

# The connection between spin and statistics

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**Abstract.** After some brief historical remarks we recall the first demonstration of the Spin-Statistics theorem by Pauli, and some further developments, like Streater-Wightman's, in the axiomatic considerations. Sudarshan's short proof then follows by considering only the time part of the kinetic term in the lagrangian: the identity  $SU(2)=Sp(1,C)$  is crucial for the argument. Possible generalization to arbitrary dimensions are explored, and seen to depend crucially on the type of spinors, obeying Bott's periodicity 8. In particular, it turns out that for space dimensions (7, 8, or 9) modulo 8, the conventional result does not automatically hold: there can be no Fermions in these dimensions.

## 1. Introduction

The idea of this work came from discussions with Sudarshan in Zaragoza some years ago. George was surprised to learn that the properties of spinors crucial for the ordinary connection between spin and statistics would not hold in arbitrary dimensions, because of periodicity 8 of real Clifford algebras. After many discussions and turnarounds, I am happy today to present this collaboration, and I also apologize for the delay, for which I am mainly responsible. In any case, the privilege to work and discuss physics with Sudarshan is a unique experience, for which I have been most fortunate.

## 2. Historical remarks

In retrospect, the first hint of the Bose statistics for "photons" comes from Plack's radiation law [1] (1900): the quantum hypothesis corpuscularizes the radiation, but by itself it leads only to Wien's law (Eintein, 1905). Without the strange correlations (first noticed by Ehrenfest in 1905) between "photons", the correct law does not obtain.

Independent light quanta lead to Wien's law (1896) for the radiation density  $\rho$  of a warmed black body with frequency  $\nu$  at temperature  $T$

$$\rho(\nu, T) = A \nu^3 \cdot \exp(-\beta \nu / T) \quad (1-1)$$

which is already unsatisfactory: for high  $T$  the density  $\rho(\nu, T)$  tends to a constant, whereas from thermodynamics one should expect a dependence linear with  $T$  [2]. This is indeed the case for the correct Planck's formula

$$\rho(\nu, T) d\nu = 2 \cdot 4\pi (\nu^2/c^2) (h \nu) [\exp(h\nu/kT) - 1]^{-1} d(\nu/c) \quad (1-2)$$

which fits experiments wonderfully (then and now). The correlation aspects of (1-2) can be interpreted also as wave-particle duality (the fluctuation formula of Einstein, 1909): it is intriguing that the Bose correlations manifest themselves in different ways.

There is no hint of the exclusion principle (statistics for Fermions) until Bohr's unsuccessful attempts (ca. 1922) to understand the periodic system of elements with the help of the correspondence principle; that the task was impossible was pointed out by Pauli, who later enunciated the correct principle. In January 1925, W. Pauli stated the *Aequivalenz Verbot* (exclusion principle). From some terms missing in atomic spectra, in particular, from the absence of the singlet state of orthohelium  $1^3S_1$ , Pauli concluded that the electrons in each atom are characterized for four quantum numbers each, and no two of them can have the four numbers equal. The periodic system, the concept of valence, chemical compounds, etc., was immediately understood. Indeed, today we realize that the exclusion principle is the true *differentiating principle* in Nature.

Still within the Old Quantum Theory, Fermi applied the exclusion principle [3] to the statistical properties of metals, obtaining satisfactory results (January, 1926). It is remarkable that the density formulas of Planck, Fermi and the classical one by Boltzmann, are so similar:

$$\rho \propto 1/[(\exp E/kT) + \varepsilon] \quad (1-3)$$

with  $\varepsilon = 0, -1$  or  $+1$  for Boltzmann, Bose, or Fermi statistics respectively.

