Physics Letters B 808 (2020) 135658

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Thermodynamics and P-v criticality of Bardeen-AdS black hole in 4*D* Einstein-Gauss-Bonnet gravity

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A R T I C L E I N F O

Article history:
Received 3 April 2020
Received in revised form 14 May 2020
Accepted 24 July 2020
Available online 29 July 2020
Editor: N. Lambert

A B S T R A C T

We consider the Gauss-Bonnet corrected Bardeen black hole solution in 4*D* AdS space-time. The solution is obtained by the limiting procedure adopted by Glavan and Lin in 4*D* Einstein-Gauss-Bonnet gravity. The general form of first law of black hole thermodynamics is utilized to calculate various thermodynamics variables. The solution exhibits P-v criticality and belongs to the universality class of van-der Waals fluid. The effect of Gauss-Bonnet coupling is investigated on critical parameters and inversion temperature.

1. Introduction

Black holes are singular solutions of General Theory of Relativity (GTR) and are completely dark objects according to laws of classical mechanics [1]. They are characterized by few parameters e.g. mass, charge and angular momentum. Application of laws of quantum mechanics opens up a new window to explore the black hole physics. The black holes seem to obey the laws of thermodynamics and one can assign temperature and entropy to them [2–5]. The temperature of the black hole is related with surface gravity and entropy is proportional to the area of event horizon. These laws are tested by their applicability to black hole solutions in wide variety of theories and seem to be universal. The investigation of microscopic degrees of freedom of some black holes in string theory also lends support to the area law of black hole entropy.

The regular black holes [6–12] have attracted the attention of physics community recently and the study of these black holes may provide a new window of physics to understand the nature of black hole singularities [13,14]. Bardeen black hole belongs to this class and it has been given the interpretation of magnetically charged black hole [7]. Thermodynamics of Bardeen black hole has posed further challenges and Smarr relation seems to be violated [15,16]. If one tries to obtain the correct relation, the area law of black hole entropy is not obeyed and the thermodynamics temperature does not seem to be in agreement with Hawking temperature [17]. However, the problem can be rectified if one considers the general form of first law of black hole thermodynamics and include the contribution of various parameters and conjugate potentials carefully [18–22].

Bardeen-AdS black hole is a straightforward generalization of this black hole, if one considers the Einstein's gravity in the presence of a negative cosmological constant. Black hole thermodynamics can be studied in extended phase space, where Gauss-Bonnet coupling is treated as an independent parameter and the cosmological constant as the pressure of the system [23–40] (see [37] for a review). It has been shown recently that Bardeen-AdS black hole belongs to the universality class of RN-AdS Black hole and P-v isotherms are similar to a van-der Waals fluid [41]. The thermodynamics and inversion curves in the Joule-Thompson throttling process are also studied and lend support for the identification of black hole interior with a van-der Waals fluid [42–46]. The Bardeen type black hole solutions are also studied in de-Sitter space [47,48]. The Gauss-Bonnet generalizations of these solutions in higher dimensions have appeared in [49,50].

Recently, a 4-dimensional theory of gravity with Gauss-Bonnet correction is introduced by the re-scaling of Gauss-Bonnet coupling $\alpha \to \frac{\alpha}{D-4}$ and taking the limit $(D-4) \to 0$ [51]. The theory possesses all the ingredients of Einstein gravity and circumvent the Lovelock theorem about Gauss-Bonnet correction in 4*D* space-time. The spherically symmetric black hole solutions are also obtained in the same paper. The generalization to other black holes has also appeared [52–61]. The Gauss-Bonnet theory in 4-D obtained by the limiting procedure may be related to the dimensionally reduced theories from higher dimensions. A reduction of this form is already carried out in D=2 [62]. In that case, the solutions of Gauss-Bonnet gravity may be related with the solutions of the dimensionally reduced theories. Here, we report the regular

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https://doi.org/10.1016/j.physletb.2020.135658

Bardeen-AdS type black hole solution with Gauss-Bonnet corrections in $4D$ space-time.

We investigate the effect of the Gauss-Bonnet coupling on the thermodynamic of the these black holes. In higher dimensions, the entropy of the black hole solutions of the Gauss-Bonnet theory has corrections proportional to Gauss-Bonnet coupling. We shall see that the $D=4$ theory also has similar corrections and they can also be obtained by taking $D \to 4$ limit of black hole entropy in $D=4+\epsilon$ theory. We also calculate the critical constants and inversion temperature and explore their dependence on Gauss-Bonnet coupling.

The paper is organized as follows. We present the Gauss-Bonnet corrected Bardeen-AdS black hole solution in next section. The black hole thermodynamics is introduced in section 3. The P-v criticality is discussed in section 4. We outline the summary and directions of future research in the concluding section.

2. Bardeen black hole solution in 4D EGB gravity

Consider the Einstein-Hilbert action coupled to nonlinear electrodynamics in the presence of Einstein-Gauss-Bonnet gravity with negative cosmological constant;

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\Lambda + \alpha\mathcal{L}_{\mathcal{GB}} - 4\mathcal{L}_m\right), \tag{1}$$

where R is the scalar curvature, $\Lambda = -6/l^2$ is the cosmological constant and α is the re-scaled $\alpha \to \frac{\alpha}{d-4}$ Gauss-Bonnet coupling constant.

