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Low energy physics from type IIB string theory

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"Een geest, een en al logica, is als een mes
dat een en al lemmet is.
Het doet de hand die het gebruikt bloeden."

Tagore

"De wijze waarop het eten wordt opgediend, is minstens even belangrijk als de wijze waarop het wordt toebereid."

Ons Kookboek, KVLV

Voorwoord

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Chapter 1

Introduction

Two thousand five hundred years ago, Plato [1] had the brilliant insight that space, time and matter as we see it might very well be only a pale shadow of a much more complex underlying reality, inaccessible by direct observation. In a beautiful paper [1], he described the following gedanken experiment. Imagine a bunch of people sitting next to each other, staring at the wall of a cave. They are heavily chained and only able to see this wall, and have been sitting there all their conscious life. Behind their back, guards have made huge fires, such that the shadows of the chained people, as well as of other objects passing in front of the fire, are visible on the wall of the cave. The entire observable reality of the chained is a two dimensional world of shadows. Direct observation provides a number of rules for the shadow dynamics, though they are intricate, require a priori specification of a large number of undetermined quantities and break down when pushed too far. It will be exceedingly difficult for those people to arrive at the insight that there is actually an underlying unifying three dimensional reality, and this will most certainly require brave new ideas and far-reaching abstract theoretical developments. As discussed in [1], it will moreover be almost impossibly hard to convince the chained of the reality, and the intrinsic beauty, of this three dimensional world, even if they were dragged out of the darkness of the cave and taken outside, into the Sunshine.

Twentieth century physics has shown Plato's insight to be astonishingly true. As years progressed, the search for the fundamental laws of physics required a picture of reality which got seemingly further and further removed from everyday observations, but at the same time got increasingly beautiful and elegant. At present, the top of this evolution is dominated by a remarkably rich and unifying theoretical construct: the theory of strings.

From the point of view of us, the chained, the central question in any candidate underlying theory is of course what it implies for our observable world. In this thesis, we will address aspects of this problem in the light of the quite drastic new developments in string theory the past few years.

1.1 Why (not) strings?

1.1.1 Why

The fundamental laws of physics needed to understand and predict the results of experiments in present day particle accelerators are to an impressively high degree of accuracy given by the so called *Standard Model* of elementary particles. This describes all known elementary particles together with the three fundamental interactions between them which are relevant at the energies which can now be reached in accelerators: the electromagnetic, weak and strong interaction.

The framework of the Standard Model is quantum field theory. In a quantum field theory, elementary particles are represented as point-like objects, interacting with each other by emitting and absorbing other particles; for example the electromagnetic force between electrons is due to exchange of photons. Each absorption or emission has a certain probability, proportional to a certain 'constant' g which only depends on the type of interaction and the energy scale[1] of the process. This constant has to be determined experimentally[2] and is called the (effective) *coupling constant* of the interaction. For electromagnetic interactions at low energies this is $g \approx 0.303$. Clearly, the smaller the coupling constant, the smaller the emission/absorption probability, and the weaker the interaction force. If, as for the electromagnetic interaction at presently accessible energies, g is sufficiently small, one can hope to calculate with good accuracy the outcome of e.g. particle collision experiments as a truncated power series in g. Such calculations are called *perturbative*, and are in most quantum field theories the only possible way to extract precise quantitative predictions. There exists an attractive diagrammatic representation of such series, due to Feynman, where one has to sum over all possible intermediate emission/absorption processes, ordered according to the number of interactions.

For electromagnetic and weak interactions, perturbation theory is useful at all accessible energies. For the strong interaction, this is not the case at low energies, but at high energies ($\gg 250$ MeV) it is. All in all, it turns out that the standard

[1] We have in mind here the center of mass energy of a particle collision experiment, say.

[2] At least for one value of the energy scale. Given all the particles species which exist in nature, the theory then provides the coupling constant at all other energies.

model gives a very good description of presently accessible particle physics, with strong predictive power, provided one plugs in a set of about twenty parameters (coupling constants and particle masses).

Despite its experimental successes, the Standard Model is very likely not the end of the story. The model shows some striking structures (such as the appearance of three fermion generations) begging for an explanation, which the theory itself cannot provide. It also contains about twenty a priori undetermined dimensionless adjustable external parameters, of which many have values which are eyebrow-raisingly unnatural in the context of the Standard Model.

The main reason to doubt about the Standard Model as the ultimate theory however, is that it fails to describe the gravitational interaction adequately. For gravity, the quantum field theory framework fails miserably, at least in perturbation theory. Roughly, one can understand this as follows. The gravitational force between two particles is proportional to their mass, and hence their energy. In other words, the effective coupling constant increases with increasing energy scale. This implies that gravity is unimportant at low energies, but on the other hand also that it rapidly grows strong at high energies. The characteristic energy scale determining the meaning of 'low' and 'high' here is the so called (four dimensional) *Planck mass* M_P, given in terms of the Newton constant G_N as

$$M_P = G_N^{-1/2} \approx 1.22 \times 10^{19} \mathrm{GeV} \qquad (1.1.1)$$

(we use units with $c = \hbar = 1$). As quantum mechanics allows arbitrarily large energy fluctuations, provided these are localized in a sufficiently small region of spacetime, and the point particle picture of quantum field theory on the other hand indeed allows two particles to be localized in an arbitrarily small region of spacetime, there is no limit on the effective gravitational interaction strength in intermediate emission/absorption processes during particle collisions. As a result, perturbation theory breaks down unless a high energy (or short distance) cutoff is artificially introduced. But this in turn introduces a large degree of arbitrariness in the theory, greatly reducing its predictive power, and annihilating its credibility as a truly fundamental theory.

There are two possible resolutions. The first one is that the problem with gravity is an artifact of perturbation theory, and that it disappears when the theory is solved exactly. Evidently, the latter is not an easy task, and this program has been unsuccessful thus far. The second possibility is that quantum field theory is simply not the right description of physics all the way down to distance scales of order M_P^{-1}, where gravity becomes important. Somehow, interactions should be 'smeared out' in a natural way, avoiding the short distance divergence of the gravitational interaction, and allowing a consistent perturbative description of gravity.

There is at present only one known way to achieve this, and that is string theory.[3] The idea behind string theory is simply to replace the different particle species, (perturbatively) interacting with each other via emission and absorption of other particles, by different vibrational modes of a single kind of string, interacting simply by splitting and joining of strings. Instead of one dimensional particle worldlines connected to each other in interaction vertices, we now get smooth 'worldsheets' swept out by the strings in spacetime. The theory is thus extremely simple in its basic ingredients, but nevertheless, when one tries to set up such a consistent theory of strings (which is not manifestly incompatible with observation), one *necessarily* finds the following rich set of features:

1. *Gravity.* Every consistent string theory contains a state with the properties of a graviton (the particle mediating the gravitational force), whose interactions reduce *at low energies* to general relativity.

2. *Finite perturbation theory.* Due to the effective smearing out of interactions, string theory gives a perturbation theory which is finite order by order. In particular, it provides a finite, unitary perturbative description of quantum gravity, in sharp contrast with field theory.

3. *Unification.* Apart from the graviton and gravity, string theory also leads to other particles and forces, including those of the Standard Model. There are also particles and interactions not present in the Standard Model, but those do not (necessarily) contradict observations. All forces are thus described on the same footing.

4. *Extra dimensions.* String theory requires a definite number of spacetime dimensions. For all theories of which consistency has been established, this number is 10. To be consistent with observations, six of those have to be so small that they cannot be resolved by present[4] day experiments. This is compatible with the consistency constraints of the theory. The extra dimensions give rise to additional particles and interactions in the four visible' dimensions, depending on the geometry of the compact internal space. Some of the allowed geometries produce the particle spectrum of the Standard Model. Recall Plato's cave.

5. *Supersymmetry.* All established consistent string theories are supersymmetric, meaning that there is a fundamental symmetry between bosonic and

[3]We use the word string theory here in a broad sense, including candidate nonperturbative extensions such as the matrix model [2].

[4]They could be discovered in the next generation of accelerator experiments however (e.g. at LHC): the appearance of TeV scale effects of extra dimensions are not excluded by present day observations.

fermionic degrees of freedom in the theory. This symmetry is clearly broken in the world as we know it, but there is quite some evidence in favor of its presence at the level of the fundamental theory, from theoretical as well as experimental considerations. There is hope that the LHC collider at CERN, under construction at the time of writing, will settle this issue.

6. *No free parameters.* String theory has no adjustable external dimensionless constants. It only has one fundamental energy scale (at least in perturbation theory), identified with the tension of the string, and called the *string scale* M_S. Sometimes the string scale is identified with the Planck scale M_P, but this is wrong in general (see below). Now though there are no external adjustable constants, there are in a certain sense 'internal' adjustable 'constants': string theory seems to allow a huge continuous family of different consistent 'vacua', with different expectation values of certain massless fields (which can be considered as 'condensates' of the corresponding massless particles). Particular examples are the metric $g_{\mu\nu}$ and the 'dilaton' field Φ. The vacuum expectation value of the exponential of the latter, $g_s \equiv \langle e^\Phi \rangle$, appears as a sort of coupling constant in string perturbation theory: a string worldsheet with γ donut holes and n boundary components gives a contribution proportional to $g_s^{-2+2g+n}$.[5] String perturbation theory is therefore accurate when g_s is small. Incidentally, the ratio of the ten dimensional Planck scale and the string scale turns out to be $M_{P,10}/M_S \sim g_s^{-1/4}$, implying that the string mass scale is always smaller than the (10D) Planck scale at weak string coupling.

7. *Uniqueness.* Though there are several distinct perturbative string theories, those are now believed to be just different perturbative expansions about different vacua of a single underlying theory, of which the fundamental formulation is not yet known.

1.1.2 Why not

Despite its theoretical successes, also string theory is very likely not the end of the story, at least not in a formulation where strings are the truly fundamental degrees of freedom. There are some objections against string theory as described above:

1. The theory is defined in an intrinsically perturbative way: scattering probabilites of particles are given by an asymptotic series in powers of the string coupling constant g_s. The kth order term in this series corresponds to a sum

[5] The number $\chi \equiv -2 + 2g + n$ is called the *Euler characteristic* of the surface.

over different possible string splitting and joining processes during the collision, where each possibility sweeps out a worldsheet in spacetime of fixed Euler characteristic $-2+2g+n=k$. As usual for a perturbative series, this only makes sense for sufficiently small coupling constant g_s. Moreover, perturbation theory will clearly fail to give an adequate description of the physics as soon as there are energies involved which are sufficiently high to produce states which are *not* in the 'elementary', perturbative spectrum of the theory. The prototype examples of such states are the magnetic monopole in 'ordinary' quantum field theory, and the black hole in general relativity. On top of that, any perturbative power series in a coupling constant g will miss contributions to the observables which do not have a Taylor expansion about $g=0$, the most notable example being the so called *instanton* contributions, which are typically of the form $e^{-\frac{1}{g^n}S_{cl}}$, with $n \geq 1$, $S_{cl} > 0$. As the string coupling constant g_s can take arbitrary values, depending on the chosen vacuum, there definitely exists a sector of the theory where the perturbative formulation breaks down completely. In absence of a non-perturbative formulation, this is evidently a serious problem.

2. Unlike particles, which can be given a non-perturbative, background independent, and at the same time very convenient description by going to quantum field theory, strings apparently do not have such a second quantized formulation, at least not of comparable elegance and simplicity. There has been substantial work on this subject [3], but the yield has been rather disappointing, and at present there is a widespread belief that second quantized strings are *not* the fundamental degrees of freedom needed to go beyond perturbative string theory.

3. In its perturbative formulation, string theory has a plethora of energetically degenerate vacua, which strongly reduce its predictive power. One hopes that supersymmetry breaking and nonperturbative effects will lift most or even all of this degeneracy, though this is far from sure.

4. Despite recent successes in explaining the thermodynamics of some near-supersymmetric black holes, including a microscopic derivation of the Beckenstein-Hawking entropy, a direct understanding of the quantum mechanics of arbitrary black holes is still lacking.

5. The extreme richness and equally extreme tightness of the theory itself hints to the existence of underlying organizing principles which are not known yet.

6. String theory fails as miserably as its predecessors in accounting for the ridiculously low experimental value of the cosmological constant, given the fact that supersymmetry should be sufficiently strongly broken to produce

particle masses of the order of those appearing in the Standard Model. This seems hard to avoid, even beyond perturbation theory, and may well be the most outstanding puzzle of present day theoretical high energy physics.

1.1.3 Beyond perturbation theory

So there are enough reasons to go on and try to get some insight in aspects of nonperturbative string theory, whatever the fundamental underlying theory will turn out to be. There has really been a lot of progress here during the past five years. This is largely due to the discovery of *string dualities* and *D-branes*,[6] making it possible to probe many aspects of nonperturbative physics within the framework of string perturbation theory.

Let us start by explaining the concept of a *Dirichlet p-brane* [4], D-brane in short. Instead of introducing its definition *ad hoc*, we will try to argue for its existence and usefulness in string theory from physical arguments. Consider therefore first a *black p-brane* in an asymptotically flat, noncompact, ten dimensional spacetime, with $p \leq 7$. This is an object infinitely extended in $p+1$ dimensions (1 time and p spatial dimensions) with the property that the gravitational force is so strong in its vicinity that anything coming too close can never escape again. More precisely, there exists a certain $8-p$ dimensional surface surrounding the object, called the *horizon*, beyond which return tickets are no longer available. A pulse signal sent out by somebody falling in will arrive to a distant observer with ever decreasing frequency (as escaping becomes increasingly difficult for the signal), eventually coming to a standstill when the person falling in reaches the horizon (so from the point of view of the external observer, the person falling in actually never passes through the horizon). One says there is an 'infinite gravitational redshift' at the horizon. Since frequency is tantamount to energy, this implies that any object with finite energy with respect to a freely falling observer *at* the horizon, will have *zero* energy with respect to a distant observer. Such black p-branes are well known solutions of classical gravity theories; they are the higher dimensional generalizations of the $p=0$ case, the black hole.

One can also consider multiple black p-brane configurations, but these will in general be unstable. Only when the gravitational attraction is exactly canceled by another, repulsive force between the branes, such a system can remain in equilibrium. In supersymmetric theories, such configurations can easily be realized by taking a number of so-called *BPS p-branes*, aligned parallel to each other, but otherwise at arbitrary relative positions. Mass and charge densities of a *BPS* p-brane satisfy a very specific relation, saturating a lower bound on the mass density for the given charge density.

[6] The suffix 'brane' stems from the word 'mem-brane'.

Imagine now two parallel BPS p-branes and a number of closed strings floating around. Some of the closed strings will be captured by the first p-brane, some by the second one, and some will float around in spacetime forever. Also, some will be captured by *both* p-branes at the same time, that is, one part of the string will be stuck to the horizon of the first brane, while the other part will be stuck to the horizon of the second brane. As argued above, the parts of the string at the horizons do not contribute to the total energy, due to the infinite redshift (at least if near horizon interactions meet some finiteness conditions). All energy resides in the part of the string stretched between the two branes. Such a (closed) string will thus effectively behave very much like two open strings with their endpoints confined to the brane horizons. Moving one of those two components off to infinity along one of the p noncompact dimensions of the brane[7], we are left with one effectively open string stretched between the p-branes.

In the following, we always choose our energy units such that the string mass scale M_S equals 1. Such units are called string units. Now suppose we send the string coupling constant g_s to zero while keeping the p-branes of fixed charge density and at fixed distance from each other. As the 10D Planck length $M_{P,10}^{-1} \sim g_s^{1/4}$, which determines the strength of gravity, goes to zero (in string units) when $g_s \to 0$, spacetime at any fixed nonzero distance from the brane becomes flat, at least if the BPS mass density does not grow too fast when $g_s \to 0$.[8] If this is the case, then when $g_S = 0$, we are simply left with flat spacetime containing two rigid $p+1$ dimensional objects between which open strings can be stretched. Apart from the stretched open strings, there can also be closed strings floating around in spacetime, as well as open strings with both endpoints on one brane. Thus to lowest order in string perturbation theory, such p-branes merely provide Dirichlet boundary conditions for string worldsheets propagating in trivial flat spacetime, hence their name: Dirichlet p-branes, or simply D-branes.

There are other ways to arrive at D-branes in string theory, for example via T-duality (see below) or the supersymmetry algebra. The idea of a D-brane being established, one can try to use the D-brane prescription in string perturbation theory in other circumstances, such as curved backgrounds, D-branes wrapped around compact dimensions, coincident D-branes and so on. Quantizing the strings ending on D-branes yields a spectrum of excitations which are interpreted as the degrees of freedom of the brane, just as closed strings in bulk spacetime yield the degrees of freedom of the bulk (including fluctuations of its geometry). As usual in string theory, quantization gives also consistency conditions on the allowed D-

[7] This is of course not possible for $p = 0$, or for branes with finite spatial extent. Those require a separate discussion, but we will not go into this here.

[8] This depends on the details of the theory. For example in type IIB string theory, this requirement excludes the NS5-brane, which indeed cannot be represented by D-branes in string perturbation theory.

brane embeddings, which have the form of equations of motion for the D-branes. As could be expected intuitively, isolated D-branes have 'ripples' in their excitation spectrum, propagating as waves over the brane, with the obvious interpretation of embedding fluctuations. For $N > 1$ coincident D-branes, the massless degrees of freedom are more exotic: instead of simple coordinate fluctuations, one finds $N \times N$ *hermitean matrix fluctuations*, suggesting the emergence of noncommutative geometry.

Thus it becomes possible to study various aspects of the dynamics of branes — objects which are clearly not in the perturbative string Fock spectrum — within the realms of string perturbation theory. Furthermore, many aspects of their low energy dynamics can be extrapolated to strong string coupling, and their mere existence provides hints for string dualities (see below) and the nonperturbative structure of the theory. D-branes thus prove to be a very powerful tool in uncovering the mysteries of nonperturbative string theory.

The other cornerstone of the progress made in understanding nonperturbative string theory is *duality*. In short, duality is the physical equivalence of two seemingly different theories. Massive interest in dualities was sparked with the seminal work of Seiberg and Witten in [5, 6], where an *exact* expression was derived for the quantum low energy effective action of $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory. Of central importance in their work was a duality between $\mathcal{N} = 2$ $SU(2)$ Yang-Mills and an $\mathcal{N} = 2$ theory of magnetic monopoles coupled to a $U(1)$ abelian gauge theory. The former description is weakly coupled when the latter is strongly coupled and vice versa, making possible accurate perturbative calculations in one picture when one is far into the nonperturbative region of the other one! Convincing evidence accumulated that similar and even more powerful dualities are present in string theory. Equivalences emerged between weakly coupled string theories on different backgrounds (this is called T-duality and includes mirror symmetry), and between weak and (extrapolated) strong coupling regimes of the same or even 'different' string theories (this is called S-duality, strong-weak coupling duality, or string-string duality, depending on context and author). Gradually, a picture developed in which all known perturbative string theories can be understood as convenient perturbative expansions about different (perturbative) vacua (or perhaps better: in different regimes) of the *same* underlying theory. With some luck, scattering amplitudes in a vacuum far outside the validity region of one perturbative string picture can simply be calculated perturbatively in another picture. This does *not* mean that for any value of the vacuum parameters, we have a suitable perturbative picture available, as there are vacua of 'intermediate' coupling where none of the perturbative string theories is adequate, but at least it limits how exotic the theory can get at strong coupling. Perhaps the situation here is, morally speaking, comparable to the description of water in different phases.[9] Depending on whether we have the solid, the liquid or the gas phase, totally different (approximate) descriptions are appropriate, though the fundamental laws are of course always the same. The fact that we have here a 'duality' between ice, water and steam does however not mean that we can all of a sudden calculate all we want about e.g. water turbulence. But for somebody who would have learned the theory of ice, the theory of water and the theory of steam as three distinct items in a difficult green textbook, the revelation that they are merely different descriptions of the same fundamental thing, would definitely be quite exciting!

T-dualities, relating two perturbative descriptions in different background geometries, are pretty well under control, and in many cases rigorously established order by order in perturbation theory. S-dualities on the other hand are equally difficult to tackle as it would be to demonstrate theoretically the presence of a single set of fundamental laws describing both ice and steam, in the absence of a theory of atoms. Consequently, S-dualities are still largely conjectural, though indirect tests and their aesthetic attractiveness have placed their existence — at least in a rough form — beyond reasonable doubt.

Another kind of duality has emerged about two years ago: the Maldacena correspondence [7]. This relates a string theory on a certain curved background to a certain *lower dimensional field theory* without gravity, thus realizing in a certain sense the holographic principle of 't Hooft [8] and Susskind [9]. This principle states in its original form that the fundamental degrees of freedom of a certain region of space should actually live on the *boundary* of this region, with about one binary degree of freedom per unit Planck area.

Given all these dualities between string theories, the prize question is of course what the underlying 'molecular' theory is. This as yet elusive theory has tentatively been given the name *M-theory*.

1.2 Strings and low energy physics

One of the main results of renormalization theory is that physics at low energies is, apart from a few parameters, independent of the details of physics at high energies. This is fortunate, since it allows us to predict the outcome of present day particle collision experiments without having to know anything about Planck scale physics, but on the other hand, it also implies that these experiments will

[9] This simple analogy of course fails on many points: we are not (necessarily) speaking about different 'phases' of string theory (there can even be regimes in which several adequate descriptions overlap) and the richness and power of string dualities is of course substantially bigger.

teach us virtually nothing about string theory or whatever it is that governs the Planck scale.

This does not mean string theory is useless at low energy scales. There are of course the twenty undetermined parameters of the Standard Model which string theory *might* provide, but we are still far from achieving this. However, string theory has turned out to be useful for low energy physics in a completely different way, namely as a powerful geometrical tool in analyzing nonperturbative aspects of quantum field theories; rather surprisingly, string theory leads to quantum field theory results which would be virtually impossible to obtain within the conventional framework of quantum field theory itself! There are two lines of attack here. One is to exploit the appearance of nonabelian Yang-Mills theories in the description of the dynamics of coincident noncompact branes, and to make use of various string theory results and dualities to derive quantum aspects of those [10]. Typical results thus obtained are exact low energy effective actions (reproducing and extending the Seiberg-Witten solution of low energy quantum $\mathcal{N} = 2$ $SU(2)$ Yang-Mills), the BPS spectrum of the theory and the phase structures of gauge theories, including confinement phases. The results are mostly restricted to supersymmetric theories and — probably related to this fact — very elegant and geometric in nature. The restriction to low energies, and to a certain extent also the restriction to supersymmetric theories, is avoided in approaches based on the Maldacena correspondence. The other line of attack [11, 12, 13, 14, 15, 16] — the one we will follow in this thesis — is to make use of the representation of massive charged (gauge) particles as branes wrapped around nontrivial cycles of the six compact dimensions of spacetime, and of various nonrenormalization theorems together with string dualities and geometrical constructions of the four dimensional low energy effective action. This approach is usually a bit more involved, but has the advantage that a larger class of field theories can be studied, and that gravity can also be included. Here as well, the results are very elegant but mostly restricted to supersymmetric theories.

1.3 Outline and summary of results

In this thesis, we will study some aspects of low energy physics extracted from string theory. We will focus on two subjects: the derivation of low energy effective actions including gravity, and low energy properties of charged (BPS) particle states. We will work almost exclusively in the context of *type IIB string theory* in a spacetime with six compact dimensions forming a *Calabi-Yau manifold*. Type IIB theory is a perturbative string theory which has a thirty two supersymmetry generators when spacetime is flat. A Calabi-Yau manifold of (real) dimension six is a space with the property that the group of transformations on vectors induced by parallel transport along closed loops is isomorphic to $SU(3)$ (instead of the 'usual' $SO(6)$). Such a manifold has a metric satisfying the vacuum Einstein equations. The direct product of four dimensional Minkowsky space with a Calabi-Yau manifold is an exact solution of the string theory equations of motion. In such a background, type IIB string theory remains invariant under eight supersymmetry generators, yielding at low energies an $\mathcal{N} = 2$ supersymmetric theory in the four noncompact dimensions. We will furthermore restrict most of the time to the so-called *vectormultiplet sector* of the theory. The reason why we make these particular choices is that they provide a setting in which exact low energy results can be obtained which are at the same time very nontrivial.

The outline of this thesis is as follows.

In *chapter 2* the concept of an effective action is introduced, both in field and in string theory. The difference and relation between the one particle irreducible and the Wilsonian effective action are outlined and the meaning of the Seiberg-Witten effective action is explained in this context. Rigid and local special geometry are defined and their central role in four dimensional $\mathcal{N} = 2$ Yang-Mills resp. supergravity actions is discussed. Following our work in [17, 18], we show in general how rigid special geometry arises as a certain limit of local special geometry. We conclude the chapter with a compendium of Seiberg-Witten theory and its generalizations. This chapter contains mainly known material, but some effort is done to tie up some loose ends in the usual Seiberg-Witten review literature.

In *chapter 3* we study in great detail Calabi-Yau compactifications of type IIB string theory. The bosonic four dimensional massless spectrum and its low energy effective action is derived. The result is well known, but we use a formalism which is more intrinsically geometric than usual. This is especially useful to discuss BPS states obtained from wrapping D3-branes around nontrivial cycles in the Calabi-Yau manifold. Before we get to this, we elaborate on how one can (pragmatically) describe a 'state' in string theory, and in particular how this relates to classical backgrounds. Next, we focus on BPS states in four dimensional $\mathcal{N} = 2$ theories and the supersymmetry multiplets in which they are organized. It is explained how they are realized as special Lagrangian embeddings of 3-branes in the Calabi-Yau manifold. Following our work in [20], we derive the precise reduction of the (bosonic part of the) wrapped D3-brane action to a BPS particle action in four dimensions. Here in addition, we also discuss the curvature couplings. As an application, we calculate the electromagnetic force between two moving BPS test particles with arbitrary charges. Finally, we discuss the problem of the determination of the type of supersymmetry multiplet to which a certain wrapped D-brane gives rise. We present an easy shortcut for this which can be applied in some cases.

Most of the remaining part of the chapter is devoted to the effective field theory picture of the BPS states, and in particular the *attractor mechanism*. We derive the spherically symmetric equations of motion in an invariant geometrical framework, slightly relaxing the usual ansatz made in the literature, and in particular we work out in detail the analogy with the dynamics of a nonrelativistic particle moving in moduli space. We use this to give an intuitive discussion of the general solution, including non-BPS black holes. We point out that the attractor mechanism does not arise as a result of damped motion (as is often claimed in the literature), but rather as a result of the finite energy condition together with the unstability of the system, as usual in soliton physics. Next we turn to the discussion of some BPS solutions, with special emphasis on the non-generic ones, and we obtain a novel solution which first appeared in our work in [21] and which will play a prominent role in chapter 5. We end this part on attractors with a discussion of the multicenter case and the presentation of some powerful techniques for solving the attractor flow equations, providing an intrinsic geometrical, Kähler gauge invariant formulation of the methods developed in [22, 23].

We conclude the chapter by presenting a discussion of the validity of D-brane and field theory pictures in this specific setting. We sketch the Maldacena correspondence is this context. When the same reasoning is applied to non-black hole BPS states in four dimensions, we seem to find a new kind of correspondence, which is reminiscent of the well known mathematical 'Nahm duality' between N monopoles and a certain $N \times N$ hermitean matrix system. We leave this as an intriguing open issue.

In *chapter 4* we study in detail how the low energy effective action of $\mathcal{N} = 2$ quantum Yang-Mills theory, weakly coupled to gravity, is obtained from type IIB string theory compactified on a Calabi-Yau manifold. This is based on the well known geometrical engineering techniques [11, 12, 13, 14]. The main novel features here are the explicit coupling to gravity and to the 'dynamical dynamically generated scale', and the fact that we derive everything directly and completely within the type IIB theory, without restricting to local considerations and/or invoking S- or T- dualities. Also the reduction of local to rigid special geometry is demonstrated explicitly. Our approach is down-to-earth but fairly general.

We start by presenting a detailed derivation of the D-brane and Calabi-Yau geometries needed to get the correct light Yang-Mills spectra in four dimensions. This involves some singularity theory. Next we prove the reduction of local to rigid special geometry, and we obtain the low energy effective action, including the (weak) coupling to gravity and the dynamical dynamically generated scale. This reproduces and extends the Seiberg-Witten solution. We go on by analyzing the various unification scales which emerge, and we make the connection with the (S-)dual heterotic string picture. To get some insight in the physical content of our results, we plug in some experimental data (though the $\mathcal{N} = 2$ models we are studying of course do not really match phenomenological observations). We end the chapter with a detailed case study of an explicit example, based on our work in [17], in which cycles, periods, monodromies and special geometry data are constructed explicitly, with results supporting our general discussion.

Finally, in *chapter 5*, we study BPS states in the effective field theory picture at weak gravity. Most results here are new. We reconsider the effective particle picture of the spherically symmetric equations of motion in the weak gravity limit. We point out an interesting $\mathcal{O}(G_N)$ correction to the effective action which is, from the microscopic quantum field theory point of view, purely due to the backreaction of *quantum* fluctuations on the dynamical scale. The weak gravity attractor flow equations are used to give a field theoretic description of Strominger's 'massless black holes', moving at the speed of light. We further specialize to the exactly rigid limit, where gravity is decoupled completely, with as prototype example $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory. Only for monopoles and elementary dyons, a spherically symmetric BPS solution exists. Some properties of these solutions are investigated, and we establish in particular the existence of stable equilibrium points of various charged test particles at finite distance from the monopole or elementary dyon core. Next, we investigate in detail what goes wrong in trying to construct spherically symmetric solutions for the other charges. A picture of those states (which include the W-bosons) as bound states of the monopole and the elementary dyon emerges, making contact with the so-called 3-pronged string representation of BPS states. We conclude the chapter with the start of a discussion on the low energy dynamics of N monopoles. We give a general formula for the moduli space metric and evaluate this for $N = 1$, producing the expected result.

A note

Though a substantial part of the results in this thesis require rather sophisticated geometrical techniques, we have tried to make the discussion everywhere as physical and intuitive as possible. Sometimes, this has been at the price of a certain loss of rigor. However, as the power and importance of good intuitive pictures in physics can hardly be overestimated, we hope the benefits outweigh the costs in this case.

Chapter 2

Low energy effective field theories

At sufficiently low[1] energies, any reasonable physical theory satisfying some basic assumptions like relativistic invariance, locality and unitarity, can be given an effective quantum field theory description (see e.g. [24, 25]). In principle, for any theory with asymptotic particle states, one can even construct a quantum effective "field theory" from which one can calculate any *quantum* scattering amplitude of the original theory at the *classical* level of the effective theory, that is, in the limit $\hbar \to 0$, or, diagrammatically, at tree level (no loops). In practice however, such quantum effective theories are usually extremely nonlocal, ugly and complicated, and extremely difficult to find (since they are equivalent to solving all amplitudes of the full quantum theory). However, again at sufficiently low[2] energies, one can approximate the full quantum effective theory by a local field theory described by an action with at most two derivative and four fermion terms (two fermions is equivalent to one derivative). One can of course refine the approximation by adding higher derivative terms. Such two derivative effective field theories are usually still very difficult to find (since they are equivalent to solving all amplitudes at very low energy), and one has to make still further approximations, usually in the form of a perturbative series in powers of the coupling constant. However, when sufficient supersymmetry is present, it turns out to be possible to find the exact two derivative low energy quantum effective action in some nontrivial cases, thanks to

[1] The meaning of 'low' is very context dependent. In string theory, it usually means 'lower than the energy of any massive state of the theory'. Generically, this is the Planck scale, but as we will see, there are important exceptions.

[2] again much lower than any relevant mass scale in the theory

Figure 2.1: The general idea behind low energy effective actions is that the effects of the virtual heavy particles can be summarized in effective interactions between the light particles, obtained by integrating out the heavy ones.

the extremely strong constraints supersymmetry puts on the form of those effective actions. For example for $\mathcal{N} = 2$ theories in four dimensions, supersymmetry imposes the structure of *special Kähler geometry* on the two derivative part of the action [26]. Seminal nontrivial examples of exact solutions are type II string theory compactified on a Calabi-Yau manifold and $\mathcal{N} = 2$ Yang-Mills theory in four dimensions (a review of both can be found for example in [15]; we do not intend to give complete references here).

Obviously, the low energy effective action of a theory provides a very powerful tool to extract various aspects of the low energy physics. Often, especially in string theory, the low energy effective theory also teaches us a lot about the structure and symmetries of the full theory itself.

Since effective field theories and their geometrical structure play a major role in this work, it is probably worth spending some time on the basics of this subject. In this chapter we will therefore review some basic facts about effective actions and special geometry. Though we will not go into much detail, we will attempt to clarify some (but not all!) conceptual issues concerning effective actions which are usually left obscure in the literature.

2.1 1PI, Wilson and Seiberg-Witten

The idea behind all low energy effective descriptions is to construct an action for the light fields which gives their quantum dynamics for low energetic excitations. One distinguishes essentially between two different kinds of effective actions: the generating functional of one particle irreducible connected diagrams, in short the

1PI effective action, and the Wilsonian effective action.

2.1.1 1PI effective action

In **field theory**, the exact 1PI effective action Γ can be defined as the Legendre transform of the logarithm of the partition function $W[J]$ (which in perturbation theory is the generating functional of connected diagrams). This is explained in any field theory textbook (e.g. [25]), so we will only briefly repeat the main features. Denoting the fields collectively by ψ^x, with x a generalized index (spacetime position and other indices), we thus define:

$$W[J] = \ln \int \mathcal{D}\psi \; e^{iS_{cl}[\psi]/\hbar + J_x \psi^x} \tag{2.1.1}$$

$$\Gamma[\Psi] = J_x[\Psi]\Psi^x - W[J[\Psi]] \tag{2.1.2}$$

where $J[\Psi]$ is given by inverting the relation

$$\Psi = \frac{\partial W}{\partial J}[J] = \langle \psi \rangle_J \tag{2.1.3}$$

that is, $J[\Psi]$ is the external current needed to induce the expectation value Ψ for the field ψ. One can show that Γ reproduces at tree level all scattering amplitudes of the original theory, that is, when (formally) used as if it were the classical action S_{cl}, Γ reproduces all original scattering amplitudes in the limit $\hbar \to 0$ [25]:

$$W[J] = \lim_{\hbar \to 0} W_\Gamma[J]. \tag{2.1.4}$$

Furthermore, the physical field expectation values (at $J = 0$) are stationary points of Γ. Thus Γ indeed deserves to be called the quantum effective action.

Now in perturbation theory any connected diagram in the expansion of $W[J]$ can be regarded as a tree, whose vertices consists of one particle irreducible subdiagrams. So in order for (2.1.4) to be correct, the perturbative expansion (in powers of some small coupling constant) of $\Gamma[\Psi]$ must be the sum of all 1PI connected diagrams with arbitrary numbers of external lines, each external line corresponding to a factor Ψ (rather than a propagator or wave function). Hence the part of Γ which can be written as a perturbative series is indeed the 1PI generating functional. Note however that our definition of Γ extends beyond perturbation theory; for example it can contain instanton contributions $\sim e^{-1/g^2}$, which are invisible in a power series expansion in the coupling constant g.

The two derivative low energy approximation of Γ can simply be obtained by putting all heavy fields to zero and expanding Γ to second order in the field momenta (or derivatives).

String theory doesn't have an analogous straightforward second quantized 'off-shell' field theoretic formulation. It has a certain (infinite) perturbative Fock particle spectrum however, to which one can associate fields as usual. The massless fields can be consistently used as background for the perturbative string worldsheet path integral, but only if they are *on shell*, that is, if they obey certain equations of motion, obtained by requiring Weyl invariance of the worldsheet theory (which is needed for its consistency).

One defines the '1PI' effective action Γ for these fields directly in perturbation theory, simply as the (or rather *a*)[3] spacetime action which at tree level reproduces the string scattering amplitudes. In particular, the quantum equations of motion, i.e. the minima of the effective action, are obtained by requiring the 'tadpole' amplitudes (amplitudes with one external particle) to vanish. Of course, to get an effective action which is the integral over a local Lagrangian density, one has to make a momentum expansion of the amplitudes, which by dimensional analysis is equivalent to an α' expansion.[4] To get something sensible, one thus has to restrict energies to be much lower than the string scale $1/\sqrt{\alpha'}$, making the effective field theory description of string theory only useful for fields much lighter than the string scale (these are all massless in a maximally symmetric vacuum of the theory).

Now in order for the above to make sense, the equations obtained from requiring worldsheet Weyl invariance up to order k in the string coupling constant, needed for setting up a consistent, finite, perturbation theory up to order k, should imply the equations of motion derived from the k-th order corrected effective action. Perhaps this statement needs first some clarification. One might think that, since Weyl invariance is a purely local property and the Weyl anomaly is a pure worldsheet UV effect, the requirement of Weyl invariance should in particular be independent of worldsheet topology and hence receive no correction beyond string tree level. This is not so. If one sums over topologies and integrates over the Riemann surface moduli space, as one should in string perturbation theory, one has to include worldsheets with arbitrarily small handles attached. These 'UV handles' contribute to the total Weyl anomaly, thus correcting the consistency equations of motion for the background (including particle mass corrections). Ignoring this contribution and simply taking the background needed for tree level conformal invariance results in divergent loop tadpoles and related amplitudes, so one really has to take the corrections to the background into account to get all amplitudes finite. This is known as the *Fischler-Susskind mechanism* (see [27] for a review). Fortunately, most supersymmetric backgrounds do not suffer from these corrections.

[3]Considering the fact that one can only directly define scattering amplitudes with all particles on shell in string theory, and on the other hand the fact that an action provides an (arbitrary) off-shell extension of these amplitudes, such an action cannot be unique.

[4]α' being the inverse of the string tension.

Let us first see how tree level conformal invariance indeed implies the tree level tadpoles to vanish. Here 'tree level' means the sphere contribution for closed and the disk contribution for open strings. The argument is taken from [28], p. 174. Consider for instance a genus 0 massless closed string tadpole amplitude, which can be represented as the expectation value $\langle V(0)\rangle_0$ of a massless closed string vertex operator V inserted at $z=0$ in a conformal field theory on the complex z-plane. Conformal invariance means in particular that this amplitude should be invariant under the rescaling $z \to \lambda z$. On the other hand, under such a transformation, a massless closed string vertex operator transforms as $V(z) \to \lambda^{-2} V(\lambda z)$. Therefore we have

$$\lambda^{-2}\langle V(0)\rangle_0 = \langle V(0)\rangle_0, \tag{2.1.5}$$

so $\langle V\rangle_0 = 0$, as we wanted to show.

For disk diagrams, the same trick can be used to show that tree level conformal invariance implies vanishing tadpoles inserted at the *boundary* of the worldsheet.

One can further argue [30] that the higher order corrections to the background needed for Weyl invariance of the full amplitudes at order k, are also precisely the corrections needed to have still vanishing tadpole amplitudes at this order. So, happily, the two ways of obtaining equations of motion from string perturbation theory give exactly the same results.

Note that the disk diagram is of order g_S^{-1}, while the sphere diagram is of order g_S^{-2}, where g_S is the string coupling constant. Therefore, at tree level in perturbation theory, the fields from the open strings will feel the fields from the closed strings, but conversely, the closed string fields will *not* feel the open string fields: the effect of their presence appears only as a correction $\sim g_S$. For theories with D-branes represented as spacetime defects on which strings can end, this implies that to lowest order in perturbation theory, one should *not* take into account the backreaction of the D-branes on the bulk fields; the bulk background should be a solution of the *sourceless* field equations. This is very fortunate, since it allows one to keep on doing flat space perturbation theory even when D-branes are present (as long as the string coupling constant is small), considering backreaction as small perturbations.

One cautionary note to end this part: from the above, one might get the impression that string theory is exactly equivalent to a certain (complicated, nonlocal) effective field theory, at all energies. In particular, its fundamendal degrees of freedom might seem to be describable as an infinite tower of fields of increasing mass, with interactions that are effectively cut off in the ultraviolet. This is not the case. Of course, the exact effective field theory will be horrendously nonlocal and can hardly be thought of as a field theory in the conventional sense. But much deeper is the fact that the construction of this 'field theory' depends completely on string perturbation theory, which is only defined as an asymptotic series, and which breaks down as soon as circumstances become too 'extreme'. In particular, perturbation theory will definitely have broken down when the circumstances are such that states (like particles) can be produced which are not in the perturbative (Fock) spectrum, e.g. in graviton-graviton scattering at energies which are sufficiently high to produce a black hole ([31] p. 209) or certain wrapped D-brane states. The latter possibility can even occur at arbitrarily low energies, as a matter of fact, even at zero energy![5] So we cannot draw any conclusion at this point about the true high energy fundamental degrees of freedom of the full nonperturbative 'string theory' (assuming such a thing exists), and certainly not that it would be just an infinite tower of fields.

Actually, we should be rather happy about this: equivalence with any field theory, even an effectively UV cutoff one, satisfying some minimal locality principles, would probably be incompatible with the holographic principle [8, 9], implying among other disastrous things that string theory would not be able to explain the Beckenstein-Hawking black hole entropy formula.

This brings us to the question: then what *is* string theory beyond perturbation theory? It is probably not a theory of (second quantized) strings. It could be M(atrix) theory [2]. It could be a more general holographic theory [7]. It could be something completely different. Nobody knows.

2.1.2 Wilsonian effective action

The idea of the Wilsonian effective action S_W is to split the degrees of freedom of our theory in 'heavy' and 'light' modes, and to 'integrate out' somehow the heavy modes so as to obtain an effective theory for the light modes alone. Thus roughly we define (in field theory):

$$e^{iS_W[\psi_{light}]} = \int \mathcal{D}\psi_{heavy}\; e^{iS[\psi]}. \tag{2.1.6}$$

The scattering amplitudes for the light modes are then obtained by quantizing the theory given by the 'classical' action S_W. The two derivative low energy Wilsonian effective action is obtained by making a two derivative approximation of S_W. As an example, the quantum field theory describing the Standard Model can be considered as a low energy Wilsonian effective theory derived from whatever

[5]This is the case for example for string theory compactified on a Calabi-Yau manifold with a conifold singularity: D-branes (which are nonperturbative states) wrapping the vanishing cycle have zero mass, so string perturbation theory breaks down at zero energy, which explains why the low energy theories of such compactifications are in fact singular. See chapters 3 and 4 for a more elaborate discussion.

will turn out the fundamental theory of nature, valid well below the energy scale where the details of this fundamental theory become important (which is at most the Planck scale, since there ordinary field theory obviously breaks down).

Clearly, there is some vagueness in this definition. For example the split in heavy and light degrees of freedom can be done by introducing a scale μ and integrating out all Fourier modes of the fields above μ, or by splitting the fields in light and heavy fields and integrating out completely the heavy ones, or in any other convenient way, depending on the case at hand. However, as we will see, for the two derivative approximation, in favorable circumstances (sufficient supersymmetry), these ambiguities do not matter.

One can also define a Wilsonian effective action in string theory[6], namely as the effective action obtained as above, but restricting the string path integral to the massive string states. This is somewhat artificial of course and indeed this Wilsonian effective action does not respect some of the symmetries of string theory. On the other hand, the 'very low' energy part (two derivative for example) has sometimes useful holomorphicity properties which the (more physical) 1PI effective action does not necessarily have ([31] p. 300).

2.1.3 Seiberg-Witten effective action

To make all this a bit clearer, and to discuss the relation between S_W and Γ (in field theory), let us consider the example of $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory in four dimensions. This theory was solved in the two derivative low energy approximation by Seiberg and Witten in [5]. We will wait to give their solution till we have discussed some necessary technicalities of special geometry, but we will already use some properties for the purpose of illustration here. For simplicity, we will only consider the bosonic degrees of freedom, but it is of course thanks to the 'balancing' fermionic degrees of freedom that this theory is so well behaved and under control at the quantum level.

The bosonic fields of $\mathcal{N}=2$ $SU(2)$ pure Yang-Mills are a complex scalar triplet ϕ^a, $a=1,\ldots,3$ and a vector triplet A^a_μ, both transforming in the adjoint of $SU(2)$ (the fermionic fields are two $SU(2)$ adjoint Weyl fermion triplets ψ^a and λ^a, but we will not consider those further here). The bosonic part of the classical

[6]In a sense of course, the '1PI' effective action for string theory *is* also Wilsonian, since it leaves out even in principle the (usually massive) nonperturbative string states, and breaks down together with perturbation theory at the energy scale of these states. Also, in practice, even the perturbative massive fields are not included, and the action is written down in a low energy expansion. Perhaps we should call the action defined here 'string perturbative Wilsonian action'.

action is

$$S = -\frac{1}{g^2}\int D\phi^a \wedge *D\bar{\phi}^a - \int \frac{1}{2g^2}F^a \wedge *F^a + \frac{\theta}{16\pi^2}F^a \wedge F^a \quad (2.1.7)$$

$$+\frac{1}{4g^2}\int d^4x(\epsilon^{abc}\phi^b\bar{\phi}^c)^2, \quad (2.1.8)$$

where

$$D\phi^a = \mathrm{d}\phi^a + \epsilon^{abc}A^b\phi^c \quad (2.1.9)$$

$$F^a = \mathrm{d}A^a + \epsilon^{abc}A^b \wedge A^c. \quad (2.1.10)$$

This theory has 'flat directions', that is, a continuous family of classical minimal energy configurations, namely

$$\phi^a = \text{const.} \quad (2.1.11)$$

$$\epsilon^{abc}\phi^b\bar{\phi}^c = 0. \quad (2.1.12)$$

One can eliminate such flat directions from the path integral by fixing ϕ at spatial infinity. Take $\phi_\infty = a\delta_3^a$. Note that, by the Higgs effect, for nonzero a, the fields ϕ^1, ϕ^2, A_μ^1, A_μ^2 and their superpartners are massive with mass $\sim |a|$. We will define our 'Wilsonian' effective action here as the action obtained by integrating out *completely* those massive fields, and integrating out the momentum (Fourier) modes of the massless fields above a scale μ. The range of validity of our effective theory is then for energies lower than μ and the W-boson mass scale $|a|$. Denote from now on with ϕ the low momentum modes of ϕ^3, and with A_μ those of A_μ^3. We thus have:

$$e^{iS_W[\phi,A]} = \int \mathcal{D}\phi^3_{p>\mu}\mathcal{D}\phi^1\mathcal{D}\phi^2 \cdots e^{iS[\phi^a,A^a,\ldots]}. \quad (2.1.13)$$

It can be shown that the supersymmetry prevents the generation of an effective potential for the massless fields. That is, the flat directions remain flat and the derivative (momentum) expansion of the bosonic part of S_W gives

$$S_W = -\int \frac{1}{g^2(\phi/\Lambda)}\mathrm{d}\phi \wedge *\mathrm{d}\bar{\phi}$$

$$-\int \frac{1}{2g^2(\phi/\Lambda)}F \wedge *F + \frac{\theta(\phi/\Lambda)}{16\pi^2}F \wedge F + \cdots \quad (2.1.14)$$

where the dots indicate higher derivative terms, and $g(\phi/\Lambda)$, $\theta(\phi/\Lambda)$ are the field-dependent effective coupling constant resp. theta-angle, which also depend on

Λ, the dynamically generated 'scale'[7] of the theory, replacing the classical $\tau = \theta/2\pi + i4\pi/g^2$ as a free parameter of the theory.

An exact expression for the two derivative part of S_W was derived by Seiberg and Witten in [5] from supersymmetry and a number of physical consistency arguments (see section 2.3). Their arguments are valid for any strictly positive value of the massless momentum cutoff scale μ, and since those arguments determine the two derivative part of S_W uniquely,[8] we conclude that the two derivative part of S_W is actually independent of μ. Since lowering μ means including more quantum corrections from the massless field loops to S_W, this is equivalent to the statement that the two derivative part of S_W does not receive quantum correction from the massless fields (at nonzero momentum). Presumably, this nonrenormalization theorem can be argued directly from supersymmetry. Note for example that there are no 1-loop diagrams of the microscopic theory involving the massless fields only. Since it is known that, thanks to supersymmetry, there are no perturbative corrections beyond one loop, this implies indeed that the only *perturbative* corrections to the effective action of the massless fields come from (single) *massive* particle loops. It is not clear to us how to extend this argument to nonperturbative corrections, so we leave this point open. See [32] however for a possible starting point to the relevant literature.

This nonrenormalisation theorem also implies that we can calculate the exact quantum scattering amplitudes of the massless particles in the (very) low energy limit simply at tree level from the two derivative part of the Wilsonian effective action S_W. Or in other words, the 1PI quantum effective action $\Gamma_W[\Psi]$ obtained by taking S_W as 'classical action' (with Ψ and J only consisting of modes with momentum below μ and only path integrating over the 'remaining' modes with momentum below μ) has the same two derivative part as the 'tree level' effective action $S_W[\Psi]$.[9] This eliminates already one of the ambiguities of the interpretation of the Seiberg-Witten effective action.

One could also wonder whether the 1PI effective action Γ_W obtained by using S_W as classical action is the same as the full 1PI effective action Γ with the massive fields put to zero. This is indeed the case: denoting all fields collectively as ψ^a

[7] Λ is a complex number, so it is not really a scale. It's rather its modulus, $|\Lambda|$, which deserves this name. We will however follow common parlance and call Λ the scale anyway. The complex character of Λ is needed because it takes over the role of the complex free parameter $\tau = \theta/2\pi + i4\pi/g^2$ of the classical theory.

[8] Actually, even if the action is written in the form independent of a scalar vacuum expectation value, their arguments only fix a *family* of actions, differing from each other by the dynamically generated scale Λ. So in the following, when we talk about 'the' Seiberg-Witten effective action, we actually mean this complete family.

