



Skew G -codes over $\mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle$ and new binary entanglement-assisted quantum error-correcting codes

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Abstract

In this paper, we explore the algebraic structure of skew- G codes over the ring $A = \mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle$. In addition to that, we bridge theoretical results with practical applications in quantum computing. More specifically, we derive a significant number of new binary EAQEC codes from the gray images of the skew- G codes over A according to Grassl's codetable [14].

Keywords Skew group rings · Skew G -code · Binary EAQEC codes · Hermitian construction

1 Introduction

Quantum error-correcting (QEC) codes are essential for advancing quantum computation and ensuring secure quantum communications. Their applications span several cutting-edge areas, including quantum signature schemes, quantum key distribution protocols, and quantum identity authentication [8, 36, 37]. The field began with Shor's pioneering work [32], which laid the groundwork for the rapid evolution of QEC theory and construction. A landmark contribution came from Calderbank et al., who established a powerful link between quantum codes and classical codes. They transformed the quest for quantum error correction into the search for additive self-orthogonal codes over the field \mathbb{F}_4 with respect to the Euclidean inner product [7]. Despite this advancement, the challenge of dual-containing codes has impeded further progress. In a groundbreaking development, Brun et al. [5] introduced entanglement-assisted

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quantum error-correcting (EAQEC) codes. These codes allow the use of classical error-correcting codes without requiring orthogonality, as long as shared entanglement is available between the sender and receiver. The quest to construct maximal EAQEC codes remains a pivotal endeavor due to their profound implications for quantum communication. This innovation has spurred the creation of numerous effective EAQEC codes, with notable examples including [6, 10, 15, 17, 19–25, 28, 34, 35, 39].

Linear codes are pivotal in coding theory due to their ability to facilitate efficient encoding and decoding processes. However, the linear structure is sometimes insufficient for obtaining good codes. This is why additional algebraic structures have been incorporated since the early days of coding theory. Among the various types of linear codes, cyclic codes are highly effective and widely applicable in error correction, data storage, and cryptography. Beyond their classic forms, cyclic codes have inspired the development of several advanced variations, including constacyclic codes, quasi-cyclic codes, skew cyclic codes, and G -codes. For more information about these codes, see [9, 12, 16] and the references therein. Skew G -codes, introduced in [11], provide a general framework for working with the aforementioned codes. They allow for the examination of codes that remain invariant under the skew action of specialized groups, such as cyclic or dihedral groups. These codes are often analyzed within the context of group rings or group algebras, especially when the alphabet is a field, revealing rich structures and applications in modern coding theory.

In the past decade, researchers have used skew codes derived from the factorization of skew polynomial rings to construct quantum codes [2, 9]. A significant computational challenge with this approach is the factorization of polynomials within the skew polynomial ring, as no computer algebra systems currently support this factorization. Despite existing algorithms in the literature for this purpose [3], our method provides a solution to this computational barrier. Instead of relying on skew polynomial rings, our approach utilizes skew G -codes based on skew group rings. This advantage makes the matrix representation of skew G -codes more practical for computations within any computer algebra system [31].

Inspired by the considerations mentioned above, this paper investigates the algebraic structure of skew- G codes over the ring $A = \mathbb{F}_4 + v\mathbb{F}_4 + u\mathbb{F}_4 + uv\mathbb{F}_4$, which is isomorphic to $\mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle$. We first determine all possible automorphisms of the ring A . We then construct all possible generator matrices for skew- G codes over the ring A . Specifically, we analyze the dihedral group and the cyclic group for two different orderings of their elements. Next, we utilize these matrices as generators for linear codes over A . Subsequently, we obtain binary EAQEC codes from the gray images of these codes, resulting in the creation of significant number of new binary codes according to Grassl's codetable. This study not only highlights the practical importance of skew- G codes but also demonstrates their potential to significantly advance quantum error correction methods.

2 Preliminaries

We start by reviewing the standard definitions and concepts that will be employed throughout this paper.

2.1 The ring $A = \mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle$

Let

$$A = \mathbb{F}_4 + v\mathbb{F}_4 + u\mathbb{F}_4 + uv\mathbb{F}_4 \cong \mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle,$$

where $v^2 = u^2 = 0$, and $uv = vu$. Consider the ideals $I_1 = \langle 0 \rangle$, $I_2 = \langle u \rangle$, $I_3 = \langle v \rangle$, $I_4 = \langle u + v \rangle$, $I_5 = \langle u, v \rangle$, and $I_6 = \langle uv \rangle$ in the ring A . It is clear that,

$$I_1 = \langle 0 \rangle \subseteq I_6 \subseteq I_2, I_3, I_4 \subseteq I_5 \subseteq A.$$

Since the ideals of A do not form a chain under inclusion, A is not a chain ring. The ring A is also a commutative Frobenius ring. Note that A has a unique maximal ideal, specifically $I_5 = \langle u, v \rangle$. Therefore, A is also a local ring.

2.2 Codes over the ring A

Consider A^n as the collection of all n -tuples over A . A code C of length n over A is defined as a nonempty subset of A^n . If C is a left (or right) A -submodule of A^n , it is referred to as a left (or right) linear code of length n over A . Each element $c = (c_0, c_1, \dots, c_{n-1})$ in C is called as a codeword. The Hamming weight of a codeword $c = (c_0, c_1, \dots, c_{n-1})$ is represented by $w_H(c)$ and is defined as the number of nonzero coordinates of c . The Hamming distance between any two codewords x and y is denoted by $d_H(x, y)$ and is defined as the number of positions where x and y differ. When the context is clear, the symbol H can be omitted, and one can simply use $w(c)$ and $d(x, y)$. The Hamming distance of a code C is given by:

$$d_H(C) = \min\{w_H(c) : 0 \neq c \in C\}.$$

Next, we investigate the topic of distance-preserving maps, specifically a gray map. For an element $x = a + ub + vc + uvd \in A$, the relationship between the Lee weight w_L and the Hamming weight w_H is given by

$$w_L(a + ub + vc + uvd) = w_H(a + b + c + d, c + d, b + d, d),$$

where $a, b, c, d \in \mathbb{F}_4$. The Lee weight of a vector $c = (c_0, c_1, \dots, c_{n-1})$ is determined by the sum of the Lee weight of its components. Following [1], a gray map

$$\psi : A \rightarrow \mathbb{F}_4$$

is defined by

$$a + ub + vc + uvd \rightarrow (a + b + c + d, c + d, b + d, d).$$

This map can be extended to

$$\Psi : A^n \rightarrow \mathbb{F}_4^{4n} \tag{1}$$

by

$$\Psi(a_0, a_1, \dots, a_{n-1}) = (\psi(a_0), \psi(a_1), \dots, \psi(a_{n-1})),$$

where $a_i \in A$. Let $c_1, c_2 \in A^n$. The Lee distance between any $c_1, c_2 \in A^n$, is given by

$$d_L(c_1, c_2) = d_H(\Psi(c_1), \Psi(c_2)).$$

The Lee distance of a code C can be expressed as follows:

$$d_L(C) = \min\{d_L(c_1, c_2) : c_1, c_2 \in C, c_1 \neq c_2\}.$$

It is evident that

1. The gray map Ψ is a linear isometry from $(A^n, \text{Lee distance})$ to $(\mathbb{F}_4^{4n}, \text{Hamming distance})$.
2. If C is a linear code over A of length n , size M and minimum Lee distance d , then $\Psi(C)$ is a linear code over \mathbb{F}_4 of length $4n$, size M and minimum Hamming distance d .

2.3 Skew G-codes

Now we recall the following definition from the [11].