In June 1925 Matrix Mechanics was established by Heisenberg, and the mystery of statistics was cleared up very soon after. The next year, both Heisenberg and Dirac independently pointed out that the unobservability of the phase of the wavefunction allows naturally for an extra sign under the permutation of two identical particles:

$$\Psi(1, 2) = \pm \Psi(2, 1) \quad (1-4)$$

For the moment, there is only an empirical rule: integer spin objects go with the  $+$  sign, half-integer ones with the  $-$  sign. For many particles, there is a neat group-theoretical interpretation:

For  $n$  "Fermions", the permutation group  $S_n$  is represented modulo the alternating group, so  $S_n/A_n = Z_2$ , whereas for "bosons", there is only the Identity representation. Other representations (parastatistics) seem to be naturally excluded.

If we accept that two anomalies *should* go together, there is even a hint for the spin-statistics connection: For spin  $1/2$  particles, the wavefunction picks up a  $-$  sign under a  $2\pi$  rotation, AND the odd transposition picks up also a  $-$  sign; the two oddities go together: Fermi statistics for spinors. For spin integer, neither of this happens. This is just a clue, not a proof, of the theorem we seek on spin and statistics.

Early uses of statistics were P. Jordan's work with second quantization, where the Fermi case forced *anticommutation* relations for quantum fields (Jordan, 1927; Jordan and Klein, 1927; Jordan and Wigner, 1928). Ehrenfest and Oppenheimer proved (1931) that the rule was compatible with compound systems, and in this way quantum statistics was successfully applied to nuclear physics. Indeed, one of the first arguments for the proton-neutron model of nuclei was the experimental statistics for Nitrogen-14.

However, nobody noticed, at the time, that the Fermi/Bose rules allowed a neat presentation of *supersymmetry*: Define Bose/Fermi operators  $a, b$  and hamiltonian  $H$  by

$$2H = \{a^\dagger, a\} + [b^\dagger, b] \quad \text{with } [a, a^\dagger] = \{b, b^\dagger\} = 1 \quad (1-5)$$

Then diagonalizing

$$H = (N_a + 1/2) + (N_b - 1/2) \text{ with } N_{a,b} = a^\dagger a, \quad b^\dagger b \quad (1-6)$$

and  $N_a = 0, 1, 2, \dots$  but  $N_b = 0$  or  $1$ . *Supersymmetry* is shown in that there is no zero-point energy, there is Bose-Fermi symmetry in the spectrum (each level is double degenerate), and also in that the SUSY operator  $Q$  is a square root of the energy:

$$Q := b^\dagger a + (b^\dagger a)^\dagger \quad \text{implies} \quad Q^2 = H \quad (1-7)$$

In the 30s there was some important work (Heisenberg, Dirac, Pauli, and Weisskopf) in which the two ways of quantization were used consistently, and even the symmetry/antisymmetry alternative helped in taming divergences (perturbation-theory infinities).

### 3. The conventional proof

Pauli prepared his first proof of the spin-statistics connection for relativistic quantum fields (an outgrowth of his study, with Fierz, of higher-spin wave equations), for the 1939 Solvay conference. The work [4] appeared in Phys Rev in 1940.

The proof relies heavily on the properties of  $SL(2, \mathbb{C})$ , the covering group of the Lorentz group; the irreducible linear representations (*irreps*) are labeled as  $D_{jk}$ , with  $2j$  and  $2k$  integers. For  $j+k$  integer, we have tensor irreps, type (+1) for both integers, type (-1) if both are half-integers. Spinors have  $j+k$  half-integer, again type (+ $\epsilon$ ) for  $j$  integer, type (- $\epsilon$ ) for  $k$  integer. The four classes multiply as Klein's *Vierergruppe*.

The essence of Pauli's argument is this: describe particles with fields  $\varphi(\mathbf{x}, t)$ ; for tensors (class +1 and -1), the *particle* density  $j_0$  is undefined ( $\propto \varphi^* \partial_0 \varphi$ ), but the *energy* density  $T_{00}$  is defined,  $\propto |\partial_0 \varphi|^2$ . So the field can describe two types of charges, but with positive energy each.