$\mathcal{L}_m$ is the action of nonlinear matter field [7], given by;

$$\mathcal{L}_m = \frac{3}{2sg^2}\left[\frac{\sqrt{2g^2F}}{1+\sqrt{2g^2F}}\right]^{\frac{5}{2}}, \tag{2}$$

where g is the magnetic monopole charge, $F = F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}$ $(\partial_\mu A_\nu - \partial_\nu A_\mu)$ is the electromagnetic field tensor. The parameter s will be fixed in terms of ADM mass of the black hole as $s = \frac{g}{2M}$.

We are interested to obtain the static spherically symmetric black hole solution of the theory described by EGB action coupled with non-linear matter Lagrangian. Let us consider the metric ansatz consistent with spherical symmetry;

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2, \tag{3}$$

where $d\Omega_2$ is the metric of 2-sphere. The ansatz for the field strength is taken as,

$$F_{\theta\phi} = g\, sin\theta. \tag{4}$$

Substituting the general form of metric and ansatz for electromagnetic field strength into action, we obtain;

$$S = \frac{1}{2}\int dt\, dr\left(-r(f(r)-1) + \alpha\frac{(f(r)-1)^2}{r} + \frac{r^3}{l^2} + \frac{g}{s}\left(1 - \frac{r^3}{(r^2+g^2)^{\frac{3}{2}}}\right)\right)', \tag{5}$$

where the prime denotes the derivative with respect to r. One can identify the integral of motion as ADM mass,

$$\frac{1}{2}\left(-r(f(r)-1) + \alpha\frac{(f(r)-1)^2}{r} + \frac{r^3}{l^2} + \frac{g}{s}\left(1 - \frac{r^3}{(r^2+g^2)^{\frac{3}{2}}}\right)\right) = M. \tag{6}$$

The above equation is a quadratic equation in $(f(r)-1)$ and can be solved easily to give,

$$f(r) = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + 4\alpha\left(\frac{2M}{(r^2+g^2)^{3/2}} - \frac{1}{l^2}\right)}\right), \tag{7}$$

where we have used $s = \frac{g}{2M}$.

In the limit $\alpha \to 0$, the two branches of the solution behave as,

$$f_+(r) = 1 + \frac{2M}{(r^2+g^2)^{3/2}} - \frac{r^2}{l^2} + \frac{r^2}{\alpha},$$
$$f_-(r) = 1 - \frac{2M}{(r^2+g^2)^{3/2}} + \frac{r^2}{l^2}.$$

It is obvious that the +ve branch does not have a smooth $\alpha \to 0$ limit. For non-zero values of α, this branch is asymptotic to AdS space in the limit $M=0$ and $1/l^2 = 0$. The −ve branch is asymptotically flat in the same limit. If we consider $M \neq 0$, the +ve branch corresponds to the AdS-Bardeen black hole solution with negative gravitational mass, whereas the −ve branch reduces to the Bardeen black hole solution with positive gravitational mass. It has been shown by Boulware and Deser [63] that the +ve branch of the solution with negative gravitational mass is unstable and leads to graviton ghosts. Henceforth, we shall consider the −ve branch of the solution in the rest of the paper.

In the limit $\alpha \to 0$, this solution becomes the Bardeen-AdS black hole. The limit $g \to 0$ corresponds to Gauss-Bonnet AdS-Schwarzschild solution. The horizon of the black hole solution is obtained as the root of the equation $f(r) = 0$,

$$1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + 4\alpha\left(\frac{2M}{(r^2+g^2)^{3/2}} - \frac{1}{l^2}\right)}\right) = 0. \tag{8}$$

This equation can be solved numerically and for suitable values of the parameters α and g the metric function, $f(r) = 0$ admits two solutions $r_\pm$. The extremal EGB Bardeen-AdS black hole is obtained if the two horizons become degenerate. Henceforth, we shall treat the outer horizon as event horizon of the black hole. The effect of Gauss-Bonnet coupling on horizon location is shown in Fig. 1. The radius of the event horizon decreases with increase in Gauss-Bonnet coupling and would be black hole may have no horizon at all for the large value of Gauss-Bonnet coupling.

The regularity of the black hole solution can be seen by the behavior of the scalar invariants, which are given by,

$$\lim_{r\to 0} R = \frac{6}{\alpha}\left(\sqrt{1 + \frac{8M\alpha}{g^3} - \frac{4\alpha}{l^2}} - 1\right),$$
$$\lim_{r\to 0} R_{\mu\nu}R^{\mu\nu} = \frac{18}{\alpha^2} + \frac{18}{\alpha^2}\left(\frac{4\alpha M}{g^3} - \frac{2\alpha}{l^2} - \sqrt{1 + \frac{8M\alpha}{g^3} - \frac{4\alpha}{l^2}}\right),$$
$$\lim_{r\to 0} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12}{\alpha^2} + \frac{12}{\alpha^2}\left(\frac{4\alpha M}{g^3} - \frac{2\alpha}{l^2} - \sqrt{1 + \frac{8M\alpha}{g^3} - \frac{4\alpha}{l^2}}\right). \tag{9}$$

These invariants have smooth $r \to 0$ limit and hence the space-time is regular everywhere.

