beam of 10^{14} protons per second could be accelerated and stacked at about 14 GeV, and RF noise introduced to increase the energy width of the stack to 1 GeV. The stack could then be brought onto the energy-loss foil slowly and uniformly over a period of a second by a phase-displacement oscillator while a new stack was being formed at 14 GeV. A beam duty factor in excess of 80% should result.

The extraction channel is not discussed beyond the current septum; magnets and magnetic channels would follow in a rather conventional manner, after the initial separation of extracted particles from circulating particles.

For simplicity, the sketches indicate particles striking the first foil as having no betatron oscillation amplitude. The operation of the system is independent of *initial* amplitudes so long as the oscillations are nearly linear. A spread of betatron amplitudes will produce the only energy spread in the external beam; approximately 100 MeV for a ± 1 cm betatron oscillation amplitude.

It is reasonable to conclude, therefore, that an FFAG accelerator will be able to provide external proton beams of high quality ranging from one turn extraction of accumulated (stacked) particles to essentially continuous 10 to 50 μA beams.

LIST OF REFERENCES

1a. Wright, B. T. Magnetic deflector for the bevatron. Rev. sci. Instrum., *25*, p. 429-31, 1954.
b. Piccioni, O., Clark, D., Cool, R., Friedlander, G. and Kassner, D. External proton beam of the cosmotron. Rev. sci. Instrum., *26*, p. 232-3, 1955.
2. O'Neill, G. K. and Woods, E. J. Intersecting beam systems with storage rings. Princeton-Pennsylvania Accelerator Project Internal Report (*). GKON-12 EJ.W-2, 1959.
3. Terwilliger, K. M. Achieving higher beam densities by superposing equilibrium orbits. See below.
4. Tuck, J. L. and Teng, L. C. Synchro-cyclotron progress report III. University of Chicago, Institute of Nuclear Studies, 1950. Chapter 8.
5. Le Couteur, K. J. The regenerative deflector for synchro-cyclotrons. Proc. phys. Soc., Lond. B, *64*, p. 1073-84, 1951.

ACHIEVING HIGHER BEAM DENSITIES BY SUPERPOSING EQUILIBRIUM ORBITS

K. M. Terwilliger (**)

Midwestern Universities Research Association, Madison, Wis. (***)

(presented by F. T. Cole)

I. INTRODUCTION

In order to do colliding-beam experiments with a reasonable counts-to-background ratio, it is desirable to have a high beam current density. Obtaining equal numbers of beam-beam collisions and beam-residual gas collisions, for example, would require a current density of about 50 A/cm^2 at 10^{-8} mm Hg. To get this density it is usually necessary to spatially superimpose a large number of injected pulses, requiring RF beam stacking [1]. In this stacking process, however, depositing a beam pulse at a

(*) Internal memoranda not generally distributed but possibly available from author.

(**) University of Michigan, Ann Arbor, Mich.

(***) Supported by the United States Atomic Energy Commission.

given energy with the RF lowers the energy of the previously stacked coasting beam by the energy width of the accelerating RF bucket, in accordance with Liouville's theorem. This decrease in energy decreases the beam radius an amount determined by the momentum compaction factor. When this decrease in radius is larger than the maximum radial betatron oscillation, the added pulses will no longer spatially superimpose on the first ones. There will be an increase in the total stacked beam, but no further increase in beam density.

It would be useful to eliminate the spatial separation of the different energy equilibrium orbits, at least in the colliding-beam region. The betatron oscillation amplitude would cease to be a limit on the achievable beam density; pulses could be spatially superposed independent of this amplitude. Such equilibrium orbit superposition is reasonable easy to achieve, at least over a limited energy range.

