi(g(\lambda')) a(g(\lambda'))^{-1} h(g(\lambda'))^{-1}. \tag{3.20}$$

Thus, (3.19) expresses the effect of noninfinitesimal mass vertex insertion as the integral over the infinitesimal steps thereto. The point of (3.19) is that the effects of the infinitesimal steps are more easily controlled than the effect of the noninfinitesimal operation.

Generalizing (3.1) one may also consider the effect of insertions of other finite operators, $N_4(\phi^4)$ and $N_4(\partial_\mu \phi \partial^\mu \phi)$, that occur in Zimmermann's [2] effective Lagrangian, and is led to generalizations [19] of the PDEs (3.7). However, these other PDEs have found no application yet since the vertex insertions then have operator dimension four, and VFs with such insertions do not have a characteristic property distinguishing them from the original VFs.

4. Asymptotic forms.

The references for this section are [16], [20], [13], and [14].

4.1 Existence of asymptotic forms.

(3.19) can be made useful only if λ can be chosen such that the r.h.s., excepting the integral, simplifies. In view of (3.17), the appropriate choice is $\lambda=0$.

From now on, we will consider, unless said otherwise, Euclidean momenta only, i.e. energy components (p_{i0}, q_{j0}) imaginary (ip_{i4}, iq_{j4}), space components real. Now from (2.12)

$$\begin{aligned}
& \Gamma(p_1 \dots p_{2n}, q_1 \dots q_{\ell} 0; m^2 \lambda^2, \bar{g}) \\
& = \lambda^{-2+2n+2\ell} \Gamma(\lambda^{-1} p_1 \dots \lambda^{-1} p_{2n}, \lambda^{-1} q_1 \dots \lambda^{-1} q_{\ell} 0; m^2, \bar{g})
\end{aligned} \tag{4.1}$$

and Weinberg's theorem [21] yields for the VF on the r.h.s. as $\lambda \rightarrow 0$ a behaviour proportional to $(\lambda^{-1})^{4-2n-\lambda(\ell+1)}(\ln \lambda)^c$, with c depending on the order considered, provided the external momenta are not such that there exists a partition of the graph into subgraphs such that the external large momenta can be contained in some subgraphs and the sum of Weinberg exponents [21] for these subgraphs is larger than the exponent for the whole graph.

In the present model, it is required that there exist a partition of the external momenta into at least three nontrivial even sets (i.e. an even number of p and an arbitrary number of q momenta) of zero sum each, to obtain a Weinberg exponent larger than the superficial one. In view of the vanishing q -momentum in (3.21), the vanishing of one even partial sum of other momenta is required to invalidate the estimate mentioned before. Such momenta sets we call exceptional, other momenta sets (e.g., generic ones) nonexceptional. In formulae, nonexceptional momenta are defined by the condition that

$$\Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2 \lambda^2, \bar{g}) = 0((\ln \lambda) \dots) \quad (4.2)$$

at least to all orders in perturbation theory, and the necessary and sufficient condition (for Euclidean momenta) for (4.2) to hold was stated before.

If (4.2) holds, then due to (3.16) the integral in (3.19) will allow $\lambda \rightarrow 0$, and thus

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} [a(g)^{n-\ell} a(g(\lambda))^{-n+\ell} h(g)^{-\ell} h(g(\lambda))^\ell \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2 \lambda^2, g(\lambda)) \\ & - i a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))] \delta_{no} \delta_{\ell 2}] \\ & \equiv \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) \end{aligned} \quad (4.3)$$

then exists at least to all orders in perturbation theory; (3.17) suggests that (in this nonexceptional case) (4.3) exists also outside of perturbation theory. In view of what was said after (3.19), we have also

$$\begin{aligned} & \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) \\ & = \Gamma^{S(g,0)}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) \end{aligned} \quad (4.4)$$

which identifies these functions as VFs of a ϕ^4 theory with that particular choice of the bare mass which, as we shall prove below, leads to a vanishing renormalized mass. We call the VFs Γ_{as} the preasymptotic ones.

4.2 Properties of asymptotic forms.

Replacing in (4.3) λ by $\lambda\lambda'$ and letting $\lambda' \rightarrow 0$ gives, due to $g(\lambda\lambda') = [g(\lambda)](\lambda')$ from (3.14),

$$\begin{aligned} & a(g)^{n-l} a(g(\lambda))^{-n+l} h(g)^{-l} h(g(\lambda))^l \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2 \lambda^2, g(\lambda)) \\ & - i a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))] \delta_{no} \delta_{l2} \\ & = \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2, g) \end{aligned} \quad (4.5)$$

which is equivalent to

$$\begin{aligned} & O_{p_{2n}, l} \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2, g) \\ & = -\kappa(g) \delta_{no} \delta_{l2} \end{aligned} \quad (4.6)$$

with $O_{p_{2n}, l}$ from (3.7b). Rewriting (4.5) as

$$\begin{aligned} & \Gamma_{as}(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_l; m^2, g) \\ & = \lambda^{4-2n-2l} a(g)^{n-l} a(g(\lambda))^{-n+l} h(g)^{-l} h(g(\lambda))^l \\ & \cdot \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2, g(\lambda)) \\ & - i a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))] \delta_{no} \delta_{l2} \end{aligned} \quad (4.7)$$

allows, because of (3.17), to read off the small-momenta behaviour of Γ_{as} , to arbitrary precision apart from the λ -independent overall factor, from the series expansions of the functions of $g(\lambda)$ involved. In particular, it follows that for $p^2 \rightarrow 0$

$$\begin{aligned} & \Gamma_{as}(p(-p), ; m^2, g) \\ & = i a(g) p^2 [1 - 2b_0^{-2} c_0 (\ln[m^2(-p^2 - i\varepsilon)^{-1}])^{-1+0} ((\ln p^2)^{-2} \ln \ln p^2)], \end{aligned} \quad (4.8)$$

showing that, as claimed, the Γ_{as} describe a zero-mass theory, with moreover the massless particles discrete, a non-perturbation theoretical result. One may also say that, in view of (4.7) and $g(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$, the preasymptotic theory is for $0 < g < g_\infty$ "asymptotically free in the infrared".

From (3.19) for $\lambda \rightarrow 0$, (4.3), and (2.12) follows

$$\begin{aligned} & \Gamma(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_l; m^2, g) \\ & = \Gamma_{as}(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_l; m^2, g) \\ & - i m^2 \lambda^{2-2n-2l} \int_0^1 d\lambda' a(g)^{n-l} a(g(\lambda'))^{-n+l} \end{aligned}$$

$$\begin{aligned}
& \cdot h(g)^{-\ell} h(g(\lambda'))^{\ell} \phi(g(\lambda')) \\
& \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_{\ell} \ 0; m^2 \lambda^{-2} \lambda'^2, g(\lambda')). \quad (4.9)
\end{aligned}$$

This shows that the Γ_{as} on the r.h.s. collects, in view of (4.2), and (3.16), for $\lambda \rightarrow \infty$ in perturbation theory all terms with factors $\lambda^{4-2n-2\ell} (\ln \lambda)^K$, $K \geq 0$, and (4.7) shows that it collects these terms only while the remainder integral in (4.9) collects all terms with factors $\lambda^{2r-2n-2\ell} (\ln \lambda^2)^K$, $K \geq 0$, $1 \leq r < \infty$.

Use of (3.19) in (4.3) allows to rewrite the limitand as

$$\begin{aligned}
& \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_{\ell}; m^2, g) \\
& - i m^2 \int_0^{\lambda} d\lambda' \lambda'^2 a(g)^{n-\ell} a(g(\lambda'))^{-n+\ell} h(g)^{-\ell} h(g(\lambda'))^{\ell} \\
& \cdot \phi(g(\lambda')) \Gamma(p_1 \dots p_{2n}, q_1 \dots q_{\ell} \ 0; m^2 \lambda'^2, g(\lambda'))
\end{aligned}$$

where, due to (3.16) and (4.2), the integral yields in its perturbation theoretical expansion only terms $\lambda^{2r} (\ln \lambda^2)^K$, $K \geq 0$, $r \geq 1$. It follows that in the limitand in (4.3) all terms not of the last-mentioned form, except λ -independent ones which yield the Γ_{as} function, cancel identically. This implies that in the limitand in (4.3), only the λ -independent terms need be treated precisely, all terms involving positive powers of $\ln \lambda^2$ being immaterial, which in turn, in conjunction with our conclusions from (4.9) proves the correctness of the "elementary recipe": If $\Gamma(p_1 \dots p_{2n}, q_1 \dots q_{\ell}; m^2 \lambda^2, g)$ is expanded for $\lambda \rightarrow 0$ in a double power series in λ^2 and $\ln \lambda^2$, then the sum of the λ -independent terms is $\Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_{\ell}; m^2, g)$, and the sum of the terms with factors $\lambda^0 (\ln \lambda^2)^K$, $K \geq 0$, is $\Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_{\ell}; m^2 \lambda^2, g)$. To obtain from (4.9) all terms with factors of form $\lambda^{2-2n-2\ell} (\ln \lambda^2)^K$, $K \geq 0$, requires the large- λ analysis of the VF in the integrand. These momenta are exceptional, however, such that (4.9) itself is not applicable for that VF. The appropriate modification of (4.9) will be discussed in Sect. 5.3.

4.3 Relation to zero-mass theory.

The zero-mass ϕ^4 theory VFs are directly constructed in perturbation theory in appropriate parametrization, analogous to the procedure of

Gell-Mann and Low [22] in QED, employing a normalization mass U and coupling constant, V , and denoted by $\Gamma_O(p_1 \dots p_{2n}, q_1 \dots q_\ell; U^2, V)$. As renormalization conditions we choose

$$\Gamma_O(00, ; U^2, V) = 0, \quad (4.10a)$$

$$\Gamma_O(p(-p), : U^2, V) \Big|_{p^2 = -U^2} = -iU^2, \quad (4.10b)$$

$$\Gamma_O(p_1 \dots p_4, : U^2, V) \Big|_{s.p.t. \text{ to } -U^2} = -iV, \quad (4.10c)$$

$$\Gamma_O(q/2 \ q/2 \ , \ -q; U^2, V) \Big|_{q^2 = -U^2} = 1, \quad (4.10d)$$

$$\Gamma_O(\ , \ q(-q); U^2, V) \Big|_{q^2 = -U^2} = 0. \quad (4.10e)$$

The normalization mass U is nonintrinsic, such that its change can be compensated by a change of normalization and of coupling constant, and an additive term for $n=0, \ell=2$. The functional relations expressing this fact are the renormalization group equations of Gell-Mann and Low [22], the differential form of which is

$$\begin{aligned} \left[U^2 \frac{\partial}{\partial U^2} + \hat{\beta}(V) \frac{\partial}{\partial V} - 2n\hat{\gamma}(V) + \ell(2\hat{\gamma}(V) + \hat{\eta}(V)) \right] \Gamma_O(p_1 \dots p_{2n}, q_1 \dots q_\ell; U^2, V) \\ = -i\delta_{no} \delta_{\ell 2} \hat{\kappa}(V) \end{aligned} \quad (4.11)$$

with parametric functions with expansions analogous to (3.8a-e). Since also the Γ_{as} are the VFs of a zero-mass theory, renormalization theory [8], [9] implies that the Γ_{as} and Γ_O must be related in the analogous fashion to (3.10),

$$\begin{aligned} \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g) \\ = Z_1(g)^{-n} Z_2(g)^{-\ell} \Gamma_O(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, V(g)) \\ + i\delta_{no} \delta_{\ell 2} f(g). \end{aligned} \quad (4.12)$$