3. Thermodynamics

Thermodynamic quantities like energy, temperature, entropy etc. can be assigned to a black hole solution. The mass of the black hole plays the role of enthalpy or heat energy of the black hole.

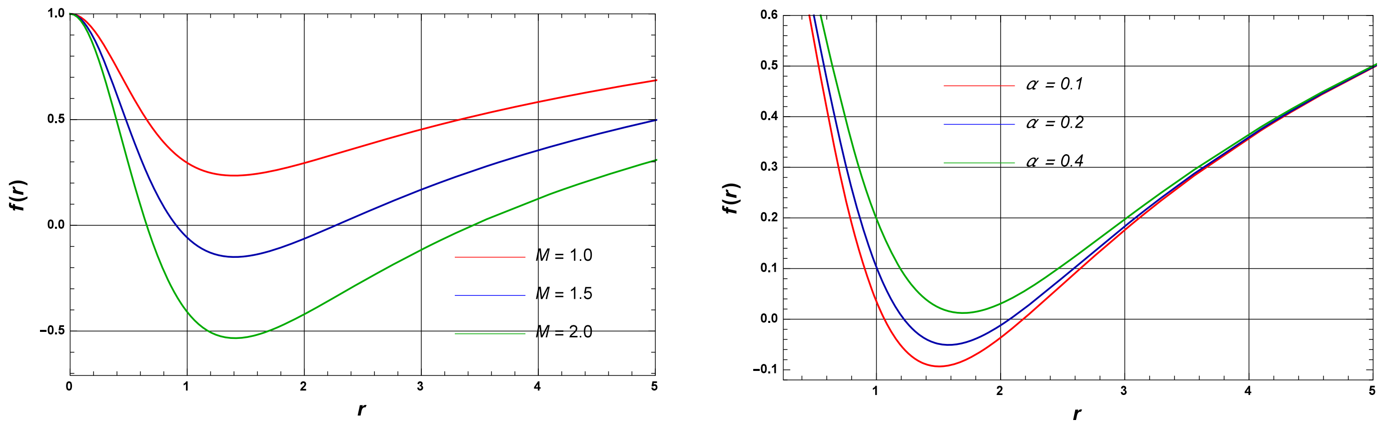

Fig. 1. Plot of metric function $f(r)$ vs r for different values of black hole mass for fixed $g = 1$ and $\alpha = 0$ (left) and for different values of Gauss-Bonnet coupling for fixed mass $M = 1$ and charge $g = 1$ (right) (and $l = 20$ in both figures).

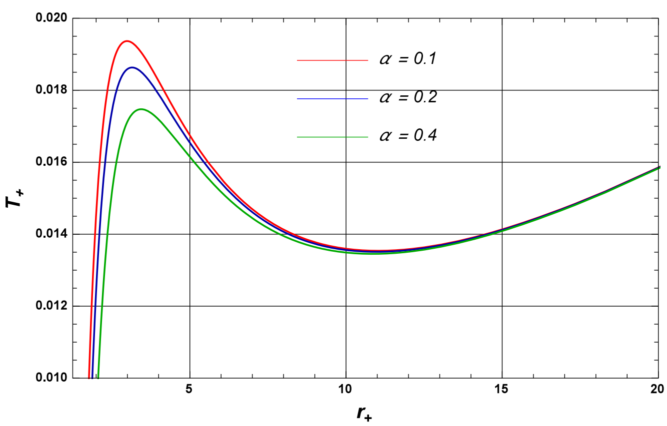

Fig. 2. Plot of temperature T_+ vs horizon radius r for different values of Gauss-Bonnet coupling α with fixed value of charge $g = 1$ (and $l = 20$).

The solution of the horizon condition $f(r_+) = 0$ gives the mass of the black hole in terms of its horizon radius.,

$$M = \frac{(g^2 + r_+^2)^{3/2}}{2l^2 r_+^4}\left(r_+^4 + l^2(r_+^2 + \alpha)\right). \tag{10}$$

The above expression reduces to the mass of the Bardeen-AdS black hole in the limit $\alpha \to 0$ as expected.

The black hole temperature, also known as Hawking temperature can be defined in terms of surface gravity of the black hole as, $T_H = \kappa/2\pi$. The surface gravity can be evaluated using the metric function $f(r)$ and is given by,

$$\kappa = \frac{1}{2} f'(r)\,|_{r=r_+}\,. \tag{11}$$

Hence, the Hawking temperature for the EGB Bardeen-AdS black hole is obtained as;

$$T_H = \frac{1}{4\pi r_+}\frac{3r_+^6 - 2g^2 l^2(r_+^2 + 2\alpha) + r_+^2 l^2(r_+^2 - \alpha)}{l^2(r_+^2 + 2\alpha)(r_+^2 + g^2)}. \tag{12}$$

The above expression reduces to the temperature of the Bardeen-AdS black hole in the limit $\alpha \to 0$. In Fig. 2 we plot the temperature T as a function of horizon radius r_+ for different values of Gauss-Bonnet coupling.

The thermodynamics of black holes solutions of non-linear electrodynamics is subtle, particularly those in which the matter Lagrangian depends on the black hole mass [18]. The Bardeen black hole belongs to this class of solutions.

