Particle Dynamics of Branes

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The work described in this thesis was performed at the Centre for Theoretical Physics of the Rijks*universiteit* Groningen. This work is part of the research program of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

Printed by Grafisch Bedrijf Ponsen & Looijen bv Wageningen, The Netherlands.

RIJKSUNIVERSITEIT GRONINGEN

Particle Dynamics of Branes

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
maandag 26 mei 2008
om 14:45 uur

door

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geboren op 1 juni 1979
te Steenwijk

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ISBN: 978-90-367-3389-2

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Chapter 1

Introduction

A long standing problem in physics is the unification of all known forces in nature. These are gravity and the strong, the weak and the electromagnetic force. The last three forces are combined in what is called the standard model of particle physics. This model has been developed by Salam, Glashow and Weinberg between 1970 and 1973. Combining the standard model with gravity is however a complicated issue. String theory is one of the most promising attempts to achieve this. The full mathematical structure of string theory is complicated and unfinished, only perturbatively the description of a string is by now well understood. A non-perturbatively description is still lacking.

As the name suggest, in string theory the fundamental objects are strings. These can be thought of as one-dimensional objects that trace out two-dimensional surfaces in time. Different vibrational modes of the string are associated to different particles and forces. One of these modes is the graviton. For this reason string theory is a candidate for unifying the known four forces in nature. The graviton belongs to the massless sector. The first massive states have masses around the Planck mass $\sim 10^{-5}$ grams or energy $\sim 10^{19}$GeV. This corresponds to lengths of about 10^{-33} centimeters. Since this is out of reach of today's accelerators, the main focus is on the massless sector.

One of the surprising things about string theory is that, although we start from a string, the massless spectrum gives rise to higher-dimensional objects. For example, think of a two-dimensional membrane. A second intriguing aspect of string theory is that it requires a ten-dimensional space-time instead of the four-dimensional universe we live in. It is therefore not surprising that we have even higher-dimensional objects than membranes, these are called branes. Branes will play a central role in this thesis. These objects tell us about non-perturbative aspects of string theory.

One way to learn more about string theory, is by looking at its low energy limit.

There is also a good physical reason to do this. Namely, the aim of string theory is to unify the standard model with gravity. However the results of experiments done so far in accelerators can be explained by the standard model. A new milestone in accelerator physics is the Large Hadron Collider (LHC) which will come online this summer at CERN. The energy scale of this accelerator is of the order of ten tera-electron-Volt (10^{12} eV). This corresponds to lengths of about 10^{-19} meters. Any deviation from the standard model possibly observed at LHC should then be explainable by the low energy limit of string theory.

The low energy limit of string theory leads to so-called supergravities. A supergravity is a classical theory which extends Einstein's theory of general relativity to fermions in such a way that bosonic degrees of freedom are related to fermionic degrees of freedom, the number of bosons equals the number of fermions. The low energy limit of string theory leads to five different supergravities, but these theories are related through a whole web of dualities. This suggests that each of these five theories are different limits of a single theory called M-theory. There is very little known about this theory.

We mentioned that string theory requires a ten-dimensional space-time. Our observable universe is only four-dimensional including time. Somehow we have to rationalize away six dimensions. The standard way is via dimensional reduction. With this we mean that we assume that the ten-dimensional universe can be considered as a direct-product space of our four-dimensional universe and a compact six-dimensional space which is of small size. The extra dimensions need to be smaller than 10^{-18} meters else we would have observed them by now[1].

Let us get back to the branes. In this thesis we are going to study branes that are solutions of the two so-called type II supergravities that follow from considering the low energy limit of string theory. The dimensions of the extended object form the worldvolume of the brane. For example, in case of the string we would have a two-dimensional worldvolume consisting out of time and one spatial direction. The other space-time dimensions form the transverse space. The main focus will be on two types of branes. If time is part of the worldvolume the brane is called a (timelike) p-brane. The p stands for the number of spatial directions of the worldvolume. In total we have $p + 1$ dimensions. If time is not part of the worldvolume it is called an (spacelike) Sp-brane. Here p stands for the number of spatial worldvolume directions minus one. In this way p refers in both cases to a $(p + 1)$-dimensional worldvolume.

To investigate these brane solutions we could try to solve the equations of motion that follow from the action of the supergravity directly. This is however not the way we are going to proceed. As the title of the thesis suggest, we are going to look at

[1]There is however a different string theory scenario where not all extra dimensions have to be small. This is the so-called brane world or Randall-Sundrum scenario [1, 2]. In this model all interactions except gravity are restricted to a four dimensional hyperplane, which represents our universe. This model has the advantage that, since gravity is spread over the whole space-time, it gives an explanation as to why the gravitational force is weak compared to the other three forces.

branes from a particle point of view. With this we mean that we are going to look at brane solutions whose dynamics depends only on one parameter, just like particles do. This parameter will be related to a coordinate of the transverse space. Because of this the worldvolume coordinates do not appear explicitly in the solutions. In this way we see that the worldvolume directions do not really matter. For this reason we first reduce over the worldvolume via dimensional reduction and then try to solve the remaining lower-dimensional equations of motion. This is the first step in reducing the problem of finding brane solutions. Since we have reduced over the worldvolume the theory is a (p=-1)-brane, such that we indeed have a zero-dimensional worldvolume. If time is part of the reduced worldvolume, the lower-dimensional theory lives in a Euclidean space-time. Such a solution is called a (-1)-brane or instanton. If time is not part of the reduced worldvolume, is it called an S(-1)-brane.

Alternatively, we will show that we can reduce over the transverse directions that are not related to the parameter describing the dynamics of the solutions. As it turns out, this way we can generate a potential in the lower-dimensional action, which we then call a massive theory. A theory is called massless if there is no potential. If the lower-dimensional theory requires a Minkowski space-time we have two different solutions. We call it a cosmology if the solution depends explicitly on time only. If it does not depend on time explicitly, we call it a domain-wall. This can be considered as the stationary version of a cosmology[2].

It is important to mention that we will only consider consistent reductions. With this we mean that we can always undo the steps of the reduction in such a way that we are guaranteed that we also have a solution of the action we started with. In this way we construct a solution of the higher-dimensional theory. This procedure is called uplifting or oxidation.

To solve the lower-dimensional equations of motion we have to make a difference between the massless and massive theories.

The massless case is the easiest. Due to the dimensional reduction, the lower-dimensional action will have a much bigger symmetry group than the action we started with (not including diffeomorphisms). This we can use to simplify our quest for brane solutions further. In solving the lower-dimensional equations of motion we will first see that we can decouple the scalar sector from the gravity sector. As as result, we solve the metric independently from the scalar fields. We will not need to look for the most general scalar field solutions. Instead we will look for much easier solutions, namely generating solutions. With this we mean that if we act with the symmetry group on this solution, we automatically find the most general solution possible.

In case we have a massive theory, a general solution is difficult to give due to the presence of the potential. There is no decoupling from the gravity sector. Instead we will show when we can write the second order equations in terms of first order

[2]In chapter 6 and 7 we consider a few examples where we have a Euclidean theory *with* a potential. We call these solutions instantons as well.

equations.

In this thesis we want to achieve the following. We are first going to show that p- and Sp-branes can be linked to lower-dimensional actions whose solutions are respectively given by instantons or S(-1)-branes if we reduce over the worldvolume of the brane. And similarly, if we reduce a p-brane or Sp-brane over all but one of its transverse directions we find a domain-wall or a cosmology. The main goal of this thesis is: *Derive the solutions that correspond to the lower-dimensional action.* In case the lower-dimensional theory is massless we look for the generating solution. For a massive theory the focus will be on re-writing the second order differential equations as first order equations. We will further see that a specific class of massive solutions behave as if there is no potential at all.

If we would uplift our solution back to the original theory we have constructed a general brane solution of the original theory. Along the way we will see that there are all kind of links between the lower- and higher-dimensional solutions.

The plan of the thesis is as follows. In the next chapter we begin with giving a short introduction to string theory. The focus will be on introducing the relevant concepts. In particular we will spend some time on the p- and Sp-branes.

In chapter 3 we are going to explain how one does a dimensional reduction. We will restrict to two different types of reductions relevant for branes. At the end of the chapter we explain how dimensional reduction can be applied to branes.

In chapter 4 we look for time-dependent Sp-branes via reducing over its worldvolume. This way we will obtain a massless theory. With the help of the generating solution and the symmetry group we will be able to construct the most general Sp-brane with deformed worldvolume.

In chapter 5 we are going to look at massive theories, i.e. cosmologies and domain-walls. Solving a theory with a potential is more complicated, but we will show that often the equations of motion can be written as first order equations. Furthermore we will show that under certain conditions the problem is basically the same as looking for solutions of a model without a potential.

There is a link between these two solutions, which is called the domain-wall / cosmology correspondence. The correspondence can be summarized roughly as stating that for every cosmology there exists also a domain-wall and *vice-versa*. In chapter 6 chapter we are going to see what happens if we put this correspondence in a supergravity setting. We will see that this leads to a new type of brane, the so-called Ep-brane. We point out a relation to Sp-branes.

All the solutions considered so far live in a Minkowski space-time. In chapter 7 we are going to consider solutions that require a Euclidean space-time i.e. instantons. In chapter 3 we show how such theories can be obtained from dimensional reduction of ordinary Lorentzian supergravities. The main focus will be on finding the generating Euclidean solutions.

The last chapter will be about conclusions and possible future research.

There are four appendices. Appendix A contains all the necessary conventions that we use for general relativity and differential geometry. Appendix B is concerned with spinors in arbitrary dimensions. This is used in chapter 6. In appendix C we give an overview of Lie groups and Lie algebras. In the last appendix we give the published papers.

Chapter 2

String Theory

In this chapter we begin with introducing the relativistic point particle and the free bosonic string. We then move to the superstring and focus on its low energy limit, obtaining supergravities. This allows us to introduce Dp- and Sp-brane solutions, which will play an important role in the coming chapters. We will not give many details, for this we refer to books and lecture notes such as [3–7].

2.1 Classical String Theory

Before discussing string theory, it is interesting to remind ourself how to describe a free relativistic particle of mass $m > 0$ in a Minkowski space-time, given by the D-dimensional flat metric $\eta_{\mu\nu} = (-1, 1, \ldots, 1)$ with line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \sum_{i=1}^{D-1}(\mathrm{d}x^i)^2 \,, \tag{2.1.1}$$

where c is the speed of light, t is the time coordinate and x^i are spatial coordinates. Since we are dealing with a free particle, we expect it to trace out a straight line in space-time. The action for such a particle is given by the shortest path

$$S = -mc\int_{\mathcal{P}} \sqrt{-\mathrm{d}s^2} = -mc\int_{\lambda_i}^{\lambda_f} \sqrt{-\eta_{\mu\nu}x'^\mu x'^\nu}\mathrm{d}\lambda \,, \tag{2.1.2}$$

where c is the speed of light, $x'^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for some parameter λ describing the curve $x^\mu(\lambda)$ and λ_i and λ_f are the values of λ at the initial and final points of the world-line $\mathcal{P}$. The presence of mc is dictated by requiring the right units for an action S. To see that this is the action we are after, we note that with the help of (2.1.1) and using $\lambda = t$ we can rewrite it as

$$S = -mc \int_{t_i}^{t_f} c\sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t\,, \tag{2.1.3}$$

where the velocity squared is given by $v^2 = \sum_{i=1}^{D-1}(\mathrm{d}x^i/\mathrm{d}t)^2$. This clearly shows that v is bounded by the speed of light c, in agreement with the special theory of relativity. Also note that a Taylor expansion around small v/c leads to the Lagrangian

$$\mathcal{L} = -mc^2 + \frac{1}{2}mv^2\,, \tag{2.1.4}$$

which is the expected expression for a non-relativistic free particle. From now on we take $c = 1$.

The action (2.1.2) can be extended to a curved space-time described by the metric $g_{\mu\nu}(x)$ via replacing the line element (2.1.1) with

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{2.1.5}$$

By extremizing the corresponding action and using for λ an affine parameter[1], we derive the equations of motion

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0\,, \tag{2.1.6}$$

with the Christoffel symbol $\Gamma^\mu_{\rho\nu}$ given by (A.2.2). This equation is called the geodesic equation. If we use $g_{\mu\nu} = \eta_{\mu\nu}$ we find the equations of motion for a free particle in Minkowski space-time. We can interpreted the action (2.1.2) as a map from the parameter space λ to an embedding in a D-dimensional space-time described by x^μ.

A string is the two-dimensional extension of this. Instead of the world-line we have a two-dimensional surface called the world-sheet Σ of the string. It is common to describe this world-sheet by the parameters (τ, σ). We then consider the mapping from the (τ, σ) world-sheet to the D-dimensional space-time described by X^μ [2].

What is the action describing a string in a Minkowski space-time? The particle is parameterized by λ, the "metric" induced on the one-dimensional world-line is given by

$$g_{\lambda\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\eta_{\mu\nu}\,, \tag{2.1.7}$$

[1] An affine parameter means that λ is related to $s = \int_{\mathcal{P}} \sqrt{-\mathrm{d}s^2}$ via $\lambda = as + b$ with $a, b \in \mathbb{R}$. This means that we parameterize the curve by the distance along the curve, $x^\mu = x^\mu(s)$.

[2] In string theory it is conventional to use capital letters for the embedding coordinates, i.e. X^μ instead of x^μ.

and we note that the determinant (Det) g of $g_{\lambda\lambda}$ appears in the integrand in (2.1.2). For the string we have the natural extension to the so-called Nambu-Goto action

$$S = -T \int \mathrm{d}^2\zeta \sqrt{-g}\,, \quad \text{with } g_{ij} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\zeta^i}\frac{\mathrm{d}X^\nu}{\mathrm{d}\zeta^j}\eta_{\mu\nu}\,, \tag{2.1.8}$$

where $\zeta^i = (\tau, \sigma)$ and $g = \mathrm{Det}(g_{ij})$. The fields X^μ are the coordinates of the string in the Minkowski space-time described by the flat metric $\eta_{\mu\nu}$. The tension T has units of force. It is the force which tries to pull the string together to a point and is therefore called the string tension. It is often rewritten as $T = 1/2\pi\alpha' = 1/l_s^2$. The parameter l_s is called the string length and introduces a fundamental scale in the theory. Instead of working with (2.1.8) it is more convenient to work with the alternative action

$$S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\zeta \sqrt{-h} h^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\,. \tag{2.1.9}$$

This is called the Howe-Tucker-Polyakov action [3]. Here h_{ij} is an independent metric on the worldvolume, independent of the induced metric g_{ij}. From the equation of motion for h_{ij} we obtain

$$g_{ij} = \frac{1}{2} h_{ij}(h^{ab} g_{ab})\,. \tag{2.1.10}$$

This can be used to show that (2.1.9) is classically equivalent to (2.1.8). From (2.1.9) it follows that h_{ij} allows for a conformal re-scaling symmetry

$$h'_{ij} = f(\zeta) h_{ij}\,, \tag{2.1.11}$$

with $f(\zeta)$ an arbitrary function of the world-sheet coordinates. This is called the Weyl re-scaling. The action (2.1.9) has two more symmetries. Namely general coordinate transformations on the world-sheet and global Poincaré transformations in D-dimensional space-time

$$X^\mu \rightarrow X^{'\mu} = \Lambda^\mu{}_\nu X^\nu + a^\mu\,, \tag{2.1.12}$$

where $\Lambda^\mu{}_\nu$ is an SO(1,$D-1$) matrix obeying $\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \eta_{\mu\nu} = \eta_{\kappa\lambda}$ and a^μ is a constant vector.

It is a well known result of two-dimensional geometry that a coordinate re-parametrization allows an arbitrary metric h_{ij} to be cast locally in a conformal flat metric

$$h_{ij} = \rho^2(\zeta)\eta_{ij}\,, \tag{2.1.13}$$

where $\eta = \mathrm{diag}(-1, 1)$ and ρ is called the conformal factor. The action (2.1.9) in this *conformal gauge* reduces to

$$S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\zeta \eta^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\,. \tag{2.1.14}$$

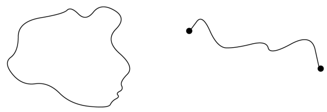


Figure 2.1.1: The figure on the left (right) represents a closed (open) string.

and (2.1.10) becomes in this gauge the constraint

$$\Big(\partial_\tau X^\mu \pm \partial_\sigma X^\mu\Big)^2 = 0\,, \tag{2.1.15}$$

where the square means a contraction with $\eta_{\mu\nu}$.