Applying here $Op_{2n, \ell}$ of (3.7b) to both sides and using (4.6) and (4.11) yields

$$\frac{dg}{\beta(g)} = \frac{dV(g)}{\hat{\beta}(V(g))}, \quad (4.13a)$$

$$Z_1(g) = \hat{a}(V(g))a(g)^{-1} \quad (4.13b)$$

$$Z_2(g) = \hat{a}(V(g))^{-1} \hat{h}(V(g))^{-1} a(g)h(g), \quad (4.13c)$$

$$f(g) = a(g)^{-2} h(g)^{-2} [\hat{k}(V(g)) - k(V)] , \quad (4.13d)$$

with definition of the hatted functions analogous to (3.18). The properties analogous to (3.11),

$$Z_1(g) = 1 + O(g^2), \quad (4.14a)$$

$$Z_2(g) = 1 + O(g), \quad (4.14b)$$

$$V(g) = g + O(g^2), \quad (4.14c)$$

$$f(g) = O(1). \quad (4.14d)$$

are compatible with (4.13) only if

$$b_0 = \hat{b}_0 \quad b_1 = \hat{b}_1 \quad c_0 = \hat{c}_0 \quad h_0 = \hat{h}_0 \quad k_0 = \hat{k}_0 \quad (4.15)$$

hold, which relations, except the second, are easily verified directly and, in conjunction with (3.13), show the universality of these coefficients. In conclusion, the Γ_{as} can be computed from the Γ_0 via (4.12) provided only the functions $\beta(g)$, $\gamma(g)$, $\eta(g)$, and $\kappa(g)$ are known (and an integration constant needed in $V(g) = \hat{\rho}^{-1}(\rho(g))$ is determined from a one-loop graph).

5. Exceptional momenta.

References to this section are [13] and [14] .

5.1 Asymptotic forms at exceptional momenta.

Exceptional momenta were defined in Sec. 4.1 as momenta sets such that (4.2) is violated in perturbation theory, and a necessary and sufficient criterion for exceptionality at Euclidean momenta was given there. If (4.2) is violated, and in perturbation theory it then is so by having terms of behaviour $\lambda^{-2}(\ln \lambda^2)^K$ or worse, then Γ_{as} does not exist at such momenta as seen from (3.19) and (4.3). The reason for non-existence of Γ_{as} in this case is the infrared (IR) singularity of the massless-theory VF at such momenta; in fact, the vanishing of a partial sum of momenta is a necessary (and actually also sufficient) condition for nonregularity of the massless-theory VF restricted to Euclidean momenta as proven by Ruelle [23] .

For Euclidean exceptional momenta, at least the "elementary recipe" described at the end of sect. 4.2 is still applicable. If the momenta allow a partition into N even sets of zero sum each, then the leading terms as $\lambda \rightarrow 0$ are of the $\lambda^{-2a}(\ln \lambda^2)^K$ type where $a = \max(N-2, 0)$, as a simple application of Weinberg's [21] criteria shows. The sum of these leading terms defines $\Gamma_{as}(p_1 \dots p_{2N}, q_1 \dots q_L; m^2 \lambda^2, g)$; the

coefficient of λ^{-2a} itself is $\Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2, g)$. We will prove this, and obtain the analoga of (4.3) and (4.7), only for the renormalization functions at exceptional momenta; in sect. 5.3 below we consider a somewhat more general case. The nontrivial exceptional momenta sets for the renormalization functions (i.e. those with $D \geq 0$) are $(p(-p)00,)$, $(p(-p), 0)$, and $(p(-p)p'(-p'),)$. For the first of these sets, (3.7) becomes

$$\begin{aligned} & \text{Op}_{4,0} \Gamma(p(-p) 00,; m^2, g) \\ &= -im^2 \phi(g) \Gamma(p(-p)00,0; m^2, g). \end{aligned} \quad (5.1)$$

We want to isolate on the r.h.s. the parts of behaviour $(\ln p^2)^c$ as $-p^2 \rightarrow \infty$, since it is these that violate (4.2) and have the same behaviour as the terms in the operand on the l.h.s. of (5.1). (2.17) yields (we omit arguments m^2, g where expendable)

$$\begin{aligned} & \Gamma(p(-p)00,0) \\ &= \Gamma(p(-p)00,) \Gamma(00,00) + \Gamma_{\text{rem}}(p(-p)00, 0) \end{aligned} \quad (5.2)$$

where the remainder term stems from the last two terms in (2.17) and, as discussed in sect. 2, has large- p behaviour $(p^2)^{-1} (\ln p^2)^c$. Now

$$\begin{aligned} & -im^2 \phi(g) \Gamma(00,00) = \text{Op}_{2,1} \Gamma(00,0) = \eta(g) \\ & \text{Op}_{2,1} = \eta(g) + m^2 \frac{\partial}{\partial m^2} + \beta(g) \frac{\partial}{\partial g} \end{aligned} \quad (5.3)$$

from (2.9) and (3.7). Using (5.2) and (5.3) in (5.1) yields

$$\begin{aligned} & \text{Op}_{2,-1} \Gamma(p(-p)00,; m^2, g) \\ &= -im^2 \phi(g) \Gamma_{\text{rem}}(p(-p)00,0; m^2, g). \end{aligned} \quad (5.4)$$

Omitting to write the analog of (3.19) for the present case, we immediately proceed to the analog of (4.3)

$$\begin{aligned} & \Gamma_{\text{as}}(p(-p)00,; m^2, g) \\ &= \lim_{\lambda \rightarrow 0} \left[a(g)^2 a(g(\lambda))^{-2} h(g) h(g(\lambda))^{-1} \Gamma(p(-p)00,; m^2 \lambda^2, g(\lambda)) \right]. \end{aligned} \quad (5.5)$$

Now from (3.18b) and (3.17) follows

$$h(g(\lambda))^{-1} \approx g(\lambda)^{-1/3} \approx b_0^{1/3} [\ln \lambda^{-2}]^{1/3} \quad (5.6)$$

such that comparison of (5.5) with (4.3) shows that

$$\Gamma_{as}(p(-p)00; m^2, g) = 0 \quad (5.7)$$

i.e. the UR singularity for this momenta configuration is a suppression to zero. (5.7) is, using (5.5) and (5.6), sharpened to

$$\Gamma^s(g, \lambda)(p(-p)00; m^2, g) \quad (5.8)$$

$$\approx h(g)^{-1} b_0^{-1/3} [\ln \lambda^{-2}]^{-1/3} \Gamma_{as}(p(-p)00; m^2, g)$$

for $\lambda \rightarrow 0$.

It is of interest to compare (5.8) with singularity behaviour derived directly in the zero-mass theory. The identity (2.17) we will read as an equation in Γ_{as} theory, or, for more direct reference to previous equations, in Γ_0 theory related to Γ_{as} theory via (4.12). It is easy to see that the last two terms on the r.h.s. of (2.17) do exist in the massless theory by observing that wherever there is a "dangerous" propagator pair $\overset{\circ}{G}$, the factor to its left or right vanishes of first order at zero relative momentum thus eliminating the UR divergence in the loop integration. Since also the l.h.s. of (2.17) exists for total momentum $q \neq 0$, and so does $\overset{q}{\Gamma}_0$, $\overset{\circ}{\Gamma}_0 \circ (\overset{\circ}{\Gamma}_0 \circ)^{-1}$ must exist in the zero-mass theory. Moreover, since the two last terms in (2.17) actually vanish (up to logarithms) of first order as $q \rightarrow 0$,

$$\lim_{q \rightarrow 0} \overset{q}{\Gamma}_0 \circ (\overset{q}{\Gamma}_0 \circ)^{-1} \equiv \overset{\circ}{\Gamma}_0 \quad (5.9)$$

exists. In the Γ_{as} theory, we evaluate the l.h.s. using (4.3), obtaining

$$\begin{aligned} & \lim_{q \rightarrow 0} \lim_{\lambda \rightarrow 0} \{ a(g)^2 a(g(\lambda))^{-2} \Gamma((p - \frac{q}{2})(-p - \frac{q}{2}) \frac{q}{2} \frac{q}{2}; m^2 \lambda^2, g(\lambda)) \cdot \\ & \cdot h(g) h(g(\lambda))^{-1} \Gamma((-\frac{q}{2})(-\frac{q}{2}), q; m^2 \lambda^2, g(\lambda))^{-1} \} \\ & = \lim_{q \rightarrow 0} \lim_{\lambda \rightarrow 0} \{ a(g)^2 h(g) a(g(\lambda))^{-2} h(g(\lambda))^{-1} \cdot \\ & \cdot [\Gamma(p(-p)00; m^2, g(\lambda)) \Gamma(00, 0; m^2 \lambda^2, g(\lambda))^{-1} \\ & + r(p, q; m^2 \lambda^2, g(\lambda)) \Gamma((-\frac{q}{2})(-\frac{q}{2}), q; m^2 \lambda^2, g(\lambda))^{-1}] \} \\ & = \Gamma_{as}(p(-p)00; m^2, g) + \lim_{q \rightarrow 0} \lim_{\lambda \rightarrow 0} O((\ln q)^c (\ln \lambda^2)^{c'}) = \end{aligned}$$

$$= \Gamma_{\underline{as}}(p(-p)00,;m^2,g) \quad (5.10)$$

where $r(\dots)$ stands for the sum of the last two terms in (2.17) and (2.9) and (5.5) have been used.

Thus,

$$\Gamma_{\underline{as}}((p-\frac{q}{2})(-p-\frac{q}{2})\frac{q}{2}\frac{q}{2},;m^2,g) \quad (5.11)$$

$$\begin{aligned} & \approx \Gamma_{\underline{as}}(p(-p)00,;m^2,g)\Gamma_{\underline{as}}((-\frac{q}{2})(-\frac{q}{2}),q;m^2,g) \\ & = \Gamma_{\underline{as}}(p(-p)00,;m^2,g)h(g)^{-1}h(g(\lambda)) \quad \Gamma_{\underline{as}}((-\frac{q}{2})(-\frac{q}{2}),q;m^2\lambda^2,g(\lambda)) \\ & \approx \Gamma_{\underline{as}}(p(-p)00,;m^2,g)h(g)^{-1}b_0^{-1/3}(\ln \frac{m^2}{-q^2})^{-1/3} \end{aligned}$$

where in order to obtain the last estimate we have set $\lambda^2 = (-q^2)/m^2$ and used (5.6).