[9] There is a loophole here: our argument does not exclude that there are extra contributions to Γ_W from loops at *zero* momentum. These and other infrared subtleties do indeed occur in generic $\mathcal{N} = 1$ supersymmetric theories, but apparently not here [32].

with a the adjoint $SU(2)$ index, we have

$$\Gamma[\Psi^a] = \int J_a[\Psi]\Psi^a - \ln \int \mathcal{D}\psi \; e^{iS_{cl}[\psi]+\int J_a\psi^a} \tag{2.1.15}$$

where $J_a[\Psi]$ is the external current needed to induce the vacuum expectation value Ψ^a for ψ^a. Now if $J_1 = J_2 = 0$ (and $a = \phi^3(\infty) \neq 0$), we always have $\langle\psi^1\rangle = \langle\psi^2\rangle = 0$, no matter what J_3 is. This follows physically from the fact that excitations of the ψ^3 fields alone do not act as sources for the ψ^1 and ψ^2 fields, and mathematically from the invariance of $S_{cl}[\psi] + \int J_a\psi^a$, the P.I. measure and the boundary condition $\phi^3(\infty) = a$ (and hence of Γ) under $\psi^1 \to -\psi^1$, $\psi^2 \to -\psi^2$ when $J_1 = J_2 = 0$, so that indeed $\langle\psi^1\rangle = -\langle\psi^1\rangle = 0$ and $\langle\psi^2\rangle = -\langle\psi^2\rangle = 0$. So the external current J needed to induce the vacuum expectation value zero for ψ^1 and ψ^2 has simply $J_1 = J_2 = 0$, that is: $J^{1,2}[\Psi^1 = \Psi^2 = 0] = 0$. Therefore (with Ψ and J^3 only consisting of modes with momentum lower than μ):

$$\begin{aligned}
& \Gamma[\Psi^3 = \Psi, \Psi^1 = \Psi^2 = 0] && (2.1.16)\\
= & \int J_3[\Psi]\Psi - \ln \int \mathcal{D}\psi^3_{p<\mu}\mathcal{D}\psi^3_{p>\mu}\mathcal{D}\psi^1\mathcal{D}\psi^2 \; e^{iS_{cl}[\psi]+\int J_3[\Psi]\psi^3} && (2.1.17)\\
= & \int J_3[\Psi]\Psi - \ln \int \mathcal{D}\psi^3_{p<\mu} \; e^{iS_W[\psi^3]+\int J_3[\Psi]\psi^3} && (2.1.18)\\
= & \Gamma_W[\Psi]. && (2.1.19)
\end{aligned}$$

Thus we conclude that the two derivative low energy parts of $\Gamma[\Psi^1 = \Psi^2 = 0]$, Γ_W and S_W are in fact identical. We can simply refer to this object as *the* two derivative low energy effective action, without having to worry about potential ambiguities in the meaning of the word. From the arguments it follows that we will always be able to do this as long as

- the massless fields do not renormalize the two derivative terms in the low energy effective action and
- pure massless field excitations do not act as sources for the massive fields.

2.1.4 Relation with beta-function

Let us further consider the example of $SU(2)$ $\mathcal{N} = 2$ Yang-Mills. An obvious question is what the relation is of the effective coupling constant $g(\phi/\Lambda)$ (which is determined by the Seiberg-Witten solution, still to be discussed) in (2.1.14) with the usual running coupling constant $\tilde{g}(M)$ in field theory, and in particular if we can see the beta function appearing. Let us see what we can say about this.

If we define the running coupling $g(M)$ of the $SU(2)$ theory e.g. as in [25] p. 125, we have that $g^2(M)$ is proportional to the single photon exchange diagram of the effective action Γ, between two charged (heavy) particles at momentum transfer $q^2 = M^2$. Therefore, since at vanishing momentum transfer, the photon propagator is obtained from the two derivative low energy part of the effective action Γ as given in (2.1.14), we have:

$$\tilde{g}(M=0) = g(a/\Lambda). \tag{2.1.20}$$

On the other hand, at weak coupling, we have for $M \gg |a|$, because of asymptotic freedom:

$$M\frac{d\tilde{g}}{dM} = \beta(\tilde{g}) = -\frac{\tilde{g}^3}{16\pi^2}(2N_c - N_f) = -\frac{\tilde{g}^3}{4\pi^2}, \tag{2.1.21}$$

hence

$$\frac{1}{\tilde{g}^2} = \frac{1}{2\pi^2}\ln\frac{M}{|\Lambda|}, \tag{2.1.22}$$

where the integration constant Λ is the dynamically generated scale introduced earlier[10]. Together, this gives a picture for the running coupling $\tilde{g}(M)$ which starts at $\tilde{g} = g(a/\Lambda)$ for low values of the energy scale M and then matches onto the asymptotic running (2.1.22) for large values of M.

If on the other hand we define the coupling $\tilde{g}(M)$ in the spirit of [25] p. 127, we should take our energy scale $M = |\langle\phi\rangle| = |a|$, and simply put

$$\tilde{g}(M) = g(\phi = M). \tag{2.1.23}$$

Clearly, both definitions are quite different in the low energy region. However, as argued in [25], p. 138, any "reasonable" definition of the coupling gives the same β function up to fourth order in the coupling constant. Therefore (because of asymptotic freedom), we expect the same asymptotic running for $M \to \infty$ for both definitions. In particular, this means we should find for large ϕ/Λ

$$\frac{1}{g^2(\phi/\Lambda)} \approx \frac{1}{2\pi^2}\ln(|\phi/\Lambda|). \tag{2.1.24}$$

Happily, this indeed turns out to be the case. Despite some claims made in the literature, much more is not known about the relation between exact beta function and the Seiberg-Witten effective action [32].

[10] without further specification at that time; here we see the emergence of its meaning, at least of its modulus. The phase of Λ is defined by removing the modulus signs and replacing the left hand side with the complexified coupling constant $\tau/4\pi$

2.2 From local to rigid special geometry

The two derivative part of the action of any four dimensional $\mathcal{N} = 2$ supersymmetric theory describing massless abelian vector multiplets is governed by special Kähler geometry: *local* special Kähler if the supersymmetry is realized locally (that is, if it is a supergravity theory), *rigid* special Kähler if it is realized only globally (that is, if it is a quantum field theory in flat space without gravity).

We give a quick review here, referring to [33, 34] for more details.

2.2.1 Local special geometry

A (local) special Kähler manifold $\mathcal{M}$ is an n complex dimensional Kähler manifold with Kähler potential of the form

$$\mathcal{K} = -\ln(-iV^t Q^{-1} \bar{V}), \tag{2.2.1}$$

where V is a certain holomorphic local section of a rank $2(n+1)$ symplectic vector bundle over $\mathcal{M}$ and Q an invertible, antisymmetric and constant matrix, the symplectic form. The Kähler metric is given by $g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^b}\mathcal{K}$, where the z^a, $a = 1, \ldots, n$ are complex coordinates on $\mathcal{M}$. Transition functions between different local sections should be such that the Kähler metric is globally well defined. The symplectic section furthermore has to satisfy the following integrability condition[11]:

$$(D_a V)^t Q^{-1} D_b V = 0, \tag{2.2.2}$$

where $D_a = \partial_a + \partial_a \mathcal{K}$.

Assuming a symplectic basis has been chosen such that Q is of standard form

$$Q = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.2.3}$$

and such that V can be split as $V = (X^I, F_I)$, $I = 1, \ldots, n+1$ with the F_I (locally) expressible as functions of the X^I, a particularly efficient device encoding all geometric quantities in special geometry is the *prepotential* F. This object is defined locally as a degree 2 homogeneous function of the X_I, given by $F(X) \equiv \frac{1}{2} X_I F^I$. We then have, thanks to the integrability condition (2.2.2), $F_I = \frac{\partial F}{\partial X^I}$ and, in 'special' coordinates $t^a \equiv X^a / X^0$ and with $\mathcal{F}(t) \equiv 1/(X^0)^2 F$,

$$e^{-\mathcal{K}} = \frac{i}{2}|X^0|^2[2(\mathcal{F} - \bar{\mathcal{F}}) - (t^a - \bar{t}^{\bar{a}})(\mathcal{F}_a + \bar{\mathcal{F}}_{\bar{a}})]. \tag{2.2.4}$$

[11] We will not address here the subtleties arising for the case $n = 1$ [33].

The curvature of $\mathcal{M}$ satisfies in these coordinates the constraint

$$R_{a\bar{b}c\bar{d}} = g_{a\bar{b}}g_{c\bar{d}} + g_{a\bar{d}}g_{c\bar{b}} - e^{2\mathcal{K}}C_{ace}g^{e\bar{f}}C_{\bar{f}\bar{b}\bar{d}}, \qquad (2.2.5)$$

with $C_{abc} = \partial_a\partial_b\partial_c\mathcal{F}$. This is yet another intrinsic geometrical object, named quite differently depending on the physical or mathematical context: Yukawa couplings in $\mathcal{N}=1$ heterotic compactifications, magnetic moments in type II $\mathcal{N}=2$ supergravity, operator product coefficients or 3-point functions in the context of conformal or topological field theory on the worldsheet and triple intersection numbers from the point of view of Calabi-Yau geometry. We will simply call it the 3-coupling.

In geometry, a *moduli space* is usually a space parametrizing a family of different possible geometrical structures, like complex structures, Kähler structures, quaternionic structures and so on. In field theory, a moduli space is a space parametrizing different possible scalar vacuum expectation values (and thus a family of degenerate vacua). If these scalars are dynamical and described by an (effective) action, the physical metric on moduli space is given by the scalar kinetic term (at zero energy).

Four dimensional $\mathcal{N}=2$ supergravity coupled to n_V vector multiplets has a n_V complex dimensional scalar moduli space $\mathcal{M}_V$ on which supersymmetry imposes local special geometry [26]. The form of the (two derivative) action is completely determined by special geometry (actually by an integral over superspace of the prepotential). The bosonic part has the following form:

$$S = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-G}R \;-\; 2g_{a\bar{b}}\,\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^{\bar{b}} \;-\; \frac{1}{4\gamma^2}\int \mathcal{F}^I \wedge \mathcal{G}_I. \qquad (2.2.6)$$

Here κ_4 is the gravitational constant, $G_{\mu\nu}$ is the spacetime metric, with determinant G and Ricci scalar R, the z^a, $a = 1,\ldots,n_V$ are the (mass dimensionless) scalar moduli fields, $\mathcal{F}^I$ ($I = 1,\ldots,n_V+1$), is the 2-form field strength of an abelian 1-form potential $\mathcal{A}^I$, γ is a constant depending on the normalisation of the $\mathcal{A}^I$ (of order one with the usual conventions), and

$$\mathcal{G}_I = \mathrm{Re}\,\mathcal{N}_{IJ}\mathcal{F}^J - \mathrm{Im}\,\mathcal{N}_{IJ} * \mathcal{F}^J, \qquad (2.2.7)$$

where $\mathcal{N}_{IJ}$ is a moduli dependent symmetric matrix:

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{(\mathrm{Im}\,F_{IK})X^K(\mathrm{Im}\,F_{JL})X^L}{X^M(\mathrm{Im}\,F_{MN})X^N}, \qquad (2.2.8)$$

with $F_{IJ} = \partial_I\partial_J F$ and F the prepotential. In order for the vector kinetic energy to be positive, $\mathrm{Im}\,\mathcal{N}_{IJ}$ should be negative definite. The 3-couplings also appear, but only in the fermionic part of the action. The fully general $\mathcal{N}=2$ two derivative action, including fermions and hypermultiplets, can be found in [35].

For type IIB string theory compactified on a Calabi-Yau manifold (which will be analyzed in detail in the next chapter), the vector multiplet scalar moduli space coincides with the complex structure moduli space of the internal Calabi-Yau manifold. This space has local special Kähler geometry with

$$\begin{aligned} V_\Sigma &= \int_{C_\Sigma} \Omega \\ Q_{\Sigma\Lambda} &= C_\Sigma \cdot C_\Lambda, \end{aligned} \tag{2.2.9}$$

where Ω is the holomorphic 3-form on the CY, $\{C_\Sigma\}_\Sigma$ is a basis of 3-cycles, and the dot denotes the intersection product.

2.2.2 Rigid special geometry

A rigid special Kähler manifold M is an r complex dimensional Kähler manifold with Kähler potential of the form

$$K = i v^t q^{-1} \bar{v}. \tag{2.2.10}$$

Here v is a holomorphic section of a rank $2r$ symplectic vector bundle with symplectic form q, and the Kähler metric is given by $g_{i\bar{j}} = \frac{\partial}{\partial u^i}\frac{\partial}{\partial \bar{u}^j} K$, where the u^i, $i = 1,\ldots,r$ are coordinates on M. Transition functions between different local sections should be such that the Kähler metric is globally well defined. The integrability condition now is:

$$(\partial_i v)^t q^{-1} \partial_j v = 0. \tag{2.2.11}$$

Assuming a symplectic basis has been chosen such that q is of standard[12] form

$$q = 2 \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.2.12}$$

and such that the v can be split as $v = (\phi^A, \phi_{D,A})$, $A = 1,\ldots,r$ with the $\phi_{D,A}$ (locally) expressible as functions of the ϕ^A, we can again define a prepotential $\mathcal{F}$. For rigid special geometry, this is no longer a homogeneous function. Using the integrability condition (2.2.11), it can be defined as a function of the ϕ^A by

$$\frac{\partial \mathcal{F}}{\partial \phi^A} = \phi_{D,A}(\phi). \tag{2.2.13}$$

[12]The factor 2 is introduced to get the standard rigid special geometry formulas, in the conventions of e.g. [5].

Again, all geometrical objects can be derived from the prepotential, for example

$$K = \mathrm{Im}\, \bar{\phi}^A \frac{\partial \mathcal{F}}{\partial \phi^A}, \tag{2.2.14}$$

and in the coordinates ϕ^A:

$$g_{A\bar{B}} = \mathrm{Im}\, \tau_{AB} \tag{2.2.15}$$

where τ_{AB} is the moduli dependent symmetric matrix given by

$$\tau_{AB} = \partial_A \partial_B \mathcal{F}. \tag{2.2.16}$$

Rigid $\mathcal{N} = 2$ supersymmetric $U(1)^r$ Yang-Mills field theories have an r complex dimensional scalar moduli space M_v on which supersymmetry imposes a rigid special geometry. As with supergravity, the form of the (two derivative) action is completely determined by rigid special geometry (again by an integral over superspace of the prepotential). The bosonic part has the following form:

$$S = -\int \frac{1}{4\pi} g_{A\bar{B}}\, \mathrm{d}\phi^A \wedge *\mathrm{d}\bar{\phi}^{\bar{B}} + \frac{1}{8\pi} F^A \wedge G_A. \tag{2.2.17}$$

Here the ϕ^A, $A = 1, \ldots, r$ are the (mass dimension 1) scalar fields, F^A is the 2-form field strength of an abelian 1-form potential A^A, and

$$G_A = \mathrm{Re}\, \tau_{AB} F^B + \mathrm{Im}\, \tau_{AB} * F^B, \tag{2.2.18}$$

with τ_{AB} as in (2.2.16). Note that in order for the kinetic energy to be positive, $\mathrm{Im}\, \tau_{AB}$ should be positive definite. Our conventions are those of [5].

The reason why we use mass dimensionful scalar fields here rather than dimensionless ones as we did for supergravity, is that we do not want to introduce a dimensionful parameter in the classical Yang-Mills theory with constant scalar metric. In the supergravity theory, we already had such a parameter: the gravitational constant. (Of course, there, we could equally well have chosen to work with dimensionful scalar fields.) However, the presence of dimensionful scalars implies that we will have to introduce a dimensionful parameter as soon as the scalar metric is no longer constant. This parameter indeed appears in quantum effective actions: it is the dynamically generated scale. So in the quantum case, since we have this scale parameter anyway, we could equally well work with dimensionless scalars. Note that with these conventions, the symplectic vector has dimension mass, and the Kähler potential dimension mass squared (while in supergravity, these quantities are dimensionless).

Rigid special geometry is realized on the complex structure moduli space of certain classes of Riemann surfaces as follows:

$$\begin{aligned} v_\sigma &= \Lambda \int_{c_\sigma} \lambda_{SW} && (2.2.19) \\ q_{\sigma\lambda} &= c_\sigma \cdot c_\lambda, && (2.2.20) \end{aligned}$$

where Λ is a scale parameter (the 'dynamically generated scale'), λ_{SW} is a certain meromorphic 1-form on the Riemann surface, and $\{c_\lambda\}_\lambda$ is a certain set of $2r$ independent 1-cycles. Not any class of Riemann surfaces endowed with a meromorphic one-form and a set of 1-cycles satisfies the axioms of rigid special geometry. It is an open question what the criterion is for a subspace of moduli space to have rigid special geometry.

Such geometric moduli spaces appear beautifully in the exact solution of the quantum low energy effective action of $\mathcal{N} = 2$ field theories, as we will see in the section 2.3.

2.2.3 The rigid limit of local special geometry

Quite generally, a rigid limit of local special geometry can be obtained as follows. Suppose there is a region in moduli space where we can choose a subset of coordinates $(u_1, \ldots, u_r)$ and symplectic vector components $(V_1, \ldots, V_{2r}) \equiv v$ such that the Kähler potential (2.2.1) can be written as

$$\mathcal{K} = -\ln\left(L^2 - i v^t q^{-1} \bar{v} + R\right), \qquad (2.2.21)$$

where L is real and independent of the u_i, q is a real, invertible and antisymmetric matrix, and R is a remainder such that

$$\frac{v^t q^{-1} \bar{v}}{L^2} \to 0, \quad \frac{R}{v^t q^{-1} \bar{v}} \to 0 \qquad (2.2.22)$$

when approaching a certain locus in this region. Then close to this locus, we can make the following expansions:

$$\begin{aligned} v &= v_0 + \ldots && (2.2.23) \\ \mathcal{K} &= -\ln L^2 + \frac{1}{L^2} i v_0^t q^{-1} \bar{v}_0 + \ldots && (2.2.24) \\ \mathcal{D}_u v &= \partial_u v_0 + \ldots && (2.2.25) \end{aligned}$$

where the dots indicate subleading terms that can be neglected in the limit under consideration. Note that (2.2.24) is, up to a u-independent term, the expression for the Kähler potential in *rigid* special geometry. Moreover, the integrability condition (2.2.2) reduces to

$$(\partial_u v_0) q^{-1} v_0 = 0, \tag{2.2.26}$$

which is precisely the integrability condition defining rigid special geometry. Thus we find that the geometry of the moduli subspace parametrized by the moduli u_i and endowed with the symplectic vector v_0, is essentially rigid special Kähler.

The limit described above will be called a *rigid limit*, and can be thought of as sending the Planck mass to infinity, effectively decoupling gravity from the degrees of freedom associated with the rigid moduli u_i. We will come back to this in great detail in chapter 4.

2.3 Low energy effective action of $\mathcal{N} = 2$ Yang-Mills theory

For generic expectation values of the scalars of a nonabelian $\mathcal{N} = 2$ super Yang-Mills theory, the gauge group is broken (at low energies, by the Higgs effect) to its Cartan subgroup $U(1)^r$, where r is the rank of the original gauge group. Therefore, from the above discussion, it follows that the exact solution of the two derivative quantum low energy effective action of such theories simply amounts to finding the correct underlying rigid special Kähler manifold.

It turns out that, for a very large class of such theories, this can be done in terms of a certain moduli space of Riemann surfaces. This was originally argued for the pure $SU(2)$ theory by Seiberg and Witten in [5] and later extended to include matter hypermultiplets [6, 36] and higher rank gauge groups [37, 38], by using physical arguments based on spectrum considerations, monodromies and asymptotic behavior from the one loop beta function. As we will deduce these low energy effective actions directly from string theory in chapter 4, we will not review those arguments here (some good reviews are e.g. [15, 16]). Instead we will just give a compendium of the relevant moduli spaces and some aspects of the corresponding physics (without justification), for a number of example theories:[13]

- *Pure $SU(2)$:*

  This is the case originally considered by Seiberg and Witten [5]. The rank of $SU(2)$ is 1. The rigid special Kähler manifold underlying the effective action

[13] For the non-expert reader: the details of the following summary are not really necessary to understand the main line of this thesis. We will make contact with Seiberg-Witten theory in chapter 4, and use some of the formulas for the $SU(2)$ moduli space metric given below in chapter 5.

is the moduli space of genus 1 curves Σ, given by the equation

$$\frac{1}{2}(z+\frac{1}{z}) = x^2 + u \tag{2.3.1}$$

where z, x are the ambient space coordinates and u is the modulus determining the vacuum:

$$u = \frac{1}{2}\langle\frac{\phi^2}{\Lambda^2}\rangle_{VAC}. \tag{2.3.2}$$

Here ϕ is the at weak coupling 'elementary' scalar field of the unbroken $U(1)$ vector multiplet and Λ is the dynamically generated scale. Fig. 2.2 shows the structure of the Seiberg-Witten Riemann surface, and fig. 2.3 gives a picture of the moduli space metric $g_{u\bar{u}}$. The set of two independent 1-cycles is a basis (α, β) of $H_1(\Sigma, \mathbb{Z})$ with $\alpha \cdot \beta = 2$. The meromorphic Seiberg-Witten 1-form is given by

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi}\, x\, \frac{\mathrm{d}z}{z}. \tag{2.3.3}$$

When α is chosen to be equal to twice the unit circle in the z plane (lifted to the Riemann surface) and β to be the cycle encircling the branch points given by $\frac{1}{2}(z+\frac{1}{z}) = u$ in the z-plane, we have $\phi = \Lambda \int_\alpha \lambda_{SW}$ and $\phi_D = \Lambda \int_\beta \lambda_{SW}$, where ϕ_D can be shown to correspond to the scalar of a *dual*, magnetic $U(1)$ massless vector multiplet, which becomes weakly coupled when the original electric theory becomes strongly coupled (see below).

The theory contains massive charged BPS particles (these are particles with minimal mass for the given charge), with electric and/or magnetic charges. A BPS particle with electric charge n and magnetic charge m has mass equal to $\sqrt{2}|n\phi + m\phi_D|$. In the conventions we use, particles in the adjoint of $SU(2)$ have integer electric and magnetic charges, while particles in the fundamental have half-integer electric (and integer magnetic) charges. By the BPS condition and the triangle inequality, the BPS particles are absolutely stable, except when ϕ_D/ϕ is real, then they are only marginally stable. Marginal stability occurs in the u-plane on a closed ellips-like line passing through $u = \pm 1$. Outside this line of marginal stability, the theory is well described in terms of the original variables, with the coupling running to zero (by asymptotic freedom) for $u \to \infty$ and running to ∞ at the singularitues $u = \pm 1$ in moduli space. Outside the curve of marginal stability, the BPS spectrum consists of the $W^\pm$ bosons (charge $(\pm 1, 0)$), a magnetic monopole (charge $(0, 1)$), which becomes massless at $u = 1$, an 'elementary' dyon (charge $(\pm 1, 1)$, depending on cut conventions), which becomes massless at $u = -1$, and dyons of charge $(n, 1)$, $n \in \mathbb{Z}$ (and of course all charge conjugates of those). Note that the latter can be obtained from the monopole

Figure 2.2: Schematic picture of the Seiberg-Witten Riemann surface, represented as a 2-sheeted cover of the z-plane. The z-plane is mapped (by $z \to \ln z$) to a strip with opposite sides identified. The two sheets of the surface are represented by two such strips, connected via the branch cuts running to infinity from $z = -u \pm \sqrt{u^2 - 1}$. The α cycle consists of two disconnected loops (one on each sheet) running from one side of the strips to the other. The β cycle runs around the branch points, encircling the hole in the picture. More details can be found in the text.

Figure 2.3: The metric $g_{u\bar{u}}$ on the $SU(2)$ Seiberg-Witten moduli space, plotted as a function of u. The spikes at $u = \pm 1$ are strong coupling singularities produced by integrating out massless charged particles.

or the elementary dyon by a monodromy $u \to e^{2\pi i} u$ about $u = \infty$. Indeed, such a monodromy transforms α and β als follows:

$$\alpha \to -\alpha \tag{2.3.4}$$

$$\beta \to -\beta + 2\alpha. \tag{2.3.5}$$

From our general discussion on effective actions, one would expect the effective action approximation to break down at points where integrated-out charged particles become massless. This is indeed the case: the moduli space has singularities at $u = \pm 1$ (see fig. 2.3).

Inside the line of marginal stability, the only stable BPS particles in the spectrum are the monopole and the elementary dyon. The theory in the neighborhood of the strong coupling point (in the original variables) $u = 1$ is well described in terms of the dual magnetic massless $U(1)$ field coupled to the light monopole. Such a theory is infrared free and becomes weakly coupled for $u \to 1$. See fig. 2.4. Analogously, close to $u = -1$, one finds essentially a dual 'dyonic' $U(1)$ theory coupled to the elementary dyon. There is no point in moduli space where the original $SU(2)$ gauge symmetry is restored.

Figure 2.4: The dual coupling constant squared $g_D^2 \sim 1/g_{\phi_D\bar{\phi}_D}$, plotted as a function of u in a neighboorhood of $u=1$. At $u=1$, the dual coupling is zero.

Approximate expressions for some special geometry quantities near $u=\infty$ are:

$$\phi/\Lambda \approx \sqrt{2u} \tag{2.3.6}$$

$$\phi_D/\Lambda \approx \frac{i}{\pi}\sqrt{2u}\ln u \tag{2.3.7}$$

$$K/|\Lambda|^2 \approx \frac{|u|}{\pi}\ln|u|^2 \tag{2.3.8}$$

$$g_{u\bar{u}}/|\Lambda|^2 \approx \frac{1}{4\pi}\frac{\ln|u|^2}{|u|} \tag{2.3.9}$$

$$g_{\phi\bar{\phi}} \approx \frac{2}{\pi}\ln\left|\frac{\phi}{\Lambda}\right|, \tag{2.3.10}$$

and near $u=1$:

$$\phi/\Lambda \approx \frac{4}{\pi} - \frac{1}{2\pi}(u-1)\ln(u-1) \tag{2.3.11}$$

$$\phi_D/\Lambda \approx \frac{i}{2}(u-1) \tag{2.3.12}$$

$$K/|\Lambda|^2 \approx \frac{1}{\pi}\mathrm{Re}\,(u-1) + \frac{1}{8\pi}|u-1|^2\ln|u-1|^{-2} \tag{2.3.13}$$

$$g_{u\bar{u}}/|\Lambda|^2 \approx \frac{1}{8\pi}\ln|u-1|^{-2} \tag{2.3.14}$$

$$g_{\phi_D \bar{\phi}_D} \approx \frac{1}{\pi} \ln \left| \frac{\phi_D}{\Lambda} \right|^{-1} . \tag{2.3.15}$$

The prepotential in the weak coupling region has the following form [5]:

$$\mathcal{F}(\phi) = \frac{i}{2\pi} \phi^2 \ln \frac{\phi^2}{\Lambda^2} + \sum_{k=1}^{\infty} c_k \left(\frac{\Lambda}{\phi} \right)^{4k} \phi^2 . \tag{2.3.16}$$

The first term is the one-loop correction while the kth term in the sum arises as a contribution from k instantons. The Seiberg-Witten solution amounts to an exact specification of all coefficients c_k.

The results for $SU(2)$ can be generalized in various directions, either by generalizing the gauge group or by adding matter. We now consider some of those generalizations.

- *Pure* $SU(N_c)$*:*

  The rank of the group is $r = N_c - 1$. The relevant Riemann surfaces form an r dimensional family of hyperelliptic genus r curves embedded in $\mathbb{C}^2$, given by the equation

$$\frac{1}{2}(z + \frac{1}{z}) = x^{r+1} + u_{r-1}\, x^{r-1} + \cdots + u_1\, x + u_0 , \tag{2.3.17}$$

  where z, x are the ambient space coordinates and the u_i are the moduli. As set of 1-cycles, we can just take any basis of 1-cycles, which wil contain indeed precisely $2r$ 1-cycles. The meromorphic 1-form is again given by (2.3.3).

- $SU(2)$ *with one massive fundamental hypermultiplet:*

  Actually this generalization requires a slight extension of the rigid special geometry framework presented in the previous section, since special geometry only governs the low energy effective action of vector multiplets. However, it turns out [6] that the low energy theory of asymptotically free or conformal invariant $\mathcal{N} = 2$ Yang-Mills theories with additional matter multiplets is still completely determined by the geometry of a certain family of Riemann surfaces endowed with a certain meromorphic 1-form. Since we will mainly restrict to the pure Yang-Mills case in this thesis anyway, we will not go into the details of this construction.

  The relevant curves for $SU(2)$ with one hypermultiplet of mass m in the fundamental of $SU(2)$, are given by [6]:

$$\frac{1}{2}(z^2 + 2mz + \frac{1}{z}) = x^2 + u , \tag{2.3.18}$$

where, as in the pure Yang-Mills case, u is identified with $\frac{1}{2}\langle\frac{\phi^2}{\Lambda^2}\rangle$). Again, the meromorphic 1-form is (2.3.3), the metric on the vector multiplet moduli space is rigid special Kähler, constructed as for the pure $SU(2)$ case, and BPS masses can be calculated simply as periods of λ_{SW} (if we consider a residue of λ_{SW} also as a period).

- *Pure E_6:*

  We include this case [39, 38] just to show that life isn't always that easy. For the E_6 exceptional group (without matter), the meromorphic 1-form is as usual, but the moduli space is given by the following 6-parameter family of genus 34 surfaces:

$$\frac{1}{2}x^6 Z^2 - Q_1 x^3 Z + Q_2 = 0, \tag{2.3.19}$$

  where

$$\begin{aligned}
Z &= z + \frac{1}{z} + u_6, \\
Q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 \\
&\quad -162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 \\
&\quad +(27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 \\
&\quad -3u_2u_5x - u_2^3, \\
Q_2 &= \frac{1}{2}(Q_1^2 - P_1^2P_2), \\
P_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 \\
&\quad -u_5x - u_2^2, \\
P_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2.
\end{aligned}$$

  Out of the 68 independent cycles spanning $H_1(\Sigma, \mathbb{Z})$, there are 12 special ones providing the 12 independent periods defining rigid special geometry. The construction of those cycles can be found in [39]. The underlying organizing structure under this apparent mess is of course group theory, here E_6 representation theory. Other unifying connections are Toda integrable systems and singularity theory. The latter connection emerges naturally from string theory. We will come back to this in chapter 4.

Many other generalizations have been constructed, for which we refer to the literature [37, 36, 38, 14].

In summary, we have argued in this chapter that we can safely talk about 'the' (two derivative) low energy effective action of a certain quantum $\mathcal{N}=2$

Yang-Mills theory. Its form is beautifully governed by the rigid special geometry of a certain geometrical moduli space of Riemann surfaces, in the sense that for example the scalar kinetic term of the low energy effective action appears with the natural special Kähler metric on this moduli space.

Chapter 3

Calabi-Yau compactifications of IIB string theory

In this chapter we will investigate some low energy aspects of type IIB string theory compactified down to four dimensions on a Calabi-Yau manifold. We will start by demonstrating how four dimensional $\mathcal{N} = 2$ supergravity, coupled to a number of vector- and hypermultiplets, arises as the low energy theory of the four dimensional massless fields. Next we will study the massive particle-like BPS states in the 4D theory, originating from 3-branes[1] wrapped around nontrivial cycles of the Calabi-Yau manifold. These BPS states can also be obtained as solitonic BPS solutions of the low energy 4D effective field theory. As we will see, these solitons have the remarkable property of being *"attractors"* for the scalars in the vector multiplets; that is, towards the center of the soliton, the scalars always flow to certain fixed values, only dependent on the electric and magnetic charges of the state. We will study the attractor solutions in quite some detail, emphasizing the physical intuition behind the equations. We conclude the chapter with a discussion of the range of validity of field theory and D-brane pictures.

Part of the material in this chapter is a review of well known facts, though we will present these in a not so conventional way and we will elaborate a bit more than usual on some conceptual issues.

[1] 1- and 5-branes cannot generate 4D BPS states in this way since there are no nontrivial one or five dimensional cycles in a Calabi-Yau manifold (at least with the definition of a CY manifold which we use).

Figure 3.1: Sketch of a Calabi-Yau compactification. The moduli of the Calabi-Yau X can vary over the noncompact, curved, four dimensional spacetime M_4.

3.1 Four dimensional low energy effective action

We will immediately turn to the tree level two derivative low energy action[2] of the massless fields in the theory. For the world sheet perspective, higher order corrections, and many other aspects which we cannot cover here, Polchinsky's book [31] is highly recommended. We will also use mostly his notations and conventions.

3.1.1 Type IIB string theory

Low energy type IIB theory has $\mathcal{N} = 2$ supersymmetry (32 supersymmetry generators) in 10 dimensions, with equal chirality. The supersymmetry algebra is

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = -2\delta^{AB} P_\mu (\Pi_+ \Gamma^\mu)_{\alpha\beta}, \tag{3.1.1}$$

[2] this means we consider the effective action for these fields to leading order in the string coupling constant $g_s \sim e^\Phi$ and the gravitational constant κ_{10} (or α').

where

$$\Pi_+ = \frac{1+\Gamma}{2} \tag{3.1.2}$$

with $\Gamma = \Gamma_0 \dots \Gamma_9$, and $A, B = 1, 2$, $\alpha, \beta = 1, \dots, 16$. The high amount of supersymmetry completely fixes the low energy theory (at tree level), which is just 10D type IIB supergravity. The massless bosonic fields are

- A scalar Φ, a 2-form potential B with field strength H_3, and the metric g (Neveu-Schwarz - Neveu-Schwarz sector).
- A scalar C_0, a 2-form potential C_2 with field strength F_3, and a 4-form potential C_4 with selfdual field strength F_5 (Ramond-Ramond sector).

We adopt conventions such that the mass dimensions of all components[3] of the potentials are zero. To write down the action, it is convenient (and it makes S-duality manifest) to define

$$\tau = C_0 + ie^{-\Phi}, \tag{3.1.3}$$

$$\mathcal{M}_{ij} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & 1 \end{pmatrix}, \tag{3.1.4}$$

$$F_3^i = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}, \tag{3.1.5}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \tag{3.1.6}$$

We will also work always in the Einstein frame, which has canonical Einstein-Hilbert term and is related to the string frame by $G_{E,\mu\nu} = e^{-\Phi/2}G_{S,\mu\nu}$, where Φ is the dilaton ($g = \langle e^{\Phi}\rangle$ is the string coupling constant). Due to the presence of the self-dual 5-form field strength, there is actually no (simple) manifestly covariant action for this theory, but the following comes close:

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,R \\ &- \frac{1}{4\kappa_{10}^2}\int \frac{1}{(\operatorname{Im}\tau)^2}\,\mathrm{d}\tau\wedge *\mathrm{d}\bar{\tau} + \mathcal{M}_{ij}\,F_3^i\wedge *F_3^j + \frac{1}{2}\tilde{F}_5\wedge *\tilde{F}_5 \\ &\qquad + \frac{\epsilon_{ij}}{2}\,C_4\wedge F_3^i\wedge F_3^j. \end{aligned} \tag{3.1.7}$$

At the classical level, one only has to impose the additional selfduality constraint $*\tilde{F}_5 = \tilde{F}_5$, *after* varying this action, to get the correct equations of motion. The

[3] so the dimension of the p-form itself is (length)^p

constant κ_{10} is related to the constant α' appearing in the worldsheet action of string perturbation theory by $\alpha' = 2^{-3/2}\pi^{-7/4}\,\kappa_{10}^{1/2}$ [31]. It can be identified as the ten dimensional gravitational constant. Note however that this scale on itself doesn't have a physical meaning: only *ratios* of mass scales are meaningful and measurable. Which combination of κ_{10} and the coupling constant we call 'the' ten dimensional gravitational constant is therefore purely conventional. Something which *does* have a physical meaning for example is the ratio of the mass of the lowest excited string oscillator state ($2g^{1/4}/\alpha'^{1/2}$ in the *Einstein* frame) and the physically measured 10D Planck mass $\kappa_{10}^{-1/4}$. This ratio is $[(4\pi)^7 g^2]^{1/8}$ and independent of frame; it tells how strong spacetime is deformed when a minimally excited string is present. Notice that the meaning of κ_{10} as used e.g. in [31] in the string frame is different from the meaning of κ_{10} in the Einstein frame: in the Einstein frame it is the physically measured gravitational constant, in the string frame it differs from the physical gravitational constant with a factor g. Confusion can always be eliminated by calculating mass ratios.

The action (3.1.7) is invariant under the following $SL(2,\mathbb{R})$ symmetry:

$$\tau \;\rightarrow\; \frac{a\tau + b}{c\tau + d}, \tag{3.1.8}$$

$$F_3^i \;\rightarrow\; \begin{pmatrix} d & c \\ b & a \end{pmatrix} F_3^i, \tag{3.1.9}$$

$$\tilde{F}_5 \;\rightarrow\; \tilde{F}_5 \tag{3.1.10}$$

$$G_{\mu\nu} \;\rightarrow\; G_{\mu\nu}. \tag{3.1.11}$$

There is strong evidence that the discrete subgroup $SL(2,\mathbb{Z})$ is actually an exact symmetry of string theory (not only at tree level and low energies): this is the type IIB S-duality group.

Type IIB string theory contains Dirichlet p-branes with odd p. We are in particular interested in the case $p = 3$, which has the following tree level low energy action for the bosonic degrees of freedom:

$$S_{D3} = S_{D3,DBI} + S_{D3,WZ}, \tag{3.1.12}$$

The first term is, apart from curvature corrections [40], of Dirac-Born-Infeld form and independent of the R-R fields. In the Einstein frame and without the curvature corrections, this is:

$$S_{D3,DBI} = -\frac{\sqrt{\pi}}{\kappa_{10}} \int d^4\xi \sqrt{-\det(h_{ab} + e^{-\Phi/2}B_{ab} + 2\pi\alpha' e^{-\Phi/2}F_{ab})}, \tag{3.1.13}$$

where h_{ab} and B_{ab} are the pull-back of spacetime metric and NS-NS B-field to the brane, and F_{ab} is the field strength of the gauge field A living on the brane: $F = \frac{1}{2} F_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b = \mathrm{d}A$.[4] The curvature corrections are of a rather complicated form (see [40]) and we will not give them here. For the cases we are interested in, these terms happen to cancel anyway (in a nontrival way however). The second term is of Wess-Zumino form. Apart from the curvature couplings, it is independent of the metric and the other NS-NS fields:

$$\begin{aligned} S_{D3,WZ} &= \frac{\sqrt{\pi}}{\kappa_{10}} \int C_4 \, + \, (2\pi\alpha' F + B_2) \wedge C_2 \\ &+ \left((2\pi\alpha' F + B_2)^2 \, + \, \frac{(2\pi\alpha')^2}{96} (\mathrm{tr} R_T^2 - \mathrm{tr} R_N^2) \right) C_0 , \end{aligned} \quad (3.1.14)$$

where R_T (R_N) is the curvature of the tangent (normal) bundle of the brane [41, 40]. There are further nonperturbative instanton corrections to S_{D3}, making this action invariant under S-duality, as it should.

3.1.2 Calabi-Yau manifolds and special geometry of the complex structure moduli space

We assume the reader is familiar with the basic facts about differential geometry and Calabi-Yau manifolds. Excellent reviews can be found in [42, 43, 29]. See also [44]. We define a Calabi-Yau n-fold as a compact n complex dimensional Kähler manifold with vanishing first Chern class and no nontrivial 1-cycles, equipped with a Ricci-flat metric, which has $SU(n)$ holonomy. A theorem of Calabi and Yau says that for any fixed complex and Kähler structure of the manifold, such a metric exists and is unique.

In the following, we restrict to the case $n = 3$. The moduli space of complex structures of a Calabi-Yau 3-fold X, equipped with arbitrary complex coordinates z^a, $a = 1, \ldots, h^{2,1}$, has (local) special geometry. We will not prove this well known fact here, but merely review the main features and introduce some notations which we will need in the sequel. Readers not interested in the technicalities of special geometry can probably skip this section.

Denote the (up to normalisation unique) holomorphic 3-form on X by Ω. The

[4] We will adopt the following general notational convention for form components: $F = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$. This is as in [31] (cf. page 450).

Kähler potential for the moduli space metric is[5]

$$\mathcal{K} = -\ln i \int_X \Omega \wedge \bar{\Omega}, \tag{3.1.15}$$

with Kähler metric

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}. \tag{3.1.16}$$

The 3-couplings (aka Yukawa couplings/magnetic moments/3-point functions/triple intersection numbers) are

$$C_{abc} = \int_X \Omega \wedge \partial_a \partial_b \partial_c \Omega. \tag{3.1.17}$$

Choose a symplectic basis of 3-cycles $\{A^I, B_I\}$, $I = 1, \ldots, h^{2,1}+1$ and denote the corresponding periods as X^I, F_I:

$$X^I = \int_{A^I} \Omega \tag{3.1.18}$$

$$F_I = \int_{B_I} \Omega. \tag{3.1.19}$$

With this choice, the prepotential is given by

$$\mathcal{F} = \frac{1}{2} X^I F_I \tag{3.1.20}$$

and (X^I, F_I) is of course nothing but the symplectic vector of special geometry. One furthermore has the property:

$$\Omega \in H^{3,0} \tag{3.1.21}$$

$$\partial_a \Omega \in H^{3,0} + H^{2,1} \tag{3.1.22}$$

$$\partial_a \partial_b \Omega \in H^{3,0} + H^{2,1} + H^{1,2} \tag{3.1.23}$$

$$\partial_a \partial_b \partial_c \Omega \in H^{3,0} + H^{2,1} + H^{1,2} + H^{0,3} \tag{3.1.24}$$

Defining the Kähler covariant derivative

$$D_a \Omega \equiv (\partial_a + \partial_a \mathcal{K}) \Omega, \tag{3.1.25}$$

[5] We attribute mass dimension zero to Ω, as well as to other abstract geometric quantities such as cohomology classes (and hence periods).

we see from (3.1.15), (3.1.22) and (3.1.16) that $D_a\Omega$, $a = 1, \ldots, h^{2,1}$, forms a basis of $H^{2,1}(X)$, and that

$$g_{a\bar{b}} = -ie^{\mathcal{K}} \int_X D_a\Omega \wedge \bar{D}_{\bar{b}}\bar{\Omega}. \tag{3.1.26}$$

Thus we can decompose any (real) harmonic 3-form Γ on X according to the Hodge decomposition theorem

$$H^3(X,\mathbb{C}) = H^{3,0}(X) + H^{2,1}(X) + H^{1,2}(X) + H^{0,3}(X) \tag{3.1.27}$$

as

$$\Gamma = \mathrm{i}\, e^{\mathcal{K}}\, \Omega \int_X \Gamma \wedge \bar{\Omega} \;-\; \mathrm{i}\, e^{\mathcal{K}}\, D_a\Omega\, g^{a\bar{b}} \int_X \Gamma \wedge \bar{D}_{\bar{b}}\bar{\Omega} \;+\; \text{c.c.}. \tag{3.1.28}$$

This decomposition is useful because it diagonalizes the Hodge star operator:

$$*\Gamma^{p,3-p}(X) = (-1)^p\, \mathrm{i}\, \Gamma^{p,3-p}(X) \tag{3.1.29}$$

It can be used to find an explicit expression in terms of central charges for the intersection and Hodge scalar products on X:

$$\int_X \Gamma_1 \wedge \Gamma_2 \;=\; 2\,\mathrm{Im}\,[-Z(\Gamma_1)\,\bar{Z}(\Gamma_2) \;+\; g^{a\bar{b}}\, D_a Z(\Gamma_1)\, \bar{D}_{\bar{b}}\bar{Z}(\Gamma_2)], \tag{3.1.30}$$

$$\int_X \Gamma_1 \wedge *\Gamma_2 \;=\; 2\,\mathrm{Re}\,[Z(\Gamma_1)\,\bar{Z}(\Gamma_2) \;+\; g^{a\bar{b}}\, D_a Z(\Gamma_1)\, \bar{D}_{\bar{b}}\bar{Z}(\Gamma_2)] \tag{3.1.31}$$

where we defined the "central charge" Z of Γ as

$$Z(\Gamma) \equiv e^{\mathcal{K}/2} \int_\Gamma \Omega = \int_X \Gamma \wedge \Omega, \tag{3.1.32}$$

and $D_a Z \equiv (\partial_a + \frac{1}{2}\partial_a\mathcal{K})Z$.[6] Here, by slight abuse of notation, we have denoted Γ and its Poincaré dual by the same symbol. If no confusion is possible, we will always do this, that is:

$$\int_X \Gamma \wedge B \equiv \int_\Gamma B \equiv \Gamma \cdot B\,, \tag{3.1.33}$$

where the dot denotes the intersection product: the number of points in the intersection, counted with signs. We will use the same notational conventions for forms and cycles of any degree. Note also the relative minus sign between the first terms in the r.h.s. of (3.1.30) and (3.1.31).

[6] In general, for an object f which transforms as $f \to \lambda^p \bar{\lambda}^q f$ under a Kähler transformation $\Omega \to \lambda\Omega$, we define $D_a f \equiv (\partial_a + p\partial_a\mathcal{K})f$ and $\bar{D}_{\bar{a}} f \equiv (\bar{\partial}_{\bar{a}} + q\bar{\partial}_{\bar{a}}\mathcal{K})f$.

3.1.3 Four dimensional massless spectrum

The four dimensional massless spectrum of type IIB string theory compactified on the Calabi-Yau manifold X can be deduced as follows (see e.g. [15]). Choose a basis σ^A, $A = 1, \ldots, h^{1,1} = b_2 = b_4$ of $H_2(X, \mathbb{Z})$ (or by Poincaré duality of $H^4(X, \mathbb{Z})$), and denote the corresponding dual basis of $H_4(X, \mathbb{Z})$ ($H^2(X, \mathbb{Z})$) by τ_A, i.e. $\tau_A \cdot \sigma^B = \delta_A^B$. As above, choose a symplectic basis $\{A^I, B_I\}$, $I = 1, \ldots, h^{2,1}+1$ of $H_3(X, \mathbb{Z})$ (or $H^3(X, \mathbb{Z})$). X has $h^{2,1}$ complex structure deformations and $h^{1,1}$ Kähler class deformations. By Torelli's theorem, the complex structure deformations can be parametrized (locally on moduli space) by the periods of the holomorphic 3-form Ω on X,

$$\Omega = X^I B_I - F_I A^I. \tag{3.1.34}$$

Note that indeed since $*\Omega = -i\Omega$ and the normalisation of Ω is irrelevant, a set of $h^{2,1}$ independent variables, e.g. $t^i = X^I/X^0$, is sufficient to parametrize (locally) all complex structures. The complexified Kähler class deformations on the other hand can be parametrized (again locally) by the periods of the complexified Kähler form $\omega \equiv B + iJ$, where B is the NS-NS 2-form potential and J the usual Kähler form on X:

$$\omega = t^A \tau_A \tag{3.1.35}$$
$$t^A = \int_{\sigma^A} \omega. \tag{3.1.36}$$

As already mentioned above, Yau's theorem states that there is a unique Ricci flat metric on X for any given complex structure and (real) Kähler class. The relation between metric fluctuations and complex structure and Kähler class deformations is given by

$$(\delta g)_{m\bar{n}} = (\delta J)_{m\bar{n}} \tag{3.1.37}$$
$$(\delta g)_{mn} = (\delta\Omega)_{m\bar{r}\bar{s}} \Omega^{\bar{r}\bar{s}}{}_n / ||\Omega||^2 \tag{3.1.38}$$

where $||\Omega||^2 = \Omega_{mnr}\bar{\Omega}^{mnr}$.

Now the 10D massless fields produce 4D massless descendants by decomposing them according to the compactification $M_{10} = M_4 \otimes X$ in the harmonic form bases constructed above:

$$\mu^4 C_4 = \mathcal{A}^I \otimes B_I - \mathcal{B}_I \otimes A^I + T^A \otimes \tau_A + S_A \otimes \sigma^A \tag{3.1.39}$$
$$\mu^2 C_2 = c^A \otimes \tau_A + C_{2,4D} \tag{3.1.40}$$
$$\mu^2 B = b^A \otimes \tau_A + B_{4D} \tag{3.1.41}$$

Here μ is an arbitrary mass scale parameter which has been introduced to give the four dimensional fields their usual dimensions (that is, our 4D forms have dimension zero). Note that the selfduality constraint on $\tilde{F}_5$ relates the 1-form $\mathcal{A}^I$ to the 1-form $\mathcal{B}_I$, and the scalar S_A to the 2-form T^A.[7]

Now dualize the 4D 2-forms B_{4D}, $C_{2,4D}$ and T^A to $\tilde{B}$, $\tilde{C}$ and $\tilde{T}^A$ respectively (so for example $d\tilde{B} = *dB_{4D}$). Then t^A and $\tilde{t}^A \equiv c^A + i\tilde{T}^A$, together with $t^0 \equiv \tilde{B} + ie^{-\Phi}$ and $\tilde{t}^0 \equiv \tilde{C} + iC_0$ form the bosonic fields of $n_H = h^{1,1} + 1$ 4D hypermultiplets[8], while the four dimensional metric G_{4D}, the vectors $\mathcal{A}^I$ and the scalars X^I (mod $X^I \simeq \lambda X^I$) form the bosonic fields of the gravity multiplet plus $n_V = h^{2,1}$ vector multiplets.

This is precisely the bosonic spectrum of four dimensional $\mathcal{N} = 2$ supergravity coupled to $n_H = h^{1,1}$ hypermultiplets and $n_V = h^{2,1}$ vectormultiplets. Apart from the precise number of vector- and hypermultiplets, this could have been deduced purely from supersymmetry. Indeed, it is known [29] that a compactification on a Calabi-Yau 3-fold preserves one quarter of the original supersymmetries, i.e. 8 supersymmetry generators, or $\mathcal{N} = 2$ in four dimensions.