Now we are going to derive the equations of motion for X^μ. Given an initial and a final condition at τ_i and τ_f, we need to vary (2.1.14) with respect to X^μ, i.e. $\delta X^\mu(\tau_i,\sigma) = \delta X^\mu(\tau_f,\sigma) = 0$. We now have to make a difference between *open* and *closed* strings, see figure 2.1.1. As the name suggests, an open string has two end-points labeled by $\sigma = 0$ and $\sigma = l$. The variation of (2.1.14) leads to

$$\delta S = T\int_\Sigma \mathrm{d}^2\zeta \Box X^\mu \delta X_\mu + T\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \partial_\sigma X^\mu \delta X_\mu|_{\sigma=0}^{\sigma=l} = 0\,. \tag{2.1.16}$$

The first term on the right-hand side of the expression above leads to the well known wave equation

$$\Box X^\mu(\tau,\sigma) = \Big(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\Big) X^\mu(\tau,\sigma) = 0\,, \tag{2.1.17}$$

with the general solution

$$X^\mu(\tau,\sigma) = X^\mu_-(\tau-\sigma) + X^\mu_+(\tau+\sigma)\,. \tag{2.1.18}$$

The subscript $-$ $(+)$ stand for right (left) moving modes on the string. There are two ways to make the second term on the right-hand side of (2.1.16) zero.

First we can choose to work with so-called *Neumann* boundary conditions. These are specified by

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,l) = 0\,. \tag{2.1.19}$$

The end-points of the strings can move freely. Alternatively we can choose to keep (some of) the end-points of the string fixed. For this we have to restrict the variations to

$$\delta X^\mu(\tau,0) = \delta X^\mu(\tau,l) = 0 \to X^\mu(\tau, 0 \text{ or } l) = \text{constant}\,. \tag{2.1.20}$$

These are called *Dirichlet* boundary conditions. Because the string is fixed in the directions where these Dirichlet conditions are applied, momentum cannot be conserved in these directions. Therefore these boundary conditions imply that the open string has to couple to a dynamical object which is called a D-brane. The name comes from the Dirichlet boundary condition and the word brane generalizes the concept of a membrane. D-branes are an important class of extended objects in string theory and have played a crucial role in understanding the non-perturbative structure of string theory.

Extended objects are in general called p-branes. Here p stands for the number of spatial directions of the extended object. The free relativistic particle we discussed before is in this language a 0-brane and the string a 1-brane. An open string that has both endpoints confined to the same Dp-brane satisfies Neumann conditions in the $(p+1)$ directions which make up the worldvolume of the brane. Note that time is considered part of this worldvolume. The $(D-p-1)$ Dirichlet conditions are transverse to this plane. An exception is the D(-1)-brane or D-instanton. This brane lives in a Euclidean background, the time coordinate has been replaced by a spatial direction and all these spatial directions are transverse to the brane. If we instead take for the time coordinate a Dirichlet condition the brane is called an Sp-brane, which has by definition a Euclidean $(p+1)$-dimensional worldvolume and time is one of the transversal coordinates [8]. This means that the time coordinate X^0 obeys a Dirichlet condition. The S stands for the spacelike worldvolume. In section 2.4 we will discuss both p- and Sp-branes.

Finally for the *closed* strings the same variation as given in (2.1.16) holds. The difference is that the end-points do not exists and we have to demand a periodic boundary condition specified by

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+l) \quad \forall \quad \mu\,. \tag{2.1.21}$$

2.2 Quantization of the Bosonic String and Curved Backgrounds

So far we have been working with classical bosonic strings in a Minkowski background $g_{\mu\nu} = \eta_{\mu\nu}$. In this background the theory can be quantized exactly. The first noteworthy feature is that to regain Lorentz covariance the space-time dimension needs to be $D = 26$. The oscillation modes for the open string lead to the following mass spectrum

- the vacuum with mass squared $M^2 = -\hbar/\alpha'$, corresponding to the tachyonic scalar T,
- the first excited state with $M^2 = 0$, corresponding to a massless vector A_μ,
- an infinite tower of massive modes.

The tachyonic particle has $M^2 < 0$, which leads to an instability of the theory[3]. The mass gap between each subsequent mass level is $\hbar/\alpha'$.

The first two closed string spectrum levels are

- the vacuum with mass squared $M^2 = -4\hbar/\alpha'$, corresponding to a tachyonic scalar T,

- the first excited state with $M^2 = 0$, consisting out of a symmetric traceless field $g_{\mu\nu}$, an anti-symmetric 2-form field $B_{\mu\nu}$ and a scalar field ϕ called the dilaton.

The field mediating the gravitational force is identified with the symmetric traceless tensor $g_{\mu\nu}$. This identification follows from the fact that the degrees of freedom of a classical D-dimensional gravitational field is carried by a symmetric, traceless tensor field with number of independent components $1/2\ D(D-3)$. The closed string spectrum shows that gravity is part of the quantized closed bosonic string. For this reason it is believed that string theory could form the basis of a theory of quantum gravity.

So far we have only considered non-interacting strings, moving in a flat Minkowski background. Just as we did for the particle, we now want to extend this to a more general background $g_{\mu\nu}(X)$. A possible starting point is the Howe-Tucker-Polyakov action (2.1.9), but now with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$. We can think of this string moving in a coherent background of gravitons [3]. We have seen that the graviton is itself an excited state of the string, so we can generalize this by also turning on backgrounds for the other two massless fields appearing in the closed string spectrum. Therefore we consider the closed string in a background consisting out of the massless states ϕ, $g_{\mu\nu}$ and $B_{\mu\nu}$. The action can be obtained in the following way. We assume at most two world-sheet derivatives and we extend the symmetries of the action (2.1.9) to this case. That means that we require general covariance on the world-sheet and in the target space, as well as local Weyl invariance. It turns out that the following action

[3]The instability can be understood as follows. Consider a scalar field ϕ with mass M^2 that depends only on the time t. The equation of motion of such a particle with potential $V = \frac{1}{2}M^2\phi^2$ is given by

$$\frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} + M^2\phi(t) = 0\,. \tag{2.2.1}$$

In case $M^2 > 0$ we see that the solution for ϕ is given by $\phi(t) = \phi_0 \sin(Mt + \alpha)$ with ϕ_0, α two integration constants. The scalar field can "sit" at $\phi = 0$ forever since it is a stable point. If on the other hand $M^2 < 0$ we see that the general solution is given by $\phi(t) = A\cosh(mt) + B\sinh(mt)$ with m given by $M^2 = -m^2$ and A, B two constants of integration. It is clear that for $A = 0$ and $|t| \to \infty$ the scalar field $|\phi|$ blows up. This time $\phi = 0$ is a maximum of the theory. So if the scalar field sits at $\phi = 0$, a small perturbation $\delta\phi$ will cause the field to start rolling down the potential [6].

will do[4]

$$S = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\zeta\sqrt{-h}\Big(h^{ij}g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu - \epsilon^{ij}B_{\mu\nu}\partial_i X^\mu \partial_j X^\nu - \alpha'\phi\mathcal{R}(h)\Big)\,, \qquad (2.2.2)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the world-sheet metric h_{ij}. This action is an example of a non-linear σ-model. A non-linear σ-model is a scalar field field theory in which the scalar fields take on values in some non-trivial manifold M. The last term in (2.2.2) plays a special role. Assume that we have a constant mode of the dilaton ϕ_0. The Gauss-Bonnet theorem [9] states that

$$\chi = \frac{1}{4\pi}\int_{\Sigma} \mathrm{d}^2\zeta\sqrt{-h}\mathcal{R}(h) = 2(1-g)\,, \qquad (2.2.3)$$

where the genus g is the number of handles of the world-sheet. One can now calculate scattering amplitudes between different string modes via the string path integral based on the action (2.2.2). From (2.2.3) we see that the amplitude of a string diagram of genus g is multiplied by $(\mathrm{e}^{\phi_0})^{2g-2}$. As a consequence every interaction will have an associated string coupling constant g_s given by the expectation value of e^{ϕ_0}. A world-sheet with genus g can therefore be seen as the g-th loop correction to string theory.

The scattering amplitude for the massless modes can be summarized by an effective action. For the closed bosonic string it is given by [10]

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\sqrt{-g}\,\mathrm{e}^{-2\phi}\Big(\mathcal{R}(g) + 4(\partial\phi)^2 - \frac{1}{2(3!)}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\,, \qquad (2.2.4)$$

where κ_0 is related to the 26-dimensional Newton's constant G_{26} via $\kappa = \kappa_0 g_s = \sqrt{8\pi G_{26}}$ and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$. Note that this is not the standard Einstein-Hilbert action as given in appendix A.4. The Ricci scalar is coupled to the dilaton in a specific way. This defines the so-called string frame $g^{(S)}_{\mu\nu}$. Via the conformal mapping $g^{(S)}_{\mu\nu} = \mathrm{e}^{\phi/2}g^{(E)}_{\mu\nu}$ we find the action in the Einstein frame (E)

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\sqrt{g^{(E)}}\Big(\mathcal{R}(g^E) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(3!)}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\,. \qquad (2.2.5)$$

We thus see that the bosonic string action leads to the effective 26-dimensional action (2.2.5). In the next section we are going to include world-sheet fermions and obtain effective actions via the method mentioned here.

[4]At the quantum level Weyl invariance is broken by an anomaly and the last term in the action (2.2.2) is needed to restore the symmetry.

2.3 Superstrings and Supergravities

The free bosonic string of the previous section has two drawbacks. First there are the tachyons signaling an instability. Secondly, there are no fermions present in the theory. To fix this we add fermions to the world-sheet and we will see that this also solves the tachyon problem. In this section we will work in the conformal gauge.

The standard way to proceed is by adding D world-sheet fermions $\psi^\mu = (\psi^\mu_+,\, \psi^\mu_-)$ to the Howe-Tucker-Polyakov action (2.1.9) via the term

$$S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\zeta\Big(\eta^{ij}\partial_i X^\mu \partial_j X_\mu + i\bar{\psi}^\mu\gamma^j\partial_j\psi_\mu\Big)\,, \tag{2.3.1}$$

where γ^j are the Γ-matrices in two dimensions

$$\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\,, \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \tag{2.3.2}$$

satisfying $\{\gamma_i, \gamma_j\} = \eta_{ij}$, see appendix B for more details. The action (2.3.1) is invariant under the following world-sheet *supersymmetry* transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu\,,\;\; \delta\psi^\mu = -i(\partial\!\!\!/ X^\mu)\epsilon\,, \tag{2.3.3}$$

where $\epsilon = (\epsilon_+\,, \epsilon_-)$ is a constant spinor. For the free theory we are considering in this section we see that the bosonic and fermionic sector decouple. We have the same bosonic solution (2.1.18). A variation with respect to $\bar{\psi}^\mu$ gives the following equations of motion

$$(\partial_\tau \mp \partial_\sigma)\psi^\mu_\pm = 0\,, \tag{2.3.4}$$

together with the boundary condition

$$\Big(\psi^\mu_+\delta\,\psi_{+\mu} - \psi^\mu_-\delta\,\psi_{-\mu}\Big)\Big|_{\sigma=0}^{\sigma=l} = 0\,. \tag{2.3.5}$$

From (2.3.4) we see that the most general solution is given by $\psi^\mu_\pm(\tau \pm \sigma)$, where ψ_+ (ψ_-) is called the left (right) mover.

Let us first focus on the open strings. Then (2.3.5) is satisfied if $\psi^\mu_+ = \pm\psi^\mu_-$ and $\delta\psi^\mu_+ = \pm\delta\psi^\mu_-$. Since an overall sign in the boundary conditions is irrelevant, we can set $\psi^\mu_+(\sigma = 0) = \psi^\mu_+(\sigma = l)$. We find the following two possibilities for an open string

$$\begin{aligned} \text{Ramond (R)}:\;\; \psi^\mu_+(l,\tau) &= \psi^\mu_-(l,\tau)\,, \\ \text{Neveu-Schwarz (NS)}:\;\; \psi^\mu_+(l,\tau) &= -\psi^\mu_-(l,\tau)\,. \end{aligned} \tag{2.3.6}$$

For the closed string we have the periodic identification for σ. This means we can impose (anti)-periodicity for the left- and right-moving component $\psi^\mu_\pm$ separately

$$\begin{aligned} \text{Ramond (R)}:\;\; \psi^\mu_\pm(0,\tau) &= \psi^\mu_\pm(l,\tau)\,, \\ \text{Neveu-Schwarz (NS)}:\;\; \psi^\mu_\pm(0,\tau) &= -\psi^\mu_\pm(l,\tau)\,. \end{aligned} \tag{2.3.7}$$

As a result we get four differen sectors for the closed string: R-R, NS-NS, R-NS and NS-R.

We can now quantize the free superstring. To regain Lorentz covariance we find that we need to require $D = 10$. To have space-time supersymmetry we apply the Gliozzi-Scherk-Olive (GSO) projection. This basically truncates the states that do not have a counterpart in the other sector. This projection also eliminates the tachyonic ground state from the spectrum. Due to the space-time supersymmetry we refer to this theory as the superstring. Choosing several combinations for the boundary conditions in the open and closed string case leads to five differen supersymmetric string theories. Namely type IIA, type IIB, Type I and heterotic $E_8 \times E_8$ and heterotic SO(32). The type IIA and type IIB are closed string theories, containing $\mathcal{N} = 2$ space-time supersymmetry. In type IIB the supersymmetry parameters have the same chirality, in type IIA they are opposite. Type I is the only open string theory and has $\mathcal{N} = 1$ supersymmetry. The heterotic theories also have $\mathcal{N} = 1$ supersymmetry, they differ in their gauge group.

2.3.1 Supergravities

In section 2.2 we mentioned that we can obtain a 26-dimensional effective action for the free bosonic string. This method can similarly be applied to the five superstring theories. As it turns out these theories are *supergravities*. This means that the symmetries of such a theory combine general coordinate transformations and local supersymmetry. That is the spinor ϵ depends on the space-time coordinates. In this section we will only write down the bosonic sector (in the string frame). In chapter 6 we will make use of the fact that these are local supersymmetric theories.

Type IIA

The action is given by the following expression

$$\begin{aligned} S_{\rm IIA} \;\; = \;\; & -\frac{1}{2\kappa_{10}^2}\int {\rm d}^{10}x\,\sqrt{|g|}\Big\{{\rm e}^{-2\phi}\Big[-\mathcal{R}-4\big(\partial\phi\big)^2+\tfrac{1}{2}H\cdot H\Big] \\ & +\tfrac{1}{2}\sum_{n=1}^{2}G^{(2n)}\cdot G^{(2n)}-\frac{1}{\sqrt{|g|}}\mathcal{L}_{\rm CS}\Big\}\,, \end{aligned} \tag{2.3.8}$$

where

$$\mathcal{L}_{\rm CS} = -\tfrac{1}{4\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}}\,, \tag{2.3.9}$$

and we have the following expressions for the field strengths

$$H = {\rm d}B\,,\quad G^{(2)} = {\rm d}C^{(1)}\,,\quad G^{(4)} = {\rm d}C^{(3)} - H^{(3)}\wedge C^{(1)}\,. \tag{2.3.10}$$

Here κ_0^2 is related to the physical coupling κ^2 via

$$\frac{1}{2\kappa_0^2} = \frac{\mathrm{e}^{2\phi_0}}{2\kappa^2} \,. \tag{2.3.11}$$

Type IIB

The action is given by the following expression

$$\begin{aligned} S_{\mathrm{IIB}} \quad = \quad & -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\Big\{\mathrm{e}^{-2\phi}\Big[-\mathcal{R}-4\big(\partial\phi\big)^2+\tfrac{1}{2}H\cdot H\Big] \\ & +\tfrac{1}{2}\sum_{n=1/2}^{3/2} G^{(2n)}\cdot G^{(2n)}+\frac{1}{4}G^{(5)}\cdot G^{(5)}-\frac{1}{\sqrt{|g|}}\mathcal{L}_{\mathrm{CS}}\Big\}\,, \end{aligned} \tag{2.3.12}$$

where

$$\mathcal{L}_{\mathrm{CS}} \quad = \quad -\frac{1}{3\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}C^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_5}C^{(2)}_{\mu_6\mu_7}\partial_{\mu_8}B_{\mu_9\mu_{10}}\,. \tag{2.3.13}$$

The scalar $C^{(0)}$ is called the axion and we have the following expressions for the field strengths

$$G^{(1)} = \mathrm{d}C^{(0)}\,, \quad G^{(3)} = \mathrm{d}C^{(2)} - H^{(3)}C^{(0)}\,. \tag{2.3.14}$$

To get the right number of degrees of freedom, we must impose that the five-form field strength is self-dual

$$G^{(5)} = *G^{(5)}\,. \tag{2.3.15}$$

This constraint is added to the equations of motion. We will not write down the other three $\mathcal{N}=1$ supergravities resulting from type I and the two heterotic string theories since we will not make use of them. They can be found in e.g. [11].

