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Relations between tree-level closed string amplitudes and mixed string amplitudes at fewer points

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Abstract. We formulated the relations between closed string amplitudes and mixed string amplitudes at tree level. In this paper we focus on fewer points of external states. We factorized the closed string amplitudes by analytical continuation of complex variables. With a careful examination of branch points, the results show that closed string amplitudes can be expressed in terms of products of mixed string amplitudes and additional terms with appropriate phase factors. In terms of five-point amplitudes, we introduced a one-parameter dependent amplitude to capture the results. Finally, some physical interpretations were provided.

1. Introduction

String scattering amplitudes are captured by worldsheet integral which can be computed in perturbation theory as a sum over all worldsheet topologies. This may refer to a genus expansion in perturbative string theory. Remarkably, at each order of genus, the wolrdsheet encapsulates all Feynmann diagrams of corresponding field theories at the same loop level at low energies. At the level of amplitudes, string theory provides us with an intriguing connection between gravity and gauge theories. It was discovered that, at tree level, closed string amplitudes can be expressed as a sum of products of open string amplitudes which are known as KTL relations [1]. Bjerrum-Bohr and Damgaard demonstrated that the monodromy relations reduce the number of independent color-ordered amplitudes from $(n-1)!$ to $(n-3)!$ [2]. Mixed string amplitudes describing the scattering process of open and closed strings, can be written in terms of linear combination of open string amplitudes [3].

In this paper, we formulated the relations between closed and mixed string amplitudes only for four and five points scattering. The technique of complex analysis was used to obtain the results.

2. String scattering amplitudes

The string scattering amplitudes are computed by the expectation value of the vertex operators. For closed strings, the vertex operators are inserted in the bulk of the worldsheet, while for open strings, they are inserted on the worldsheet's boundary. We will give you the general expressions of each type of amplitude at the tree level. There are three kinds of amplitudes, i.e. closed string amplitudes, open string amplitudes, and mixed string amplitudes.

The general expression for closed string amplitudes takes the form [4]

$$\mathcal{A}_n^{cl} = C_{S^2}(2\pi)^D \delta^D(\sum_i k_i) \int \frac{|z_{ab}z_{ac}z_{bc}|}{dz_a dz_b dz_c} \prod_{i=1}^{n} d^2 z_i \prod_{1\le j<l\le n} |z_j - z_l|^{\alpha' k_j\cdot k_l} F_n, \quad (1)$$

where C_{S^2} is a normalization constant, $z_{ij} = z_i - z_j$ and the function F_n contains polarization and kinematic factors whose value depends on external states. The points z_a, z_b and z_c are freely fixed to arbitrary points in the complex plane due to the conformal symmetry.

The general expression for open string amplitudes is

$$\mathcal{A}_n^{op} = C_{D_2}(2\pi)^D \delta^D(\sum_i k_i) \sum_{(a_1,\dots,a_n)\in S_n/\mathbb{Z}_n} \mathrm{Tr}\{T^{a_1}\cdots T^{a_n}\}\mathcal{A}(a_1,\dots,a_n). \quad (2)$$

The color-ordered amplitudes are given by [4]

$$\mathcal{A}(a_1,\dots,a_n) = \int \prod_{i=1}^{n} dx_i \frac{|x_{ab}x_{ac}x_{bc}|}{dx_a dx_b dx_c} \prod_{i=1}^{n-1} \Theta(x_{a_{i+1}} - x_{a_i}) \prod_{1\le i<j\le n} |x_i - x_j|^{2\alpha' k_i\cdot k_j} F_n, \quad (3)$$

where $\Theta(y)$ is the Heaviside function, $x_{ij} = x_i - x_j$, and F_n is again an external-state-dependent factor.

The expression for mixed string amplitudes, describing a scattering of $n-2$ open strings and one closed string, is [3]

$$\mathcal{M}_n(1,\dots,n-2;q_1,q_2) = C_{D_2}(2\pi)^D \delta(\sum_{i=1}^{n-2} p_i + q_1 + q_2) \int \prod_{i=1}^{n-2} dx_i \prod_{1\le r<s\le n-2} |x_r - x_s|^{2\alpha' p_r p_s + n_{rs}}$$
$$\times \int d^2 z (z-\bar{z})^{2\alpha' q_1 q_2 + n} \prod_{i=1}^{n-2} (x_i - z)^{2\alpha' p_i q_1 + n_i} (x_i - \bar{z})^{2\alpha' p_i q_2 + \bar{n}_i}, \quad (4)$$

where C_{D_2} is a normalization constant, n_{rs}, n_i, $\bar{n}_i$, and n are some integers determined by external states.