For later use, we record here the $D = 4$, $\mathcal{N} = 2$ supersymmetry algebra in its most general form including central charges of point-like objects:

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = -2\delta^{AB} P_\mu \Gamma^\mu_{\alpha\beta} - 2i\epsilon^{AB}(\mathrm{Re}\,\tilde{Z}\,\delta_{\alpha\beta} + \mathrm{Im}\,\tilde{Z}\,\Gamma^5_{\alpha\beta}), \tag{3.1.42}$$

where $A, B = 1, 2$ and $\alpha, \beta = 1, \ldots, 4$ (Majorana rep.). $\tilde{Z}$ is the central charge, which commutes with all other generators of the super-Poincaré group. We added the tilde to distinguish this operator from the geometric object Z as it was introduced in (3.1.32). We will later see that Z and $\tilde{Z}$ are in fact simply proportional to each other.

3.1.4 Four dimensional action

Since in this thesis, we will only be interested in the physics of the four dimensional massless vector multiplets, we will take all four dimensional massless hypermultplet fields to be trivial (constant scalars). It can be checked from the type IIB action (3.1.7) and the dimensional reduction formulas (3.1.39)-(3.1.41) that this is a consistent truncation of the theory: the vector and gravitational fields do not appear as sources of the hypermultiplet fields.

[7] So only half of these 4D fields are independent; which fields we choose as fundamental degrees of freedom is a matter of choice, different choices are related by duality transformations.

[8] We don't bother about the precise grouping and normalisation of these scalars here, since we will not consider the massless hypermultiplets in detail in the remaining part of this thesis.

In that case the dimensional reduction of the action (3.1.7) yields, taking into acount[9] the selfduality constraint on F_5 and using (3.1.39), (3.1.38) and special geometry of the complex structure moduli space:

$$S_{4D} = \frac{1}{2\kappa_4^2}\int_{M_4} d^4x\sqrt{-G}R \;-\; 2g_{a\bar{b}}\,\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^{\bar{b}} \;-\; \frac{1}{4\gamma^2}\int_{M_4} \mathcal{F}^I \wedge \mathcal{G}_I, \qquad (3.1.43)$$

where

$$\kappa_4^2 \;=\; \kappa_{10}^2/\mathrm{Vol}(X) \qquad (3.1.44)$$

$$\gamma \;=\; \kappa_{10}\mu^4 = \kappa_4\sqrt{\mathrm{Vol}(X)}\mu^4 \qquad (3.1.45)$$

(with μ defined in (3.1.39)), $\mathcal{F}^I = \mathrm{d}\mathcal{A}^I$, and $\mathcal{G}_I = \mathrm{d}\mathcal{B}_I$ is to be considered as a function of $\mathcal{A}^I$, given by the selfduality constraint on F_5:

$$\mathcal{F}^I \otimes B_I - \mathcal{G}_I \otimes A^I = *_4\mathcal{F}^I \otimes *_X B_I - *_4\mathcal{G}_I \otimes *_X A^I. \qquad (3.1.46)$$

Here $*_4$ denotes the Hodge star operator on the spacetime manifold M_4 and $*_X$ the one on the Calabi-Yau manifold X. Integrating this equation over A^I, B_I and using (3.1.31) gives a system of equation which can be used to express the $\mathcal{G}_I$ in terms of the $\mathcal{F}^I$:

$$\mathcal{G}_I = \mathrm{Re}\,\mathcal{N}_{IJ}\mathcal{F}^J - \mathrm{Im}\,\mathcal{N}_{IJ} * \mathcal{F}^J, \qquad (3.1.47)$$

where $\mathcal{N}_{IJ}$ is a moduli dependent symmetric matrix. We will not need its explicit form, but give it here anyway for completeness:

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{(\mathrm{Im}\,F_{IK})X^K(\mathrm{Im}\,F_{JL})X^L}{X^M(\mathrm{Im}\,F_{MN})X^N} \qquad (3.1.48)$$

with $F_{IJ} = \partial_I\partial_J F$ and F the prepotential as given in (3.1.20). We will discuss the range of validity of this action in section 3.4.

While the action is only determined up to a choice of symplectic basis (A^I, B_I), the electromagnetic energy is unambiguously defined once a space/time decomposition has been chosen. Introducing the notation

$$\mathrm{d}\mathbb{A} \equiv \mathbb{F} \equiv \mu^4 F_5 = \mathcal{F}^I \otimes B_I - \mathcal{G}_I \otimes A^I, \qquad (3.1.49)$$

(where the second equality is just (3.1.39) with the hypermultiplets taken to be constant) and denoting the spatial components of $\mathbb{F}$ as $\mathbb{F}_s$, the electromagnetic energy density is simply given by:

$$\mathcal{H}_{em}\mathrm{d}t = \frac{1}{4\gamma^2}\int_X \mathbb{F}_s \wedge *_{10}\mathbb{F}_s \qquad (3.1.50)$$

[9]Because of the subtleties associated to the selfduality constraint, the action for the vectors (which come from the selfdual 5-form) is actually determined by requiring that the correct equations of motion are reproduced, and by matching the expressions for e.g. the energy to get the right normalisation. See also [45].

This can be given a more explicit expression using special geometry. Denote

$$\eta = e^{\mathcal{K}/2} \int_X \mathbb{F}_s \wedge \Omega, \tag{3.1.51}$$

then, from (3.1.28):

$$\mathcal{H}_{em} \mathrm{d}t = \frac{1}{2\gamma^2}(\eta \wedge *_4 \bar{\eta} \ + \ g^{a\bar{b}} D_a \eta \wedge *_4 \bar{D}_{\bar{b}} \bar{\eta}). \tag{3.1.52}$$

The low energy theory thus obtained is, as expected, $\mathcal{N} = 2$ supergravity coupled to n_V vectormultiplets, with scalar manifold given by the complex structure moduli space of X. The action (3.1.43) is indeed identical to (2.2.6). In principle, (3.1.43) is just the classical low energy effective action, and though its form is fixed by supersymmetry, there could still be an infinite number of very complicated string loop and worldsheet instanton corrections. However, a miracle happens. From considerations of supersymmetry and special geometry [26], it can be shown that there cannot be any couplings between massless vector- and hypermultiplets (at the two derivative low energy level). Since the dilaton Φ in type II theory is in a hypermultiplet, and since the string coupling constant is given by e^{Φ}, there can therefore be no string loop corrections (to the vectormultiplet action, at this low energy level). Furthermore, since string worldsheet instantons always come with factors $\sim \exp \int B + iJ$, which again contain hypermultiplet fields[10], there can also be no worldsheet instanton corrections. So we conclude that the two derivative low energy effective action of $D = 4, \mathcal{N} = 2$ supergravity coupled to a number of *vector* multiplets *does not receive any quantum corrections.* It is exact. The importance of this result cannot be overestimated. It allows us to extract *exact* quantum results from classical geometry, and is at the core of such diverse things as curve counting via mirror symmetry [46], exact solutions of quantum field theories [14] and possibly even the solution of Hilbert's twelfth problem [47].

3.2 BPS states from wrapped 3-branes

3.2.1 States in string theory

String theorists tend to be a bit sloppy about the meaning of the word "state". This is in part because it is simply not clear what the full set of states of string

[10]In type IIA theory on the other hand, these are vector multiplet fields, and indeed, the 4D low energy vector multiplet action obtained from type IIA string theory does receive this kind of corrections.

theory is, and in part just a manifestation of the often convenient habit of abusing terminology in physics. We will try to explain this word in the context of string theory a bit more precisely here, though we do *not* intend to give a full, solid definition.

Stated abstractly, a state is (roughly) just something which associates expectation values to physical observables. In a quantum theory, usually, but not necessarily, states are given a Hilbert space representation. In string theory one usually talks about states with this Hilbert space framework in mind, though a more general framework might actually be more appropriate (or even necessary), as string theory in its present formulation does not allow a constructive definition of its full underlying Hilbert space. This is simply because, unlike in quantum field theory, the nonperturbative fundamental degrees of freedom of string theory are not known.[11]

Note that we are talking about states of the full spacetime theory here, *not* about the states of one of the specific worldsheet theories arising in perturbation theory from the string path integrals. Those two are not entirely disconnected however. For example the states of, say, the type IIB worldsheet theory on a cylinder in a flat ten dimensional background provide the different one-particle states of the type IIB spacetime theory 'expanded' around the flat 'vacuum'. These are the quanta of the spacetime degrees of freedom. Incidentally, replacing worldsheets by worldlines, the same applies of course to field theory (though there one needs different types of worldline theories to get different types of particle species). When the flat spacetime in which we were considering our type IIB example also contains a D-brane, we can analogously consider the states of the worldsheet theory on a strip with boundaries glued to the D-brane (cf. chapter 1). This gives a spectrum of 1-particle states living on the brane, which are the quanta of the brane degrees of freedom: ripples and so on.

As in field theory, one can consider scattering experiments, with several widely separated and effectively free particles in the far past and future, interacting for a finite time with each other in the laboratory. String theory in its original formulation does nothing more than giving a prescription how to calculate such scattering amplitudes as a perturbative series (ordered according to worldsheet topology), and the main good news here is that the results are free of UV divergencies. Now we can consistently formulate all this with strings propagating in various background geometries, with or without D-branes, and with various other fields turned on, possibly including solitonic configurations, at least provided all those objects satisfy certain equations of motion. The important thing to remember is however

[11] There does exist a sort of a second quantized *string field theory*[3], but there are quite some objections against it, and it is now widely believed that this is not the proper description of nonperturbative string theory [30, 31].

that in string theory, a priori, *scattering amplitudes are all we have to probe the physics*. We could equivalently say that the physical observables we have at our disposal (in this perturbative framework) are built out of particle annihilation and creation operators in the far past and future.

What is the meaning of the classical backgrounds then? They are definitely not directly observable. However, different backgrounds will in general give different scattering amplitudes (i.e. different expectation values of our observables), so they do define different states. Of course, by the usual path integral saddle point argument, at sufficiently large time and distance scales, the scattering amplitudes will be very much as we would expect from classical particles propagating in the given classical background. Usually in string scattering theory, the background is taken to be time independent and stable, and then one refers to the corresponding state as a (perturbative) ‘vacuum. Note that, though it may still seem odd that for example a D-particle at rest and localized at a certain point in space, could give rise to a consistent quantum state in perturbative string theory, this actually is not that strange: when the string coupling g is sent to zero, the mass of the D-particle diverges in string units, so the quantum mechanical spreading of its wave function is indeed suppressed (with respect to the string scale) when $g \to 0$. At finite g however, the use of such classical particle-like[12] solutions having finite mass is questionable in perturbative string scattering theory. We will encounter precisely such particle-like states. However, as their low energy properties, which we will study, turn out to be independent of g (in the Einstein frame), we can always imagine $g = 0$, so that the mass of the particle diverges in string units and we don't have to face this complication.

Apart from varying the background within a given perturbative string theory (like type I, type IIA, type IIB, $E_8 \times E_8$ heterotic, and so on), we can also change the perturbative string theory itself. The picture which emerged during the past five years is that after such a change we are *not* looking at another fundamental theory, but merely at another perturbative description of the same underlying theory. For example it can be proven that type IIA perturbative string theory on a space with one compact dimension of radius R in string units and string coupling g, has actually precisely the same physics as type IIB string theory on a space with one compact dimension of radius $1/R$ and string coupling g/R. A IIA string winding n times around the compact circle corresponds to a IIB string with momentum n along the circle. Scattering amplitudes are likewise mapped to each other. So, at least to all orders in perturbation theory, we have an identification of the (IIA,R,g) and the (IIB,$1/R$,g/R) state. This is T-duality. It shows very clearly the fact that the background geometry doesn't have an absolute meaning, it is just part of a (nonunique) state label. Similar equivalences between IIA and

[12]For higher noncompact branes there is no problem of this kind.

Figure 3.2: An artist view of the coninuum of states described in the text, including M-theory. What the picture wants to emphasize is the connectedness of this space, the different perturbative theories being no more than different suitable perturbative descriptions in different 'corners' of the space.

IIB exist when D-branes are added (even branes in IIA corresponding to certain odd ones in IIB), or when the geometry is made more complicated, for example Calabi-Yau compactifications, where the map is given by mirror symmetry.

Note that the above IIA and IIB perturbative string theories between which the T-duality map acts are actually both at the same time sensible perturbative descriptions *only* if $g \ll 1$ and $g \ll R$. So in some regions of the labels IIA is accurate, in some IIB, in some both, and in some none. Another kind of duality, which is much harder to test as it interchanges strong and weak coupling and hence requires always a sort of extrapolation of the theories away from perturbation theory, is S-duality. An example is the $SL(2,\mathbb{Z})$-duality of type IIB theory, already mentioned in section 3.1. Another example is the duality between heterotic theory on T^4 and type IIA theory on $K3$. Most of the tests of S-dualities rely on low energy extrapolations combined with supersymmetry constraints.

One thus arrives at a picture of one fundamental theory with a coninuum of states, of which some 'corners' have a good description in terms of one perturbative string theory, and others in terms of another. Some states might have no perturbative string description whatsoever. However, thanks to various S-dualities, the extreme strong coupling limits are often dual to weak coupling and hence perturbative string limits. The most notable exception is the strong coupling limit of type IIA theory. This is the magical, mysterious, matricious M-theory, which at low energies reduces to *eleven* dimensional supergravity! [48] It does certainly not have a consistent perturbative string description. How to do quantum physics in this regime is therefore not clear. The number one candidate for this is the matrix model of Banks, Fischler, Shenker and Susskind [2]. The picture one thus gets for this coninuum of states is shown in fig. 3.2.

3.2.2 BPS branes

An important step towards understanding nonperturbative aspects of strings has been the discovery that often, perturbative particle states in one perturbative string picture are realized as D-branes in another, S-dual, perturbative string picture. This is very useful, because strong coupling aspects of these states in one picture can then be calculated simply at weak coupling in the other picture. We will see examples of this below. Till now, quantitative results have been mainly obtained for BPS states, which have a certain amount of supersymmetry and therefore relatively controllable quantum corrections. We will especially be interested in such BPS states arising in type IIB Calabi-Yau compactifications which have a particle (or black hole) interpretation in the 4D low energy effective theory.

Let us therefore briefly recall what is meant by a BPS state in a 4D $\mathcal{N} = 2$ theory. Consider a state with definite central charge $\tilde{Z}$ and energy-momentum P_μ, and go to a Lorentz frame where the spatial momentum P_i is zero (and $P_0 = M$). On such a state, the algebra (3.1.42) becomes

$$\{Q^A_\alpha, Q^{B\dagger}_\beta\} = 2M\delta^{AB}\delta_{\alpha\beta} + 2i\epsilon^{AB}(\mathrm{Re}\,\tilde{Z}\,\Gamma^0_{\alpha\beta} + \mathrm{Im}\,\tilde{Z}\,(\Gamma^0\Gamma^5)_{\alpha\beta}). \tag{3.2.1}$$

The left hand side, considered as a matrix in $(A\alpha, B\beta)$ is non-negative definite, so the eigenvalues $M \pm |\tilde{Z}|$ of the right hand side have to be non-negative as well, implying

$$M \geq |\tilde{Z}|. \tag{3.2.2}$$

A state is called BPS when its mass is minimal for a given $\tilde{Z}$, i.e. $M = |\tilde{Z}|$. From the algebra, it then follows that half of the 8 supercharges annihilate the state, so we keep $\mathcal{N} = 1$ supersymmetry. The remaining four supercharges can be taken together [31] to form 2 complex fermionic operators, say b_1 and b_2, furnishing together a spin 1/2 representation of the little group ($SO(3)$ for a massive state) and satisfying the fermionic oscillator algebra

$$\{b_i, b_j^\dagger\} = \delta_{ij}, \tag{3.2.3}$$

The smallest representation of this algebra is four dimensional, so any BPS state (in a $D = 4$, $\mathcal{N} = 2$ theory) will be part of a multiplet of at least four states, degenerate in mass (since the $b_i^\dagger$ commute with P_0). One can construct such a multiplet by starting from one or more fermionic oscillator ground states (also called fermionic oscillator vacua)[13] $|0\rangle_r$, labeled by an index r, *which are supposed to furnish again a representation of the little group* and to satisfy $b_i|0\rangle_r = 0$, and subsequently acting on those with one or more of the creation operators $b_i^\dagger$. For single particle states, this gives typically one of the following BPS multiplets (for $M > 0$):

- a half-hypermultiplet: this has one spin 0 fermionic oscillator vacuum, 4 states. The 'spin content' is $(0^2, \frac{1}{2})$.
- a full hypermultiplet: two spin 0 fermionic oscillator vacua of opposite charge, 8 states. Spin content is $(0^4, \frac{1}{2}^2)$.
- a vectormultiplet: spin 1/2 fermionic oscillator vacuum, 8 states. Spins: $(0, \frac{1}{2}^2, 1)$.
- (BPS massive) 'gravity' multiplet: spin 3/2 fermionic oscillator vacuum, 16 states: $(1, \frac{3}{2}^2, 2)$.

Such BPS multiplets can be seen as Goldstone multiplets of broken supersymmetry.

Note that these BPS multiplets are shorter than generic massive multiplets: a generic massive state breaks all supersymmetry and is therefore part of a multiplet of at least $2^4 = 16$ states. This is why in an $\mathcal{N} = 2$ supersymmetric vacuum, a particle which is in a short BPS multiplet at vanishing coupling, is expected to stay BPS ($M = |\tilde{Z}|$) when interactions become stronger, at least as long as it exists as a stable state: the opposite would imply a discontinuous jump upwards in the number of states in the multiplet. This argument and analogous reasonings underly many results in strongly coupled string theory, since it allows one to extend some quantities obtained in the D-brane picture at weak string coupling (where the D-brane can just be represented as an object on which strings can end, without having to take into account backreaction on the ambient space), to the strong coupling regime.

Let us turn now to the D-brane representation of these 4D BPS states in type IIB string theory compactified on a Calabi-Yau manifold. Type IIB theory contains only odd branes, and since the only odd dimensional nontrivial cycles in a Calabi-Yau have dimension 3, the only possibility to get particle-like BPS states in four dimensions from D-branes, is to wrap a 3-brane about a 3-cycle in the Calabi-Yau, in such a way that one half of the supersymmetries remains

[13] the different fermionic oscillator vacua are usually also connected to each other by creation/annihilation operators constructed from certain fermionic zero modes, see e.g. [49].

intact. More precisely, the latter condition means that it should be possible to compensate the action of one half of the 8 supersymmetries of the given Calabi-Yau background, on the (fermionic) fields of the 3-brane worldvolume theory, by a worldvolume κ-transformation. We will call such an embedding *supersymmetric*. This problem was studied by Becker, Becker and Strominger in [50], resulting in the following criterion for the embedding f of the 3-brane in the Calabi-Yau, to be supersymmetric:

1. The pullback of the Kähler form to the brane should vanish:
$$f^*J = 0. \tag{3.2.4}$$
2. The pullback of the holomorphic 3-form should have constant phase:
$$f^*\Omega = e^{i\alpha}\rho(\xi)d^3\xi, \tag{3.2.5}$$
where α is a constant, the ξ^i, $i = 1, \ldots, 3$ are coordinates on the spatial part of the 3-brane, and ρ is a real positive function on the brane.

In the mathematics literature, a submanifold satisfying these conditions is known as a *special Lagrangian submanifold*. It can be shown [50] that a 3-brane satisfying these conditions has minimal volume in a given homology class (see also below). The analysis of [50] implicitly assumes triviality of all background fields, including the worldsheet $U(1)$ field strength, except for the metric and selfdual 5-form field strength (since the brane itself is a source for these fields). This is compatible with the assumptions we made earlier[14]. Considering the known curvature corrections to the D3-brane action [40], one could worry about the fact that these curvature couplings could cause the wrapped brane to act as a source for C_0 and Φ. This is not the case for supersymmetric branes, as can be shown directly from the expressions in [40], at least if the property holds that a supersymmetric cycle in a Calabi-Yau manifold is totally geodesic. The only terms to worry about are those in the DBI part of the action, since those in the WZ part vanish automatically because of the direct product structure of the brane. Under the assumption of being totally geodesic, using the fact that normal and tangent bundle of a supersymmetric cycle in a Calabi-Yau manifold are isomorphic [51], we find that the various terms in the DBI part as given in [40] cancel exactly.

Not every homology class has a supersymmetric representative. The existence of such a representative is equivalent to the existence in the 4D effective theory of a BPS state with charges corresponding to the homology class under consideration. Unfortunately, it is not easy to establish existence of supersymmetric cycles in a generic Calabi-Yau.

[14] We did not say anything about the worldsheet $U(1)$ field strength yet, but it is clear that turning on this field would add extra energy to the state, which would move it away from the BPS bound and break the remaining supersymmetry — not what we want.

3.2.3 Mass and charge of wrapped 3-branes

Let us now compute the mass and charges of such a BPS particle obtained from wrapping 3-branes about Calabi-Yau 3-cycles. This can be most easily done by studying how the D3-brane action (3.1.12) reduces to a particle action in four dimensions.

We consider the brane as a 'probe', that is, we neglect backreaction on the spacetime fields. As discussed in sections 2.1.1 and 3.2.1, this is what we should do if we want to use the branes as Dirichlet-branes in lowest order string perturbation theory. In particular this means that the moduli are taken to be constant over the four dimensional spacetime. When one does take into account backreaction, the moduli at the worldline are (for a BPS state) always at their so-called attractor values (see section 3.3), and the 'mass' one thus finds by reduction of the D3 action is not the actual mass of the state, but only the 'bare mass' of the minimal brane, as measured by an observer at $r = 0$ (which is the horizon when the BPS state is a black hole). From the point of view of an observer at infinity, the complete[15] mass of the state is in the surrounding fields (as will be shown in section 3.3). It is perhaps surprising that the (bare) mass in the probe brane picture is the same as the mass of the fields when the full backreaction is taken into account. However, this is just a consequence of supersymmetry: it is the BPS mass formula at work.

We will deal with the backreaction issue in section 3.3, and discuss the validity of the D-brane and effective field theory pictures in section 3.4. For now, we simply put the branes in a fixed background.

Aspects of the relation between wrapped D3-branes and black hole solutions are studied e.g. in [52, 53]. Boundary states for wrapped D3-branes were studied in [54].

Now with our assumptions about the background fields, the D3-brane action (3.1.12) becomes:

$$S_{D3} = -\frac{\sqrt{\pi}}{\kappa_{10}} \int d^4\xi \sqrt{-\det h_{ab}} + \frac{\sqrt{\pi}}{\kappa_{10}} \int C_4. \qquad (3.2.6)$$

We first consider the DBI part. Dimensional reduction gives

$$S_{DBI} = -\frac{\sqrt{\pi}}{\kappa_{10}} \int ds\, V_{D3}(s), \qquad (3.2.7)$$

where s is the proper time measured with the 4D metric and V_{D3} is the volume of the spatial (wrapped) part of the brane:

$$V_{D3} = \int d^3\xi \sqrt{\det h_{ij}}. \qquad (3.2.8)$$

[15]the bare mass, if any, is redshifted away

Denote with y^m the holomorphic coordinates on the Calabi-Yau manifold X. Then in general

$$h_{ij} = G_{m\bar{n}} \frac{\partial y^m}{\xi^{(i}} \frac{\partial \bar{y}^n}{\xi^{j)}}, \tag{3.2.9}$$

but if the first BPS condition $f^* J = 0$ is satisfied, we can drop the symmetrization:

$$h_{ij} = G_{m\bar{n}} \frac{\partial y^m}{\partial \xi^i} \frac{\partial \bar{y}^n}{\partial \xi^j}, \tag{3.2.10}$$

and

$$\det h_{ij} = \det G_{m\bar{n}} \left| \det \frac{\partial y^m}{\partial \xi^i} \right|^2. \tag{3.2.11}$$

Now note that

$$\|\Omega\|^2 = \Omega_{mnk} \bar{\Omega}^{mnk} = 6(\det G_{m\bar{n}})^{-1} |\Omega_{123}|^2. \tag{3.2.12}$$

Using this equation and the fact that Ω is covariantly constant over X (so $\|\Omega\|^2$ is simply a constant), we find:

$$e^{-\mathcal{K}} = i \int_X \Omega \wedge \bar{\Omega} = 48 \|\Omega\|^2 V_X, \tag{3.2.13}$$

where V_X is the volume of the Calabi-Yau X. Plugging this back into (3.2.12), we obtain

$$\det G_{m\bar{n}} = 8 \cdot 6^2 e^{\mathcal{K}} V_X |\Omega_{123}|^2. \tag{3.2.14}$$

Using this in (3.2.11) to compute $\det h_{ij}$, we get the following expression for the volume (3.2.8) of the 3-brane (if the first BPS condition is satisfied):

$$V_{D3} = 2\sqrt{2}\sqrt{V_X} e^{\mathcal{K}/2} \int |f^* \Omega|. \tag{3.2.15}$$

So we find the inequality

$$V_{D3} \geq 2\sqrt{2}\sqrt{V_X} e^{\mathcal{K}/2} | \int f^* \Omega|, \tag{3.2.16}$$

where equality is satisfied if and only if the phase of $f^* \Omega$ is constant, which was precisely the second BPS condition. So, recalling (3.1.32), we conclude that for a BPS 3-brane wrapped around a 3-cycle Γ in X, the DBI part (3.2.7) of the D3-brane action reduces to

$$S_{DBI} = -\frac{2\sqrt{2\pi}\sqrt{V_X}}{\kappa_{10}} \int |Z(\Gamma)| ds = -\frac{2\sqrt{2\pi}}{\kappa_4} \int |Z(\Gamma)| ds. \tag{3.2.17}$$

Therefore, the mass of the 4D BPS particle is:

$$M_{BPS} = \frac{2\sqrt{2\pi}}{\kappa_4}|Z(\Gamma)| = \frac{1}{\sqrt{G_N}}|Z(\Gamma)|, \tag{3.2.18}$$

with G_N the four dimensional Newton constant. This also suggests the following relationship between the geometric 'central charge' Z and the central charge $\tilde{Z}$ from the supersymmetry algebra (3.1.42):

$$\tilde{Z} = \frac{1}{\sqrt{G_N}} Z. \tag{3.2.19}$$

This can indeed be verified from the representation of the supersymmetry algebra on the type IIB fields, but we will not go into this.

The reduction of the WZ part of the D3-brane action is straightforward. Recalling the notation (3.1.49), we get from (3.2.6) and (3.1.39):

$$S_{WZ} = \frac{\sqrt{\pi}}{\mu^4 \kappa_{10}} \int \mathbb{A}, \tag{3.2.20}$$

so if our D-brane is wrapped around a cycle $\Gamma = n_I A^I + m^I B_I$, we find (recalling the definition $\gamma = \kappa_{10}\mu^4$ given under (3.1.43)):

$$S_{WZ} = \frac{\sqrt{\pi}}{\gamma} \int \Gamma \cdot \mathbb{A} = \frac{\sqrt{\pi}}{\gamma} \int n_I \mathcal{A}^I + m^I \mathcal{B}_I. \tag{3.2.21}$$

So if we consider the $\mathcal{A}$-vectors (as opposed to the $\mathcal{B}$-vectors) as the elementary massless excitations of the theory, we can identify[16] the electric charges with winding numbers about the A-cycles, and magnetic charges with winding numbers about the B-cycles.

From the equations of motion obtained from the total action $S = S_{4D} + S_{D3}$, with S_{4D} as given in (3.1.43), it follows that the electromagnetic field produced by a static brane wrapped around Γ at $r = 0$, in a spherically symmetric background, is in the notation (3.1.49), given by

$$\mathbb{F} = \omega \otimes \Gamma + *_4\omega \otimes *_X\Gamma, \tag{3.2.22}$$

where

$$\omega = \frac{\gamma}{\sqrt{4\pi}} \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{3.2.23}$$

[16] Note that this is purely a conventional identification, there is a priori no invariant distinction between A and B cycles, and no invariant distinction between electric and magnetic charges. This manifests itself as the electric-magnetic duality of low energy theory.

Here θ and ϕ are spherical coordinates about $r = 0$. This can be expressed in components using (3.1.49) and (3.1.31). More interestingly, we can use the same formulae together with (3.2.21) to calculate the electromagnetic force on a test particle with charges Γ_1, moving with velocity $\vec{v}$ in the field of a static charge Γ_2. From (3.2.21) and (3.1.49) , we find this is

$$\vec{F} = \frac{1}{2}(\Gamma_1 \cdot *\Gamma_2)\,\frac{\vec{e}_r}{r^2} + \frac{1}{2}(\Gamma_1 \cdot \Gamma_2)\,\frac{\vec{e}_r}{r^2} \times \vec{v}. \tag{3.2.24}$$

Here the dot denotes as usual the intersection product: $\Gamma \cdot \Gamma' = \int_X \Gamma \wedge \Gamma'$, which can be calculated for linear combinations of integral cycles and their Hodge duals using (3.1.31). In particular, for two particles with colinear charges $\Gamma_i = Q_i \Gamma$, we thus obtain

$$\vec{F} = Q_1 Q_2 V(z,\bar{z}) \frac{\vec{e}_r}{r^2}, \tag{3.2.25}$$

with

$$V(z,\bar{z}) \equiv \frac{1}{2}\Gamma \cdot *\Gamma = |Z|^2 + g^{a\bar{b}} D_a Z \bar{D}_{\bar{b}} \bar{Z} = |Z|^2 + 4 g^{a\bar{b}} \partial_a |Z| \bar{\partial}_{\bar{b}} |Z|, \tag{3.2.26}$$

with Z the central charge of Γ.

3.2.4 Spin

It is a much more difficult problem to determine the 4D spin of the particles obtained from wrapped branes, or more precisely, to determine the kind of $\mathcal{N} = 2$ supermultiplet one obtains by wrapping a brane about a particular cycle. In principle, this could be done by investigating the spectrum of strings ending on the D-brane, but in practice this is way too involved.

When the homology class under consideration has a unique supersymmetric representative, there is usually no problem: in that case, one expects a unique fermionic oscillator ground state (which must have spin zero since higher spin implies a degenerate ground state), on which one can build a half-hypermultiplet by acting with combinations of the 4 broken supersymmetries, as outlined above. The other half of the hypermultiplet is obtained by CPT conjugation, which simply reverses the orientation of the brane (and hence the charge).

The situation becomes problematic when there is a continuous family (or moduli space) of supersymmetric cycles in a given homology class. The superpartners of these moduli are fermionic zeromodes which can be used to build up a degenerate fermionic oscillator vacuum, on which one can then build $\mathcal{N} = 2$ multiplets in the usual way. However, the degeneracy and spin of these fermionic oscillator vacua, and hence the kind of multiplet, is a priori undetermined, and in principle a detailed analysis of supersymmetric quantum mechanics on the moduli space is needed to figure this out. This is analogous to the determination of the spin of e.g. a monopole in field theory, which can also be quite nontrivial (see e.g. [49]).

Fortunately, in some cases there is a shortcut. We would like to present a particularly simple and useful one here. The idea is the following. Suppose we can *increase* the number of remaining supersymmetries, e.g. from 4 to 8 generators (of the original 32 generators of type IIB string theory), by a change of the compactification geometry away from the brane. For example, if the compactification manifold is a $K3$ fibration over a certain base manifold B and the 3-brane a $K3$ 2-cycle fibration over a nontrivial circle in B, we can put the brane in a $T^2 \times K3$ compactification ($B \to T^2$) to double the supersymmetry. Often, more supersymmetry makes it easier to deduce the multiplet(s) produced by the brane, because there are fewer possible multiplets, and because the compactification geometry is usually simpler as the holonomy group becomes smaller. Suppose we thus manage to find the multiplet in the case with higher supersymmetry. As usual, this multiplet can be constructed with creation/annihilation operators which are combinations of the supercharges broken purely by the presence of the brane. Those supercharges are spinors invariant under the holonomy of the compactification manifold. Now when we restore the original compactification geometry, the holonomy group increases and some of these supercharges will no longer be invariant under it. Consequently, some of the states in the original extended susy BPS multiplet will become noninvariant under the holonomy and leave the multiplet of BPS states.[17] States remaining invariant will stay to form a multiplet of our original supersymmetry algebra.[18] [19] A state which always satisfies this is the top spin state of the extended multiplet, e.g. the vector in an $\mathcal{N}=4$ vector multiplet or the graviton in an $\mathcal{N}=8$ gravity multiplet. Indeed, this state is always invariant under the local Lorentz group of the compactification manifold (since there is only one highest spin state in a supermultiplet, and the spin value has to remain invariant under internal symmetries, this state must be in a singlet of the local Lorentz group of the compactification manifold), so a fortiori under the internal holonomy group. On a more pedestrian level, we could simply say that this is because the top spin field 'has all its indices in the noncompact spacetime'.

So in the cases where such a trick is possible, the brane produces at least

[17] However, by adding one ore more suitable extra fermionic zero-modes which might be induced by the change of compactification, this noninvariance can be canceled again, leading to more multiplets. To analyze this requires a more sophisticated approach, involving twists of the Lorentz algebra, see e.g. the third reference in [55].

[18] It is possible that the brane produces still more multiplets, by the mechanism discussed in the previous footnote.

[19] Note that nothing guarantees the original fermionic oscillator vacuum itself to stay in the multiplet, since the original annihilation operators can get mixed up with the original creation operators under the enlarged holonomy group.

the $\mathcal{N} = 2$ multiplet labeled by this highest spin state: a (massive BPS) vector multiplet if the extended multiplet is of vector type, or a (massive BPS) gravity multiplet if the extended multiplet is of gravity type. In the above example of the $K3$ fibration, we would get a vector multiplet. We will discuss this and other examples in much more detail in the next chapter.

3.3 BPS states from the 4D effective action: attractors

At least for sufficiently low energies, that is, for slowly varying field expectation values, it should also be possible to find the BPS states of our theory as solutions of the four dimensional low energy effective action. Furthermore, there should also be a nontrivial representation of the four broken supersymmetry generators on the fields, generating a 'multiplet of BPS solutions' (which upon quantization should reproduce the quantum multiplet structure discussed earlier in section 3.2.2). Thus, the two complex Grassmann parameter family of BPS solutions generated out of a single spherically symmetric ('spin 0') BPS solution will correspond to a half-hypermultiplet, while the family of BPS solutions corresponding to a vectormultiplet will be generated out of a more intricate family of BPS solutions.[20]

For an example of how the generation of superpartner solutions works concretely in $\mathcal{N} = 2$ supergravity, see [56], where a "hypermultiplet of solutions" is constructed from a spherically symmetric black hole solution. Another example is the $\mathcal{N} = 2$ $SU(2)$ monopole, which is worked out in detail, including non-BPS bound states, in [57]. In [58], some physical properties of the superpartner solutions are studied.

In this thesis, we will restrict ourselves mainly to static spherically symmetric BPS solutions, which thus should correspond to BPS hypermultiplets of the four dimensional $\mathcal{N} = 2$ theory.

[20] It is not entirely clear to us how exactly this family should look like, but it will definitely not be a spherically symmetric solution, since this corresponds to a hypermultiplet. We will call them 'spin 1/2 fermionic oscillator vacuum solutions', in analogy with the spin 1/2 fermionic vacuum state used to build the quantum vector multiplet. At this point, this is just a name.

3.3.1 Static spherically symmetric configurations

We take a general[21] static spherically symmetric ansatz for the metric:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(\frac{1}{f(r)^2}dr^2 + r^2 d\Omega_2^2), \tag{3.3.1}$$

or, changing variables $r = c/\sinh c\tau$, $f(r) = h(\tau)\cosh c\tau$:

$$ds^2 = -e^{2U}dt^2 + e^{-2U}(\frac{1}{h^2}\frac{c^4}{\sinh^4 c\tau}d\tau^2 + \frac{c^2}{\sinh^2 c\tau}d\Omega_2^2). \tag{3.3.2}$$

Here $U(r)$ and $f(r)$ are functions of the radial coordinate r, to be determined by the (BPS) equations of motion. We require asymptotic flatness of spacetime, so $U \to 0$ and $f, h \to 1$ at spatial infinity ($r \to \infty$, $\tau \to 0$). We furthermore assume the moduli and electromagnetic fields to be spherically symmetric and produced by a source with charge $\Gamma \in H^3(X, \mathbb{Z})$ at $r = 0$, that is, according to (3.2.22),

$$\mathbb{F} = \omega \otimes \Gamma + *_4\omega \otimes *_X\Gamma, \tag{3.3.3}$$

where $\omega = \frac{\gamma}{\sqrt{4\pi}} \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = *_4(\frac{\gamma}{\sqrt{4\pi}}\frac{e^{2U}}{h}\, \mathrm{d}\tau \wedge \mathrm{d}t)$. Note that with this ansatz, the equations of motion for $\mathbb{F}$ are automatically satisfied.

The metric (3.3.2) has Ricci scalar

$$R = 2he^{2U}\frac{\sinh^4 c\tau}{c^4}[-\ddot{U} + h(\dot{U}^2 - c^2) - 2c\frac{\cosh c\tau}{\sinh c\tau}\dot{h} + \frac{c^2}{\sinh^2 c\tau}(h - \frac{1}{h})] \tag{3.3.4}$$

Putting all fermionic fields to zero and plugging all ansätze for the bosonic fields into (3.1.43), we find for the effective action[22] modulo boundary terms:

$$\begin{aligned} S_{4D}/T \quad = \quad & -\frac{1}{2G_N}\int_0^\infty \mathrm{d}\tau\, \{h(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - c^2) + G_N\frac{e^{2U}}{h}V(z,\bar{z}) \\ & -\frac{c^2}{\sinh^2 c\tau}(h + \frac{1}{h} - 2)\}, \end{aligned} \tag{3.3.5}$$

[21] This ansatz is more general than the one used in [59]. However, it reduces to the latter by the equations of motion, as we show here.

[22] The action $S_{4D}/T = -\frac{1}{4\gamma^2}\int \mathcal{F}^I \wedge \mathcal{G}_I + \cdots$ is only determined up to symplectic duality transformations (i.e. up to choice of symplectic basis (A^I, B_I)). However, to get a manifestly consistent reduced action principle at fixed field strength $\mathbb{F}$, $\mathcal{F}^I = \mathrm{d}\mathcal{A}^I$ should only appear in the action such that the action is not varied via changes of $\mathcal{F}^I$ when the other fields are varied. With the ansatz we use, this is the case when we choose our basis such that $\Gamma \cdot B_I = 0$, and then the electromagnetic part of the action is equal to the electromagnetic energy (3.1.52), leading to the above expression for S_{4D}.

where $T = \int dt$ is the elapsed time and the dot denotes $\frac{d}{d\tau}$. The "scalar potential" $V(z,\bar{z})$ is derived from (3.1.52), (3.3.3) and (3.1.32):

$$V(z) = |Z|^2 + g^{a\bar{b}} D_a Z \bar{D}_{\bar{b}} \bar{Z} = |Z|^2 + 4 g^{a\bar{b}} \partial_a |Z| \bar{\partial}_{\bar{b}} |Z|. \tag{3.3.6}$$

Note that, as usual for static solutions, the action per unit time is equal to minus the (potential) energy (possibly up to boundary terms).

Now the equation of motion for $h(\tau)$ obtained from variations of (3.3.5) corresponding to radial diffeomorphisms (*i.e.* $\delta f(\tau) = \epsilon(\tau) \dot{f}(\tau)$ for a function f), acting on the fields U, z^a and h, together with the δh equation of motion, actually implies $h = 1 + \frac{k}{c^2} \tanh^2 c\tau$, with k an arbitrary parameter. Solutions with different values of k are related by a radial diffeomorphism corresponding to a change of the — till now arbitrary — constant c. Let us now gauge fix this remaining diffeomorphism by putting $k = 0$ (so $h = 1$).

Varying h on the other hand implies at $h = 1$ the constraint:

$$\dot{U}^2 + g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} - G_N e^{2U} V(z) = c^2. \tag{3.3.7}$$

However, since at $h = 1$, by τ-translational invariance of the reduced action (3.3.5), the left hand side is a conserved quantity along τ-translations anyway, this just determines the value of c (from the boundary conditions), leaving no nontrivial constraint independent of the other equations of motion. Bearing this in mind, we can simply put $h \equiv 1$ and write (3.3.5) as

$$S_{4D}/T = -\frac{1}{2G_N} \int_0^\infty \mathrm{d}\tau \, \{\dot{U}^2 + g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} - c^2 + G_N e^{2U} V(z)\}, \tag{3.3.8}$$

or, by completing squares, as:

$$\begin{aligned} S_{4D}/T \quad &= \quad \frac{1}{2G_N} \int_0^\infty \mathrm{d}\tau \, \{(\dot{U} \pm \sqrt{G_N} e^U |Z|)^2 + \|\dot{z}^a \pm 2\sqrt{G_N} e^U g^{a\bar{b}} \bar{\partial}_{\bar{b}} |Z|\|^2 - c^2\} \\ &\quad \pm \frac{e^U}{\sqrt{G_N}} |Z| \, \Big|_{\tau=\infty}^{\tau=0}. \end{aligned} \tag{3.3.9}$$

3.3.2 BPS solutions: the attractor flow equations

We will first try to get some intuition in the general solution of the model, and then proceed to the more precise and technical study of the equations.

Looking at the form of the action (3.3.8), we see that our system is equivalent to a particle moving on $\mathbb{R} \times \mathcal{M}$, with $\mathbb{R}$ parametrized by U and $\mathcal{M}$ the complex

Figure 3.3: Sketch of the potential $-e^{2U}V$ in which the effective particle is moving, plotted as a function of e^{2U} and the moduli z. For generic initial conditions, the particle will run away to infinity, producing an unphysical (singular, antigravitating) supergravity solution (trajectory a). If $-\dot{U}(\tau = 0)$ is sufficiently large, the particle keeps on moving all the way to $U = \infty$, at least if $\dot{z}(\tau = 0)$ is at the same time tuned such that $\dot{z}(\tau = \infty) = 0$. This corresponds to a finite energy, (nonextremal) black hole solution with ADM mass $M_{ADM} = -\dot{U}(\tau = 0)/\sqrt{G_N}$ (trajectory b). The critical trajectory, with minimal M_{ADM} for given initial z, barely reaching the top of the potential at $\tau = \infty$, corresponds to an extremal black hole (trajectory c).

structure moduli space of X, subject to the potential $-e^{2U}V$. The minus sign in front of the potential is crucial. The parameter $\tau \in \mathbb{R}^+$ plays the role of time.

Contrary to what is usually claimed in the literature, the motion on this space is *not* damped, as energy is conserved according to (3.3.7). Close to local maxima of $e^{2U}V$, we expect oscillatory solutions. Close to local minima of $e^{2U}V$, we expect unstable behavior and runaway solutions, *except* when the initial conditions are fine-tuned such that the particle climbs the hill with just enough energy to reach the top, ending its motion there (see fig. 3.3). Such finite particle action solutions correspond to finite energy field configurations in our original field theory: they are the solitons we are looking for!

Since critical points (minima in particular) of $e^{2U}V$ are thus of central importance to find finite energy solutions, let us study those a bit closer. Clearly, critical points of $|Z|$ are also critical points of V, with $V_{cr} = |Z_{cr}|^2$. The converse is *not* necessarily true. A counterexample is given e.g. by taking (essentially) $Z = z + O(z^2)$ and $g^{z\bar{z}} = 1 + O(z^2)$, which has $\partial_z V = 0$ but $\partial_z Z \neq 0$ at $z = 0$. We will mainly focus on critical points of Z, as they will turn out to be the relevant ones for BPS solutions. Using some special geometry identities, one can show [59] that at critical points of $|Z|$ we furthermore have $(\partial_a \partial_b V)_{cr} = (\partial_a \partial_b |Z|)_{cr} = 0$ and $(\partial_a \bar{\partial}_{\bar{b}} V)_{cr} = 4(\partial_a \bar{\partial}_{\bar{b}} |Z|)_{cr} = 2 g_{a\bar{b}} |Z_{cr}|^2$. Therefore, since the metric $g_{a\bar{b}}$ is positive definite, such critical points are always minima. Happily, minima are precisely what we want.

For finite U, minima of V are only minima of $e^{2U} V$ if $Z_{cr} = 0$, that is, when the 3-cycle Γ under consideration has a vanishing point in moduli space. This case will be discussed in section 3.3.3 and in chapter 5. In the generic case however, $Z_{cr} \neq 0$ and in order for $e^{2U} V$ to be critical, we need $U \to -\infty$. Therefore, recalling $\tau \to \infty$ corresponds to $r \to 0$, finite energy solutions must have $U = -\infty$ at $r = 0$. Typically, this gives a black hole with horizon at $r = 0$. Now $U = -\infty$ is *always* a critical point of $e^{2U} V$, for any finite z. Therefore, for sufficiently large $-\dot{U}(\tau = 0)$, the motion in the U direction has no turning point and the particle runs all the way to the $U = -\infty$ top of the $-e^{2U} V$ hill, at least if we tune $\dot{z}(\tau = 0)$ such that at the same time $\dot{z} \to 0$ when $\tau \to \infty$ (if not, the motion will in general explode due to unstability of the system, and the particle will fall back). Hence in this case we always find a finite energy solution, with no preferred moduli values at the horizon. This is not surprising, since the ADM mass of the solution is given by $M_{ADM} = -\dot{U}/\sqrt{G_N}$, and we indeed expect (nonextremal) black hole solutions for any sufficiently large value of M_{ADM}. Now for fixed initial values of the moduli, when we start to lower $-\dot{U}$, at a certain point, $-U$ will fail to reach infinity for all initial moduli velocities and consequently finite energy solutions no longer exist. The critical trajectory, with minimal ADM mass for the given moduli at $\tau = 0$, will have $\dot{U} = 0$ at $\tau = \infty$ and we expect[23] it to just reach a minimum of V there (since minimal V means minimal 'push-back' force $-\partial_U(-e^{2U} V) = 2e^{2U} V$ in the positive U direction, leaving no room to lower the initial $|\dot{U}|$ even more while still reaching the top of the mountain in the U direction). Consequently, for such a critical solution we have both kinetic and potential particle energy equal to zero at $\tau = \infty$, implying $c = 0$ according to (3.3.7). Thus we expect to find solutions with minimal mass for the given charge and moduli at infinity, which furthermore are extremal ($c = 0$). Again, this is not surprising: such solutions are simply the extremal black holes of $\mathcal{N} = 2$ supergravity!

From the above argument, we thus expect that the moduli at $r = 0$ (the

[23] This limit is of course a bit subtle, but we will see below that this intuition is indeed confirmed by the equations.

horizon) will be fixed at a critical point of V, invariant under continuous variations of the moduli at infinity. This is the famous *attractor mechanism* [60]. The attractor property has nothing to do with damping of the effective particle motion, as often wrongly stated. Quite the contrary: it arises from the finite energy condition thanks to the *unstability* of the particle motion, in a way which is actually very common in soliton physics.

Note that *extremal* black holes (i.e. solutions with $c = 0$) are *not* necessarily BPS. If we have a critical point of V at $z = z_*$ which is not a critical point of Z, and we take say $z(\tau = 0) = z_*$ (so $z =$ const.), straightforward solution of the equations of motion derived from (3.3.8) yields the $c = 0$ solution $U = -\ln(1 + \sqrt{G_N V}\tau)$, with $\sqrt{G_N} M_{ADM} = \sqrt{V} = \sqrt{|Z|^2 + 4\partial^a|Z|\partial_a|Z|} > |Z|$. The converse is true however: BPS black holes are always extremal.

Let us see now how we get the BPS solutions from the equations.

From (3.3.9), it is clear that the reduced action at fixed values of c and the boundary moduli, has a minimum when

$$\begin{aligned} \dot{U} &= -\sqrt{G_N} e^U |Z| && (3.3.10) \\ \dot{z}^a &= -2\sqrt{G_N} e^U g^{a\bar{b}} \bar{\partial}_{\bar{b}} |Z|. && (3.3.11) \end{aligned}$$

Equation (3.3.7) then implies $c = 0$, so $r = 1/\tau$ and

$$ds^2 = -e^{2U} dt^2 + e^{-2U} d\vec{x}^2. \qquad (3.3.12)$$

which has Ricci curvature[24]

$$\begin{aligned} R^t{}_t &= e^{2U}\tau^4 \ddot{U} \\ R^\tau{}_\tau &= e^{2U}\tau^4 (2\dot{U}^2 - \ddot{U}) \\ R^\theta{}_\theta = R^\phi{}_\phi &= e^{2U}\tau^4(-\ddot{U}) \end{aligned} \qquad (3.3.13)$$

Assuming asymptotic flatness, *i.e.* $U \to 0$ at spatial infinity, these solutions saturate the BPS bound

$$M = \frac{1}{\sqrt{G_N}} |Z(\tau = 0)|. \qquad (3.3.14)$$

Here we have dropped the $\tau = \infty$ boundary term since (3.3.10) and (3.3.11) imply that both e^U and $|Z|$ are monotonously decreasing functions (see also equation (3.3.18) below) satisfying the estimate $e^U|Z| \leq \min\{|Z(0)|/(1+\sqrt{G_N}|Z(\infty)|\tau), |Z|\}$,

[24]The nonextremal metric with $c \neq 0$ is obtained from this one simply by replacing the factor τ^4 by $\sinh^4 c\tau / c^4$.

and hence $e^U|Z| \to 0$ when $\tau \to \infty$. Note that this estimate also implies that when $Z(\infty) \neq 0$, the solution is a black hole with horizon at $r = 0$. Indeed, from the form of the metric (3.3.2) and direct analysis of equation (3.3.10) in the limit $\tau \to \infty$, we then get an $AdS_2 \times S^2$ near horizon metric

$$ds^2 = -\frac{1}{G_N|Z(\infty)|^2\tau^2}dt^2 + G_N|Z(\infty)|^2\tau^2 d\vec{x}^2, \tag{3.3.15}$$

with horizon area

$$A = 4\pi G_N|Z(\infty)|^2, \tag{3.3.16}$$

and hence with thermodynamic Beckenstein-Hawking entropy

$$S_{BH} = \frac{A}{4G_N} = \pi|Z(\infty)|^2. \tag{3.3.17}$$

Choosing the other sign possibility in (3.3.9), does not give an acceptable[25] solution: now e^U and $|Z|$ are *increasing* functions, satisfying the estimate $e^U|Z| \geq |Z(\tau_*)|/(1-\sqrt{G_N}|Z(\tau_*)|\tau)$ for any fixed τ_*, hence any nontrivial solution develops a singularity at finite distance from the origin, and has infinite action and energy. Furthermore, these solutions would be gravitationally repulsive. They correspond to the unstable solutions falling down the hill along BPS trajectories in the effective particle picture. Note however that in a *finite region* of spacetime, preventing τ to run to infinity, such solutions might be acceptable and possibly important.[26]

Equations (3.3.10) and (3.3.11) are called the *attractor flow equations.* This is because their solutions converge to fixed moduli values at $\tau = \infty$, namely to those values for which $|Z|$ is minimal. Indeed, from (3.3.11), we see:

$$\frac{d}{d\tau}|Z| = -4\sqrt{G_N}e^U g^{a\bar{b}}\,\partial_a|Z|\,\bar{\partial}_{\bar{b}}|Z| < 0, \tag{3.3.18}$$

so the moduli will flow "down the hill" till a minimal value of $|Z|$ is reached (with vanishing norm of its gradient). This is intuitively clear in the brane picture, since we expect the 3-brane to "pull" the moduli such that its volume is as small as possible. It is of course also precisely what we expected from the intuitive particle picture described above.