11d Supergravity

Although Lorentz covariance requires superstrings to live in $D = 10$, there does exist a supergravity in $D = 11$ [12]. Its bosonic action is given by

$$\begin{aligned} S = & -\frac{1}{4\kappa_{11}^2}\int \mathrm{d}^{11}x\,\sqrt{|g|}\Big[-\mathcal{R}+\frac{1}{2}G^{(4)}\cdot G^{(4)}\Big] \\ & -\frac{1}{4\kappa_{11}^2}\int \mathrm{d}^{11}x\,\frac{1}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\,, \end{aligned} \tag{2.3.16}$$

we see that it has a 3-form gauge potential related to $G_{\mu\nu\rho\sigma} = 4\,\partial_{[\mu}\,C_{\nu\rho\sigma]}$. It is a theory with $\mathcal{N}=1$ supersymmetry.

2.3.2 T-duality

Let us go back to the bosonic string in 26 dimensions. We assume that the 25th coordinate X^{25} has the topology of a circle S^1 with radius R. This can be achieved by imposing that all points along this direction are identified if they differ by $2\pi R$.

For the closed string we have to modify the boundary condition (2.1.21) as

$$X^{25}(\tau, \sigma + l) = X^{25}(\tau, \sigma) + 2\pi R m\,, \tag{2.3.17}$$

where the integer m now indicates how many times the closed string is wrapped around the circle. This leads to quantized momentum along this direction

$$p^{25} = \frac{k}{R}\,, \tag{2.3.18}$$

with k an integer. This follows from $\mathrm{e}^{ip^{25}X^{25}}$ together with the boundary condition (2.3.17). The mass spectrum is given by

$$M^2 \propto \left(\frac{k^2}{R^2} + \frac{m^2 R^2}{\alpha'^2}\right). \tag{2.3.19}$$

It is invariant under the inversion of the radius with a simultaneous interchange of k with m

$$R \to \frac{\alpha'}{R}\,, \quad k \to m\,. \tag{2.3.20}$$

This transformation is called T-duality.

The surprising thing is that if we apply T-duality to type IIA string theory we will end up with type IIB string theory. This can be shown for example by noting that when both type II supergravities are reduced to nine dimensions the same action appears. This winding is a stringy effect, in field theories particles cannot wrap around a compact dimension.

2.3.3 S-duality

This type of duality is a strong-weak duality. Let us show this for type IIB supergravity. We combine the dilaton ϕ and axion $C^{(0)}$ in a complex scalar τ via

$$\tau = C^{(0)} + ie^{-\phi}\,. \tag{2.3.21}$$

The scalar part of (2.3.12) containing ϕ and $C^{(0)}$ can be written as

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{2\phi}(\partial C^{(0)})^2 = -\frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2}\,, \tag{2.3.22}$$

where τ_2 is the imaginary part of τ. Note that we are working in the Einstein frame. Define now the following fractional linear transformation on τ

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad \text{with } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\,. \tag{2.3.23}$$

We can group these numbers in a SL(2, $\mathbb{R}$) transformation Λ

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{2.3.24}$$

Under this SL(2, $\mathbb{R}$) transformation the full type IIB supergravity is invariant if the two 2-form potentials transform as a doublet

$$\begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix} \to \Lambda \begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix}\,, \tag{2.3.25}$$

while the 4-form transforms as a singlet. Assume that we have a background in which $C^{(0)}$ vanishes. An S-duality transformation is the specific SL(2, $\mathbb{R}$) transformation with $a = d = 0$ and $b = -c = 1$ such that

$$\phi \to -\phi\,, \quad C^{(2)} \to B^{(2)}\,, \quad B^{(2)} \to -C^{(2)}\,. \tag{2.3.26}$$

This means that the string coupling g_s goes to $1/g_s$. If g_s is small initially, (2.3.26) maps the theory from weak to strong coupling.

The five string theories we mentioned above turn out to be related to one another by dualities such as T- and S-duality, see for example [11]. This suggest that these five theories represent various limits of one single fundamental theory, called M-theory. The idea is that the 11d supergravity (2.3.16) is one of the low energy approximation of M-theory.

2.4 Brane Solutions

In this section we will discuss two different types of brane solutions belonging to type II supergravities. First we discuss time-independent p-branes, after that we will look at time-dependent Sp-branes.

2.4.1 p-branes

In section 2.1 we have introduced D-branes as objects arising due to the Dirichlet boundary conditions applied to open strings. We will now focus on the type II supergravity actions and show that the Dp-branes are a special class of p-branes.

In the previous section we have seen that both type II theories and 11d supergravity contain higher rank gauge fields $C^{(n+1)}$

$$\begin{aligned} \text{IIA} \quad &: \quad \left\{C^{(1)}_{\mu}, C^{(3)}_{\mu_1\mu_2\mu_3}\right\} , \\ \text{IIB} \quad &: \quad \left\{C^{(0)}, C^{(2)}_{\mu_1\mu_2}, C^{(4)}_{\mu_1\cdots\mu_4}\right\} , \\ \text{11d} \quad &: \quad \left\{C^{(3)}_{\mu_1\mu_2\mu_3}\right\} . \end{aligned} \tag{2.4.1}$$

For a charged 0-brane (or a point particle) we know that the coupling to a one-form gauge field $A^{(1)}$ is of the form

$$S = -mc\int_{\mathcal{P}} \mathrm{d}s + q\int_{\mathcal{P}} A^{(1)} - \frac{1}{2}\int_{\mathcal{M}} *\mathrm{d}A^{(1)} \wedge \mathrm{d}A^{(1)} , \tag{2.4.2}$$

where q is the electric charge of the 0-brane, $A^{(1)} = A_\mu \mathrm{d}x^\mu$ the one-form gauge field, $\mathcal{M}$ the space-time manifold and $\mathcal{P}$ the world-line. We see that the 0-brane couples naturally to a one-form gauge field $A^{(1)}$.

The existence of the higher rank $C^{(n+1)}$ gauge field suggests a coupling to a higher-dimensional object, namely a $(p = n)$-brane instead of a 0-brane. In section 2.1 we mentioned that a p-brane is a $(p+1)$-dimensional object in space-time. The coupling of a $(n+1)$-rank gauge field to a $(p = n)$-brane generalizes to

$$T\int_{\Sigma} \mathrm{d}^{n+1}\zeta \frac{1}{(n+1)!}\partial_{a_1}X^{\mu_1}\ldots\partial_{a_{n+1}}X^{\mu_{n+1}}A_{\mu_1\ldots\mu_{n+1}}\varepsilon^{a_1\ldots a_{n+1}} . \tag{2.4.3}$$

Here T is called the brane tension and Σ is the $(n+1)$-dimensional worldvolume of the brane. The expression (2.4.3) is called the Wess-Zumino (WZ) term[5].

From the type II supergravities (in the Einstein frame) it is clear that there are in general couplings between the field strengths and the dilaton. The action for a p-branes is given by

$$S = \frac{1}{2\kappa^2}\int\Big(*\mathcal{R} - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{a\phi} * \mathrm{d}A_{n+1} \wedge \mathrm{d}A_{n+1}\Big) , \tag{2.4.4}$$

where we allow for an arbitrary dilaton coupling parameter a. The equations of motion together with the Bianchi identity are given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{n+1}{2(D-2)((n+2)!)}g_{\mu\nu}\mathrm{e}^{a\phi}F^2_{n+2} + \frac{1}{(n+1)!2}\mathrm{e}^{a\phi}(F^2_{n+2})_{\mu\nu} , \tag{2.4.5}$$

$$\mathrm{d}\Big(*\mathrm{e}^{a\phi}F_{n+2}\Big) = (-)^{(D-n)}2\kappa^2 * J^{(n+1)} , \quad \mathrm{d}F_{n+2} = 0 , \tag{2.4.6}$$

[5] Besides the Wess-Zumino term, we can also add the D-brane low energy effective worldvolume action. This term is called the Dirac Born-Infeld action [13].

$$\Box\phi = \frac{a}{(n+2)!2}F_{n+2}^2 e^{a\phi}\,, \tag{2.4.7}$$

where $J^{(n+1)}$ follows from the variation of the WZ-term (2.4.3), $\Box$ is given by (A.2.8) and F_{n+2}^2 and $(F_{n+2}^2)_{\mu\nu}$ are given by (A.4.9).

From (2.4.6) we see that $\mathrm{d} * J^{(n+1)} = 0$, so we define the electric charge Q_e as

$$Q_e = \frac{1}{\sqrt{2\kappa^2}}\int_{S^{D-n-2}} * e^{a\phi}F^{(n+2)}\,, \tag{2.4.8}$$

where S^{D-n-2} is the higher-dimensional sphere surrounding the brane.

We will ignore the WZ-term and focus on the bulk action (2.4.4). This action has an electric/magnetic duality. To see this we define the dual field strength $\tilde{F}^{(D-n-2)}$ via Hodge duality $\tilde{F}^{(D-n-2)} = * e^{a\phi}F^{(n+2)}$. The equations of motion (2.4.5-2.4.7) are invariant under the following "duality transformations"

$$a\phi \to -a\phi\,, \quad (n+2) \to (D-n-2)\,, \quad F^{(n+2)} \to \tilde{F}^{(D-n-2)}\,. \tag{2.4.9}$$

Under this duality the Bianchi identity and equation of motion for $F^{(n+2)}$ swap their role. This means that there exists also a magnetic solution with charge

$$Q_m = \frac{1}{\sqrt{2\kappa^2}}\int_{S^{n+2}} F^{(n+2)}\,. \tag{2.4.10}$$

The action (2.4.3) therefore has both an electric solution with a $(n+1)$-dimensional worldvolume and a magnetic solution with a $(D-n-3)$-dimensional worldvolume. Let us present these solutions in some detail.

We denote the worldvolume coordinates of an arbitrary p-brane by x^i with $i = 0, 1\ldots p$ and the coordinates of the space transverse to the brane by y^a with $a = p+1, \ldots D-1$. We assume that the worldvolume has Poincaré symmetry $\mathrm{ISO}(1,p)$ and the transverse space $\mathrm{SO}(D-p-1)$. The following Ansätze will do

$$\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}x^i\mathrm{d}x^j\eta_{ij} + e^{2B(r)}\mathrm{d}y^a\mathrm{d}y^b\delta_{ab}\,, \quad \phi(r)\,, \tag{2.4.11}$$

with $r = \sqrt{y^a y^b \delta_{ab}}$ and A, B are arbitrary functions. As discussed above, there are now two different solutions, namely an electric $(p = n)$-brane or a magnetic $(p = (D-n-4))$-brane. Solving the equations of motion gives the following metric

$$\mathrm{d}s^2 = h^{\frac{-4(D-p-3)}{(D-2)\Delta}}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j + h^{\frac{4(p+1)}{(D-2)\Delta}}\delta_{ab}\mathrm{d}y^a\mathrm{d}y^b\,, \tag{2.4.12}$$

where the parameter Δ is given by [14]

$$\Delta = a^2 + 2\frac{(p+1)(D-p-3)}{D-2}\,. \tag{2.4.13}$$

The electric ($p = n$)-brane is given by [14, 15]

$$e^{\phi} = h^{2a/\Delta}\,, \quad F^{(n+2)} = \frac{2}{\sqrt{\Delta}} \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^n \wedge \mathrm{d}h(r)^{-1}\,. \tag{2.4.14}$$

For the magnetic solution we prefer to work with transversal coordinates given by

$$\delta_{ab}\mathrm{d}y^a \mathrm{d}y^b = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{D-p-2}\,. \tag{2.4.15}$$

Here $\mathrm{d}\Omega^2_m$ is the metric on the S^m sphere see (A.2.15). The magnetic ($p = (D-n-4)$)-brane is described by

$$e^{\phi} = h^{-2a/\Delta}\,, \quad F^{(n+2)} = Q\frac{2}{\sqrt{\Delta}}\sqrt{g_{S^{(n+2)}}}\mathrm{d}\theta^1 \wedge \ldots \wedge \mathrm{d}\theta^{n+2}\,, \tag{2.4.16}$$

where $g_{S^{(n+2)}}$ is the determinant of the metric on $S^{(n+2)}$. The function $h(r)$ is the harmonic function of the transverse space. For $p < D - 3$ we have

$$h = 1 + \frac{Q}{(D-p-3)r^{D-p-3}}. \tag{2.4.17}$$

In both cases Q is related to either the electric or magnetic charge of the brane. Note that the metric approaches Minkowski space-time when r goes to infinity. For the special case that $p = (D - 3)$ we have a logarithmic harmonic function, while if $p = (D - 2)$ we have a linear harmonic function.

It can be shown that the electric Q_e and magnetic charge Q_m satisfy a Dirac quantization condition [11, 16, 17]

$$Q_e Q_m = 2\pi n\,, \quad n = \text{integer}\,. \tag{2.4.18}$$

This is the generalization of Dirac's quantization for electric and magnetic monopoles. The $\mathrm{SL}(2, \mathbb{R})$ symmetry we mentioned earlier (2.3.23), is broken down to $\mathrm{SL}(2, \mathbb{Z})$ in the quantum theory due to this charge quantization.

2.4.2 Dp-branes

So far we have discussed general p-branes. To make contact to the type II supergravities we choose $D = 10$ and $a = (3 - p)/2$ [6]. We defined Dp-branes in section 2.1 as hyperplanes on which open strings can end. They are a special class of p-brane solutions with a coupling to the RR-potentials and satisfy Dirichlet boundary conditions along their transversal spacelike coordinates[7]. The above discussion of p-branes

[6] Note that this is a consistent truncation of the type II supergravities, with this we mean that any solution of the truncated theory is also a solution of the full theory.

[7] A different Dp-brane picture is given in [18]. Here one considers a stable Dp-brane as a tachyonic kink solution on the worldvolume of an unstable D$(p + 1)$-brane.

showed us that a $(n+1)$-gauge potential leads to a $(p=n)$- or a $(p=(D-n-4))$-brane. From (2.3.8) we see that in type IIA we have odd-form gauge potentials. These give rise to D0-, D2-, D4- and D6-branes. In type IIB there are D1-, D3-, D5- and D7-branes coupled to even-form potentials. The D3-brane is special in that its dual is the D3-brane itself. It is a dyonic solution, which means that it carries both electric and magnetic charge. The field strength should be self-dual, see (2.3.15)[8].

There are two special branes that play an important role in this thesis. First there are the $p=(D-2)$-branes. These are called domain-walls, since the brane has only one transverse direction, separating space-time into two regions. The corresponding field strength is a zero form. Such a term can for example be added to type IIA supergravity, obtaining massive IIA [19].

From the field content of type IIB we see that there is also an axion present, this leads to a D$(p=-1)$-brane or the D(-1)-instanton. As the name suggest, we have a zero-dimensional worldvolume and all the transversal directions are spatial. This branes lives in a Euclidean background and is dual to the D7-brane.

Finally note that in both type II theories there is also the NS-NS 2-form $B_{\mu\nu}$. This couples to the fundamental F1-string and its dual the NS5-brane.

One of the successes of the D-branes is the Maldacena conjecture [20–22] or AdS/CFT correspondence. This correspondence arises from considering the near horizon limit of a N D3-branes in which we consider the region close to $r=0$, where the metric has geometry $\mathrm{AdS}_5 \times \mathrm{S}^5$. Here AdS_5 is a five-dimensional Anti-de Sitter space and S^5 a five sphere. On the other hand, far away from the D3-brane we have free bulk supergravity. From the D3-brane action perspective, the dynamics far away from the brane is also free bulk supergravity. However near the brane we have a supersymmetric SU(N) gauge theory. The conjecture is that $\mathcal{N}=4$ SU(N) super-Yang-Mills theory in 3+1 dimensions is the same as (or dual to) type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [22].

All these branes can be shown to preserve half of their supersymmetries. Such solutions have the property that they fulfill some first-order differential equations which arise from demanding that the fermion supersymmetry transformations vanish. These first-order equations are now referred to as Bogomol'nyi or BPS equations, named after Bogomol'nyi [23] and Prasad and Sommerfield [24]. Witten and Olive showed in [25] the relation to preserved supersymmetry of solitons in supersymmetric theories. Nowadays the term BPS–equation is used for first order equations of motion that are found by rewriting the action as a sum of squares. In general supersymmetric solutions belong to this class. Stationary non-extremal (see below) and time-dependent solutions cannot preserve supersymmetry in ordinary supergravity theories. Nonetheless, we will later see that these solutions often can be found from first–order equations. Even more, we will see that a time-dependent solution sometimes does preserve supersymmetry if we embed it in a so-called star supergravity [26].

[8] This can be obtained from our solution if we replace $F_5 \to \frac{1}{2}(F_5 + *F_5)$.

Such a theory is closely linked to the supergravity theories of subsection 2.3.1.