3. Relations between closed string and mixed string amplitudes

3.1. 4-point relation

We start with 4-point closed string amplitudes. By choosing $z_a = \frac{i}{2}$, $z_b = -\frac{i}{2}$, and $z_c = \infty$, according to equation (1), the 4-point closed string amplitude takes the form

$$\mathcal{A}_4^{cl} = C_{S^2} \int d^2 z \left| z - \frac{i}{2} \right|^{\alpha' p_1 p_2} \left| z + \frac{i}{2} \right|^{\alpha' p_1 p_3}. \quad (5)$$

By writing $z_j = x_j + iy_j$, we obtain

$$\mathcal{A}_4^{cl} = C_{S^2} 2i \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (x + iy - \frac{i}{2})^{\alpha_{12}} (x - iy + \frac{i}{2})^{\alpha_{12}} (x + iy + \frac{i}{2})^{\alpha_{13}} (x - iy - \frac{i}{2})^{\alpha_{13}}, \quad (6)$$

where $\alpha_{ij} = \frac{\alpha'}{2} p_i p_j$. Branch points of y are at $\pm\frac{1}{2} \pm ix$. For short, we refer to the integrand as I. We deform the y-contour using the contour path shown in figure 1. The amplitude becomes

$$\mathcal{A}_4^{cl} = C_{S^2} 2i \left[\int_{-\infty}^{\infty} dx (-\int_{C_2} dy - \int_{T_1+T_2} dy - \int_{T_3+T_4} dy) I \right]. \quad (7)$$

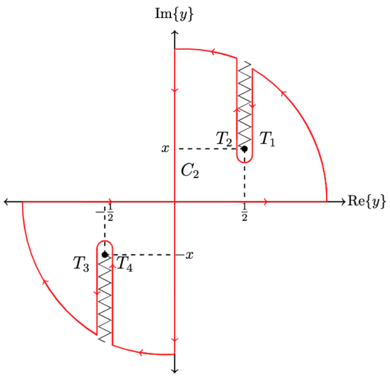

Figure 1. 4-point case.

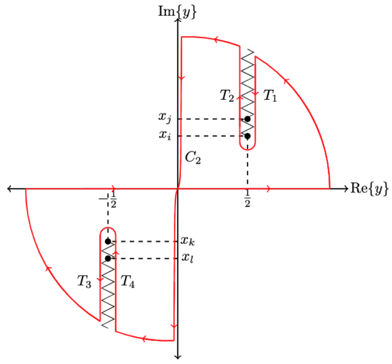

Figure 2. 5-point case.

Along C_2, y transforms as $y \to iy$. We defined new variables as $\eta = x + y$ and $\xi = x - y$. The amplitudes takes the form

$$\mathcal{A}_4^{cl} = C_{S^2}\Big[(-1)\int_{-\infty}^{\infty} d\eta(\eta + \frac{i}{2})^{\alpha_{12}}(\eta - \frac{i}{2})^{\alpha_{13}} \int_{-\infty}^{\infty} d\xi(\xi - \frac{i}{2})^{\alpha_{12}}(\xi + \frac{i}{2})^{\alpha_{13}} \\ + [\int_{-\infty}^{\infty} dx(-\int_{T_1+T_2} dy - \int_{T_3+T_4} dy)I]\Big]. \quad (8)$$

The first part corresponds to the product of two 4-point mixed string amplitudes. Along the path $T_1 + T_2$ and $T_3 + T_4$, we separate x into 2 cases: $x > 0$ and $x < 0$. To avoid crossing branch cut, we have to put appropriate phase factors.

$$\begin{aligned}\mathcal{A}_4^{cl} = C_{S^2}\Big[&(-1)\mathcal{M}_4(1,4;p_2,p_3) \times \mathcal{M}_4(1,4;p_2,p_3) \\ &- 8iS_{12}\int_0^{\infty} dx \int_x^{\infty} dy(y-x)^{\alpha_{12}}(y+x)^{\alpha_{12}}(x-y+i)^{\alpha_{13}}(x+y-i)^{\alpha_{13}} \\ &- 8iS_{13}\int_0^{\infty} dx \int_x^{\infty} dy(y-x)^{\alpha_{13}}(y+x)^{\alpha_{13}}(x-y+i)^{\alpha_{12}}(x+y-i)^{\alpha_{12}}\Big],\end{aligned} \quad (9)$$

where S_{ij} is $\sin(\alpha_{ij}\pi)$.