II. SUPERPOSITION OF EQUILIBRIUM ORBITS

The linearized radial equation of motion of a particle of momentum $P+\Delta P$ about the equilibrium orbit of a particle of momentum P is [2]

$$\frac{d^2x}{ds^2}+\left(\frac{1-n(s)}{\rho(s)^2}\right)x=\frac{1}{\rho(s)}\frac{\Delta P}{P}, \tag{1}$$

where s, n and ρ are measured along the P equilibrium orbit. The equilibrium orbit of the particle of momentum $P+\Delta P$ is of course the particular solution to the above equation. For conventional weak focusing or AG machines, the average equilibrium orbit displacement is

$$\langle x\rangle=\frac{r}{\nu_x^2}\frac{\Delta P}{P}$$

while for a scaling FFAG machine, with field $H\sim r^k$,

$$\langle x\rangle=\frac{r}{k+1}\frac{\Delta P}{P}.$$

Now, since Eq. (1) is linear, if a particular solution for a given ΔP is designed to be zero or tangent to $x=0$ at a given azimuth, all equilibrium orbits will automatically be zero or tangent at that azimuth. So the possibility of superposition is obvious; it is just necessary to find a convenient linear modification of the conventional fields.

The different equilibrium orbits can obviously be made tangent at particular azimuths by adding a field modification such that the off-momentum equilibrium orbits undergo a forced oscillation about their average displacement, with an amplitude equal to this average displacement. One simple method of producing a forced oscillation equal to the average displacement consists of adding a field gradient perturbation having a Fourier component $\Delta H=[H'_M\cos M\theta]x$, where M is an integer reasonably close to ν_x, and x again is the displacement from the selected equilibrium orbit. This selected equilibrium orbit, of course, will be unchanged by the perturbation. The perturbation could be produced with pole-face windings or quadrupole lenses [(*)].

The effect of such a gradient type field perturbation on the equilibrium orbits is easily calculated, to a first approximation, giving some insight into the process. Measuring all distances in terms of r_0 = circumference/2π, writing $\theta=s/r_0$ and taking $P=\langle H\rangle_{\text{orbit}}=1$, Eq. (1) is

$$\frac{d^2x}{d\theta^2}+\left(\frac{1-n}{\rho^2}\right)x=\frac{\Delta P}{\rho}. \tag{2}$$

With the additional perturbing field

$$\Delta H=[H'_M\cos M\theta]\,x,$$

this becomes

$$\frac{d^2x}{d\theta^2}+\left[\frac{1-n}{\rho^2}+H'_M\cos M\theta\right]x=\frac{\Delta P}{\rho}, \tag{3}$$

n and ρ here having their original unperturbed values of Eq. (2). Taking x_2 as the particular solution to Eq. (2), and x_3, the desired equilibrium orbit, as the particular solution to Eq. (3), expand about x_2 in Eq. (3) by defining a difference orbit $x_4=x_3-x_2$. Then Eq. (3) gives the equation for x_4:

$$\frac{d^2x_4}{d\theta^2}+\left[\frac{1-n}{\rho^2}+H'_M\cos M\theta\right]x_4=-[H'_M\cos M\theta]x_2. \tag{4}$$

To obtain an approximate solution to Eq. (4) replace

(*) Another technique for achieving superposition, suggested by Courant, is to drive the off-momentum equilibrium orbits by modulating $\varrho(s)$ of (1) with a periodicity close to ν_x, like the proposal of Vladimirskij and Tarasov for eliminating the transition energy.

the left-hand side with a harmonic oscillator of the same frequency ν and x_2 by its average value $\langle x_2 \rangle$. Then

$$\frac{d^2 x_4}{d\theta^2} + \nu^2 x_4 = -[H'_M \cos M\theta]\langle x_2 \rangle \, , \qquad (5)$$

which gives as the approximate particular solution to Eq. (4) :

$$x_4 \cong \frac{[H'_M \cos M\theta]\langle x_2 \rangle}{M^2 - \nu^2} \, . \qquad (6)$$

Since the correct forced equilibrium orbit is given by $x_3 = x_2 + x_4$, replacing x_2 and x_4 by their approximate values gives

$$x_3 \cong \langle x_2 \rangle + \frac{[H'_M \cos M\theta]\langle x_2 \rangle}{M^2 - \nu^2} = \langle x_2 \rangle \left[1 + \frac{H'_M \cos M\theta}{M^2 - \nu^2} \right] . \qquad (7)$$

So to this approximation, the off-momentum equilibrium orbits can all be made tangent to zero at periodic angles. If the angles were chosen to be $\theta_m = \frac{2\pi m}{M}$, for example, the perturbation coefficient would be

$$H'_M = -(M^2 - \nu^2) \, . \qquad (8)$$

The addition of the perturbing field $[H'_M \cos M\,\theta]x$ will open up stopbands at $\nu = M/2$, $2\,M/2$, $3\,M/2$,... and will cause a perturbation of the original ν. The tune-changes and the stopbands, away from the $M/2$ region, however, are small; so bringing M to the closest or next to closest integer to ν should not prove troublesome [3].