In the literature these supersymmetric branes are also called extremal. The word extremal comes from the fact that the branes obey a relation between the mass and the charge of the D-branes. To be precise, when the mass equals the charge a brane is called extremal [27,28][9]. When this is not the case the brane is called non-extremal.

Non-extremal p-branes

There are two standard types of deformations of the extremal p-brane. In the literature these are called type 1 and type 2 non-extremal p-branes [30].

For type 1 deformations the form of the D-dimensional metric Ansatz remains the same as in the extremal case, namely

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\mathrm{d}x^i\mathrm{d}x^j\eta_{ij} + \mathrm{e}^{2B}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2})\,, \tag{2.4.19}$$

where A and B are functions of r only. For the extremal case we discussed above these two functions are related via

$$X = (p+1)A + (D-p-3)B = 0\,. \tag{2.4.20}$$

For the type II supergravity D-branes this relation follows from the requirement that the brane solutions preserve some unbroken supersymmetry see e.g. [15, 27]. For the non-extremal type 1 deformations we have $X \neq 0$.

The type 2 deformation begins from a modified form for the metric Ansatz [31,32], namely

$$\mathrm{d}s^2 = \mathrm{e}^{2A}(-\mathrm{e}^{2f}\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^j\delta_{ij}) + \mathrm{e}^{2B}(\mathrm{e}^{-2f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2})\,. \tag{2.4.21}$$

Here f is a function of r only and in this case the relation $X = 0$ still holds.

Although both type 1 and type 2 deformations introduce an additional function, namely X or f, the way in which they enter the metric Ansatz is quite different. The two become equivalent when $p = 0$.

2.4.3 Sp-branes

In the previous section we discussed p-branes. A natural question is what happens if we choose a Dirichlet condition for the time-coordinate [8]. Since time is then no longer part of the worldvolume, we have a brane with a Euclidean worldvolume. Such branes are called *spacelike* branes or S-branes for short.

[9] However, an extremal brane is not necessarily supersymmetric, see for example [29].

Just as for a p-brane, an S-brane is carried by a metric, a dilaton and a $(p+2)$-form field strength. An Sp-brane will have a $(p+1)$-dimensional Euclidean worldvolume and $D-p-1$ transversal coordinates of which one is the time coordinate.

In [8] the S-branes were first introduced. The metric of an Sp-brane describes a time-dependent geometry which schematically looks like

$$\mathrm{d}s^2 = -\mathrm{e}^{2A(t)}\mathrm{d}t^2 + \mathrm{e}^{2B(t)}\delta_{ab}\mathrm{d}y^a\mathrm{d}y^b + \mathrm{e}^{2C(t)}\mathrm{d}\mathbb{H}^2_{D-p-2}\,. \tag{2.4.22}$$

The δ_{ab} makes sure that we have a Euclidean worldvolume with symmetry ISO$(p+1)$. The transverse space consists out of time and a $(D-p-2)$-dimensional hyperbolic space, see appendix A.2. This gives the symmetry SO$(D-p-2,1)$. In [8] it was argued that this is the required symmetry for an S-brane Ansatz. In the gauge where $A=C$ the Ansatz looks like that of the metric of a p-brane (2.4.11). In [33–39] many different S-brane solutions are given.

By now this has been extended to many different time-dependent Ansätze. Such as Ansätze where the hyperbolic space has been replaced by a compact sphere. It is unlikely that this has anything to do with the original Sp-branes of [8]. For example, when t goes to infinity these solutions do not even approach flat Minkowski space any more. Nonetheless, in this thesis we will define Sp-branes in this generalized sense. That is, we call a time-dependent solution carried by a metric, possibly a field strength and a scalar field an Sp-brane.

Due to the time-dependence the S-branes belonging to type II supergravities are not supersymmetric. As a result the solutions are more complicated to write down. We will not discuss general Sp-branes as we did for the p-branes. Instead we will focus on S(-1)-branes. In this section we will consider the S(-1)-brane belonging to type IIB supergravity. This brane can be considered as the time-dependent 'twin' of the Euclidean D(-1)-instanton.

The action for the S(-1)-brane follows from the truncation of type IIB supergravity to its scalar sector (for $D=10$)

$$S = \int \mathrm{d}^D x\sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2\Big)\,, \tag{2.4.23}$$

where ϕ is the dilaton and we denote the axion with χ instead of $C^{(0)}$. Observe that we can introduce a metric G_{ij} on the scalar manifold. With this we mean that we can write the scalar part of (2.4.23) as follows

$$\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2 \equiv \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j\,. \tag{2.4.24}$$

If we consider ϕ and χ as coordinates, we can read of what G_{ij} has to be. In general it depends on the scalar fields. To show that G_{ij} indeed describes a metric on the scalar manifold, consider the following action with N scalar fields ϕ^i

$$S = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g}\,G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{2.4.25}$$

The scalar fields ϕ^i define a local map from a space-time parameterized by the coordinate x^μ to a N-dimensional Riemannian manifold parameterized by ϕ^i

$$\phi : \mathcal{M}_{\text{space-time}} \to \mathcal{M}_{\text{scalar}} : \quad x^\mu \to \phi^i(x) \,. \tag{2.4.26}$$

Let us show that G_{ij} indeed transforms as a metric under coordinate transformations on $\mathcal{M}_{\text{scalar}}$

$$G_{ij}\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x^\nu} = G_{ij}\frac{\partial\phi^i(\phi')}{\partial\phi'^k}\frac{\partial\phi^j(\phi')}{\partial\phi'^l}\frac{\partial\phi'^k}{\partial x^\mu}\frac{\partial\phi'^l}{\partial x^\nu} \equiv G'_{kl}\frac{\partial\phi'^k}{\partial x^\mu}\frac{\partial\phi'^l}{\partial x^\nu} \,. \tag{2.4.27}$$

We see that G_{ij} transforms as a bilinear.

The equations of motion that follow from (2.4.23) are given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\mathrm{e}^{2\phi}\partial_\mu\chi\partial_\nu\chi \,, \tag{2.4.28}$$

$$\partial_\mu\Big(\mathrm{e}^{2\phi}\sqrt{-g}g^{\mu\nu}\partial_\nu\chi\Big) = 0 \,, \tag{2.4.29}$$

$$\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) - \sqrt{-g}\mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi = 0 \,. \tag{2.4.30}$$

When $p = -1$ all space is transverse so the part containing B is not present in the Ansatz (2.4.22). We choose the gauge where $\mathrm{e}^{2C} = t^2$ and we generalize the Ansatz to

$$\mathrm{d}s^2 = -f(t)^2\mathrm{d}t^2 + t^2\Big(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big) \,, \tag{2.4.31}$$

such that it is valid for all k. The part between brackets describes for $k = 0$ flat space, for $k = +1$ a sphere and for $k = -1$ a hyperboloid, see appendix A.2. Only for $k = -1$ there is an expected string theory interpretation. This follows from the fact that when t goes to infinity the metric describes a flat Minkowski space-time only for $k = -1$ (if f approaches one), see for example (2.4.32). The two scalars depend only on t.

In section 3.5 we will show a way to solve the equations of motion (2.4.28-2.4.30) in a structured way. Basically this comes down to realizing that the scalar fields trace out a geodesic on the scalar manifold described by the metric G_{ij}. For now let us just give the metric solution

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{\frac{||v||^2}{2(D-1)(D-2)}t^{-2(D-2)} - k} + t^2\Big(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big) \,. \tag{2.4.32}$$

Here $||v||^2$ is a strictly positive number. This constant will turn out to be the affine velocity labelling the geodesic. The scalar fields are given by

$$\begin{aligned}\phi(t) &= \log\Big(c_1\cosh(||v||h + c_2)\Big) \,,\\ \chi(t) &= \pm\frac{1}{c_1}\tanh(||v||h + c_2) + c_3 \,.\end{aligned} \tag{2.4.33}$$

Here c_i are constants of integrations and the harmonic function h is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)} \log\left|\sqrt{a}t^{2-D} + \sqrt{at^{2(2-D)} - k}\right| + c\,, \qquad a = \frac{||v||^2}{2(D-1)(D-2)}\,, \tag{2.4.34}$$

with c a constant.

The link between the two scalar fields and the geodesic is due to the following relation

$$(\partial_h \phi)^2 + \mathrm{e}^{2\phi}(\partial_h \chi)^2 = ||v||^2\,. \tag{2.4.35}$$

We give an explanation of this in section 3.5.

For $k = -1$ we have the S(-1)-brane of type IIB supergravity [33]. For $k = 0$ the brane describes a so-called power-law universe in FLRW-coordinates. With this we mean that after a coordinate transformation we find that the metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big)\,. \tag{2.4.36}$$

Here $a(\tau)$ is called the scale factor for an obvious reason. For the S(-1)-brane solution we discuss here we find that $a(t) = \tau^p$ with $p = 1/(D-1)$. Such a scale factor is called a power-law.

In case $k = 1$ we cannot really consider it as an S(-1)-brane, actually the metric (2.4.36) describes a transition from a Big Bang to a Big Crunch for a closed universe [40]. For example, in three dimensions we easily derive that $a(\tau)^2 \propto (||v||^2/4 - \tau^2) > 0$. At $\tau = \pm||v||/2$ the Ricci scalar blows up and hence we see that this describes a Big Bang to a Big Crunch universe. A different reason to use the FLRW metric instead of (2.4.32) is that the latter has a coordinate singularity for some finite t. In the FLRW frame the metric covers the whole manifold, but finding explicit expressions for $h(\tau)$ and $a(\tau)$ is difficult in general.

In chapter 4 we will use the relation between geodesics and the scalar fields to write down the most general Sp-brane with a deformed worldvolume.

A supersymmetric brane obeys the extremality condition (2.4.20), i.e. $X = 0$. In [41] it was shown that if one demands that the extremality condition also holds for an Sp-brane with $k = -1$ one finds that the resulting field strength is *imaginary*. This shows that the extremality condition cannot be satisfied for real S-brane solutions. However, in chapter 6 we will see that there is a different interpretation possible for the imaginary solution.

In subsection 2.4.2 we mentioned the AdS/CFT correspondence. For S-branes this leads to a proposed dS/CFT correspondence [42, 43]. Since the worldvolume of an S-brane is Euclidean, the field theory will be a Euclidean conformal field theory. Unlike the AdS/CFT correspondence there is not much support yet for the dS/CFT one.

Chapter 3

Dimensional Reduction of Branes

In the previous chapter we mentioned that the superstring requires a ten-dimensional space-time. To make a connection to our four-dimensional universe we introduce in this chapter dimensional reduction to link a higher-dimensional theory to a lower-dimensional one. We will restrict to torus reductions and reductions on maximally symmetric Einstein spaces.

In the last section we show that via dimensional reduction over the worldvolume of a brane we obtain a link between p-branes and instantons, and similarly between Sp-branes and S(-1)-branes. If on the other hand we reduce over the maximally symmetric transverse space of a brane, we generate a potential. This way we have a relation between branes and domain-walls and cosmologies. These observations form the basis for the rest of the thesis.

Some useful references about Kaluza–Klein reductions are [44, 45].

3.1 Dimensional Reduction

Consider a free scalar field $\hat{\phi}$ in $\hat{D} = D + 1$ dimensions, depending on the coordinates $x^{\hat{\mu}} = (x^{\mu}, y)$. We put hats on the fields when they are in $D + 1$ dimensions. What happens if we reduce the theory to D dimensions via compactifying the coordinate y on a circle S^1 of radius R? The first thing we can do is expand the scalar field $\hat{\phi}$ in a Fourier series. Due to the circle we have to impose the following boundary condition on the scalar field

$$\hat{\phi}(x^{\mu}, 0) = \hat{\phi}(x, 2\pi R) \,. \qquad (3.1.1)$$

We expand $\hat{\phi}$ as

$$\hat{\phi}(x^\mu, y) = \sum_n \phi_n(x) \mathrm{e}^{iny/R} \,. \tag{3.1.2}$$

We obtain a discrete spectrum of fields $\phi_n(x)$ with quantized momentum $k = n/R$, see (2.3.18). The equation of motion for $\hat{\phi}$ is

$$\hat{\Box}\hat{\phi} = (\Box + \partial_y \partial^y)\hat{\phi} \,. \tag{3.1.3}$$

Using (3.1.2) we see that the lower-dimensional scalar fields $\phi_n(x)$ obey

$$\Box\phi_n(x) - \frac{n^2}{R^2}\phi_n(x) = 0 \,. \tag{3.1.4}$$

We thus see that $\phi_0(x)$ is a massless field in D dimensions and that the other modes are massive fields with masses $m^2 = n^2/R^2$. The usual Kaluza–Klein approach is to assume that the radius R of S^1 is very small, in which case the masses of the modes with $n \neq 0$ will be enormous. This holds for general compact internal spaces and fields $\hat{\phi}$. This is the physical reason as to why we can choose to work with fields independent of y, since when we take the radius R of the extra dimension to be small so that we do not observe it, the massive fields ϕ_n become extremely massive and will not play a role in the effective D-dimensional theory. On the level of the equations of motion (3.1.4) this means that we truncate the fields $\phi_{n>0}$. These massive fields are knows as Kaluza–Klein states and when they are truncated we see from (3.1.2) that $\hat{\phi}$ is independent of the extra dimension y.

Let us therefore assume that all higher-dimensional fields are independent of the coordinate y. For a scalar field we just have that $\phi(x) = \hat{\phi}(x)$, but for fields that transform non-trivially under coordinate transformations we need to do something more. A vector will give $\hat{A}_{\hat{\mu}} = (A_\mu(x), \chi(x))$, i.e. it gives a D-dimensional vector $A_\mu(x)$ and a scalar field $\chi(x)$. The metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ will give rise to a D-dimensional metric $g_{\mu\nu}$, a vector $A_\mu(x)$ and a scalar field $\varphi(x)$, since we can write it as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \varphi \end{pmatrix} \,. \tag{3.1.5}$$

If we would start in $D+1$ dimensions with a scalar field, a vector and a metric, we would have in D dimensions three scalar fields, two vectors and one metric.

Not all compact manifolds are allowed, since a reduction must be consistent. This means that a solution of the lower-dimensional theory is also a solution of the higher-dimensional theory via tracing back the steps of the reduction. This procedure is called uplifting or oxidation.

An example of an inconsistency is choosing the scalar field φ to be a constant in the reduction (3.1.5). This follows from the equation of motion for φ, i.e. the details of the interactions between the various lower-dimensional fields prevent the truncation of the scalar φ. A different inconsistency can appear from the truncation of the massive modes in the expansion (3.1.2) and similarly for the metric and other fields. It can be that turning off the massive modes is not allowed, although one can show that this is not a problem for our S^1 reduction. For example, suppose we had kept all the modes in the Fourier expansion. It could have been that from the resulting equations of motion for these higher modes we would have found that it is not allowed to take the modes with $\phi_{n>0}$ to be zero. In more complicated Kaluza–Klein reductions, the issue of the consistency of the truncation to the massless sector is a tricky one. For these reasons it is best to reduce the equations of motion instead of the action. However in the examples we will consider this consistency is know to be in order.

If the massive sector of a certain reduction cannot be consistently truncated this does not automatically mean that the reduction is of no use. Assuming that the massive modes are very heavy, it is probable that the massive modes have negligible interactions with the massless sector because they are so heavy. So even in the case that the massive modes cannot be consistently truncated, leaving them out of the theory at low energies might still be a good approximation.

We will also use dimensional reduction for a different reason. We will show that reducing a theory over some of its dimensions leads to a theory which is easier to solve than the original one. Via uplifting the solution back to $\hat{D}$ dimensions we have generated a solution of the higher-dimensional system. This way we have constructed a solution-generating technique. It is clear that for this to work, the reduction must be consistent to be sure that it leads to a solution of the higher-dimensional theory.

So far we have focussed on a compact internal manifold such as S^1. Later we will see examples where the internal manifold is not compact. The only thing we require is that we can consistently truncate to the massless sector, although there is no physical motivation as to why we should do this truncation. For the solution-generating technique this turns out to be useful. Such a reduction is called a non-compactification, in contrast to a compactification on a compact manifold.

3.2 Torus Reduction of Gravity

As a first example we will work out the sphere reduction we introduced in the previous section. From (3.1.5) we see that the metric of the higher-dimensional theory is given by $\mathrm{d}s^2_{D+1} = \mathrm{d}s^2_D + \mathrm{d}\phi^2 + 2A_\mu \mathrm{d}x^\mu \mathrm{d}y$. We could use this Ansatz and plug it in the higher-dimensional Einstein-Hilbert action

$$S = \int \mathrm{d}^{D+1}x\sqrt{-\hat{g}}\,\hat{\mathcal{R}}\,. \tag{3.2.1}$$

The resulting action would not look familiar, for example we will not find the standard Einstein-Hilbert term. We would have to introduce some field redefinitions to fix this. To avoid this we will use the following $(D+1)$-dimensional metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\phi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\phi}(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)^2\,, \tag{3.2.2}$$

with

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\,, \quad \beta = -(D-2)\alpha\,. \tag{3.2.3}$$

These constants are chosen such that the lower-dimensional action immediately contains the Einstein-Hilbert term. The conventions we use for the Christoffel symbol are given in appendix A.2. Plugging (3.2.2) in (3.2.1) leads to the following D-dimensional Lagrangian

$$\mathcal{L} = *\mathcal{R}_D - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{-2(D-1)\alpha\phi} * \mathrm{d}A \wedge \mathrm{d}A\,. \tag{3.2.4}$$

Due to the coupling between ϕ and A we cannot take ϕ to be zero.