3.2. 5-point relation

For 5-point scattering, we also start from the closed string amplitude, but this time we choose to fix the conformal degrees of freedom differently which are $z_2 = \frac{i}{2}$, $z_3 = -\frac{i}{2}$, $y_1 = y_4 = y$, and $z_5 = x_5$. Accordingly, the amplitude takes the form

$$\begin{aligned}\mathcal{A}_5^{cl} = C_{S^2}\int dx_1 dx_2 dx_5 dy \left|z_1 - \frac{i}{2}\right|^{\alpha' p_1 p_2}\left|z_1 + \frac{i}{2}\right|^{\alpha' p_1 p_3}|z_1 - z_4|^{\alpha' p_1 p_4}|z_1 - x_5|^{\alpha' p_1 p_5} \\ \left|z_4 - \frac{i}{2}\right|^{\alpha' p_2 p_4}\left|x_5 - \frac{i}{2}\right|^{\alpha' p_2 p_5}\left|z_4 + \frac{i}{2}\right|^{\alpha' p_3 p_4}\left|x_5 - \frac{i}{2}\right|^{\alpha' p_3 p_5}|z_4 - x_5|^{\alpha' p_4 p_5}\mathcal{F},\end{aligned} \quad (10)$$

where $\mathcal{F}$ is a branch-free function and contains kinematic factors. Using a similar technique to the 4-point case, we deform the y-contour using the contour path shown in figure 2. We can rewrite the closed string amplitudes as

$$\mathcal{A}_5^{cl} = \mathcal{A}_{5,C_2}^{cl} + \mathcal{A}_{5,T_{1,2,3,4}}^{cl}, \quad (11)$$

Figure 3. $y_1 > 0$ and $y_4 > 0$.

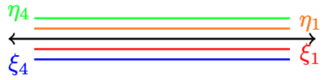

Figure 4. $y_1 < 0$ and $y_4 < 0$.

Figure 5. $y_1 > 0$ and $y_4 < 0$.

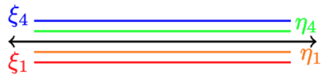

Figure 6. $y_1 < 0$ and $y_4 > 0$.

where $\mathcal{A}^{cl}_{5,C_2}$ and $\mathcal{A}^{cl}_{5,T_{1,2,3,4}}$ are obtained from the integral with the contour-y being along C_2 and T_1, T_2, T_3, T_4 respectively. Along C_2, y transforms as $y \to ie^{-i\epsilon}y \sim iy + \epsilon y$ and we introduce new variables as $\eta_i = x_i + y$ and $\xi_i = x_i - y$. To separate x_5 variable, we used the binomial expansion. But we have one constraint that is $\eta_1 - \eta_4 + \xi_4 - \xi_1 = 0$, we have to put Dirac's delta function into the integration, we then obtain