For spinors, the opposite is true, as it is plain already from the Dirac equation:

$$\text{Particle density definite, } \rho = \psi^* \psi \quad (2-1)$$

$$\text{Energy undefined, } E = \pm \sqrt{(k^2 + m^2)} \quad (2-2)$$

The deep and general reason for the proof is this: there is an *allowed* symmetry (close to the later CPT symmetry) in the equations of motion which changes the sign of the complex particle density for tensors, and changes the sign of the energy density for Fermions. All this holds in the *first quantized* theory.

Then Pauli argues that second quantization comes to the rescue and it is necessary in both cases (pair creation was well established by 1939; in particular, the Pauli-Weisskopf 1934 "anti-Dirac" paper established systematically pair creation for bosons), but in the tensor case commutators will respect indefinite charge and definite energy, whereas in the spinor case anticommutators will restore indefinite charge (so electrons and positrons are described by components of the same field), and recover definite energy (as excitations from the vacuum of either charge sign, hence positive energy: second quantization oversteps hole theory).

Implementation of the rule implies also causality: physical magnitudes (= observables) commute outside each other's light cones.

Pauli's argument deals only with *free* particles, without interactions, but does not restrict only to first order equations. The final remark of Pauli's Phys. Rev. paper [4] is worth repeating:

"... in our opinion, the connection between spin and statistics is one of the most important application of special relativity theory"

Is this true? Keep tuned!

Pauli should indeed feel very proud of his proof, because it is a satisfactory solution of a problem which he himself triggered with the discovery of the exclusion principle.

For other contributions to the issue in the early 40s see the book by Sudarshan and Duck (1997) [3].

#### 4. Some further developments

It was to be expected that the “renormalization” revolution in Quantum Field Theory in 1947/49 would shed some light on the statistics issue; indeed that was the case.

J. Schwinger in his 1951 papers [5,6], seems to derive the theorem by imposing a sort of  $\Theta$  = CPT invariance. Time reversal is antiunitary and it is interpreted also as rotation in a fictitious 4-5 biplane; after some analysis JS concluded that this implies *anticomutation* rules for spinors; the fifth dimension is hidden already in the  $\gamma_5$  matrix (the whole Dirac algebra generates in fact the conformal group,  $O(2, 4)$ ). An important advance of this work is that the commutation/anticommutation alternatives are *derived*, not postulated. Still, in Schwinger’s approach no interactions are contemplated.

In the later work of Lüders and Zumino (from 1954 on) a reversal of arguments occurred: the spin-statistics connection was used to *prove* the CPT Theorem. In his contribution to a *Festschrift* for Niels Bohr, Pauli (1955) incorporated interactions [7]. He proved the CPT theorem based in three main assumptions: 1) The usual Spin-Statistics connection, 2) Invariance under the connected Lorentz group, and 3) Locality: Fields are *finite*-dimensional tensors and spinors, and interaction terms occur with at most *finite* order derivatives.

We are approaching now 1956, when Parity symmetry was seen to be violated. Sudarshan took a leading role in establishing the V – A form of the weak interactions, incorporating automatically parity violation. It is curious that the early arguments emphasize the masslessness of the neutrinos, whereas the true argument is that only one helicity (of either neutrinos or other Fermions) enters in the interaction. So the argument holds even today, with massive neutrinos.

Among the further work up to the Streater-Wightman book [8] (1964) we note the Lüders-Zumino work of 1958. They disentangled the spin-statistics theorem from the CPT theorem. They started from the axioms of field theory, including uniqueness of the vacuum and positive metric in state space. Burgoyne’s proof (1958) is similar, but includes arbitrary spins. The march of the proof is: first prove the Spin-Statistics connection, then the CPT invariance follows fairly easily.

Axiomatization takes a leading role in the full treatment exposed in the Streater-Wightman book. Indeed, the authors claim that

“In this book, we have eliminated all theorems whose proof are non-existent.”