The equilibrium orbits must superpose, with a correctly designed gradient perturbation, only as long as Eq. (3) remains the radial equation of motion, that is, as long as the equation stays linear in the quantities x and ΔP. In a perfect linear field AG machine the radial equation stays essentially linear in x, and the x tune is independent of amplitude. The coefficient of x, however, depends on the momentum ΔP. An increase in ΔP will change ν generally decreasing it; in any event it will change $|M^2 - \nu^2|$, hence the amplitude of the forced oscillations, and the orbits will no longer exactly superimpose, as seen from Eq. (7). In a non-linear but scaling FFAG machine, the tune is independent of ΔP, but depends on the x amplitude, so there is a change in $|M^2 - \nu^2|$ for the larger driven oscillations, and the orbits for large ΔP again will not exactly superpose. For both types of machine it should be possible, of course, to make the perturbation slightly non-linear to compensate for the change in $M^2 - \nu^2$, giving exact superposition over a larger range of ΔP. Since the perturbation itself is usually small, this added non-linearity should not appreciably affect the frequencies.

III. EXAMPLE OF SUPERPOSED ORBITS IN AN AG STORAGE RING

As an illustration, a field gradient perturbation giving superposed equilibrium orbits was designed for an AG machine with the MURA IBM-704 computer, using the approximate result of the previous section, Eq. (8), as a guide. The unperturbed magnetic field, expanded about a circle, was

$$H = 1 - [300 \cos 32\theta]x \, ,$$

which gives the linearized on-momentum radial equation and tune

$$\frac{d^2 x}{d\theta^2} + [1 - 300 \cos 32\theta]x = 0 \, , \quad \nu_x = 7.27 \, . \qquad (9)$$

Nearly exact superposition for small ΔP was achieved with the added field gradient perturbation

$$\Delta H = H'_M \cos M\theta x = -9.7 \cos 8\theta x \, ,$$

giving the new on-momentum equation and tune

$$\frac{d^2 x}{d\theta^2} + [1 - 300 \cos 32\theta - 9.7 \cos 8\theta]x = 0 \, , \quad \nu_x = 7.25 \, . \qquad (10)$$

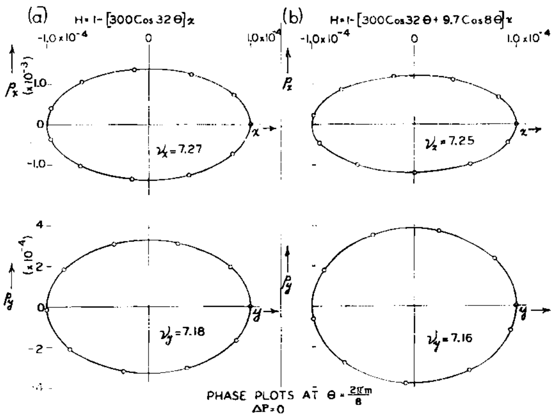

Fig. 1. Betatron oscillation phase plots, (a) without and (b) with the field gradient perturbation. The plots are made at azimuths $\theta_m = \frac{2\pi m}{8}$, 8 points per revolution. These phase plots are for on-momentum orbits.

Sample betatron oscillation phase plots, with and without the field gradient perturbation, are presented in Fig. 1. As expected, the perturbation has only a slight effect on the betatron oscillation frequencies and phase plots.

To get superposition using the approximation (8) of the previous section, one would calculate the perturbation coefficient

$$-H'_M = M^2 - \nu_x^2 = 8^2 - 7.25^2 = 11.4$$

reasonably close to the computer designed value $-H'_M = 9.7$.