To describe the flow in moduli space, it is sometimes convenient to introduce a 'scale' variable $\mu = e^{-U}/\sqrt{G_N}$ in terms of which the attractor flow equations

[25] at least not for our purposes; in [61], it is discussed in what sense such solutions could still be meaningful.

[26] a comparable situation is perhaps the occurrence of exponential "tunneling" solutions of Maxwell's equations between two dielectrics.

take the form

$$\frac{d\mu}{d\tau} = |Z| \quad (3.3.19)$$

$$\mu\frac{dz^a}{d\mu} = -g^{a\bar{b}}\bar{\partial}_{\bar{b}} \ln |Z|^2, \quad (3.3.20)$$

suggesting an interpretation as a renormalisation group flow in moduli space [47].

The attractor flow equations can also be obtained from the requirement of conservation of one half of the supersymmetries [60, 62, 22]. Solutions and generalisations have been discussed for example in [47, 62, 22, 23, 63, 64, 21].

Before we go on, we would like to make the following side remark (which is further not consequential for this thesis). Suppose we restrict the region of spacetime which we consider to $r > \epsilon$ ($\tau < 1/\epsilon$), with ϵ very small. The action principle tells us that for *any* value of the fields on $\tau = 1/\epsilon$ and $\tau = 0$, we can find a solution of the equations of motion. Suppose we keep z (and U) fixed at $\tau = 0$ and $\tau = 1/\epsilon$, deviating a finite amount from the attractor values at the latter point. Now let $\epsilon \to 0$. Because of the instability of the effective particle system, the solution $(U,z)_\epsilon$ of the equations of motion given these boundary values will be arbitrarily close to the attractor flow solution $(U,z)_{attr}$ for τ not too close to $1/\epsilon$. More precisely, for any *fixed and finite* τ we have $\lim_{\epsilon\to 0}(U,z)_\epsilon(\tau) = (U,z)_{attr}(\tau)$. Also, some closer analysis of the action (3.3.8) shows that *for black holes*, the action of the deviating solution approaches that of the attractor solution: $\lim_{\epsilon\to 0} S[(U,z)_\epsilon] = S[(U,z)_{attr}] = -T|Z(\tau = 0)|/\sqrt{G_N}$ (essentially because the deviating solution only differs significantly from the attractor one close to $r = \epsilon$, but there the factor e^U is very small, damping out all difference in action[27]). On the other hand, the value of the moduli at $r = \epsilon \to 0$ (hence at the horizon) of $\lim_{\epsilon\to 0} z_\epsilon$ will, by construction, *differ* from the attractor values by a finite amount. The limiting solution is discontinuous and therefore not acceptable as a smooth classical solution, but it indicates the presence of degrees of freedom at the horizon of which the excitations have zero energy as seen from any finite radius. The solutions deviating from the attractor flow infinitesimally close the horizon would be as important as the attractor flows themselves in a Feynman path integral. This can be understood as an infinite redshift effect. The degenerate excitations on the horizon give the black hole an entropy, which by extensivity we expect to be proportional to the *area* A of the horizon — in agreement with the Beckenstein-Hawking formula. Unfortunately, when one actually tries to count the excitations on the horizon in this way, the result is divergent when arbitrarily short wavelengths are allowed. A cutoff of the order of the Planck scale is needed.

[27]this is not the case for non-black hole solutions

String theory should effectively provide such a cutoff in principle, but going into the subject of string theory counting of black hole states [65] would lead us way to far afield, so we will drop the subject here.

3.3.3 Existence of the BPS state

We can trust the low energy supergravity approximation if the curvature stays sufficiently small. From (3.3.13) and the attractor flow equations, it follows that this is the case if and only if

$$G_N{}^2 e^{4U} \tau^4 V(z) \ll 1. \tag{3.3.21}$$

Note that, as long as the curvature is everywhere finite, we can always make it arbitrarily small by adding (equal) charges at $r = 0$. Indeed, obviously, the solution $\Phi \equiv (U, z^a)$ to the attractor flow equations for multiple charge $N\Gamma$ can be obtained from the solution for single charge Γ as

$$\Phi_N(\tau) = \Phi_1(N\tau). \tag{3.3.22}$$

The curvature then scales correspondingly as

$$R_N(\tau) = \frac{1}{N^2} R_1(N\tau), \tag{3.3.23}$$

so we can always make the curvature, if finite, arbitrarily small (such that it satisfies (3.3.21)) by taking N large. The same argument holds for the derivatives of the curvature (and of all the other fields). So in general, we can expect supergravity to be reliable in the large N limit, which is of course no surprise.

Let us see what the attractor flow equations can teach us about the existence of a BPS hypermultiplet[28] with given charge Γ and given moduli at spatial infinity. There are three possibilities:

1. $|Z|$ has a nonvanishing local minimum, say at $z = z_*$. The flow gradient field (given by the r.h.s. of the attractor flow equations) in moduli space is shown schematically in fig. 3.4. In this case, as outlined above, we get a black hole solution with near horizon metric given by (3.3.15). The condition (3.3.21) for our supergravity approximation to be valid, translates then to $|Z(z_*)| \gg 1$, which can always be obtained by taking a sufficient amount of charges. Therefore, in this case, the low energy theory establishes the existence of a BPS hypermultiplet of the given charge.

[28] A spherically symmetric solution gives a hypermultiplet, as we argued earlier. The nonexistence of a static spherically symmetric solution does however not exclude the existence of non-spherically symmetric solutions, and hence of multiplets built on those solutions.

Figure 3.4: Typical flow gradient field in moduli space (represented by the z-plane) close to a generic attractor point with finite Z_{crit}. The gradient vectors vanish at the attractor point.

We argued earlier that the existence of a BPS state is equivalent to the existence of a supersymmetric representative in the given homology class, so a remarkable connection emerges here between existence of special Lagrangian embeddings in Calabi-Yau manifolds and existence of solutions to the attractor equations [47]. More concretely, in the case at hand, this connection predicts the existence of a special Lagrangian representative in homology classes Γ which have nonzero minimal $|Z(\Gamma)|$. From the mathematical point of view, this is a rather nontrivial conjecture. But one can go even further: the degeneracy of the black hole ground state predicted by the Beckenstein-Hawking entropy formula grows for large Z according to (3.3.17) as $\exp[\pi|Z_{attr}|^2]$. This degeneracy supposedly is (roughly) equal to the dimension of the moduli space of the wrapped 3-branes, that is, the moduli space of special Lagrangian embeddings of the given homology class [66]. The mathematical consequences of this statement are enormous, as was realized and worked out by Moore in [47], and further in [67]. It could even lead to a solution of Hilbert's twelfth problem [47]. We will not go deeper into this fascinating but very difficult subject however.

2. Z has a zero at a regular point $z = z_*$ in moduli space. Regularity implies that the holomorphic period $Y = \int_\Gamma \Omega$ then must have a zero at $z = z_*$. The holomorphic equation $X(z) = 0$ in a regular neighborhood of z_*, actually defines a complex codimension one set of zeros of Z in moduli space. Take

Figure 3.5: Flow gradient field in the transversal Y-plane close to a regular locus $Y=0$ in moduli space where $Z=0$. The gradient vectors do not vanish at the attractor point.

Y as the coordinate transverse to the zero locus.[29] Close to $Y=0$, the attractor flow is approximately given by

$$\dot{U} = -k_1 e^U |Y| \tag{3.3.24}$$
$$\dot{Y} = -k_2 e^U Y/|Y|, \tag{3.3.25}$$

where k_1 and k_2 are certain positive constants. See fig. 3.5. The approximate solution for small Y of this system is

$$\arg Y = \text{const.} \tag{3.3.26}$$
$$|Y| = k_3 - k_4 \tau \tag{3.3.27}$$
$$U = -k_5 + k_6 |Y|^2, \tag{3.3.28}$$

where the k_i are again positive constants, whose precise relation to each other and to k_1 and k_2 is not important here.

We've got a problem here: from (3.3.27), it follows that we reach $Y=0$ at *finite* τ, and that beyond this point, we cannot continue the solution! The flow breaks down; a spherically symmetric BPS solution to the equations

[29]Here we are assuming Y does not have a double zero in the point under consideration (we classify such a point as singular). If it does, the analysis and conclusions will change drastically. See item (3).

of motion does not exist in this case. Continuing the flow by just keeping $Y = 0$ in the interior does not yield a solution of the equations of motion. Continuing the flow by inverting the sign of the r.h.s. of the attractor flow equations does not give an acceptable solution either, because it is highly singular and antigravitating, as discussed earlier. Note that if the metric component $g^{Y\bar{Y}}$ has nonvanishing gradient at $Y = 0$ (which is the generic case), the point under consideration will not even be a stationary point of V, so, from the effective particle picture, we do not expect it to be a true attractor point for *any* solution, BPS or not.

On the other hand, the curvature (and all its derivatives) stays everywhere finite, so (at least for large charge N) we can trust this low energy analysis. We are therefore led to conclude that there does *not* exist a BPS hypermultiplet with the given charge in this case (that is, for moduli at infinity sufficiently close to the $Y = 0$ locus).

Fortunately, this is exactly what we would expect physically. Indeed, if such a BPS hypermultiplet would exist, we would have a massless charged hypermultiplet in a vacuum with $Y = 0$. This should however give logarithmic corrections to the dynamics of the $U(1)$ vector multiplet, and in particular there should be a singular one-loop correction to the moduli space geometry, as in [68], producing a singularity at $Y = 0$, which would contradict our initial regularity assumption. So this all fits nicely.

Finally note that these considerations do *not* exclude the existence of an extremal ($c = 0$) but non-BPS solution to the equations of motion with the given charge and moduli at spatial infinity (actually, by lowering the ADM mass of a nonextremal black hole, one eventually expects to end up with an extremal one, so extremal solutions should generically exist).

3. Z has a zero at a singular or boundary point of moduli space. In this case, the analysis is much more subtle an depends strongly on the case at hand. We give three examples here, following and extending [47].

Example 1

Consider a one parameter family of Calabi-Yau manifolds near a 'large complex structure' limit[30]. Denoting the modulus with t and taking the large complex structure limit to be at $\operatorname{Im} t \to \infty$, the Kähler potential and metric at large $\operatorname{Im} t$ for this case can be taken to be

$$e^{-\mathcal{K}} = (\operatorname{Im} t)^3 \tag{3.3.29}$$

[30]The precise geometrical meaning of this is not important at this point. The reader not familiar with such terminology can follow the example starting from the given expressions for Kähler potential and periods.

Figure 3.6: Flow gradient field in the $(e^U, \mathrm{Im}\, t)$-plane for a cycle with vanishing Z in the large complex structure limit $\mathrm{Im}\, t \to \infty$.

$$g_{t\bar{t}} \;=\; \frac{3}{4}(\mathrm{Im}\, t)^{-2}. \tag{3.3.30}$$

There are four independent periods, (asymptotically) proportional to $1, t, t^2, t^3$. Combinations of the first two periods have vanishing central charge in the limit $\mathrm{Im}\, t \to \infty$:

$$Z = \frac{q_0 + q_1 t}{\sqrt{(\mathrm{Im}\, t)^3}} \to 0. \tag{3.3.31}$$

Let us take $q_1 = 0$ for simplicity. The more general case yields qualitatively the same results, though some exponents etc. in the solutions are different. The attractor flow equations in this limit are then

$$\dot{U} \;=\; -q_0 \sqrt{G_N} e^U (\mathrm{Im}\, t)^{-3/2} \tag{3.3.32}$$

$$\dot{t} \;=\; 2iq_0 \sqrt{G_N} e^U (\mathrm{Im}\, t)^{-1/2} \tag{3.3.33}$$

(fig. 3.6) with large τ asymptotics:

$$e^U \;\sim\; \tau^{-1/4} \tag{3.3.34}$$

$$t \;\sim\; i\,\tau)^{1/2} \tag{3.3.35}$$

This solution has a naked singularity: the curvature diverges at the origin of spacetime ($\tau \to \infty$). At a finite radius, the supergravity approximation breaks down. So in principle, we cannot draw conclusions about the existence in the full string theory of a BPS hypermultiplet with the given charge (close to the large complex structure limit). However, the (physically undesirable) appearance of a naked singularity for all values of the parameters leads us rather to believe that such a hypermultiplet does *not* exist. This would also be in agreement with physical expectations based on the brane picture and mirror symmetry. Indeed, a 3-brane wrapped around the cycle considered here corresponds in the type IIA theory on the mirror Calabi-Yau to a 0-brane [51].[31] A 0-brane in a background with 32 supercharges (e.g. a torus compactification) is in a BPS massive gravity multiplet (256 states). By the trick discussed at the end of section 3.2.4, we conclude that it should also give rise to a BPS massive gravity multiplet in our Calabi-Yau background (8 supercharges). This has 16 states, with a *spin 3/2* fermionic oscillator vacuum. Therefore, we indeed do not expect a valid spherically BPS solution of the bosonic low energy effective action for this charge (as this would give a hypermultiplet). It would be interesting to check whether a nonsingular non-spherically symmetric multiplet of BPS solution *can* be constructed. We have not done this.

Example 2

For our second example, we consider a vanishing cycle near a conifold locus in moduli space. Since the internal Calabi-Yau degenerates at this point, we cannot necessarily trust the supergravity approximation. However, the results obtained in this case are interesting, so we will ignore this potential problem and proceed. For simplicity, we will again consider only one modulus, z, which we take to be the period of the vanishing cycle. Then the Kähler potential and metric for $z \to 0$ can be taken to be:

$$e^{-\mathcal{K}} = k_1^2 + \frac{1}{2\pi}|z|^2 \ln |z|^2 + k_2 \mathrm{Re}\, z \qquad (3.3.36)$$

$$g_{z\bar{z}} = \frac{1}{2\pi k_1^2} \ln |z|^{-2}, \qquad (3.3.37)$$

where k_1 and k_2 are positive constants. This is similar to the geometry near the singularities $u = \pm 1$ in the Seiberg-Witten plane, discussed in the previous chapter (see also fig. 2.3). The central charge of N times the vanishing cycle close to $z = 0$ is

$$Z = \frac{N}{k_1} z. \qquad (3.3.38)$$

[31] Actually it is not necessary to go to the mirror for this argument: the 3-brane considered here is a 3-torus, which as a wrapped brane in a 6-torus compactification would also give the $\mathcal{N} = 8$ BPS massive gravity multiplet in four dimensions. However, the "trick" part of the argument is perhaps clearer in the 0-brane picture, and furthermore it is nice to see how the mirror symmetry conjecture works out fine again.

Figure 3.7: The potential $V(z,\bar{z})$ for the conifold cycle as a function of z near its minimum at the conifold point $z=0$. The gradient vectors vanish at the attractor point.

The form of the corresponding potential $V(z,\bar{z})$ near $z=0$ is shown in fig. 3.7. The flow gradient field in the z-plane is given in fig. 3.8. The attractor flow equations in this limit are

$$\dot{U} = -\frac{1}{k_1}\sqrt{G_N}Ne^U|z| \qquad (3.3.39)$$

$$\dot{z} = -2\pi k_1\sqrt{G_N}Ne^U\frac{z}{|z|}\frac{1}{\ln|z|^{-2}}, \qquad (3.3.40)$$

with solution (approximately for $z\to 0$) given by:

$$\arg z = \text{const.} \qquad (3.3.41)$$

$$|z|\ln|z|^{-1} = |z_0|\ln|z_0|^{-1} - \pi k_1\sqrt{G_N}N\,\tau \qquad (3.3.42)$$

$$U = \frac{1}{2\pi k_1^2}(|z|^2\ln|z|^{-2} - |z_0|^2\ln|z_0|^{-2}). \qquad (3.3.43)$$

Again, the flow terminates at finite distance from the origin, namely at

$$1/r = \tau = \tau_* \equiv \frac{1}{\pi k_1\sqrt{G_N}N}\,|z_0|\ln|z_0|^{-1}. \qquad (3.3.44)$$

However, this time the flow *can* be smoothly[32] continued beyond this point,

[32]actually only once continuously differentiable, but this can be interpreted as an artifact of the two derivative low energy approximation we are using

Figure 3.8: Flow gradient field for the vanishing conifold cycle in the transversal z-plane close to the conifold locus $z = 0$ in moduli space.

just by taking $z(\tau) = 0$ and $U(\tau) = U(\tau_*)$ for $\tau \geq \tau_*$ (fig. 3.9). Indeed, $\dot{U}, \dot{z} \to 0$ when $\tau \nearrow \tau_*$, as can be seen immediately from the attractor flow equations. Also, the curvature is easily seen to stay everywhere finite (in fact arbitrarily small for large N or small z_0). Only the higher derivatives of curvature and modulus diverge close to τ_*. So strictly speaking, the low energy approximation breaks down there. Nevertheless, we believe that higher derivative corrections will not alter qualitatively the conclusions; we expect those merely to smooth out the solution, e.g. by replacing the constant "inner" solution by an almost-constant exponentially decreasing one.[33] In any case, as long as we stick to the low energy approximation, this solution is perfectly well behaved and physically acceptable. Therefore, we are here rather led to conclude that such a BPS hypermultiplet indeed exists. Again, this is in agreement with physical expectations: starting with the work of Strominger [68], one has argued from various points of view that there should indeed be a BPS hypermultiplet near the conifold with charge corresponding to the vanishing cycle.[34] It is this hypermultiplet which is held responsible

[33]Morally speaking, this is analogous to the fact that we do not expect the motion of a falling piece of inelastic mud to the surface of the earth, to be influenced much by higher derivative corrections, though its idealized motion definitely has divergent derivative of acceleration at the 'attractor point' (the surface of the earth).

[34]The issue whether the $N \geq 2$ states can be bound (i.e. of single particle type) cannot directly be addressed here. Of course, if there exist an $N = 1$ state, there will exist an *unbound* N-particle state for any N (see also next section), but to check for *bound* states, one should find out more

Figure 3.9: The space dependence of the modulus z for the conifold attractor. Inside a critical radius $r = r_*$, the modulus is constant and equal to its attractor value $z = 0$.

for the appearance of the conifold singularity in moduli space (hence in the low energy dynamics): in the low energy action as given earlier, this hypermultiplet is, as all other massive states, integrated out. However, precisely at the conifold point this hypermultiplet becomes *massless* and we should no longer integrate it out: if we still do so, the one loop correction of this hypermultiplet produces a (logarithmic) singularity in the low energy effective action of the other massless fields.

Note that an alternative solution of the equations of motion is obtained by inverting the signs in the r.h.s. of the attractor flow equations for $\tau > \tau_*$. However, again, as discussed earlier, such a solution is very unphysical: it is antigravitating, has singular curvature at finite radius, and infinite energy — in stark contrast with the smooth finite energy solution found by stopping the flow.

'Conifold transitions' are not possible either in this case, since this requires more than one cycle vanishing at the same time [69].

The unphysical 'inverted flow' solution has appeared a number of times, in various guises, in the literature (e.g. [64, 23]), causing quite some confusion. The finite energy solution was missed because of the fact that the earlier

about the dynamics, e.g. in a low energy soliton moduli space approximation. It has been argued from the precise form of the loop corrections to the moduli space geometry [68] that $N \geq 2$ BPS bound states do not exist in this case.

analysis was done in a formalism and with coordinates on moduli space which precisely become ill defined at the conifold singularity of moduli space. To get an idea what goes wrong, consider the (nonholomorphic) coordinates

$$x = \operatorname{Im}(e^{-i\alpha} z) \tag{3.3.45}$$

$$y = \operatorname{Im}(e^{-i\alpha} iz \ln z^{-1}), \tag{3.3.46}$$

where $\alpha = \arg z(\tau = 0)$. These are (roughly) the natural coordinates arising in the formalism of [23] as the imaginary parts of a symplectic period basis in a particular Kähler gauge. The advantage of these coordinates is that they are harmonic functions on spacetime for BPS solutions: indeed, using $\Delta f(\tau) \sim \partial_\tau^2 f(\tau)$ and (3.3.42) we see that trivially $\Delta x = \Delta y = 0$. One would then be tempted to conclude that the solutions are either constant or $1/r$ divergent. This would correspond to an 'inverted flow' solution. For our solution with a flow terminating at $1/r = \tau_*$, we have $x = \text{const.} = 0$ and $y \sim \tau_* - \frac{1}{r}$, but the latter only for $r > 1/\tau_*$. For $r < 1/\tau_*$ we have $y = 0$. Our solution is smooth (continuously differentiable) when expressed in good coordinates on moduli space, but not when expressed in the y coordinate, simply because y is *not* a good coordinate around the conifold point $y = 0$!

The proper treatment of the (exterior) conifold attractor flow, and in particular the fact that the flow terminates, appeared for the first time in [47]. In [21], we observed that the flow could be smoothly continued to the interior, and the physical acceptability of this solution (and its generalization to weakly gravitating effective Yang-Mills BPS states) was emphasized. Recently, analogous 'spontaneous cut-off' solutions have emerged in the context of Dirac-Born-Infeld brane dynamics [70, 71].

Properties of this and similar solutions will be discussed in detail in chapter 5.

Example 3

As a third example we consider a zero of order $1 + \gamma$ of Z at a point with nondegenerate metric, say $Z = z^{1+\gamma}$ and $g_{z\bar{z}} = 1$. Again, starting sufficiently close to $Z = 0$ will give only small variation of the metric function U, while a short calculation, analogous to the above examples, gives for the central charge itself if $\gamma \neq 1$:

$$|Z| = [(\gamma^2 - 1)\tau + |Z_0|^{\frac{1-\gamma}{1+\gamma}}]^{\frac{1+\gamma}{1-\gamma}}, \tag{3.3.47}$$

and if $\gamma = 1$:

$$|Z| = |Z_0| e^{-4\tau} \tag{3.3.48}$$

The phase is again constant. For $-1 < \gamma \leq 0$ we find a flow which stops at finite τ and cannot be continued. These cases also correspond to (divergent) maxima of V in stead of the required minima. For $0 < \gamma < 1$ the flow stops at finite τ but can be continued as $Z = 0$ to the interior region. For $\gamma \geq 1$, the flow runs smoothly all the way to $\tau = \infty$. At least for sufficiently small Z_0, none of these solutions is a black hole.

It could be interesting to compare these solutions with the perturbative heterotic BPS string states which should correspond to it by heterotic - type II duality. This would extend the results in [72].

3.3.4 Multicenter case

The previous discussion is readily extended to the extremal multicenter case *with equal charges*[35], by introducing an effective "radial" coordinate

$$\tau \equiv \frac{1}{N} \sum_{i=1}^{N} \tau_i, \tag{3.3.49}$$

where i runs over the N different centers and τ_i is defined as τ in the previous discussion, relative to the ith center. Surfaces of equal τ can be considered as equipotential surfaces for the multi-source configuration (fig. 3.10). The ansatz for the metric is the extremal

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} d\vec{x}^2. \tag{3.3.50}$$

The electromagnetic field is given by superposition and has exactly the same form as (3.3.3), with $\omega \equiv \sum_i \omega_i = N *_4 (\gamma \frac{e^{2U}}{\sqrt{4\pi}} \, \mathrm{d}\tau \wedge \mathrm{d}t)$. The scalar fields are supposed to be functions of τ only.

Since the complete setup is formally the same as for the spherically symmetric case, so are the attractor flow equations. Therefore, everything said about the (extremal) spherically symmetric case applies to the (extremal equal charge) multicenter case as well.

It is easy to see that the total force between equal static BPS particles indeed vanishes. The magnitude of the (repulsive) static electromagnetic force is, from (3.2.25), given by

$$\frac{1}{r^2}(|Z|^2 + 4 g^{a\bar{b}} \partial_a |Z| \bar{\partial}_{\bar{b}} |Z|). \tag{3.3.51}$$

[35] Multicenter solutions with charges corresponding to different elements of $H^3(X, \mathbb{Z})$, and in particular with mutually nonlocal charges (=nonzero intersection product), are much more difficult to study, and their existence is not clear.

Figure 3.10: Some surfaces of equal τ in the 2-center case.

The first term is compensated by gravity, the second one by scalar exchange.

One could also contemplate replacing the function τ with an arbitrary harmonic function. However, in order to get nonsingular solutions of the attractor flow equations, this function should be bounded from below. Therefore all consistent harmonic functions are equivalent to a distribution of equal charge sources. In particular single center harmonic functions containing spherical harmonics do not give consistent solutions.

3.3.5 Attractor technology

Here we would like to give some formulas which can be useful for solving the attractor flow equations (3.3.10) - (3.3.11). The results in this section provide among other things an intrinsic, Kähler gauge independent formulation of the formalism developed in [22, 23].

We will put the Newton constant $G_N \equiv 1$ throughout this section.

Using $\partial_i |Z| = \frac{1}{2} e^{i\alpha} D_i Z$, with $\alpha = \arg Z$ (and $D_i = \partial_i + \frac{1}{2}\partial_i \mathcal{K}$), we can rewrite the attractor flow equations as

$$e^{-i\alpha} \dot{U} = -e^U \bar{Z} \tag{3.3.52}$$
$$e^{-i\alpha} \dot{z}^a = -e^U g^{a\bar{b}} \bar{D}_{\bar{b}} \bar{Z}, \tag{3.3.53}$$

From this, a straightforward but slightly tedious calculation shows that the central charge Z' of a charge Γ' satisfies:

$$\frac{d}{d\tau}(e^U e^{-i\alpha} Z') = -e^{2U}[Z'\bar{Z} + g^{a\bar{b}} D_a Z' \bar{D}_{\bar{b}} \bar{Z}] \tag{3.3.54}$$

and

$$\frac{d}{d\tau}(e^{-U}e^{-i\alpha}Z') = -[-Z'\bar{Z} + g^{a\bar{b}}D_a Z'\bar{D}_{\bar{b}}\bar{Z}]. \tag{3.3.55}$$

Taking the real part of (3.3.54) and the imaginary part of (3.3.55) and using (3.1.30)-(3.1.31) gives the following nice geometric formulae:

$$\frac{d}{d\tau}\mathrm{Re}\,(e^{U}e^{-i\alpha}Z') = -\frac{1}{2}e^{2U}\Gamma' \cdot *\Gamma \tag{3.3.56}$$

$$\frac{d}{d\tau}\mathrm{Im}\,(e^{-U}e^{-i\alpha}Z') = -\frac{1}{2}\Gamma' \cdot \Gamma. \tag{3.3.57}$$

Since the intersection product is a topological invariant, the second of these equations can readily be integrated:

$$\mathrm{Im}\,(e^{-i\alpha}Z') = e^{U}\{-\frac{1}{2}\Gamma' \cdot \Gamma\,\tau + [\mathrm{Im}\,(e^{-U}e^{-i\alpha}Z')]_{\tau=0}\}. \tag{3.3.58}$$

This is a very powerful identity, since by choosing for Γ' a basis of 3-cycles Γ_Λ, it gives, provided U is known, the *solution* of the attractor equations in terms of coordinates $\mathrm{Im}\,(e^{-i\alpha}Z_\Lambda)$ on moduli space (where some care has to be taken to check that the flow does not pass through a singularity where it could stop, as in some of the examples above). By choosing a (position dependent) normalisation of Ω such that Z is real and $\mathcal{K} = 2U$ (this is a choice of Kähler gauge[36]), and writing $Z_\Lambda = e^{\mathcal{K}/2}X_\Lambda$, (3.3.58) becomes for $\Gamma' = \Gamma_\Lambda$ and $\Gamma = N^\Lambda\Gamma_\Lambda$:

$$\mathrm{Im}\,X_\Lambda = -\frac{1}{2}Q_{\Lambda\Sigma}N^\Sigma\,\tau + e^U[\mathrm{Im}\,X_\Lambda]_{\tau=0}, \tag{3.3.59}$$

where $Q_{\Lambda\Sigma} = \Gamma_\Lambda \cdot \Gamma_\Sigma$ is the intersection matrix. This expression can be compared with [22, 23].

Note that (3.1.30) implies at a nonsingular[37] Γ attractor point, we have:

$$\mathrm{Im}\,(\bar{Z}Z_\Lambda) = -\frac{1}{2}Q_{\Lambda\Sigma}N^\Sigma. \tag{3.3.60}$$

This equation in principle determines the position of the regular attractor points in moduli space. It is therefore often called the *attractor equation*.

3.4 Validity of the D-brane and field theory pictures

To end this chapter, we would like to make some comments on the domain of validity of the effective field theory (supergravity) and the D-brane pictures of

[36] Note that this gauge choice is singular at a black hole horizon, where $U \to -\infty$.

[37] At a conifold singularity, the gradient term in (3.1.30) does not necessarily vanish.

the dynamics of nonperturbative states. We will also briefly touch the Maldacena conjecture and speculate on a possible extension.

The D-brane picture in the sense of Polchinski's prescription (spacetime defects where strings can end) is in general expected to be a good description of the physics when the string coupling constant g is sufficiently small such that string perturbation theory is adequate. In case we have a stack of a large number N of D-branes, we must have that gN is small, since the sum over the different branes makes gN rather than g the effective coupling in brane amplitudes [31], as in the large N 't Hooft expansion of field theories.

The low energy field theory picture on the other hand is expected to be a good description for sufficiently slowly varying fields, which is in general expected to be the case when the system under consideration is 'macrosocopic', i.e. at large N.

So one would expect the regime where the D-brane picture is valid and the regime where the field theory picture is valid to be more or less complementary in general.

These statements are of course rather vague and imprecise. In particular the meaning of 'large' and 'small' is not really clear. Let us therefore try to make this more precise in the case of type IIB Calabi-Yau compactifications.

Massive states limiting the field theory picture

Consider the effective action (3.1.43). In general, in order for a low energy effective field theory of the massless fields to makes sense, all relevant energy scales have to be much smaller than the lightest massive states in the full theory which couple to the massless fields. Higher derivative terms in the effective action are suppressed by inverse powers of these masses.

Now which of those light massive states can we possibly have in type IIB string theory compactified on a Calabi-Yau manifold? Candidates are fundamental (F) strings, D-strings, wrapped D3-branes (either completely around a 3-cycle or partially around a 2-cycle, producing an effective string in four dimensions), wrapped NS5- and D5-branes (partially, around a 4-cycle), wrapped $(1,0)$ or $(0,1)$ D7-branes (partially, around the full Calabi-Yau), and of course also simply excitations of the 10D massless particles in the internal space X (Kaluza-Klein states).[38] Actually it is not entirely clear whether we should include partially wrapped brane states leading to strings in four dimensions. This is in the spirit of the 'self-dual

[38] One can also consider more generally (p,q) strings and D7-branes, and other bound states, but it is sufficient to consider the 'extremes' (i.e. $(1,0)$ and $(0,1)$) to single out the lightest ones (which of those two will be the lightest depends on whether g is larger or smaller than 1).

strings' arising in IIB K3 compactifications from partially wrapped 3-branes [73], used in [66, 74] to count black hole microstates. We don't know of any reason not to include such states here as well. Mirror symmetry also supports this point of view. Indeed, type IIB D-strings living in four dimensional spacetime are mapped to certain partially wrapped D4-branes in the mirror type IIA theory. It is then natural to consider arbitrary partially wrapped type IIA D4-branes, which are in turn mapped back to certain wrapped D3-, D5- and D7-branes in type IIB. On the other hand, it is not clear that we can simply treat those objects as effective strings and deduce features of their excitation spectrum accordingly. Nevertheless, we will do so in the following and see what comes out.

Denote the IIB string coupling constant e^{Φ} with g. The mass of the lowest excited F-string states is, in the string frame, of the order of $1/\sqrt{\alpha'}$ (where $\alpha' = 2^{-3/2}\pi^{-7/4}\,\kappa_{10}^{1/2}$), hence in the Einstein frame (in which we have been working), $M_{S,F} \sim g^{1/4}\kappa_{10}^{-1/4}$. The Einstein mass of an excited D-string is $M_{S,D1} \sim g^{-1/4}\kappa_{10}^{-1/4}$. The mass of the lightest excited (in the 4D effective string sense) partially wrapped 3-branes is $M_{S,D3} \sim k_2 {V_X}^{1/6}\kappa_{10}^{-1/2}$, where V_X is the volume of the Calabi-Yau X and k_2 is a dimensionless factor depending on the Kähler moduli of X. That of a completely wrapped D3-brane is $M_{0,D3} \sim c{V_X}^{1/2}\kappa_{10}^{-1}$, where c is the modulus of the central charge of the lightest 3-cycle (depending on the complex structure moduli). For a partially wrapped D5-brane we have $M_{S,D5} \sim k_4 g^{1/4}{V_X}^{1/3}\kappa_{10}^{-3/4}$, where k_4 again is a dimensionless factor depending on the Kähler structure moduli, for its NS5 analogon we have $M_{S,NS5} \sim k_4 g^{-1/4}{V_X}^{1/3}\kappa_{10}^{-3/4}$ and for a partially wrapped $(1,0)$ / $(0,1)$ D7-brane $M_{S,D7(1,0)} \sim g^{1/2}{V_X}^{1/2}\kappa_{10}^{-1}$ resp. $M_{S,D7(0,1)} \sim g^{-1/2}{V_X}^{1/2}\kappa_{10}^{-1}$. Finally the mass of a typical low Kaluza-Klein excitation in X is $M_{KK} \sim V_X^{-1/6}$.

Expressed in units where the four dimensional Planck length $\kappa_4 = \kappa_{10}{V_X}^{-1/2}$ equals 1, and introducing the dimensionless ratios

$$\rho \equiv \frac{{V_X}^{1/6}}{\kappa_{10}^{1/4}} = \left(\frac{{V_X}^{1/6}}{\kappa_4}\right)^{1/4} \tag{3.4.1}$$

$$\sigma \equiv \frac{{V_X}^{1/6}}{M_{S,F}^{-1}} = g^{1/4}\rho \tag{3.4.2}$$

(which can be considered as the 'radius' of X in 10D Planck units resp. F-string units), this becomes:

$$\begin{array}{rcrcl} M_{KK} & \sim & \rho^{-4} & = & g\sigma^{-4} \\ M_{S,F} & \sim & g^{1/4}\rho^{-3} & = & g\sigma^{-3} \\ M_{S,D1} & \sim & g^{-1/4}\rho^{-3} & = & g^{1/2}\sigma^{-3} \\ M_{S,D3} & \sim & k_2\rho^{-2} & = & k_2 g^{1/2}\sigma^{-2} \\ M_{S,D5} & \sim & k_4 g^{1/4}\rho^{-1} & = & k_4 g^{1/2}\sigma^{-1} \\ M_{S,NS5} & \sim & k_4 g^{-1/4}\rho^{-1} & = & k_4\sigma^{-1} \\ M_{S,D7(1,0)} & \sim & g^{1/2} & = & g^{1/2} \\ M_{S,D7(0,1)} & \sim & g^{-1/2} & = & g^{-1/2} \\ M_{0,D3} & \sim & c & = & c. \end{array} \qquad (3.4.3)$$

In 10D Planck units, all this has to be multiplied by $\kappa_{10}^{1/4}/\kappa_4 = \rho^3 = g^{-3/4}\sigma^3$. Note the invariance of the spectrum under $g \to g^{-1}$, which is of course just type IIB S-duality.[39] We can therefore assume $g < 1$.

As a warm-up example, we see for instance that for fixed energies in 4D or 10D Planck units, the four dimensional effective field theory description breaks down when $\rho \to \infty$ or $\sigma \to \infty$. This is evident, since this limit corresponds to decompactification. A perhaps more surprising feature is that for fixed Calabi-Yau moduli (i.e. fixed k_2, k_4, c and ρ or σ), and again fixed energies *in Planck units*, the four dimensional effective field theory description *always* breaks down in the weak string coupling limit $g \to 0$, simply because e.g. the fundamental string mass becomes much lighter than the 4D Planck mass.

The limit $g \to 0$ with fixed σ and fixed X moduli

When $g \to 0$ with σ and the other compactification moduli fixed, corresponding to vanishing string coupling with a fixed internal space size compared to the string scale, the IIB perturbative string states with fixed KK and oscillator excitatition number (masses proportional M_{KK} and $M_{S,F}$) have vanishing mass compared to the nonperturbative wrapped D-brane states. The *bulk* dynamics of the fields at energies finite with respect to the string scale is in this case perfectly well decribed by IIB string perturbation theory. At energies far below the lowest massive string state ($M_{S,F}$) and the lowest Kaluza-Klein state (M_{KK}), the (two derivative) low energy 4D field theory description[40] of the *bulk* fields is moreover perfectly good as well.

[39] Actually, we have assumed for simplicity that the R-R scalar $C_0 = 0$ in deriving our mass formulas. A nonzero value would make the mass formulas (and S-duality) a bit more complicated, without altering qualitatively the conclusions however.

[40] where the action is, since we are in the weak string coupling regime, the 4D *string tree level* effective action. This can differ from the dimensionally reduced 10D action by worldsheet

The description of the dynamics of the nonperturbative states on the other hand is more subtle. In the limit under consideration, the Compton wavelengths of the nonperturbative objects become vanishingly small compared to the string scale $M_{S,F}^{-1}$, so they are seen as localized objects from the perturbative string point of view. Moreover, when $g \to 0$ with fixed D-brane charges, since the string scale becomes much larger than any other relevant distance scale in the 10D theory, interactions between different D-branes and backreaction of D-branes on the bulk fields vanish at any in string units fixed distance from the branes, at least if this interaction and backreaction fall off with distance.[41] Therefore one can set up a good perturbative description of the nonperturbative states as D-branes represented as spacetime defects on which strings can end, simply superposed on the original background, without backreaction. Interactions between the branes and backreaction on the bulk fields are then taken into account as perturbative corrections. Subsequent terms in the perturbative expansion with increasing number of string boundary components on the branes become smaller with powers of gN when $\sigma > 1$ and powers of $\sigma^{-3}gN$ when $\sigma < 1$, where N is the order of magnitude of the brane charges under consideration. The factor N comes from summing over the different branes on which the strings can end. The factor σ^{-3} can be understood from the fact that when σ is small, the Dirichlet boundary conditions eliminate for every string boundary component 3 translational degrees of freedom in X.[42] This factor is also the reason why the four dimensional gravitational backreaction of a wrapped 3-brane is independent of σ, though its mass compared to the string scale $M_{S,F}$ grows as σ^3 when σ is increased.

The stringy (type IIB) perturbative description of these states is therefore valid as long as

$$g N \sigma^{-3} \ll 1 \text{ for } \sigma < 1, \quad gN \ll 1 \text{ for } \sigma > 1 \tag{3.4.4}$$

In particular, this implies that in this regime, the dynamics of N coincident 3-branes totally wrapped around a certain supersymmetric 3-cycle Γ, at energies well below $M_{S,F} = g\sigma^{-3}/\kappa_4$, will be given by a certain twisted [75] supersymmetric $U(N)$ Yang-Mills theory on $\mathbb{R} \times \Gamma$. For generic cycles Γ, the KK modes of the fields on the brane will have energies of the order of the KK modes of the bulk fields in X, i.e. $M_{KK} \sim \sigma^{-1} M_{S,F}$, so for energies below this scale as well (automatic if σ is of order 1 or less), the low energy dynamics will actually be this Yang-Mills

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instanton corrections $\sim e^{-k_2\sigma^2}$, corresponding to eucledian string world sheets wrapping nontrivial 2-cycles in X. Such corrections only affect the hypermultiplet part of the two derivative low energy effective action, since the instanton factors involve hypermultiplet moduli, and those are by supersymmetry forbidden to couple to the vectors at the two derivative level [26].

[41] This is not the case if there are confining interactions present.

[42] When σ is large and the different endpoints are close to each other those degrees of freedom are eliminated automatically by the tension of the string.

theory dimensionally reduced[43] to the worldline of the corresponsing 4D particle, that is, a matrix model. The D3-brane worldvolume Yang-Mills coupling constant is [31] $g_{(3)}^2 = 2\pi g$, so the coupling constant of the matrix model is, using (3.2.15),

$$g_{(0)}^2 = V_{D3} g_{(3)}^2 = \frac{\pi}{\sqrt{2}} \frac{1}{\sigma^3 |Z|} \, g \, M_{S,F}^3, \tag{3.4.5}$$

where Z is the central charge of Γ. The effective perturbative expansion parameter for N coincident branes at energy scale E will be $g_{(0)}^2 N E^{-3}$. If we stay in the low energy ($E < M_{S,F}$) regime, the dynamics of the branes will keep on being described by the $U(N)$ matrix model even when N is increased such that (3.4.4) is no longer valid, though the model is effectively strongly coupled now and perturbation theory can no longer be used.

What about the 4D effective field theory description of these states? Consider again the case of N coincident totally wrapped BPS 3-branes. As we have seen in the sections on attractors, the spatial gradient of the dimensionless fields — and hence the energy scale E — of the corresponding effective field theory solution scales as N^{-1}. More precisely, as the total field energy $M_{0,D3}$ of such a state is generically of order 1 in 4D Planck units, we have $E \sim N^{-1} \kappa_4^{-1}$. As discussed above, in the limit under consideration, validity of the 4D field theory picture requires E to be much less than $M_{S,F}$ and M_{KK}, that is,

$$g N \sigma^{-3} \gg 1 \text{ for } \sigma < 1, \quad g N \sigma^{-4} \gg 1 \text{ for } \sigma > 1. \tag{3.4.6}$$

Comparison with (3.4.4) learns that for $\sigma < 1$, the 4D field theory and string perturbative regimes are precisely complementary. For $\sigma > 1$, there is an additional 'gap': this is the regime where 10D field theory is the good description. Note that for the dynamics of N coincident branes at energies far below $M_{S,F}$ and M_{KK}, we have two valid descriptions if (3.4.6) is satisfied and $g \to 0$: 4D effective field theory and a (strongly coupled) $U(N)$ matrix model. As the relevant energy scale of this regime of simultaneous validity goes to zero with g compared to the 4D Planck scale, the relevant effective field excitations (in the generic $c \sim 1$ case) are forced to vanish everywhere except infinitesimally close to the horizon, where the diverging red shift factor allows finite field excitations with arbitrarily low energies. As the geometry is $AdS_2 \times S^2$ there, we thus arrive in this limit at a duality between low energy field theory on $AdS_2 \times S^2$ and a strongly coupled $U(N)$ matrix model. Similar considerations hold in higher dimensions. This is the celebrated Maldacena conjecture [7].[44]

[43] This 'reduction' can be nontrivial, as the embedding geometry of the 3-brane can induce field twists and additional zeromodes.

[44] Actually, precisely in the AdS_2 case this duality turns out to be much more subtle than in

The $c \to 0$ case

In this thesis, we will be mainly interested in BPS states produced by totally wrapped 3-branes with mass $M_{0,D3}$ very much smaller than the 4D Planck mass, that is, the $c \to 0$ case. We have already seen the example of a brane wrapped around a vanishing conifold cycle in section 3.3.3. In chapter 5, we will study such states in detail. Gravity decouples in this limit, and the states are rigid field theory states, not black holes.

If g, σ and the remaining Kähler moduli are kept fixed while sending $c \to 0$, the lightest states in the spectrum are the totally wrapped 3-branes of mass $\sim c$ in 4D Planck units. Type IIB string perturbation theory has clearly broken down at the energy scale of these states (however, via heterotic-type II duality, heterotic perturbation theory can be used at energies sufficiently above this scale, as the wrapped 3-brane then correspond to weakly coupled perturbative string states in the heterotic picture; this will be discussed in more detail in chapter 4). The field theory picture of N coincident BPS particles on the other hand is valid when the energy scale of the field variations is much lower than the mass c/κ_4, which amounts simply to $N \gg 1$ when only light charges ($M \sim c/\kappa_4$) are considered, and to $cN \gg 1$ when generic charges ($M \sim 1/\kappa_4$) are involved.

If one wants to keep type IIB perturbation theory valid in a certain energy range as well, one should take g down to zero together with c, in such way that moreover $g/c \to 0$. Then perturbation theory is valid for energies finite with respect to the string scale. Note that this energy range necessarily goes to zero with respect to the mass of the 3-brane states. Taking $\sigma = 1$ for simplicity, the condition for perturbation theory to be valid in the presence of N coincident wrapped D3-branes is now

$$gN \ll 1, \tag{3.4.7}$$

while the condition for the field theory picture to be valid becomes

$$\frac{g}{c} N \gg 1. \tag{3.4.8}$$

In their respective regimes, the perturbative description yields a $U(N)$ matrix model for the dynamics while the field theory description can be truncated to two derivatives, provided the energy scale E is far below the lightest massive states, i.e.:

$$E \ll \frac{g}{\kappa_4} = \frac{g}{c} M_{0,D3}. \tag{3.4.9}$$

higher dimensions [76]. This is related to the fact that unlike their noncompact higher dimensional generalisations, the 4D BPS particles have also arbitrarily low energy modes corresponding to slow translational motion away from each other.

Note that (3.4.7) and (3.4.9) can easily be satisfied simultaneously. At energy scales given by (3.4.9) the dynamics of rigid QFT states in the large N limit can thus be described by a weakly coupled matrix model. This seems to be a new kind of Maldacena-like correspondence, giving a holographic description of a rigid quantum field theory (at least at low energies). Interestingly, the Atiyah-Donaldson-Drinfeld-Hitchin-Manin-Nahm-Nakajima construction of the moduli space metric of N (nonabelian nonsupersymmetric $SU(2)$ Yang-Mills) monopoles [77] is based precisely on a (purely mathematical) correspondence with a certain (bosonic) $U(N)$ matrix system, which is of the form one would expect for the correspondence we propose here! We believe this deserves further study.

Other limits

The limits considered in the previous discussion were adapted to type IIB F-string perturbation theory, most naturally expressed in terms of the radius σ of X compared to the F-string scale. From (3.4.3), one sees that S-duality is manifest when all masses are expressed in terms of ρ, the radius of X expressed in 10D (Einstein frame) Planck units. A natural limit to consider in these coordinates is $g \to 0$ with fixed ρ. Assuming partially wrapped D-branes behaving effectively like strings in four dimensions indeed exist, (3.4.3) shows that in this limit, type IIB string perturbation theory breaks down, since the lightest massive states in the spectrum are now partially wrapped $(1,0)$ 7-branes rather than strings. Low energy effective field theory will be valid at energies far below this mass scale, $g^{1/2}\kappa_4^{-1}$. However, even at zero energy, the hypermultiplet[45] action will have instanton corrections which — unlike in the fixed σ case — do not vanish when $g \to 0$: euclidean NS5, D5, D3, D1, F and D(-1) branes (and their (p,q) combinations) can wrap X, a 4-cycle, a 2-cycle or a point. Their contributions are weighted roughly with the following $e^{-S_{cl}}$ factors:

$$\begin{array}{rlcl}
D(-1): & e^{-1/g} & & \\
F: & e^{-k_2\sigma^2} & = & e^{-k_2\rho^2 g^{1/2}} \\
D1: & e^{-k2\sigma^2/g} & = & e^{-k_2\rho^2 g^{-1/2}} \\
D3: & e^{-k4\sigma^4/g} & = & e^{-k_4\rho^4} \\
D5: & e^{-\sigma^6/g} & = & e^{-\rho^6 g^{1/2}} \\
NS5: & e^{-\sigma^6/g^2} & = & e^{-\rho^6 g^{-1/2}}
\end{array} \tag{3.4.10}$$

As it should, in the $g \to 0$, σ fixed limit where string perturbation theory is supposed to be valid, the nonperturbative instanton corrections are exponentially

[45] As the nonrenormalization theorem for the two derivative low energy effctive action of the vector multiplet action does not rely on perturbation theory, the exactness of the classical vector multiplet action continues to hold in this regime as well.

small. The only surviving instanton contributions are those from euclidean F-strings wrapped around nontrivial 2-cycles , which are included in perturbation theory. They become arbitrarily small in the large radius limit $\sigma \to \infty$. When on the other hand ρ is fixed while $g \to 0$, instanton corrections from eucledian wrapped F-strings, D5-branes and possibly (for small ρ) D3-branes are important.

It is not clear if there exists a weakly coupled dual formulation of the theory in this regime.

Another interesting limit is the one arising from the M-theory matrix model, mirrored to type IIB [78], which amounts to sending g and σ to zero with $\sigma \sim g^{2/3}$. The relevant energy scale here is the energy of states with finite eleven dimensional light cone energy in eleven dimensional Planck units, $E_{DKPS} \equiv g^{1/3} M_F$. The field theory breaks down at this scale in the limit under consideration, due to e.g. partially wrapped D5-branes. Indeed, physics is (supposedly) not described by a field theory in this regime, but by a $U(N)$ matrix model describing the dynamics of N 4D particles obtained by wrapping a 3-brane around a certain supersymmetric T^3 3-cycle in X.