The result in (5.11) is essentially identical in form with (5.8): Rather than letting the mass (by change of the bare mass only) become small at exceptional momenta, we can, with the same effect to leading order, let within the preasymptotic theory the momenta approach exceptional ones. This holds also in the other cases discussed later. Replacing in (5.5) λ by $\lambda\lambda'$ and letting $\lambda' \rightarrow 0$ yields

$$\begin{aligned} & \Gamma_{\underline{as}}(p(-p)00,;m^2,g) \\ & = a(g)^2 a(g(\lambda))^{-2} h(g)h(g(\lambda))^{-1} \Gamma_{\underline{as}}(p(-p)00,;m^2\lambda^2 g(\lambda)) \quad (5.12) \end{aligned}$$

differing characteristically from (4.5). The considerations at the end of sect. 4.2 on the limitand in (4.3), based on (3.19), are extensible to (5.5) on the basis of the integrated form of (5.4) and lead to the validity of the "elementary recipe", as described at the beginning of this section, for the exceptional-momenta set here considered.

The homogeneous PDE for $\Gamma_{\underline{as}}(p(-p)00,)$ implied by (5.4) can also be derived using that this function is a SD expansion coefficient, see (2.17) or (5.31) below, and that the corresponding composite operator, $N_2(\phi^2)$, appearing in a Green's function gives rise merely to an additive γ -term, see (3.7), and by comparing expansions before and after mass vertex insertion. This derivation is more flexible than the one given here, which uses the explicit Green's function form of the coefficient, since the γ -terms relating to composite operators [2] are obtainable [24] using renormalization theory alone. Of particular interest for applications are light cone expansions, and by the method

just indicated homogeneous PDEs for (simple linear combinations of) the coefficient functions of the leading light-cone singularity in certain models were derived by Christ, Hasslacher, and Mueller [25]. From (2.20), in the interpretation described there, Mueller [15] has also derived a homogeneous transformation law for the asymptotic forms of the singularity-coefficient functions $\hat{\Gamma}^0$.

We turn to the set $(p(-p), 0)$ and consider

$$Op_{2,1}\Gamma(p(-p), 0; m^2, g) = -im^2\phi(g)\Gamma(p(-p), 00; m^2, g). \quad (5.13)$$

The expansion formula for the VF on the r.h.s. is (2.18), yielding

$$\Gamma(p(-p), 00) = \Gamma(p(-p)00,)\Gamma(, 000) + \Gamma_{rem}(p(-p), 00) \quad (5.14)$$

similar to (5.2). From (3.7) follows

$$-im^2\phi(g)\Gamma(, 000) = Op_{0,2}\Gamma(, 00) + i\kappa(g) = i\kappa(g) \quad (5.15)$$

in view of (2.13). Thus (5.13) becomes

$$\begin{aligned} Op_{2,1}\Gamma(p(-p), 0; m^2, g) &= \\ &= i\kappa(g)\Gamma(p(-p)00,; m^2, g) - im^2\phi(g)\Gamma_{rem}(p(-p), 00; m^2, g). \end{aligned} \quad (5.16)$$

Now (3.18c) implies

$$Op_{0,2}[a(g)^{-2}h(g)^{-2}k(g)] = \kappa(g)$$

such that, using (5.4), (5.16) can be written

$$\begin{aligned} Op_{2,1}[\Gamma(p(-p), 0; m^2, g) \\ - ia(g)^{-2}h(g)^{-2}k(g)\Gamma(p(-p)00,; m^2, g)] \\ = -im^2\phi(g)[\Gamma_{rem}(p(-p), 00; m^2, g) \\ - ia(g)^{-2}h(g)^{-2}k(g)\Gamma_{rem}(p(-p)00, 0; m^2, g)] \end{aligned} \quad (5.17)$$

There follows, as in earlier cases, a limit relation and the transformation formula, rewritten with the help of (5.12),

$$\begin{aligned} \Gamma_{as}(p(-p), 0; m^2, g) \\ = h(g)^{-1}h(g(\lambda))\Gamma_{as}(p(-p), 0; m^2\lambda^2, g(\lambda)) \\ + ia(g)^{-2}h(g)^{-2}[k(g) - k(g(\lambda))]\Gamma_{as}(p(-p), 00; m^2, g) \end{aligned} \quad (5.18)$$

equivalent to

$$\begin{aligned} & \text{Op}_{2,1} [\Gamma_{\underline{as}}(p(-p), 0; m^2, g) \\ & - ia(g)^{-2} h(g)^{-2} k(g) \Gamma_{\underline{as}}(p(-p)00; m^2, g)] = 0, \end{aligned} \quad (5.19)$$

or to (5.16), with Γ replaced by $\Gamma_{\underline{as}}$ and Γ_{rem} set zero. The reinterpretation of the (suppressed) mass limit relation in terms of

$$\Gamma_{\underline{as}}^0 = \lim_{q \rightarrow 0} [\Gamma_{\underline{as}}^q - \Pi_{\underline{as}}^q \circ \Gamma_{\underline{as}}^0] \quad (5.20)$$

on the basis of (2.18), along the lines of our treatment of (5.9), we here omit to avoid tedious repetition. We only list the analog of (5.8)

$$\begin{aligned} \Gamma_{\underline{as}}^{s(g, \lambda)}(p(-p), 0; m^2, g) & \approx -\frac{1}{3} (\ln \lambda^{-2})^{1/3} b_0^{1/3} \cdot \\ & \cdot a(g)^{-2} h(g)^{-2} i \Gamma_{\underline{as}}(p(-p)00; -^2, g) \end{aligned} \quad (5.21)$$

for $\lambda \rightarrow 0$, showing

$$\Gamma_{\underline{as}}(p(-p), 0; m^2, g) = \infty \quad (5.22)$$

in contrast to (5.7), and the analog of (5.9)

$$\Gamma_{\underline{as}}^{qp} = \Gamma_{\underline{as}}^{op} + \Pi_{\underline{as}}^q \circ \Gamma_{\underline{as}}^{op} + O(q (\ln q)^c). \quad (5.23)$$

This formula gives a singularity at $q \rightarrow 0$ precisely analogous to the one in (5.21) because of

$$\Pi_{\underline{as}}^q \approx -ia(g)^{-2} h(g)^{-2} b_0^{1/3} \left[\ln \left(\frac{m^2}{-q} \right) \right]^{1/3} \quad (5.24)$$

as follows from (4.7) and (3.18c).

Finally, we consider the exceptional momenta set $(p(-p)p'(-p'))$, assuming for simplicity $p \neq \pm p'$. In

$$\begin{aligned} & \text{Op}_{4,0} \Gamma(p(-p)p'(-p'), m^2, g) \\ & = -im^2 \phi(g) \Gamma(p(-p)p'(-p'), 0; m^2, g) \end{aligned} \quad (5.25)$$

we use (2.17) twice, obtaining

$$\begin{aligned} \Gamma(p(-p)p'(-p'), 0) &= \Gamma(p(-p)00,) \Gamma(, 000) \Gamma(p'(-p')00,) + \\ & + \Gamma_{\text{rem}}(p(-p)p'(-p'), 0) \end{aligned}$$

which yields, by similar steps as led to (5.19),

$$\begin{aligned} & \text{Op}_{4,0} [\Gamma_{\underline{as}}(p(-p)p'(-p'); m^2, g) \\ & - ia(g)^{-2} h(g)^{-2} k(g) \Gamma_{\underline{as}}(p(-p)00; m^2, g) \Gamma_{\underline{as}}(p'(-p')00; m^2, g)] = 0. \end{aligned}$$

The analog of (5.9) is now

$$\overset{q}{\Gamma}_{as} = \overset{o}{\Gamma}_{as} + \overset{o}{\Gamma}_{as} \overset{q}{\Pi}_{as} \overset{o}{\Gamma}_{as} + O(q (\ln q)^C). \quad (5.26)$$

One can give a construction of the Γ_{as} functions from the zero-mass-theory VFs Γ_o without any limiting process. To this end one first defines analogs $\overset{o}{\Gamma}_o$ of the Γ_{as} , by using formulae analogous to (5.11), (5.20), and (5.26). E.g. (5.9) now becomes using (4.12)

$$\begin{aligned} \overset{o}{\Gamma}_o &= \lim_{q \rightarrow 0} [\overset{q}{\Gamma}_o (\overset{q}{\Gamma}_o)^{-1}] \\ &= Z_1(g) Z_2(g)^{-1} \lim_{q \rightarrow 0} [\overset{q}{\Gamma}_{as} (\overset{q}{\Gamma}_{as})^{-1}] \\ &= Z_1(g) Z_2(g)^{-1} \overset{o}{\Gamma}_{as} \end{aligned} \quad (5.27)$$

and $\overset{o}{\Gamma}_o$ in turn is computed from (2.17) read for Γ_o VFs with q finite. Similarly, one first determines from (5.20) and (5.26), with the help of (4.12), the (finite) relation between $\overset{o}{\Gamma}_{as}$, $\overset{o}{\Gamma}_{as}$ and $\overset{o}{\Gamma}_o$, $\overset{o}{\Gamma}_o$, respectively, and limit-free formulae for the latter from (2.18) and (2.19).

In perturbation theory, one can simplify the computation of $\overset{o}{\Gamma}_o$, $\overset{o}{\Gamma}_{as}$, and $\overset{o}{\Gamma}_o$ by exploiting the "elementary recipe". (This procedure uses an idea of G.t'Hooft.) One writes the Feynman integrals for $\overset{o}{\Gamma}_o$, $\overset{o}{\Gamma}_o$, and $\overset{o}{\Gamma}_o$. The diagrams with no two-particle link $\overset{o}{G}$ in the channel carrying zero-momentum are finite and contribute in full to $\overset{o}{\Gamma}_o$ etc. Diagrams with b $\overset{o}{G}$ - links give rise to expressions

$$\overset{b}{\Pi} \int dk_i (-k_i^2 - i0)^{-2} f(k_1 \dots k_b p \dots) \quad (5.28)$$

where all integrations, except the ones related to the links, are considered carried out, and $p \dots$ stands for the external momenta. $f(k_1 \dots k_b p \dots)$ vanishes, due to renormalization and the absolute convergence of all (Euclidean) integrals achieved thereby, for any $k_i \rightarrow \infty$. Now

$$\begin{aligned} &f(k_1 \dots k_b p \dots) \\ &= \overset{b}{\Pi} \{ [0_{ik_i} - c(k_i, U) 0_{io}] + c(k_i, U) 0_{io} \} \cdot \\ &\quad \cdot f(k_1 \dots k_b p \dots) \end{aligned} \quad (5.29a)$$

where

$$c(k_i, U) \equiv U^2 [U^2 - k_i^2 - i0]^{-2}, \quad (5.29b)$$

0_{ir} means to set the i^{th} momentum r , and U is the normalization momentum in (4.10). Any lone factor 0_{io} gives in (5.28) rise to an

integral

$$U^2 \int dk_i^2 (-k_i^2 - i0)^{-2} [U^2 - k_i^2 - i0]^{-2}$$

which is discarded, leaving as contribution to $\underline{I}_0^0$ etc. from the diagram considered only

$$\prod_{i=1}^b \int dk_i (-k_i^2 - i0)^{-2} [0_{ik_i} - c(k_i, U) 0_{i0}] \} f(\dots p \dots)$$

which is UR and UV convergent. The correctness of this prescription follows from

$$U^2 \int dk \{ [- (k + \frac{1}{2}q)^2 - i0] [- (-k + \frac{1}{2}q)^2 - i0] [U^2 - k^2 - i0] \}^{-1} \\ = \text{const} \ln [U^2 (-q^2 - i0)^{-1}] + O(U^{-2} q^2)$$

and the fact that the terms to be subtracted from $\underline{I}_0^0$ in (5.27), and similarly in the Γ_0 -theory analoga of (5.20) and (5.26), are, because of (4.10d-e), power series in $\ln [U^2 (-q^2 - i0)^{-1}]$ without constant term.