Let us first consider the simpler case with $\Lambda = 0$ and $\alpha = 0$. One can easily verify that,

$$dM \neq T_H\, dS + \Phi_q\, dq; \tag{13}$$

where, S is the Bekenstein-Hawking entropy of the black hole and Φ_q is the potential conjugate of the charge q.

It means that the first law of black hole thermodynamics can not be applied in its simple form and some additional input is required. The problem was investigated by a number of authors [18–21] and can be cured, if one considers the general form of the first law of thermodynamics in the form,

$$dM = T\, dS + \psi_q\, dq + \phi_{a_i}\, da_i\,. \tag{14}$$

Here, a_i denote the parameters associated with non-linear electrodynamics Lagrangian density and ϕ_{a_i} are the corresponding conjugate potentials defined as,

$$\phi_{a_i} = \frac{1}{4\pi}\int \left(\frac{\partial \mathcal{L}_m}{\partial a_i}\right) d^3r. \tag{15}$$

The non-linear Lagrangian density, $\mathcal{L}_m$ of the Bardeen solution contains two parameters, black hole mass M and the magnetic charge $q = g$ and the first law can be written as [20],[1]

$$dM = T\, dS + \Phi_q\, dq + \phi_M\, dM, \tag{16}$$

which can be written in the form,

$$(1 - \phi_M)\, dM = T\, dS + \Phi_q\, dq. \tag{17}$$

This form of the first law was obtained by Ma et al. [18], who considered the contribution of non-linear electrodynamics stress-tensor in the first law. The explicit form of, ϕ_M can be obtained using the definition (15) and is given by,

$$\phi_M = 1 - \frac{r_+^3}{(r_+^2 + g^2)^{3/2}}. \tag{18}$$

Let us return to the case with $\Lambda \neq 0$ and $\alpha \neq 0$. The general form of the first law applied to Bardeen-AdS black hole can be written in the form,

$$(1 - \phi_M)\, dM = T\, dS + V\, dP + \Phi_q\, dq + \Phi_\alpha\, d\alpha\,; \tag{19}$$

[1] In fact, this form can also be anticipated from the Euclidean version of the action (5). The upper limit of radial integration gives the asymptotic mass M, and the lower limit gives the terms which can be identified with the entropy etc. and one additional term $\phi_M M$.

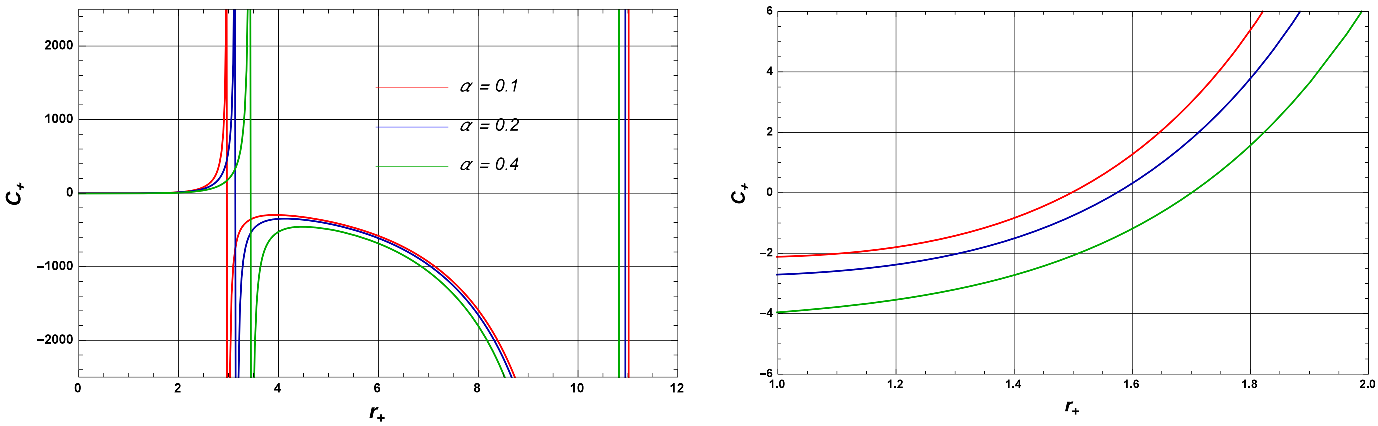

Fig. 3. Plot of specific heat C_P vs horizon radius r_+ for different values of Gauss-Bonnet coupling α with fixed value of charge $g = 1$ (left) Insat of the diagram on the left (right) ($l = 20$).