They establish four group of axioms, including asymptotic completeness, a postulate necessary for the collision theory of Ruelle and Haag. They proceed to prove first the CPT theorem, shown to be equivalent to some identities among the vacuum expectation values, identities that necessarily hold in every field theory of *local* fields. Excluding explicitly parastatistics, they then show that the usual connection holds, in the sense of commutators/anticommutators of field operators at spacelike distances. The proof, irreproducible, relies heavily in analysis with several complex variables.

R. Jackiw has a pessimistic attitude [9] towards those “axiomatic” approaches, and we sympathize:

“All the axiomatic progress in Quantum Field Theories in 1+1 dimensions has been unable to discover the most important new phenomenon: these theories might describe: *solitons*, as special excitations of the field, with capital physical implications ” (1976).

In Spain we say: “Matando moscas a cañonazos”...

#### 5. SUDARSHAN’S SHORT PROOF

The essence of Sudarshan’s proof [10] of the spin-statistics theorem (since 1968) is that the whole affair is really a nonrelativistic problem, or rather that it is the rotation part  $SO(3)$  of the Poincaré group which is crucial for the proof. In fact, spinors as irreps of the covering groups

SU(2) of the 3-dimensional rotation group already exist! For  $n \geq 2$ , SO(n) is connected but for  $n > 2$  its universal cover, Spin(n), is a double covering. The essential nontrivial homotopy group of symmetries of Nature is already revealed in the pure space rotation part. This topological feature, allowing genuine projective representations, seems to be the culprit for the existence of Fermions and the attendant exclusion principle. Although there are cases in which the covering group SU(2) shows up already in classical physics [11], the crucial projective nature of implementation of symmetry in Quantum Mechanics makes up for at least two types of realizations of the rotation group.

We shall see that although the argument can go in a nonrelativistic setting, one needs really a full quantum field theory for the proof to proceed. For the actual proof, Sudarshan and Duck [3] describe the theory through a Lagrangian, and look at the kinematic part only, as Schwinger already did in 1951, see above [5]. The crucial observation is a Cartan identity among Lie groups:

$$\text{Spin}(3)=\text{SU}(2)=\text{Sp}(1,\text{C})(\text{unitary symplectic}) \quad (\text{IV-1})$$

Hence, the group action leaves invariant an antisymmetric bilinear form. But *even* products of this form leave a symmetric bilinear form invariant; indeed for the square, we get the natural quadratic form for the s=1 irrep of the SO(3) group. The complete argument follows now:

2j odd:  $D_j$  irreps of SU(2) are pseudoreal, *symplectic*, faithful, even  
(2j+1) dimensional  
(IV-2)

2j even:  $D_j$  irreps of SU(2) are real, faithful of SO(3), *orthogonal*, odd  
(2j+1) dimensional (IV-3)

Let us try to construct a Lagrangian, which should be a scalar: from above, the spinorial part should pair the fields antisymmetrically to obtain scalars, whereas the tensor part should pair them symmetrically, for the same reason. Write now all fields as a “long” field  $\xi(x)$ , somewhat in the Schwinger spirit, hermitean ( $\xi = \xi^\dagger$ ). As Lagrangian density we take, with  $A' = \partial A / \partial t$

$$\mathcal{L} = (1/2) \sum_{rs} K^0_{rs} (\xi_r \xi_{s'} - \xi_{r'} \xi_s) + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \quad (\text{IV-4})$$

where  $\mathcal{L}_2$  includes the space derivatives in the kinetic term,  $\mathcal{L}_3$  is the mass term, and  $\mathcal{L}_4$  includes the interaction terms.

For our analysis the time derivative term of the kinetic part  $\mathcal{L}_1$  will suffice - another advantage of Sudarshan’s treatment. Antisymmetrization avoids a possible total derivative, and the labels r,s include, of course, all fields in the game, Bose and/or Fermi.