5.2 Asymptotic behaviour near exceptional momenta.

From the fact that the set of nonexceptional-momenta sets is open, it is easily derived that in the same way as for non-exceptional momenta, the l.h.s. of (4.9) is, for $\lambda \rightarrow \infty$, approximated by the r.h.s.,

$$\Gamma((\lambda p_1 + r_1) \dots (\lambda p_{2n} + r_{2n}), (\lambda q_1 + s_1) \dots (\lambda q_\ell + s_\ell))$$

is approximated by Γ_{as} of the same arguments, for $p_1 \dots q_\ell$ nonexceptional. However the Γ_{as} are analytic (or, if an odd partial sum of momenta vanishes, still continuous) at nonexceptional momenta, which implies that in the Γ_{as} function to the same logarithmic accuracy the r, s , momenta can be set zero, such that

$$\Gamma((\lambda p_1 + r_1) \dots (\lambda p_{2n} + r_{2n}), (\lambda q_1 + s_1) \dots (\lambda q_\ell + s_\ell)) \quad (5.30) \\ = \Gamma(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_\ell) + O(\lambda^{3-2n-2\ell} (\ln \lambda)^2)^c$$

if no partial sum of the $p \dots q$ vanishes, and with a possibly less sharp bound on the difference if the $p \dots q$ are nonexceptional but an odd partial sum of momenta vanishes.

The usual power-counting argument applied to (2.17), however, yields

$$\Gamma((\lambda p + r_1)(-\lambda p + r_2)r_3r_4) \quad (5.31) \\ = \Gamma(\lambda p(-\lambda p)00)\Gamma(r_3r_4, (r_1+r_2)) + O(\lambda^{-1}(\ln \lambda)^c)$$

recalling (2.9). Likewise, (2.18) becomes

$$\Gamma((\lambda p + r_1)(-\lambda p + r_2)r_3) \quad (5.32) \\ = \Gamma(\lambda p(-\lambda p), 0) + \Gamma((r_1+r_2)r_3)\Gamma(\lambda p(-\lambda p)00) + O(\lambda^{-1}(\ln \lambda)^c)$$

and (2.19)

$$\begin{aligned} & \Gamma((\lambda p + r_1)(-\lambda p + r_2)(\lambda p' + r_3)(-\lambda p' + r_4),) \\ & = \Gamma(\lambda p(-\lambda p)\lambda p'(-\lambda p'),) + \Gamma(\lambda p(-\lambda p)00,). \\ & \cdot \Gamma(\lambda p'(\lambda p')00,)\Gamma((r_1 + r_2)(r_3 + r_4)) + 0(\lambda^{-1}(\ln \lambda)^C). \end{aligned} \quad (5.33)$$

(5.31-3) show how (5.30) is violated for particular exceptional sets of momenta p, q . In these formulae we have achieved a separation of λ -dependent terms, in which λ appears simply as scale factor to the momenta, from λ -independent terms. Obviously, in the first terms we can, to the accuracy of these formulae, replace the Γ by Γ_{as} . Consider now reading (2.17), which is an identity, for the Γ_{as} theory. Then all functions become Γ_{as} ones but the ratio $\overset{\circ}{I}^0 (\overset{\circ}{I}^0)^{-1}$ is to be replaced by $\overset{\circ}{I}_{as}^0$. This follows from (5.5). Alternatively, we may use that all other terms in (2.17) have a $q \rightarrow 0$ limit when we first replace in (2.17) everywhere total momentum zero by q (note that "zero" total momentum also enters the formula (2.7) for W) and then let $q \rightarrow 0$ using (5.10). In any case, comparing the new r.h.s. with the one of (5.31) we find

$$\begin{aligned} & \Gamma((\lambda p + r_1)(-\lambda p + r_2)r_3r_4,) \\ & - \Gamma_{as}((\lambda p + r_1)(-\lambda p + r_2)r_3r_4,) = 0 ((\ln \lambda)^C) \end{aligned} \quad (5.34)$$

rather than $0(\lambda^{-2}(\ln \lambda)^C)$ as it would have been for nonexceptional momenta $p_1 \dots p_4$ in the growing parts. - Note that for $r_3 + r_4 = 0$, $r_3 \neq 0$ the Γ_{as} function in (5.34) would even be infinite as follows from (5.26) and (5.24). - By the same argument as just used one derives analoga of (5.34) from (5.32) and (5.33).

5.3 Application to finite-mass corrections.

We examine to what extent (4.9) can be used to determine large-momenta behaviour (in the sense of overall scaling, for the reduction of other cases to this one, see the previous sect. 5.2). The first term on the r.h.s. may be rewritten using (4.7); however, since $g(\lambda)$ increases as λ increases (at least for $0 < g < g_\infty$) no estimate can be obtained from (4.7) unless new, relatively speculative, assumptions are made. We defer these to sect. 6 and examine the integral in (4.9), the "mass correction" term. λ becoming large is for the last VF in (4.9) the problem of large exceptional-momenta behaviour. The analog of (4.9) for this momenta set is the integrated form of the appropriate generalization of (5.17) we first derive. We have

$$\begin{aligned}
& Op_{2n, \ell+1} \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell 0; m^2, g) \\
& = -im^2 \phi(g) \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell 00; m^2, g).
\end{aligned} \tag{5.35}$$

For $p_1 \dots q_\ell$ large, the short-distance expansion analogous to (5.14) derived from (2.17) is

$$\begin{aligned}
& \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell 00) \\
& = \Gamma'(p_1 \dots p_{2n} 00, q_1 \dots q_\ell) \Gamma(, 000) \\
& + \Gamma_{\text{rem}}(p_1 \dots p_{2n}, q_1 \dots q_\ell 00)
\end{aligned} \tag{5.36a}$$

with

$$\begin{aligned}
& \Gamma'(p_1 \dots p_{2n} 00, q_1 \dots q_\ell) = \Gamma(p_1 \dots p_{2n} 00, q_1 \dots q_\ell) \\
& + \sum_{\text{partitions}} \Gamma(\dots 0, \dots) G(\Sigma'(-\Sigma'),) \Gamma(\dots, \dots) \dots \\
& \dots G(\Sigma''(-\Sigma''),) \Gamma(\dots 0, \dots).
\end{aligned} \tag{5.36b}$$

Here the sum is over all different partitions of the momenta into non-empty even sets such that the two zero-momenta are in the first and the last set; the sets are the arguments of VFs ordered in a chain, with connecting propagators transferring momenta as obvious, in a graphical picture, from momentum conservation. That (5.36) is the correct generalization of (5.14) is seen best by drawing a number of diagrams, which we leave to the reader. (It is important to use that the momenta $p_1 \dots q_\ell$ are nonexceptional.)

We next consider Γ' and form

$$\begin{aligned}
& Op_{2n+2, \ell} \Gamma'(p_1 \dots p_{2n} 00, q_1 \dots q_\ell; m^2, g) \\
& = -im^2 \phi(g) \{ \Gamma(p_1 \dots p_{2n} 00, q_1 \dots q_\ell 0; m^2, g) + \dots \}
\end{aligned} \tag{5.37}$$

where the dotted terms stem from the partitioned terms in (5.36b) and are in the limit under discussion negligible compared to the Γ' on the l.h.s. of (5.37) while the explicitly written VF is not. For it, we have the analog of (5.36a), derived similarly,

$$\begin{aligned}
& \Gamma(p_1 \dots p_{2n} 00, q_1 \dots q_\ell 0) \\
& = \Gamma'(p_1 \dots p_{2n} 00, q_1 \dots q_\ell) \Gamma(00, 00) \\
& + \Gamma_{\text{rem}}(p_1 \dots p_{2n} 00, q_1 \dots q_\ell 0).
\end{aligned} \tag{5.38}$$

Putting (5.35-38) together and using (5.3), (5.15) and the relation following (5.16) yields the generalization of (5.17)

$$Op_{2n, \ell+1} \square \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell 0; m^2, g) - \tag{5.39}$$

$$\begin{aligned}
& -ia(g)^{-2}h(g)^{-2}k(g)\Gamma'(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2, g) \\
& = -im^2\phi(g) \left[\Gamma_{\text{rem}}(p_1\dots p_{2n}, q_1\dots q_\ell^{00}; m^2, g) \right. \\
& \quad \left. -ia(g)^{-2}h(g)^{-2}k(g) (\Gamma_{\text{rem}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell^0; m^2, g) + \dots) \right]
\end{aligned}$$

where the dots are the same terms as in (5.37), and (5.37) in the form

$$\begin{aligned}
& Op_{2n, \ell-1} \Gamma'(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2, g) \quad (5.40) \\
& = -im^2\phi(g) \left[\Gamma_{\text{rem}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell^0; m^2, g) + \dots \right]
\end{aligned}$$

is used. (5.39) and (5.40) are the analoga of (3.7) needed now and converted by integration into the analoga of (4.9) with Γ_{as} functions that solve the corresponding homogeneous PDEs. There follows the generalization of (5.18)

$$\Gamma_{\text{as}}(p_1\dots p_{2n}, q_1\dots q_\ell^0; m^2, g) \quad (5.41)$$

$$\begin{aligned}
& = a(g)^{n-\ell-1} a(g(\lambda))^{-n+\ell+1} h(g)^{-\ell-1} h(g(\lambda))^{\ell+1} \\
& \cdot \Gamma_{\text{as}}(p_1\dots p_{2n}, q_1\dots q_\ell^0; m^2\lambda^2, g(\lambda))
\end{aligned}$$

$$+ i \left[k(g) - k(g(\lambda)) \right] a(g)^{-2} h(g)^{-2} \Gamma'_{\text{as}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2, g)$$

along with the generalization of (5.12)

$$\Gamma'_{\text{as}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2, g) \quad (5.42)$$

$$= a(g)^{n-\ell+1} a(g(\lambda))^{-n+\ell-1} h(g)^{-\ell+1} h(g(\lambda))^{\ell-1} .$$

$$\Gamma'_{\text{as}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2\lambda^2, g(\lambda)) .$$