Chapter 4

Quantum Yang-Mills + gravity from IIB strings

Till about 1995 [48], type II string theories were considered to be incapable of describing phenomenologically interesting nonabelian gauge theories, since their bulk perturbative spectrum can only carry a very limitited range of gauge groups — too limited for the standard model, as it turns out. For example, as we have seen in detail in chapter 3, type IIB compactified on a Calabi-Yau manifold has typically only a $U(1)^{h^{2,1}}$ abelian gauge theory sector. Heterotic and type I theories on the other hand support big nonabelian gauge groups, and therefore those were considered to be the only possibly phenomenologically relevant theories.

This turned out to be completely wrong. With the advent of string dualities and D-branes, probing nonperturbative aspects of string theory, it became clear that actually, type II theories even do better: they *do* describe nonabelian gauge theories, and often *even their exact (low energy) quantum dynamics*. The first indications of this remarkable fact emerged [79] from Heterotic - type II duality [80, 81]: evidence accumulated suggesting that (nonperturbatively completed) heterotic string theory compactified on $T^2 \times K3$ is equivalent to certain IIA (and hence IIB by mirror symmetry) Calabi-Yau compactifications. Since the former theory has nonabelian vectors, so should the latter. The obvious question is then where the nonabelian vectors are hiding in type II theory. The answer turned out to be beautifully simple: those are precisely the states obtained by wrapping D-branes about nontrivial cycles in the compactification manifold! [55]. Once this was realised, a magical window was opened to the derivation of quantum field theory results from string theory. For example, this insight can be used to obtain

Figure 4.1: A sketch of a Calabi-Yau manifold which, at low energies in a compactification of type II string theory, gives rise to a (nonabelian) gauge theory weakly coupled to gravity. Branes wrapped around a generic cycle Γ will have typically Planck scale masses. However, after suitable tuning of the moduli, some cycles γ can become much lighter than the Planck mass.

the exact quantum two derivative low energy effective action for a plethora of four dimensional $\mathcal{N} = 2$ field theories [11, 12, 13, 14] (a good review can be found in [15]). Indeed, by tuning the compactification moduli such that the masses of a number of the branes corresponding to nonabelian vectors becomes very small compared to the Planck mass (fig. 4.1), one expects their low energy dynamics to be governed by an $\mathcal{N} = 2$ nonabelian Yang-Mills theory (possibly with additional matter).

Now we only get an effective action for the massless, unbroken $U(1)^r$ vectors from string theory, but at energies well below the mass of the wrapped branes, this is sufficient. On the other hand, as we have seen in chapter 3, the straightforwardly obtained 4D two derivative low energy effective action of type IIB string theory compactified on a Calabi-Yau manifold, does *not* receive any quantum corrections. So if we manage to identify the (nonabelian) quantum field theory governing the dynamics of the light branes, we have obtained the exact quantum two derivative low energy effective action for the massless fields in this theory!

The price we have to pay for exact quantum results is the restriction to low energies. Indeed, as discussed in chapter 2, perturbative string theory, and hence the effective actions extracted from it, breaks down at energies of states not in the perturbative spectrum. Usually these states have masses of the order of the Planck mass, but precisely in the case at hand, they are much lower, since the nonperturbative wrapped D-brane states have to provide the gauge particles. So there is really no hope to extend these exact results in the perturbative type II picture to energies higher than the lightest gauge particle masses, not even in principle. (However, this is precisely where the perturbative heterotic picture takes over!)

In many cases, the procedure outlined above reduces the local special geometry of the Calabi-Yau moduli space to the rigid special geometry of the moduli space of a certain class of Riemann surfaces, reproducing and extending the Seiberg-Witten solution of $\mathcal{N} = 2$ quantum Yang-Mills theory. Furthermore, many features of quantum field theory have a beautiful geometrical interpretation in this framework, and this provides quite elegant solutions to problems which would be hard to tackle with ordinary field theory techniques, like for example the existence and stability of BPS states [12, 15, 16] or confinement [82].

Note that such a reduction from local to rigid is, by our discussion in chapter 2, necessary if we want to extract rigid quantum field theory results from string theory. Decoupling gravity is tantamount to going to a rigid limit of special geometry. This limit will therefore play a central role in this chapter.

A very large class of $N = 2, d = 4$ quantum field theories (and even more exotic theories) can be "engineered" and solved geometrically in this way. The usual procedure [13, 14] is to find a local IIA model which in the rigid limit produces the field theory to be solved; to map this IIA theory to an equivalent IIB theory using local mirror symmetry; and finally to solve this IIB theory exactly (in the low energy limit) using classical geometry. One argues that the restriction to local models and local mirror symmetry — where "local" means that one only considers a certain small region of the Calabi-Yau manifold — is allowed roughly because the relevant (light brane) degrees of freedom are all localized well inside that region. The drawback of such strictly local considerations is that one cannot see directly how the rigid limit is obtained in the low energy string theory, and, more important, that one looses all information about the coupling of the gauge theory to gravity and the rest of the 'ambient string theory'. In the literature, the focus has consequently been almost entirely on the decoupled rigid field theory aspects of this construction.

We will directly work in the type IIB theory, without assuming a priori that we can restrict ourselves to local considerations. The reduction of local to rigid special geometry will be demonstrated explicitly. This will allow us to derive the lowest order coupling of the effective quantum field theory to gravity and the other remaining string theory degrees of freedom, which, as we will show, has a universal form. Some interesting physics will be derived from this as well, like the gravitational backreaction of quantum fluctuations of the gauge theory, and the dynamics of the dynamical dynamically generated scale.

Apart from the 'geometric engineering' strategy discussed above, there are several other methods to extract nonperturbative quantum field theory results from string theory [10, 7]. Unlike the bulk spacetime approach of geometrical engineering, these alternatives all have as starting point the realization of gauge theories on (multiple) D-brane world volumes. In [10], one considers various embeddings of 'large' branes into (possibly compactified) ten dimensional spacetime, where time together with three noncompact spatial directions of the brane form the four dimensional spacetime on which the quantum field theory under study is defined. By making use of various properties of M-theory and/or brane dynamics, one derives the low energy quantum gauge theory dynamics in a fairly straightforward and elegant way. In [7], an exact equivalence is conjectured between four dimensional $\mathcal{N} = 4$ large N $SU(N)$ Yang-Mills theory and type IIB string theory on $AdS_5 \times S^5$ with N units of F_5-flux. The string perturbative expansion corresponds to a certain $1/N$ expansion of the Yang-Mills theory. This remarkable correspondence makes it even possible to go beyond the low energy approximation of the quantum Yang-Mills theory.

Though these alternative approaches are often more powerful, or at least simpler, than the approach we will follow, they are not very useful for our purposes, since nothing can be learned from them about the interplay between Yang-Mills theory and gravity and other stringy degrees of freedom. This is because brane worldvolume theories never contain gravity. From the point of view of physics, this is the main advantage of our approach (and approaches based on bulk spacetime compactifications in general).

This chapter is organized as follows. We first discuss the geometric structures needed to get nonabelian Yang-Mills theories in four dimensions. We will argue that Calabi-Yau manifolds which are locally of 'near ALE fibration' form give rise at low energies to a $\mathcal{N} = 2$ Yang-Mills theory weakly coupled to gravity. Next we consider this weak gravity limit and derive its universal form. Finally, we comment on the interpretation of the low energy effective theory thus obtained, including the effects of gravity, and compare with the dual heterotic picture. To illustrate and clarify the general results, we conclude this chapter with the detailed analysis of an explicit example. This example inspired the general arguments, and if the latter would get too obscure to the reader at a certain point, it could be helpful to have a look at the example first.

In the next chapter, we will use these results to discuss BPS states, and in particular the attractor equations, in the weak gravity limit.

Figure 4.2: The compactification $T^2 \times K3$. A 3-cycle can be constructed as a product of a K3 2-cycle σ and a torus 1-cycle γ.

4.1 Near ALE fibrations and the light BPS spectrum

4.1.1 Vector branes

From the previous discussion and chapter 3, it follows that we have to look for Calabi-Yau manifolds with 3-cycles which, by wrapping 3-branes around them, give rise to light massive BPS vector multiplets in four dimensions. To find out what kind of 3-cycles produces vector multiplets, we are going to use the trick explained in chapter 3 (section 3.2.4). Thus, let us first try to find out which cycles produce a vector multiplet in an $\mathcal{N} = 4$ compactification.[1] To get $\mathcal{N} = 4$ in four dimensions by compactification of type IIB string theory in the way described in chapter 3, we have to take the compact manifold to be $T^2 \times K3$ (fig. 4.2). Since K_3 has no nontrivial odd dimensional cycles, all nontrivial 3-cycles in $T^2 \times K3$ have to be the product of a torus 1-cycle γ and a $K3$ 2-cycle σ. Denoting the embedding of σ in the $K3$ by f_σ, the conditions (3.2.4) and (3.2.5) to have a

[1] A massive BPS $\mathcal{N} = 4$ vector multiplet has 16 states, with spin content $(0^5, \frac{1}{2}^4, 1)$.

supersymmetric 3-cycle become

$$f_\sigma^* J_{K3} = 0 \qquad (4.1.1)$$
$$\arg f_\sigma^* \Omega_{K3} = \alpha = \text{const.} \qquad (4.1.2)$$

Here Ω_{K3} and $J_{K3} = i g_{m\bar{n}} dz^m \wedge d\bar{z}^n$ are the the holomorphic 2-form resp. Kähler form on $K3$. By a suitable choice of the phase of Ω_{K3}, we can take $\alpha = 0$. To get better insight in the meaning of these conditions, let us recall some basic facts about $K3$ geometry (for a review, see [43]). A $K3$ manifold (defined as the up to diffeomorphisms unique two dimensional Calabi-Yau manifold) has actually 3 independent, anticommuting, covariant constant complex structures $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$. Denote the associated 2-forms by J_i, that is, $J_{i,\mu\nu} = g_{\mu\rho} \mathcal{J}_{i\,\nu}^{\,\rho}$. A choice of complex structure specifies the Kähler form J_{K3} and the holomorphic 2-form Ω_{K3} in terms of these, for example

$$J_{K3} = J_1; \quad \Omega_{K3} = J_2 + i J_3, \qquad (4.1.3)$$

or any other choice corresponding to a rotation of the J_i.[2] Now let us go back to our original $K3$. There we already assumed a certain choice of complex structure, but we can also consider another (rotated) one, defined by specifying (for $\alpha = 0$)

$$J'_{K3} = \operatorname{Re} \Omega_{K3}; \quad \Omega'_{K3} = \operatorname{Im} \Omega_{K3} - i J_{K3}. \qquad (4.1.4)$$

In the primed complex structure, the condition for a supersymmetric cycle thus becomes simply

$$f_\sigma^* \Omega'_{K3} = 0. \qquad (4.1.5)$$

If we choose complex coordinates (z'_1, z'_2) compatible with the primed complex structure and describe σ locally by writing z'_2 as a function of z'_1 and $\bar{z}'_1$, (4.1.5) immediately gives $\frac{\partial z'_2}{\partial \bar{z}'_1} = 0$. So the condition for a supersymmetric cycle simply reduces to holomorphicity of the embedding of σ with respect to the primed complex structure.

Now, in contrast to special Lagrangian embeddings of 3-cycles in Calabi-Yau 3-folds, holomorphic embeddings of 2-cycles in $K3$ are extensively studied and very well understood in the mathematics literature. A mathematical result [83] which is very important for us is that a $K3$ 2-homology class with self-intersection $2g-2$ ($g \geq 0$) has a g (complex) parameter family of holomorphic representatives of genus g.[3] The area of these curves is minimal in the given homology class and

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[2] Note that for any Kähler form J, we have (by a short calculation) $\int_{K3} J \wedge J = 8 V_{K3}$. Therefore (4.1.3) implies for a given $K3$ volume V_{K3} a choice of normalisation (modulo phase) of Ω_{K3}: $\int \Omega \wedge \bar{\Omega} = 16 V_{K3}$.

[3] This implies that the 2-brane moduli space has complex dimension $2g$: g from the embedding deformations, and g from the flat $U(1)$ brane worldvolume gauge field deformations. Such brane ground state degeneracies can be shown to reproduce exactly the Bekenstein-Hawking entropy formula for black holes [74]!

given by[4]

$$A_\sigma = \frac{1}{2}\int_\sigma J'_{K3} = \frac{1}{2}|\int_\sigma \Omega_{K3}|. \tag{4.1.6}$$

In particular, for cycles with self-intersection -2, we have a *unique* holomorphic representative which is a minimal area sphere. One can also show that the volume of cycles with $g \geq 1$ is actually bounded from below on K3 moduli space (with lower bound proportional to $\sqrt{g}$ [74]). Since we are looking for branes which can be made aribtrary light, this leaves us with the $g = 0$ cases as candidates for branes representing gauge particles.

Does such a 3-cycle give rise to a $\mathcal{N} = 4$ vector multiplet in four dimensions? The easiest way to see this is to perform a T-dualtity on the T^2 in the direction of γ. This maps IIB to IIA and our 3-brane to the 2-brane given by σ, but leaves the physics (and in particular the type of multiplet which our brane gives in four dimensions) invariant. Let us first consider the theory as IIA compactified to six dimensions on $K3$. This has $\mathcal{N} = (1,1)$ supersymmetry in six dimensions. Since for a fixed position in the 6D spacetime, σ has a unique supersymmetric representative, the 2-brane has a unique (zero mode) fermionic oscillator ground state (where the fermionic oscillator creation and annihilation operators are composed of the supersymmetries broken by the brane, cf. chapter 3). Therefore this ground state (or 'vacuum') must be in the trivial representation of the $SO(4) \subset SO(1,5)$ little group. Consequently, the six dimensional $\mathcal{N} = (1,1)$ supersymmetry multiplet of $16 = 2^4$ states built on this ground state is a (6D) (massive BPS) vector multiplet. When we compactify further on T^2, this 6D vector multiplet trivially reduces to a 4D $\mathcal{N} = 4$ (massive BPS) vector multiplet, which is precisely what we were looking for!

All in all, we conclude that the light four dimensional $\mathcal{N} = 4$ vector multiplets are produced in type IIB compactified on $T^2 \times K3$ by 3-branes wrapped around the product of a T^2 1-cycle and a $K3$ 2-cycle with self-intersection -2 (which is a minimal area sphere).

Now what about $\mathcal{N} = 2$ compactifications? How can we get our light vector multiplets there? We follow the reasoning outlined in chapter 3 (section 3.2.4). We transform our $\mathcal{N} = 4$ $T^2 \times K3$ compactification to an $\mathcal{N} = 2$ compactification by changing the geometry away from a set of $\mathcal{N} = 4$ vector 3-branes $\gamma \times \sigma_i$. This can be done for example by making the moduli of $K3$ position dependent on T^2, replacing the direct product by a $K3$ *fibration.* We can furthermore change the basis B of the fibration away from γ to e.g. a sphere or a higher genus Riemann surface, or change the $K3$ geometry away from the σ_i to another 2-fold K (fig.

[4]With a more general normalisation of Ω, the right hand side is replaced by $2V_{K3}^{1/2}|\int_\sigma \Omega_{K3}|/(\int \Omega \wedge \bar{\Omega})^{1/2}$.

Figure 4.3: We break $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=2$ by deforming the $T^2 \times K3$ direct product geometry to a (Calabi-Yau) fibration geometry with fiber K and base B.

4.3). Of course, the resulting manifold should be Calabi-Yau in order to have an $\mathcal{N}=2$ compactification. We furthermore take care that our 3-cycles remain nontrivial and that the geometry in their neighbourhood does not change too much; let's say we assume that the geometry in a neighborhood of the 3-cycles can be continuously deformed to the original product geometry in this neighborhood, and that we take the deformation parameters sufficiently small, such that we still have a supersymmetric representative of topology $S^1 \times S^2$ for each of our 3-cycles. Thus at least some of the 16 BPS states in the $\mathcal{N}=4$ multiplet will survive the change of compactification geometry. Not all however, since the holonomy group of the compactification manifold has been enlarged from $SU(2)$ to $SU(3)$, and therefore some of the states (which are defined by annihilation/creation operators composed of the 16 supercharges invariant under the $SU(2)$, but not necessarily the $SU(3)$ holonomy) will leave the multiplet. As argued in section 3.2.4, it is not guaranteed that e.g. the original fermionic osillator ground state remains in the multiplet, but certainly the highest spin state of the multiplet does, that is, the vector. So we are again left with at least an $\mathcal{N}=2$ massive BPS vector multiplet!

A massive BPS $\mathcal{N}=2$ vector multiplet has 8 states. The fermionic oscillator ground state is a spin $\frac{1}{2}$ doublet. In the context of the above argument, this doublet corresponds to two states of the original 16 states in the $\mathcal{N}=4$ multiplet which remain invariant in passing from $SU(2)$ to $SU(3)$ holonomy, and which are annihilated by the annihilation operators formed with the $\mathcal{N}=2$ supercharges. Note that this ground state degeneracy is not in contradiction with our earlier assertion that a unique supersymmetric representative gives a spin 0 fermionic oscillator ground state: the representative is *not* unique here, since a 3-brane with topology $S^1 \times S^2$ has at least one real modulus corresponding to turning on a flat $U(1)$ brane worldvolume gauge field along the S^1 component γ, and one corresponding to embedding deformation of the brane [51]. The deformations can be thought of as a shift of γ in the base B of the fibration. A nontrivial brane moduli space indeed typically gives rise to such a multiplet of fermionic oscillator ground states (with respect to the fermionic creation/annihilation operators made from the $\mathcal{N}=2$ susy generators), but a direct computation of this requires a detailed analysis of supersymmetric quantum mechanics on the moduli space (analogous to monopole moduli space analysis). From general considerations, we expect however that in a background with 8 unbroken supercharges (as is the case here), the number of fermionic oscillator ground states is equal to the number of holomorphic forms on moduli space (satisfying certain boundary conditions if there is a boundary). If we take our base manifold B to be a *sphere*, the moduli space will be one complex dimensional and topologically a cilinder, so indeed we expect no more than two harmonic forms (the constant function and a holomorphic one form). Therefore, in this case[5], we expect precisely one vector multiplet for every 3-cycle of the kind described above.

This vector multiplet can be made arbitrarily light by tuning the moduli such that the 2-cycles σ_i in K remain arbitrarily small in a neighborhood of γ in the base B. What does K look like in the neighborhood of these almost vanishing spheres? Consider first the case where we have just one sphere. Then in the limit of a zero size sphere, we have an A_1 singularity [84]. Any complex deformation of an A_1 singularity which is locally Calabi-Yau is locally given by the following equation in $\mathbb{C}^3$:

$$w^2 + y^2 + x^2 = u, \tag{4.1.7}$$

where u is the deformation parameter and x, y, w are coordinates on $\mathbb{C}^3$. This space is an (A_1) ALE (Asymptotically Locally Euclidean) space. The 2-sphere is given by the intersection with $\text{Im}\,[u^{-1/2}(x,y,w)] = 0$. Taking the natural choices $\Omega \equiv \frac{dx \wedge dy}{w}$ and $J = i(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dw \wedge d\bar{w})$, it is easy to see that this representant is also the supersymmetric one. Its area is proportional to $|u|$.

If we have more spheres, the limiting singularity and the local geometry de-

[5]If B is a torus with $K3$ fibre independent of the base, we get an extra $\mathcal{N}=2$ hypermultiplet, as this is simply the $\mathcal{N}=4$ case. The correspondence of states with holomorphic forms does not longer hold because of the enlarged number of supersymmetries. We have not analyzed other cases.

Figure 4.4: The deformed D_4 $(SO(8))$ singularity has a basis of four vanishing spheres intersecting each other as indicated.

pend on the way those spheres intersect. Mathematicians discovered a beautiful correspondence between singularities and Lie algebras [84]. In this correspondence, nontrivial spheres in the deformed singularity are identified with Lie algebra roots, and sphere intersections with (minus) the inner product of the roots.[6] Corresponding to the set of positive roots, there is a set of 'postive spheres' in terms of which all other spheres can be expressed (as linear combination in homology) with only negative or only positive coefficients. The possible intersections, hence the possible singularities, can be encoded by Dynkin diagrams: a dot indicates a sphere, and each line connecting two dots indicates an intersection. Fig. 4.4 shows as an example the D_4 $(SO(8))$ case. In the cases which we are considering (singularities appearing by collapsing spheres in $K3$), all spheres have self-intersection -2 and mutual intersections 0 or 1. Therefore the corresponding Lie algebra will be simply laced. Actually, as it turns out, it will be of A-D-E type (such singularities are called *simple* [84]). For a singularity of type S, the local geometry is given by complex deformations of $W_S = 0$, where

$$W_{A_k} = w^2 + y^2 + x^{k+1} \tag{4.1.8}$$
$$W_{D_k} = w^2 + y^2 x + x^{k-1} \tag{4.1.9}$$
$$W_{E_6} = w^2 + y^3 + z^4 \tag{4.1.10}$$
$$W_{E_7} = w^2 + y^3 + yx^3 \tag{4.1.11}$$

[6] Note that, as it should for this correspondence to make sense, the set of spheres is indeed only a subset of the total integer homology group: for a homology class to have a (supersymmetric) sphere representative, its self-intersection has to be -2 (at least in the cases relevant to us, by the result stated under (4.1.6)).

Figure 4.5: A 2-sphere can be constructed as a circle fibration over an open path, with vanishing circle fibres at the endpoints.

$$W_{E_8} \quad = \quad w^2 + y^3 + x^5. \tag{4.1.12}$$

It is sufficient to consider deformations up to polynomial orders strictly lower than $\partial_z W_S$ $(z = w, y, x)$ since higher order deformations can locally be absorbed in a coordinate transformation. For example, the A_{n-1} deformations are

$$w^2 + y^2 + x^n + U_{n-2}x^{n-2} + \cdots + U_1 x + U_0 = 0. \tag{4.1.13}$$

The 2-cycles in this space can be constructed as follows. First note that the space defined by (4.1.13) at fixed x, consists of 2 copies of the complex plane connected with a throat (so this space has cylinder topology). The nontrivial 1-cycle around the throat vanishes when $P(x) \equiv x^n + U_{n-2}x^{n-2} + \cdots + U_1 x + U_0 = 0$. So by transporting this nontrivial 1-cycle along a path in the x-plane running from one zero x_i to another zero x_j of $P(x)$, we obtain a 2-cycle σ_{ij} with S^2 topology (fig. 4.5). We can carry out this construction for all $n(n-1)$ oriented pairs of zeros of $P(x)$, so we expect to find $n(n-1)$ different oriented minimal 2-spheres, of which $n-1$ are also homologically independent. Note also that

$$\sigma_{ij} \cdot \sigma_{kl} = \delta_{il} + \delta_{jk} - \delta_{ik} - \delta_{jl}. \tag{4.1.14}$$

In general [84], the number r_S of linearly independent homology classes in the space given by $W_S = 0$ equals the rank of the corresponding Lie algebra S, e.g. $r_{A_k} = k$. This is equal to the number of deformation moduli (one can consider the periods of a basis of 2-cycles to be the moduli). The number of (supersymmetric) spheres, counting separately opposite orientations, equals the number of roots of the corresponding Lie algebra, i.e. the dimension of the adjoint representation minus r_S.

An interesting property of the nontrivial 2-spheres of the space given by W_S is that they are permuted by Weyl reflections of the Lie algebra S under monodromies in the deformation moduli space [84]. Indeed, by the Picard-Lefschetz formula, a 2-cycle α transforms as follows under a monodromy about a locus in moduli space where the two-sphere σ_i vanishes:

$$\alpha \to \alpha + \sigma_i \cdot \alpha. \tag{4.1.15}$$

Consequently, the 3-brane states $|i\rangle$ constructed from the 2-cycles σ_i will be only physically distinguishable up to Weyl transformations. Since Weyl transformations are precisely the residual gauge transformations which one has after spontaneous breaking of the gauge symmetry, the nonabelian gauge theory supposedly describing the dynamics of the (massive) vector multiplets corresponding to the $|i\rangle$ *must have a gauge group with Lie algebra S*. The massless vector multiplets corresponding to the unbroken $U(1)^{r_S}$ gauge group (which is generated by the Cartan subalgebra) are the vector multiplets associated to the deformation moduli (cf. chapter 3). Note also that the results of chapter 3 imply that the massive vector multiplet states $|i\rangle$ have precisely the right charges under this unbroken $U(1)^{r_S}$ which one would expect for a gauge group with Lie algebra S. To see this, choose the symplectic basis (A^I, B_I) of 3-cycles such that a subset of the A-cycles are given by $A^i = \gamma \times \sigma_i$ (by this we mean that A^i is a fibration over the nontrivial 1-cycle γ in the base B, with fibre σ_i). From the discussion around (3.2.21), it follows that the vector multiplets corresponding to 3-branes wrapped around $\gamma \times n^i \sigma_i$ have $U(1)$ charge n^i w.r.t. the massless vector A^i_μ.

Thus we arrive at the following picture for Calabi-Yau compactifications of type IIB string theory. Suppose we have a Calabi-Yau X which *locally* can be continuously deformed to a product of a cylinder and an S type ALE space. Then the 3-brane states, obtained by wrapping the brane on the nontrivial cylinder 1-cycle and the ALE 2-spheres (deformed to X,[7]) together with the massless vector multiplets from the perturbative string spectrum, fill out a nonabelian vector multiplet of the Lie algebra S. This vector multiplet can be made arbitrarily light by shrinking the ALE 2-spheres. Therefore, when the spheres are small, we expect the dynamics of this vector multiplet, at energies much smaller than the Planck scale, to be given by an $\mathcal{N} = 2$ Yang-Mills theory (possibly with matter coming from other 3-branes in X) with gauge group algebra S, weakly coupled to gravity. In the limit of vanishing 2-spheres, gravity decouples completely from the vector dynamics. In particular, since the two derivative low energy effective action

[7] We assume the deformation sufficiently small such that the supersymmetric representative exists throughout the deformation.

(3.1.43) of the massless vector multiplets in type IIB compactified on a Calabi-Yau does not receive quantum corrections (cf. chapter 3), we expect to reproduce the Seiberg-Witten solution for the low energy effective action of quantum $\mathcal{N} = 2$ Yang-Mills theory. This will indeed turn out to be the case. Moreover, we will be be able to extend their result by deriving the coupling of this theory to gravity.

Figure 4.6: Construction of a nontrivial 3-sphere in X. The different elements are explained in the text.

Figure 4.7: Distribution over the base B of the points where a 2-sphere in the fiber vanishes. The base manifold B, locally a cylinder, is represented as the punctured complex plane, with γ mapped to the unit circle. The 2-sphere σ_i (here $i = 1, 2, 3$) vanishes in precisely 2 points, $P(\sigma_i)$ and $P'(\sigma_i)$, one inside and one outside γ.

4.1.2 Hyper branes

In general, the nontrivial fibration structure of our Calabi-Yau manifold X, as opposed to the trivial direct product structure of the $\mathcal{N} = 4$ $T^2 \times K3$ compactification, yields new nontrivial 3-cycles, and therefore new massive charged (BPS) particles in the spectrum. Typically, this goes as follows. Because of the nontrivial dependence of the fibre K on the position in the base B, some nontrivial 2-cycles σ in K will vanish at one or more points $P_j(\sigma)$ in the base. One can then construct a 3-cycle in X as a fibration with fiber σ over a path π in B starting and ending on elements of $\{P_j\}_j$, such that the fibre σ vanishes at the endpoints (fig. 4.6). If σ has the topology of a 2-sphere, the resulting 3-cycle will have S^3 topology (this is analogous to the construction of the S^2 in fig. 4.5). Assuming there exists a supersymmetric representative for this 3-cycle, we thus find another $\mathcal{N} = 2$ BPS multiplet. Now since supersymmetric deformations of 3-cycles on Calabi-Yau manifolds are in one to one correspondence with harmonic 3-forms on the brane worldvolume [51], the supersymmetric embedding of such a brane with S^3 topology is unique. By the same reasoning as used in chapter 3, section 3.2.4, and in this chapter in the $\mathcal{N} = 4$ case, we thus expect a unique $\mathcal{N} = 2$ fermionic oscillator vacuum in four dimensions, which has necessarily spin 0 (since it should support a representation of the little group $SO(3)$). Consequently, it is in a half-hypermultiplet. The conjugate half-hypermultiplet is obtained by reversing the orientation of the brane.

Generically, if we tune the moduli of X such that the $S^1 \times S^2$ vectormultiplet branes become light, also some S^3 hypermultiplet branes will become light, namely (at least) those constructed with the vanishing (ALE) 2-spheres in K (and a finite path π in B). So the hypermultiplet content of the four dimensional field theory describing the light BPS states will be determined by the dependence of the local ALE (i.e. the singularity deformation parameters) on the base B.

For concreteness, suppose we want to obtain pure $SU(n)$ $\mathcal{N} = 2$ Yang-Mills. Then we should have the following properties of the hypermultiplet spectrum:

1. From semiclassical considerations, it is known that at weak but nonzero coupling, there are BPS hypermultiplets with nonzero magnetic charge, namely the monopole and dyon hypermultiplets.

2. By definition, there are no purely electrically charged hypermultiplets.

Translated to the brane picture of the spectrum, this means:

1. There should be some light S^3 branes, and so, by the above construction, some points $P_j(\sigma_i)$ in B where an ALE 2-sphere σ_i vanishes.

2. For a given σ_i, the $P_j(\sigma_i)$ should be distributed such that necessarily, the interconnecting paths π *intersect* the path γ (which was the base for the vector multiplet 3-cycles), since no intersection means no magnetic charge. In other words, there should be precisely one zero of σ_i at both sides of γ (fig. 4.7), and there should be no other nontrivial 3-spheres except those constructed from a path interconnecting those pairs of zeros.

We already know that in order to have an $SU(n)$ vector multiplet, the local ALE fibration should have an A_{n-1} fibre. The dependence of the fibre on the base points is given by specifying the U_i as (analytic) functions of a coordinate ζ on the base. We choose the coordinate ζ such that the local cylinder structure of B is represented as the punctured ζ-plane, and the nontrivial 1-cycle wrapping the cylinder is given by the unit circle $|\zeta| = 1$. From the discussion below (4.1.13), it follows that the points in the base where an ALE 2-sphere degenerates are the zeros of the discriminant $\Delta(u(\zeta))$ of $P(x) \equiv x^n + U_{n-2}(\zeta)x^{n-2} + \cdots + U_1(\zeta)x + U_0(\zeta)$, where two zeros of $P(x)$ coincide. From elementary considerations about polynomial zeros, one can see that the only possibility to satisfy the above two requirements to have a pure $\mathcal{N} = 2$ Yang-Mills spectrum in this case, is to have all $U_i(\zeta) \equiv u_i$ independent of ζ, except $U_0(\zeta)$, which should be given by

$$U_0(\zeta) = \frac{b}{2}(\zeta + \frac{1}{\zeta}) + u_0, \tag{4.1.16}$$

where $b(\neq 0)$ and u_0 are constants. The constant b can be put to 1 by a suitable rescaling of coordinates and moduli.

Indeed, the symmetry $\zeta \to 1/\zeta$ guarantees a pairing of zeros on the two sides of γ (the unit circle $|\zeta| = 1$), and with the above ζ dependence of the U_i, there are precisely $n-1$ *homologically independent* 2-spheres in the ALE fibre which vanish each at a point inside the unit circle. (fig. 4.7 shows a typical $n = 4$ configuration). Higher powers of ζ or ζ-dependence of the U_i with $i \geq 1$ would introduce more than $n-1$ zeros of the discriminant on one side of the unit circle, with (since the dimension of the ALE 2-homology is $n-1$) necessarily some of the corresponding 2-cycles homologically dependent on the others, which would make it possible to construct S^3 $3-cycles$ not intersecting the unit circle.

With this choice of ζ-dependence, there are $n-1$ homologically independent S^3 3-cycles, precisely as many as there are $S^1 \times S^2$ 3-cycles. Together they form a set of $2(n-1)$ independent 3-cycles with nondegenerate intersection matrix (which can be put in standard symplectic form after suitable recombination of the basis elements), and they can be taken to form part of a basis of 3-cycles of the full Calabi-Yau.

This ALE fibration can be written in nonsingular polynomial form as

$$W = \zeta(y^2 + w^2 + x^n + \cdots + u_0) + b(\zeta^2 + 1) = 0. \tag{4.1.17}$$

Considering this as a local approximation for an algebraic Calabi-Yau manifold, the unique holomorphic 3-form on this manifold is (in the patch $w \neq 0$) given by

$$\Omega = \nu' \, \frac{d\zeta \wedge dx \wedge dy}{\partial_w W} = \frac{\nu'}{2} \frac{d\zeta}{\zeta} \wedge \frac{dx \wedge dy}{w}, \tag{4.1.18}$$

where ν' is an arbitrary normalisation factor. Note that sending b to zero in (4.1.16) corresponds to the deformation to the trivial direct product fibration considered in section 4.1.1. The dyon hypermultiplets become infinitely massive in the limit $b \to 0$, so this should correspond to turning off the Yang-Mills coupling (since in semiclassical Yang-Mills theory, dyon masses are proportional to $1/{g_{YM}}^2$).

Analogous considerations can be made for the other A-D-E gauge groups.

Thus we conclude that in order to have a pure Yang-Mills spectrum in four dimensions, the dependence of the ALE fiber on the base coordinate ζ must be given by (4.1.16), with all other U_i constant.

4.1.3 Reduction to Seiberg-Witten periods

We consider again the pure $SU(N)$ case. Consider a 'Yang-Mills' 3-cycle γ_{ij} constructed as a 2-sphere fibration over a path γ in the ζ-plane, with the 2-sphere fibre σ_{ij} constructed as under (4.1.13) from a path in the x-plane running between zeros x_i and x_j of

$$P(x, \zeta) = x^N + u_{N-2}x^{N-2} + \cdots + u_0 + \frac{1}{2}(\zeta + \frac{1}{\zeta}). \tag{4.1.19}$$

With the holomorphic 3-form Ω given by (4.1.18), the period of γ_{ij} is then

$$\int_{\gamma_{ij}} \Omega = \pi i \nu' \int_{\gamma} (x_j - x_i) \frac{d\zeta}{\zeta}. \tag{4.1.20}$$

Here we used that $\oint dy (y^2 - k^2)^{-1/2} = 2\pi i$ for a contour encircling the cut between $y = \pm k$ in the y-plane. Equation (4.1.19) describes precisely the genus $N - 1$ $SU(N)$ Seiberg-Witten Riemann surface, as given in (2.3.17). Moreover, if we denote by $\hat{\gamma}_{ij}$ the 1-cycle on this Riemann surface obtained as the lift of γ to the sheet x_j minus the lift of γ to the sheet x_i, we find

$$\int_{\gamma_{ij}} \Omega = 2\sqrt{2}\pi^2 i \nu' \int_{\hat{\gamma}_{ij}} \lambda_{SW}, \tag{4.1.21}$$

where $\lambda_{SW} = \frac{1}{2\sqrt{2}\pi} x \frac{d\zeta}{\zeta}$ is nothing but the Seiberg-Witten meromorphic 1-form (2.3.3). Thus (in the local ALE fibration approximation of the Calabi-Yau X) our light periods reduce precisely to the Seiberg-Witten periods! Note also that (4.1.14) implies that, up to a sign, the intersection product is conserved under the map $\gamma_{ij} \to \hat{\gamma}_{ij}$:

$$\gamma_{ij} \cdot \gamma'_{kl} = (\gamma \cdot \gamma')(\delta_{il} + \delta_{jk} - \delta_{ik} - \delta_{jl}) = -\hat{\gamma}_{ij} \cdot \hat{\gamma}'_{kl}. \qquad (4.1.22)$$

4.2 The weak gravity limit

We will now study in more detail the procedure of tuning the Calabi-Yau moduli such that the branes wrapped around the 'Yang-Mills cycles' (i.e. the small cycles producing the Yang-Mills spectrum) become very light compared to the Planck mass. From (3.2.18), it follows that this is equivalent to tuning the periods of those cycles to be much smaller than the periods of the 'generic' cycles. Assuming a specific but rather general form of the Calabi-Yau, we will deduce the behavior of the periods in such a limit, and from this we will obtain the general form of the Kähler potential on the complex structure moduli space in this limit. We call this limit the 'weak gravity limit', since the Yang-Mills dynamics on which we focus will be arbitrarily weakly coupled to gravity in this limit.

We will restrict to the pure $SU(N)$ Yang-Mills case, though generalizations are clearly possible.

Consider an arbitrary compact algebraic Calabi-Yau manifold X, embedded in a projective space and given in a certain affine coordinate patch with coordinates (ζ, x, y, w) by a polynomial equation

$$W(\zeta, x, y, w) = 0. \qquad (4.2.1)$$

To get a manifold X which is locally given by the pure $SU(N)$ ALE fibration (where we take 'local' to mean small x, y, w and finite ζ) with the Yang-Mills (ALE) periods much smaller than the others, we assume W to be of the form:

$$W = \zeta(W_{A.F.} + W'(x, y, w)), \qquad (4.2.2)$$

where

$$W_{A.F.} = w^2 + y^2 + x^N + L^{-2/N} u_{N-2}\, x^{N-2} + \cdots + L^{-1+1/N} u_1\, x + L^{-1} u_0 + L^{-1} \frac{1}{2}(\zeta + \frac{1}{\zeta}) \qquad (4.2.3)$$

with L very large and the u_i finite, and where W' is a polynomial containing only terms of order x^{N+1}, w^3, y^3, $L^{-1-\epsilon}$ and higher, possibly still depending on moduli different from the u_i (which will not be important in what follows).[8] Indeed, by rescaling $w = L^{-1/2}\tilde{w}$, $y = L^{-1/2}\tilde{y}$ and $x = L^{-1/N}\tilde{x}$, we see that for finite tilde variables (which correspond to small non-tilde variables x, y, w), we get the required local geometry (4.1.17):

$$W \approx \zeta W_{A.F.} = L^{-1}[\zeta(\tilde{w}^2 + \tilde{y}^2 + \tilde{x}^N + u_{N-2}\,\tilde{x}^{N-2} + \cdots + u_1\,\tilde{x} + u_0) + \frac{1}{2}(\zeta^2 + 1)]. \tag{4.2.4}$$

For obvious reasons, we will call the moduli u_i the *rigid moduli.*

The holomorphic 3-form on the other hand is (in a patch with $\partial_w W \neq 0$):

$$\Omega = \nu\,\frac{d\zeta \wedge dx \wedge dy}{\partial_w W} = \nu\,\frac{d\zeta}{\zeta} \wedge \frac{dx \wedge dy}{2w + \partial_w W'} \tag{4.2.5}$$

where ν is an arbitrary normalisation factor. For finite tilde variables indeed reduces to (4.1.18) (with $\nu' = L^{-1/N}\nu$):

$$\Omega \approx L^{-1/N}\frac{\nu}{2}\,\frac{d\zeta}{\zeta} \wedge \frac{d\tilde{x} \wedge d\tilde{y}}{\tilde{w}} \tag{4.2.6}$$

We will assume that it is possible to construct a basis of 3-cycles for X entirely localized in the coordinate patch parametrized by finite ζ, x, y, w. This puts some constraints on the global structure of X. For example, it excludes the possibility that X is a fibration over a higher genus Riemann surface (of which ζ would then parametrize only a certain coordinate patch). Roughly, this assumption thus says that X is a fibration over a sphere (the ζ-plane), but actually we don't have to be that precise[9], since how exactly X is compactified 'at infinity' (w.r.t. the coordinates x, y, w, ζ) does not matter for our purposes.

Now choose a (maximal) set $\{\gamma_\sigma\}_\sigma$ of $2(N-1)$ independent Yang-Mills 3-cycles, which are compact on the *ALE* fibration, i.e. finite in the tilde variables. Denote the (nondegenerate) intersection matrix by $q_{\sigma\lambda} = \gamma_\sigma \cdot \gamma_\lambda$. Extend this basis with 3-cycles Γ_Σ to a basis for $H_3(X, \mathbb{Q})$, in such way that $\Gamma_\Sigma \cdot \gamma_\sigma = 0$. Since $q_{\sigma\lambda}$ is nondegenerate, this is always possible (however, in general it is *not* possible to construct such a basis for $H_3(X, \mathbb{Z})$). Denote the intersection matrix of the Γ cycles by $Q_{\Sigma\Lambda}$.

Note that since there are no intersections of the Γ cycles with the γ cycles, there is no obstruction for deforming the Γ cycles away from the region of small

[8]This can be generalized to cases where W' also depends on ζ, but to keep the — already quite technical — arguments a bit transparent, we will not consider this here. We will see an example of this in section 4.4 however.

[9]for example, the fibration structure does not necessarily have to be extendable to the points at infinity w.r.t. the coordinates (x, y, w, ζ).

Figure 4.8: If a cycle Γ fibred over a path in the ζ-plane encircles (or stretches between) branch points moving to infinity ($\zeta = 0, \infty$) when $L \to \infty$, this cycle will be "stretched" to infinity.

(x, y, w) (finite tilde variables), where the γ cycles are localized. We will furthermore assume that we can keep the Γ cycles at finite values of x, y, w when $L \to \infty$ (which is not a strong assumption).

The γ periods $\int_\gamma \Omega$ can be seen from (4.2.6) to be proportional to $L^{-1/N}$. As explained in section 4.1.3, they can be reduced to Seiberg-Witten periods. Furthermore, in that section it was also explained that the intersection form of those cycles equals (minus) the corresponding intersection form on the Seiberg-Witten Riemann surface.

We now turn to the Γ periods. These can be divergent when $L \to \infty$, but since for $L \to \infty$ the Γ cycles stay finite in x, y, w and away from the singularity locus (finite tilde variables) so that $\partial_w W/\zeta = 2w + \partial_w W'(w, x, y)$ is bounded from below on Γ, the only potential source of divergencies is the fact that X factorizes (in the coordinate patch under consideration) as a direct product of the punctured ζ plane and a complex 2-fold (W/ζ becomes independent of ζ) when $L \to \infty$: consequently, some Γ cycles can (and will) be "stretched" to infinity in the ζ-plane (fig. 4.8). Since W is polynomial in ζ, such cycles will be typically stretched to values of ζ proportional to certain positive ($\zeta \to \infty$) or negative ($\zeta \to 0$) powers of L. So we see from (4.2.5), since we assumed w, x, y to stay finite and away from the singularity locus on the Γ cycles, that these stretched cycles will at most be logarithmically divergent, that is

$$\int_{\Gamma_\Sigma} \Omega = a_\Sigma \ln L + b_\Sigma + \cdots \tag{4.2.7}$$

where a_Σ and b_Σ could a priori still be dependent on all moduli different from L. We will now show however that, if the normalisation factor ν is u-independent, the leading order u-dependent term is actually at most proportional to $L^{-2/N}\ln L$, so that a_Σ and b_Σ must be *independent of the rigid moduli* u_i. Indeed, using (4.2.2), (4.2.3) and (4.2.5), we find

$$\frac{\partial}{\partial u_k}\int_\Gamma \Omega = -\int_\Gamma \Omega \frac{1}{\partial_w W}(\partial_w - \frac{\partial_w^2 W}{\partial_w W})\frac{\partial W}{\partial u_k} \tag{4.2.8}$$

$$= L^{-1+k/N}\int_\Gamma \Omega \frac{\partial_w^2 W}{(\partial_w W)^2}\,\zeta\, x^k \tag{4.2.9}$$

$$= L^{-1+k/N}\int_\Gamma \Omega \frac{2+\partial_w^2 W'}{(2w+\partial_w W')^2}\, x^k \tag{4.2.10}$$

Again because Γ stays finite in x, y, w and away from the singularity locus of finite tilde variables (so that $\partial_w W/\zeta$ stays bounded from below), the integral factor in the r.h.s. of (4.2.10) can at most be proportional to $\ln L$, and since $k \leq N-2$, the full period at most to $L^{-2/N}\ln L$.[10] This is what we wanted to show. Now since u-independence of the a_Σ and b_Σ will turn out to be essential to extract the rigid limit as described in general in section 2.2.3, we will thus take the normalisation factor ν to be u-independent.

Combining all this to compute the form of the Kähler potential (3.1.15), we find

$$\mathcal{K} = -\ln(i\int_X \Omega\wedge\bar\Omega) \tag{4.2.11}$$

$$= -\ln(iQ^{\Sigma\Lambda}\int_{\Gamma_\Sigma}\Omega\int_{\Gamma_\Lambda}\bar\Omega + iq^{\sigma\lambda}\int_{\gamma_\sigma}\Omega\int_{\gamma_\lambda}\bar\Omega) \tag{4.2.12}$$

$$= -\ln(a\ln|L|^2 + b + |L|^{-2/N}K(u,\bar u) + \cdots) \tag{4.2.13}$$

$$\approx -\ln(a\ln|L|^2 + b) - \frac{|L|^{-2/N}}{a\ln|L|^2+b}K(u,\bar u). \tag{4.2.14}$$

Here a and b are u-independent real constants (because a_Σ and b_Σ are), $a > 0$,

$$K(u,\bar u) \equiv 8\pi^4|\nu|^2 iq^{\sigma\lambda}\int_{\hat\gamma_\sigma}\lambda_{SW}\int_{\hat\gamma_\lambda}\bar\lambda_{SW}, \tag{4.2.15}$$

(here we used (4.1.21) and (4.2.6), and the dots include u-independent terms of nonzero order in $L^{-1/N}$ and u-dependent terms higher than second order in $L^{-1/N}$. The reason for the absence in (4.2.13) of terms proportional to $\ln L\ln\bar L$ or

[10] Note that this argument fails for the γ periods because on these $\partial_w W$ is *not* bounded from below, and indeed the u-derivatives of those periods are proportional to $L^{-1/N}$!

$\ln(L/\bar{L})$ is the fact that $e^{-\mathcal{K}} = i\int_X \Omega \wedge \bar{\Omega}$ must be invariant under the monodromy $L \to e^{2\pi i N} L$. The presence of the divergent term can also be inferred from the form of Ω and the fact that X degenerates to the direct product of an infinite cylinder (the punctured ζ-plane) and a compact manifold. The integral over the cylinder (parametrized by ζ) gives a logarithmically divergent factor. Note that this divergent term would be absent for compactifications on e.g. the direct product of a torus and $K3$ (yielding $\mathcal{N} = 4$ in four dimensions). Such $a = 0$ cases could be discussed along the same lines, though some features will be qualitatively different.

Note that the form of the Kähler potential we find is precisely the one needed to have a reduction from local to rigid special geometry for the u-moduli, as was explained in chapter 2, section 2.2.3. Clearly $K(u, \bar{u})$ is closely related to the $SU(N)$ Seiberg-Witten Kähler potential $K_{SW}(u, \bar{u})$, but let us make this more precise. From (4.1.22), (2.2.10) and (2.2.19), it follows that

$$K(u, \bar{u}) = -8\pi^4 |\nu|^2 |\Lambda|^{-2} K_{SW}(u, \bar{u}), \tag{4.2.16}$$

where Λ is the *scale* of the gauge theory, which appeared as an arbitrary parameter in chapter 2, equation (2.2.19): it is the proportionality factor between the dimensionful Yang-Mills scalars and the dimensionless geometric periods (recall that the Kähler potential in rigid Yang-Mills theory has dimension mass squared, that is, expressed in the u variables, it has an overal factor $|\Lambda|^2$). Λ is fixed here in terms of L and the four dimensional Planck length κ_4 by requiring the Yang-Mills fields to have canonical kinetic terms, as in (2.2.17). Comparing the scalar kinetic terms in (2.2.17) and (3.1.43), we see that this puts

$$|\Lambda|^2 = \frac{32\pi^5 |\nu|^2}{\kappa_4^2} \, \frac{|L|^{-2/N}}{a \ln |L|^2 + b}. \tag{4.2.17}$$

Alternatively, one could compare the BPS mass formulas in rigid Yang-Mills theory and supergravity (cf. chapter 3), or, to fix the phase of Λ as well, compare central charges:

$$\frac{Z_{sugra}}{\sqrt{G_N}} \equiv \sqrt{2} Z_{susy}. \tag{4.2.18}$$

This gives

$$\Lambda = \frac{4\pi^2 \sqrt{2\pi}\nu}{\kappa_4} \, \frac{L^{-1/N}}{(a \ln |L|^2 + b)^{1/2}}. \tag{4.2.19}$$

Large L therefore corresponds to a gauge theory scale which is small compared to the Planck scale, hence sending $L \to \infty$ while keeping Λ finite (and $\kappa_4 \to 0$ in the way prescribed by (4.2.17)) corresponds to decoupling gravity from the finite energy nontrivial gauge theory dynamics. The relationship with κ_{10} and the volume of X is given by (3.1.44).

So all in all we find

$$\frac{1}{\kappa_4^2}\mathcal{K} = -\frac{1}{\kappa_4^2}\ln(a\ln|L|^2 + b) + \frac{1}{4\pi}K_{SW}(\Lambda, u, \bar{u}) + \cdots . \tag{4.2.20}$$

Note that the terms we have dropped are proportional to positive powers of κ_4.

Thus we see how the Seiberg-Witten solution for the two derivative low energy effective action of the massless fields in $\mathcal{N} = 2$ Yang-Mills indeed beautifully emerges from purely geometric concepts in type IIB string theory. But we have more now: we have a powerful geometric representation of the massive BPS states as supersymmetric 3-branes wrapped around Calabi-Yau 3-cycles, and we have a consistent embedding of quantum low energy Yang-Mills theory in string theory, including gravity.