$$\begin{aligned}\mathcal{A}^{cl}_{5,C_2} = C_{S^2} \sum_{a,b,c,d=0}^{\infty} \binom{\alpha_{15}}{a}\binom{\alpha_{45}}{b}\binom{\alpha_{15}}{c}\binom{\alpha_{45}}{d} \int dl \int dx_5 \left|x_5 - \frac{i}{2}\right|^{2\alpha_{25}} \left|x_5 + \frac{i}{2}\right|^{2\alpha_{35}} \\ (-x_5)^{a+b+c+d} \Big[\int d\eta_1 d\eta_4 e^{il(\eta_1-\eta_4)} (\eta_1 - \frac{i\epsilon y_1}{2})^{\alpha_{15}-c} (\eta_4 - \frac{i\epsilon y_4}{2})^{\alpha_{45}-d} (\eta_1 + \frac{i}{2} - \frac{i\epsilon y_1}{2})^{\alpha_{12}} \\ (\eta_1 - \frac{i}{2} - \frac{i\epsilon y_1}{2})^{\alpha_{13}} (\eta_4 + \frac{i}{2} - \frac{i\epsilon y_4}{2})^{\alpha_{24}} (\eta_4 - \frac{i}{2} - \frac{i\epsilon y_4}{2})^{\alpha_{34}} (\eta_1 - \eta_4)^{\alpha_{14}} \mathcal{F}_1(\eta_i) \Big] \\ \Big[\int d\xi_1 d\xi_4 e^{il(\xi_4-\xi_1)} (\xi_1 - \frac{i}{2} + \frac{i\epsilon y_1}{2})^{\alpha_{12}} (\xi_1 + \frac{i}{2} + \frac{i\epsilon y_1}{2})^{\alpha_{13}} (\xi_4 - \frac{i}{2} + \frac{i\epsilon y_4}{2})^{\alpha_{24}} \\ (\xi_4 + \frac{i}{2} + \frac{i\epsilon y_4}{2})^{\alpha_{34}} (\xi_1 + \frac{i\epsilon y_1}{2})^{\alpha_{15}-a} (\xi_4 + \frac{i\epsilon y_4}{2})^{\alpha_{45}-b} (\xi_1 - \xi_4)^{\alpha_{14}} \mathcal{F}_2(\xi_i) \Big],\end{aligned} \tag{12}$$

where $\alpha_{ij} = \frac{\alpha'}{2} p_i p_j$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are also branch free-part of each variable. The integral of x_5 variable can be captured by the beta function. Then, we define the integrals in the square parentheses as one-parameter dependent amplitudes, i.e. $\mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l)$. Therefore, the amplitude becomes

$$\begin{aligned}\mathcal{A}^{cl}_{5,C_2} = C_{S^2} \sum_{a,b,c,d=0}^{\infty} \binom{\alpha_{15}}{a}\binom{\alpha_{45}}{b}\binom{\alpha_{15}}{c}\binom{\alpha_{45}}{d} (\frac{1}{4})^{\alpha_{25}+\alpha_{35}} (-1)^{a+b+c+d} e^{2\pi i(\alpha_{25}+\alpha_{35})} \\ \times \sin(\pi(\alpha_{25} + \alpha_{35})) \frac{B(1 + \alpha_{25} + \alpha_{35}, \frac{1}{2} - \alpha_{25} - \alpha_{35} - \frac{a+b+c+d}{2})}{2} \\ \times \int dl \mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l) \times \mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l).\end{aligned} \tag{13}$$

We can rewrite the newly defined amplitude $\mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l)$ in terms of color-ordered one-parameter dependent amplitudes. To do so, we need to consider all possible contours of ξ_i and η_i shown in figures 3-6. This can be done by inserting an identity, $1 = \Theta(y_1)\Theta(y_4) + \Theta(-y_1)\Theta(-y_4) + \Theta(-y_1)\Theta(y_4) + \Theta(y_1)\Theta(-y_4)$. However, all the four cases give us the same expression for $\mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l)$ which is

$$\begin{aligned}\mathcal{M}_5(k_1, k_4, k_5; p_2, p_3; l) = e^{i\pi\alpha_{14}} \mathcal{M}_5(5, 1, 4; 2, 3; l) + \mathcal{M}_5(5, 4, 1; 2, 3; l) \\ + e^{-i\pi(\alpha_{15}+\alpha_{45}-\alpha_{14})} \mathcal{M}_5(1, 4, 5; 2, 3; l) + e^{-i\pi(\alpha_{15}+\alpha_{45})} \mathcal{M}_5(4, 1, 5; 2, 3; l) \\ + e^{-i\pi(\alpha_{15}-\alpha_{14})} \mathcal{M}_5(1, 5, 4; 2, 3; l) + e^{-i\pi\alpha_{45}} \mathcal{M}_5(4, 5, 1; 2, 3; l).\end{aligned} \tag{14}$$