Definition 1 Let R be a finite commutative Frobenius ring and let $G = \{g_1, g_2, \dots, g_n\}$ be a fixed listing of elements of a finite group G . Let $\varphi : G \rightarrow \text{Aut}(R)$ be a group homomorphism. The skew group ring of G over R induced by φ is the ring of formal sums

$$R \rtimes_{\varphi} G = \left\{ \sum_{i=1}^n \alpha_i g_i : \alpha_i \in R \right\},$$

where the addition operation is component-wise and multiplication is given by

$$\alpha_i g_i *_{\varphi} \beta_j g_j = \alpha_i \varphi(g_i)(\beta_j) g_i g_j$$

and then extending linearly. A skew G -code is an ideal (left or right) in the skew group ring $R \rtimes_{\varphi} G$.

Let $w = \sum_{i=1}^n \alpha_i g_i \in R \rtimes_{\varphi} G$. Then the matrix of w has the following form:

$$\sigma_{\varphi}(w) = \begin{pmatrix} \varphi_1(\alpha_{g_1^{-1}g_1}) & \varphi_1(\alpha_{g_1^{-1}g_2}) & \varphi_1(\alpha_{g_1^{-1}g_3}) & \cdots & \varphi_1(\alpha_{g_1^{-1}g_n}) \\ \varphi_2(\alpha_{g_2^{-1}g_1}) & \varphi_2(\alpha_{g_2^{-1}g_2}) & \varphi_2(\alpha_{g_2^{-1}g_3}) & \cdots & \varphi_2(\alpha_{g_2^{-1}g_n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_n(\alpha_{g_n^{-1}g_1}) & \varphi_n(\alpha_{g_n^{-1}g_2}) & \varphi_n(\alpha_{g_n^{-1}g_3}) & \cdots & \varphi_n(\alpha_{g_n^{-1}g_n}) \end{pmatrix}. \tag{2}$$

We can associate each element in the skew group ring $R \rtimes_{\varphi} G$ with an element in R^n by the canonical map

$$\Theta : R \rtimes_{\varphi} G \rightarrow R^n,$$

such that $\Theta(\sum_{i=1}^n \alpha_{g_i} g_i) = (\alpha_{g_1}, \dots, \alpha_{g_n})$. Define the following code over the ring R for a given element $w \in R \rtimes_{\varphi} G$.

$$C(w, \varphi) = \langle \sigma_{\varphi}(w) \rangle = \Theta((R \rtimes_{\varphi} G) *_{\varphi} w) \\ = \{(\beta_1, \dots, \beta_n)\sigma_{\varphi}(w) \mid (\beta_1, \beta_2, \dots, \beta_n) \in R^n\}$$

Specifically, the code is formed by taking the row space of $\sigma_{\varphi}(w)$ over the ring R . We do not claim that the rows of the matrix $\sigma_{\varphi}(w)$ are necessarily linearly independent; in general, they will not be. If they are, then the generated code is the ambient space.

2.4 Dual of codes

For any vectors $x = (x_0, x_1, \dots, x_{n-1})$ and $y = (y_0, y_1, \dots, y_{n-1})$ of length n over finite fields or rings, the Euclidean inner product is defined as

$$\langle x, y \rangle = \sum_{i=0}^{n-1} x_i y_i,$$

and the Hermitian inner product is defined as

$$\langle x, y \rangle_H = \sum_{i=1}^n x_i \overline{y_i}.$$

The vectors $x, y \in \mathbb{F}_4^n$ are termed Euclidean orthogonal if $\langle x, y \rangle = 0$ and Hermitian orthogonal if $\langle x, y \rangle_H = 0$. Let C be a linear code. Its Euclidean dual code C^{\perp} consists of the set of vectors that are orthogonal to C under the usual inner product. Its Hermitian dual code C^{\perp_H} consists of the set of vectors that are orthogonal to C under the Hermitian inner product. A code C is termed Euclidean self-dual if $C = C^{\perp}$ and Hermitian self-dual if $C = C^{\perp_H}$. In particular, if $C \subseteq C^{\perp}$ ($C \subseteq C^{\perp_H}$), then C is called a Euclidean (Hermitian) self-orthogonal code. The Euclidean hull of a linear code C is defined as

$$\text{Hull}(C) = C \cap C^\perp$$

and the Hermitian hull of a linear code C is defined as

$$\text{Hull}_H(C) = C \cap C^{\perp_H}.$$

When the context is clear, one can omit the term "Euclidean" or "Hermitian," as we will do in the rest of the paper.

2.5 Entanglement-assisted quantum error correction codes

An $[[n, k, d; c]]_q$ EAQEC code over the finite field \mathbb{F}_q is designed to encode k logical qubits into n physical qubits using c copies of maximally entangled states. The effectiveness of such a code is assessed by two key metrics: the rate $\frac{k}{n}$ and the net rate $\frac{k-c}{n}$. In particular, when $c = 0$, the EAQEC code reduces to a standard stabilizer code, which is a well-known type of quantum codes without the use of entanglement assistance. Thus, EAQEC codes can be viewed as a broader class of quantum codes that generalize standard stabilizer codes by incorporating entanglement.

Brun et al. [5] demonstrated that catalytic codes can be obtained when the net rate of an entanglement-assisted quantum code is positive. This implies that the net rate is a crucial measure of the performance of an EAQEC code, highlighting how effectively entanglement is utilized in the code construction. Furthermore, Qian and Zhang [29] employed the net rate to evaluate the performance of EAQEC codes derived from linear binary codes, showing its significance in practical scenarios.

When $c = n - k$, the EAQEC code is referred to as a maximal EAQEC code. The concept of maximal entanglement EAQEC codes was introduced by [19], where it was established that these codes can achieve the entanglement-assisted quantum capacity of a depolarizing channel, demonstrating their optimal performance in specific quantum communication contexts.

Proposition 2 ([26], Proposition 1) *If C is an $[[n, k, d]]_{q^2}$ code, then there exists an $[[n, \kappa, \delta; c]]_q$ EAQECC Q with*

$$c = k - \dim_{\mathbb{F}_{q^2}}(\text{Hull}_H(C)), \quad \kappa = n - 2k + c.$$

$$\delta = \begin{cases} \text{wt}(C^{\perp_H}), & \text{if } C^{\perp_H} \subseteq C; \\ \text{wt}(C^{\perp_H} \setminus (C \cap C^{\perp_H})), & \text{otherwise} \end{cases}.$$

The code Q in Proposition 2 is nondegenerate or pure if

$$\delta = d(C^{\perp_H}) = \text{wt}(C^{\perp_H}).$$

Proposition 3 ([27], Theorem 6) *Q be an $[[n, \kappa, \delta; c]]_q$ EAQECC constructed by a corresponding linear code C with ℓ -dimensional Hermitian hull from Proposition 2. The code Q implies the existence of an $[[n - s, \kappa, \geq \delta; c + s]]_q$ EAQECCs for each $s \in \{1, \dots, \ell\}$.*

Proposition 4 ([27], Theorem 7) *Let Q be a pure $[[n, \kappa, \delta; c]]_q$ EAQECC constructed via a corresponding linear code C with ℓ -dimensional Hermitian hull by Proposition 2. The code Q implies the existence of an EAQECC Q with parameters $[[n - s, \kappa + s, \geq \delta - s; c]]_q$ for each $s \in \{1, \dots, \ell\}$.*

3 Skew G-codes over the ring A

In this section, we will examine various classes of groups and construct the skew generator matrix for these groups over the ring A . We begin by determining the number of automorphism of the ring A .