Now for *tensor* fields, scalar (invariant) implies symmetrical products, and the antisymmetry in the  $\xi$ ’s forces the numerical kernel  $K^0$  to be antisymmetric as well. For spinor fields, of course, the opposite conclusion obtains:

$$K^0_{rs} \text{ is } (r, s) \text{ antisymmetric for both } r,s \text{ tensor labels} \quad (\text{IV-5})$$

$$K^0_{rs} \text{ is } (r, s) \text{ symmetric if both } r,s \text{ are spinor labels} \quad (\text{IV-6})$$

Of course, there is no spinor-tensor mixing: either superselection rules, or better,  $D_{\text{tens}} \otimes D_{\text{spin}}$  will *never* contain the scalar  $D_0$ .

To clinch the argument one has to show, in this lagrangian formalism, that for *commuting* fields the numerical kernel  $K_{rs}^0$  must be antisymmetric (in r,s), so conmutators should go with bosons, whereas for *anticommuting* fields the kernel should be antisymmetric (in r,s), so Fermi fields should anticommute.

To show this, Sudarshan relies on Schwinger's quantum action principle (1951) [5], which unifies equations of motion and Noether's theorem for conserved quantities in the most beautiful way. Let us consider variations on the action brought about by variations on the fields  $\xi$  only, so we need to consider only the space integral:

$$\begin{aligned} \xi &\rightarrow \xi + \delta\xi_k \Rightarrow \\ 2 \delta I(\delta\xi) &= \int d^3\mathbf{x} \sum K_{rs}^0 (\xi_r \delta\xi_s - (\delta\xi_r) \xi_r) \end{aligned} \quad (\text{IV-7})$$

Since the infinitesimal variation of the action transforms the field variables infinitesimally, we require

$$[\xi_a, \delta I(\delta\xi)] = i \hbar \xi_a \quad (\text{IV-8})$$

or

$$2i\hbar \delta\xi_a = \int d^3\mathbf{x} [\xi_a, \sum K_{rs}^0 (\xi_r \delta\xi_s - \delta\xi_r \xi_s)] \quad (\text{IV-9})$$

Let us discuss now the two alternatives. If the fields  $\xi$  are *bosonic* fields,  $\delta\xi$  is supposed to commute with everything. It follows straightforwardly, expanding the commutator, keeping the variations  $dx$  at the end, and adding space labels  $\xi(\mathbf{x})$  etc., that  
(see Sudarshan & Shaji [12])

$$[\xi_a, \delta I(\delta\xi)] \neq 0 \Rightarrow K_{rs}^0 = -K_{sr}^0 \text{ (Bose)} \quad (\text{IV-10})$$

which imply the usual canonical conmutation relations

$$[\xi_a(\mathbf{y}), \Pi_j(\mathbf{x})] = i \hbar \delta^3(\mathbf{x}-\mathbf{y}) \delta_{aj} \quad (\text{IV-11})$$

introducing the conjugate momenta

$$\Pi_k \equiv \partial \mathcal{L} / \partial \dot{\xi}_k = K_{kr}^0 \xi_r \quad (\text{IV-12})$$

For anticommuting fields, i.e. supposing the variations  $d\xi$  anticommute with Fermi fields, it is fairly clear we shall obtain the opposite result [12] after reordering viz:

$$\{\xi_a, \delta I(\delta \xi)\} \neq 0 \Rightarrow K_{rs}^0 = + K_{sr}^0 \text{ (Fermi)} \quad (\text{IV-13})$$

which, in turn, leads to anticommutation, i.e. the symmetric rule between fields and conjugate momenta.

“We have seen that the rotational invariance of the kinematical part of the Lagrangian density requires that the matrix  $K_{rs}^0$  be antisymmetric for integral spin fields while it will be symmetric for half-integral spin fields. The action principle of Schwinger requires  $K_{rs}^0$  to be antisymmetric for commuting (Bose) fields, and symmetric for anticommuting (Fermi) fields:

The Spin-Statistics connection obtains!”

Negative-norm states might appear, and have to be declared “outlaws”. Also, even in our nonrelativistic setting the coupling of creation and annihilation operators appear with the same strength (Kirchoff’s principle).