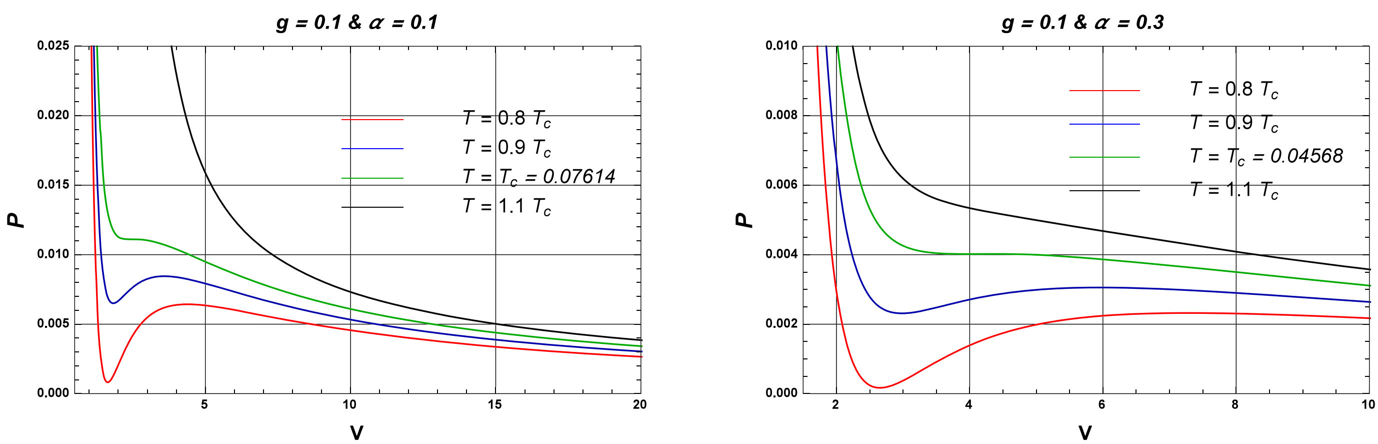

Fig. 4. Plot of pressure P vs specific volume v for different values of temperature for fixed value of magnetic charge $g = 0.1$ and Gauss-Bonnet couplings $\alpha = 0.1$ (left) and $\alpha = 0.3$ (right).

where, $P = -\frac{\Lambda}{8\pi}$ and V and Φ_α are the conjugate potentials of the pressure and Gauss-Bonnet coupling respectively.

The thermodynamics variables appearing in the above expression are given by,

$$T = \frac{3r_+^6 - 2g^2l^2(r_+^2 + 2\alpha) + r_+^2 l^2(r_+^2 - \alpha)}{4\pi l^2 r_+(r_+^2 + 2\alpha)(r_+^2 + g^2)}; \quad V = \frac{4}{3}\pi r_+^3;$$

$$P = \frac{3}{8\pi l^2};$$

$$S = \pi r_+^2 + \pi\alpha + 2\pi\alpha \ln\frac{r_+^2}{L^2}; \qquad \Phi_q = \frac{3g(l^2(r_+^2 + \alpha) + r_+^4)}{(r_+ l^2(r_+^2 + g^2)};$$

$$\Phi_\alpha = \frac{1}{2r_+} - 2\pi T \ln\frac{r_+^2}{L^2} - \pi\alpha T; \qquad \phi_M = 1 - \frac{r_+^3}{(r_+^2 + g^2)^{3/2}}. \tag{20}$$

These variables obey the Smarr relation of the form,

$$M = 2\tilde{T}\,S - 2\tilde{V}\,P + \tilde{\Phi}_q q + 2\tilde{\Phi}_\alpha \alpha; \tag{21}$$

where, $\tilde{T} = \frac{T}{(1-\phi_M)}$; $\tilde{V} = \frac{V}{(1-\phi_M)}$ and $\tilde{\Phi}_i = \frac{\Phi_i}{(1-\phi_M)}$ etc.

Here, few comments are due about the form of the entropy. The leading term gives the standard Bekenstein-Hawking area law. The other two terms can also be obtained from the regularized expression of entropy in $D = 4 + \epsilon$ dimensions in the limit $\epsilon \to 0$. The logarithmic term in the entropy has to be regulated by some length scale L. In quantum theory of gravity a natural candidate exists for this length scale and presumably, this may be identified with the Planck length, l_P.

In order to analyze the local stability of the solutions, we consider the heat capacity of the black hole given by,

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,q,\alpha} \tag{22}$$

Using the expression of the entropy and temperature as above, the specific heat can be expressed in terms of r_+. The expression is too lengthy and we omit it here. In Fig. 3 we show the plot of C_P as a function of horizon radius r_+ for different values of Gauss-Bonnet coupling.

Let us discuss the stability of the solution for $\alpha = 0.1$. The heat capacity is negative for $r_+ < 1.5$ as the temperature becomes negative, which is unphysical and no black hole horizon exists. The upper limit corresponds to extremal black hole and has zero temperature. In the range $1.5 < r_+ < 3$ the heat capacity is positive and it corresponds to the stable small black hole solution. The upper limit of r_+ corresponds to maxima of temperature. The heat capacity becomes negative beyond this value until we reach the minima of the temperature for $r_+ = 11$ approx. This region corresponds to unstable black hole solution with intermediate mass. Specific heat becomes positive beyond this value again and this region corresponds to stable large black hole solution. Thus, the system undergoes a first order phase transition from small black hole to large black hole. Similar considerations apply to other values of Gauss-Bonnet coupling as well.