Theorem 5 *The number of the automorphisms of the ring A is 4.*

Proof Let θ be an automorphism of the ring A . The automorphism θ is determined by a permutation of the set $\{u, v\}$ and the Frobenius automorphism $\sigma : \mathbb{F}_4 \rightarrow \mathbb{F}_4$, defined by $\sigma(r) = r^2$. If we consider the permutation of the set $\{u, v, uv\}$ and assume $\theta(u) = uv$, we find that $\theta(u)\theta(v) = uv \cdot u = 0$ or $\theta(u)\theta(v) = uv \cdot v = 0$, whereas $uv \neq 0$. Hence, θ cannot be a bijection. Consequently, only the permutation of the set $\{u, v\}$ determines an automorphism for the ring A and the cardinality of this set is 2. It is well-known that $\text{Aut}(\mathbb{F}_4) = \{\text{id}, \sigma\}$, which is a cyclic group of order 2. As a result, $|\text{Aut}(A)| = 2 \cdot 2 = 4$. We can express all 4 automorphisms of the ring A as follows:

$$\begin{aligned} \theta_1(a + bu + cv + duv) &= a + bu + cv + duv, \\ \theta_2(a + bu + cv + duv) &= \sigma(a) + \sigma(b)u + \sigma(c)v + \sigma(d)uv, \\ \theta_3(a + bu + cv + duv) &= a + cu + bv + duv, \\ \theta_4(a + bu + cv + duv) &= \sigma(a) + \sigma(c)u + \sigma(b)v + \sigma(d)uv. \end{aligned}$$

□

Remark 6 1. It is clear that the order of the each element of $\text{Aut}(A)$ except the identity is 2. Therefore, $\text{Aut}(A)$ is an abelian group.

2. $\text{Aut}(A) = \{id = \theta_1, \theta_2, \theta_3, \theta_4\}$ is a Klein-4 group under the composition map. This is the smallest non-cyclic group and isomorphic to the dihedral group of order (cardinality) 4. Cayley’s table can be given by:

	θ_1	θ_2	θ_3	θ_4
θ_1	θ_1	θ_2	θ_3	θ_4
θ_2	θ_2	θ_1	θ_4	θ_3
θ_3	θ_3	θ_4	θ_1	θ_2
θ_4	θ_4	θ_3	θ_2	θ_1

3.1 The cyclic group C_n of order n

We now determine all possible group homomorphisms between the cyclic group C_n of order n and the automorphism group A .

Proposition 7 *Let C_n be a cyclic group of order n and $\varphi : C_n \rightarrow \text{Aut}(A)$ be a group homomorphism. Then*

1. *If n is even, then there exists 4 group homomorphisms.*
2. *If n is odd, then there exists only a trivial group homomorphism.*

Proof Let $\varphi : C_n \rightarrow \text{Aut}(A)$ be a group homomorphism where $C_n = \langle c \rangle$ is a cyclic group of order n , that is,

$$C_n = \{e, c, c^2, \dots, c^{n-1}\}.$$

A fundamental property of group homomorphisms tells us that the order of the image of an element divides the order of the element. Additionally, we know that the order of each element in $\text{Aut}(A)$, except for the identity, is 2. The order of c^k is given by $n/\text{gcd}(k, n)$.

1. If n is even, a non-trivial group homomorphism φ exists if 2 divides n . The non-trivial homomorphisms are as follows for $i = 2, 3, 4$:

$$\{\varphi_1, \dots, \varphi_n\} = \{id, \theta_i, id, \theta_i, \dots, id, \theta_i\}.$$

2. If n is odd, 2 does not divide n , so only the trivial homomorphism exists:

$$\{\varphi_1, \dots, \varphi_n\} = \{id, \dots, id\}.$$

□

We now design the skew generator matrix $\sigma_\varphi(w)$ for a cyclic group of even order. Since the skew generator matrix of a cyclic group of odd order is trivial, we will not consider odd orders.

Proposition 8 *Let $w \in A \rtimes_\varphi C_{2n}$, then the generator matrix $\sigma_\varphi(w)$ has the following form:*

$$\mathcal{G}_{1,j} = \sigma_\varphi(w) = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{2n} \\ \theta_j(a_{2n}) & \theta_j(a_1) & \theta_j(a_2) & \cdots & \theta_j(a_{2n-1}) \\ a_{2n-1} & a_{2n} & a_1 & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \cdots & a_2 \\ \theta_j(a_2) & \theta_j(a_3) & \theta_j(a_4) & \cdots & \theta_j(a_1) \end{pmatrix},$$

where $a_i \in A$ and $\theta_j \in \text{Aut}(A)$.

Proof Let $\varphi : C_{2n} \rightarrow \text{Aut}(A)$ be a group homomorphism and

$$\{a_1, a_2, \dots, a_{2n}\} = \{e, c, c^2, \dots, c^{2n-1}\}. \tag{3}$$

Then by Proposition 7, we get $\varphi(c) = \theta_j$ for $1 \leq j \leq 4$. Then we get

$$\{\varphi_1, \dots, \varphi_{2n}\} = \{id, \theta_j, \dots, id, \theta_j\}, \tag{4}$$

since $\theta_j^{2k} = id$ and $\theta_j^{2k+1} = \theta_j$ with the ordering of elements in Eq. 3. Combining Eqs. 2 and 4 give the generator matrix $\mathcal{G}_{1,j}$. \square

It is also possible to obtain different generator matrices by different orderings. Let C_{2n} be the cyclic group of order $2n$ with the ordering of the elements as follows:

$$\{a_1, a_2, \dots, a_{2n}\} = \{1, c^2, \dots, c^{2n}, c, c^3, \dots, c^{2n-1}\}. \tag{5}$$

Then by [12],

$$\sigma(w) = \begin{pmatrix} A & B \\ D & A \end{pmatrix}, \tag{6}$$

where $A = cir(e, c^2, \dots, c^{2n})$, $B = cir(c, c^3, \dots, c^{2n-1})$ and $D = cir(c^{2n-1}, c, \dots, c^{2n-3})$ are $n \times n$ circulant matrices.

Proposition 9 Let C_{2n} be a cyclic group with the ordering of elements in Equation 5 and $w \in A \rtimes_{\varphi} C_{2n}$. Then the generator matrix $\sigma_{\varphi}(w)$ has the following form:

$$\mathcal{G}_{2,j} = \sigma_{\varphi}(w) = \begin{pmatrix} A & B \\ D & A \end{pmatrix}, \text{ where,}$$

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ \theta_j(a_n) & \theta_j(a_1) & \dots & \theta_j(a_{n-1}) \\ a_{n-1} & a_n & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_3 & a_4 & \dots & a_2 \\ \theta_j(a_2) & \theta_j(a_3) & \dots & \theta_j(a_1) \end{pmatrix}, \quad B = \begin{pmatrix} a_{n+1} & a_{n+2} & \dots & a_{2n} \\ \theta_j(a_{n+2}) & \theta_j(a_{n+3}) & \dots & \theta_j(a_{n+1}) \\ a_{n+3} & a_{n+4} & \dots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n} & \dots & a_{n-2} \\ \theta_j(a_{2n}) & \theta_j(a_{n+1}) & \dots & \theta_j(a_{2n-1}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} a_{2n} & a_{n+1} & a_{n+2} & \dots & a_{2n-1} \\ \theta_j(a_{2n-1}) & \theta_j(a_{2n}) & \theta_j(a_{n+1}) & \dots & \theta_j(a_{2n-2}) \\ a_{2n-2} & a_{2n-1} & a_{2n} & \dots & a_{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n+2} & a_{n+3} & a_{n+4} & \dots & a_{n+1} \\ \theta_j(a_{n+1}) & \theta_j(a_{n+2}) & \theta_j(a_{n+3}) & \dots & \theta_j(a_{2n}) \end{pmatrix},$$

where $a_i \in A$ and $\theta_j \in Aut(A)$.

Proof Proof is the similar to the proof of Proposition 8. \square

Remark 10 There exist three non-trivial skew generator matrices for a cyclic group of even order. If we take θ_1 , then the generator matrix is just a cyclic generator matrix, not a skew generator matrix.

The next Example illustrates Proposition 9.