4.3 Unification scales and the heterotic picture

In order to compare our results with standard treatments of gauge theory physics in heterotic string theory (see e.g. [31] chapter 18), we assume $\ln L \gg \frac{b}{2a}$ and rewrite (4.2.20) with

$$S \equiv \ln L^2 + \frac{b}{a} \approx \ln L^2 \tag{4.3.1}$$

as

$$\frac{1}{\kappa_4^2}\mathcal{K} \approx -\frac{1}{\kappa_4^2}\ln(S + S^*) + \frac{1}{4\pi}K_{SW}(|\Lambda|, u, \bar{u}) + \cdots , \tag{4.3.2}$$

(where we have dropped an irrelevant term $\sim \ln a$) and

$$\Lambda = \frac{1}{\kappa_4}\, k\, \frac{\exp[-S/(2N)]}{\sqrt{\operatorname{Re} S}}, \tag{4.3.3}$$

where

$$k \equiv \frac{4\pi^2\sqrt{2\pi}\nu}{\sqrt{a}}. \tag{4.3.4}$$

Note that $|k|$ is independent of the choice of normalisation of Ω since $\sqrt{a}$ scales as $|\nu|$ under rescaling of Ω. It depends on the details of the chosen model for X however, including moduli different from L and the u_i, if those are present.

The asymptotic running of the pure $\mathcal{N} = 2$ $SU(N)$ Yang-Mills coupling (which can be extracted from the solution of the low energy effective theory we have found, agreeing with the well-known 1 loop $\mathcal{N} = 2$ beta function) is then given (for energy scales $M \gg \Lambda$) by:

$$\frac{1}{g_{YM}^2(M)} \approx \frac{2N}{8\pi^2}\ln\frac{M}{\Lambda} \tag{4.3.5}$$

$$= \frac{1}{8\pi^2}\mathrm{Re}\,S - \frac{2N}{8\pi^2}\ln\frac{M_{SU}}{M}, \tag{4.3.6}$$

where

$$M_{SU} \equiv \frac{1}{\kappa_4}\,|k|\,(\mathrm{Re}\,S)^{-1/2}. \tag{4.3.7}$$

The mass scale M_{SU} can be interpreted as the *string gauge unification scale*: if we would have several $SU(N)$ gauge groups induced by the Calabi-Yau compactification[11], this would be the value of M where the different gauge groups have equal coupling, equal to

$$g_{YM}^2(M_{SU}) = \frac{8\pi^2}{\mathrm{Re}\,S}, \tag{4.3.8}$$

to be compared with [31], eq. (18.6.7). Note that, from a phenomenological point of view, the string gauge unification scale is not necessarily equal to the scale M_{GUT} where the standard model group $SU(3)\times SU(2)\times U(1)$ unifies to a simple group (like $SU(5)$), though this would be quite pleasing of course.

The string gauge unification scale is closely related (but not equal) to the *string scale* M_S, which can be extracted from the low energy gauge physics as ([31], eq. (18.3.1)):

$$M_S = \frac{g_{YM}(M_{SU})}{\sqrt{2}\kappa_4} = \frac{2\pi}{\kappa_4\sqrt{\mathrm{Re}\,S}} = \frac{2\pi}{|k|}M_{SU}. \tag{4.3.9}$$

Expressions similar to the above formulas typically appear in the context of four dimensional gauge theory physics obtained from (heterotic) string theory. Compare for example (4.3.2) with (18.8.16a) and (4.3.6) with (18.8.2) in [31]. In heterotic string theory S is the dilaton-axion scalar: $S = e^{-2\Phi_4} + ia$, where Φ_4 is the four dimensional heterotic dilaton and a is the axion ([31], eq. (16.4.10)). This is a manifestation of the duality between heterotic and type II string theory [79]. Let us have a short closer look at this correspondence. Consider for example

[11]Though we did not consider this case, it can be obtained straightforwardly by replacing the polynomial in x in (4.2.3) by a polynomial which has two or more finitely separated bunches of closely spaced zeros. We will see an example of this in the next section. The unification scale is then the energy scale of the gauge theory branes when the u moduli are taken so large that all zeros are more or less equally spaced. Generically, this is indeed close to the string scale, but if it happens that two bunches of zeros are already quite near each other at order 1 u values, the corresponding two gauge groups can already get unified at considerably lower energies.

the expression for the rigid prepotential in the $SU(2)$ case, equation (2.3.16). Substituting $\Lambda = M_{SU} e^{-S/4}$ gives

$$\mathcal{F}_{rig}(\phi) = \frac{i}{4\pi}\phi^2 [S - 4 \ln \frac{M_{SU}}{\phi} + \sum_{k=1}^{\infty} c_k \left(\frac{M_{SU}}{\phi}\right)^{4k} \exp(-kS)]. \qquad (4.3.10)$$

Since $\mathrm{Re}\, S$ is identified with $e^{-2\Phi_{4,het}} \sim \frac{1}{g^2_{S,het}}$, we see that, from the heterotic point of view, we have an infinite series of (1 loop) perturbative and nonperturbative quantum corrections!

In heterotic string theory, the string scale M_S is related to the constant α' appearing in string perturbation theory: $\alpha'_{het} = M_S^{-2}$. It is *not* directly related however to the constant α' appearing in type IIB perturbation theory: $\alpha'_{IIB} = 2^{-3/2}\pi^{-7/4}\,\kappa_{10}^{1/4} = 2^{-1}\pi^{-5/4}\,(V_X/\mathrm{Re}\, S)^{1/4}\, M_S^{-1/2}$ (we used (3.1.44) and (4.3.9) for the second equality). So it is the dual heterotic string scale α'_{het} rather than the type IIB string scale α'_{IIB} which appears naturally in the low energy gauge physics we are studying. This can be understood from the fact that type IIB perturbation theory breaks down here at energies higher than the mass of the lightest massive gauge particles (since these particles are non-perturbative states in type IIB), while heterotic perturbation theory is valid for energies above the gauge particle masses and all the way up to (and beyond) the unification scales (it can break down at low energies however because the gauge coupling becomes strong in the infrared for rigid moduli of order 1). So the relevant perturbative string theory for energies near M_S is heterotic, hence the relation with the heterotic α'.

Now though these $\mathcal{N} = 2$ theories are probably not directly relevant to phenomenology, let us nevertheless consider some experimental data [31], to get a more concrete idea of the physical meaning of the above results. Measurements of the running of couplings together with the (experimentally quite plausible) assumption of minimal $SU(5)$ supersymmetric unification[12] suggest $M_{GUT} \approx 10^{16.1\pm0.3}$ GeV and $g^2_{YM}(M_{GUT}) \approx 0.50$. Let us assume $M_{SU} = M_{GUT}$ and see what we get. From (4.3.9), using the unified coupling value and $\kappa_4^{-1} = 2.4 \times 10^{18}$ GeV, we find $M_S \approx 1.2 \times 10^{18}$ GeV. Now in order for $M_{SU} = \frac{k}{2\pi} M_S$ to be equal to M_{GUT} within the error bars, we must have $0.03 < k < 0.13$. Using the relation between M_S and M_{SU} obtained in the context of heterotic string orbifold compactifications ([31] eq. (16.4.36)), one obtains $k = 2.717\ldots$. This is clearly too big. Some possible solutions are explained in [31]. In the example we will work out in the next section, we will however find $k = 0.122\ldots$, which *is* compatible with experiment. From (4.3.8), again using the unified coupling value, we get $\mathrm{Re}\, S \approx 158$. Thus, generalizing (4.3.3) in an obvious way to arbitrary Yang-Mills theories (possibly

[12] This is the minimal GUT supersymmetric extension of the $SU(3) \times SU(2) \times U(1)$ standard model. In this model, the gauge group is $SU(5)$ at energy scales $M > M_{GUT}$.

with matter), we find for the scale of such a theory (taking the experimentally most favored $k = 0.07$):

$$\Lambda = e^{158/b}\, 1.3 \times 10^{16}\text{GeV}, \tag{4.3.11}$$

where b is the beta function coefficient of the theory ($b = N_f - 2N_c$ for $\mathcal{N} = 2$ $SU(N_c)$ with N_f flavors). Since k is dependent on the details of the chosen Calabi-Yau manifold, considerations of this kind might put (together with spectrum considerations) constraints on candidates to find realistic type II Calabi-Yau compactifications. But since such compactifications seem to have too much supersymmetry to make them realistic anyway, and since a serious analysis of the phenomenology would lead us too far afield (and by insufficient expertise of the author) we will leave this subject here.

4.3.1 Dynamics of the dynamical dynamically generated scale

In the following, we will consider Calabi-Yau manifold X which gives rise at low energies to an arbitrary asymptotically free gauge theory with simple gauge group, and assume the Calabi-Yau manifold X has no other complex structure moduli than the rigid moduli u_i and S. Furthermore we will assume $S \gg 1$, as in 4.3. Generalizing (4.3.3) in an evident way to arbitrary gauge theories, with beta-function coefficient $b < 0$, gives the following expression for the dynamically generated scale Λ:

$$\Lambda = \frac{1}{\kappa_4}\, k\, \frac{\exp(-S/|b|)}{\sqrt{\operatorname{Re} S}}. \tag{4.3.12}$$

Since S is a dynamical field, we see that the dynamically generated scale itself is dynamical; it can vary over spacetime. Let us see what we can say about its dynamics.

We will assume S is always large (if not, the whole weak gravity approximation breaks down). Then the contribution from spacetime variation of the denominator of (4.3.12) to spacetime derivatives $\partial_\mu \Lambda$ is suppressed by a factor $(\operatorname{Re} S)^{-1} \ll 1$ w.r.t. the contribution of the exponential. We will assume that this factor is sufficiently small such that this contribution can be neglected. Then we can rewrite (4.3.12) as

$$\Lambda = M_{SU} \exp(-S/|b|), \tag{4.3.13}$$

with the string unification scale M_{SU} as in (4.3.7), assumed to be a constant.

Plugging this in the expression (4.3.2) for the Kähler potential, we get

$$\frac{1}{\kappa_4^2}\mathcal{K} \approx -\frac{1}{\kappa_4^2} \ln(\ln |M_{SU}/\Lambda|^2) + \frac{1}{4\pi}|\Lambda|^2 \tilde{K}_{SW}(u, \bar{u}) + \cdots, \tag{4.3.14}$$

where $K_{SW}(|\Lambda|, u, \bar{u}) = |\Lambda|^2 \tilde{K}_{SW}(u, \bar{u})$ and we have dropped the constant $-\ln|b|$. The corresponding effective action (3.1.43) for the graviton and the scalars is

$$\begin{aligned} S &= \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-G}\,R - \frac{2}{(\ln|M_{SU}/\Lambda|^2)^2}\left|\frac{\mathrm{d}\Lambda}{\Lambda}\right|^2 \\ &- \frac{1}{4\pi}\int |\Lambda|^2\partial_i\bar{\partial}_j\tilde{K}_{SW}\mathrm{d}u^i\wedge *\mathrm{d}\bar{u}^j \qquad (4.3.15)\\ &+2\mathrm{Re}\,(\Lambda\partial_i\tilde{K}_{SW}\mathrm{d}u^i\wedge *\mathrm{d}\bar{\Lambda}) + \tilde{K}_{SW}\;|\mathrm{d}\Lambda|^2\,. \end{aligned}$$

From this we see that generically, the Λ-field will couple to the gauge theory fields, but almost[13] as weakly as gravity, because of the factor $1/\kappa_4^2$ in front of the Λ kinetic term. Therefore in circumstances where gravity effects are very small (for instance in accelerator experiments), the effects of having a dynamic Λ will be very small as well (i.e. $d\Lambda/\Lambda \ll 1$). Note also that there is actually *no* coupling between the gauge theory fields and Λ when K_{SW} is the *classical* Kähler potential and if one uses the period variables $\phi^i = \Lambda\int_{\alpha^i}\lambda_{SW}$ instead of the u_i variables, as follows immediately from (4.3.14) together with $K_{SW,class} = \eta_{i\bar{j}}\phi^i\bar{\phi}^j$ where $\eta_{i\bar{j}}$ is constant. From the microscopic Yang-Mills field theory point of view, the backreaction of the gauge theory fields on the gauge theory scale is thus entirely a (stringy) quantum effect.

In a certain sense, one can also interpret the spacetime variation of the scale as a spacetime variation of an effective Newton constant with constant scale. To see this, write

$$e^\sigma \equiv \Lambda/\langle\Lambda\rangle, \qquad G'_{\mu\nu} \equiv \exp(2\mathrm{Re}\,\sigma)G_{\mu\nu}, \qquad (4.3.16)$$

and substitute this in (4.3.15):

$$\begin{aligned} S &= \int \frac{1}{2(\exp(\mathrm{Re}\,\sigma)\kappa_4)^2}[d^4x\sqrt{-G'}R' - (-6 + \frac{2}{(\ln|\frac{M_{SU}}{\Lambda}|^2)^2})\,\mathrm{d}\sigma\wedge *'\mathrm{d}\bar{\sigma}] \\ &-\frac{1}{4\pi}\int |\langle\Lambda\rangle|^2[\partial_i\bar{\partial}_j\tilde{K}_{SW}\mathrm{d}u^i\wedge *'\mathrm{d}\bar{u}^j \qquad (4.3.17)\\ &+2\mathrm{Re}\,(\partial_i\tilde{K}_{SW}\mathrm{d}u^i\wedge *'\mathrm{d}\bar{\sigma}) + \tilde{K}_{SW}\;\mathrm{d}\sigma\wedge *'\mathrm{d}\bar{\sigma}]. \qquad (4.3.18) \end{aligned}$$

So we see κ_4 gets replaced by an 'effective' $\exp(\mathrm{Re}\,\sigma)\kappa_4$. A review on the physical consequences of a varying Newton constant can be found in [85]. This interpretation is perhaps less natural here considering the coupling of σ to the scalar fields (including hypermultiplet scalars) this rescaling induces. Of course, this situation (and the action) is very similar to what one has with the usual dilaton scalar in the role of σ. In the light of the discussion on heterotic - type II duality in section 4.3, this is not surprising at all.

[13] somewhat stronger actually because of presence of the extra factor $\ln|M_{SU}/\Lambda|^2$.

Finally note that dynamical gauge theory scales (and hence dynamical gauge coupling constants) could produce interesting, and perhaps measurable, physical effects. In particular, considering the time dependendence of the matter density in the universe, and the fact that the gauge theory scale couples to this according to the picture we have developed, one would expect the gauge coupling constant(s) to have some time variation. There are indeed some (cosmological) experimental indications that this is the case [86]. One could also contemplate the possibility of strong spatial variations of gauge theory scales near black holes. This would certainly produce rather spectacular effects, since some particles might suddenly become unstable by a change of the scales, leading to violent explosions, gamma bursts, dark matter production, black hole jets, and all that. Or, more realistically perhaps, fine structure constant fluctuations (waves) might be measured by equipment of similar sensitivity as gravitational wave detectors. Considering the fact that extremely high precision measurements of the fine structure constant are possible, such experiments might be worthwhile.

4.4 An explicit example

In this section, we will study in detail an explicit example [17] to illustrate (and perhaps clarify) some of the topics we have discussed more or less in general in this chapter.

4.4.1 Fibration structure

We take as example the Calabi-Yau manifold $X^*_{1,1,2,8,12}[24]$. The details of the general construction of algebraic Calabi-Yau manifolds will not be important for us. We will define this manifold simply as the hypersurface $W=0$ with

$$\begin{aligned} W &= \frac{B}{24}(x_1^{24}+x_2^{24})+\frac{1}{12}x_3^{12}+\frac{1}{3}x_4^3+\frac{1}{2}x_5^2 \\ &\quad -\psi_0\, x_1x_2x_3x_4x_5-\frac{1}{6}\psi_1\,(x_1x_2x_3)^6-\frac{1}{12}\psi_s\,(x_1x_2)^{12} \end{aligned} \tag{4.4.1}$$

and identifications

$$(x_1,x_2,x_3,x_4,x_5) \simeq (\lambda x_1,\lambda x_2,\lambda^2 x_3,\lambda^8 x_4,\lambda^{12}x_5) \tag{4.4.2}$$
$$\simeq (\alpha\, x_1,\alpha^{-1}x_2,x_3,x_4,x_5) \tag{4.4.3}$$
$$\simeq (\alpha^{-1}x_1,\alpha^{-1}x_2,\alpha^2 x_3,x_4,x_5) \tag{4.4.4}$$

where $\lambda\in\mathbb{C}^*$ and $\alpha=\exp(2\pi i/24)$. It is also understood that all singular points induced by the identifications (i.e. the fixed points) are blown up. This introduces additional Kähler moduli, which are however not important for us, as we will only deal with the complex structure moduli. Note that these singularities lie all on the locus $x_1x_2x_3 = 0$. As it turns out, we can for our purposes simply ignore this locus and always work in the patch $x_1x_2x_3 \neq 0$ (as we will do from now on). $X^*_{1,1,2,8,12}[24]$ is the mirror of $X_{1,1,2,8,12}[24]$, which is defined by $W' = 0$ with $W'(x_1, x_2, x_3, x_4, x_5)$ the general polynomial invariant under (4.4.2) with $\lambda = \alpha$, with identifications (4.4.2) but without (4.4.3) and (4.4.4). Similarly, W could have been defined as the general polynomial invariant under (4.4.2) with $\lambda = \alpha$ as well as under (4.4.3) and (4.4.4). This indeed yields, up to isomorphisms, (4.4.1). For an introduction to algebraic Calabi-Yau manifolds, see [29] p. 441 or [42]. See also [44].

The space $X^*_{1,1,2,8,12}[24]$ has $h^{2,1} = 3$. The complex structure moduli space is hence parametrized by 3 coefficients of W. We choose to work in the gauge $\psi_S = -1$ (instead of the more usual $B = 1$).

This Calabi-Yau manifold (as well as its mirror) is a $K3$ *fibration*, with $K3$ fibre $X^*_{1,1,4,6}[12]$ (resp. $X^*_{1,1,4,6}[12]$). The change of variables needed to exhibit this is

$$x_1 = x_0^{1/2}\zeta^{1/24} \tag{4.4.5}$$
$$x_2 = x_0^{1/2}\zeta^{-1/24} \tag{4.4.6}$$

which is well defined because of the identifications (4.4.3)-(4.4.4). This gives as defining polynomial for the $K3$ at fixed ζ:

$$\begin{aligned} W &= \frac{1}{12}B'x_0^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 \\ &\quad -\psi_0 x_0x_3x_4x_5 - \frac{1}{6}\psi_1(x_0x_3)^6 \end{aligned}$$

where

$$B' = \frac{B}{2}(\zeta + \frac{1}{\zeta}) + 1 \tag{4.4.7}$$

and the following identifications remain:

$$(\zeta, x_0, x_3, x_4, x_5) \simeq (\zeta, \lambda x_0, \lambda x_3, \lambda^4 x_4, \lambda^6 x_5) \tag{4.4.8}$$
$$\simeq (\zeta, \alpha^{-2}x_0, \alpha^2 x_3, x_4, x_5). \tag{4.4.9}$$

The $K3$ fibre itself is an elliptic fibration. Actually, it can be fibered in two ways. The most evident one, which also works for the mirror manifold and is suitable for discussing the large complex structure limit, can be obtained in complete analogy with the above construction. The fibre is $X^*_{1,2,3}[6] = X_{1,2,3}[6]$

and there are no identifications left. The other possibility does not apply to the mirror (where it would lead to a genus 5 fibre) but is more suitable for discussing the rigid limit. Both possibilities are exhibited by the following change of variables:

$$x_0 = y_0^{1/2}\xi^{-1/12} \quad (4.4.10)$$

$$x_3 = y_0^{1/2}\xi^{1/12} \quad (4.4.11)$$

$$x_4 = x + \frac{1}{2}\psi_0^2\, y_0^2 \quad (4.4.12)$$

$$x_5 = y + \psi_0\, y_0 x + \frac{1}{2}\psi_0^3\, y_0^3, \quad (4.4.13)$$

again well defined because of the identifications. The projective 'gauge' symmetry acts on these variables as

$$(\zeta, \xi, y_0, x, y) \simeq (\zeta, \xi, \lambda y_0, \lambda^2 x, \lambda^3 y). \quad (4.4.14)$$

There are no discrete identifications left. Fix the projective gauge symmetry by putting $y_0 = 1$ (which can be done because we work in the patch $y_0 = x_1x_2x_3 \neq 0$). This gives

$$W = \frac{1}{2}y^2 + \frac{1}{12}(\xi + \frac{B'}{\xi}) + \frac{1}{6}P(x), \quad (4.4.15)$$

where

$$P(x) = 2x^3 - \frac{3}{2}\psi_0^4 x - \frac{1}{2}(\psi_0^6 + 2\psi_1). \quad (4.4.16)$$

The first fibration is obtained by considering the ξ plane to be the base manifold, the second one by taking the x plane instead. Indeed, in both cases (4.4.15) defines a (punctured[14]) torus for any fixed generic base point. From now on, we will consider only the second fibration.

The holomorphic 3-form on the Calabi-Yau manifold is

$$\Omega^{(3,0)} = \frac{1}{(2\pi i)^3}\frac{\mathrm{d}\zeta}{\zeta} \wedge \mathrm{d}x \wedge \frac{1}{y}\frac{\mathrm{d}\xi}{\xi}. \quad (4.4.17)$$

Note that $\Omega^{(1,0)} \equiv \frac{1}{2\pi i}\frac{1}{y}\frac{\mathrm{d}\xi}{\xi}$ is the (up to a constant factor unique) holomorphic 1-form on the torus fibre, and $\Omega^{(2,0)} \equiv \frac{1}{2\pi i}\mathrm{d}x \wedge \Omega^{(1,0)}$ the holomorphic 2-form on the $K3$.

[14]The punctures can be removed (and the tori compactified) by going to a projective description of the torus fibres. This yields $X_{1,2,3}[6]$ as the torus fibre for the first fibration, and $X^*_{1,1,2}[4]$ for the second fibration.

Figure 4.9: Branch points, cuts and cycles of the torus fibre.

4.4.2 Construction of cycles and periods

We will explicitly construct a basis of 3-cycles and their corresponding periods. To achieve this, we exploit the fibration structure of the model; first we study the cycles and periods of the torus fibre, then those of the $K3$, and finally those of the Calabi-Yau manifold itself. The explicit construction of the cycles allows us to compute monodromies and intersection forms in a straightforward way. For the torus and $K3$ fibres, we furthermore obtain closed expressions of the periods in terms of hypergeometric functions.

Torus cycles and periods

Branch points: If we consider (4.4.15) to be an equation for y as a function of ξ, we get the torus as a 2-sheeted cover of the ξ plane. The sheets coincide at the branch points, which come in pairs symmetric under $\xi \leftrightarrow \frac{B'}{\xi}$, and are located at

$$\xi = -P \pm \sqrt{P^2 - B'} \tag{4.4.18}$$

and at

$$\xi = 0, \infty. \tag{4.4.19}$$

Cycles: In the following, $\sqrt{z}$ denotes the positive square root of z: $\sqrt{z} > 0$ (and $\sqrt{-1} = i$). Define β to be the 1-cycle $|\xi| = |B'|^{1/2}$ passing counterclockwise through the point ($\xi = i\sqrt{B'}, y = \frac{i}{\sqrt{3}}\sqrt{P}$), and α the shortest cycle encircling the pair of branch points (4.4.18), with orientation such that $\alpha \cdot \beta = +1$. This is shown in figure 4.9.

Singularities and vanishing cycles: The torus degenerates when two branch points coincide. There are two possibilities:

- $\mathbf{P^2 - B' = 0}$: the branch points (4.4.18) coincide and α vanishes. The locus in the (ζ, x)-plane where this occurs is a *genus 5* Riemann surface, as can be seen by substituting the expressions for $P(x)$ and $B'(\zeta)$. We denote this surface by Σ. Since α collapses to a point on Σ, this surface can be lifted trivially to the full Calabi-Yau. By slight abuse of notation, we denote the lifted surface by Σ as well. Thus Σ is the locus of elliptic fibre singularities of the Calabi-Yau. We could also view the full 10D spacetime $M_4 \times CY$ as an elliptic fibration. Then the locus of elliptic fibre singularities gets promoted to a 5+1 dimensional manifold $M_4 \times \Sigma$.
- $\mathbf{B' = 0}$: one of the branch points (4.4.18) coincides with the branch point $\xi = 0$, and β vanishes. The surface where this occurs consists of two copies of the x-plane at fixed ζ positions, and is denoted by Σ'. Again, this surface can be lifted to the full Calabi-Yau or promoted to a $5 + 1$ dimensional submanifold of spacetime.

Cuts: There are jumps (or cuts) in the definition of α at values of $P/\sqrt{B'}$ where there are two homologically different cycles encircling the two branch points with equal minimal length. This occurs when the branch points (4.4.18) are colinear, i.e. when $\frac{P}{\sqrt{B'}}$ is imaginary (type A cut). Jumps in the definition of β occur when the branch points lie on the circle $|\xi| = |B'|^{1/2}$, i.e. when $\frac{P}{\sqrt{B'}} = 0$ lies on the real interval $[-1, 1]$ (type B cut). There is yet another type of cut, for both α and β, namely where our prescription for the orientation of β (and hence α) is ambiguous. For fixed B', this is the case when P is real and negative (type C cut).

The cut structure in the $P/\sqrt{B'}$-plane is shown in fig. 4.10. The transformation rules for continuous transport of torus cycles across the cuts (yielding the monodromies) are:

$$A, A': \quad \alpha \to \alpha + 2\beta \tag{4.4.20}$$

$$B, B': \quad \beta \to \beta + \alpha \tag{4.4.21}$$

$$C: \quad \alpha, \beta \to -\alpha, -\beta \tag{4.4.22}$$

Figure 4.10: Singularities and cuts in the $P/(B')^{1/2}$-plane.

Notice that the origin of the $P/\sqrt{B'}$-plane is not really singular, as the mondromy about this point is in fact trivial.

Periods: The following expressions for the periods can be obtained by direct integration. Denoting $k_\mp^2 = \frac{1}{2}(1 \mp \frac{P}{\sqrt{B'}})$, we find:

$$\int_\alpha \Omega^{(1,0)} = \frac{2\sqrt{6}}{\pi}(\pm\sqrt{B'})^{-1/2}\ \mathbf{K}(k_\mp^2) \tag{4.4.23}$$

for $\frac{P}{\sqrt{B'}}$ close to ± 1, and

$$\int_\beta \Omega^{(1,0)} = \frac{\sqrt{6}}{\pi}(\pm\sqrt{B'})^{-1/2}\, k_\mp^{-1}\ \mathbf{K}(k_\mp^{-2}) \tag{4.4.24}$$

for $\frac{P}{\sqrt{B'}}$ far from ± 1. Here $\mathbf{K}(u) = \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2},1;u)$ is the complete elliptic integral of the first kind. These expressions can be extended to other values of $P/\sqrt{B'}$ by standard analytic continuation.

Figure 4.11: (Elliptic) fibre singularities, cuts and cycles of the $K3$ manifold.

K3 cycles and periods

As explained earlier, the $K3$ fibre (at a fixed generic value of ζ, and hence of $\sqrt{B'}$) is itself an elliptic fibration, with base parametrized by x. Accordingly, the relevant 2-cycles of the $K3$ fibre, that is, those which are in the transcendental lattice, can be constructed as circle fibrations over certain paths in the x-plane, where the circle is a 1-cycle in the torus fibre.

Points with degenerating elliptic fibre: The x-plane can be viewed as a 3-fold covering of the $P/\sqrt{B'}$-plane considered in 4.4.2, as ζ is fixed and P is of degree 3 in x. Therefore, in the x-plane, there are 3 copies of every ingredient (cuts, singularities, ...) of fig. 4.10. This is shown in fig. 4.11. In particular, the $g = 5$ Riemann surface Σ, on which α vanishes, intersects the x-plane (having a fixed value of ζ) in 6 points. As Σ splits in two branches $\Sigma_{\pm}$ corresponding to the solutions of $P(x)/\sqrt{B'} = \pm 1$, we can divide those six points accordingly in two groups of three, which we label by $1^+, 2^+, 3^+$ and $1^-, 2^-, 3^-$. Choose numbering such that the type B cuts connect i^+ with i^- (see figure).

Cycles: The idea is to construct $K3$ 2-cycles as circle fibrations by transporting a certain torus 1-cycle c along a path γ in the x-plane. The path γ can either be a closed loop without monodromy for c, or an open path terminating on the points $i^{\pm}$, with c vanishing at both endpoints. The first possibility will produce a torus and the second a sphere. This gives us the following 2-cycles (see fig. 4.11):

- $\mathbf{s}^+_{\mathbf{ij}}$: $c = \alpha$ (at i^+) and γ running between i^+ and j^+. This is a sphere.
- $\mathbf{s}^-_{\mathbf{ij}}$: $c = \alpha$ (at i^-) and γ running between i^- and j^-; again a sphere.
- $\mathbf{t}_{\mathbf{ij}}$: $c = \beta$ (at i^+) and γ a closed path encircling i^+, i^-, j^+, j^-. This is a torus.

Note that it is not possible to construct a 2-cycle by taking γ to run from i^+ to j^- because such a path necessarily passes through an A type cut, so we cannot have α as circle fibre at both endpoints.

Actually we haven't given a precise description yet of the above cycles in general, only for the specific case of fig. 4.11. To define this set in the general case, we can proceed as follows: first, we require the cycles to be compatible with the above description including the homological relations and intersections. Any set of cycles obtained by continuation from the specific set of fig. 4.11 will satisfy this. Because continuation is in general not uniqely defined (due to monodromies), there are still many possibilities, corresponding to the choice of cuts. We will not try to fix the remaining ambiguity here in general, but assume that in any case a prescription is adopted such that $s^{\pm}_{ij}$ vanishes whenever $i^{\pm}$ approaches $j^{\pm}$.

At the level of homology, we have the following relations:

$$\begin{aligned} t_{ij} &= s^+_{ij} - s^-_{ij} &\quad (4.4.25)\\ s^{\pm}_{12} + s^{\pm}_{23} + s^{\pm}_{31} &= 0 &\quad (4.4.26)\end{aligned}$$

This implies that of the 2-cycles constructed above, only four are independent.

Intersections: Combining the the intersections of the paths γ in the base (as shown in fig. 4.11) with the known intersections of the torus 1-cycles (taking into account their transformation when passing a cut), all 2-cycle intersections can be calculated in a straightforward way:

$$\begin{array}{lllllllll} s^{\pm}_{12}\cdot s^{\pm}_{12} &=& -2 & s^{+}_{12}\cdot s^{-}_{12} &=& -2 & s^{\pm}_{12}\cdot t_{12} &=& 0\\ s^{\pm}_{12}\cdot s^{\pm}_{23} &=& 1 & s^{+}_{12}\cdot s^{-}_{23} &=& 2 & s^{\pm}_{12}\cdot t_{23} &=& 1\\ s^{+}_{12}\cdot s^{-}_{31} &=& 0 & s^{\pm}_{12}\cdot t_{31} &=& -1 & t_{ij}\cdot t_{kl} &=& 0 \end{array} \quad (4.4.27)$$

and all cyclic permutations hereof. The first two equations can be summarized as

$$s^{\pm}_{ij}\cdot s^{\pm}_{kl} = \delta_{il} + \delta_{jk} - \delta_{ik} - \delta_{jl}. \quad (4.4.28)$$

Note that this is precisely (minus) the intersection of the 0-cycles $j^{\pm} - i^{\pm}$ and $l^{\pm} - k^{\pm}$.

Singularities and vanishing cycles: The $K3$ degenerates when two (or more) of the points $1^\pm, 2^\pm, 3^\pm$ coincide. There are several possibilities (corresponding to the marked points in fig. 4.12):

- $\mathbf{B' = 0}$: here $1^+ = 1^-$, $2^+ = 2^-$, $3^+ = 3^-$, and the tori t_{ij} degenerate to lines (recall that β vanishes at $B' = 0$).
- $\mathbf{B' = (\psi_0^6 + \psi_1)^2}$: Keeping in mind that we have defined $\sqrt{B'}$ to have positive real part, there are two possibilities:
  - if $\mathrm{Re}\,(\psi_0^6 + \psi_1) < 0$, two of the zeros $1^+, 2^+, 3^+$ of $P = +\sqrt{B'}$ coincide, and the corresponding sphere s_{ij}^+ vanishes.
  - if $\mathrm{Re}\,(\psi_0^6 + \psi_1) > 0$, two of the zeros $1^-, 2^-, 3^-$ of $P = -\sqrt{B'}$ coincide, and the corresponding sphere s_{ij}^- vanishes.

  In each case, we call the vanishing sphere at this point v_a.
- $\mathbf{B' = \psi_1^2}$: Again, there are two cases: if $\mathrm{Re}\,\psi_1$ is positive (negative), there is a vanishing s_{ij}^- (s_{ij}^+). Call this vanishing sphere v_b.

Define $t_a \equiv t_{ji}$ if $v_a = s_{ij}^\pm$, and similarly for t_b. The tori t_a and t_b degenerate at $B' = 0$. The set (v_a, v_b, t_a, t_b) forms a basis of the Picard lattice, which has rank four here.

We have constructed different bases depending on the signs of $\mathrm{Re}\,(\psi_0^6 + \psi_1)$ and $\mathrm{Re}\,\psi_1$. We therefore also expect the corresponding intersection matrix to depend on these signs. To calculate this matrix when, say, $\mathrm{Re}\,(\psi_0^6 + \psi_1) > 0$ and $\mathrm{Re}\,\psi_1 > 0$, we consider the case where ψ_1 is very close to $\psi_1 + \psi_0^6$ (which is the case shown in fig. 4.11). Then it is easy to see, by direct inspection of the roots of $P(x)^2 - B'$, that v_a and v_b are to be identified with s_{12}^- and s_{23}^-, if one chooses suitable numbering of the roots. This identification, together with (4.4.27), provides the complete intersection matrix. An analogous procedure can be followed for the other cases.

The results are:

- If $\mathrm{Re}\,\psi_1$ and $\mathrm{Re}\,(\psi_1 + \psi_0^6)$ have the same sign:

$$\mathcal{I} = \begin{pmatrix} -2 & 1 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.4.29}$$

  We recognise the $SU(3)$ Cartan matrix in the upper block and therefore call this part of muduli space the "$SU(3)$ sector".

- If the real parts of $\psi_1 + \psi_0^6$ and ψ_1 have opposite sign:

$$\mathcal{I}' = \begin{pmatrix} -2 & 0 & 0 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.4.30}$$

  Here we recognize the $SU(2) \times SU(2)$ Cartan matrix; accordingly we call this part of moduli space the "$SU(2) \times SU(2)$ sector".

Notice that our division of the moduli space in an $SU(3)$ and an $SU(2)\times SU(2)$ sector is dependent on the sign convention for $\sqrt{B'}$ (except on the subspace $\psi_1^2 = (\psi_1 + \psi_0^6)^2$). Therefore, though the convention we have taken is quite natural, especially when we are close to a rigid limit, one can not expect the boundary between these sectors to have any physical significance[15] However, we shall see that there exists a certain region inside the $SU(3)$ sector (close to the $SU(3)$ rigid limit) where a 4D low energy observer indeed sees $SU(3)$ Yang-Mills physics (weakly) coupled to gravity, and similarly for $SU(2) \times SU(2)$. Outside these regions, 4D low energy physics might not look at all like a particular nonabelian gauge theory.

Monodromies From the explicit construction of the cycles and the Picard-Lefschetz formula, it is easy to calculate the monodromies about the singular points $B' = 0$, $B' = (\psi_1 + \psi_0^6)^2$, $B' = \psi_1^2$ and $B' = \infty$. The monodromies are of the form

$$\begin{pmatrix} v_a \\ v_b \\ t_a \\ t_b \end{pmatrix} \to M \begin{pmatrix} v_a \\ v_b \\ t_a \\ t_b \end{pmatrix}. \tag{4.4.31}$$

The resulting monodromy matrices are:

- About $B' = (\psi_1 + \psi_0^6)^2$
  - In the $SU(3)$ sector of moduli space ($\mathrm{Re}\psi_1 \cdot \mathrm{Re}(\psi_1 + \psi_0^6) > 0$):

$$M = T_a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.32}$$

[15] A physically significant definition of e.g. the $SU(3)$ sector would be the region of moduli space where BPS states exist which can be identified as $SU(3)$ gauge bosons. Unfortunately, for a generic point of moduli space, the existence of these states is very difficult to check analytically, if not impossible.

– In the $SU(2) \times SU(2)$ sector of moduli space $(\mathrm{Re}\,\psi_1 \cdot \mathrm{Re}\,(\psi_1 + \psi_0^6) < 0)$:

$$M = T_a' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.33}$$

- About $B' = \psi_1^2$: In the $SU(3)$ sector:

$$M = T_b = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.34}$$

  In the $SU(2) \times SU(2)$ sector:

$$M = T_b' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.35}$$

- About $B' = 0$: In both sectors, this is

$$M = B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.36}$$

The matrices T_a, T_a', T_b and T_b' have Jordan form $\mathrm{diag}(-1,1,1,1)$, hence an expansion of the periods in a variable z around the corresponding singularity has terms of the form z^n and $z^{1/2+n}$. B on the other hand has Jordan form

$$B_{Jordan} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.37}$$

so period expansions have terms z^n and $z^n \ln z$.

The matrices T_a and T_b generate the group $\mathcal{S}_3$, the Weyl group of $SU(3)$, while T_a' and T_b' generate $\mathcal{S}_2 \times \mathcal{S}_2$, the Weyl group of $SU(2) \times SU(2)$.

Periods An integral representation for the $K3$ periods can be given by making use of the elliptic fibration structure and (4.4.23)-(4.4.24). Note in particular that when $i^{\pm}$ and $j^{\pm}$ come close to each other, we have

$$\int_{s_{ij}^{\pm}} \Omega^{(2,0)} \approx \frac{\sqrt{6}}{2\pi i}(\pm\sqrt{B'})^{-1/2}\int_{i^{\pm}}^{j^{\pm}} \mathrm{d}x = \frac{\sqrt{6}}{2\pi i}(\pm\sqrt{B'})^{-1/2}(x_{j^{\pm}} - x_{i^{\pm}}), \qquad (4.4.38)$$

where the $x_{i^{\pm}}$ are simply found by solving $P(x) = \pm\sqrt{B'}$.

Though not really necessary for the discussion of the rigid limit, it is possible, using Picard-Fuchs techniques, to find a closed expression for a basis of (nonintegral) periods of the $K3$, namely [17]:

$$\vec{\vartheta} = \begin{pmatrix} \vartheta_{12} \\ \vartheta_{21} \\ \vartheta_{11} \\ \vartheta_{22} \end{pmatrix} = -\frac{4\pi}{3}\begin{pmatrix} \xi_1(r)\,\xi_2(s) \\ \xi_2(r)\,\xi_1(s) \\ \xi_1(r)\,\xi_1(s) \\ \xi_2(r)\,\xi_2(s) \end{pmatrix} . \qquad (4.4.39)$$

where

$$\begin{aligned} \xi_1(u) &= B_1\, u^{-1/6}\, {}_2F_1\left(\frac{1}{6},\frac{1}{6},\frac{1}{3};\frac{1}{u}\right) \\ \xi_2(u) &= B_2\, u^{-5/6}\, {}_2F_1\left(\frac{5}{6},\frac{5}{6},\frac{5}{3};\frac{1}{u}\right) , \end{aligned} \qquad (4.4.40)$$

with

$$B_1 = \frac{\Gamma(\frac{2}{3})}{\Gamma^2(\frac{5}{6})} ; \qquad B_2 = \frac{\Gamma(-\frac{2}{3})}{\Gamma^2(\frac{1}{6})} ; \qquad B_1 B_2 = -\frac{\sqrt{3}}{4\pi} , \qquad (4.4.41)$$

and

$$r = \frac{1}{2} + \frac{\sqrt{(\psi_0^6+\psi_1)^2 - B'} - \sqrt{\psi_1^2 - B'}}{2\psi_0^6} \qquad (4.4.42)$$

$$s = \frac{1}{2} + \frac{\sqrt{(\psi_0^6+\psi_1)^2 - B'} + \sqrt{\psi_1^2 - B'}}{2\psi_0^6} . \qquad (4.4.43)$$

For a derivation of this, we refer to [17].

The monodromies of these solutions can be calculated using the well known continuation formulae of the hypergeometric functions. By comparing monodromies, one obtains the expression of the periods of v_a, v_b, t_a and t_b in this basis up to an overall factor. The overall factor can be determined by comparing asymptotic expansions in a limit where these can be calculated easily on both sides, e.g. in a

Figure 4.12: ($K3$) fibre singularities, cuts and cycles of the CY manifold.

rigid limit (see later). The result for the $SU(3)$ sector is, with $\omega = e^{2i\pi/3}$:

$$\begin{pmatrix} v_a \\ v_b \\ t_a \\ t_b \end{pmatrix} = S_v \begin{pmatrix} \vartheta_{12} \\ \vartheta_{21} \\ \vartheta_{11} \\ \vartheta_{22} \end{pmatrix} \quad ; \qquad S_v = \begin{pmatrix} -i\omega & i\omega^2 & 0 & 0 \\ -i & i & 0 & 0 \\ \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} . \tag{4.4.44}$$

CY cycles and periods

Since the Calabi-Yau 3-fold under consideration is a $K3$ fibration, one can construct the CY 3-cycles as $K3$ cycle fibrations over paths in the base manifold (the ζ-plane). Denote the path in the base by γ and the $K3$ cycle which is transported along γ by c; γ can be open, with c vanishing at both endpoints, or closed, with trivial monodromy for c. The corresponding CY period is given by

$$\frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta} \theta(\zeta), \tag{4.4.45}$$

where $\theta(\zeta)$ denotes the period of c in the $K3$ fibre above ζ.

A basis of 3-cycles can be constructed as follows (see fig. 4.12):

- $\mathbf{V_{v_a}}$: $c = v_a$ and γ running between the two solutions of $\frac{B}{2}(\zeta + \frac{1}{\zeta}) + 1 = (\psi_0^6 + \psi_1)^2$. This is an S^3.
- $\mathbf{V_{v_b}}$: $c = v_b$ and γ running between the two solutions of $\frac{B}{2}(\zeta + \frac{1}{\zeta}) + 1 = \psi_1^2$. Again an S^3.
- $\mathbf{T_{v_a}}$: $c = v_a$ and γ the unit circle. This has topology $S^1 \times S^2$.
- $\mathbf{T_{v_b}}$: same as T_{v_a}, but with $c = v_b$.
- $\mathbf{V_{t_a}}$: $c = t_a$ and γ running between the two solutions of $\frac{B}{2}(\zeta + \frac{1}{\zeta}) + 1 = 0$. Topology: $S^2 \times S^1$.
- $\mathbf{V_{t_b}}$: same as V_{t_a}, but with $c = v_b$.
- $\mathbf{T_{t_a}}$: $c = t_a$ and γ the unit circle. Topology: T^3.
- $\mathbf{T_{t_b}}$: same as $\mathbf{T_{t_a}}$, but with $c = t_b$.

Formal connection with brane picture

We can also consider the CY cycles constructed above as circle fibrations over certain '2-branes' in the (x, ζ) space. These 2-branes can either be closed with trivial monodromy for the circle fibre, or open and ending on the fivebrane $M_4 \times \Sigma$ (α fibre) or $M_4 \times \Sigma'$ (β fibre). The topology of the 2-brane can be either a disc (gives S^3 CY 3-cycle), a cylinder ($S^1 \times S^2$ 3-cycle) or a torus (T^3 3-cycle). This is of course very similar to the 2/5 brane picture 'solutions' of field theories via M theory [10, 87]. There are several other similarities, like the conditions for a brane to be supersymmetric and the topology - multiplet type correspondence, but we will not discuss them here. One could wonder whether this connection could hint to a generalisation of the usual derivation of effective field theories without gravity from branes in a trivial background, to effective field theories with some gravitational corrections from branes in a nontrivial background. We will not pursue this question here however, also because it is not clear what the physical basis would be for such gravitational corrections from branes, since branes do not have a dynamical intrinsic metric.

4.4.3 The $SU(3)$ rigid limit

Definition

The $SU(3)$ rigid limit is reached when three sheets of the Riemann surface Σ coincide, or equivalently, when the CY considered as an elliptic fibration acquires an I_3 singularity according to the Kodaira classification (i.e. a curve of A_2 singularities). This occurs when simultaneously $B=0$, $\psi_0=0$ and $\psi_1=\pm 1$. However, as some periods are ill defined in this limit, we have to specify how we approach this point. We set $\epsilon \equiv L^{-1}$ and take

$$B = \epsilon b \quad (4.4.46)$$

$$\psi_1 = 1+\frac{\epsilon}{2}\lambda_1 \quad (4.4.47)$$

$$\psi_0 = \epsilon^{1/6}\lambda_0. \quad (4.4.48)$$

and let $\epsilon \to 0$ while keeping b, λ_1 and λ_0 finite. It will become clear that this prescription is indeed essentially the one given in equation (4.2.3) of the general discussion. Note that a rescaling $(b,\lambda_1,\lambda_0) \to (\mu^6 b, \mu^6\lambda_1, \mu\lambda_0)$ yields the same rigid limit; hence these variables give a projective description of the residual moduli space in the rigid limit.

The above choice of ϵ dependence keeps the branch points in the ζ-plane with vanishing v_a or v_b at finite positions, resp. given by:

$$\frac{1}{2}(\zeta+\frac{1}{\zeta}) = \frac{1}{b}\lambda_1 + O(\epsilon) \quad (4.4.49)$$

$$\frac{1}{2}(\zeta+\frac{1}{\zeta}) = \frac{1}{b}(\lambda_1+2\lambda_0^6) + O(\epsilon) \quad (4.4.50)$$

while the branch points with vanishing t_a and t_b, given by

$$\frac{1}{2}(\zeta+\frac{1}{\zeta}) = \frac{1}{\epsilon b}, \quad (4.4.51)$$

are sent to infinity. This choice gives rise to light BPS states (namely D-3-branes wrapped around the basis cycles V_{v_a}, V_{v_b}, T_{v_a}, T_{v_b}) which can be identified as the massive gauge bosons and dyons of the pure $N=2$ $SU(3)$ Yang-Mills theory. Furthermore we will show that in this limit, local special geometry indeed reduces to $SU(3)$ rigid special geometry on the rigid moduli space parametrized by b, λ_1 and λ_0, justifying the above choice of ϵ-dependence.

Choice of period basis

We start with the basis of CY 3-cycles as defined in section 4.4.2. Close to the rigid limit, we clearly are well inside what we earlier called the $SU(3)$ region of moduli space. Thus the intersection matrix of v_a, v_b, t_a, t_b is given by eq. (4.4.29).

From this, it follows that the full CY intersection matrix is:

$$Q = \begin{pmatrix} 0 & -1 & -2 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.4.52}$$

where the order of the basis cycles is the order in which they were defined. As the periods of v_a and v_b for finite values of ζ become small in the rigid limit (we will show this explicitly below), we expect the first four CY basis cycles to give rise to the light D-brane states corresponding to gauge bosons and dyons. To isolate the contribution of those to the Kahler potential, we have to redefine the last four basis elements such that the intersection matrix becomes block diagonal. To accomplish this, we put

$$V'_{t_a} = V_{t_a} + \frac{1}{3}V_{v_a} + \frac{2}{3}V_{v_b} - \frac{1}{3}T_{v_a} \tag{4.4.53}$$

$$V'_{t_b} = V_{t_b} - \frac{2}{3}V_{v_a} - \frac{1}{3}V_{v_b} - \frac{1}{3}T_{v_b} \tag{4.4.54}$$

$$T'_{t_a} = T_{t_a} + \frac{1}{3}T_{v_a} + \frac{2}{3}T_{v_b} \tag{4.4.55}$$

$$T'_{t_b} = T_{t_b} - \frac{2}{3}T_{v_a} - \frac{1}{3}T_{v_b}. \tag{4.4.56}$$

The new (noninteger) basis has intersection matrix

$$Q' = \begin{pmatrix} 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{7}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{7}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \end{pmatrix}, \tag{4.4.57}$$

To deduce the form of the ϵ-dependence of the periods, it is sufficient to calculate their monodromies under $\epsilon \to e^{2\pi I}\epsilon$. The calculation of these monodromies is a bit tedious, but can be done from the construction of the cycles without knowing any detailed expressions for the periods. The result is (denoting the periods with the corresponding caligraphic letters):

$$\begin{pmatrix} \mathcal{V}_{v_a} \\ \mathcal{V}_{v_b} \\ \mathcal{T}_{v_a} \\ \mathcal{T}_{v_b} \end{pmatrix} \to \omega \begin{pmatrix} \mathcal{V}_{v_a} \\ \mathcal{V}_{v_b} \\ \mathcal{T}_{v_a} \\ \mathcal{T}_{v_b} \end{pmatrix} \tag{4.4.58}$$

where $\omega = e^{2\pi i}$, and

$$\begin{pmatrix} \mathcal{V'}_{t_a} \\ \mathcal{V'}_{t_b} \\ \mathcal{T'}_{t_a} \\ \mathcal{T'}_{t_b} \end{pmatrix} \to \omega \begin{pmatrix} -1 & -1 & 2 & 2 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V'}_{t_a} \\ \mathcal{V'}_{t_b} \\ \mathcal{T'}_{t_a} \\ \mathcal{T'}_{t_b} \end{pmatrix} \tag{4.4.59}$$

It is convenient to go to a basis of periods for which this monodromy matrix is in Jordan form. Therefore we replace again the last four basis elements, by

$$\mathcal{T}_1 = \mathcal{V}'_{t_b} - \omega^2 \mathcal{V}'_{t_a} \tag{4.4.60}$$
$$\mathcal{V}_1 = \mathcal{T}'_{t_b} - \omega^2 \mathcal{T}'_{t_a} \tag{4.4.61}$$
$$\mathcal{T}_{\omega^2} = \mathcal{T}'_{t_b} - \omega \mathcal{T}'_{t_a} \tag{4.4.62}$$
$$\mathcal{V}_{\omega^2} = \mathcal{V}'_{t_b} - \omega \mathcal{V}'_{t_a} \tag{4.4.63}$$

The new basis indeed has the ϵ-monodromy in Jordan form:

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{V}_1 \\ \mathcal{T}_{\omega^2} \\ \mathcal{V}_{\omega^2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & -2\omega^2 & \omega^2 \end{pmatrix} \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{V}_1 \\ \mathcal{T}_{\omega^2} \\ \mathcal{V}_{\omega^2} \end{pmatrix}. \tag{4.4.64}$$

This implies the following behaviour:

$$\mathcal{V}_{v_a}, \mathcal{V}_{v_b}, \mathcal{T}_{v_a}, \mathcal{T}_{v_b} = \epsilon^{1/3}\,\text{analytic} \tag{4.4.65}$$
$$\mathcal{T}_1 = \text{analytic} \tag{4.4.66}$$
$$\mathcal{V}_1 = -2\frac{\ln\epsilon}{2\pi i}\mathcal{T}_1 + \text{analytic} \tag{4.4.67}$$
$$\mathcal{T}_{\omega^2} = \epsilon^{2/3}\,\text{analytic} \tag{4.4.68}$$
$$\mathcal{V}_{\omega^2} = -2\frac{\ln\epsilon}{2\pi i}\mathcal{T}_{\omega^2} + \epsilon^{2/3}\,\text{analytic} \tag{4.4.69}$$

Small periods and the Seiberg-Witten curve

In the region of moduli space under consideration, the paths in the x-plane defining the $K3$ 2-cycles v_a and v_b stretch between sheets of Σ_-, the branch of the $g = 5$

Riemann surface Σ given by the equation $\Sigma_- : P(x) = -\sqrt{B'(\zeta)}$. Expressed in the variables b, λ_1, λ_0 and ϵ, this equation becomes:

$$2x^3 - \frac{3}{2}\epsilon^{2/3}\lambda_0^4 x - \frac{\epsilon}{2}(\lambda_0^6 + \lambda_1) + \sqrt{\frac{\epsilon b}{2}(\zeta + \frac{1}{\zeta}) + 1} - 1 = 0 \qquad (4.4.70)$$

or, rescaling $x = \epsilon^{1/3}\tilde{x}$ and expanding the square root, for finite ζ:

$$\tilde{x}^3 - \frac{3}{4}\lambda_0^4\tilde{x} - \frac{1}{4}(\lambda_0^6 + \lambda_1) + \frac{b}{8}(\zeta + \frac{1}{\zeta}) + O(\epsilon) = 0 \qquad (4.4.71)$$

which is the equation for the genus 2 $SU(3)$ Seiberg-Witten Riemann surface (2.3.17), if we put $b = 4$, $u_0 = -\frac{1}{4}(\lambda_0^6 + \lambda_1)$ and $u_1 = -\frac{3}{4}\lambda_0^4$. Thus we see that in the $SU(3)$ rigid limit, a genus 2 branch of our general genus 5 Riemann surface Σ degenerates and produces the Seiberg-Witten surface, with punctures 'at infinity', where the rest of the genus 5 surface is attached.