What is left is to determine $\mathcal{A}^{cl}_{5,T_{1,2,3,4}}$. It turns out that there are eight cases to consider regarding the values of the variables x_1 and x_4. In this paper, we will show only the expression for the case where $0 < x_1 < x_4$ which is given by

$$\mathcal{A}^{cl}_{5,T_{1,2,3,4}} = C_{S^2} \int dx_5 I_1 \int_0^\infty dx_4 \int_0^{x_4} dx_1 (-S_2 \int_{x_4}^\infty dy I_2 - S_{12} \int_{x_1}^{x_4} dy I_2 - S_3 \int_{x_4}^\infty dy I_3 - S_{13} \int_{x_1}^{x_4} dy I_3) + \text{all possible cases,} \tag{15}$$

where $S_i = \sin(\pi(\alpha_{1i} + \alpha_{i4}))$ and $S_{ij} = \sin(\pi(\alpha_{ij}))$. The sine factors arise from a careful examination of brunch cuts. The integrands I_1, I_2 and I_3 are given by

$$I_1 = (x_5 - \frac{i}{2})^{\alpha_{25}} (x_5 + \frac{i}{2})^{\alpha_{25}} (x_5 - \frac{i}{2})^{\alpha_{35}} (x_5 + \frac{i}{2})^{\alpha_{35}}, \tag{16}$$

$$I_2 = (x_1 - y)^{\alpha_{12}} (x_1 + y)^{\alpha_{12}} (x_1 + y - i)^{\alpha_{13}} (x_1 + y - i)^{\alpha_{13}} (x_1 - x_4)^{2\alpha_{14}} (x_1 - y - x_5 + \frac{i}{2})^{\alpha_{15}} (x_1 + y - x_5 - \frac{i}{2})^{\alpha_{15}} (x_4 - y - x_5 + \frac{i}{2})^{\alpha_{45}} (x_4 + y - x_5 - \frac{i}{2})^{\alpha_{45}} (x_4 - y)^{\alpha_{24}} (x_4 + y)^{\alpha_{24}} (x_4 - y + i)^{\alpha_{34}} (x_4 + y - i)^{\alpha_{34}} \mathcal{F}, \tag{17}$$

$$I_3 = (x_1 - y + i)^{\alpha_{12}} (x_1 + y - i)^{\alpha_{12}} (x_1 - y)^{\alpha_{13}} (x_1 + y)^{\alpha_{13}} (x_1 - y - x_5 + \frac{i}{2})^{\alpha_{15}} (x_1 + y - x_5 - \frac{i}{2})^{\alpha_{15}} (x_4 - y)^{\alpha_{34}} (x_4 + y)^{\alpha_{34}} (x_4 - y - x_5 + \frac{i}{2})^{\alpha_{45}} (x_1 - x_4)^{2\alpha_{14}} (x_4 + y - x_5 - \frac{i}{2})^{\alpha_{45}} (x_4 - y + i)^{\alpha_{24}} (x_4 + y - i)^{\alpha_{24}} \mathcal{F}. \tag{18}$$

The remaining terms are similar to that of equation (15) but with different integral regions and sine factors.

4. Conclusion

By using the analytical continuation of complex variables, we can factorize the tree-level closed string amplitudes. For the 4-point case, tree-level closed string amplitudes can be expressed in terms of product of two mixed string amplitudes, describing a scattering of two open strings and one closed string plus the additional terms. The expression is shown in equation (9). For the 5-point case, tree-level closed string amplitudes can be expressed in terms of product of the beta function and two 5-point one-parameter dependent mixed string amplitudes, which are explicitly shown in equation (13). Similar to the 4-point case, the full expression comes with the additional terms. In both cases, the additional terms are obtained from the integrals along the branch cuts when we analytically continue the real variables into the complex plane.

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References

[1] Kawai H, Lewellen D C and Tye S H H 1986 *Nucl. Phys.* B **269** 1–23
[2] Bjerrum-Bohr N E J, Damgaard P H and Vanhove P 2009 *Phys. Rev. Lett.* **103** 161602 (*Preprint* 0907.1425)
[3] Stieberger S and Taylor T R 2016 *Nucl. Phys.* B **903** 104–17
[4] Green M B, Schwarz J H and Witten E 2012 *Superstring Theory Vol. 1: 25th Anniversary Edition* (Cambridge: Cambridge University Press)