Inserting (5.41) with use of (5.42) into (4.9) yields

$$\Gamma(\lambda p_1\dots \lambda p_{2n}, \lambda q_1\dots \lambda q_\ell; m^2, g) \quad (5.43)$$

$$= \Gamma_{\text{as}}(\lambda p_1\dots \lambda p_{2n}, \lambda q_1\dots \lambda q_\ell; m^2, g)$$

$$- im^2\lambda^{2-2n-2\ell} a(g)^{n-\ell} a(g(\lambda))^{-n+\ell+1} h(g)^{-\ell} h(g(\lambda))^{\ell+1} .$$

$$\cdot \left[\int_0^1 d\lambda' {}^2\phi(g(\lambda')) a(g(\lambda'))^{-1} h(g(\lambda'))^{-1} \right] .$$

$$\cdot \left[\Gamma_{\text{as}}(p_1\dots p_{2n}, q_1\dots q_\ell^0; m^2, g(\lambda)) \right.$$

$$\left. - ik(g(\lambda)) a(g(\lambda))^{-2} h(g(\lambda))^{-2} \Gamma'_{\text{as}}(p_1\dots p_{2n}^{00}, q_1\dots q_\ell; m^2, g(\lambda)) \right]$$

$$+ i \left[\int_0^1 d\lambda' {}^2\phi(g(\lambda')) a(g(\lambda'))^{-1} h(g(\lambda'))^{-1} k(g(\lambda')) \right] .$$

$$\cdot [a(g(\lambda))^{-2} h(g(\lambda))^{-2} \Gamma'_{\text{as}}(p_1 \dots p_{2n} \ 00, q_1 \dots q_\ell; m^2, g(\lambda))] \} \\ + \dots$$

where the same linear combination of VFs as in (5.39) appears. The corrections indicated by dots involve integrals over the integrals over the r.h.s. of (5.39) and (5.40). These latter can be analyzed with the help of short-distance formulae [2] extended to higher accuracy than in (5.36) and (5.38), but this need not be pursued here. The case $n=1, \ell=0$ of (5.43) was derived in [13], the formula for $\ell=0, n$ arbitrary given in [26]. Note that the complication in the computation of the "finite mass corrections" in this section stems only from the exceptional momenta, or the UR divergence of a mass vertex insertion into a massless theory involving scalar particles, which prohibit in (4.9) the trivial iterative use of the rewritten form of (4.9)

$$a(g(\lambda'))^{-n+\ell} h(g(\lambda'))^\ell \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2 \lambda^{-2} \lambda'^2, g(\lambda')) \\ = a(g(\lambda))^{-n+\ell} h(g(\lambda))^\ell \Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g(\lambda)) \\ - i m^2 \lambda^{-2} \int_0^{\lambda'^2} d\lambda''^2 \phi(g(\lambda'')) a(g(\lambda''))^{-n+\ell} h(g(\lambda''))^\ell \\ \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell \ 0; m^2 \lambda^{-2} \lambda''^2, g(\lambda''))$$

valid only for nonexceptional momenta. In this connection cp. [27].

6. Question of true large-momenta behaviour.

References to this section are [28] and sect. IV. of [13].

6.1 Assumptions relating to fixed points.

As remarked in sect. 5.3, the $\lambda \rightarrow \infty$ behaviour of the first term on the r.h.s. of (4.9), or of the r.h.s. of (4.7), is obtainable only upon making assumptions having no basis in perturbation theory. The usual assumptions are

There exists g_∞ in $(0, +\infty)$ such that

$$\lim_{g \uparrow g_\infty} \int^g dg' \beta(g')^{-1} = +\infty, \quad (6.1)$$

and $\beta(g) > 0$ for $0 < g < g_\infty$

$$\gamma(g) \rightarrow \gamma(g_\infty) \text{ continuously as } g \uparrow g_\infty, \quad (6.2)$$

and likewise for $\eta(g), \kappa(g), \phi(g)$

$$\begin{aligned}
& \Gamma_{as}(p(-p), ; m^2, g) \Big|_{g \uparrow g_\infty} \text{ exists for some } p^2 < 0, \\
& \Gamma_{as}(p_1 \dots p_4, ; m^2, g) \Big|_{g \uparrow g_\infty} \text{ exists for some Euclidean set } p_1 \dots p_4 \quad (6.3) \\
& \text{and } \Gamma_{as}\left(\frac{1}{2}q, \frac{1}{q}q, (-q); m^2, g\right) \Big|_{g \uparrow g_\infty} \text{ for some } q^2 < 0.
\end{aligned}$$

We emphasize that it is not necessary that $g_\infty < \infty$, though (6.2) and (6.3) appear less natural if $g_\infty = \infty$. The limits in (6.3) can be taken as defining renormalization conditions for a massless theory

$$\Gamma_{as}(\dots, \dots; m^2, g_\infty) \equiv \Gamma_{GL}(\dots, \dots; m^2). \quad (6.4)$$

(6.2) gives

$$a(g)^{-1} a(g(\lambda)) = \lambda^{4\gamma(g_\infty)} e^{R(g, \lambda)} \quad (6.5a)$$

where

$$R(g, \lambda) = 2 \int_1^{\lambda^2} \lambda'^{-2} d\lambda'^2 [\gamma(g(\lambda')) - \gamma(g_\infty)] \quad (6.5b)$$

Now choose $\delta(\epsilon)$ so small that $|\gamma(g') - \gamma(g_\infty)| < \epsilon$ for $|g' - g_\infty| < \delta(\epsilon)$.

Then

$$\begin{aligned}
|R(g, \lambda)| \leq 2 \Big| \int_{g_\infty - \delta(\epsilon)}^{g_\infty - \delta(\epsilon)} dg' \beta(g')^{-1} [\gamma(g') - \gamma(g_\infty)] \Big| \\
+ 2\epsilon \int_{g_\infty - \delta(\epsilon)}^g dg' \beta(g')^{-1} + 2\epsilon \ell n \lambda^2, \text{ for } g(\lambda) > g_\infty - \delta(\epsilon).
\end{aligned}$$

Herein only the last term is λ -dependent, and has arbitrarily small coefficient, such that

$$\lim_{\lambda \rightarrow \infty} [R(g, \lambda) (\ell n \lambda)^{-1}] = 0. \quad (6.5c)$$

Similarly one proves that

$$h(g)^{-1} h(g(\lambda)) = \lambda^{2\eta(g_\infty)} e^{R'(g, \lambda)} \quad (6.6a)$$

with

$$\lim_{\lambda \rightarrow \infty} [R'(g, \lambda) (\ell n \lambda)^{-1}] = 0. \quad (6.6b)$$

Thus

$$\begin{aligned}
& \Gamma(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_\ell; m^2, g) \quad (6.7) \\
& = \lambda^{4-2n-2\ell-4(n-\ell)\gamma(g_\infty)+2\ell\eta(g_\infty)} \\
& \cdot e^{(-n+\ell)R(g, \lambda)+\ell R'(g, \lambda)} \Gamma_{GL}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2) \\
& + i \delta_{no} \delta_{\ell 2} a(g)^{-2} h(g)^{-2}.
\end{aligned}$$

$$\bullet \left\{ \begin{array}{ll} \left[4\gamma(g_\infty) + 2\eta(g_\infty) \right]^{-1} \lambda^{8\gamma(g_\infty)+4\eta(g_\infty)} & \\ \cdot e^{2R(g,\lambda)+2R'(g,\lambda)} \kappa(g_\infty) & \text{if } 2\gamma(g_\infty)+\eta(g_\infty)>0 \\ \hline (\ln \lambda^2) \kappa(g_\infty) e^{2R(g,\lambda)+2R'(g,\lambda)} & \text{if } 2\gamma(g_\infty)+\eta(g_\infty)=0 \\ \hline \left[-k(g) + k(g_\infty) \right] & \text{if } 2\gamma(g_\infty)+\eta(g_\infty)<0 \end{array} \right.$$

- + (A) correction term from $g(\lambda) \neq g_\infty$ argument of Γ_{as}
- + (B) correction term from massive $\neq$ massless theory
- + (C) correction term for $\delta_{no} \delta_{l2}$ part.

We note that (6.4) gives, using (4.6) (or, more precisely, (6.13) below)

$$\begin{aligned} & \Gamma_{GL}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2 \lambda^2) \\ &= \lambda^{4(n-l)\gamma(g_\infty)-2l\eta(g_\infty)} \Gamma_{GL}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2) \\ &- \delta_{no} \delta_{l2} i \kappa(g_\infty) \frac{1}{4} (2\gamma(g_\infty)+\eta(g_\infty))^{-1} (\lambda^{4(2\gamma(g_\infty)+\eta(g_\infty))})_{-1} \end{aligned} \quad (6.8)$$

i.e. the m -dependence of the Gell-Mann-Low limit theory VFs Γ_{GL} is, apart from the $\delta_{no} \delta_{l2}$ term, merely the one enabling the Γ_{GL} to be scale invariant but with anomalous dimensions (see below) and to keep their mass dimension $4-2n-2l$.

The (B) term in (6.7) is the r.h.s. of (5.43) apart from the first term. We shall later give an argument why also Γ_{as} and Γ'_{as} occurring there should have $g(\lambda) \rightarrow g_\infty$ limits. Using (6.5) and (6.6) we find that the correction term becomes for $\lambda \rightarrow \infty$ small relative to the main term if

$$-1 < 2\gamma(g_\infty) + \eta(g_\infty) < 1. \quad (6.9)$$

It is obvious that the (A) part in (6.7) will become small relative to the main term if $\Gamma_{as}(\dots, \dots; m^2, g(\lambda))$ goes to $\Gamma_{GL}(\dots, \dots; m^2)$ continuously as we, of course, must assume to give meaning to Γ_{GL} . A more precise estimate can be given only if assumptions on the rate of approach are made. If one then assumes left-differentiability of $\beta(g)$ at g_∞ , from (3.14), written as

$$\begin{aligned} \ln \lambda^2 &= \int_g^{g(\lambda)} dg' \{ \beta(g')^{-1} - (g' - g_\infty)^{-1} [\beta'(g_\infty)]^{-1} \} \\ &+ [\beta'(g_\infty)]^{-1} [\ln(g_\infty - g) - \ln(g_\infty - g(\lambda))] \end{aligned} \quad (6.10a)$$

we obtain the estimate

$$g_\infty - g(\lambda) \approx f(g) \lambda^{2\beta'(g_\infty)}, \quad (6.10b)$$

and estimates from appropriate modifications of (6.10) if $\beta'(g_\infty)=0$ but some higher derivate unequal zero, cp. [29]. (6.10b) would have to be used to estimate the (A) term.