For finite ζ we furthermore have, using (4.4.38):

$$\int_{s_{ij}^{\pm}} \Omega^{(2,0)} = \frac{\sqrt{6}}{2\pi}\epsilon^{1/3}(\tilde{x}_{j^-} - \tilde{x}_{i^-}) + O(\epsilon^{4/3}). \qquad (4.4.72)$$

So for a CY cycle $\gamma \wedge s_{ij}^-$ obtained by transporting s_{ij}^- along a path γ in the ζ-plane, we find

$$\int_{\gamma\wedge s_{ij}^-} \Omega^{(3,0)} = \epsilon^{1/3}\frac{\sqrt{3}}{\pi i}\int_{\gamma_{j^-} - \gamma_{i^-}} \lambda_{SW}, \qquad (4.4.73)$$

where as in (2.3.3)

$$\lambda_{SW} = \frac{1}{2\sqrt{2}\pi}\,\tilde{x}\,\frac{\mathrm{d}\zeta}{\zeta} \qquad (4.4.74)$$

and

$$\gamma_{i^-} = \gamma \text{ lifted to sheet } i \text{ of } \Sigma_-. \qquad (4.4.75)$$

Note that $\gamma_{j^-} - \gamma_{i^-}$ is always a closed cycle on the SW Riemann surface (single loop if γ is open, double loop if closed). Thus, since λ_{SW} is also precisely the Seiberg-Witten meromorphic one-form, we find that the first 4 CY periods are nothing but the Seiberg-Witten periods. Note that these periods are indeed 'small', as they are proportional to $\epsilon^{1/3}$. Comparison with (4.1.21) furthermore yields $\nu = \epsilon^{-1/3}\nu' = -\frac{\sqrt{3}}{2\sqrt{2}\pi^3}$.

Finally, using (4.4.28) and the comment below it, it is not difficult to show that the intersection matrix of the 4 small Calabi-Yau basis cycles is precisely equal to minus the intersection matrix of the corresponding four SW cycles. This is crucial to show the reduction of the local Kahler potential to the rigid Kahler potential (see below).

It is clear that higher order ϵ-corrections (gravity and other stringy effects) to e.g. the SW BPS mass formula can in principle be calculated systematically in this setup. There are two sources of corrections: the surface itself is corrected as well as the meromorphic one-form.

Large periods

From (4.4.46)-(4.4.48), we see that the regular part of the CY periods should become independent of the rigid moduli b, λ_1, λ_0 in the limit $\epsilon \to 0$, since on one hand the regular part of those periods should be continuous in B, ψ_1, ψ_0 (at their limiting values $0, 1, 0$) while on the other hand B, ψ_1 and ψ_0 themselves become independent of the rigid moduli when $\epsilon \to 0$. This, together with (4.4.66)-(4.4.67) implies in particular:

$$\mathcal{T}_1 = k_1 + O(\epsilon) \tag{4.4.76}$$

$$\mathcal{V}_1 = -2\frac{\ln\epsilon}{2\pi i}(k_1 + O(\epsilon)) + k_2 + O(\epsilon) \tag{4.4.77}$$

with k_1, k_2 constants independent of the rigid moduli (and $k_1 \neq 0$). Alternatively, one could follow the argument outlined in the general discussion to establish the independence of the rigid moduli of these constants.

Though not really necessary for the conclusions, it is possible to calculate k_1 from the exact solutions of the $K3$ periods given earlier. The result is $k_1 = \sqrt{3}i\omega^2\frac{B_1}{B_2}$, with B_1 and B_2 as in (4.4.41).

Kähler potential and reduction to rigid SG

The Kähler potential is expressed in terms of the periods as follows (in the second line the small/large periods and corresponding intersection matrices are denoted as in the general treatment):

$$\begin{aligned}
\mathcal{K} &= -\ln\left(i\int_{CY}\Omega\wedge\bar{\Omega}\right) \\
&= -\ln\left(iQ^{\Sigma\Lambda}\int_{\Gamma_\Sigma}\Omega\int_{\Gamma_\Lambda}\bar{\Omega} + iq^{\sigma\lambda}\int_{\gamma_\sigma}\Omega\int_{\gamma_\lambda}\bar{\Omega}\right) \\
&= -\ln\left(2\,\mathrm{Im}\,(\mathcal{V}_1\bar{\mathcal{T}}_1 + \mathcal{V}_{\omega^2}\bar{\mathcal{T}}_{\omega^2}) + \frac{14}{\sqrt{3}}(|\mathcal{T}_{\omega^2}|^2 - |\mathcal{T}_1|^2) + |\epsilon|^{2/3}K\right) \\
&= -\ln\left(\frac{\ln|\epsilon|^{-2}}{\pi}|k_1|^2 + 2\,\mathrm{Im}\,(\bar{k_1}k_2) - \frac{14}{\sqrt{3}}|k_1|^2 + |\epsilon|^{2/3}K + O(\epsilon^{4/3})\right).
\end{aligned}$$

This is precisely of the form (4.2.13), with $L = \epsilon^{-1}$, $a = \frac{|k_1|^2}{\pi}$, $b = 2\,\mathrm{Im}\,(\bar{k}_1 k_2) - \frac{14}{\sqrt{3}}|k_1|^2$ and K as in (4.2.15) with $|\nu|^2 = \frac{3}{8\pi^6}$. We can therefore proceed exactly as in the general discussion (which we will not repeat here). Note in particular that we find for the $SU(3)$ scale

$$\Lambda = \sqrt{\frac{3}{\pi}}\,\frac{1}{\kappa_4}\,\frac{L^{-1/N}}{(a \ln |L|^2 + b)^{1/2}}. \tag{4.4.78}$$

Moreover, since we have the exact value of k_1, we can also compute

$$\sqrt{a} = \sqrt{\frac{3}{\pi}}\left|\frac{B_1}{B_2}\right| = 4\sqrt{\pi}\frac{\Gamma^2(\frac{2}{3})}{\Gamma^4(\frac{5}{6})} \approx 8.0075, \tag{4.4.79}$$

and hence the constant $k = 2\pi M_{SU}/M_S$ determining the ratio of string and string gauge unification scales, defined in (4.3.4), is

$$k = \left|\frac{B_2}{B_1}\right| = \frac{\sqrt{3}}{4\pi}\frac{\Gamma^4(\frac{5}{6})}{\Gamma^2(\frac{2}{3})} \approx 0.122035. \tag{4.4.80}$$

Chapter 5

Attractors at weak gravity

Having studied the brane representation of BPS states in weak gravity quantum Yang-Mills theory, one could wonder how these states look from the effective field theory point of view. Are they also attractors, as the $\mathcal{N} = 2$ black holes described in chapter 3? From the general discussion there, one would expect they are. But are they black holes then? Have they naked curvature singularities? Or are they smooth solutions with only weak backreaction on the spacetime metric, like classical nonabelian Yang-Mills monopoles? Physically, one would expect the latter. Indeed, as long as the fields are sufficiently slowly varying — which, recalling the general discussion in chapter 3, can always be arranged by taking the charge N of the state sufficiently large — the effects of the massive nonabelian degrees of freedom are taken into account to an arbitrarily good approximation in the quantum (abelian) effective action. Furthermore the nonabelian gauge bosons are always massive in an $\mathcal{N} = 2$ theory, and the monopoles/dyons decouple from the massless fields in the infrared, so we don't expect the effective action description to break down in a fatal way anywhere if the charge is sufficiently large. Thus we expect at least curvature singularity free solutions of the effective action corresponding to semi-classical monopoles. Only then we can really say gravity can be consistently decoupled from the effective theory, something which we expect to be the case for an asymptotically free theory.

These arguments are of course rather heuristic, but we will see that, in a quite subtle way, our expectations are indeed confirmed. By careful analysis of the attractor flow equations in the weak gravity limit, we will be able to show that we obtain indeed monopole solutions free of physical singularities, with finite mass entirely residing in the fields, thereby solving some paradoxes and puzzles which were encountered in earlier studies of this subject [64, 23, 89]. Not unexpectedly, these solutions are very similar to the conifold states we discussed in section 3.3.3.

Note that the above physical arguments do *not* hold for $\mathcal{N}=4$ Yang-Mills: this theory *has* points in moduli space where the gauge bosons become massless, and is *not* asymptotically free. The two derivative low energy abelian effective action for an $\mathcal{N}=4$ Yang-Mills theory is indeed classical, and it is well known that classical abelian gauge theories do not have singularity-free monopole solutions. The discrepancy in this respect with the full nonabelian theory can be understood from the fact that at the attractor point of the flows in moduli space , the gauge bosons are massless and the coupling constant finite, so that they can definitely not be neglected near the monopole core.

The outline of this chapter is as follows. We first derive the form of the effective reduced action (3.3.5) for spherical solutions in the weak gravity limit, which in principle should yield all static spherically symmetric solutions, BPS or not. To sharpen our intuition, we give again an interpretation in terms of a particle moving in a certain potential. We then restrict to BPS solutions, in other words, the attractor flow equations in the weak gravity limit. We discuss properties of the solutions we find, and comment and speculate on possible extensions of our results.

An extensive review on *classical* dyons coupled to gravity can be found in [90], while aspects of quantum dyons without gravity are studied e.g. in [91, 89, 92] and in particular by Chalmers, Rocek and von Unge in [93]. Some related work can be found in [63].

5.1 Attractor flow equations in the weak gravity limit

In the following, we will make the same assumptions as in section 4.3.1. In particular, we will assume the ratio of gauge and Planck scales $\Lambda\kappa_4$ to be very small. We will also frequently make use of the notations and formulas introduced there and elsewhere in chapter 4.

5.1.1 Spherically symmetric configurations at weak gravity

The weak gravity metric on moduli space, derived from (4.3.14) is (as can also be read off ffrom (4.3.15)):

$$g_{\Lambda\bar{\Lambda}} \approx \frac{1}{(\ln|M_{SU}/\Lambda|^2)^2}\frac{1}{|\Lambda|^2}+\frac{\kappa_4^2}{4\pi}K_{sw} \tag{5.1.1}$$

$$g_{i\bar{j}} \approx \frac{\kappa_4^2}{4\pi}\, g_{sw,i\bar{j}} \tag{5.1.2}$$

$$g_{i\bar{\Lambda}} \approx \frac{\kappa_4^2}{4\pi}\frac{1}{\bar{\Lambda}}\,\partial_i K_{sw}. \tag{5.1.3}$$

where $g_{sw,i\bar{j}} \equiv \partial_i\bar{\partial}_{\bar{j}} K_{sw}$ is the Seiberg-Witten metric on rigid moduli space and the string unification scale M_S, as introduced in section 4.3, is given by $M_{SU}^2 = \frac{1}{\kappa_4^2}\frac{|k|^2}{\mathrm{Re}\,S} \approx \frac{1}{\kappa_4^2}\frac{|k|^2}{|b|\ln(|\Lambda|^{-1}\kappa_4^{-1})}$, with b the beta function coefficient of the gauge theory under consideration and k a model dependent constant. In the approximation we are making, the inverse metric is:

$$g^{\Lambda\bar{\Lambda}} \approx (\ln|M_{SU}/\Lambda|^2)^2\,|\Lambda|^2 \tag{5.1.4}$$

$$g^{i\bar{j}} \approx \frac{4\pi}{\kappa_4^2}\, g_{sw}^{i\bar{j}} \tag{5.1.5}$$

$$g^{i\bar{\Lambda}} \approx -\bar{\Lambda}(\ln|M_{SU}/\Lambda|^2)^2\, g_{sw}^{i\bar{j}}\bar{\partial}_j K_{sw}. \tag{5.1.6}$$

We consider one of the 'light' 3-cycles γ. The relation between the supergravity central charge $Z(\gamma)$ and the Seiberg-Witten central charge $Z_{sw}(\gamma)$ is given by (4.2.18):

$$Z = \sqrt{2G_N}Z_{sw} = \frac{\kappa_4}{2\sqrt{\pi}}Z_{sw}. \tag{5.1.7}$$

With this information, we can calculate for a given charge the potential V as defined in (3.3.6):

$$V \approx \|\nabla Z_{sw}\|_{sw}^2 + 2G_N(\ln|M_{SU}/\Lambda|^2)^2\,\chi\,|Z_{sw}|^2 + 2G_N|Z_{sw}|^2, \tag{5.1.8}$$

where

$$\chi \equiv 1 - \langle\nabla\ln|Z_{sw}|^2, \nabla K_{sw}\rangle_{sw} \tag{5.1.9}$$

and we have denoted the scalar product (norm) on the Seiberg-Witten moduli space by $\langle\cdot,\cdot\rangle_{sw}$ ($\|\cdot\|_{sw}$), that is: $\langle\nabla f, \nabla h\rangle_{sw} = \mathrm{Re}\,[g_{sw}^{i\bar{j}}\partial_i f\bar{\partial}_{\bar{j}}\bar{h}]$ and $\|\nabla f\|_{sw}^2 = \langle\nabla f, \nabla f\rangle_{sw}$. It is straightforward to check that $\chi = 0$ when the moduli space geometry is 'classical', that is when K_{sw} is quadratic and the metric flat. From the microscopic quantum field theory (or heterotic) point of view, the second term in the right hand side of 5.1.8 can therefore be interpreted as a 'gravitational (i.e. $\sim G_N$) backreaction' correction to the potential[1] which is caused by loop and nonperturbative quantum field theory effects. χ also vanishes in the ultraviolet free (vanishing electric coupling) limits of moduli space ($\phi \to \infty$ in $SU(2)$ case), since the Kähler potential becomes effectively quadratic there. However, it does not vanish (it even diverges) for the infrared free (vanishing dual magnetic coupling)

[1] Recall the physical meaning of the potential V is the energy density of the electromagnetic fields (including the graviphoton).

Figure 5.1: $-\chi|Z_{sw}|^2$ plotted as a function of u in a neighborhood of $u=1$ for a purely magnetic charge.

limits ($\phi_D \to 0$ in the $SU(2)$ case) as the Kähler potential does not become effectively quadratic there (see for example (2.3.13)). On the other hand, the full second term in V diverges in the ultraviolet free limits, while it is zero in the infrared free limits if the charge under consideration is that of the particle with vanishing mass in this limit (in fact, the complete potential then vanishes, as it should of course for the energy density of the fields of a zero momentum massless particle). In general, χ can be positive or negative, depending on the relative angle of the gradients of K and $\ln|Z|^2$, but at weak electric coupling, it is always positive. In fig. 5.1, $-\chi|Z_{sw}|^2$ is plotted as a function of u in a neighborhood of $u=1$ for a purely magnetic charge. Note also that the factor $2G_N(\ln|M_{SU}/\Lambda|^2)^2$ in front of χ is equal to $\frac{8\pi^2}{G_N M_S^4 b^2}$ with M_S the string scale and b the beta-function coefficient of the gauge theory. Hence in a large N (i.e. large b) limit with fixed string and Planck scale, this term will be suppressed as well.

We will come back to this term and give a geometrical interpretation below.

Finally, we plug the expressions for the moduli space metric in the reduced effective action (3.3.5) (with $h=1$) for general static spherically symmetric solutions, and obtain:

$$S_{4D}/T \qquad = -\frac{1}{2G_N}\int_0^\infty \mathrm{d}\tau \, \{\dot{U}^2 - c^2 + \frac{|\dot{\Lambda}/\Lambda|^2}{(\ln|M_{SU}/\Lambda|^2)^2}\}$$

$$-\int_0^\infty \mathrm{d}\tau \, \{g_{sw,i\bar{j}} \dot{u}^i \dot{\bar{u}}^{\bar{j}} + (\partial_i K_{sw} \dot{u}^i \, \frac{\dot{\bar{\Lambda}}}{\bar{\Lambda}} + c.c.) + \frac{1}{2} e^{2U} V\} \quad (5.1.10)$$

with V as in (5.1.8).

From this action, by completion of squares, or simply from (3.3.10) and (3.3.11), by substitution of (5.1.4)-(5.1.6), one obtains the weak gravity approximation to the attractor flow equations:

$$\dot{U} = -\sqrt{2} G_N \, e^U \, |Z_{sw}| \quad (5.1.11)$$

$$\dot{\Lambda}/\Lambda = -\sqrt{2} G_N \, e^U \, (\ln |M_{SU}/\Lambda|^2)^2 \, f(u, \bar{u}) \quad (5.1.12)$$

$$\dot{u}^i = -\sqrt{2} \, g_{sw}^{i\bar{j}} \, e^U \, \bar{\partial}_{\bar{j}} |Z_{sw}|, \quad (5.1.13)$$

where

$$f(u, \bar{u}) \equiv \left(1 - g_{sw}^{i\bar{j}} \partial_i K_{sw} \bar{\partial}_{\bar{j}} \ln |Z_{sw}|^2\right) \, |Z_{sw}| \quad (5.1.14)$$

As expected, the variation of the scale Λ and the metric red shift factor e^U vanishes when $G_N \to 0$. The factor f appearing in the r.h.s. of (5.1.12) is closely related to χ in (5.1.9): $\chi |Z_{sw}| = \mathrm{Re}\, f$. Analogous to the analysis of χ earlier, it can be seen that f vanishes in any 'classical' limit of moduli space, where the coupling of the charge Γ to the electromagnetic field vanishes. Decomposing λ_{SW} cohomologically as $\lambda_{SW} = \lambda_{SW}^{(1,0)} \oplus \lambda_{SW}^{(0,1)} = c^i \partial_i \lambda \oplus \tilde{c}^i \overline{\partial_i \lambda}$, some elementary manipulations give the following geometrical interpretation to f:

$$f = e^{i\alpha} \overline{\int_\gamma \lambda^{(0,1)}}. \quad (5.1.15)$$

Here α is the phase of Z. This expression makes it obvious that f vanishes in the classical case, since λ_{SW} itself is a holomorphic (1,0)-form then. From the microscopic nonabelian field theory point of view, the space variation of the gauge theory scale Λ is therefore entirely a quantum effect, in the presence of gravity (that is, string theory).

Massless and Schwarzschild charged black holes

The weak gravity solution can be used to study Strominger's massless 'black holes'[2] [68]. If we take Z to be already zero at spatial infinity, the attractor flow equations gives simply flat space with constant moduli (but still the — energyless — electromagnetic field of a point charge). We shouldn't expect more of course for a massless particle at rest! However, we can let the rest mass approach zero and

[2]This name is purely historical. They are just particles, not black holes.

simultaneously boost the solution along the x-axis to approach the speed of light, keeping the energy $E = \sqrt{2}|Z(\tau = 0)|/\sqrt{1 - v^2}$ fixed. When $\gamma \equiv 1/\sqrt{1 - v^2} \to \infty$, the boosted metric is given by

$$ds^2 = dt^2 - dx^2 + 4\gamma^2 U (dt - dx)^2 - dy^2 - dz^2, \qquad (5.1.16)$$

and, for nonzero $x - t$, $\tau \to \frac{1}{\gamma|x-t|} \equiv \frac{\sigma}{\gamma}$. Denoting $U' \equiv \gamma^2 U$ and $Y \equiv \sqrt{2}\gamma Z$, we have $|Y(\tau = 0)| = E$, and from the weak gravity attractor flow equations (5.1.11)-(5.1.13):

$$\begin{aligned} \Lambda &= \text{const.} & (5.1.17) \\ \frac{dU'}{d\sigma} &= -G_N |Y| & (5.1.18) \\ \frac{d|Y|}{d\sigma} &= -\frac{k}{\ln(\gamma|Y|^{-1})} \to 0, & (5.1.19) \end{aligned}$$

with k a positive constant. This implies $|Y| = \text{const.} = E$ and $U' = -G_N E \sigma$.

So we find a simple but nontrivial solution, with $Z = 0$ everywhere, the electromagnetic field strength of a point particle boosted to the speed of light, and a "shockwave metric" (due to the combined effect of expansion of the core region in the rest frame and longitudinal Lorentz contraction while taking the limit):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{4G_N E}{|x - t|}(dx - dt)^2. \qquad (5.1.20)$$

This is the Aichelburg-Sexl metric for a massless particle [94]. It is locally flat and can be brought to standard form by changing coordinates as $t' = t \mp 2GE \ln|t - x|$, $x' = x \pm 2GE \ln|t - x|$, where the upper (lower) sign is to be used for $t - x > 0$ ($t - x < 0$).

At conifold points of moduli space ($Z(\gamma) = 0$), the low energy effective field theory also allows γ-charged Schwarzschild black holes with arbitrary mass, with moduli fixed at the conifold, as can be seen immediately from the action (3.3.8). Their extremal limit are the Strominger massless BPS states. This freedom corresponds to the freedom to add a bare mass to the massless monopoles at the $u = 1$ point in Seiberg-Witten moduli space, thus breaking (in field theory) explicitly supersymmetry to $\mathcal{N} = 1$. From the point of view of the effective particle motion, this corresponds to a trajectory on the top of the $-V$ potential, with arbitrary initial velocity in the U direction, as was already indicated in fig. 5.2.

5.1.2 Rigid limit

If we send G_N all the way to zero (i.e. if we take the rigid limit), the dynamics of the gravitational field and the scale decouple and we can consistently put $U = 0$, $\Lambda = \text{const.}$. Then (5.1.10) becomes simply

$$S_{4D}/T = -\int_0^\infty \mathrm{d}\tau \, \{g_{sw,i\bar{j}}\dot{u}^i\dot{\bar{u}}^{\bar{j}} + \frac{1}{2}\|\nabla Z_{sw}\|^2_{sw}\}, \tag{5.1.21}$$

which could also have been obtained directly from the rigid effective action of course. By completing squares, one finds immediately the attractor flow equations in the rigid limit:

$$-S_{4D}/T = \pm\sqrt{2}(|Z_{sw}(0)| - |Z_{sw}(\infty)|) + \int_0^\infty \mathrm{d}\tau \, \|\dot{u}^i \pm \sqrt{2}g^{i\bar{j}}_{sw}\bar{\partial}_{\bar{j}}|Z_{sw}|\|^2, \tag{5.1.22}$$

which is minimized when

$$\dot{u}^i = -\sqrt{2}g^{i\bar{j}}_{sw}\bar{\partial}_{\bar{j}}|Z_{sw}|. \tag{5.1.23}$$

The other sign again leads to an unphysical infinite energy solution. A very useful property of the rigid attractor flows is the fact that the phase of the central charge is constant along the flow, as can be verified by a short calculation. Essentially, this is thanks to the holomorphicity of Z in the rigid limit.

The rigid limit of the formulas of section 3.3.5 is also very useful. In particular, equation (3.3.58) becomes

$$\mathrm{Im}\,(e^{-i\alpha}\sqrt{2}Z'_{sw}) = \frac{1}{2}\gamma\cdot\gamma'\,\tau + [\mathrm{Im}\,(e^{-i\alpha}\sqrt{2}Z')]_{\tau=0}, \tag{5.1.24}$$

with $e^{i\alpha}$ the (constant) phase of Z_{sw}.

The rigid limit of (3.1.30)-(3.1.31) is of practical interest as well:

$$\gamma_1\cdot\gamma_2 = 2\,\mathrm{Im}\,[g^{i\bar{j}}_{sw}\,\partial_i Z_{sw,1}\,\bar{\partial}_{\bar{j}}\bar{Z}_{sw,2}] \tag{5.1.25}$$

$$\gamma_1\cdot *\gamma_2 = 2\,\mathrm{Re}\,[g^{i\bar{j}}_{sw}\,\partial_i Z_{sw,1}\,\bar{\partial}_{\bar{j}}\bar{Z}_{sw,2}] \tag{5.1.26}$$

5.1.3 Solutions in the rigid limit

Since the $\mathcal{O}(G_N)$ corrections are small, it is sufficient in first approximation to investigate the exactly rigid case. We will do so by reconsidering the analysis in section 3.3.2. Again the system is equivalent to a particle moving in a certain potential, $-V$ in the exactly rigid case, $-e^{2U}V$ if we do consider the gravity perturbation. Crucial again are thus the critical points of V. In the following, we will drop the subscript 'SW' to indicate rigid quantities, since we will restrict to the rigid case anyway for now.

Critical points

Evidently, critical points of Z are automatically minima of $V = \|\nabla Z\|^2$, with critical value $V_{cr} = 0$. Also, since Z is analytic, and since we already know from section 3.3.2 that $|Z|$ can only have minima, Z must be zero at its critical points. Alternatively, one can see this as follows. From (5.1.26) it follows that $\int \gamma \wedge * \gamma = 0$ at a critical point of Z, where γ is the cycle under consideration. Since the Hodge product is positive definite, this implies that γ is a vanishing cycle, which has $Z = 0$ (at finite points in moduli space).

In the pure $SU(2)$ Yang-Mills case (completely decoupled from gravity), the converse is also true, in the sense that minima of V must be minima of $|Z|$. Indeed, for a central charge $Z = n\phi + m\phi_D$, we have $V = \frac{|n+m\tau|^2}{\operatorname{Im}\tau} = \frac{m^2}{\operatorname{Im}\tilde{\tau}}$ with $\tau = \partial_\phi \phi_D$ and $\tilde{\tau} = -\frac{m}{n+m\tau}$.[3] Minima of V are therefore maxima of $\operatorname{Im}\tilde{\tau}$. Since $\tilde{\tau}$ is analytic, this maximum has to be a pole, so $0 = V = \|\nabla Z\|^2$ at this point, that is, it is a critical point of Z as well (and by the previous argument also a zero of Z). It is not clear to us if and how this generalizes to higher rank gauge groups.

Anyway, let us focus now on critical points of V which are also critical points of Z. Then, since $V_{cr} = 0$, the potential $-e^{2U}V$ is flat in the U direction at the critical point, as shown in fig. 5.2.

Rigid BPS solutions

As argued above, the cycle γ must vanish at the critical point. Therefore, we expect the situation close to a critical point to be very similar to our discussion of the conifold attractor in section 3.3.3. Take the example of $SU(2)$ Yang-Mills. The candidates for spherically symmetric BPS solutions are the charges which can have vanishing mass: here the monopole and the elementary dyon[4]. Note that these are the *only* candidates: there will be no spherically symmetric BPS solution for example for a purely electrical charge (a W-boson). This is a good thing, since the W-bosons are in a BPS vector multiplet, and by the arguments of section 3.3, we do not expect a rotationally invariant bosonic solution for $\mathcal{N} = 2$ BPS vector multiplets. We will come back to this later. A quick glance at (2.3.13) and (3.3.36) learns that close to $u = \pm 1$, the geometry is indeed that of a conifold point. The exact potential $-V$ for γ a magnetic charge in a neighborhood of $u = 1$ is given in fig. 5.3. Note that we have indeed a maximum for $-V$. Though the top is a sharp spike, the details of the equations are such that we can find a continuously

[3] We assume $m \neq 0$. The case $m = 0$ is completely analogous.

[4] In the usual cut conventions (i.e. all cuts as much as possible to the left in the u plane), the elementary dyon has charge $\pm(1, \pm 1)$, the relative sign between electric and magnetic charges depending on how the singularity is approached (+ if from $\operatorname{Im} u > 0$).

Figure 5.2: The effective particle potential $-e^{2U}V$ near a critical point in the weak gravity limit. The potential is flat in the U direction (equal to zero) at the critical point of V. Trajectory a is a typical weakly gravitating BPS solution. Trajectory b is a charged Schwarschild black hole with vanishing electromagnetic energy in a $Z = 0$ vacuum. Trajectory c correspond to a generic non-BPS black hole. V itself is drawn as a smooth potential for the sake of the picture, but actually, it will have a sharp spike-like top in general.

Figure 5.3: Potential $-V$ for the effective particle in a neighborhood of $u = 1$ in the Seiberg-Witten plane.

Figure 5.4: Value of $|u-1|$ as a function of the radial distance for N units of magnetic charge, with $u(r=\infty)=2+i$. The attractor point is reached at $r=r_*=1.56N\Lambda$. Though the radial derivative of the solution does not look continuous on this picture, it actually *is*, but the drop to zero slope happens over a very small distance.

differentiable BPS (attractor flow) solution for the effective particle motion, ending at the top, as we have seen for the conifold case in section 3.3.3. For flows not close to the critical point, numerical analysis is needed. As an example, the numerically integrated flow for N units of magnetic charge, with $u(r=\infty)=2+i$ is shown in fig. 5.4. The mass of such a monopole is $0.942N\Lambda$ (the mass of the gauge boson at this point in moduli space is 2.947Λ). The attractor point is reached at the finite radius $r=r_*=1.56N/\Lambda$. This core radius increases when $u(r=\infty)$ moves towards the attractor point. Inside the region $r<r_*$, the modulus u is constant, $u=1$. Only the electromagnetic field $\mathbb{F}$ is nonvanishing here, since it is as always given by (3.3.3) (here with $e^U=h=1$). However, this electromagnetic field *contains no energy*, as can readily be seen by recalling the fact that $\frac{1}{8\pi r^4}V$ is nothing but the electromagnetic field energy density ϵ. A plot of the e.m.f. energy density as a function of radial distance for our example is given in figs. 5.5 and 5.6. Note that this is half of the total energy density for BPS solutions, as implied e.g. by (3.3.7) (or directly from the attractor equations).

An exact formula for the core radius of a charge N magnetic monopole can be obtained from (5.1.24) together with $\alpha\cdot\beta=2$ and the fact that $Z'(\tau_*)=\frac{4}{\pi}$ for

Figure 5.5: Electromagnetic energy density ϵ as a function of the radial coordinate for the flow of fig. 5.4. This is half the total energy density. The density drops to zero inside the core $r < r_*$. The dotted line is the energy density the electromagnetic field would have if the moduli were kept constant. This diverges of course at $r = 0$.

Figure 5.6: Energy distribution over space, as in fig. 5.5. Most of the energy is localized in a shell surrounding the core.

$\gamma' = \alpha$ (as can be read of from (2.3.11)):

$$r_* = \frac{N}{\Lambda} \frac{|\phi_D(0)|}{\sqrt{2}\mathrm{Im}\,[\bar{\phi_D}(0)(\phi(0) - \frac{4}{\pi})]}. \tag{5.1.27}$$

When u is large (i.e. weak electric coupling), this reduces approximately to

$$r_* \approx \frac{N}{\Lambda} \frac{1}{2\sqrt{|u|}} = \frac{N}{\sqrt{2}|\phi|}, \tag{5.1.28}$$

which is exactly equal to the core radius of a classical nonabelian 't Hooft-Polyakov monopole.

Force fields and potentials

Apart from the energy, another measurable property of an electromagnetic field is the force it exerts on a test particle. For a test particle of unit magnetic charge in the field of a magnetic monopole, this is given by (3.2.25) as $F = NV(u(r))/r^2$, plotted for our numerical example in fig. 5.7. The force drops to zero inside the core, in agreement with the fact that the 'dual' electromagnetic coupling constant is zero at $u = 1$ (in fact, it is not difficult to see that V is actually proportional to the dual coupling).

More generally, also for more general groups, from (3.2.21) and (3.2.22), it follows that the electromagnetic force, on a test particle with charge γ_2, in the field of a particle at rest with charge γ_1, is given by (3.2.24):

$$\vec{F}_{e.m.} = \frac{1}{2}\gamma_1 \cdot *\gamma_2 \frac{\vec{e}_r}{r^2} + \frac{1}{2}\gamma_1 \cdot \gamma_2 \frac{\vec{e}_r}{r^2} \times \vec{v}, \tag{5.1.29}$$

where $\vec{v}$ is the velocity of the test particle. Using (5.1.26), this becomes:

$$\vec{F}_{e.m.} = \langle \nabla Z_1, \nabla Z_2 \rangle \frac{\vec{e}_r}{r^2} + \frac{1}{2}\gamma_1 \cdot \gamma_2 \frac{\vec{e}_r}{r^2} \times \vec{v} \tag{5.1.30}$$

If the fields satisfy the attractor flow equations for the charge γ_1 of the particle at rest, a short calculation using (5.1.23) shows that the force can be nicely derived from a potential:

$$\vec{F}_{e.m.} = -\vec{\nabla}\left(-\sqrt{2}\mathrm{Re}\,(e^{-i\alpha_1} Z_2)\right) - \frac{\gamma_1 \cdot \gamma_2}{2}\vec{\nabla}\left(\frac{1}{r}\right) \times \vec{v}, \tag{5.1.31}$$

where α_1 is the (constant) phase of Z_1. For colinear charges, i.e. $\gamma_1 \sim \gamma_2$, the potential is equal to $\mp\sqrt{2}|Z_2|$, the sign depending on the relative sign of the charges

Figure 5.7: Static electromagnetic force F as felt by a test particle of unit magnetic charge, in function of the radial distance from the monopole of fig. 5.4. This is half the total force felt by a static test particle with unit magnetic charge opposite to the charge of the monopole at $r = 0$. The force drops to zero inside the core, in agreement with the fact that the 'dual' electromagnetic coupling constant is zero at $u = 1$. The dotted line shows the force if the moduli would be kept constant.

(minus for equal signs). The attractor flow equations imply this potential to be strictly monotonous till the core is reached, and approximately Coulomb at large r.

On the other hand, the force on the test particle (γ_2) due to scalar exchange is easily seen from (3.2.17) (giving in the rigid limit $S_{DBI} = \int M ds = \int \sqrt{2}|Z|ds$) to be

$$\vec{F}_{sc} = -\vec{\nabla}[\sqrt{2}|Z_2|]. \tag{5.1.32}$$

Therefore the total static potential W for the test particle γ_2 in the BPS field of γ_1 is

$$W = \sqrt{2}[|Z_2| - \mathrm{Re}\,(e^{-i\alpha_1} Z_2)]. \tag{5.1.33}$$

Denoting the (variable) phase of Z_2 with α_2, we can rewrite this as:

$$W = 2\sqrt{2}\sin^2(\frac{\alpha_2 - \alpha_1}{2})\,|Z_2|. \tag{5.1.34}$$

Note that W is always positive, and that it will reach a minimum (zero) when $\alpha_1 = \alpha_2$ (it is of course flat zero when the charges are positively colinear). This result is interesting: it shows that *when the attractor flow of γ_1 passes through a point where the phases of Z_1 and Z_2 are equal, there is a stable point for the*

Figure 5.8: The static force potential W for a positive electric test particle in the field of a positive charge N magnetic monopole, with modulus at infinity equal to $-1.2i$. The minimum of W is located at the radius where the flow intersects the curve of marginal stability.

test particle γ_2 *in the field of* γ_1. Note that in a vacuum with equal phases of Z_1 and Z_2, the heaviest of the two particles is only marginally stable w.r.t. decay to the other (plus a particle with complementary charge of course). For the $SU(2)$ case, this means that such minimum, if it exists, is always located at the radius where the flow intersects the curve of marginal stability. A numerically integrated example is given in fig. 5.8. The appearance of stable points suggest the possibility of bound states. We will come back to this later.

Returning to the $SU(2)$ example, we get for the static electromagnetic force coefficients linear combinations of the following cycle products: $\beta\cdot*\beta = -1/\mathrm{Im}\,\tau^{-1}$, $\beta\cdot*\alpha = \mathrm{Re}\,\tau/\mathrm{Im}\,\tau$, $\alpha\cdot*\alpha = 1/\mathrm{Im}\,\tau$. From e.g. the approximate expressions (2.3.11) and (2.3.14), it is straightforward to check that also the e.m. force on a *static* electric test particle $\sim \alpha\cdot*\beta$ [5] will drop to zero at a monopole core. However, the magnetic force on a moving electric particle will stay finite, since always $\alpha\cdot\beta = 2$. The e.m. force between two Coulomb electric test particles on the other hand is proportional to $\alpha\cdot*\alpha$, which diverges when approaching the core region — a signal of electric charge confinement at $u = 1$.

[5] Note that this is due to a nonzero theta-angle $\mathrm{Re}\,\tau$.

Figure 5.9: The potential V for an electric charge (α cycle). V is multi-valued when one allows encircling singularities of moduli space (such that α transforms to another cycle, $\alpha + 2n\beta$ in this case). There are no extrema at finite u.

Electric BPS states and suicidal flows

Having found these nice solutions for magnetic monopoles and elementary dyons, one of course wonders what happens for the other BPS states which exist at weak coupling: the $(n, 1)$ dyons, and the pure electric charges. Let us consider for example the electric particle (corresponding to the α cycle). The potential V does not have a minimum at finite u for this charge. This follows from the argument that the cycle under consideration *must* vanish in a critical point of V, plus the fact that neither the α cycle, nor any of the cycles generated by monodromy from it, vanish in a point of moduli space. For example in a neighborhood of $u = 1$, the potential V for an electric charge is shown in fig. 5.9, which also elucidates probably how it can be that V has no extrema at finite points of (the covering space of) moduli space for this charge.

There is a minimum of V at $u = \infty$ (in the rigid approximation; it can indeed be checked from (2.3.9) that $\lim_{u\to\infty} V = 0$), but since (5.1.22) implies that the energy of a solution is always larger than $\sqrt{2}|Z(\infty) - Z(0)|$, the energy of a solution flowing to infinity is infinite, so certainly not acceptable as a BPS state. This does not exclude solutions flowing to large values of u when gravity is turned on again, only that these solutions then will involve gravity in an essential way. It can indeed be seen from (5.1.8) that the minimum at infinity will get displaced to finite moduli values when the Newton constant G_N is turned on. The corresponding solution will be a black hole, and will not be BPS. It cannot be identified with the field theory vector multiplet.

If we try to solve the attractor flow equations for $\gamma = \alpha$, we find a 'false' or 'suicidal' flow, as in case 2 in section 3.3.3, terminating on a regular point of the line of marginal stability where $Z = 0$. We will discuss this phenomenon in more detail for the $(n, 1)$ dyons.

All in all we conclude for $SU(2)$ that *there are no electrically charged, spherically symmetric BPS solutions at weak gravity.* We should be happy with this of course, since the electric BPS states are in a vector multiplet, and as explained in section 3.3, we do not expect a spherically symmetric solution without fermionic excitations in this case.

We can understand the unevitability of this situation purely from asymptotic freedom and the unstability of the electrically charged particles at strong coupling. Asymptotic freedom means the coupling grows when the mass of the electric particle decreases. On the other hand, attractor flows tend to minimal mass, hence maximal coupling in this case. Given the fact that the particle does not exist at strong coupling, it is therefore impossible to have a legitimate attractor solution. Turning this argument around, we conclude that the field theory describing BPS hypermultiplets at weak gravity *must be infrared free.* This is also what one expects from the general $\mathcal{N} = 2$ beta function results.

Of course, there *does* exist a purely electrical BPS multiplet in the $\mathcal{N} = 2$ $SU(2)$ theory. So how is it realized in the effective field theory? We believe it should be considered as a bound state of an elementary dyon and a monopole with opposite magnetic charges. Evidence for this proposal is the charge matching, the stability of the constituents at all moduli values and the appearance of a potential as in fig. 5.8 for the force on a test dyon in a monopole field and vice versa. This suggests one should look for a 2-centered BPS solution with charges $(1, 1)$ resp. $(0, -1)$ in the centers. We have not been able to find such a solution however. Other possibilities worth looking at are solutions with fermions turned on (of which some should give a spin zero solution according to the $\mathcal{N} = 2$ multiplet logic) and solution with charge distributed over a surface in stead of concentrated in points. The connection with 3-pronged strings (see below) and the '3/7 brane picture' of $\mathcal{N} = 2$ theories could provide useful hints towards the solution of this problem.

Below, for the higher dyons, we will present a quantum mechanical picture of such bound states which is quite attractive, and makes contact with the 3-pronged string picture.

Figure 5.10: Monopole attractor flows to $u = 1$ in the $\mathrm{Re}\, u > 0$ part of moduli space. The fat line represents a cut.

Higher dyons

The higher dyons, with electric charge $|n| \geq 2$ and magnetic charge $|m| = 1$, present a puzzle. In general, the same problems as for the purely electric charge arise; a spherically symmetric BPS solution does not exist.[6] On the other hand, such BPS states can be obtained at weak coupling either from a monopole (for even n) or an elementary dyon (for odd n) by performing a number of times the monodromy $u \to e^{2\pi i} u$, as can be seen from (2.3.4)-(2.3.5). It seems quite unlikely that the attractor flow would all of a sudden cease to exist at a certain point while we are continuously varying the moduli at infinity as $u \to e^{2i\phi} u$, $\phi \in \mathbb{R}$, for a large initial value of u. Nevertheless, this is exactly what happens.

To see this, consider the flows to a $(0, 1)$ monopole core while we are thus varying the modulus at infinity. In the part of moduli space with $\mathrm{Re}\, u > 0$, nothing can go wrong, as can be seen from fig. 5.10. But when we keep on varying counterclockwise the modulus at infinity — which we assumed to be large — we get into the $\mathrm{Re}\, u < 0$ part, where eventually (when exactly depends on the precise choice of cut position), the cycle under consideration will be assigned

[6] Depending on where one puts the cuts in moduli space, there can be a limited range of moduli at infinity for which such solutions still exist for charges obtained from the monopole or elementary dyon by a single monodromy about $u = \infty$. Such objects should rather be considered as elementary dyons or monopoles however, since the corresponding flows necessarily pass through this cut again, such that sufficiently close to the attractor point, the charge is again a multiple of the vanishing monopole or dyon charge. See also fig. 5.11.

Figure 5.11: Monopole attractor flows to $u = 1$ in the $\mathrm{Re}\, u < 0$ part of moduli space. The dotted lines indicate the 'false flows' which one obtains when one tries to pull the flows through the $u = -1$ singularity. The fat straight line to the right of the singularity represents a cut. In the usual conventions, there is also a cut to the left, but we have not indicated this one her.

charge $(2, -1)$. So let us follow the flows further there in fig. 5.11. Here something *does* happen. Everything goes well till we reach the flow touching the $u = -1$ singularity. When we try to 'pull' the flow through the singularity, i.e. when we rotate the modulus at infinity a little bit further, a catastrophe happens: the flow now passes through the cut at the *right* side of $u = -1$, such that its charge changes identity w.r.t. the neighboring flow at its left hand side, and in particular is no longer a 'good' monopole charge. The solution breaks down; the flow turns into a 'false' one, again terminating on a regular point of the line of marginal stability, as in case 2 in section 3.3.3. Let us clarify this a bit.

First, consider fig. 5.12, where the potential $V(u, \bar{u})$ for the effective particle describing the $u(\tau)$ dynamics in this case is plotted as a function of u. The indicated trajectory corresponds to the fattened flow in fig. 5.11. When moving the starting point of the particle more and more to the right (corresponding to counterclockwise rotating the modulus at infinity), at a certain point, the particle will no longer be able to reach the top of the monopole potential (at least not in a BPS way, that is, such that its total energy is zero). This makes it intuitively clear why the spherically symmetric BPS solution ceases to exist.

One can also understand this at the level of the flows. The attractor flows are gradient flows for the function $|Z|$, that is, they flow down the hill towards minimal

Figure 5.12: The potential $V(u,\bar{u})$ for the effective particle describing the $u(\tau)$ dynamics, in the situation shown in fig. 5.11. The indicated trajectory corresponds to the fattened flow in fig. 5.11. When moving the starting point of the particle more and more to the right, at a certain point, it will no longer be able to reach the top of the monopole potential (at least not in a BPS way, that is, such that its total energy is zero).

Figure 5.13: $|Z|$ as a function of u for the flows of fig. 5.11. The indicated flow corresponds to the fattened flow in fig. 5.11. Beyond a critical flow, 'down the hill' no longer leads to the monopole zero of Z.

$|Z|$. In fig. 5.13, $|Z|$ is plotted as a function of u for the case at hand. Again, the indicated flow corresponds to the fattened flow in fig. 5.11. Rotating the modulus at infinity, at a certain point (namely at the flow touching $u = -1$), 'down the hill' no longer leads to the monopole zero at $u = 1$, but to a certain regular zero of Z (on the line of marginal stability), as shown in fig. 5.14 (a). This is a false flow however, in that it does not correspond to a genuine solution of the equations of motion. Indeed, as one can see in fig. 5.14 (b), the effective particle potential is not extremal at this point; it is just an ordinary regular point, and the particle will simply continue its journey down the potential hill there, corresponding to an 'inverted' BPS flow, which, as we have seen several times already, leads to an unphysical, singular, infinite energy solution.

Now if the higher dyons are not realized as spherically symmetric BPS solutions, what are they then? And how can they be continuously connected to the spherically symmetric monopole or the elementary dyon? Again, we believe they are realized as bound states of monopoles and elementary dyons. In the case at hand, this would be two elementary dyons of charge $(1, -1)$ bound to the original charge $(0, 1)$ monopole.

There are good reasons to believe in this. First of all, when considering the dyons as test particles in the monopole field, or vice versa, we find in the situation at hand a stable equilibrium at finite distance, as in fig. 5.8. Furthermore, quantum mechanically, such a bound state would indeed be continuously connected

Figure 5.14: Left (a): $|Z|$ as a function of u for the continuation of the flows of fig. 5.11 beyond the critical flow. This connects smoothly to the lower sheet of fig. 5.13. The indicated flow corresponds to one of the dotted lines in fig. 5.11. It is a false flow, terminating at a regular zero of Z. Right (b): the corresponding effective particle potential. Here we see clearly that the flow is indeed a false one, not corresponding to a solution of the equations of motion. The dotted lines show how the true particle motion continues beyond the false attractor point.

to the monopole, in the following sense. Suppose we have a monopole with $u(r)$ infinitesimally close to the critical flow. Now deform $u(r)$ to a trajectory in moduli space which lies just at the other side of the critical flow (this does not have to be a solution to the equations of motion and in particular will be excited in energy above the BPS bound). Denote the radius at which $u = -1$ in the critical flow by r_c. Now imagine that just before the move a virtual monopole-antimonopole pair was created, to be destroyed again just after the move, and that the monopole happened to be at $r > r_c$ when the critical trajectory was crossed, while the antimonopole was at $r < r_c$. Then the spacetime trajectory of the monopole antimonopole pair, mapped to moduli space via $u(r,t)$, encircles the point $u = -1$. Consequently, there is a monodromy on the monopole charge, of which the net result is that we are left with two (almost massless) elementary dyons when the monopole-antimonopole pair is destroyed again! Now if the presence of the dyon charge manages to relax the energy of the complete configuration back to its BPS bound, this state born out of a quantum fluctuation will actually survive as a stable state, which we can deform further maintaining the BPS property by further rotating the moduli at infinity!