Example 11 Let us take the ordering in Equation 5 and consider the automorphism θ_2 . We construct the skew cyclic generator matrix $\mathcal{G}_{2,2}$ in Proposition 9 over the ring A for the element

$$s = (\omega^2 uv + u + \omega v)e + (\omega^2 uv + \omega^2 u + 1)c^2 + (u + \omega^2)c + (uv + \omega)c^3 \in A \rtimes_{\varphi} \mathcal{C}_4.$$

The generator matrix obtained from s is calculated as follows:

$$\mathcal{G}_{2,2} = \begin{pmatrix} \omega^2 uv + u + \omega v & \omega^2 uv + \omega^2 u + 1 & u + \omega^2 & uv + \omega \\ \omega uv + \omega u + 1 & \omega uv + u + \omega^2 v & uv + \omega^2 & u + \omega \\ uv + \omega & u + \omega^2 & \omega^2 uv + u + \omega v & \omega^2 uv + \omega^2 u + 1 \\ u + \omega & uv + \omega^2 & \omega uv + \omega u + 1 & \omega uv + u + \omega^2 v \end{pmatrix}.$$

The linear code $C(s, \varphi)$ over \mathbb{F}_4 generated from the gray image of $\mathcal{G}_{2,2}$ has parameters $[16, 14, 2]$. These are the best-known parameters according to Grassl’s codetable [14].

3.2 The dihedral group D_ℓ of order 2ℓ

Recall that the dihedral group of order 2ℓ is defined as:

$$D_\ell = \langle a, b \mid b^\ell = a^2 = e, ab = b^{-1}a \rangle,$$

where e is the identity element of the group.

Proposition 12 Let D_ℓ be a dihedral group of order 2ℓ and $\varphi : D_\ell \rightarrow \text{Aut}(A)$ be a group homomorphism. Then

1. If ℓ is even, then there exists 16 group homomorphisms.
2. If ℓ is odd, then there exists 4 group homomorphisms.

Proof We focus only on the images of the generators b and a of the dihedral group D_ℓ , as all other images can be derived from these.

1. If ℓ is even, then the orders of the generators b and a are even. Since $\text{Aut}(A)$ is a Klein-4 group, the order of each element except the identity is 2. The order of the image of an element must divide the order of the element. Therefore, we can map b to any of the θ_i . There are 4 choices for b . Similarly, there are 4 choices for a . Consequently, there are $4 \times 4 = 16$ choices.

- If ℓ is odd, then we can map b only to $\theta_1 = id$ and a can be mapped to any of the θ_i . Consequently, there are $1 \times 4 = 4$ choices.

□

We now design the skew generator matrix $\sigma_\varphi(w)$ for a dihedral group of even order. Let \mathcal{D}_ℓ be the dihedral group of order 2ℓ with the ordering of the elements as follows:

$$\{1, b, b^2, \dots, b^{\ell-1}, a, ab, \dots, ab^{\ell-1}\}. \tag{7}$$

Then by [12],

$$\sigma(w) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{8}$$

where

$$A = \text{cir}(\alpha_1, \alpha_b, \dots, \alpha_{b^{\ell-1}})$$

is an $\ell \times \ell$ circulant matrix and

$$B = \text{rcir}(\alpha_a, \alpha_{ab}, \dots, \alpha_{ab^{\ell-1}})$$

is an $\ell \times \ell$ reverse circulant matrix.

Proposition 13 *Let \mathcal{D}_ℓ be the dihedral group of order 2ℓ for $\ell > 2$ with the ordering given in Equation (7). Then the generator matrix $\sigma_\varphi(w)$ of $w \in A \rtimes_\varphi \mathcal{D}_\ell$ has the following forms:*

- If ℓ is odd, then
- If ℓ is odd, then

$$\mathcal{G}_{3,j} = \sigma_\varphi(w) = \begin{pmatrix} A & B \\ \theta_j(B) & \theta_j(A) \end{pmatrix}, \text{ where}$$

$$A = \text{circ}(a_1, a_2, a_3, \dots, a_\ell),$$

$$B = \text{rcirc}(a_{\ell+1}, a_{\ell+2}, a_{\ell+3}, \dots, a_{2\ell}),$$

$$\theta_j(A) = \text{circ}(\theta_j(a_1), \theta_j(a_2), \dots, \theta_j(a_\ell)),$$

$$\theta_j(B) = \text{rcirc}(\theta_j(a_{\ell+1}), \theta_j(a_{\ell+2}), \dots, \theta_j(a_{2\ell})),$$

and $a_i \in A$ and $\theta_j \in \text{Aut}(A)$.

- If ℓ is even, then

$$\mathcal{G}_{4,j,i} = \sigma_\varphi(w) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ where}$$

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_\ell \\ \theta_j(a_\ell) & \theta_j(a_1) & \dots & \theta_j(a_{\ell-1}) \\ a_{\ell-1} & a_\ell & \dots & a_{\ell-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_3 & a_4 & \dots & a_2 \\ \theta_j(a_2) & \theta_j(a_3) & \dots & \theta_j(a_1) \end{pmatrix}, B = \begin{pmatrix} a_{\ell+1} & a_{\ell+2} & \dots & a_{2\ell} \\ \theta_j(a_{\ell+2}) & \theta_j(a_{\ell+3}) & \dots & \theta_j(a_{\ell+1}) \\ a_{\ell+3} & a_{\ell+4} & \dots & a_{\ell+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2\ell-1} & a_{2\ell} & \dots & a_{\ell-2} \\ \theta_j(a_{2\ell}) & \theta_j(a_{\ell+1}) & \dots & \theta_j(a_{2\ell-1}) \end{pmatrix},$$

$$C = \begin{pmatrix} \theta_i(a_{\ell+1}) & \theta_i(a_{\ell+2}) & \dots & \theta_i(a_{2\ell}) \\ \theta_k(a_{\ell+2}) & \theta_k(a_{\ell+3}) & \dots & \theta_k(a_{\ell+1}) \\ \theta_i(a_{\ell+3}) & \theta_i(a_{\ell+4}) & \dots & \theta_i(a_{\ell+2}) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_i(a_{2\ell-1}) & \theta_i(a_{2\ell}) & \dots & \theta_i(a_{\ell-2}) \\ \theta_k(a_{2\ell}) & \theta_k(a_{\ell+1}) & \dots & \theta_k(a_{2\ell-1}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} \theta_i(a_1) & \theta_i(a_2) & \dots & \theta_i(a_\ell) \\ \theta_k(a_\ell) & \theta_k(a_1) & \dots & \theta_k(a_{\ell-1}) \\ \theta_i(a_{\ell-1}) & \theta_i(a_\ell) & \dots & \theta_i(a_{\ell-2}) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_i(a_3) & \theta_i(a_4) & \dots & \theta_i(a_2) \\ \theta_k(a_2) & \theta_k(a_3) & \dots & \theta_k(a_1) \end{pmatrix}.$$

Proof Let $\varphi : \mathcal{D}_\ell \rightarrow \text{Aut}(A)$ be a group homomorphism. It is enough to determine the images of generators of groups.

- Let ℓ be odd. Then by Proposition 12 $\varphi(b) = \theta_1 = id$ and $\varphi(a) = \theta_j$ for $1 \leq j \leq 4$. Then we get

$$\{\varphi_1, \dots, \varphi_\ell, \varphi_{\ell+1}, \dots, \varphi_{2\ell}\} = \{id, \dots, id, \theta_j, \dots, \theta_j\} \tag{9}$$

with the ordering of elements given in Equ. 7. Combining Eqs. 2, 8 and 9 give the generator matrix $\mathcal{G}_{3,j}$.