Sample off-momentum equilibrium orbits, with and without the linear perturbation, are shown in Fig. 2. The perturbed equilibrium orbit is clearly just a driven oscillation about an average displacement, as indicated earlier. The maximum radial excursion of this driven orbit has been made as small as possible by appropriate phasing of the two gradient terms in the magnetic field. The positive maximum of the perturbing gradient, hence the maximum amplitude of the driven equilibrium orbit, is made to occur at the middle of a negative gradient sector.

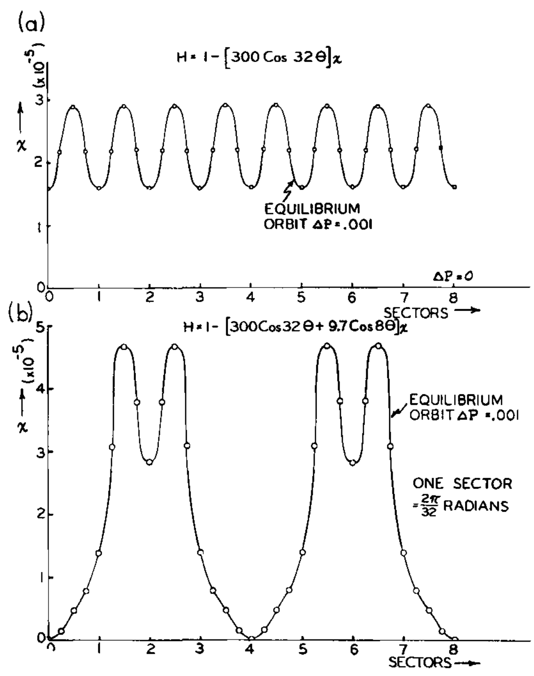

Fig. 2. Off-momentum equilibrium orbits (a) without and (b) with the perturbation.

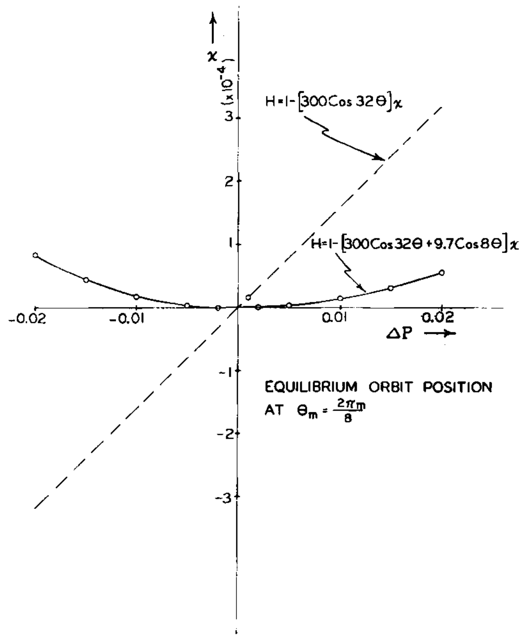

Fig. 3. Unperturbed and perturbed equilibrium orbit positions as a function of momentum, plotted for the azimuths $\theta_m = \frac{2\pi m}{8}$.

Fig. 3 presents the effect of non-linearities on superposition of the driven orbits. Plotted are perturbed and unperturbed equilibrium orbit positions at the azimuths $\theta_m = \frac{2\pi m}{8}$, corresponding to the minima of Fig. 2b, as a function of the momentum deviation ΔP. As ΔP increases, the superposition is less exact, the separation from $x = 0$ being approximately a quadratic function of ΔP. The reason, mentioned earlier, is that increasing ΔP decreases ν_x, hence increases $|M^2 - \nu_x^2|$; this decreases the amplitude of the driven oscillations and they no longer get back to $x = 0$. The quadratic dependence follows from Eq. (7), since the change in ν_x is approximately proportional to ΔP. This change in ν_x does not appreciably affect the positions of the unperturbed equilibrium orbits, hence their displacements remain proportional to ΔP. These displacements from $x = 0$ are considerably more than those of the perturbed orbits, even for quite large ΔP.