How would such a bound state look like quantum mechanically? The following picture has some satisfying features, though it is not clear how serious one can take it. Fix the overall translational zeromode[7] by putting the origin of our coordinate frame at any of the particle positions. Say we put it at the monopole position. We expect the wave function of the dyons to be concentrated at $r = r_{ms}$, the radius where the u modulus is at the line of marginal stability, since there lies the minimum of the potential for the dyon considered as a test particle in the monopole background. A charge distribution with a $(0,1)$ monopole at $r = 0$ and 2 units of $(1,-1)$ dyon charge smeared out over a spherical shell at $r = r_{ms}$ will produce a BPS field configuration which is a $(2,-1)$ attractor flow for $r > r_{ms}$ and a $(0,1)$ flow for $r < r_{ms}$.[8] See fig. 5.15. The mass of the dyonic shell is $2\sqrt{2}|Z_{1,-1}(ms)|$, with $Z_{1,-1}(ms)$ the dyon central charge at the marginal stability point $u(r_{ms})$. The mass in the fields outside the shell is according to (5.1.22) equal to $\sqrt{2}(|Z_{2,-1}(\infty)| - |Z_{2,-1}(ms)|)$, with $Z_{2,-1}(\infty)$ the central charge at $r = \infty$ of $(2,-1)$, and $Z_{2,-1}(ms)$ the same thing at the marginal stability point. The mass of the fields inside the shell is in obvious notation $\sqrt{2}|Z_{0,1}(ms)|$. Thanks to the fact that the phases of $Z_{1,-1}(ms)$ and $Z_{0,1}$ are *equal* at the marginal stability point, the total mass of the system (assuming supersymmetry eliminates the zero point energies) adds up to $\sqrt{2}|Z_{2,-1}(\infty)|$, that is, the BPS bound is saturated! The

[7] This should be considered as giving a boundary condition to calculate expectation values of fields with the effective field theory. Also for the case of a single monopole, the position had to be fixed before we could use the effective field theory. In the same spirit, the moduli at infinity had first to be fixed, as they have a flat potential.

[8] The flow for $r > r_{ms}$ also determines the value of r_{ms}.

Figure 5.15: Two possible representations of the BPS bound state of two dyons and a monopole, depending on how we eliminate the translational zeromode. Left: monopole at the center, dyons in a shell. Right: dyons at the center, monopole in a shell. The attractor flows are indicated by arrows and labeled by the charges they correspond to.

dyonic shell can also be represented as a charge $2(1,-1)$ flow 'inserted' at $r = r_{ms}$ in the $\{(2,-1),(1,0)\}$ flow, starting from the marginal stability point. This is how the field expectation values would look around the dyons if we would have started by taking the dyons at the origin of our coordinate frame in stead of the monopole (in that case the monopole would be represented as a flow insertion). This can all be summarized nicely in a '3-pronged flow' picture as in fig. 5.16.

Analogously, $(n, \pm 1)$ dyons are represented as bound states of n elementary dyons and $n \pm 1$ monopoles, with opposite magnetic charge.

Note that the construction implies that as the modulus at spatial infinity approaches the line of marginal stability, the distance between the components of the bound state will grow (since marginal stability is reached closer and closer to $r = \infty$), till eventually, when one reaches the MS line, the components are at infinite distance and the state decays in its components. This is a very pleasing physical picture of the decay of these states, which is indeed known to occur when the vacuum crosses the curve of marginal stability in moduli space! [5] Also, the above construction of bound states is clearly not possible inside the MS curve, again in agreement with physical expectations.

There is a remarkable connection with a completely different picture of $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory in string theory: in type IIB string theory, one can reproduce the Seiberg-Witten effective action as the low energy effective action of a noncompact 3-brane in the background of a $(0,1)$ and a $(1,\pm 1)$ 7-brane, placed at finite distance from each other [95]. The geometry transverse to the 7-branes

Figure 5.16: The '3-pronged flow' picture representing a charge $(2,-1)$ bound state of a monopole and two dyons. The state can be considered as a $(2,-1)$ flow followed by a $(0,1)$ flow, with a $2(1,-1)$ flow 'inserted' at the transition point (at marginal stability), or as $(2,-1)$ followed by $2(1,-1)$ with a $(0,1)$ insertion. The picture is the same in both cases. The total energy is just the sum of the energies of the different flows, and saturates the BPS bound. The dotted line indicates the false flow obtained by continuing the original $(2,-1)$ flow. 'MS' indicates the line of marginal stability.

turns out to be precisely the Seiberg-Witten geometry. BPS states correspond to $(0,1)$ or $(1,\pm1)$ strings stretching between the 3-branes and the corresponding 7-branes, the former giving rise to monopoles, the latter to elementary dyons [96]. The minimal energy stretched strings turn out to coincide with the attractor flows in the transverse plane. Precisely at the critical flow considered above, also the single stretched string ceases to be the minimal energy solution. Instead, the so-called *three pronged* strings further represent the BPS states [97, 98]. Such 3-pronged strings look exactly as the 3-pronged flows in fig. 5.16 (which was of course also inspired by the string picture). The pure electric charge corresponds to a three-pronged string as well in this picture. Incidentally, the 'spontaneously cut-off' solutions we found for the monopole and the dyon were recently reproduced in the 3/7 brane picture in [70]. There are some similarities with [71] as well.

Charges not in the BPS spectrum

Quantum $\mathcal{N}=2$ $SU(2)$ Yang-Mills theory is believed not to have other BPS states, apart from those having (a multiple of) the charges we have considered thus far. Can we see this from the low energy effective field theory?

Clearly, none of these charges will have a spherically symmetric BPS solution. For many of them, a 'bound state' construction as e.g. for the $(n,1)$ dyons is furthermore not possible, because an interaction potential with stable points at finite distance, as in fig. 5.8, does not arise. However, it seems that the above considerations do not exclude bound states which are not actually in the BPS spectrum, though more careful study could perhaps achieve this.

A similar problem arises in the 3-pronged string picture of BPS states. There this is solved by lifting string theory to M-theory.

It would be interesting to study non-BPS states of the field theory in this context. These have attracted quite some interest recently [99]. The relation with the non-BPS bound states found in [57] could also be worth investigating.

5.2 Multi monopole dynamics

As explained in section 3.3.4, the results for spherically symmetric solutions can immediately be extended to equal charge N center solutions, simply by replacing τ by the potential

$$\tau = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{r_i}, \tag{5.2.1}$$

Figure 5.17: The boundary of the core of an N-monopole configuration is simply given by the equipotential surface $\tau = \frac{1}{N}\sum_{i=1}^{N} 1/r_i = r_*$, where r_* is the one monopole core radius.

where r_i is the distance to the ith center. An example of how such a configuration can look like is given in fig. 5.17.

An interesting and important question is what can be said about the low energy dynamics of such N-center configurations. A beautiful and very powerful tool to tackle this problem is the moduli space approximation [100]. This amounts to separating collective coordinates from fluctuations, integrating out the fluctuations and considering the limit of small time derivatives of the collective coordinates. This produces an effective action for the collective coordinates x^a, which will be of the general form

$$S_{coll} = \frac{1}{2}\int dt\, G_{ab}(x)\frac{dx^a}{dt}\frac{dx^b}{dt}. \tag{5.2.2}$$

The collective coordinates parametrize a space called the moduli space (not to be confused with the moduli spaces encountered thus far), and $G_{ab}(x)$ can be considered to be the metric on this space. The action (5.2.2) is then simply the action for nonrelativistic geodesic motion on moduli space.

The moduli space metric for Reisner-Nordstrom black holes was obtained in [101]. Some aspects of the classical $\mathcal{N} = 2$ $SU(2)$ Yang-Mills monopole moduli space were discussed in [102].

For a classical $SU(2)$ BPS N monopole configuration, the collective coordinates can be identified with the monopole positions together with an internal angle for each monopole, associated with the unbroken $U(1)$ gauge symmetry. If the angles can effectively be considered as compact S^1 dimensions of moduli space, one would expect these modes to freeze out when doing quantum mechanics and considering the extreme low energy limit, as there will be a mass gap to the excitations of the angles. The internal angle is in a certain sense still visible in the low energy theory however: its excitations correspond to the $(n,1)$ dyons. So (if the mass gap indeed exists) we expect only the monopole positions to remain as collective coordinates. This is indeed what we seem to get.

Our aim is thus to obtain the moduli space metric on our collective coordinate space $\mathcal{M} = \mathbb{R}^{3N}/\mathcal{S}_N$, in the $U(1)$ low energy Seiberg-Witten theory. Denote the vector potential as in (3.1.49) by $\mathbb{A}$. For a static multicenter monopole BPS solution, we have from (3.3.3), generalized to the multicenter case:

$$\vec{\mathbb{A}} = \vec{A}_{Dir} \otimes \beta, \tag{5.2.3}$$

where $\vec{A}_{Dir}$ is the Dirac multimonopole vector potential, satisfying

$$\vec{\nabla} \times \vec{A}_{Dir} = \vec{\nabla}\tau. \tag{5.2.4}$$

Here we have put for convenience $\gamma/\sqrt{4\pi} \equiv 1$ in (3.3.3).

We expect from general considerations [102] that, in the gauge $\vec{\nabla} \cdot \vec{A} = 0$, the action (5.2.2) is here

$$S_{coll} = \frac{1}{4\pi} \int dt \int d^3x \left\| \frac{du}{dt} \right\|^2 + \left\| \frac{d\vec{\mathbb{A}}}{dt} \right\|^2 \tag{5.2.5}$$

$$= \frac{1}{4\pi} \int dt \int d^3x \, g_{u\bar{u}} \left| \frac{du}{dt} \right|^2 + \frac{1}{2}\beta \cdot *\beta \left(\frac{d\vec{A}}{dt} \right)^2 \tag{5.2.6}$$

$$= \frac{1}{4\pi} \int dt \int d^3x \, g_{u\bar{u}} \left| \frac{du}{dt} \right|^2 + g^{u\bar{u}} |\partial_u \phi_D|^2 \left(\frac{d\vec{A}}{dt} \right)^2 \tag{5.2.7}$$

$$\tag{5.2.8}$$

We have no direct proof for this however. A short calculation involving the rigid attractor flow equations (5.1.23) transforms the latter expression to

$$S_{coll} = \frac{1}{2} \int dt \frac{\sqrt{2}}{4\pi} \int d^3x \, (-\frac{d|Z_{sw}|}{d\tau}) \, [(\frac{d\tau}{dt})^2 + (\frac{d\vec{A}}{dt})^2] \tag{5.2.9}$$

The quantity between square brackets is purely geometrical and can in principle be calculated with elementary methods for arbitrary monopole positions and velocities. The effect of the interactions with the integrated out massive charged

Figure 5.18: Weight function $-\frac{d|Z_{sw}|}{d\tau}$ for a two monopole configuration.

particles resides entirely in the weighting of the integral over space with the factor $-\frac{d|Z_{sw}|}{d\tau}$, which can be interpreted as a mass density per unit of τ. The heighest weight is thus given to a shell surrounding the core of the N monopole solution, as illustrated for $N = 2$ in fig. 5.18.

For a single particle at the origin, moving at speed v, say in the z direction, one finds $(\frac{d\tau}{dt})^2 = \tau^4 v^2 \cos^2\theta$ and $(\frac{d\vec{A}}{dt})^2 = \tau^4 v^2 \sin^2\theta$. Hence the collective coordinate action is simply

$$S_{coll} = \frac{1}{2}\int dt\, \sqrt{2}|Z_{sw}(\tau = 0)|, \tag{5.2.10}$$

which is indeed what one would expect for a particle of mass $\sqrt{2}|Z_{sw}(\tau = 0)|$. Note that the central charge is evaluated at spatial infinity, *not* at the particle worldline. That is, the mass appearing in front of the action is not the bare mass, but the true, physical, quantum corrected exact mass of the monopole.

Studying the dynamics of the rigid BPS solutions in moduli space approximation, and for example comparing this with [101, 103], would be very interesting. We will keep this for another occasion however.

Appendix A

Samenvatting

Niets is wat het lijkt. Als er iets is dat het besluit mag zijn van de twintigste eeuwse natuurkunde is dat het wel. De zoektocht naar de fundamentele wetten van de natuur leidde tot een beeld van de 'werkelijkheid' dat steeds verder verwijderd scheen van de dagdagelijkse realiteit, maar tegelijk steeds mooier en eleganter werd. Op dit ogenblik wordt de top van die evolutie gedomineerd door een merkwaardig rijke en unificerende theoretische constructie: stringtheorie.

Hoe bevreemdend efficient esthetische overwegingen ook gebleken zijn bij het ontrafelen van dit beeld, de werkelijke waarde van om het even welke kandidaat fundamentele theorie zal steeds bepaald blijven door wat die theorie ons kan leren over onze observeerbare wereld. Een veel gehoord cliché over stringtheorie is dat ze enkel relevant is voor processen waarbij (quantum-)gravitatie een belangrijke rol speelt. Aangezien dit pas het geval is bij een energieschaal die zich verhoudt tot de energie bereikbaar in de huidige deeltjesversnellers zoals de afstand aarde-zon zich verhoudt tot de dikte van een blad papier, lijkt de situatie hier vrij hopeloos. Dat cliché is evenwel fout, om volgende redenen:

1. Er bestaat een twintigtal 'elementaire' natuurkonstanten, zoals deeltjesmassa's en koppelingsconstanten, die alleen zullen kunnen voorspeld worden door een theorie die een consistente beschrijving geeft van de natuurwetten tot op willekeurig hoge energieschalen, zoals wellicht stringtheorie.

2. Het is *niet* noodzakelijk zo dat de schaal waarop specifieke stringfysica zichtbaar wordt dezelfde is als die waarop de gravitatiekracht belangrijk wordt; dit kan al bij veel lagere energie het geval zijn. In principe is het zelfs mogelijk dat er stringfysica tevoorschijn zal komen bij de eerstvolgende generatie acceleratorexperimenten.

3. Stringtheorie blijkt *als kader* ook bijzonder nuttig te zijn voor de beschrijving en analyse van lage energie fysica die traditioneel door quantumveldentheorie wordt gegeven. De recente doorbraken in niet-perturbatieve stringtheorie hebben geleid tot nieuwe resultaten in (supersymmetrische) quantumveldentheorie die met traditionele technieken zo goed als onbereikbaar zouden geweest zijn. Het succes van stringtheorie hierin is vooral te danken aan het diep meetkundige, unificerende beeld dat ze levert voor tal van veldentheoretische objecten. Deze opmerkelijke resultaten tonen aan dat stringtheorie méér is dan gewoon gravitatie in een quantumkleedje; de theorie leidt tot een nieuw begripskader van de natuurkunde in haar geheel.

Deze thesis sluit aan bij het derde punt. We bekijken een specifieke klasse van modellen, namelijk type IIB stringtheorie gecompactificeerd op Calabi-Yau manifolds, en we bestuderen in dit kader een aantal aspecten van de lage energie fysica die uit deze modellen voortvloeit. We spitsen ons voornamelijk toe op twee onderwerpen: de afleiding van exacte lage energie quantum effectieve acties van niet-abelse $\mathcal{N} = 2$ supersymmetrische Yang-Mills theorieën, inclusief koppeling aan gravitatie, en de studie van lage energie eigenschappen van BPS toestanden (BPS toestanden zijn massieve, geladen, deeltjesachtige toestanden, elementair of niet, die minimale energie hebben gegeven hun lading en daardoor absoluut stabiel zijn; het prototype voorbeeld is de magnetische monopool in spontaan gebroken ijktheorieën).

De belangrijkste nieuwe resultaten in deze thesis zijn, *in order of appearance*:

- de uitwerking tot een nuttige analysetechniek van de analogie tussen sferisch symmetrische oplossingen van de vier dimensionele lage energie effectieve actie en de beweging van een niet-relativistisch deeltje in een bepaalde potentiaal op de moduliruimte (scalar manifold) van de theorie.
- de directe, expliciete afleiding van Yang-Mills lage energie effectieve acties in het kader van speciale meetkunde, volledig binnen IIB string theorie, zonder gebruik te maken van T- of S-dualiteiten.
- de fysische interpretatie van de parameters in de algemene limietprocedure van locale naar rigide speciale meetkunde.
- de exacte koppeling van de effectieve quantum Yang-Mills theorie aan gravitatie en aan de ijktheorie schaal, die blijkt dynamisch te worden wanneer de gravitatiekracht verschillend is van nul.
- de beschrijving van quantum Yang-Mills BPS toestanden bij zwakke gravitatie, met bijzonder aandacht voor de overgang van ‘elementaire’ naar gebonden toestanden.

- de eerste stappen in de analyse van lage energie multimonopool dynamica in $\mathcal{N} = 2$ quantum Yang-Mills theorieën, inclusief een mogelijke connectie met een (veralgemeende) Nahm constructie via een Maldacena-type correspondentie.

Deze samenvatting volgt de struktuur van de thesis, met de bedoeling dat het mogelijk wordt om bepaalde delen afzonderlijk in meer detail te gaan lezen in de thesis zelf. We richten ons in deze samenvatting in eerste instantie op de geinteresseerde niet-expert. We zullen om die reden ook relatief veel aandacht besteden aan de omkadering van de resultaten.

Hoofdstuk 1: Inleiding

De inleiding is ook in eerste instantie op de geinteresseerde niet-expert gericht. We motiveren de introductie van stringtheorie als fundamentele theorie en bespreken de voornaamste sterke punten en tekortkomingen in de huidige (perturbatieve) formulering ervan. Vervolgens introduceren en bespreken we de hoekstenen van de recente vooruitgang in het inzicht in niet-perturbatieve aspecten van de theorie: D-branen en dualiteit. Tenslotte wordt het gebruik van stringtheorie als kader voor lage energie fysica toegelicht. We besluiten het hoofdstuk met een samenvatting van de resultaten in deze thesis. In wat volgt nemen we aan dat de begrippen ingevoerd in deze inleiding gekend zijn door de lezer.

Hoofdstuk 2: Lage energie effectieve veldentheorie

In dit hoofdtuk voeren we het concept van een (lage energie) *effectieve actie* in, zowel in string- als in veldentheorie. Het onderscheid tussen de '1PI' effective actie en de Wilson effectieve actie wordt geschetst, en we leggen uit wat de betekenis is van de Seiberg-Witten effectieve actie in deze context. Rigide en locale speciale meetkunde worden gedefinieerd en hun centrale rol voor $\mathcal{N} = 2$ effectieve acties wordt toegelicht. Ons werk in [17, 18] volgend, leggen we uit hoe rigide speciale meetkunde tevoorschijn komt als een bepaalde limiet van locale speciale meetkunde. We besluiten het hoofdstuk met een compendium van Seiberg-Witten theorie en haar veralgemeningen. Dit hoofdstuk bevat voornamelijk welbekend materiaal, maar we doen een inspanning om een aantal losse eindjes in de gebruikelijke Seiberg-Witten review literatuur aan elkaar te knopen.

Het is voor de geinteresseerde niet-expert die wil weten wat precies de fysische inhoud is van de effectieve veldentheorieën die beschouwd worden doorheen deze thesis, wellicht nuttig hier even uit te weiden en een aantal elementen uit deze korte samenvatting in meer detail toe te lichten.

Bij voldoende lage energieën heeft om het even welke fysische quantumtheorie die aan een aantal basisprincipes voldoet een beschrijving als een effectieve veldentheorie [24], waarbij de *lichte* elementaire deeltjes van de theorie de quanta van de beschouwde velden zijn, en de (quantum gecorrigeerde) bewegingsvergelijkingen van die velden gegeven worden door extremalisatie van een effectieve actie. Dit is een functionaal die gegeven wordt door de integraal over de ruimte-tijd van een bepaalde (niet noodzakelijk lokale) uitdrukking in de velden. Bij voldoende lage energie kan men deze uitdrukking meestal goed benaderen door termen met ten hoogste twee afgeleiden (dit komt neer op een momentumexpansie tot tweede orde). De effecten van de hoge energie vrijheidsgraden zijn 'uitgeïntegreerd' en bepalen enkel de precieze vorm van de effectieve actie.

De theorieën die we bestuderen in deze thesis hebben $\mathcal{N}=2$ *supersymmetrie*, dat wil zeggen dat hun deeltjesspectrum en dynamica invariant is onder een bepaalde symmetriegroep die buiten translaties en Lorentz transformaties ook transformaties bevat die bosonische en fermionische velden op een bepaalde manier met elkaar vermengen. Bovendien beperken we ons in eerste instantie tot de beschrijving van de lage energie vrijheidsgraden die in een zogenaamde massaloze *vectormultiplet* representatie van deze supersymmetrie-groep vallen, al dan niet gekoppeld aan gravitatie. Een dergelijk vector multiplet omvat twee spin[1] 0 deeltjes (de scalar, beschreven door één complex scalair veld), een spin 1 deeltje (het foton, beschreven door een vectorpotentiaal) en twee spin $\frac{1}{2}$ deeltjes. Wanneer ook (super)gravitatie beschouwd wordt, hebben we in het spectrum van de theorie ook een spin 2 deeltje (het graviton, beschreven door de ruimte-tijd metriek $G_{\mu\nu}$), een spin 1 deeltje (het gravifoton, beschreven door een vectorpotentiaal) en twee spin $\frac{3}{2}$ deeltjes.

Men kan aantonen [26] dat $\mathcal{N}=2$ supersymmetrie vereist dat het bosonische stuk van de lage energie (quantum) effectieve actie van een aantal aan supergravitatie gekoppelde neutrale[2] vectormultipletten, beperkt tot maximaal twee afgeleiden, *noodzakelijk* de volgende vorm heeft:

$$S = S_{graviton} + S_{scalar} + S_{foton}, \tag{1}$$

waarbij $S_{graviton}$ de gebruikelijke Einstein-Hilbert gravitationele actie is:

$$S_{graviton} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-G}\, R \tag{2}$$

[1]Aangezien het hier over massaloze deeltjes gaat, komen met een 'spin s' deeltje twee toestanden overeen met heliciteit $\pm s$ (als $s>0$).

[2]Dit zijn ook de relevante lage energie vrijheidsgraden voor spontaan gebroken niet abelse $\mathcal{N}=2$ Yang-Mills theorieën, aangezien de niet-neutrale vector multipletten massief worden door de spontane ijksymmetriebreking.

(R is de scalaire kromming , G de determinant van de metriek $G_{\mu\nu}$, en G_N de Newton constante), en waarbij de actie voor de scalars de volgende vorm heeft:

$$S_{scalar} = -\frac{1}{8\pi G_N}\int d^4x\,\sqrt{-G}\,g_{a\bar{b}}(z)\,\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}}. \quad (3)$$

Hierbij is z^a, $a = 1, 2, \ldots$ het complexe scalaire veld van het a-de vectormultiplet, en $g_{a\bar{b}}(z, \bar{z})$ een hermitische, positief definiete matrix die in het algemeen afhankelijk is van de z^a, en de interpretatie van een metriek heeft op de 'scalaire manifold' $\mathcal{M}$.[3] Supersymmetrie vereist dat deze metriek van een zeer bijzondere vorm is, namelijk een zogenaamde *locale speciale Kähler metriek.* Voor de technische definitie van speciale Kähler meetkunde verwijzen we naar de thesis. Essentieel komt het erop neer dat de metriek afleidbaar is van een Kähler potentiaal, die op haar beurt afleidbaar is van een bepaalde *meromorfe* (eventueel meerwaardige) functie op $\mathcal{M}$ die de *prepotentiaal* genoemd wordt.

De actie S_{foton} voor de fotonen en het gravifoton heeft ook een zeer specifieke vorm, eveneens afleidbaar van de prepotentiaal, maar daarvoor verwijzen we naar de thesis zelf. Hetzelfde geldt voor het fermionische stuk van de actie.

Wanneer een theorie zonder gravitatie wordt beschouwd, krijgen we een analoge vorm voor de effectieve actie, met analoge beperkingen, maar nu met als relevante meetkundige structuur *rigide speciale meetkunde.*

Het is duidelijk dat speciale meetkunde zeer sterke beperkingen oplegt op de mogelijke effectieve acties. Sterk genoeg, zo bleek uit het baanbrekend werk van Seiberg en Witten [5], om tot niet-triviale *exacte oplossingen* te kunnen komen. Uit het bovenstaande volgt dat het vinden van een exacte oplossing neerkomt op het vinden van de relevante speciale Kähler manifold. Seiberg en Witten ontdekten dat de speciale Kähler manifold die de exacte oplossing van de lage energie effectieve actie van $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theorie geeft, niets anders is dan de 1 complex dimensionale *moduliruimte* die de complexe structuren op een genus 1 Riemann oppervlak (een torus) parametrizeert, met de speciale Kähler metriek gegeven in termen van integralen over niet-samentrekbare lussen van een bepaalde meromorfe 1-vorm op dat Riemann oppervlak.

De Seiberg-Witten oplossing gaf het startschot voor een massale zoektocht naar het antwoord op twee voor de hand liggende vragen:

- Hoe kan de enigmatische verschijning van een *meetkundige* moduliruimte in veldentheorie begrepen worden?
- Hoe kan dit veralgemeend worden naar andere (ijk)theorieën?

[3]De globale structuur van deze manifold hangt af van het concrete fysische model dat beschouwd wordt.

Het antwoord op de eerste vraag, en op die manier ook een systematische aanpak van de tweede vraag, werd op een bijzonder mooie manier gegeven door *stringtheorie*. Ijktheorieën hebben vaak een geometrische oorsprong in stringtheorie, en dit leverde de link tussen de exacte oplossingen van lage energie effectieve acties enerzijds en meetkundige structuren anderzijds. Het unificerende geometrische kader van stringtheorie gaf bovendien ook een elegante en nuttige meetkundige interpretatie aan tal van veldentheoretische objecten, zoals bijvoorbeeld BPS toestanden.[4] Deze laatste blijken in stringtheorie dikwijls voorgesteld te kunnen worden door D-branen (cf. hoofdstuk 1) gewikkeld rond niet-triviale cycles (gesloten niet samentrekbare lussen, oppervlakken, volumes, etc.) in de manifold die de zes kompakte ruimte-tijd dimensies vormt.

In het overblijvende deel van deze thesis bestuderen we hoe de exacte oplossingen van lage energie effectieve acties precies tevoorschijn komen uit stringtheorie, en analyseren we een aantal implicaties voor lage energiefysica (inclusief gravitatie), voor unificatie en voor de dynamica van BPS toestanden.

Hoofdstuk 3: Calabi-Yau compactificaties van type II stringtheorie

In dit hoofdstuk bestuderen we uitvoerig de compactificatie van type IIB string theorie op een Calabi-Yau manifold X (fig. A.1). Dit geeft een $\mathcal{N} = 2$ supersymmetrische theorie in vier dimensies. Het bosonische massaloze spectrum in vier dimensies en de overeenkomstige effectieve actie worden afgeleid. De vectormultiplet sector hiervan heeft de vorm bepaald door speciale meetkunde zoals besproken in hoofdstuk 2. De specifieke speciale Kähler manifold $\mathcal{M}$ is hier bepaald door klassieke meetkunde: het is de moduliruimte van complexe structuren van de Calabi-Yau manifold X. De metriek is bepaald als volgt. Op een Calabi-Yau manifold bestaat er een unieke holomorfe 3-vorm Ω, die holomorf afhangt van de complexe structuur moduli z^a van X (de z^a vormen tevens de scalairen van de 4D vectormultipletten). De metriek op $\mathcal{M}$ is dan

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}. \qquad (4)$$

waarbij de Kähler potentiaal $\mathcal{K}$ gegeven is door

$$\mathcal{K} = -\ln(i \int_X \Omega \wedge \bar{\Omega}) \qquad (5)$$

De integraal in het rechterlid kan berekend worden in termen van de integralen van Ω over een (homologie-)basis van niet-triviale 3-cycles (de zogenaamde *perioden*)

[4] De reden waarom men zich hier beperkt tot BPS toestanden is dat die een deel van de supersymmetrie van het vacuum behouden, net genoeg zo blijkt om nog een aantal exacte uitspraken te kunnen doen (zoals bijvoorbeeld een (quantum) exacte massaformule).

Figuur A.1: Schets van een Calabi-Yau compactificatie. De complexe structuur moduli van de compacte Calabi-Yau manifold X kunnen varieren over de niet-compacte, gekromde, vier dimensionale ruimte-tijd M_4. Deze modulivelden vormen de scalairen van de vectormultipletten in de vierdimensionale theorie.

van X. De andere elementen in de 4D lage energie effectieve actie worden ook uitgedrukt in termen van Calabi-Yau meetkunde. Een belangrijke eigenschap van type IIB Calabi-Yau compactificaties is dat bovendien de vector multiplet sector van de *klassieke* lage energie actie (tot tweede orde in de afgeleiden) *exact* is; quantumcorrecties zijn verboden door supersymmetrie.

Vervolgens (sectie 3.2) spitsen we ons toe op BPS toestanden in $\mathcal{N} = 2$ supersymmetrische theorieën in vier dimensies en de verschillende multipletten waarin ze georganizeerd kunnen zijn. Deeltjesachtige BPS toestanden in vier dimensies kunnen uit type IIB stringtheorie bekomen worden door D3-branen rond niet-triviale 3-cycles te wikkelen. Er zijn geen andere mogelijkheden omdat type IIB stringtheorie alleen oneven BPS D-branen heeft, en er geen niet-triviale 1- of 5-cycles in een Calabi-Yau manifold bestaan. De BPS voorwaarde legt op dat de inbedding van de D3-braan in X 'speciaal Lagrangiaans' is, hetgeen o.a. impliceert dat het volume ervan minimaal is. Ons werk in [20] volgend, leiden we de preciese reductie af van de D3-braan actie tot een deeltjesactie in vier dimensies, waaruit onmiddellijk de massa van deze toestanden volgt. Voor een D3-braan gewikkeld rond een 3-cycle Γ is dit

$$M = \frac{1}{\sqrt{G_N}} |Z(\Gamma)|, \tag{6}$$

waarbij

$$Z(\Gamma) = e^{\mathcal{K}/2} \int_\Gamma \Omega, \tag{7}$$

met $\mathcal{K}$ gedefinieerd in (5). Deze uit klassieke meetkunde bekomen formule is meteen ook quantummechanisch exact, omdat correcties verboden zijn door supersymmetrie. Als toepassing berekenen we de (quantum exacte) electromagnetische kracht tussen twee bewegende BPS testdeeltjes met willekeurige lading. Tenslotte bespreken we het probleem van de bepaling van het type supersymmetrie multiplet waartoe een bepaalde gewikkelde D3-braan aanleiding geeft. Dit hangt af van inbedding en topologie van de 3-braan. We stellen een eenvoudige shortcut voor die een aantal gevallen kan worden toegepast.

De rest van het hoofdstuk (sectie 3.3) is in hoofdzaak gewijd aan het effectieve veldentheorie beeld van de BPS toestanden, dat is, de BPS toestanden beschouwd als oplossingen van de lage energie effectieve actie. Centraal hier staat het *attractor mechanisme*, ontdekt in [60]: de scalairen van de vectormultipletten vloeien steeds naar vaste waarden wanneer men naar het centrum van de BPS oplossing toe gaat (d.i. de horizon wanneer dit een zwart gat is). Die vaste waarden worden bepaald door de lading, maar zijn onafhankelijk van de waarde van de scalairen op oneindig. We leiden in een invariant meetkundig formalisme de gereduceerde effectieve actie af voor sferisch symmetrische oplossingen, vertrekkend van een ansatz die wat algemener is dan gebruikelijk in de literatuur, en in het bijzonder werken we in detail de analogie uit met een niet relativistisch deeltje bewegend in een bepaalde potentiaal op de moduliruimte.

Meer in detail ziet dit eruit als volgt. De ruimte-tijd metriek is van de vorm

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\{(1+(c/r)^2)^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{8}$$

met c een constante. Voor een BPS toestand afkomstig van een D3-braan gewikkeld rond een 3-cycle Γ is de gereduceerde actie (vgl. (3.3.8) in thesis):

$$S_{red} = -\frac{1}{2G_N}\int_0^\infty \mathrm{d}\tau\, \{\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - c^2 + G_N e^{2U} V_\Gamma(z)\}, \tag{9}$$

waarbij de 'tijd' τ gerelateerd is aan de radiele coördinaat r via $r = c/\sinh c\tau$, een punt de afgeleide naar τ voorstelt, c de 'energie van de beweging' is ($c = 0$ voor BPS oplossingen), en de effectieve 'potentiaal' gegeven is door $-G_N e^{2U} V_\Gamma(z)$, met

$$V_\Gamma(z) = |Z(\Gamma)|^2 + 4\, g^{a\bar{b}}\, \partial_a|Z(\Gamma)|\, \bar{\partial}_{\bar{b}}|Z(\Gamma)|. \tag{10}$$

G_N is nog steeds de Newton constante en $g_{a\bar{b}}$ de metriek (4) op de complexe structuur moduli ruimte. We gebruiken deze equivalentie met een bewegend deeltje voor een intuitieve analyse van de algemene oplossing, inclusief niet-BPS zwarte

Figuur A.2: Schets van de potentiaal $-e^{2U}V$ waarin het effectieve deeltje beweegt, uitgezet als functie van e^{2U} en de moduli z. Traject (a) (generieke beginvoorwaarden) correspondeert met een onfysiche supergravitatie-oplossing (sterk singulier en antigraviterend). Traject (b) correspondeert met een niet-BPS zwart gat. Traject (c) correspondeert met een BPS zwart gat.

gaten (fig. A.2). We maken zo ook manifest dat het attractor mechanisme niet veroorzaakt wordt door dissipatieve demping (zoals dikwijls beweerd wordt in de literatuur), maar integendeel juist als gevolg van een eindige energie voorwaarde samen met de sterke instabiliteit van het systeem, zoals gewoonlijk in solitonfysica. Het (ruwe) verband tussen deze instabiliteit en de entropie van zwarte gaten wordt kort geschetst.

BPS oplossingen hebben $c = 0$ en voldoen aan de zogenaamde *attractor flow* vergelijkingen:

$$\dot{U} = -\sqrt{G_N}\, e^U\, |Z| \tag{11}$$

$$\dot{z}^a = -2\sqrt{G_N}\, e^U\, g^{a\bar{b}}\, \bar{\partial}_{\bar{b}}|Z|. \tag{12}$$

De moduli convergeren voor $\tau \to \infty$ steeds naar een welbepaalde waarde, enkel afhankelijk van de keuze van Γ, namelijk die waarde die $|Z(\Gamma)|$ minimaliseert.

In sectie 3.3.3 bekijken we een de oplossingen van de attractor flow vergelijkingen voor een aantal modellen, met speciale nadruk op de niet-generieke gevallen, en we bekomen in het bijzonder een totnogtoe onopgemerkt gebleven oplossing [21], overeenkomend met een D3-braan gewikkeld over een zogenaamde *conifold* cycle (voorbeeld 2). Dit is een niet triviale 3-cycle die tot nul volume kan krim-

Figuur A.3: Een schets van een Calabi-Yau manifold die in vier dimensies een niet-abelse $\mathcal{N} = 2$ Yang-Mills theorie zwak gekoppeld aan gravitatie oplevert. Branen gewikkeld rond een generieke cycle Γ hebben typisch Planck schaal massa's. Door fijnregeling van de moduli kunnen sommige cycles γ echter veel lichter gemaakt worden dan de Planck massa, en zo aanleiding geven tot de benodigde zwak graviterende Yang-Mills deeltjes.

pen door variatie van de moduli. Deze oplossing zal een prominente rol spelen in hoofdstuk 5. We eindigen dit deel over attractors met een discussie in 3.3.4 van het multicenter geval en de presentatie in 3.3.5 van een aantal krachtige technieken om de attractor flow vergelijkingen op te lossen, daarbij een intrinsiek meetkundige, Kähler ijk invariante formulering van de methodes ontwikkeld in [22, 23] leverend.

We sluiten dit hoofdstuk af met een analyse van de geldigheidsdomeinen van de D-braan en veldentheorie beelden van de BPS toestanden. We schetsen de beroemde Maldacena correspondentie in deze kontekst. Wanneer dezelfde redenering toegepast wordt op BPS toestanden die geen zwarte gaten zijn, schijnen we een nieuw soort correspondentie te vinden, die sterk doet denken aan de welbekende wiskundige 'Nahm dualiteit' tussen N monopolen en een bepaald $N \times N$ hermitisch matrix systeem. We laten dit als een intrigerende opening naar verder onderzoek.

Hoofdstuk 4: Quantum Yang-Mills + gravitatie uit IIB strings

In dit hoofdstuk bestuderen we in detail hoe de lage energie effectieve actie van $\mathcal{N} = 2$ quantum Yang-Mills theorie, zwak gekoppeld aan gravitatie, bekomen wordt vanuit type IIB string theorie gecompactificeerd op een Calabi-Yau manifold X. We baseren ons hier op welbekende 'geometrical engineering' technieken [11, 12, 13, 14]. De voornaamste nieuwe elementen in onze behandeling zijn de expliciete koppeling aan gravitatie en aan de 'dynamische dynamisch gegenereerde schaal' van de quantum Yang-Mills theorie, en het feit dat we alles rechtstreeks en volledig binnen type IIB string theorie afleiden, zonder ons te beperken tot locale overwegingen of gebruik te maken van string S- of T-dualiteiten. Ook de reductie van locale tot rigide speciale meetkunde wordt expliciet aangetoond.

De basisidee is als volgt (fig. A.3). Wanneer we een D3-braan wikkelen rond een generieke cycle Γ van X krijgen we typisch een BPS deeltje met massa van de orde van de Planck massa in de vier dimensionale theorie, dus zeker geen goede kandidaat voor een ijkdeeltje dat slechts zwak interageert met de gravitatiekracht. Door fijnregeling van de complexe structuur moduli van X, namelijk door X op een bepaalde manier bijna-singulier te maken, kunnen sommige cycles γ echter veel lichter gemaakt worden dan de Planck massa. Als we dit bovendien op een manier doen die ervoor zorgt dat de overeenkomstige BPS deeltjes in vier dimensies precies het massieve BPS spectrum opleveren van een of andere $\mathcal{N} = 2$ Yang-Mills theorie met ijkgroep G, met in het bijzonder een aantal massieve vector multipletten die samen een irreducibele representatie dragen van de Weyl groep van G (geinduceerd door monodromieën in de moduliruimte), dan kan aangenomen worden dat de dynamica van deze massieve BPS toestanden samen met de massaloze vectormultipletten die we al hadden, gegeven wordt door die $\mathcal{N} = 2$ Yang-Mills theorie met ijkgroep G. Anderzijds hadden we vanuit type IIB string theorie al de *exacte* quantum lage energie effectieve actie (tot tweede orde in de afgeleiden) van de massaloze vector multipletten gevonden (cf. hoofdstuk 3). Dus we bekomen op deze manier bijna zonder inspanning de exacte oplossing van de lage energie quantum effectieve actie van de beschouwde Yang-Mills theorie!

Het moeilijkste deel van de constructie is dus te bepalen welke Calabi-Yau manifolds, en welke limieten van de complexe structuurmoduli, overeenkomen met een bepaalde ijktheorie. Hieraan is sect. 4.1 gewijd. We beargumenteren dat een ijkgroep met Lie algebra S, waarbij S van type A_k, D_k of E_k is, bekomen wordt bij een Calabi-Yau die lokaal de structuur heeft van een (licht) gedeformeerde S-type singulariteit [84] gefibreerd over een cylinder. Een S-type singulariteit is ruwweg een singulariteit bekomen door in een twee complex dimensionale variëteit een aantal niet triviale 2-sferen die elkaar snijden volgens het Dynkin diagram van de Lie algebra S (zie fig. A.4 voor het voorbeeld van een D_4 (of so(8)) singulariteit), tot punten te laten krimpen. Dit geeft een toch wel zeer mooie correspondentie tussen de classificatie van singulariteiten en de classificatie van ijkgroepen.

In sect. 4.2 bestuderen we de zwakke gravitatielimiet, waarbij we de 3-cycles corresponderend met de ijktheorie-deeltjes laten degeneren tot volume (=massa) nul. We bewijzen de reductie van locale tot rigide speciale meetkunde in deze limiet en bekomen de volledige, exacte quantum effectieve actie, inclusief de koppeling aan gravitatie en aan de dynamisch geworden dynamisch gegenereerde ijktheorie

Figuur A.4: De gedeformeerde D_4 (of $so(8)$) singulariteit heeft een basis van vier tot punten krimpende sferen die elkaar snijden zoals aangegeven door het D_4 Dynkin diagram.

schaal. Dit reproduceert en veralgemeent de Seiberg-Witten oplossing. In het bijzonder wordt de metriek op de rigide sector van de moduliruimte gegeven door de speciale meetkunde van een bepaalde klasse Riemann-oppervlakken voorzien van een zekere meromorfe vorm λ_{sw}, beide éénduidig bepaald uit de (lokale) Calabi-Yau meetkunde. De lichte 3-cycles γ op de Calbi-Yau manifold corresponderen met welbepaalde 1-cycles γ op het Riemann oppervlak.

In sectie 4.3 analyseren we de verschillende unificatieschalen die tevoorschijn komen, en we maken de connectie met het (S-)duale heterotische beeld expliciet. Om wat inzicht te krijgen in de fysische inhoud van onze resultaten, beschouwen we wat experimentele data (hoewel de $\mathcal{N}=2$ theorieën die we bekijken uiteraard niet volledig overeenkomen met observaties). Tenslotte bestuderen we de dynamica van de dynamische dynamisch gegenereerde schaal Λ. We stellen vast dat deze ontkoppelt van de ijktheorie wanneer de verhouding tussen ijkschaal en Planckschaal nul is, óf wanneer de koppelingsconstante van de ijktheorie nul is. We concluderen dus dat de beïvloeding van de schaal door de Yang-Mills-vrijheidsgraden vanuit microscopisch veldentheorie standpunt puur een quantum (stringy) effect is.

De lage energie quantum effectieve actie die we vinden voor het graviton, de Yang-Mills massaloze scalars u^i en de schaal Λ van de Yang-Mills theorie is

$$
\begin{aligned}
S &= \frac{1}{2\kappa_4^2}\int d^4x\,\sqrt{-G}\,R - \frac{2}{(\ln|M_{SU}/\Lambda|^2)^2}\left|\frac{\mathrm{d}\Lambda}{\Lambda}\right|^2 \\
&- \frac{1}{4\pi}\int |\Lambda|^2\,\partial_i\bar{\partial}_j\tilde{K}_{SW}\,\mathrm{d}u^i\wedge *\mathrm{d}\bar{u}^j \qquad (13)
\end{aligned}
$$

$$+2\,\mathrm{Re}\,(\Lambda\partial_i\tilde{K}_{SW}\,\mathrm{d}u^i\wedge *\mathrm{d}\bar{\Lambda})+\tilde{K}_{SW}\,\left|\mathrm{d}\Lambda\right|^2.$$

Hier is $\tilde{K}_{SW}$ de dimensieloze (veralgemeende) Seiberg-Witten Kähler potentiaal voor de beschouwde Yang-Millstheorie, gegeven in termen van de speciale meetkunde van een welbepaalde klasse Riemann oppervlakken. M_{SU} is de (in goede benadering constante) *string unificatie schaal*:

$$M_{SU}^2\approx\frac{1}{\kappa_4^2}\frac{|k|^2}{|b|\ln(|\Lambda|^{-1}\kappa_4^{-1})}, \tag{14}$$

met $\kappa_4^2=8\pi G_N$, b de beta-functie coefficient van de beschouwde ijktheorie en k een Calabi-Yau model afhankelijke constante.

In sectie 4.4 werken we in uitvoerig detail een expliciet voorbeeld uit voor een specifieke Calabi-Yau: we onderzoeken een limiet van de $X_{24}[1,1,2,8,12]$ Calabi-Yau manifold die in vier dimensies pure $SU(3)$ Yang-Mills theorie zwak gekoppeld aan gravitatie oplevert. Cycles, perioden en monodromieën worden expliciet geconstrueerd. De resultaten ondersteunen de algemene argumentatie. Voor de waarde van k in de formule voor M_{SU}^2 hierboven geeft dit voorbeeld de waarde $k=\frac{\sqrt{3}}{4\pi}\frac{\Gamma^4(\frac{5}{6})}{\Gamma^2(\frac{2}{3})}\approx 0.122035$.

Hoofdstuk 5: Attractoren bij zwakke gravitatie

Hier bestuderen we de BPS toestanden bij zwakke gravitatie in het effectieve veldentheorie beeld. Het gros van de resultaten in dit hoofdstuk zijn nieuw. We herbeschouwen het equivalente effectieve deeltje bewegend in een potentiaal op de moduliruimte om intuitief inzicht te krijgen in de sferisch symmetrische oplossingen. We isoleren en bespreken een interessante $\mathcal{O}(G_N)$ 'correctieterm' in de gereduceerde effectieve actie die vanuit het microscopische quantumveldentheorie standpunt volledig afkomstig is van de interactie tussen de dynamische ijktheorieschaal en de quantum fluctuaties van de ijktheorie vrijheidsgraden. De zwakke gravitatielimiet van de attractor flow vergelijkingen wordt afgeleid en gebruikt om een expliciete beschrijving te geven van Strominger's massaloze 'zwarte gaten', bewegend aan de snelheid van het licht.

Omdat de $\mathcal{O}(G_N)$ correcties klein zijn voor de BPS toestanden in het spectrum van de Yang-Mills theorie, en in het bijzonder niet van belang voor de kwalitatieve eigenschappen van de oplossingen, specialiseren we verder tot de volledig rigide limiet $G_N=0$, met als prototype voorbeeld $SU(2)$ Yang-Mills theorie. Gravitatie en schaal ontkoppelen volledig, zodat U en Λ constant gesteld kunnen worden. De attractor flow vergelijkingen vereenvoudigen in dit $SU(2)$ geval tot

$$\dot{u}=-\sqrt{2}\,g_{sw}^{u\bar{u}}\,\bar{\partial}_{\bar{u}}|Z_{sw}|. \tag{15}$$

Figuur A.5: Energiedichtheid ϵ als functie van de radiale coordinaat r voor magnetische lading N met $u(r=\infty)=2+i$, numerisch geintegreerd. De dichtheid wordt nul in een kern met straal $r_* = 1.65 N\Lambda$. De streepjeslijn toont de energiedichtheid die het electromagnetische veld zou hebben indien de moduli niet dynamisch zouden zijn (zoals in Maxwell electromagnetisme). Dit divergeert uiteraard voor $r \to 0$.

Hier is $g_{sw}^{u\bar{u}}$ de inverse metriek op de Seiberg-Witten moduli ruimte en Z_{sw} de periode van de meromorfe Seiberg-Witten 1-vorm bij modulus u, over de beschouwde 1-cycle γ: $Z_{sw} = \Lambda \int_\gamma \lambda_{sw}$. Een interessante eigenschap van de rigide limiet is dat de faze van Z_{sw} constant is langs de flow.

Enkel voor magnetische monopolen en elementaire dyonen (dit zijn precies de BPS toestanden die massaloos kunnen worden voor bepaalde waarden van de moduli) vinden we een sferisch symmetrische oplossing. Deze zijn van dezelfde vorm als de eerder al besproken 'conifold oplossingen' (sectie 3.3.3, voorbeeld 2). Ze zijn continu differentieerbaar, maar hebben een eindige energieloze 'kern' waarin de modulus constant is, gelijk aan zijn attractorwaarde (fig. A.5). De totale massa is eindig en zit volledig in de velden rond de kern. De straal van de kern (en de hele schaal van de oplossing eigenlijk) is proportioneel met de lading N van de toestand. Voor geen enkele waarde van de lading vormen deze toestanden zwarte gaten; door het attractor mechanisme geraakt de massa nooit geconcentreerd binnen haar Schwarzschild straal!

We onderzoeken een aantal eigenschappen van deze oplossingen en vinden o.a. dat de krachtpotentiaal voor een testdeeltje afkomstig van een D3-braan gewikkeld rond een cycle γ_t, in het BPS veld van een deeltje afkomstig van een D3-braan

Figuur A.6: De kern van een kubische 8-monopool configuratie.

gewikkeld rond een cycle γ_0, gegeven is door

$$W(r) = 2\sqrt{2}\sin^2(\frac{\alpha_t(r) - \alpha_0}{2})\,|Z_t(r)|, \qquad (16)$$

waarbij α_0 de (constante) faze is van Z_0 en α_t de (variërende) faze van Z_t in het BPS veld van γ_0. Wanneer de attractor flow van γ_0 dus door een punt passeert waarin de fazes van Z_t en Z_0 identiek zijn[5], vinden we een stabiele afstand voor een testdeeltje γ_t in het veld van γ_0. Dit suggereert sterk de mogelijkheid van gebonden toestanden.

Vervolgens bestuderen we in detail wat er — gelukkig, zoals we beargumenteren — fout loopt wanneer we proberen een sferisch symmetrische BPS oplossing te construeren voor puur electrisch geladen deeltjes (het W-boson multiplet) en de hogere dyonen. De puzzel stelt zich het scherpst bij de dyonen, aangezien die door continue monodromietransformaties met elkaar verbonden zijn. We komen zo tot een aantrekkelijk beeld van een overgang naar een gebonden toestand van monopolen en elementaire dyonen. Het W-boson is in dit beeld bijvoorbeeld een gebonden toestand van 1 monopool en 1 elementair dyon, met tegengestelde magnetische lading. Op deze manier wordt ook (kwantitatief) contact gemaakt met de '3-pronged' string representatie van BPS toestanden. Een interessante vraag is of dit mechanisme kan veralgemeend worden naar BPS zwarte gaten.

We besluiten dit hoofdstuk in sect. 5.2 met de eerste stappen in een analyse van de lage energie dynamica van N monopolen (fig. A.6). We geven een algemene

[5] dit is een punt waarop het zwaarste deeltje slechts marginaal stabiel is voor verval naar het lichtste plus nog iets, dus dit punt ligt noodzakelijk op de zogenaamde lijn van marginale stabiliteit in de moduliruimte.

formule voor de N monopool moduli ruimte metriek en evalueren dit voor $N = 1$, hetgeen het verwachte resultaat oplevert. Een verdergaande analyse van de multimonopool metriek zou bijzonder interessant zijn, zeker in het licht van een mogelijke connectie met een (veralgemeende?) Nahm constructie via de eerder aangehaalde Maldacena-type correspondentie.

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