- Let ℓ be even, then $\varphi(b) = \theta_j$ and $\varphi(a) = \theta_i$ by Proposition 12, for $1 \leq i \leq 4$ and $1 \leq j \leq 4$. Then we get

$$\{\varphi_1, \dots, \varphi_\ell, \varphi_{\ell+1}, \dots, \varphi_{2\ell}\} = \{id, \theta_j, \dots, id, \theta_j, \theta_i, \theta_k, \dots, \theta_i, \theta_k\} \tag{10}$$

where $\theta_k = \theta_j\theta_i$, with the ordering of elements given in Equation 7. Combining Equations 2, 8 and 10 give the generator matrix $\mathcal{G}_{4,i,j}$.

□

Let \mathcal{D}_ℓ be the dihedral group of order 2ℓ with the ordering of the elements as follows

$$\{1, b, b^2, \dots, b^{\ell-1}, ba, b^2a, \dots, b^{\ell-1}a, a\}. \tag{11}$$

Then by [10],

$$\sigma(w) = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix},$$

where $A = \text{cir}(\alpha_1, \alpha_b, \dots, \alpha_{b^{\ell-1}})$ and $B = \text{cir}(\alpha_a, \alpha_{ba}, \dots, \alpha_{b^{\ell-1}a})$ are the $\ell \times \ell$ circulant matrices and A^T and B^T denote the transpose matrices of A and B , respectively.

Proposition 14 *Let \mathcal{D}_ℓ be the dihedral group of order 2ℓ for $\ell > 2$ with the ordering given in Equation (11). Then the generator matrix $\sigma_\varphi(w)$ of $w \in A \rtimes_\varphi \mathcal{D}_\ell$ has the following forms:*

1. *If ℓ is odd, then*

$$\mathcal{G}_{5,i} = \sigma_\varphi(w) = \begin{pmatrix} A & B \\ \theta_i(B^T) & \theta_i(A^T) \end{pmatrix}, \text{ where}$$

$$A = \text{circ}(a_1, a_2, a_3, \dots, a_n),$$

$$B = \text{circ}(a_{n+1}, a_{n+2}, a_{n+3}, \dots, a_{2n}),$$

$$\theta_i(A^T) = \text{circ}(\theta_i(a_1), \theta_i(a_n), \dots, \theta_i(a_2)),$$

$$\theta_i(B^T) = \text{circ}(\theta_i(a_{n+1}), \theta_i(a_{2n}), \dots, \theta_i(a_{n+2})).$$

2. *If ℓ is even, then*

$$\mathcal{G}_{6,i,j} = \sigma_\varphi(w) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ where}$$

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ \theta_j(a_n) & \theta_j(a_1) & \dots & \theta_j(a_{n-1}) \\ a_{n-1} & a_n & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_3 & a_4 & \dots & a_2 \\ \theta_j(a_2) & \theta_j(a_3) & \dots & \theta_j(a_1) \end{pmatrix}, B = \begin{pmatrix} a_{n+1} & a_{n+2} & \dots & a_{2n} \\ \theta_j(a_{2n}) & \theta_j(a_{n+1}) & \dots & \theta_j(a_{2n-1}) \\ a_{2n-1} & a_{2n} & \dots & a_{2n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+3} & a_{n+4} & \dots & a_{n+2} \\ \theta_j(a_{n+2}) & \theta_j(a_{n+3}) & \dots & \theta_j(a_{n+1}) \end{pmatrix},$$

$$C = \begin{pmatrix} \theta_k(a_{n+1}) & \theta_k(a_{2n}) & \dots & \theta_k(a_{n+2}) \\ \theta_i(a_{n+2}) & \theta_i(a_{n+1}) & \dots & \theta_i(a_{n+3}) \\ \theta_k(a_{n+3}) & \theta_k(a_{n+2}) & \dots & \theta_k(a_{n+4}) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_i(a_{2n}) & \theta_i(a_{2n-1}) & \dots & \theta_i(a_{n+1}) \end{pmatrix}$$

and

$$D = \begin{pmatrix} \theta_k(a_1) & \theta_k(a_n) & \dots & \theta_k(a_2) \\ \theta_i(a_2) & \theta_i(a_1) & \dots & \theta_i(a_3) \\ \theta_k(a_3) & \theta_k(a_2) & \dots & \theta_k(a_4) \\ \vdots & \vdots & \vdots & \vdots \\ \theta_i(a_n) & \theta_i(a_{n-1}) & \dots & \theta_i(a_1) \end{pmatrix}.$$

Proof Proof is similar to the proof of Proposition 13. □

The next example illustrates Proposition 13.

Example 15 Let us take the ordering in Equation 7 and consider the automorphism θ_4 . We construct the skew cyclic generator matrix $\mathcal{G}_{3,4}$ in Proposition 13 over the ring A for the element

$$s = (uv + u + wv) + (u + v)b + (uv + wu + w^2v + 1)b^2 + (w^2uv + w^2)a + (wuv + w)ab + (wuv + w^2v + w)ab^2 \in A \rtimes_{\varphi} \mathcal{D}_4.$$

The generator matrix obtained from s is calculated as follows:

$$\mathcal{G}_{3,4} = \begin{pmatrix} uv + u + wv & u + v & uv + wu + w^2v + 1 & w^2uv + w^2 & wuv + w & wuv + w^2v + w \\ uv + wu + w^2v + 1 & uv + u + wv & u + v & wuv + w & wuv + w^2v + w & w^2uv + w^2 \\ u + v & uv + wu + w^2v + 1 & uv + u + wv & wuv + w^2v + w & w^2uv + w^2 & wuv + w \\ wuv + w & w^2uv + w^2 & w^2uv + wu + w^2 & uv + w^2u + v & u + v & uv + wu + w^2v + 1 \\ w^2uv + w^2 & w^2uv + wu + w^2 & wuv + w & uv + wu + w^2v + 1 & uv + w^2u + v & u + v \\ w^2uv + wu + w^2 & wuv + w & w^2uv + w^2 & u + v & uv + wu + w^2v + 1 & uv + w^2u + v \end{pmatrix}.$$

The linear code $C(s, \varphi)$ over \mathbb{F}_4 generated from the gray image of $\mathcal{G}_{3,4}$ has parameters $[24, 15, 6]$. This is the best-known minimum distance for the code with this length and dimension according to Grassl’s codetable [14].

4 Duals of skew G-codes

We explore the algebraic characteristics of dual codes derived from skew cyclic codes over A .

Proposition 16 For any integer n , C is a skew G -code of length n over A if and only if C^{\perp_H} is a skew G -code over A .

Proof The proof follows by [11, Theorem 2.5] since A is a finite commutative Frobenius ring and [11, Theorem 2.5] is valid for Hermitian inner product. □

Let $Hull(w) = C(w, \varphi) \cap C(w, \varphi)^{\perp_H}$. If $Hull(w) = C(w, \varphi)$ then $C(w, \varphi)$ is a self-dual code by definition.

Lemma 17 Let $w \in R \rtimes_{\varphi} G$, then $Hull(w)$ is a skew G -code.

Proof According to Proposition 16, $C(w, \varphi)^{\perp_H}$ is a skew G -code. Consequently, if $\mathbf{x} \in C(w, \varphi)$, then $g_i *_{\varphi} \mathbf{x} \in C(w, \varphi)$ for any $g_i \in G$ and if $\mathbf{x} \in C(w, \varphi)^{\perp_H}$, then $g_i *_{\varphi} \mathbf{x} \in C(w, \varphi)^{\perp_H}$ for every $g_i \in G$. Therefore, if $\mathbf{x} \in Hull(w)$, then $g_i *_{\varphi} \mathbf{x} \in Hull(w)$ for all $g_i \in G$. □

Next theorem demonstrates that gray images of Hermitian self-dual codes are self-dual under the map defined in Equation 1.