6.2 Anomalous dimensions and conformal invariance.

(6.7) shows that

$$\dim \phi = 1 + 2\gamma(g_\infty) \quad (6.11a)$$

and

$$\dim N_2(\phi^2) = 2 + 4\gamma(g_\infty) + 2\eta(g_\infty) \quad (6.11b)$$

are the dynamical short-distance dimensions of these operators in the sense of Wilson [30]. Positivity of the metric requires

$$\gamma(g_\infty) \geq 0 \quad (6.12a)$$

and

$$4\gamma(g_\infty) + 2\eta(g_\infty) \geq -1. \quad (6.12b)$$

(6.11) are also the dimensions of these operators (with $N_2(\phi^2)$ appropriately redefined to avoid UR divergences) in the scale invariant GL - theory with VFs Γ_{GL} . Schroer [24] has shown that this theory, if it exists, is also conformal invariant. Conformal invariance should be helpful in determining the anomalous dimensions, though the schemes so far developed (see references in [26] and [31]) seem not to suggest yet any constructive procedure. We merely remark that also ϕ^4 theory can be written as a trilinear-coupling theory, as considered in [26] and [31], by letting $g^{1/2} \phi^2 = \psi$ and using a two-field Lagrangian such as

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{24} \psi^2 - \frac{1}{12} g^{1/2} \phi^2 \psi.$$

If momenta are exceptional, then (6.7) does not hold. Then the analog of the first term on the r.h.s. is to be determined from the transformation properties, like (5.41) and (5.42), of the corresponding exceptional asymptotic forms. The simplest exceptional momenta set is $(p(-p)00,)$, and from (5.12), (6.5a) and (6.6a) we find a change of power law by a factor relative to the nonexceptional-momenta case, generic $(p_1 \dots p_4,)$. In all other cases, there are inhomogeneous transformation laws, e.g. in (5.42) it is the linear combination of functions Γ'_{as} that transforms, asymptotically, apart from the $e^{R(\lambda, g)}$ factors by a power law, and in (5.39), leading to (5.41), it is another linear combination that transforms homogeneously, leading to an inhomogeneous transformation law for $\Gamma'_{as}(p_1 \dots p_{2n}, 0q_1 \dots q_k; m^2, g)$ alone, and thus of the corresponding Γ_{GL} function.

These inhomogeneous transformation laws under scale transformation of

all but the simplest Γ_{GL} lead to the appearance of logarithms if certain degeneracies in scale dimensions occur: If in

$$\{m^2 [\partial/\partial m^2] - \sum_1 \gamma\} \Gamma_{GL}(1) = \Gamma_{GL}(2)$$

$\sum_2 \gamma = \sum_1 \gamma$ holds, then the transformation law of $\Gamma_{GL}(1)$ will involve at least a logarithm; if already $\Gamma_{GL}(2)$ transforms with a logarithm, $\Gamma_{GL}(1)$ will with $(\text{logarithm})^2$ etc., and we are faced with non-fully-reducible representations [32] of the dilatation group. E.G. in (5.16), in view of (5.4), degeneracy takes place if $2\gamma(g_\infty) + \eta(g_\infty) = 0$, i.e. if the dimension of $N_2(\phi^2)$ is canonical, and the logarithm is explicit in (5.18) in form of

$$\lim_{g \rightarrow g_\infty} \{a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))]\} \text{prop. } \ln \lambda$$

then, which is the same logarithm as occurs in (6.7). The degeneracy here resides in $4 - \dim N_2(\phi^2) = \dim N_2(\phi^2)$.

Irrespective of the transformation law under scale transformation, the exceptional-momenta VFs do not transform in any simple way under conformal transformations, because the exceptional-momenta Γ_{GL} are extracted from the ordinary Γ_{GL} at exceptional momenta as discussed in sect. 5.1, however, the property that a partial sum of momenta be zero is not preserved under conformal transformations. Namely, the property is brought about by e.g. integrating a collection of arguments in fixed relative position over all space-time, and the integration measure over space-time changes under conformal transformations that are not Poincaré transformation. A related observation is the one in [33] that a restriction like $p^2=0$ is not preserved under conformal transformation, since the infinitesimal generators of these transformations do not commute with p^2 .

6.3 Consistency check of assumptions.

It appears desirable to examine (6.3) more closely, since g_∞ is a peculiar value of the coupling constant for the Γ_{as} theory. To this end, we consider the definition of Γ_{as} , (4.9) with $\lambda=1$. It is plausible that the l.h.s. has a limit as $g \rightarrow g_\infty$ since g_∞ is not a particular value of the coupling constant for the massive theory. If the Γ have limits for $g \rightarrow g_\infty$, this also yields a rationale for (6.2) since the parametric functions are expressible in terms of the massive-theory VFs using (3.7) at the momenta occurring in (1.17a-c), (2.9), (2.13).

On the r.h.s., under the integral we note that (6.5b) gives

$$\lim_{g \uparrow g_\infty} R(g, \lambda) = 0$$

due to

$$|R(g, \lambda)| < 2\epsilon \ln \lambda^2 \quad \text{if} \quad g > g_\infty - \delta(\epsilon),$$

such that from (6.5a)

$$\lim_{g \uparrow g_\infty} [a(g)^{-1} a(g(\lambda))] = \lambda^{4\gamma(g_\infty)}, \quad (6.13a)$$

and similarly (6.6) yields

$$\lim_{g \uparrow g_\infty} [h(g)^{-1} h(g(\lambda))] = \lambda^{2\eta(g_\infty)}. \quad (6.13b)$$

(4.9) becomes

$$\begin{aligned} & \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g_\infty) \\ &= \Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g_\infty) \\ & - i m^2 \int_0^1 d\lambda^2 \lambda^{4(-n+\ell)\gamma(g_\infty)+2\ell\eta(g_\infty)} \phi(g_\infty) \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2 \lambda^2, g_\infty). \end{aligned} \quad (6.14)$$

Problematic is here only the convergence of the integral at 0, which leads to the large-momenta-behaviour problem for the exceptional-momenta VF we will now discuss.

To this end we only consider the behaviour of $\Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2 \lambda^2, g_\infty)$ since corrections hereto lead essentially merely to repetitions of the present argument. We use (5.41) in the form

$$\begin{aligned} & \Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2 \lambda^2, g) \\ &= a(g)^{n-\ell-1} a(g(\lambda^{-1}))^{-n+\ell+1} h(g)^{-\ell+1} h(g(\lambda^{-1}))^{\ell+1} \\ & \cdot \{ \Gamma_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2, g(\lambda^{-1})) \\ & - i [k(g(\lambda^{-1})) - k(g)] a(g(\lambda^{-1}))^{-2} h(g(\lambda^{-1}))^{-2} \\ & \cdot \Gamma'_{\text{as}}(p_1 \dots p_{2n}, q_1 \dots q_\ell^0; m^2, g(\lambda^{-1})) \} \end{aligned} \quad (6.15)$$

for $g \uparrow g_\infty$, $g(\lambda^{-1}) \uparrow g_\infty$ for any fixed λ , and the argument that the Γ_{as} and Γ'_{as} in the r.h.s. hereby have limits is parallel to the one we are engaged in presently, based, however, on (5.39) and (5.40), and would lead again to the same condition (6.9) we shall find here. Indeed, taken the existence of these limits for granted, the small- λ behaviour of the r.h.s. of (6.15) is given by the one of the a, h , and k factors, which, using (6.13) and (3.18), almost cancels the λ powers in (6.14), and the convergence condition is identical with (6.9).

We conclude that if (6.9) holds, then the assumption that all Γ_{as} and

Γ_{as} havelimits for $g \dagger g_\infty$ is consistent, and then the "finite mass corrections" discussed in sect. 6.1 indeed are corrections only. (6.9) was derived as consistency condition in [13].

It supports this consistency argument that the Γ_{as} can be constructed from the Γ_{as} by formulae of sect. 2.2, as discussed in sect. 5.1, without any limit processes involved. E.g., read (2.17) for the Γ_{as} theory; in (5.10) we showed that $\overset{\circ}{I}^0(\overset{\circ}{I}^0)^{-1}$ is then to be replaced by $\overset{\circ}{I}_{\text{as}}^0$. Letting then in (2.17) $g \dagger g_\infty$ gives, for any generic momenta, $\overset{\circ}{I}_{\text{GL}}^0$ expressed in terms of Γ_{GL} functions. Hereby in the construction of the subtracted B or of W no UR divergences are to be expected since (6.12a) states that the (anomalous) dimension of ϕ is larger than the canonical one! (A related fact is the absence of UR divergences in conformal invariant theory, see [34].)

7. General renormalizable theory.

7.1 PDEs for the massive theory.

Let the complete renormalizable Lagrangian involve F fields, G dimensionless coupling constants $g_1 \dots g_G$ or (g) (i.e., coefficients of terms of operator dimension four in the Lagrangian, bosons counted with dimension 1, fermions with dimension $\frac{3}{2}$, and derivatives with dimension 1), and M(possibly generalized [30]) mass terms with coefficients $m_{1U} \dots m_{MU}$ of mass dimension ≥ 1 .

Let M_i be the i^{th} finite (generalized) mass term, employing the appropriate normal product [2]. Then the substitution analogous to (3.1)

$$L \rightarrow L^{s_1 \dots s_M} = L - \sum_{i=1}^M s_i M_i \quad (7.1)$$

results in new VFs

$$\Gamma \rightarrow \Gamma^{s_1 \dots s_M}(p \dots; (m), (g)) \quad (7.2)$$

$$= \prod_{j=1}^F Z_j \left[(s), \left(\frac{m}{m_1} \right), (g) \right]^{n_j}$$

$$\cdot \Gamma(p \dots; [m_1 \mu_1((s), \left(\frac{m}{m_1} \right), (g))] \dots [m_M \mu_M((s), \left(\frac{m}{m_1} \right), (g))] ,$$

$$\bar{g}_1((s), \left(\frac{m}{m_1} \right), (g)) \dots \bar{g}_G((s), \left(\frac{m}{m_1} \right), (g))\}$$

analogous to the VF-form of (3.3). Here the s_i are dimensionless, the m_i are the renormalized counterparts of the m_{iU} , $\left(\frac{m}{m_1} \right)$ is the set

of the $M-1$ mass ratios, and n_j is the number of j -type field arguments. Since, for brevity, we do not use composite-field arguments, only multiplicative renormalizations are needed. All functions $Z \dots, \mu \dots$, and $\bar{g}$ can be computed in renormalized perturbation theory, whereby only expansions in the factors of genuine interaction terms need be made. (See, e.g. Appendix C of reference [13], with the formula following (C.7) replaced by the correct (3.20).) Differentiating the $\Gamma^{(s)}$ with respect to the $M s_i$ at $s_i = 0$ yields, using the Schwinger action principle [6], M inhomogeneous PDEs generalizing (3.7).

7.2 PDEs for the massless theory.

We now go to the unique asymptotic forms Γ_{as} of the Γ of the massive theory by, e.g., the elementary recipe of sect. 4.2. which here becomes: Expand $\lambda^{-D(\Gamma)} \Gamma(\lambda p \dots; (m), (g))$ in powers of λ^{-1} and $\ln \lambda$ for $\lambda \rightarrow \infty$, where $D(\Gamma)$ is the mass dimension, equalling the superficial divergence, of Γ ; the λ -independent term in this double power series is $\Gamma_{as}(p \dots; (m), (g))$. For nonexceptional momenta, (generalized) mass vertex insertion damp the asymptotic behaviour such that

$$\begin{aligned} \Gamma_{as}^{s_1 \dots s_M}(p \dots; (m), (g)) &= \Gamma_{as} \circ \dots \circ (p \dots; (m), (g)) \\ &\equiv \Gamma_{as}(p \dots; (m), (g)). \end{aligned}$$

Thus we obtain from (7.2) in obvious abbreviation

$$\begin{aligned} \Gamma_{as}(p \dots; (m), (g)) & \tag{7.3} \\ &= \Pi Z_j [\dots]^{n_j} \Gamma_{as}(p \dots; (m\mu), (\bar{g})) \end{aligned}$$

analogous to (4.5) but in different parametrization. Differentiating (7.3) with respect to the $M s_i$ at $s_i = 0$ yields the M homogeneous PDEs corresponding to the inhomogeneous ones obtained from (7.2).