The proof is remarkable; among other things, it nearly fulfills the expectation that “the two pathologies go together”, as stated above: Fermions are strange animals in two compensating ways!

Incidentally, all the modern work of Sudarshan and collaborators on the spin-statistics issue seems to be prompted by the remarks of Feynman, collected by Neuenschwander as a question [13], about the lack of a simple and neat proof of this fundamental relation. An answer close to our “two pathologies cancel” is in [14].

## 6. Higher dimensions

Does the spin-statistics connection hold in arbitrary dimensions? In one and two ( $= 1, 1$ ), the situation is special and we shall not touch on it; in particular in  $(1, 1)$ , Fermi-Bose conversion is possible. The quantized solitons in that case are automatically Fermions. What about higher dimensions, which appear so conspicuously in modern physics, related to supersymmetry, string theory, etc.? Very few papers address directly that question. We do [15], which we follow here.

We recall first the behaviour of the rotation or spin groups in higher space dimensions. We have, modulo 8:

| Spin(3)     | Spin(4)            | Spin(5) | Spin(6) | Spin(7,8,9) | Spin(10) |
|-------------|--------------------|---------|---------|-------------|----------|
| SU(2)=Sp(1) | SU(2) <sup>2</sup> | Sp(2)   | SU(4)   | SU(4,5,5)   | SU(6)    |
| H           | H                  | H       | C       | R,R,R       | C        |

An irrep is type R if equivalent to a real one; type C if complex, and type H if equivalent to conjugate but not real.

Now the H (or quaternionic) type yields the Id irrep in the *antisymmetric* (symplectic) square of the spin irrep, like the simple SU(2) case. Hence the situation in  $\dim (3, 4, 5) \bmod 8$  is identical to the conventional 4-dim  $(1, 3)$  with space group SO(3).

In the real R type case, one obtains the Id irrep in the symmetric part of the square of the spin irrep. The complex case, type C, is somewhat intermediate (see below).

We conclude that the Id irreps for the space rotation group in dimensions 7, 8 and 9 mod 8 are always symmetric, so we cannot have spinor field theories. It does go through in dimensions 3, 4, and 5. In SO(4), there is another complication, as the center of the spin group is  $Z_2 \times Z_2$  (Klein’s Vierergruppe), and there are no faithful irreps: hence, it is not automatic that the spinor wavefunction acquires a “-” sign under a  $2\pi$  space rotation.

As for the complex case,  $\dim 2, 6 \bmod 8$ , one can always arrange the Id irrep to be in the antisymmetric part, but it is not mandatory, as the product is not the square, but the *modulus* square (D times conjugate). If written in real form, we get the Id irrep twice, one in the symmetric, another in the antisymmetric part. The “right” statistics is optional. As for the

reasoning from the Action principle, it seems not to depend on dimension, so we do not reconsider it any further.

We see that in Superstring theory, which works in  $10 = (1, 9)$  dimensions, the rotation group is in the “bad” class, and so is the little group for massless particles ( $O(8)$ ). We have not developed consequences of this in the actual presentations in string theory. As for 11-dim supergravity and also in M-Theory, one works in  $11 = (1, 10)$  dimensions: good connection is possible, if optional. The massless case (group  $O(9)$ ) is bad. Finally, in F-Theory (Vafa, 1996) we are in  $12 = (2, 10)$  dimensions, so good connection is optional for  $O(10)$ ; and  $O(11)$ , if pertinent, is good (H type).

## 7. Concluding Remarks

We see that the proof, when it holds, even in higher dimensions, is really very simple. It needs only the temporal part of the kinetic term of the Lagrangian, and it is very much in the spirit and query of Neuenschwander [13], namely, to obtain a simple and natural proof. And from Bott’s 8-periodicity, we find the special cases of space dimensions  $8n-1$ ,  $8n$ , and  $8n+1$ , where the usual proof does not go through.

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