For an estimate of the current density attainable, take the above AG machine to be a 10 GeV storage ring of 50 m radius, used in conjunction with a 10 GeV accelerator of the same tune and radius. With an injection energy of 50 MeV, the final current density for one pulse, after betatron damping, is about 0.6 A/cm^2; the density is assumed to be limited by space charge repulsion at injection, including a factor of 1/4 for RF bunching. Now from Fig. 3 it is apparent that equilibrium orbits of particles in the momentum range $\Delta P = \pm 0.01$ superpose almost exactly, within a Δx of 2×10^{-5} or 1 mm with the linear perturbation. As indicated earlier, this Δx could be made even smaller by making the perturbation slightly non-linear. These particles in this momentum range will have a total energy spread of 200 MeV. If the original energy resolution at 50 MeV injection was 50 keV, the energy spread of one pulse at full energy would be $\Delta E_{inj} \dfrac{f}{f_{inj}} = 150$ keV ($\Delta E/f =$ constant, with perfect RF handling). The number of pulses (within $\Delta P = \pm 0.01$) which can be superposed is then 200/0.150 = 1330. With the current density per pulse of 0.6 A/cm^2, this gives a total of 800 A/cm^2. Two colliding beams of this density, taking an interaction cross-section of 25 mb, would have a total *p-p* interaction rate of 4×10^7 interactions/cm^3/s, and, at 10^{-8} mm Hg, a gas collision background from both beams of 2×10^6/cm^3/s. There will, of course, be some decrease in this density due to RF mishandling. With a factor of 4 decrease in beam density in RF phase space, there could be 1330/4 = 330 superpositions and 200 A/cm^2, with the same energy spread. The *p-p* interaction rate density would then be 2.5×10^6/cm^3/s. with a background from residual gas of 5×10^5/cm^3/s. Although the net current density is quite high, the total current can be restricted to quite small values by using small betatron oscillation amplitudes. With 200 A/cm^2, if the betatron amplitudes were restricted to $\pm$ 1 mm, the total current would be about 8 A. To get 200 A/cm^2 without equilibrium orbit superposition would require, assuming the same RF losses, that the 330 pulses be spatially superimposed by the betatron oscillations. Now assuming a momentum compaction factor of 50, the 200 MeV energy spread will correspond to a radial separation of the equilibrium orbits of 4×10^{-4}, and a necessary final damped betatron oscillation (for any superposition of the extreme energies) of 2×10^{-4}, or 1 cm at 50 m radius. The current density per pulse of 0.6 A/cm^2, the 1 cm betatron amplitude, and the 330 pulses would then require the large total current of 630 A. To get an appreciable fraction of the current at the 200 A/cm^2 density would require even larger total currents.

Although the example has been worked out for an AG storage ring, according to the analysis of the previous section the perturbation should have the same effect in an FFAG accelerator. The only difference should be that the tune will change with amplitude and not energy. In an FFAG machine the linear region, where tune does not appreciably change, is of the order 3×10^{-4}. With a k of 200 this corresponds to a ΔP of $\pm$ 0.06, compared to the AG case of $\Delta P = \pm 0.01$; thus six times as many pulses could be superposed as in the previous AG design (with six times as large an energy spread). However, the two-way FFAG design usually has a low ν_y ($\sim$ 4 compared to 7 for the AG case) and a large radius (125 m compared to 50) so the net current density per pulse, proportional to ν_y/r^2 from space charge limits at injection, is down by the factor $\sim 1/10$, giving a net current density about 1/2 of the storage ring case.

LIST OF REFERENCES

1a. Symon, K. R. and Sessler, A. M. Methods of radio frequency acceleration in fixed field accelerators with applications to high current and intersecting beam accelerators. CERN Symp. 1956. *1*, p. 44-58.

b. Terwilliger, K. M., Jones, L. W. and Pruett, C. H. Beam stacking experiments in an electron model FFAG accelerator. Rev. sci. Instrum., *28*, p. 987-97, 1957.

2. Courant, E. D. and Snyder, H. S. Theory of the alternating-gradient synchrotron. Ann. Phys. New York, *3*, p. 1-48, 1958.

3. A convenient chart giving stability boundaries of the Mathieu equation, from which tune changes can be estimated, is given in McLachlan, N. W. Theory and application of Mathieu functions. Oxford, Clarendon Press, 1951.