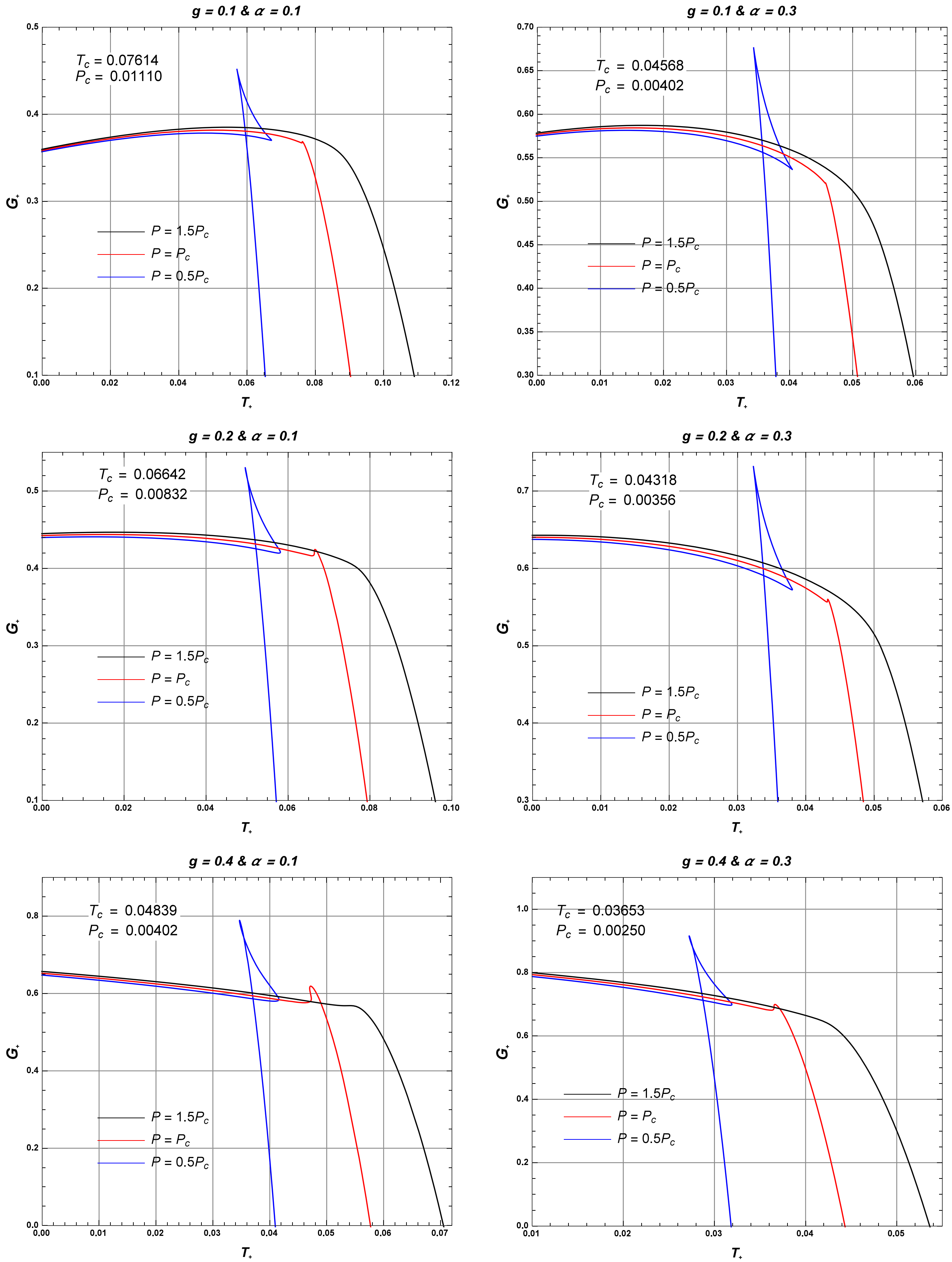

Fig. 5. Plot of Gibbs free energy G vs temperature T for magnetic charge $g = 0.1$, 0.2, 0.4 and Gauss-Bonnet couplings $\alpha = 0.1$ (left) and $\alpha = 0.3$ (right).

4. P-v criticality

The charged AdS black hole (RN-AdS) is known to belong to the universality class of a van-der Walls gas and it undergoes a first order phase transition similar to liquid-gas transition [26]. The $P - v$ isotherms are also similar and the ratio $\frac{P_c v_c}{T_c} = \frac{3}{8}$, agrees perfectly with the van-der Waals fluid. The volume used in this case is the specific volume $v = 2r_+ l_P^2$, where l_P is the Planck length. We shall put $l_P^2 = 1$ in all subsequent discussion. The Bardeen black hole isotherms are also studied recently and shown to be similar to a van-der Waals fluid [41].

Consider the equation of state written in terms of the specific volume,

$$P = \left(\frac{T}{v}(1+\frac{4g^2}{v^2}) + \frac{1}{2\pi v^2} - \frac{4g^2}{\pi v^4}\right) + \frac{8\alpha}{v^2}\left(\frac{T}{v}(1+\frac{4g^2}{v^2}) - \frac{1}{4\pi v^2} - \frac{4g^2}{\pi v^4}\right) \quad (23)$$

The isotherms of EGB Bardeen-AdS black hole are shown in Fig. 4.

We also plot the Gibbs free energy as a function of black hole temperature for three different values of pressure. It shows a 'swallow tail' behavior below the critical pressure, a characteristic of first order phase transition. The non-linearity of the solution is controlled by the parameter $s = g/2M$. Here, we have taken $M = 1$ and plotted the free energy for different values of magnetic charge. The plots for different values of magnetic charge are qualitatively similar but the values of critical parameters change significantly. Similar observations can be made about the dependence on the Gauss-Bonnet coupling (Fig. 5).