Theorem 18 *The gray image $\Psi(C)$ of a Hermitian self-dual code C of length n over A is Hermitian self-dual.*

Proof It suffices to demonstrate that the extended gray map preserves orthogonality, then the result follows from the size of the image. We first note that $\langle \mathbf{v}, \mathbf{u} \rangle_H = \langle \overline{\mathbf{u}}, \overline{\mathbf{v}} \rangle_H$. Let

$$\alpha_1 = \mathbf{a}_1 + u\mathbf{b}_1 + v\mathbf{c}_1 + uv\mathbf{d}_1 \text{ and } \alpha_2 = \mathbf{a}_2 + u\mathbf{b}_2 + v\mathbf{c}_2 + uv\mathbf{d}_2,$$

where $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1, \mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2, \mathbf{d}_2 \in \mathbb{F}_4^n$, be two codewords of length n over A such that they are orthogonal. Then, $\langle \alpha_1, \alpha_2 \rangle_H = 0$. That is:

$$\begin{aligned} &\langle \mathbf{a}_1, \mathbf{a}_2 \rangle_H + u(\langle \mathbf{a}_1, \mathbf{b}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{b}_1} \rangle_H) + v(\langle \mathbf{a}_1, \mathbf{c}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{c}_1} \rangle_H) + \\ &uv(\langle \mathbf{a}_1, \mathbf{d}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{d}_1} \rangle_H) + \langle \mathbf{b}_1, \mathbf{c}_2 \rangle_H + \langle \overline{\mathbf{b}_2}, \overline{\mathbf{c}_1} \rangle_H = 0, \end{aligned} \tag{12}$$

which implies,

$$\begin{aligned} &\langle \mathbf{a}_1, \mathbf{a}_2 \rangle_H = 0, \\ &\langle \mathbf{a}_1, \mathbf{b}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{b}_1} \rangle_H = 0, \\ &\langle \mathbf{a}_1, \mathbf{c}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{c}_1} \rangle_H = 0, \\ &\langle \mathbf{a}_1, \mathbf{d}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{d}_1} \rangle_H + \langle \mathbf{b}_1, \mathbf{c}_2 \rangle_H + \langle \overline{\mathbf{b}_2}, \overline{\mathbf{c}_1} \rangle_H = 0. \end{aligned} \tag{13}$$

Now we consider the inner product of $\Psi(\alpha_1)$ and $\Psi(\alpha_2)$, where

$$\begin{aligned} \Psi(\alpha_1) &= (\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{c}_1 + \mathbf{d}_1, \mathbf{c}_1 + \mathbf{d}_1, \mathbf{b}_1 + \mathbf{d}_1, \mathbf{d}_1), \\ \Psi(\alpha_2) &= (\mathbf{a}_2 + \mathbf{b}_2 + \mathbf{c}_2 + \mathbf{d}_2, \mathbf{c}_2 + \mathbf{d}_2, \mathbf{b}_2 + \mathbf{d}_2, \mathbf{d}_2). \end{aligned}$$

Then, by using the properties of Hermitian inner product and Equation 13 we get

$$\begin{aligned} \Psi(\alpha_1)\Psi(\alpha_2) &= \langle (\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{c}_1 + \mathbf{d}_1, \mathbf{c}_1 + \mathbf{d}_1, \mathbf{b}_1 + \mathbf{d}_1, \mathbf{d}_1), \\ &\quad (\mathbf{a}_2 + \mathbf{b}_2 + \mathbf{c}_2 + \mathbf{d}_2, \mathbf{c}_2 + \mathbf{d}_2, \mathbf{b}_2 + \mathbf{d}_2, \mathbf{d}_2) \rangle_H \\ &\quad + \langle (\mathbf{c}_1 + \mathbf{d}_1), (\mathbf{c}_2 + \mathbf{d}_2) \rangle_H + \langle \mathbf{b}_1 + \mathbf{d}_1, (\mathbf{b}_2 + \mathbf{d}_2) \rangle_H + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_H \\ &= \langle \mathbf{a}_1, \mathbf{a}_2 \rangle_H + \langle \mathbf{a}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{a}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{a}_1, \mathbf{d}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{b}_1} \rangle_H \\ &\quad + \langle \mathbf{b}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{b}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{b}_1, \mathbf{d}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{c}_1} \rangle_H + \langle \overline{\mathbf{b}_2}, \overline{\mathbf{c}_1} \rangle_H \\ &\quad + \langle \mathbf{c}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{c}_1, \mathbf{d}_2 \rangle_H + \langle \overline{\mathbf{a}_2}, \overline{\mathbf{d}_1} \rangle_H + \langle \mathbf{d}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{c}_2 \rangle_H \\ &\quad + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{c}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{c}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_H \\ &\quad + \langle \mathbf{b}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{b}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_H \\ &= 2(\langle \mathbf{b}_1, \mathbf{b}_2 \rangle_H + \langle \mathbf{b}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{c}_1, \mathbf{c}_2 \rangle_H + \langle \mathbf{c}_1, \mathbf{d}_2 \rangle_H + \langle \mathbf{d}_1, \mathbf{c}_2 \rangle_H \\ &\quad + \langle \mathbf{d}_1, \mathbf{b}_2 \rangle_H) = 0. \end{aligned}$$

It is observed that if two codewords are orthogonal then so are their gray images. \square

Remark 19 Proposition 16, Lemma 17 and Theorem 18 are also true for Euclidean inner product. The proof is similar.

5 Computational results

In this section, we construct new binary EAQEC codes by using Proposition 2 Proposition 3 and Proposition 4 from the gray images of the generator matrices that are given in Sect. 3. All the upcoming computational results are obtained by using the software package MAGMA [4]. In this section, we conducted a random computer search for linear codes over the ring A by using Magma software [4]. We can explain how we searched codes as follows:

1. We first randomly generate the vector v of length n over the ring A .
2. Next, we construct the skew generator matrix \mathcal{G}_i of size $n \times n$ from Sect. 3. It is important to note that while \mathcal{G}_i generates the code, it may not necessarily contain the minimal number of rows required to generate the code. In other words, we do not assume any kind of independence among the rows.
3. Additionally, we obtain three $n \times n$ matrices by simply multiplying each row of G by u , v and uv , respectively.
4. Subsequently, we construct the matrix M vertically stacking these four generator matrices, here M is $4n \times n$ matrix.
5. Then, we apply the gray map Ψ to the each component of the matrix M ($\Psi(M)$ has $4n$ rows and $4n$ columns).
6. Then, we obtain a linear code C over \mathbb{F}_4 from the matrix that is constructed in the previous step.
7. Finally, use the Propositions 2, 3 and 4 to construct EAQEC codes over \mathbb{F}_2 .

We tabulate the construction matrices \mathcal{G}_i , parameters of the gray images of codes and parameters of the binary EAQEC codes in Tables 1, 2, 3, 4, 5, 6 and 7. All generator matrices over the ring A and quaternary generator matrices that are used to construct the binary EAQEC codes listed in this paper can be found online at [30].