Let us now consider the reduction over a n-torus $\mathbb{T}^n$, i.e. $\mathbb{T}^n = S^1 \times \ldots \times S^1$ (n times). This reduction can be obtained from further reducing pure gravity on a series of circles. For each step the reduction of the metric generates a Kaluza–Klein vector A^i and a Kaluza–Klein scalar ϕ^i. The vectors that are already present from an earlier reduction give rise to a lower-dimensional vector and a scalar called axion. After reduction on an $n-$torus we count n vector fields A^m, n dilaton scalars ϕ^m that correspond to the radii of the circles and $n(n-1)/2$ axions χ^α. The dilatons and axions together parameterize the *coset* $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n) = \mathbb{R}\times\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$. The concept of a coset will be discussed in more detail in section 3.4.

Instead of doing a circle by circle reduction, it is easier to do the torus reduction in one step. Similar to the case of S^1 (3.1.5) we now write

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu^n \\ A_\nu^m & \varphi_{mn} \end{pmatrix}, \tag{3.2.5}$$

with φ_{mn} a symmetric strictly positive definite matrix of scalars, it contains the $n(n+1)/2$ axions and dilatons. It is positive definite since we are reducing over a Euclidean torus. The indices μ,ν (m,n) run from $1,\ldots,D$ $(D+1,\ldots,D+n)$. To avoid having to introduce field redefinitions in the lower-dimensional theory, we define $\varphi_{mn} = \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}$, with $\mathrm{e}^{2\beta\varphi}$ the determinant of φ_{mn}. This means that φ determines the volume of the torus. It is therefore called the volume modulus or the *breathing mode.* The $(D+n)$-dimensional Ansatz that gives us the Einstein-Hilbert action in D dimensions is

$$\mathrm{d}s_{D+n}^2 = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}(\mathrm{d}y^m + A^m)\otimes(\mathrm{d}y^n + A^n)\,, \tag{3.2.6}$$

with

$$\alpha^2 = \frac{n}{2(D+n-2)(D-2)}\,, \quad \beta = -\frac{(D-2)\alpha}{n}\,. \tag{3.2.7}$$

In doing this reduction we need the inverse of the metric Ansatz. This can best be achieved by using the vielbeine $\hat{e}_{\hat{\mu}}^{\ \underline{\hat{a}}}$ related to the metric as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}^{\ \underline{\hat{a}}}\hat{e}_{\hat{\nu}}^{\ \underline{\hat{b}}}\hat{\eta}_{\underline{\hat{a}\hat{b}}}\,, \quad \hat{\eta}_{\underline{\hat{a}\hat{b}}} = \mathrm{diag}(-1,1,\ldots,1)\,, \tag{3.2.8}$$

see appendix A.2.1. Indices raised and lowered with $\hat{\eta}$ are underlined for clarity here. For the metric Ansatz (3.2.6) we derive that

$$\hat{e}_{\hat{\mu}}^{\ \underline{\hat{a}}} = \begin{pmatrix} \mathrm{e}^{\alpha\varphi}e_{\mu}^{\ \underline{\nu}} & \mathrm{e}^{\beta\varphi}A_{\mu}^{p}L_{p}^{\ \underline{m}} \\ 0 & \mathrm{e}^{\beta\varphi}L_{n}^{\ \underline{m}} \end{pmatrix}. \tag{3.2.9}$$

Here L is the vielbein of the torus

$$L_{m}^{\ \underline{k}}L_{n}^{\ \underline{l}}\hat{\eta}_{\underline{kl}} = \mathcal{M}_{mn}\,, \tag{3.2.10}$$

and $\underline{k}, \underline{l}$ run from $D+1,\ldots,D+n$. Since the vielbein (3.2.9) is upper triangular it can easily be inverted. From this we find the inverse metric

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \hat{e}_{\underline{\hat{a}}}^{\ \hat{\mu}}\hat{e}_{\underline{\hat{b}}}^{\ \hat{\nu}}\hat{\eta}^{\underline{\hat{a}\hat{b}}}\,, \tag{3.2.11}$$

where $\hat{e}_{\underline{\hat{a}}}^{\ \hat{\mu}}$ is the inverse vielbein. The inverse metric can be written in terms of $\mathcal{M}_{ab}$ again.

If we plug (3.2.6) in the $(D+n)$-dimensional Einstein-Hilbert action we find the Lagrangian

$$\mathcal{L} = *\mathcal{R}_D - \frac{1}{2}*\mathrm{d}\varphi\wedge\mathrm{d}\varphi + \frac{1}{4}*\mathrm{d}\mathcal{M}_{mn}\wedge\mathrm{d}\mathcal{M}^{mn} - \frac{1}{2}\mathrm{e}^{2(\beta-\alpha)\varphi}\mathcal{M}_{mn}*\mathrm{d}A^m\wedge\mathrm{d}A^n\,. \tag{3.2.12}$$

Here $\mathcal{M}^{mn}$ means the inverse, i.e. $\mathcal{M}^{mn} = (\mathcal{M}^{-1})_{mn}$. It is consistent to put A^m to zero.

As mentioned, the scalar field φ is called the breathing mode since it describes the overall volume of the torus. The scalars in $\mathcal{M}$ can be interpreted as shape-moduli of the torus. These scalars parameterize the coset $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Together with the breathing mode φ we have the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n) = \mathbb{R}\times\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

3.2.1 Torus Reduction over Time

Above we considered the Euclidean torus reduction of pure gravity. If we want to include time in the dimensional reduction as well we need to reduce over a n-torus with a Lorentzian signature $\mathbb{T}^{n-1,1}$. The Ansatz (3.2.6) is still valid, but with the difference that now Det $\mathcal{M} = -1$ due to the Lorentzian signature of the torus. To take care of this we replace $\mathcal{M}$ (3.2.10) by

$$\mathcal{M} = L\eta L^T\,, \quad \eta = \mathrm{diag}(-1, 1, \ldots, 1)\,. \tag{3.2.13}$$

The reduction leads to the Lagrangian (3.2.12) but now with $\mathcal{M}$ given by (3.2.13). The scalar coset parameterizes $\mathrm{GL}(n, \mathrm{I\!R})/\,\mathrm{SO}(n-1, 1)$. We have a non-compact version of the $\mathrm{SO}(n)$-subgroup due to the reduction over the Lorentzian torus.

Something special happens if we reduce down to three dimensions. Due to Hodge duality (A.3.10) we can dualize *all* gauge potentials A^m to scalars. Schematically this goes as

$$\partial_\mu \tilde{\phi} \propto \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}\,, \tag{3.2.14}$$

where $F = \mathrm{d}A^m$ and $\epsilon_{\mu\nu\rho}$ the three-dimensional epsilon tensor see (A.3.7). The three-dimensional gauge potential A^m can be described by the new scalar field $\tilde{\phi}^m$. If we do this for all the gauge potentials this leads to extra scalar fields in three dimensions. As a result, there is a symmetry enhancement since it can be shown that the extra scalars combine with the existing scalars into the coset $\mathrm{SL}(n+1, \mathrm{I\!R})/\,\mathrm{SO}(n-1, 2)$. There is no decoupled $\mathbb{R}$ in this case.

For the reduction over a Euclidean torus from $3+n$ to three dimensions we have the coset $\mathrm{SL}(n+1, \mathrm{I\!R})/\,\mathrm{SO}(n+1)$.

Due to the non-compactness of the subgroup, such as $\mathrm{SO}(n-1, 1)$, the theory will contain ghosts. A ghost is an axion field with the opposite sign for the kinetic term in the Lagrangian. For future use let us discuss the ghost content for the theory with scalar coset $\mathrm{GL}(p+q)/\,\mathrm{SO}(p, q)$.

For a general coset $\mathrm{GL}(p+q)/\,\mathrm{SO}(p, q)$ the number of ghosts is pq. For the Kaluza–Klein moduli spaces this can be seen as follows. When one considers a reduction over time then there are two possible origins for ghosts. Ghost fields χ^Λ appear as the time-component of a one-form $\hat{A}^\Lambda$ in the higher dimension, that is, $\hat{A}^\Lambda = \chi^\Lambda \mathrm{d}t + A^\Lambda$. Alternatively, extra ghost fields appear in three dimensions upon dualisation of the one-forms. The extra minus sign is due to the fact that the three-dimensional theory is Euclidean. Therefore, imagine we reduce Einstein gravity in $D+n$ dimensions to $D+1$ dimensions over a spacelike torus and then perform a subsequent reduction over a timelike circle, then the $n-1$ Kaluza–Klein vectors in $D+1$ dimensions give $n-1$ ghostlike axions. This fits with the fact that the scalar coset is $\mathrm{GL}(n)/\,\mathrm{SO}(n-1, 1)$. If $D = 3$ then we can further dualise those $n-1$ descendants of the Kaluza–Klein vectors to $n-1$ ghostlike axions, thereby doubling the number of ghosts. The Kaluza–Klein vector that appears from the last timelike reduction does not dualise to a ghost but to a normal axion since that vector appeared with a wrong sign in three dimensions. This indeed explains why there are $2(n-1)$ ghosts in $\mathrm{SL}(n+1)/\,\mathrm{SO}(n-1, 2)$.

3.3 Maximally Symmetric Compactification

The torus reduction we considered so far did not generate a potential V in the lower-dimensional theory. In this section we want to show an example where this does happen.

For this let us consider gravity with the following metric Ansatz

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\varphi} g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\varphi} g_{mn}(y)\mathrm{d}y^m \mathrm{d}y^n \,, \tag{3.3.1}$$

with the same α and β as before but note that g_{mn} depends on the coordinates y^m. The only scalar field is the breathing mode $\varphi(x)$. The indices μ, ν run from $1, \ldots, D$ and m, n run from $D+1, \ldots, D+n$. To see what kind of condition g_{mn} has to satisfy we work out the higher-dimensional Ricci tensor

$$\begin{aligned} \hat{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{\mu\nu} + a(\partial\varphi)^2 g_{\mu\nu} + b\partial_\mu\varphi\partial_\nu\varphi + c\nabla_\mu\partial_\nu\varphi - \alpha g_{\mu\nu}\Box\varphi \,, \\ \hat{\mathcal{R}}_{mn} &= \mathcal{R}_{mn} - \beta \mathrm{e}^{2(\beta-\alpha)\varphi} g_{mn}\Box\varphi \,, \\ \hat{\mathcal{R}}_{\mu m} &= \hat{\mathcal{R}}_{m\mu} = 0 \,. \end{aligned} \tag{3.3.2}$$

The coefficients a, b and c are given by

$$a = -(D-2)\alpha^2 - n\beta\alpha \,, \quad b = (D-2)\alpha^2 + 2n\alpha\beta - \beta^2 n \,, \quad c = -(D-2)\alpha - n\beta \,. \tag{3.3.3}$$

Let us now focus on the internal manifold metric g_{mn}. We assume that g_{mn} is a n-dimensional *Einstein space*. Such a space is defined by

$$\mathcal{R}_{mn} = d\, g_{mn} \to \mathcal{R}_n = d\, n \,, \tag{3.3.4}$$

with d a constant and $\mathcal{R}_n$ the Ricci scalar of the n-dimensional internal manifold. From (A.2.18) we see that if we have a sphere S^n $(k = +1)$, a hyperboloid $\mathbb{H}^n$ $(k = -1)$ or flat space $\mathbb{E}^n$ $(k = 0)$ that $d = (n-1)k$. From the Einstein equation $\hat{\mathcal{R}}_{mn} = 0$ (3.3.2) we see that, if g_{mn} is one of these three Einstein spaces that this becomes an equation of motion for φ coupled to some potential V.

To be precise, the n-dimensional field equations (3.3.2) can be derived from reducing the $(D+n)$-dimensional Einstein-Hilbert action

$$\int \sqrt{-\hat{g}}\hat{\mathcal{R}} = \mathrm{Vol}(\mathcal{M}_n) \int \sqrt{-g_D}\Big(\mathcal{R}_D - \frac{1}{2}(\partial\varphi)^2 + \mathrm{e}^{2(\alpha-\beta)\varphi}\mathcal{R}_n\Big) \,, \tag{3.3.5}$$

where $\mathrm{Vol}\,\mathcal{M}_n$ is the volume of the internal manifold and we ignore the total derivative $\Box\varphi$. When g_{mn} belongs to one of the three Einstein spaces discussed above $\mathcal{R}_n$ simplifies to $d\,n$. This means that we can identify the potential V as [46]

$$V(\varphi) = -kn(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi} \,. \tag{3.3.6}$$

In case we restrict to positive potentials, we see from (3.3.6) that $k = -1$. However, $\mathbb{H}^n$ is not a compact space and $\mathrm{Vol}(\mathcal{M}_n)$ is not finite. To resolve this we mention that $\mathbb{H}^n$ can be seen as the coset $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, just as the sphere S^n is the coset $\mathrm{SO}(n+1)/\mathrm{SO}(n)$. To make $\mathbb{H}^n$ compact, we can mod out with a discrete non-compact symmetry. Since the metric is local, it does not care about topological issues such as discrete identifications.

To make contact with S- and p-branes, we add to the higher-dimensional theory a $(p-1)$-form gauge potential and a dilaton ϕ

$$\mathcal{L} = *\hat{\mathcal{R}} - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{a\phi} * \mathrm{d}A_{p-1} \wedge \mathrm{d}A_{p-1}\,. \tag{3.3.7}$$

The equations of motion and Bianchi identity are

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{p-1}{2(D-2)(p!)}g_{\mu\nu}\mathrm{e}^{a\phi}F_p^2 + \frac{1}{(p-1)!2}\mathrm{e}^{a\phi}(F_p^2)_{\mu\nu}\,, \tag{3.3.8}$$

$$\mathrm{d}\Big(*\mathrm{e}^{a\phi}F_p\Big) = 0\,, \quad \mathrm{d}F_p = 0\,, \tag{3.3.9}$$

$$\Box\phi = \frac{a}{p!2}F_p^2\mathrm{e}^{a\phi}\,. \tag{3.3.10}$$

The Ansatz for the metric is again (3.3.1), but now we have that $D = p$. For the field strength we use a Freund-Rubin [47] like Ansatz

$$F_p = f\mathrm{e}^{(D\alpha-n\beta)\varphi-a\phi}\epsilon_D(x^\mu)\,, \quad \phi = \phi(x^\mu)\,. \tag{3.3.11}$$

Here f is a constant and ϵ_D is the D-dimensional epsilon tensor (A.3.7). This expression for the field strength is in agreement with the equation of motion for F_p and its Bianchi identity (3.3.9). One can now again reduce the higher-dimensional equations of motion (3.3.8-3.3.10) by using the Ansätze for the field strength and metric. As it turns out, these lower-dimensional equations can be derived from the following Lagrangian

$$\mathcal{L}_D = \sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - V(\phi,\varphi)\Big)\,, \tag{3.3.12}$$

where the scalar potential V now gets a positive contribution from the p-form flux

$$V(\phi,\varphi) = \frac{f^2}{2}\mathrm{e}^{2(D-1)\alpha\varphi-a\phi} - kn(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi}\,. \tag{3.3.13}$$

Here the first term comes from the flux and the second term we already found in (3.3.6). We see that the flux adds a positive contribution to the potential. In case $k = 0$ and if the scalar fields can be fixed this leads to a positive cosmological constant Λ. Due to the charge quantization condition (2.4.18) Λ would be quantized.

It is important to mention that in the above reduction we reduced the equations of motion and not the action. Had we reduced the latter we would have found the wrong sign in front of the flux contribution. This is because filling in an Ansatz means filling in on-shell information and that can lead to problems. An Euler-Lagrange variation with respect to the remaining (unfixed) degrees of freedom can be inconsistent. For this reason it is better to reduce the equations of motion and then see what kind of action leads to these equations of motion. For the torus reduction discussed in the previous section such a problem does not appear.