Let us calculate the critical constants determined by the conditions,

$$\left(\frac{\partial P}{\partial v}\right)_T = 0 \quad (24)$$

$$\left(\frac{\partial^2 P}{\partial v^2}\right)_T = 0. \quad (25)$$

In the absence of Gauss-Bonnet corrections, this gives,

$$P_c = \frac{0.00116}{g^2}; \qquad v_c = 7.94\, g \qquad T_c = \frac{0.025}{g}. \quad (26)$$

The ratio,

$$\frac{P_c\, v_c}{T_c} = 0.368, \quad (27)$$

is slightly smaller than van-der Waals ratio 0.375. We have tabulated the values of this ratio for different values of the Gauss-Bonnet coupling below.

In the extended phase space, the comparison of the black hole with a van der Walls fluids gives reasonable answers as seen above. Let us consider the Joule-Thomson coefficient given by,

$$\mu = \left(\frac{\partial T}{\partial P}\right)_H = \frac{v}{C_P}(\beta T - 1)\,, \quad (28)$$

where β is the coefficient of volume expansion. We can obtain the inversion temperature by setting $\mu = 0$,

$$T_i = v\left(\frac{\partial T}{\partial v}\right). \quad (29)$$

The expression can be evaluated and can be expressed in terms of horizon radius r_+,

$$T_i = \frac{2g^2(r^2+2\alpha)^2 + r^4 A + g^2 r^2 B}{4\pi r(r^2+2\alpha)^2(r^2+g^2)^2}, \quad (30)$$

Table 1
The value of ratio $P_c v_c/T_c$ and T_i/T_c for different value of Gauss-Bonnet coupling α.

α	$P_c v_c/T_c$	T_i/T_c
0	0.368	0. 6747
0.10	0.3679	0.6776
0.50	0.368	0.6831
1.0	0.3681	0.6870

where,

$$A = 8\pi P r^6 + 5r^2\alpha + 2\alpha^2 - r^4(1 - 48\pi P\alpha), \quad (31)$$

$$B = 24\pi P r^6 + 31r^2\alpha + 22\alpha^2 + r^2(7 + 80\pi P\alpha). \quad (32)$$

We tabulate the ratio between the minimum inversion temperature and the critical temperature below for different values of the Gauss-Bonnet coupling. These values are less than the value for the van-der Wall fluid, $\frac{T_i}{T_c} = 0.75$ (Table 1).

5. Summary

In this paper, we have studied the thermodynamics of EGB Bardeen-AdS black in $4D$. The general solution is obtained and its horizon structure is analyzed as a function of Gauss-Bonnet coupling. We have followed the procedure of [18,22] to obtain correct thermodynamic variables and this is contrary to the claims of entropy and volume modification as reported in the literature. The analysis of heat capacity indicates the first order phase transition from small to large black hole. This is also confirmed by plot of Gibbs free energy as a function of black hole temperature. The P-v criticality is used to obtain the critical constants for different Gauss-Bonnet coupling. The ratio of inversion temperature with critical temperature is also obtained. It would be interesting to explore the $P_c v_c/T_c$ and T_i/T_c ratio further and why do they differ from van-der Waals fluid.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors gratefully acknowledge the support of their home institutions.

References

[1] R. Penrose, Riv. Nuovo Cimento 1 (1969) 252;
S.W. Hawking, W. Israel (Eds.), General Relativity, an Einstein Centenary Volume, CUP, 1979.
[2] J.D. Bekenstein, Phys. Rev. D 7 (1973) 2333; Phys. Rev. D 9 (1974) 3292.
[3] R.M. Wald, Living Rev. Relativ. 4 (2001).
[4] S. Carlip, Int. J. Mod. Phys. D 23 (2014) 1430023.
[5] A.C. Wall, arXiv:1804.10610 [gr-qc].
[6] J. Bardeen, in: Proceedings of GR5, Tiflis, U.S.S.R., 1968.
[7] E. Ayon-Beato, A. Garcia, Phys. Rev. Lett. 80 (1998) 5056; Gen. Relativ. Gravit. 31 (1999) 629; Phys. Lett. B 493 (2000) 149; Gen. Relativ. Gravit. 37 (2005) 635.
[8] K.A. Bronnikov, Phys. Rev. D 63 (2001) 044005.
[9] S.A. Hayward, Phys. Rev. Lett. 96 (2006) 031103.
[10] O.B. Zaslavskii, Phys. Rev. D 80 (2009) 064034.
[11] J.P.S. Lemos, V.T. Zanchin, Phys. Rev. D 83 (2011) 124005.
[12] O. Miskovic, R. Olea, Phys. Rev. D 83 (2011) 024011.
[13] A.D. Sakharov, Sov. Phys. JETP 22 (1966) 241.
[14] E.B. Gliner, Sov. Phys. JETP 22 (1966) 378.
[15] D.A. Rashid, arXiv:hep-h/9702087.
[16] N. Breton, Gen. Relativ. Gravit. 37 (2005) 643.
[17] M. Sharif, W. Javed, J. Korean Phys. Soc. 57 (2010) 217, arXiv:1007.4995.