These homogeneous PDEs have the property of admitting a solution obeying (7.3), with (s) -independent l.h.s., which is in particular independent of the way one might consider the point $(s_1 \dots s_M)$ being reached from the origin in s -space in infinitesimal steps. Furthermore, in renormalized-parameter space every point can be taken as starting point with (7.3) valid. These properties imply that the M PDEs are in involution [35].

We now choose (for convenience only) the (s) such that in the massive theory all renormalized masses to generalized mass terms vanish and that the ratios of the remaining renormalized ordinary masses take fixed values, e.g. one. This then fixes $M-1$ of the s_i in terms of one and there remains in (7.3) only a one-parameter freedom, which is the fa-

miliar one for a massless theory due to lack of intrinsic mass scale. Expressed differently, we have formed the (in perturbation theory, unique) linear combination of PDEs that has as mass differentiation $\sum_{i=1}^M m_i [\partial/\partial m_i]$ which leaves mass ratios unchanged, and moreover have chosen convenient values for these ratios. The set of praeasymptotic theories Γ_{as} so restricted is isomorphic to the set of massless theories Γ_o constructed by renormalization conditions appropriate to a massless theory, analogous to the construction in sect. 4.3. These theories are parametrized, besides by the nonintrinsic overall mass scale, only by dimensionless renormalized coupling constants $V_1 \dots V_G$ since all the renormalization conditions introducing (ordinary and generalized) masses in the massive theory are replaced by the vanishing of the corresponding Γ_o VFs at zero momenta. Indeed, would any of the generalized mass term parameters at zero momentum be nonzero, there would result UR divergences in the attempt to construct the Γ_o in perturbation theory, similar to the UR divergences one has in perturbation theory of massless ϕ^4 theory in less than four dimensions [36]. There is a relation, analogous to (4.12), between any Γ_{as} and a particular Γ_o , whereby the originally $M+G$ free parameters are effectively reduced to $1+G$ as described before, the $M-1$ PDEs besides the overall-scaling one only describing how to convert from any mass ratios to the conveniently chosen ones.

Thus, to study large-momenta behaviour of all massive-theory VFs it suffices (in perturbation theory at least) to study the family of massless theories with VFs Γ_o . These satisfy the PDEs analogous to (4.11)

$$\{U^2 [\partial/\partial U^2] + \sum_{i=1}^G \hat{\beta}_i((V)) [\partial/\partial V_i]\} \quad (7.4)$$

$$- \sum_{j=1}^F n_j \gamma_j((V)) \} \Gamma_o(p \dots; U^2, (V)) = 0$$

with characteristics described by the G ordinary DEs

$$U^{-2} dU^2 = \hat{\beta}_1((V))^{-1} dV_1 = \dots = \hat{\beta}_G((V))^{-1} dV_G. \quad (7.5)$$

These equations are the starting point of fixed point discussions such as Wilson's [28]. Computable large-momenta behaviour is obtained if there are solutions of (7.5) that lead, for $U \rightarrow \infty$, into the origin in V space. Solutions of (7.5) that lead into the origin for $U \rightarrow 0$ yield computable behaviour of appropriate massless theories in the small-momenta limit.

8. Remarks on some renormalizable theories.

This section is intended to introduce the reader to the principal facts concerning other theories than ϕ^4 and to list the relevant references without attempting completeness. Some of this material may be superseded by the time this article appears in print.

8.1 QED and gluon theory.

In QED, there is only one local mass-term insertion available, $N_3(\bar{\psi}\psi)$ in the notation of Zimmermann [2]. The PDEs analogous to (3.7), but without $N_3(\bar{\psi}\psi)$ arguments, were derived in [16]. Hereby, consistency with Ward identities as a constraint allowed to compute the γ -term related to the electromagnetic field: $e^2 \langle (\psi \dots \psi \bar{\psi} \dots \bar{\psi} A_{\mu_1} \dots A_{\mu_\ell})_+ \rangle$ and correspondingly $e^2 K(p \dots, q \dots, k_1 \dots k_\ell; m, e)$, which are functions of e^2 only, require no γ -term for A-type arguments. The coefficient of $\partial/\partial e^2$, if the mass term in the PDEs is $m[\partial/\partial m]$,

$$\beta(e^2) = (6\pi^2)^{-1} e^4 + O(e^6), \quad (8.1)$$

is gauge invariant as follows from the gauge invariance of (ordinary, and afortiori the praeasymptotic) VFs with only (≥ 4) A-type arguments. The operator $N_3(\bar{\psi}\psi)$ depends on the gauge merely through (convention dependent) overall normalization.

Only the Landau gauge, with photon propagator transverse, is stable under mass vertex insertion. Adler [29] has considered PDEs also in other gauges where, to stay in the same gauge or to have a simple gauge change, nonlocal gauge-changing insertions have to be made along with the $N_3(\bar{\psi}\psi)$ one. If the photon propagator is proportional to

$g_{\mu\nu} - c \square^{-1} \partial_\mu \partial_\nu$, with $c e^2$ - independent, then the gauge-change compensating term is soft (i.e. negligible asymptotically) such that the praeasymptotic Γ_{as} , or corresponding functions Γ_o , obey PDEs as simple as in Landau gauge as first observed by Wilson [28].

The theory Γ_o , see sect. 4.3, for QED is the one analyzed by Gell-Mann and Low [22]. The analog $\hat{\beta}(\hat{e}^2)$ of $\hat{\beta}(V)$, is (with mass term as for (8.1), and $z = \hat{e}^2/4\pi$, in their notation) $8\pi\psi(z)$, and has been computed to order z^3 by Rosner [37]. While the coefficients of $\hat{e}^4$ and $\hat{e}^6$ are identical (see (4.15)) to those in (8.1), a comparison of the coefficients $\hat{e}^8$ and e^8 , respectively, was made by de Rafael and Rosner [38].

It is noteworthy that e , although it carries over from Γ to Γ_{as} via the preserved Ward identities, is not a conventional charge since the Γ_{as} boson propagator cannot be normalized conventionally: Since the first term in (8.1) is positive, in massless QED (i.e. in the praeasymptotic or the Γ_0 theory) the small-Euclidean-momenta behaviour can be computed precisely, and one finds [20] that the boson propagator does not have a pole at $k^2=0$. Indeed, the conventional charge is zero [20] in view of the analoga in QED or gluon theory of (4.3) and (3.17).

In connection with the Gell-Mann-Low limit (in the sense of (6.4)) of QED it is tempting to use conformal invariance. However, for scale-invariant QED conformal invariance can be expected only modulo a gauge transformation [39]. (E.g., local gauge-invariance-breaking terms in the Lagrangian violate the conditions [40] for conformal invariance to be a consequence of scale invariance, since they involve derivative coupling. More generally, no Poincaré invariant gauge can be conformal invariant since for canonical vector field dimension only the longitudinal propagator is conformal invariant.) Adler [41] has proposed a class of gauges that are conformal invariant but which appear difficult to exploit. (See, e.g. [42] and references given there.) For a discussion of the eigenvalue condition for α proposed by Adler [29] see [44].

Adler [29] also considered using a finite photon mass, and this "gluon theory" has been analyzed in detail by Lowenstein and Schroer [45]. These authors in particular have used the PDEs to give a rigorous perturbation theoretical proof [46] of the Adler-Bardeen theorem [47] concerning the axial-vector-current anomaly. Gluon theory has two dimensionless parameters, e and the mass ratio, and falls under the scheme discussed in sect. 7 with $M=2$, $G=1$ (an independent $(A_\mu A^\mu)^2$ coupling being excluded as violating gauge invariance): The praeasymptotic theory is isomorphic to the one of QED.

The Ward identities are the same in the massive and the praeasymptotic theory. While the value of the bare charge is a property of the Γ_0 theory alone [22], its relation to the charge e appearing in the Ward identity is in gluon theory mass-ratio-and convention dependent. There are similar mass-ratio dependences also for other asymptotic quantities as follows from sect. 7

8.2. Yukawa theories.

The parametric functions of pseudoscalar meson theory were computed to lowest order, and the corresponding eqs. (7.5) with $G=2$ discussed, by

Ginzburg [48]. Zee [49] showed that for a class of simple Yukawa theories, the origin (in the two-coupling-constant space of these models) cannot be an UV stable fixed point. Coleman and Gross [18] gave an elegant proof that no renormalizable theory not involving non-abelian gauge fields can have the origin in coupling constant space as an UV fixed point. In both discussions, it was used that the renormalized coupling constant to a quartic Bose self interaction must be non-negative, on the basis of an instability argument applicable otherwise.

8.3 Nonabelian gauge field theories.

In view of the failure of other (unobjectional, cp. [50]) models to have an UV stable fixed point at the origin (which would lead to computable large-momenta behaviour) it was of interest [51] to consider Yang-Mills theories. Politzer [52], and Gross and Wilczek [53] showed that for simple gauge groups and no, or not too many, fermions, zero-coupling is a UV stable fixed point, by computing b_0 in

$$\hat{\beta}(f^2) = b_0 f^4 + b_1 f^6 + \dots \quad (8.2)$$

where f is the gauge coupling constant of the mass less theory and $\hat{\beta}(f^2)$ the analog of $\hat{\beta}(V)$ in sect. 4.3, and finding it to be negative under the stated condition. For $SU(2)$, this result had already been announced by G.'t Hooft [54]. Under such circumstances, the large-Euclidean-momenta behaviour of (off-shell) VFs is characterized by fractional powers of logarithms, related to (3.17) but with $\lambda \rightarrow \infty$ rather than $\lambda \rightarrow 0$. Such models are now called "asymptotically free (in the UV)". Gross and Wilczek [53][55] also discussed the effect of Higgs-scalars on the UV stability of the origin in the (now at least two-dimensional) coupling constant space, and found that, though asymptotically free models still exist, none of them appears viable (for further discussion, see [55]), and interest presently has shifted to Nambu-Jona-Lasinio type spontaneous gauge symmetry breaking (no elementary scalar fields introduced, cp. [52]) and to speculations on implications of unbroken gauge symmetries [56].