Table 1 Parameters of new EAQEC codes from the gray images of skew G-codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{1,2}$	Proposition 2	$[40, 30, 4]_4$	$[[40, 0, 20; 20]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 2	$[40, 28, 7]_4$	$[[40, 0, 18; 16]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[30, 0, \geq 18; 26]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[31, 0, \geq 18; 25]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[32, 0, \geq 20; 28]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[32, 0, \geq 18; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[33, 0, \geq 20; 27]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[33, 0, \geq 18; 23]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[40, 30, 4]_4$	$[[34, 6, \geq 14; 20]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[34, 0, \geq 20; 26]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[34, 0, \geq 18; 22]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[40, 30, 4]_4$	$[[35, 5, \geq 15; 20]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[35, 0, \geq 20; 25]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[35, 0, \geq 18; 21]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[40, 28, 7]_4$	$[[36, 4, \geq 14; 16]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[36, 0, \geq 20; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[36, 0, \geq 18; 20]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[37, 0, \geq 20; 23]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[37, 0, \geq 18; 19]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[38, 0, \geq 20; 22]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 28, 7]_4$	$[[38, 0, \geq 18; 18]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[40, 30, 4]_4$	$[[39, 0, \geq 20; 21]]_2$	New

Table 2 Parameters of new EAQEC codes from the gray images of skew G -codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{1,2}$	Proposition 2	$[48, 36, 6]_4$	$[[48, 0, 24; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[38, 0, \geq 24; 34]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[39, 0, \geq 24; 33]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[41, 0, \geq 24; 31]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[48, 36, 6]_4$	$[[42, 6, \geq 18; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[42, 0, \geq 24; 30]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[48, 36, 6]_4$	$[[43, 5, \geq 19; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[43, 0, \geq 24; 29]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 4	$[48, 36, 6]_4$	$[[44, 4, \geq 20; 24]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[44, 0, \geq 24; 28]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[45, 0, \geq 24; 27]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[46, 0, \geq 24; 26]]_2$	New
$\mathcal{G}_{1,2}$	Proposition 3	$[48, 36, 6]_4$	$[[47, 0, \geq 24; 25]]_2$	New

Table 3 Parameters of new EAQEC codes from the gray images of skew G-codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{1,3}$	Proposition 2	$[24, 22, 2]_4$	$[[24, 0, 18; 20]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 4	$[24, 22, 2]_4$	$[[22, 2, \geq 16; 20]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[24, 22, 2]_4$	$[[23, 0, \geq 18; 21]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 2	$[64, 60, 2]_4$	$[[64, 0, 44; 56]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 52, 2]_4$	$[[57, 0, \geq 22; 47]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 52, 2]_4$	$[[58, 0, \geq 22; 46]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 52, 2]_4$	$[[59, 0, \geq 22; 45]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 4	$[64, 60, 2]_4$	$[[60, 4, \geq 40; 56]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 52, 2]_4$	$[[60, 0, \geq 22; 44]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 4	$[64, 60, 2]_4$	$[[61, 3, \geq 41; 56]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 52, 2]_4$	$[[61, 0, \geq 22; 43]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 4	$[64, 60, 2]_4$	$[[62, 2, \geq 42; 56]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 60, 2]_4$	$[[62, 0, \geq 44; 58]]_2$	New
$\mathcal{G}_{1,3}$	Proposition 3	$[64, 60, 2]_4$	$[[63, 0, \geq 44; 57]]_2$	New

Table 4 Parameters of new EAQEC codes from the gray images of skew G -codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{1,4}$	Proposition 2	$[24, 18, 4]_4$	$[[24, 0, 14; 12]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 2	$[24, 20, 3]_4$	$[[24, 0, 16; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 18, 4]_4$	$[[19, 0, \geq 14; 17]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 18, 4]_4$	$[[20, 4, \geq 10; 12]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 20, 3]_4$	$[[20, 4, \geq 12; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 18, 4]_4$	$[[20, 0, \geq 14; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 18, 4]_4$	$[[21, 3, \geq 11; 12]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 20, 3]_4$	$[[21, 3, \geq 13; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 18, 4]_4$	$[[21, 0, \geq 14; 15]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 20, 3]_4$	$[[21, 0, \geq 16; 19]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 18, 4]_4$	$[[22, 2, \geq 12; 12]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[24, 20, 3]_4$	$[[22, 2, \geq 14; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 18, 4]_4$	$[[22, 0, \geq 14; 14]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 20, 3]_4$	$[[22, 0, \geq 16; 18]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 18, 4]_4$	$[[23, 0, \geq 14; 13]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[24, 20, 3]_4$	$[[23, 0, \geq 16; 17]]_2$	New

Table 5 Parameters of new EAQEC codes From the gray images of skew G-codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{1,4}$	Proposition 2	$[32, 24, 4]_4$	$[[32, 0, 16; 16]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[25, 0, \geq 16; 23]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[26, 0, \geq 16; 22]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[27, 0, \geq 16; 21]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[28, 0, \geq 16; 20]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[29, 0, \geq 16; 19]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 24, 4]_4$	$[[31, 0, \geq 16; 17]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[32, 28, 2]_4$	$[[28, 4, \geq 14; 24]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 4	$[32, 24, 4]_4$	$[[29, 3, \geq 15; 24]]_2$	New
$\mathcal{G}_{1,4}$	Proposition 3	$[32, 28, 2]_4$	$[[29, 0, \geq 18; 27]]_2$	New

Table 6 Parameters of new EAQEC codes from the gray images of skew G -codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{3,3}$	Proposition 2	$[40, 36, 3]_4$	$[[40, 0, 28; 32]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 2	$[40, 34, 4]_4$	$[[40, 0, 24; 28]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 2	$[40, 38, 2]_4$	$[[40, 0, 30; 36]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[40, 30, 4]_4$	$[[31, 0, \geq 20; 29]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 32, 4]_4$	$[[32, 8, \geq 14; 24]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 30, 4]_4$	$[[33, 7, \geq 15; 24]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 34, 4]_4$	$[[34, 6, \geq 18; 28]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 34, 4]_4$	$[[35, 5, \geq 19; 28]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 34, 4]_4$	$[[36, 4, \geq 20; 28]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 32, 4]_4$	$[[36, 4, \geq 18; 24]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 36, 3]_4$	$[[37, 3, \geq 25; 32]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[40, 36, 3]_4$	$[[37, 0, \geq 28; 35]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[40, 38, 2]_4$	$[[38, 2, \geq 28; 36]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[40, 36, 3]_4$	$[[39, 0, \geq 28; 33]]_2$	New

Table 7 Parameters of new EAQEC codes from the gray images of skew G-codes over the ring A

Generator Matrix	EAQEC Construction Formula	\mathbb{F}_4 images	EAQEC Codes	[14]
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 40, 4]_4$	$[[43, 0, \geq 20; 37]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 40, 4]_4$	$[[44, 0, \geq 20; 36]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 42, 4]_4$	$[[46, 0, \geq 22; 38]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 40, 4]_4$	$[[49, 0, \geq 20; 31]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 42, 4]_4$	$[[49, 0, \geq 22; 35]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 42, 4]_4$	$[[50, 0, \geq 22; 34]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 42, 4]_4$	$[[51, 0, \geq 22; 33]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 42, 4]_4$	$[[53, 0, \geq 22; 31]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 4	$[56, 54, 2]_4$	$[[54, 2, \geq 40; 52]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 52, 2]_4$	$[[55, 0, \geq 28; 49]]_2$	New
$\mathcal{G}_{3,3}$	Proposition 3	$[56, 54, 2]_4$	$[[55, 0, \geq 42; 53]]_2$	New

6 Conclusion

In conclusion, in this paper we examine the algebraic properties of skew- G codes over the ring $A = \mathbb{F}_4 + v\mathbb{F}_4 + u\mathbb{F}_4 + uv\mathbb{F}_4$. By utilizing the gray map on the generator matrices of these skew- G codes, we derive binary EAQEC codes. The results reveal that, our research has successfully yielded binary EAQEC codes from the gray images of skew- G codes, leading to the development of significant number of novel binary codes according to Grassl's codetable [14]. The results indicate that skew construction matrices are an effective resource for generating new EAQEC codes. We anticipate that our findings will inspire further research into developing additional new EAQEC codes with improved parameters by exploring the non-commutative structure of group rings over various finite rings. For future work, we recommend investigating the construction matrix introduced in this study across different groups and alphabet sizes to discover new EAQEC codes of varying lengths.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