[18] M. Ma, R. Zhao, Class. Quantum Gravity 31 (2014) 245014.
[19] L. Balart, S. Fernando, Mod. Phys. Lett. A 32 (39) (2017) 1750219.
[20] Y. Zhang, S. Gao, Class. Quantum Gravity 35 (2018) 145007.
[21] L. Gulin, I. Smolic, Class. Quantum Gravity 35 (2018) 025015.
[22] D.V. Singh, S. Siwach, arXiv:1909.11529 [hep-th].
[23] D. Kastor, S. Ray, J. Traschen, Class. Quantum Gravity 26 (2009) 195011.
[24] M. Cvetic, G.W. Gibbons, D. Kubiznak, C.N. Pope, Phys. Rev. D 84 (2011) 024037.
[25] B.P. Dolan, Class. Quantum Gravity 28 (2011) 125020; Class. Quantum Gravity 28 (2011) 235017; Phys. Rev. D 84 (2011) 127503.
[26] D. Kubizňák, R.B. Mann, J. High Energy Phys. 1207 (2012) 033.
[27] S. Gunasekaran, D. Kubiznak, R.B. Mann, J. High Energy Phys. 11 (2012) 110.
[28] A. Belhaj, M. Chabab, H. El Moumni, M.B. Sedra, Chin. Phys. Lett. 29 (2012) 100401.
[29] S.H. Hendi, M.H. Vahidinia, Phys. Rev. D 88 (2013) 084045.
[30] E. Spallucci, A. Smailagic, Phys. Lett. B 723 (2013) 436.
[31] S-W. Wei, Y-X. Liu, Phys. Rev. D 87 (2013) 044014.
[32] R.G. Cai, L.M. Cao, L. Li, R.Q. Yang, J. High Energy Phys. 1309 (2013) 005.
[33] J.X. Mo, W.B. Liu, Phys. Lett. B 727 (2013) 336.
[34] J.X. Mo, G.Q. Li, W.B. Liu, Phys. Lett. B 730 (2014) 111.
[35] C.V. Johnson, Class. Quantum Gravity 31 (2014) 205002.
[36] B.P. Dolan, Mod. Phys. Lett. A 30 (2015) 1540002.
[37] D. Kubiznak, R.B. Mann, M. Teo, Class. Quantum Gravity 60 (2017) 63001.
[38] D. Hansen, D. Kubiznak, R.B. Mann, J. High Energy Phys. 1701 (2017) 047.
[39] B.R. Majhi, S. Samanta, Phys. Lett. B 773 (2017) 203.
[40] C.H. Nam, Eur. Phys. J. C 78 (2018) 581.
[41] A.G. Tzikas, Phys. Lett. B 788 (2019) 219.
[42] R.V. Maluf, J.C.S. Neves, Phys. Rev. D 97 (2018) 104015; Int. J. Mod. Phys. D 28 (03) (2018) 1950048.
[43] C. Li, C. Fang, M. He, J. Ding, J. Deng, Mod. Phys. Lett. A 34 (2019) 1950336.
[44] K.V. Rajani, C.A. Rizwan, A.N. Kumara, D. Vaid, K.M. Ajith, arXiv:1904.06914 [gr-qc].
[45] J. Pu, S. Guo, Q.Q. Jiang, X.T. Zu, Chin. Phys. C 44 (3) (2020) 035102.
[46] N. Chen, arXiv:2003.00247 [gr-qc].
[47] D.V. Singh, N.K. Singh, Ann. Phys. 383 (2017) 600.
[48] S. Fernando, Int. J. Mod. Phys. D 26 (2017) 1750071.
[49] D.V. Singh, S.G. Ghosh, S.D. Maharaj, Ann. Phys. 412 (2019) 168025.
[50] A. Kumar, D.V. Singh, S.G. Ghosh, Eur. Phys. J. C 79 (2019) 275.
[51] D. Glavan, C. Lin, Phys. Rev. Lett. 124 (2020) 081301.
[52] R.A. Konoplya, A.F. Zinhailo, arXiv:2003.01188 [gr-qc].
[53] M. Guo, P.C. Li, arXiv:2003.02523 [gr-qc].
[54] P.G.S. Fernandes, arXiv:2003.05491 [gr-qc].
[55] A. Casalino, A. Colleaux, M. Rinaldi, S. Vicentini, arXiv:2003.07068 [gr-qc].
[56] Shao-Wen Wei, Yu-Xiao Liu, arXiv:2003.07769 [gr-qc].
[57] R.A. Konoplya, A. Zhidenko, arXiv:2003.07788 [gr-qc].
[58] K. Hegde, A.N. Kumara, C.A. Rizwan, K.M. Ajith, Md Sabir Ali, arXiv:2003.08778 [gr-qc].
[59] R. Kumar, S.G. Ghosh, arXiv:2003.08927 [gr-qc].
[60] S.G. Ghosh, S.D. Maharaj, arXiv:2003.09841 [gr-qc].
[61] Yu-Peng Zhang, Shao-Wen Wei, Yu-Xiao Liu, arXiv:2003.10960 [gr-qc].
[62] R.B. Mann, S.F. Ross, Class. Quantum Gravity 10 (1993) 1405.
[63] D.G. Boulware, S. Deser, Phys. Rev. Lett. 55 (1985) 2656.