8.4 A note on asymptotically free theories.

For a reason to be explained later, it is of interest to discuss how "asymptotic freedom" (in Yang-Mills theory without scalar fields) comes about. Consider the summation of "leading logarithms" for a VF (i.e., of largest powers of $\ln \lambda$, under overall scaling of all momenta by λ ,

for $\lambda \rightarrow \infty$). From sect. 4 and (8.2) we know that the characteristic quantity is

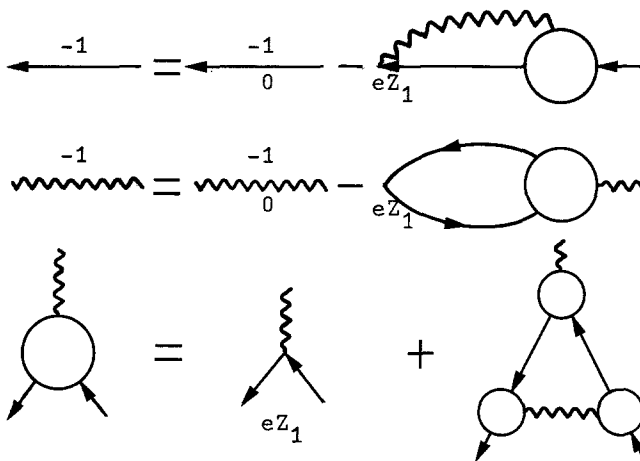
$$f^2(\lambda) = f^2 [1 - f^2 b_0 \ln \lambda^2 + o(f^2 \ln [f(\lambda)^2 f^{-2}])]^{-1} \quad (8.3)$$

where we have already indicated the effect of corrections.

If $b_0 > 0$, then for $\lambda^2 \approx \exp [f^{-2} b_0^{-1}]$ $f^2(\lambda)$ becomes arbitrarily large and thus the correction term in the denominator in (8.3) arbitrarily large relative to the other terms, and (8.3) becomes inconclusive.

If, however, $b_0 < 0$, then as $\lambda^2 \rightarrow \infty$ the correction term is small relative to the dominant term in the denominator, rendering the conclusion $f^2(\lambda) \approx (-b_0)^{-1} (\ln \lambda^2)^{-1}$ reliable. The smallness of $f^2(\lambda)$ means that all graphs not contributing to leading logarithms become suppressed upon their suitable partial summation, even more strongly (though only by inverse logarithms) than the sum of leading logs itself, which is, e.g., an alternating series formally summed to (8.3) (without correction) or to certain, possibly broken, powers of that $f^2(\lambda)$.

Landau and coworkers [57] obtained the result (8.3) in QED for $e^2(m^{-2}\Lambda^2)$, which is the bare charge squared computed with a cutoff Λ , with m e.g. the electron mass. The correction term in the denominator was neglected due to the smallness of e^2 (Bogoliubov and Shirkov [8] pointed out that this is inconsistent for $\lambda \rightarrow \infty$, $b_0 > 0$). (8.3) was obtained by determining the graphs that would give leading logarithms and computing these logarithms by studying the integral equations obeyed by the formal sums of these graphs, which are simplified forms of the correct coupled integral equations for the Green's functions. The equations are, in QED,



(8.4)

analogous to (1.14-16), of which only the vertex equation is approximate. Landau [57] concluded that QED and, similarly, pseudoscalar

meson theory, were theories inconsistent in the UV regime.

Now in the asymptotically free case, the existence of the praeasymptotic theory Γ_{as} outside of perturbation theory is less plausible than when the origin is UR stable, as in the case in ϕ^4 theory with $g > 0$. Namely, the defining integral, (4.9) with $\lambda=1$, then leads at the lower integration limit to increasing coupling constant. If, e.g., $\beta(f^2)$ stays negative for all f^2 for which the theory exists up to f_{max}^2 , say, and

$$\int_{f^2}^{f_{max}^2} df'^2 \beta(f'^2)^{-1} > -\infty,$$

then the praeasymptotic theory does not exist although it does, of course, in perturbation theory, however, conclusions concerning it and transcending perturbation theory, like use of the analog of (4.7) in conjunction with (8.3), are then doubtful. In such a case, one might attempt to relate the renormalization group directly to the large-momenta behaviour of Γ without the intermediary of Γ_{as} , as indicated in [50], provided the massless theory Γ_0 exists; however, Γ_0 and Γ_{as} are closely related. So if neither exists (outside of perturbation theory), in order to benefit from "asymptotic freedom" one could take recourse to (appropriate adaptations of) Landau's direct estimates [57], which do not presuppose explicitly the existence of the massless theory.

A rather simpler way to avoid the use of the praeasymptotic theory Γ_{as} is the following, suggested by G. Parisi: for simplicity, consider ϕ^4 theory with $g < 0$, as suffices to show the method. With (2.12), (3.19) yields

$$\begin{aligned} & \lambda^{-4+2n+2\ell} \Gamma(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_\ell; m^2, g) \\ &= a(g)^{n-\ell} a(g(\lambda))^{-n+\ell} h(g)^{-\ell} h(g(\lambda))^\ell \\ & \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2, g(\lambda)) \\ &+ i m^2 \int_{\lambda^{-2}}^1 dx^2 a(g)^{n-\ell} a(g(\lambda x))^{-n+\ell} h(g)^{-\ell} h(g(\lambda x))^\ell \\ & \cdot \phi(g(\lambda x)) \Gamma(p_1 \dots p_{2n}, q_1 \dots q_\ell; m^2 x^2, g(\lambda x)). \end{aligned} \quad (8.5)$$

For large λ , since $g(\lambda) \approx -b_0^{-1}(\ln \lambda^2)^{-1}$, for the first term on the r.h.s. of (8.5) the perturbation expansions in $g(\lambda)$ are adequate. One must now assume that the integrand is for small x bounded by $\text{const } x^{-2\epsilon}$,

with $\epsilon < 1$, this estimate being needed, however, for a VF and for factors a, h with argument $g(\lambda x) \leq g$ only, such that it is weaker than

the condition (4.2) that implied the existence of Γ_{as} . We now split the integration region into $\lambda^{-2} \dots \lambda^{-2\epsilon'}$ and $\lambda^{-2\epsilon'} \dots 1$, with $0 < \epsilon' < 1$. The first part yields $\text{const}(\lambda^{-2\epsilon'(1-\epsilon)} - \lambda^{-2(1-\epsilon)})$ which is negligible relative to the negative powers of logarithms one is here interested in. In the remaining part, since $\lambda x \leq \lambda^{1-\epsilon'} \rightarrow \infty$, perturbation expansion in $g(\lambda x)$ of the functions with this argument is adequate. A typical term is then estimated by

$$\int_{\lambda^{-2\epsilon'}}^1 x^{-2\epsilon} [\ln(\lambda x)]^{-k} dx^2$$

$$= \frac{1}{(\ln \lambda)^k} \sum_{r=0}^{\infty} (-1)^r (\ln \lambda)^{-r} \binom{-k}{r} \int_{\lambda^{-2\epsilon'}}^1 x^{-2\epsilon} (\ln x)^r dx^2$$

where, up to an error $O(\lambda^{-2\epsilon'(1-\epsilon)} (\ln \lambda) \dots)$ the integrals can be extended to lower limit zero. In the cases $n=2, \ell=0$ and $n=1, \ell=1$, the contribution from the integral in (8.5) is smaller by $(\ln \lambda)^{-1}$ than the homogeneous part; in all other cases, it is comparable to the homogeneous part, whereby for $n=1, \ell=0$ (8.5) should be used upon differentiation with respect to p^2 .

Closely related to these considerations is that the asymptotic expansions in inverse powers of logarithms obtained from formulae analogous to (4.5) and (3.17) satisfy generalized unitarity ([58], and second reference of [11]) in the sense of formal expansions. This is so because the Γ_{as} , whether or not they exist outside of perturbation theory, are themselves VFs of a Lagrangian field theory that satisfies in perturbation theory generalized unitarity. (Alternatively, one may consider the limit process (4.3) directly applied to the equations expressing generalized unitarity.) Some calculations hereto were given in [14] and [59], for praeasymptotic ϕ^4 theory with $g > 0$ which is asymptotically free in the infrared: the same calculations but with $g < 0$ then apply to large-momenta regime.

8.5 Uses of computable large-momenta behaviour.

In asymptotically free theories the large-momenta behaviour is computable, up to momenta-independent overall factors, in the deep-Euclidean region (i.e., for nonexceptional momenta) directly, and generally in the Euclidean region if appropriate exceptional-momenta techniques are used. This covers the singular coefficients in short-distance [13] and in light-cone expansions [25], (see sect. 5.1) and allows to give an explanation of Bjorken scaling as first observed by Parisi [60],

in the $(\phi_i \phi_i)^2$ theory, with $g < 0$ however, and later proposed by Gross and Wilczek [53] (see also [61] and [62]) for asymptotically free nonabelian gauge theory for which so far no similar objection [18] as against the scalar theory is known.

The other case where asymptotic freedom has found application, e^+e^- annihilation [60] [63], involves the two-current correlation function at time-like momentum, however, since this function depends on only one momentum squared and has nonnegative spectral function, the behaviour at large positive values of that invariant can relatively confidently be obtained from the one at large negative values. Inclusive e^+e^- - annihilation has in ϕ^4 (or, more realistically, $(\phi_i \phi_i)^2$) theory been made accessible to renormalization group methods by Mueller [15] by subjecting the expansion (2.20) in the interpretation described there to the technique of sect. 5.1. It can be hoped that this process can be analyzed [64] successfully in nonabelian gauge field theory.

There is, so far, no other application known of "asymptotic freedom". The difficulty is the abundance of UR logarithms [65] in theories with vector bosons, which renders Minkowskian momenta exceptional more easily than in theories without such bosons. E.g. in (massive-photon) QED, summation of leading logarithms leads for the electric form factor of the electron to ([66] [67] [68], see also [69])

$$F(q^2) = \exp \left[-(16\pi^2)^{-1} e^2 \left(\ln \left(\frac{-q^2}{m^2} \right) \right)^2 \right],$$

a formula apparently unrelated to any renormalization group result since for each factor e^2 it involves a "double logarithm". It is obvious that a broad study of infrared effects is here required.

CONCLUSION

Renormalizable field theories are definable by the formal sums of perturbation series. Alternative definitions are in terms of seemingly intractable infinite systems of integral equations (e.g., [5]), or via sequences of limit procedures (finite volume, UV cutoff) possibly to be followed by difficult analytic continuations (for a presentation of these matter, see [70]) with so far, in four dimensions, uncontrollable effects.

Given the definition by series that, in at least some cases, have been shown to be divergent, the investigation of properties of the separate terms in those series is unrewarding. The remaining way then to obtain possibly relevant information is by reordering the perturbation series.

The systematic ways hereto found so far are the renormalization group, and operator product (and related) expansions. Combining these two, physical large-momenta problems can be treated insofar as they are reducible, by some expansion (e.g. by light cone expansion as in deep-inelastic e-p scattering [25]), or by highly plausible assumptions (as in e^+e^- annihilation [63]) or these two suitably combined (as in [15]) to effectively a large-Euclidean-momenta problem. Only for asymptotically free theories does one thereby arrive at predictions, precise up to momenta-independent factors. In all other cases, assumptions on the existence of fixed points [28] and determination of corresponding anomalous dimensions are still required. In the last problem, conformal invariance might be of help, whereas it is difficult to directly apply conformal invariance in situations involving exceptional momenta. A better understanding of these latter and of the infrared phenomena they are connected with is clearly desirable.

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