An interesting generalization was considered in the papers [48–50]. We will use the result of [45]. Consider the internal manifold as a product of M different spaces $\mathcal{M}_i$

$$\mathcal{M}_{\text{int}} = \Pi_{i=1}^{M} \times \mathcal{M}_i \,. \tag{3.3.14}$$

The dimensions of each internal spaces is $\mathcal{M}_i = n_i$, obeying the sum $\sum_i n_i = n$. Each space $\mathcal{M}_i$ is assumed to be an Einstein space

$$(\mathcal{R}_i)_{A_i\,B_i} = k_i(n_i - 1)(g_i)_{A_i\,B_i} \,. \tag{3.3.15}$$

The generalization of the metric Ansatz (3.3.1) is

$$\mathrm{d}s^2_{D+n} = \mathrm{e}^{2\alpha\varphi(x)} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\varphi(x)} \sum_{i=1}^{M} X_i(x)(g_i)_{A_iB_i}\mathrm{d}y^{A_i}\mathrm{d}y^{B_i} \,. \tag{3.3.16}$$

To consider φ as the field that determines the overall volume of the internal manifold we have to require that

$$\Pi_{i=1}^{M} X_i^{n_i} = 1 \,. \tag{3.3.17}$$

With this condition we see from (3.3.16) that the only x-dependence of the determinant of the internal manifold is given by $\varphi(x)$. Due to (3.3.17) we have only $M-1$ independent X^i. It is therefore convenient to write

$$X_i = \mathrm{e}^{-\vec{\beta}_i\cdot\vec{\phi}} \,, \tag{3.3.18}$$

where $\vec{\beta}_i \cdot \vec{\phi} = \sum_{I=1}^{M-1} \beta_{iI}\phi_I$, with $\vec{\beta}_i$ a constant $(M-1)$-dimensional vector. In this case we see that we have M scalar fields in total. From (3.3.17) we find that the vectors $\vec{\beta}_i$ satisfy

$$\sum_{i=1}^{M} n_i \beta_{iI} = 0 \,. \tag{3.3.19}$$

The reduced Einstein equations for the metric Ansatz (3.3.16) can be derived from the action

$$S = \int *\mathcal{R} - \frac{1}{2} * \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\sum_{I=1}^{M-1} *\mathrm{d}\phi^I \wedge \mathrm{d}\phi^I - V(\varphi, \vec{\phi}) \,, \tag{3.3.20}$$

with the multi-exponential potential

$$V(\varphi, \vec{\phi}) = -\mathrm{e}^{2(\alpha-\beta)\varphi} \sum_{i=1}^{M} k_i n_i (n_i - 1) \mathrm{e}^{\vec{\beta}_i \cdot \vec{\phi}} . \tag{3.3.21}$$

3.4 Coset Geometry

When we discussed the torus reduction in section 3.2 we found out that the scalar fields parameterize the Riemannian coset $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{SO}(n)$. In this section we want to give a general discussion about cosets G/H with H the maximal compact subgroup of G. In particular we will show how we can obtain for such cosets a metric G_{ij}. We comment later on the case when H is non-compact due to a reduction over a Lorentzian torus.

Let us begin by defining what we mean with a coset G/H. Let G be a group with a subgroup H. The coset space G/H is the set of elements $[g]$ of G with the equivalence relation

$$[g] = [g'] \quad \text{if} \quad g' = g\, h\,, \tag{3.4.1}$$

where h is an element of H. As an example, let G be a Lie group and H any subgroup of G. We can form the coset G/H. This coset space admits a differentiable structure and G/H becomes a manifold M with dim G/H=dim G-dim H.

Assume now that G is a Lie group which acts transitively on a manifold M. That means that given any point $p \in M$, the action of G on p allows us to go to all the points of M. Such a manifold is called *homogeneous*. For example, let $H(p)$ be an isotropy group of $p \in M$, then G/H is a homogenous space. In fact, if G, $H(p)$ and M satisfy certain requirements it can be shown that $G/H(p)$ is diffeomorphic to M [51]. As an example consider the Lie group $SO(3)$ acting transitively on S^2. The isotropy group H is $SO(2)$ and as a result we have that $SO(3)/SO(2) \cong S^2$.

Let us focus on coset manifolds of the form G/H with H the maximal compact subgroup. We want to define a metric G_{ij} on G/H, which fixes the kinetic term for the scalar fields

$$e^{-1}\mathcal{L} \propto G(\phi)_{ij} \partial\phi^i \partial\phi^j\,. \tag{3.4.2}$$

This metric will not be unique, but we make the demands that the isometry group must be G and that we have invariance under local H-transformations. We mention [44, 52–54] as a few examples about the use of cosets in supergravity.

If we succeed in parameterizing G/H with some coordinates y^i, then a coset representative $L(y)$ is a representation of G/H with the extra condition that if $y \neq y'$ then there cannot exist an element h of H for which $L(y) = L(y')h$. This is because of the coset requirement (3.4.1). On the other hand, for a given y there exist multiple $L(y)$ since for all $h \in H$, $L(y)h$ is an equivalent representation of the same coset element.

It is not difficult to construct a coset representative using the Lie algebras $\mathfrak{G}$ and $\mathfrak{H}$ of G and H respectively. Since H is a subgroup of G we have the decomposition $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{F}$, with $\mathfrak{F}$ the complement of $\mathfrak{H}$ in $\mathfrak{G}$. For a given representation of the algebra $\mathfrak{G}$ we define a coset representative via

$$L(y) = \exp(y^i \mathbf{f_i}) \,, \tag{3.4.3}$$

where the $\mathbf{f_i}$ form a basis of $\mathfrak{F}$ in some representation of $\mathfrak{G}$. This defines correctly a representative since if we assume $L(y) = L(y')h$ we find that $y = y'$ as is required for a representative.

We will be interested in a decomposition $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{F}$ which can be done in such a way that

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H} \,, \quad [\mathfrak{F}, \mathfrak{H}] \subset \mathfrak{F} \,, \quad [\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{H} \,. \tag{3.4.4}$$

Such a coset is called a *symmetric space*.

To derive the metric we define a Lie algebra valued one-form from the coset representative $L(y)$ via

$$L^{-1}\mathrm{d}L \equiv E + \Omega \,, \tag{3.4.5}$$

where E takes values in $\mathfrak{F}$ and Ω in $\mathfrak{H}$. We notice that $L^{-1}dL$ is invariant under left multiplication with an y-independent element $g \in G$. Multiplying L from the right with local elements $h \in H$ results in

$$E \to h^{-1}\, E\, h \,, \qquad \Omega \to h^{-1}\, \Omega\, h + h^{-1}\mathrm{d}h \,. \tag{3.4.6}$$

In supergravity the parameters y^i are scalar fields that depend on the space-time coordinates $y^i = \phi^i(x)$. The one-form $L^{-1}\mathrm{d}L$ can be written out in terms of coset-coordinate one-forms $\mathrm{d}\phi^i$ which themselves can be pulled back to space-time coordinate one-forms $\mathrm{d}\phi^i = \partial_\mu \phi^i \mathrm{d}x^\mu$. Now we can write

$$L^{-1}\mathrm{d}L = E_\mu \mathrm{d}x^\mu + \Omega_\mu \mathrm{d}x^\mu \,. \tag{3.4.7}$$

Under the ϕ-dependent H-transformations $h(\phi(x))$ we have that $\Omega_\mu \to h^{-1}\Omega_\mu h + h^{-1}\partial_\mu h$ and $E_\mu \to h^{-1} E_\mu h$. We see that E_μ is covariant under local H-transformations and Ω_μ transforms like a connection. Using this connection Ω_μ we can make the following H-covariant derivative on L and L^{-1}

$$D_\mu L = \partial_\mu L - L\Omega_\mu \,, \qquad D_\mu L^{-1} = \partial_\mu L^{-1} + \Omega_\mu L^{-1} \,. \tag{3.4.8}$$

To find a kinetic term for the scalars we notice that the object

$$\mathrm{Tr}[D_\mu L D^\mu L^{-1}] = -\mathrm{Tr}[E_\mu E^\mu] \,, \tag{3.4.9}$$

has all the right properties as it contains single derivatives on the scalars, it is a space-time scalar, it is invariant under rigid G transformations and under local H-transformations. Thus,

$$e^{-1}\mathcal{L}_{\mathrm{scalar}} = -\mathrm{Tr}[E_\mu E^\mu] \equiv -\tfrac{1}{2} G(\phi)_{ij} \partial_\mu \phi^i \partial^\mu \phi^j \,. \tag{3.4.10}$$

So far we have been completely general, that is we did not specify the coordinates ϕ^i nor the representation. Let us therefore make a connection to section 3.2, i.e. we focus on a coset where H is the maximal compact subgroup SO(n) of G in the fundamental representation. The Lie algebra of SO(n) is the vector space of antisymmetric matrices and we have the split

$$E = \frac{L^{-1}\mathrm{d}L + (L^{-1}\mathrm{d}L)^T}{2}\,, \qquad \Omega = \frac{L^{-1}\mathrm{d}L - (L^{-1}\mathrm{d}L)^T}{2}\,, \tag{3.4.11}$$

and a calculation shows that

$$e^{-1}\mathcal{L}_{\text{scalar}} = -\mathrm{Tr}[E^2] = +\tfrac{1}{4}\mathrm{Tr}[\partial\mathcal{M}\partial\mathcal{M}^{-1}] = -\tfrac{1}{2}G(\phi)_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{3.4.12}$$

Here $\mathcal{M}$ is the SO(n)-invariant matrix

$$\mathcal{M} = LL^T\,. \tag{3.4.13}$$

Under the global isometry group G it transforms as

$$\mathcal{M} \to \mathcal{M}' = g\mathcal{M}g^T\,, \quad g \in G\,. \tag{3.4.14}$$

To find the metric $G(\phi)_{ij}$ from (3.4.12) we still need to use an explicit realization of the Lie algebra. This means a choice for the coordinate frame on G/H. However we found that (3.4.12) still has local H-invariance. We use these h-transformations to bring L in a 'nice' form for computations, i.e. we make a gauge choice.

Let us look for the 'nice' gauge in case H is the maximal compact subgroup of G. The gauge we will be using is due to the Iwasawa decomposition [44, 55]. This states that every element g in the Lie group G can be obtained by exponentiating the lie algebra $\mathfrak{G}$ as follows

$$g = g_N g_C g_H\,, \tag{3.4.15}$$

where g_N is the exponentiation of the positive-root part of the algebra $\mathfrak{G}$, $g_\mathcal{C}$ the exponentiation of the Cartan subalgebra and g_H is the maximal compact subgroup H in G. In appendix C we present a short overview of Lie algebras and Lie groups.

For the algebra of G we denote the Cartan generators by H_I with $I = 1,\ldots,r$ and r is the rank of the algebra. All the positive root generators are denoted by E_α. The commutation relations read

$$[H_I, H_J] = 0\,, \quad [H_I, E_\alpha] = \alpha_I E_\alpha\,, \quad [E_\alpha, E_\beta] = N(\alpha,\beta)E_{\alpha+\beta}\,. \tag{3.4.16}$$

The last line is to be understood as follows. If $\alpha+\beta$ is not a root we have $N(\alpha,\beta) = 0$, else we have $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$. We call the algebra formed by H_I and the positive root generators E_α the *Borel subalgebra*. For the Borel Lie algebra the matrix representation can be chosen such that all elements of it are upper-triangular [55]. We can then parameterize the coset elements in this gauge as

$$L = \exp[\mathfrak{s}]\,, \tag{3.4.17}$$

with $\mathfrak{s} = \mathcal{C} \oplus \sum E_\alpha$, the sum is over all the positive roots α. To be precise, as a representative L we take

$$L = \Pi_\alpha \exp[\chi^\alpha E_\alpha] \Pi_I \exp[\frac{1}{2}\phi^I H_I] , \tag{3.4.18}$$

where the ϕ^I are called the dilatons and χ^α the axions. The number of dilatons is given by the rank of $\mathfrak{g}$ and the number of axions is given by the number of positive roots.

In terms of the Iwasawa decomposition (3.4.15) we see that our coset representative is written in terms of $L = g_N g_\mathcal{C}$. If we now multiply this representative from the left with an element $g \in G$ and make use of the Iwasawa decomposition we see that we must be able to write $gL(y)$ as

$$gL(y) = L(y')g_H , \tag{3.4.19}$$

where $L(y') = g'_N g'_\mathcal{C}$. We can now use a local $h \in H$ to remove g_H such that we are back in the Borel gauge. This is the gauge obtained via exponentiating the Borel subalgebra.

What we have done here is only valid in case G is so-called *maximally non-compact*. A group G is maximally non-compact if the Iwasawa decomposition allows the representative to be given by *all* the Cartan generators.

In general the Iwasawa decomposition ensures the existence of a solvable Lie algebra[1] *Solv*, that is a real semisimple Lie algebra $\mathfrak{G}$ of a group G can be written as $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a solvable Lie algebra consisting out of the non-compact part of the Cartan generators and a subset of the positive root generators [53, 55]. One can then similarly use this solvable algebra as the basis for the representative. For the case we have a maximally non-compact G the solvable gauge is called the Borel gauge. In case not all Cartan generators are in $\mathfrak{s}$ we call G non-maximally non-compact. See [52] for a discussion of this in the case of dimensionally reduced heterotic supergravity.

3.4.1 The Coset $\mathrm{SL}(n, \mathbb{R})/\,\mathrm{SO}(n)$

Now we specify to $G = \mathrm{SL}(n, \mathbb{R})$, which has rank $n-1$ and $\mathrm{SO}(n)$ as its maximal compact subgroup. The number of positive roots is $n(n-1)/2$. There are therefore $n-1$ dilatons ϕ^I and $n(n-1)/2$ axions χ^α. The Cartan generators are given in terms of the weights $\vec{\beta}$ of $\mathrm{SL}(n, \mathbb{R})$ in the fundamental representation

$$(\vec{H})_{ij} = (\vec{\beta}_i)\delta_{ij} . \tag{3.4.20}$$

[1] A solvable Lie algebra is defined as follows. Let $\mathfrak{G}^0 = \mathfrak{G}$ and for $k > 0$ we define $\mathfrak{G}^{k+1} = [\mathfrak{G}^k, \mathfrak{G}^k]$. If for finite n this series terminates, i.e. $\mathfrak{G}^n = 0$, then we call the Lie algebra $\mathfrak{G}$ solvable.

The weights can be taken to obey the following algebra

$$\sum_i \beta_{iI} = 0\,, \quad \sum_i \beta_{iI}\beta_{iJ} = 2\delta_{IJ}\,, \quad \vec{\beta}_i \cdot \vec{\beta}_j = 2\delta_{ij} - \frac{2}{n}\,. \tag{3.4.21}$$

The first of these identities holds in all bases since it follows from the tracelessness of the SL generators. The second and third identity can be seen as convenient normalizations of the generators. The positive step operators E_{ij} are all upper triangular and a handy basis is that they have only one non-zero entry $[E_{ij}]_{ij} = 1$. The negative step operators are the transpose of the positive. The SO(n) algebra is spanned by the following combinations

$$\frac{1}{\sqrt{2}}(E_\beta - E_{-\beta})\,. \tag{3.4.22}$$

The action will generically look complicated but when all axions are set to zero L is diagonal $L = \text{diag}[\,\exp(-\frac{1}{2}\vec{\beta}_i \cdot \vec{\phi})]$ and the action becomes

$$+\tfrac{1}{4}\text{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} = -\tfrac{1}{4}(\sum_i \beta_{iJ}\beta_{iI})\partial\phi^I\partial\phi^J = -\tfrac{1}{2}\delta_{IJ}\partial\phi^I\partial\phi^J\,. \tag{3.4.23}$$

This action describes $n-1$ dilatons that parameterize the flat scalar manifold $\mathbb{R}^{n-1}$. Above we have set all the axions χ^α to zero, this should be consistent with the equations of motion that follow from (3.2.12)

$$\partial_\mu(\sqrt{-g}\mathcal{M}^{-1}\partial^\mu\mathcal{M}) = 0\,. \tag{3.4.24}$$

In general one can truncate a set of scalar fields that parameterize a scalar manifold M to a smaller set of scalar fields that parameterize a submanifold $M' \subset M$ if M' is a totally geodesically submanifold [56]. This means that any geodesic in M' is also a geodesic in M.

The simplest example is SL$(2,\mathbb{R})/$ SO(2). The algebra is given by

$$[H, E_2] = 2E_2\,, \quad [H, E_{-2}] = -2E_{-2}\,, \quad [E_2, E_{-2}] = H\,. \tag{3.4.25}$$

The two-dimensional fundamental representation is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.4.26}$$

From which we find the coset representative

$$L = \exp[\chi E_2]\exp[\frac{1}{2}\phi H] = \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2}\chi \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix}, \tag{3.4.27}$$

which leads to the kinetic term

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,. \tag{3.4.28}$$

This is the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$-coset of type IIB supergravity[2]. In section 2.3.3 we found that the $\mathrm{SL}(2,\mathbb{R})$ extends to the whole Lagrangian, leading to S-duality. We refer to [45] for a realization of the somewhat less trivial example $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ with five scalar fields.