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References

1. Ashraf, M., Rehman, W., Mohammad, G., Asim, M.: On reversible codes over a non-chain ring. *Comput. Appl. Math.* **42**(6), 269 (2023)
2. Aydin, N., Abualrub, T.: Optimal quantum codes from additive skew cyclic codes. *Discret. Math. Algorithms Appl.* **8**(3), 1650037 (2016)
3. Aydin, N.: Some new linear codes from skew cyclic codes and computer algebra challenges. *Appl. Algebra Eng., Commun. Computing.* **30**, 185–91 (2019)
4. Bosma, W., Cannon, J., Playoust, C.: The magma algebra system. I. the user language. *J. Symbolic Comput.* **24**, 235–265 (1997)
5. Brun, T.A., Devetak, I., Hsieh, M.H.: Correcting quantum errors with entanglement. *Science* **314**, 436–439 (2006)
6. Bowen, G.: Entanglement required in achieving entanglement-assisted channel capacities. *Phys. Rev. A* **66**, 052313 (2002)

7. Calderbank, A.R., Rains, E.M., Shore, P.W., Sloane, N.J.A.: Quantum error correction via codes over $\text{GF}(4)$. *IEEE Trans. Inform. Theory* **44**(4), 1369–1387 (1998)
8. Chen, Z., Zhou, K., Liao, Q.: Quantum identity authentication scheme of vehicular ad-hoc networks. *Int. J. Theor. Phys.* **58**(1), 40–57 (2019)
9. Dinh, H.Q., Bag, T., Upadhyay, A.K., Bandi, R., Tansuchat, R.: A class of skew cyclic codes and application in quantum codes construction. *Discrete Math.* **344**(2), 112189 (2021)
10. Dougherty, S.T., Korban, A., Sahinkaya, S., Ustun, D.: Group matrix ring codes and constructions of self-dual codes. *AAECC* **34**, 279–299 (2023)
11. Dougherty, S.T., Sahinkaya, S., Yildiz, B.: Skew G -codes. *J. Algebra Appl.* **22**(02), 2350053 (2023)
12. Dougherty, S.T., Gildea, J., Taylor, R., Tylshchak, A.: Group rings, G -codes and constructions of self-dual and formally self-dual codes". *Des., Codes Cryptogr.* **86**(9), 2115–2138 (2018)
13. Galindo, C., Hernando, F., Matsumoto, R., Ruano, D.: Entanglement-assisted quantum error-correcting codes over arbitrary finite fields. *Quantum Inf. Process* **18**(4), 116 (2019). <https://doi.org/10.1007/s11128-019-2234-5>
14. Grassl, M.: "Bounds on the minimum distance of linear codes and quantum codes." Online available at <http://www.codetables.de> Accessed on 2024-08-02
15. Guenda, K., Jitman, S., Gulliver, T.A.: Constructions of good entanglement-assisted quantum error correcting codes. *Des. Codes Cryptogr.* **86**(1), 121–136 (2018)
16. Huffman, W.C., Pless, V.: *Fundamentals of error-correcting codes*. Cambridge University Press, Cambridge (2003)
17. Hu, P., Liu, X.: EAQEC codes from two distinct constacyclic codes. *Quantum Inf Process* **22**, 100 (2023). <https://doi.org/10.1007/s11128-023-03849-x>
18. Lai, C.Y., Brun, T.A., Wilde, M.: Duality in entanglement-assisted quantum error correction. *IEEE Trans. Inf. Theory* **59**(6), 4020–4024 (2013)
19. Lai, C.Y., Ashikhmin, A.: Linear programming bounds for entanglement-assisted quantum error correcting codes by split weight enumerators. *IEEE Trans. Inf. Theory* **64**(1), 622–639 (2018)
20. Lai, C.Y., Brun, T.A.: Entanglement-assisted quantum error correcting codes with imperfect ebits. *Phys. Rev. A* **86**, 032319 (2012)
21. Liu, X., Yu, L., Hu, P.: New entanglement-assisted quantum codes from k -Galois dual codes. *Finite Fields Appl.* **55**, 21–32 (2019)
22. Liu, H., Liu, X.: New EAQEC codes from cyclic codes over $Z_4 + vZ_4$. *Quantum Inf Process* **23**, 65 (2024). <https://doi.org/10.1007/s11128-024-04277-1>
23. Liu, X., Hu, P.: New QEC and EAQEC codes from repeated-root cyclic codes of length over finite fields. *Quantum Inf Process* **23**, 164 (2024). <https://doi.org/10.1007/s11128-024-04374-1>
24. Liu, R., Li, S., Shi, M.: Construction of linear codes with various Hermitian hull dimensions and related EAQECs. *Adv. Math. Commun.* **19**(2), 588–603 (2025). <https://doi.org/10.3934/amc.2024013>
25. Lu, L., Li, R., Ma, W., Ma, Y., Liu, Y., Cao, H.: Entanglement-assisted quantum MDS codes from constacyclic codes with large minimum distance. *Finite Fields Appl.* **53**, 309–325 (2018)
26. Luo, G., Ezerman, M.F., Grassl, M., Ling, S.: Constructing quantum error-correcting codes that require a variable amount of entanglement. *Quantum Inf. Process.* **23**(1), 4 (2023)
27. Luo, G., Sok, L., Ezerman, M. F., Ling, S.: On propagation rules for entanglement-assisted quantum codes. In 2023 12th International Symposium on Topics in Coding (ISTC) (pp. 1-5). *IEEE*(2023, September)
28. Qian, J., Zhang, L.: On MDS linear complementary dual codes and entanglement-assisted quantum codes. *Des. Codes Cryptogr.* **86**(7), 1565–1572 (2018)
29. Qian, J., Zhang, L.: Entanglement-assisted quantum codes from arbitrary binary linear codes. *Des. Codes Cryptogr.* **77**(1), 193–202 (2015)
30. Sahinkaya, S., Ustun, D.: Generator matrices of "Skew G -Codes over $\mathbb{F}_4[u, v]/\langle u^2, v^2, uv - vu \rangle$ and New Binary Entanglement-Assisted Quantum Error-Correcting Codes", <https://sites.google.com/view/serap-sahinkaya/generator-matrices>
31. Sahinkaya, S., Korban, A., Ustun, D.: Maximal entanglement-assisted quantum error correction codes from the skew group ring $\mathbb{F}_4 \rtimes_{\varphi} G$ by a heuristic search scheme. *Quantum Inf Process* **21**, 156 (2022)
32. Shor, P.W.: Scheme for reducing decoherence in quantum memory. *Phys. Rev. A. Gen. Phys.* **52**(4), 2493–2496 (1995)
33. Steane, A.M.: Enlargement of Calderbank-Shor-Steane quantum codes. *IEEE Trans. Inf. Theory* **45**, 2492–2495 (1999)

34. Wang, J., Li, R., Lv, J., Guo, G., Liu, Y.: Entanglement-assisted quantum error correction codes with length $n = q^2 + 1$. *Quantum Inf. Process.* **18**, 1–21 (2019)
35. Wilde, M.M., Hsieh, M.H., Babar, Z.: Entanglement-assisted quantum turbo codes. *IEEE Trans. Inf. Theory* **60**, 1203–1222 (2014)
36. Xiao, H., Zhang, Z.: Subcarrier multiplexing multiple-input multiple-output quantum key distribution with orthogonal quantum states. *Quantum Inf. Process.* **16**(13), 1–18 (2017)
37. Xin, X., He, Q., Wang, Z., Yang, Q., Li, F.: Efficient arbitrated quantum signature scheme without entangled states. *Mod. Phys. Lett. A* **34**(21), 1950166 (2019)
38. Yu, Zhu: New binary quantum codes from group rings and skew group rings. *Int J Theor Phys* **63**, 27 (2024). <https://doi.org/10.1007/s10773-023-05545-z>
39. Zhang, X.: QEC and EAQEC codes from cyclic codes over non-chain rings. *Quantum Inf Process* **21**, 389 (2022). <https://doi.org/10.1007/s11128-022-03734-z>

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