From the above we can construct the Lorentzian version of this coset, that is $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. This gives rise to the D(-1)-instanton of type IIB. As explained in section 3.2.1 we can use the same expression for L, but need to modify $\mathcal{M}$ to $L\eta L^T$ with $\eta = (-1,1)$. We find the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,. \tag{3.4.29}$$

Indeed we see that the metric is no longer positive definite.

It is important to mention that for maximally non-compact G the Borel gauge only covers the whole manifold if the subgroup H is its maximal compact subgroup. As we saw from the torus reduction this is no longer true when we reduce over time. The Borel gauge does not cover the whole manifold any more as shown in [57].

This can also be seen explicitly for the coset (3.4.29). We can rewrite it as

$$\mathcal{L} = -\frac{1}{2}\mathrm{e}^{-2\phi}(\partial\mathrm{e}^{\phi})^2 + \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,, \tag{3.4.30}$$

which is the metric on two-dimensional anti-de Sitter space (AdS_2) in terms of the coordinates (e^{ϕ},χ). It is known that these coordinates do not cover the whole manifold. Whereas we can rewrite (3.4.28) as Euclidean AdS_2, which does cover the whole manifold.

It is therefore better not to rely on the Borel gauge for this kind of computation at all, but it seems that a general good gauge choice (a gauge that can always be imposed) is unavailable [57]. To avoid this problem with the Borel gauge, we will work in chapter 7 on the level of $\mathcal{M}$ directly when we discuss the generating solution for instantons.

3.4.2 Maximally Extended Supergravities

Let us now see what happens if we consider the dimensional reduction of type IIA and type IIB on a n-torus and eleven-dimensional supergravity on a $(n+1)$-torus. As it turns out, we find the unique maximal supergravities in $D \leq 10$. That is D-dimensional supergravities, whose supersymmetry is the maximal allowed by the

[2] The minus sign in the exponent can be removed via the field redefinition $\phi \to -\phi$.

	Minkowskian	Euclidean
$D = 10$	SO(1,1)	SO(1,1)
$D = 9$	$\frac{\mathrm{GL}(2,\mathrm{I\!R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{GL}(2,\mathrm{I\!R})}{\mathrm{SO}(1,1)}$
$D = 8$	$\frac{\mathrm{SL}(3,\mathrm{I\!R})}{\mathrm{SO}(3)} \times \frac{\mathrm{SL}(2,\mathrm{I\!R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{SL}(3,\mathrm{I\!R})}{\mathrm{SO}(2,1)} \times \frac{\mathrm{SL}(2,\mathrm{I\!R})}{\mathrm{SO}(1,1)}$
$D = 7$	$\frac{SL(5,\mathrm{I\!R})}{\mathrm{SO}(5)}$	$\frac{\mathrm{SL}(5,\mathrm{I\!R})}{\mathrm{SO}(3,2)}$
$D = 6$	$\frac{\mathrm{SO}(5,5)}{\mathrm{S}[\mathrm{O}(5)\times\mathrm{O}(5)]}$	$\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5,\mathbb{C})}$
$D = 5$	$\frac{\mathrm{E}_{6(+6)}}{USp(8)}$	$\frac{\mathrm{E}_{6(+6)}}{USp(4,4)}$
$D = 4$	$\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}(8)}$	$\frac{E_{7(+7)}}{\mathrm{SU}^*(8)}$
$D = 3$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}(16)}$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}^*(16)}$

Table 3.4.1: *Cosets for maximal supergravities in Minkowskian and Euclidean signatures.*

space-time dimension D. The reason that we obtain maximal supergravities is that a torus reduction does not break supersymmetry.

One can classify these (maximal) supergravities by the scalar field interactions in the Lagrangian. Just as for example the $\mathrm{SL}(2,\mathrm{I\!R})/\mathrm{SO}(2)$ coset specifies the Lagrangian (3.4.28). The scalar fields parameterize a Riemannian manifold whose geometry fixes the interactions terms in the supergravity Lagrangian. We summarize the scalar manifolds of the maximally extended supergravities that appear after dimensional reduction of 11-dimensional supergravity on a torus in table 3.4.1 [58]. For future use we show it both for Minkowskian and Euclidean maximal supergravities[3]. The cosets G/H in the left column are all maximally non-compact since G is the maximal non-compact real slice of a semi-simple algebra and H is the maximal compact subgroup. Since H is compact the metric is strictly positive definite and the coset is Riemannian. The cosets G/H' in the right column only differ in the isotropy group H' which is some non-compact version of H and as a consequence G/H' is not Riemannian.

There is a third class of maximally extended supergravities which we did not put in the table. Namely the so-called *star supergravities* [58, 59]. These are Lorentzian theories, but do have a non-compact isotropy group H. We will meet these theories in

[3]The relation between supergravity theories and geometries is not always completely one-to-one. For example, in section 3.3 we have seen that a reduction can also generate a potential V. This potential is in general not determined by the geometry.

chapter 6 and show how they are related to the Minkowski theories in the left column of table 3.4.1 for the case $D = 10$.

3.5 From Branes to "Particles"

As we discussed in section 2.4, many supergravity solutions have the structure of p-branes. That is, they are charged electrically under a $(p+1)$-form gauge potential A_{p+1} or magnetically under a $(D-p-3)$-form gauge potential A_{D-p-3}, where D is the space-time dimension of the supergravity theory. Another characteristic of brane solutions is that the brane geometry has a flat $(p+1)$-dimensional worldvolume, see for example (2.4.12).

In general two different kinds of brane solutions are considered; timelike p-branes that are related to the string theory D-branes [60] (or M-branes) or spacelike p-branes (known as S-branes) who are conjectured to describe time-dependent phenomena in string theory [8]. Timelike p-branes have a Lorentzian worldvolume and are stationary solutions whereas Sp-branes have a Euclidean worldvolume and are explicitly time-dependent. The metrics are given by[4]

$$\begin{aligned} \text{timelike brane:} \qquad & \mathrm{d}s_D^2 = \mathrm{e}^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2B(r)}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2}), \\ \text{spacelike brane:} \qquad & \mathrm{d}s_D^2 = \mathrm{e}^{2A(t)}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2B(t)}(-\mathrm{d}t^2 + t^2\mathrm{d}\mathbb{H}^2_{D-p-2}), \end{aligned} \qquad (3.5.1)$$

where A, B are arbitrary functions and δ, η are respectively the Euclidean and the Lorentzian metric. There exist less symmetric solutions that break the worldvolume symmetries ($\mathrm{ISO}(p,1)$ and $\mathrm{ISO}(p+1)$) and the transversal symmetries ($\mathrm{SO}(D-p-1)$ and $\mathrm{SO}(D-p,1)$). There are two standard ways to achieve this. First there are extra functions multiplying the $\mathrm{d}x\mathrm{d}x$-terms on the worldvolume. Secondly, there are off-diagonal terms that mix worldvolume directions with transversal directions ($\mathrm{d}x\mathrm{d}\theta$), like for rotating timelike branes or twisted spacelike branes [39].

Solutions that are carried by a metric and scalars alone have a simpler mathematical structure then those solutions that are carried by non-trivial p-form potentials. At first sight, there is only a restricted class of brane solutions that can be found as solutions of a scalar-metric Lagrangian of the type

$$\mathcal{L} = \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j - V(\phi)\Big)\,, \qquad (3.5.2)$$

where G_{ij} is the metric on moduli space and $V(\phi)$ is a scalar potential. If we regard a scalar potential V as a 0-form "field strength" then it can couple magnetically to $(D-2)$-branes, thus domain-walls (timelike) and cosmologies (spacelike) based on the reasoning explained in section 2.4.

[4] We choose $A = C$ in (2.4.22).

On the other hand $\partial\phi$ is a 1-form field strength and can therefore couple magnetically to $(D-3)$-branes and electrically to (-1)-branes. Note that for timelike (-1)-branes the worldvolume is zero-dimensional and the transverse space covers the whole space. Timelike (-1)-branes are solutions of Euclidean supergravity, i.e. they are instantons[5].

Apart from the $(D-3)$-branes (like the IIB 7-branes) the scalars only depend on one coordinate and the Ansatz is given by

$$\mathrm{d}s_D^2 = \epsilon f(r)^2 \mathrm{d}r^2 + g(r)^2 g_{ab}^{D-1} \mathrm{d}x^a \mathrm{d}x^b , \quad \phi^i = \phi^i(r) . \tag{3.5.3}$$

The function f corresponds to the gauge freedom of re-parameterizing the r-coordinate. For $\epsilon = -1$ the radial coordinate corresponds to time $(r = t)$ and g_{ab} is a metric on a Euclidean maximally symmetric space (the three possible FLRW geometries). When $\epsilon = +1$ (3.5.3) this describes an instanton geometry with r the direction of the tunneling process. For $\epsilon = +1$ and g_{ab}^{D-1} a Lorentzian maximally symmetric space (AdS, Minkowski or dS) (3.5.3) is a domain-wall geometry with r the transversal distance from the wall. The difficulty with $(D-3)$-branes is that these solutions depend on one complex coordinate rather that on one real coordinate. For this reason we do not consider $(D-3)$-branes in this thesis.

Let us now explain that also the other p-brane solutions can be related to the Lagrangian (3.5.2).

The worldvolume of a p-brane corresponds to Killing directions of space-time, and for that to be valid the matter fields do not depend on the worldvolume coordinates. This implies that one can "dimensionally reduce" a p-brane over its worldvolume[6]. In the dimensionally reduced theory the p-brane then corresponds to a (-1)-brane, since the worldvolume is zero-dimensional. These reductions are the torus reductions we described in section 3.2. Comparing (3.5.1) with (3.2.6) we see that we have to turn off the Kaluza–Klein vectors. Furthermore, the worldvolume of the theory is identified with $\mathcal{M}_{mn}$. We see that in general $\mathcal{M}_{mn}$ breaks the worldvolume symmetries (ISO$(p,1)$ and ISO$(p+1)$), since we will obtain extra terms multiplying the $\mathrm{d}x\mathrm{d}x$-terms on the worldvolume. If we reduce to $D = 3$ we dualize all Kaluza–Klein vectors to scalars, see (3.2.14). These Kaluza–Klein vectors will lead to off-diagonal terms that mix worldvolume directions with transversal directions $\mathrm{d}x\mathrm{d}\theta$.

We see that p- and Sp-branes reduced over their worldvolume lead to a system containing gravity and scalar fields only! In the case of timelike branes this is probably best known for the correspondence between four-dimensional black holes (0-branes) and three-dimensional instantons [56, 61]. We refer to [62] for a similar discussion in the case of spacelike p-brane solutions in maximal supergravity.

[5] In chapter 6 and 7 we consider a few examples where we have a Euclidean theory *with* a potential. We call these solutions instantons as well.

[6] We put dimensionally reduce between "" since the worldvolume is not compact.

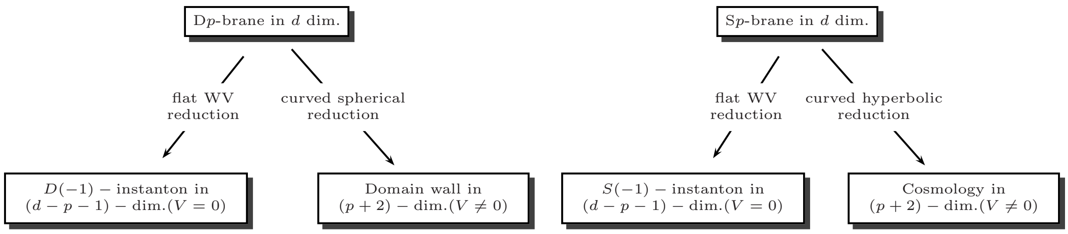


Figure 3.5.1: Starting from a brane in a Lorentzian d-dimensional space-time we show four possible reductions. The worldvolume reduction of a Dp-brane leads to an instanton in the lower-dimensional theory. Reducing over its transverse space gives a domain-wall. Starting from an Sp-brane, the worldvolume reduction leads to an S(-1)-brane, while the reduction over its transverse space leads to a cosmology.

If we instead compare the metric Ansätze (3.5.1) with that of (3.3.1) we see that it should also be possible to reduce a brane solution over its transversal space; a $(D-p-2)$-sphere ($\mathrm{d}\Omega^2_{D-p-2}$) for timelike branes and a $(D-p-2)$-hyperboloid ($\mathrm{d}\Sigma^2_{D-p-2}$) in case of a spacelike brane.

As we showed in section 3.3, after such a reduction the reduced brane is a $(D-2)$-brane that couples to a non-zero scalar potential $V(\phi)$. In case of timelike p-branes this is the known procedure to obtain brane solutions via uplifting domain-wall solutions of gauged supergravities [63]. In case of spacelike branes this is known from the fact that some (accelerating) cosmological quintessence-like solution obtained from hyperbolic reductions lift up to S-branes as shown in many papers (see for instance [64, 65]).

So if we start with gravity alone in the higher-dimensional space-time we see that the branes (3.5.1), when reduced over their worldvolume or over their transverse space, lead to the general Lagrangian (3.5.2). Furthermore, the Ansätze for the metric and scalar fields are given by (3.5.3). Everything depends only on one parameter r. This is like the physics of a "particle"! This explains the title of the thesis, 'Particle Dynamics of Branes'. In the next chapter we will see that the inclusion of a higher-dimensional $(p+2)$-form field strength leads to extra axions in the lower-dimensional theory. So again we have only scalar fields. In figure 3.5.1 we have summarized the various reductions.

As said, an action containing a metric and scalar fields alone have a simpler mathematical structure than those solutions that are carried by non-trivial p-form potentials. When we have solved the lower-dimensional (scalar) equations of motion we can lift up the solution to the original theory. This way we have a solution carried by a non-trivial p-form potential as well. Let us discuss the situation with and without a potential separately.

(−1)-branes

In case $V = 0$ we have (-1)-branes. The metric Ansatz is given by (3.5.3). The trick in solving the equations of motion is that the scalar part of (3.5.2) describes geodesic motion on the scalar manifold. To see this we re-parameterize the coordinate r as the harmonic function $h(r)$ via

$$\mathrm{d}h(r) = g^{1-D} f \mathrm{d}r \,, \tag{3.5.4}$$

then the scalar part of the action becomes

$$S = \int G_{ij}(\phi)\partial_h \phi^i \partial_h \phi^j \mathrm{d}h \,. \tag{3.5.5}$$

From this it follows that the solution describes geodesic motion on the moduli space with h as an affine parameter. Namely, in section 2.1 we have seen that the variation of the action $S \propto \int \sqrt{-g_{\mu\nu}x'^{\mu}x'^{\nu}}\mathrm{d}s$ leads to the geodesic equation (2.1.6), where a prime means a derivative with respect to the geodesic length s. One can show that the action

$$S \propto \int g_{\mu\nu}x'^{\mu}x'^{\nu}\mathrm{d}s \,, \tag{3.5.6}$$

leads to the same geodesic equation. Comparing this with (3.5.5) we see that $G_{ij}(\phi)$ takes on the role of $g_{\mu\nu}$, the scalar fields that of x^{μ} and h replaces the affine parameter s. We see that in case there is no potential, the scalar fields trace out geodesics on the scalar manifold. From this we know that the affine velocity $||v||^2$ defined by

$$||v||^2 = G_{ij}\partial_h \phi^i \partial_h \phi^j \,, \tag{3.5.7}$$

is a constant.

The Einstein equation for a (-1)-brane is given by

$$\mathcal{R}_{rr} = \tfrac{1}{2} G_{ij}\partial_r \phi^i \partial_r \phi^j \,, \qquad \mathcal{R}_{ab} = 0 \,. \tag{3.5.8}$$

For the metric (3.5.3) we derive that the Ricci tensor is given by

$$\begin{aligned} \mathcal{R}_{ab} &= -\epsilon\Big\{\frac{\mathrm{d}}{\mathrm{d}r}[\frac{g\dot{g}}{f^2}] + \frac{g\dot{g}\dot{f}}{f^3} + (D-3)\frac{\dot{g}^2}{f^2}\Big\} g^{D-1}_{ab} + \mathcal{R}^{D-1}_{ab} \,, \\ \mathcal{R}_{rr} &= (D-1)\Big\{-(\frac{\ddot{g}}{g}) + \frac{\dot{g}\dot{f}}{gf}\Big\} \,, \end{aligned} \tag{3.5.9}$$

where a dot refers to a derivative with respect to r. Combining the Einstein equations together with (3.5.7) we deduce the following first-order equation

$$\dot{g}^2 = \frac{||v||^2}{2(D-2)(D-1)} f^2 g^{4-2D} + \epsilon k f^2 \,. \tag{3.5.10}$$

A solution exists when the right-hand side remains positive. There is no equation of motion for f since it corresponds to the re-parametrization freedom for r. We thus see that the metric can be solved without having to know anything about the scalar field solutions!

We now have to make a difference between the reduction of space- and timelike branes. The former gives rise to a coset with a compact isotropy group. This means that the metric G_{ij} will be positive definite and hence $||v||^2 > 0$. So we have only spacelike geodesics. Via uplifting the S(-1)-brane we obtain a (fluxless) S-brane. *We thus have an S(-1)-brane / Sp-brane map.* This will be the subject of the next chapter.

For the timelike branes on the other hand the isotropy group is non-compact, for example SO$(n-1,1)$ instead of SO(n). Because of this G_{ij} will not be a positive definite metric on the scalar manifold and $||v||^2$ can be zero, positive or negative. We call these respectively lightlike, spacelike or timelike geodesics. We will discuss these (-1)-branes in chapter 7. *This way we obtain a (-1)-brane / p-brane map.*

Domain-walls and cosmologies

Let us now discuss what happens if $V \neq 0$. We know that the presence of the potential leads to domain-walls (timelike) and cosmologies (spacelike). Due to the potential there is a priori no reason to assume that these solutions are still geodesics of the scalar manifold. Under certain conditions however this turns out to be the case. Namely when we have a so-called scaling solution, see subsection 5.1.3. Let us illustrate this. A scaling solution has the property that if we calculate the *on-shell* potential V and kinetic energy $T = \frac{1}{2}G_{ij}\partial_r\phi^i\partial_r\phi^j$ we find that they have the same r-dependence. Effectively, we can consider $T + V$ as some new T only. From what we discussed above, we know that this means geodesic motion. Of course, filling in on-shell information is rarely a consistent procedure. We analyze this in chapter 5. There we will also show that scaling solutions are important since they correspond to the so-called critical points of autonomous differential equations governing the evolution of cosmologies. The critical points say a lot about the general evolution of a cosmology.

Besides that both type of solutions couple to a potential, their Lorentzian Ansätze also look very much the same. In [66] it was first noted that for a given domain-wall one also finds a cosmology. This was worked out in detail in [67] and is called the domain-wall / cosmology correspondence. We give a summary of this correspondence in chapter 6.

Chapter 4

Massless Time-Dependent Solutions

In this chapter we are going to look for time-dependent solutions of the Lagrangian (3.5.2) without a potential V. We are then discussing S(-1)-branes belonging to the action

$$S = \int \mathrm{d}^D x \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2} G_{ij} \partial\phi^i \partial\phi^j\Big)\,. \tag{4.0.1}$$

We will restrict to scalar manifold metrics G_{ij} which belong to maximally non-compact cosets G/H with H its maximal compact subgroup. If we consider solutions that depend only on the time, $\vec{\phi}$ is a geodesic on the scalar manifold as we explained in section 3.5. To find the most general geodesic we are going to construct a solution-generating technique.

From section 3.5 we know that the above action can be obtained from reducing gravity together with a dilaton and a p-form over a Euclidean torus. If we oxidize the time-dependent geodesic solution back to the original higher-dimensional theory we will obtain a (fluxless) Sp-brane. This leads to the S(-1)-brane / Sp-brane map.

The work in this section is done together with E. A. Bergshoeff, W. Chemissany, T. Van Riet and M. Trigiante [68, 69].

4.1 S(-1)-brane Geometries

We want to look for solutions belonging to the action (4.0.1) which only depend on the time coordinate t. The Ansätze for the time-dependent S(-1)-brane is given by (3.5.3)

$$\mathrm{d}s_D^2 = -f^2(t)\mathrm{d}t^2 + g^2(t) g_{ab}^{D-1} \mathrm{d}x^a \mathrm{d}x^b\,, \qquad \phi^i = \phi^i(t)\,. \tag{4.1.1}$$

In section 3.5 we showed that the scalar part of (4.0.1) leads to a geodesic on the scalar manifold with affine parameter the harmonic function h. In terms of this affine parameter the velocity $||v||$ is a strictly positive constant

$$||v||^2 = G_{ij}\partial_h\phi^i\partial_h\phi^j > 0\,. \tag{4.1.2}$$

Via combining the scalar field equations and the Einstein equations we deduced that the metric can be found from solving (3.5.10). If we choose to work in the gauge where $g^2 = t^2$ we find that the Einstein equations (3.5.10) give the following D-dimensional metric

$$\mathrm{d}s_D^2 = -\frac{\mathrm{d}t^2}{a\,t^{-2(D-2)} - k} + t^2\mathrm{d}\Sigma_k^2\,, \qquad a = \frac{||v||^2}{2(D-1)(D-2)}\,, \tag{4.1.3}$$

while the scalar fields trace out geodesic curves with the harmonic function $h(t)$ as affine parameter. The harmonic function is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)}\log\left|\sqrt{a}t^{2-D} + \sqrt{at^{2(2-D)} - k}\right| + c\,. \tag{4.1.4}$$

We take $c = 0$ in what follows since it just corresponds to a shift in the affine parameter h.

Now that we have solved the metric, we proceed by explaining how one can find the scalar field geodesics.

4.2 A Solution-Generating Technique

To discuss geodesic curves it is useful to introduce coordinates (scalar fields) on the moduli space. As explained in section 3.4, we use the solvable gauge which for maximally non-compact manifolds G/H coincides with the Borel gauge. In the Borel gauge the scalar fields are divided in dilatons ϕ^I and axions χ^α. This is done by choosing the coset element as follows

$$L = \Pi_\alpha \exp\,[\chi^\alpha E_\alpha]\Pi_I \exp\,[\tfrac{1}{2}\Phi^I H_I]\,, \tag{4.2.1}$$

where H_I are the Cartan generators of the Lie algebra of G and the E_α are the positive root operators. The number of Cartan generators is the rank r of the Lie algebra of G and for the cosets listed in table 3.4.1 the rank is $r = 11 - D$. The number of axions equals the dimension of the isotropy group H for maximally non-compact cosets since the Lie algebra of H is spanned by the combinations $E_\alpha - E_{-\alpha}$.

Our approach to understand all the geodesic curves is by constructing *the generating solution.* By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group

G generates all other geodesics from the generating solution. Below we will explain that for cosets G/H where G is a maximally non-compact real slice of a complex semi-simple group and H is the maximal compact subgroup, the generating solution can be taken to be the straight line through the origin carried by the dilaton fields

$$\boxed{\phi^I(h) = v^I\, h\,, \qquad \chi^\alpha = 0\,, \quad I = 1,\dots r\,.} \tag{4.2.2}$$

Here h is the harmonic function. This solution contains r arbitrary integration constants v^I, with r the rank of G. This theorem applies to all the cosets in the left column of table 3.4.1.

Since the straight line solution is the generating solution, by definition G-transformations on this solution generate all the other geodesic curves. The number of independent constants in G is the dimensions of G which is $r + 2\dim H$. In total this gives us $2\,r + 2\dim H$ arbitrary (integration) constants as expected since there are $r + \dim H$ scalars (coordinates) for which we have to specify the initial place and velocity. The number of dilatons is given by r, the number of axions by $\dim H$. However this counting exercise is no proof since it might be that the action of G does not create independent integration constants or if the solutions lie in disconnected areas. The latter is the case for the cosets in the right column of table 3.4.1. There the straight line solution is not generating since the affine velocity is positive

$$||v||^2 = \sum (v^I)^2 > 0\,. \tag{4.2.3}$$

The affine velocity is invariant under G-transformations and by transforming the straight line we only generate spacelike geodesics. But cosets with non-compact isotropy H have metrics with indefinite signature and therefore allow for spacelike, lightlike and timelike geodesics. In chapter 7 we derive the generating solutions for cosets with non-compact isotropy group $\mathrm{SO}(p,q)$.

Let us repeat the proof of (4.2.2) as given in [68][1]. In the Borel gauge the geodesic equation is

$$\ddot\phi^I + \Gamma^I_{JK}\dot\phi^J\dot\phi^K + \Gamma^I_{\alpha J}\dot\chi^\alpha\dot\phi^J + \Gamma^I_{\alpha\beta}\dot\chi^\alpha\dot\chi^\beta = 0\,, \tag{4.2.4}$$

$$\ddot\chi^\alpha + \Gamma^\alpha_{JK}\dot\phi^J\dot\phi^K + \Gamma^\alpha_{\beta J}\dot\chi^\beta\dot\phi^J + \Gamma^\alpha_{\beta\gamma}\dot\chi^\beta\dot\chi^\gamma = 0\,. \tag{4.2.5}$$

Since $\Gamma^I_{JK} = 0$ and $\Gamma^\alpha_{JK} = 0$ at points for which $\chi^\alpha = 0$ a trivial solution is given by

$$\phi^I = v^I\, t\,, \qquad \chi^\alpha = 0\,, \tag{4.2.6}$$

for some parameter t. How many other solutions are there? A first thing we notice is that every global G-transformation $\Phi \to \tilde\Phi$ brings us from one solution to another

[1] See also the appendix of [70] for earlier remarks.

solution. Since G generically mixes dilatons and axions we can construct solutions with non-trivial axions in this way. We now prove that in this way *all* geodesics are obtained and this depends on the fact that G is maximally non-compact with H the maximal compact subgroup of G.

Consider an arbitrary geodesic curve $\Phi(t)$ on G/H. The point $\Phi(0)$ can be mapped to the origin $L = \mathbb{1}$ using a G-transformation, since we can identify $\Phi(0)$ with an element of G and then we multiply the geodesic curve $\Phi(t)$ with $\Phi(0)^{-1}$, generating a new geodesic curve $\Phi_2(t) = \Phi(0)^{-1}\Phi(t)$ that goes through the origin. The origin is invariant under H-rotations but the tangent space at the origin transforms under the adjoint of H. One can prove that there always exists an element $k \in H$, such that $\mathrm{Adj}_k\dot{\Phi}_2(0) \in$ CSA [71]. Therefore $\dot{\chi}_2^\alpha = 0$ and this solution must be a straight line. So we started out with a general curve $\Phi(t)$ and proved that the curve $\Phi_3(t) = k\Phi(0)^{-1}\Phi(t)$ is a straight line. If we take $t = h$ it follows that the scalar fields are given by (4.2.2).

4.3 Spacelike Branes

In this section we consider the time-dependent (-1)-brane solutions in D dimensions and their uplift to general Sp-branes in $D + p + 1$ dimensions.

Sp-branes are solutions of the following action

$$\mathcal{L} = \sqrt{-g}\Big(\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2(p+2)!}\mathrm{e}^{b\phi}F_{p+2}^2\Big)\,, \tag{4.3.1}$$

with b the dilaton coupling constant. The reduction Ansatz for the metric is as in section 3.2

$$\mathrm{d}s^2_{D+p+1} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_D + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}\mathrm{d}z^n \otimes \mathrm{d}z^m\,, \tag{4.3.2}$$

where

$$\alpha^2 = \frac{p+1}{2(D+p-1)(D-2)}\,, \qquad \beta = -\frac{(D-2)\alpha}{p+1}\,. \tag{4.3.3}$$

The matrix $\mathcal{M}$ and the scalar φ are the moduli of the $(p+1)$-torus and depend on the D-dimensional coordinates. In particular $\mathcal{M}$ is a positive-definite symmetric $(p+1)\times(p+1)$ matrix with unit determinant and the modulus φ controls the overall volume. For a dimensional reduction over a Euclidean torus the scalars parameterize $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$ where φ belongs to the decoupled $\mathbb{R}$-part and $\mathcal{M}$ is the $\mathrm{SL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$ part. More precisely $\mathcal{M} = LL^T$ where L is the vielbein matrix of the internal torus and it also plays the role of the coset representative of $\mathrm{SL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$.

The reduction of a $(p+1)$-form A^{p+1} over a $(p+1)$-torus gives a scalar χ and various other forms of lower degree in D dimensions since

$$A^{p+1} = \sum_{i=0}^{p+1} A^{(i)}(x)\mathrm{d}z^{i+1} \wedge \mathrm{d}z^{i+2} \wedge \ldots \wedge \mathrm{d}z^{p+1}\,. \tag{4.3.4}$$

Here the $A^{(i)}$ are the gauge potentials of rank i. If one of the non-trivial forms in the series is a $(D-2)$-form, it can be dualized to a scalar field χ_2 in the lower dimension. Since we start from an electric Ansatz, we can have magnetic flux only if we have a dyonic solution in $D+p+1$ dimensions. This gives the constraint $p+2 = D-1$. Non-zero values for χ and χ_2 imply then respectively non-zero electric and magnetic flux. The reduction Ansatz for electrical solutions is $\hat{A} = \chi(x)\mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^{p+1}$.

The reduced D-dimensional Lagrangian is

$$\mathcal{L} = \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} - \tfrac{1}{2}\mathrm{e}^{b\phi+2(D-2)\alpha\varphi}(\partial\chi)^2\Big\}\,. \tag{4.3.5}$$

If the scalar fields in $\mathcal{M}$ are non-trivial then $\mathcal{M} \neq \mathbb{1}$ and the $\mathrm{ISO}(p+1)$ worldvolume symmetries of the brane becomes smaller. The fact that we are able to write down the most general solution with a deformed worldvolume illustrates the power of our approach.

After an appropriate SO(2)-rotation of the two dilatons φ and ϕ we get the more familiar Lagrangian for the scalars that parameterize $\mathbb{R} \times \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$

$$\mathcal{L} = \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\varphi')^2 - \tfrac{1}{2}(\partial\phi')^2 - \tfrac{1}{2}\mathrm{e}^{c\phi'}(\partial\chi)^2 + \tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1}\Big\}\,, \tag{4.3.6}$$

where the $'$ denotes that the scalars are rotated versions of the original scalars and where the radius of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ part is given by

$$c = \sqrt{b^2 + 2\tfrac{(D-2)(p+1)}{D+p-1}}\,. \tag{4.3.7}$$

The $\mathrm{SL}(2,\mathbb{R})$ transformations Ω work in a non-linear fashion on ϕ' and χ, but on the level of the scalar matrix

$$\mathcal{M}_2 = \mathrm{e}^{\frac{c}{2}\phi'}\begin{pmatrix} \frac{c^2}{4}\chi^2 + \mathrm{e}^{-c\phi'} & \frac{c}{2}\chi \\ \frac{c}{2}\chi & 1 \end{pmatrix}, \tag{4.3.8}$$

the transformation is $\mathcal{M}_2 \to \Omega\mathcal{M}_2\Omega^T$.

4.3.1 Pure Gravity

We start by considering Sp-brane solutions of pure gravity. The corresponding S(-1)-brane is given by geodesics on $\mathrm{GL}(p+1,\mathbb{R})/\,\mathrm{SO}(p+1)$. In section 4.2 we showed that the most general geodesic solution is given by the most general $\mathrm{SL}(p+1,\mathbb{R})$-transformation of the "straight line" through the origin (which is therefore the generating solution):

$$\varphi = v^{\varphi} h + c^{\varphi}\,, \qquad \phi^I = v^I h\,, \tag{4.3.9}$$

where I runs from 1 to p and $v^{\varphi}\,, c^{\varphi}$ and v^I are integration constants and h is given by (4.1.4). The ϕ^I are the dilaton scalars of $\mathrm{SL}(p+1,\mathbb{R})/\,\mathrm{SO}(p+1)$. The dilatons are related to the diagonal components of the metric $\mathcal{M}$ on the internal space via $\mathcal{M} = LL^T$ with L given by (4.2.1).

In case all axions are truncated we have that

$$\mathcal{M} = \mathrm{diag}(\exp[\beta_{iI}\phi^I])\,, \tag{4.3.10}$$

where the $\vec{\beta}_i$ are the weights of $\mathrm{SL}(p+1,\mathbb{R})$ in a suitable basis where they obey (3.4.21). The affine velocity follows from (4.2.3) and is given by $||v^2|| = (v^{\varphi})^2 + \sum_I (v^I)^2$.

Uplifts

Since the scalar field matrix transforms as $\mathcal{M} \to \Omega\mathcal{M}\Omega^T$ with $\Omega \in \mathrm{SL}(p+1,\mathbb{R})$ we notice that we only need to uplift the straight line geodesic since all other geodesics are just Ω-transformations which can be absorbed by redefining the torus coordinates $\mathrm{d}\vec{z}' = \Omega\mathrm{d}\vec{z}$. The higher-dimensional geometries we find depend on the curvature k of the lower-dimensional FLRW-space.

- For flat FLRW-spaces ($k=0$) the uplift becomes the Kasner solution [68]

$$\mathrm{d}s^2 = -\tau^{2p_0}\mathrm{d}\tau^2 + \sum_{a=1}^{D-1}\tau^{2p_a}(\mathrm{d}x^a)^2 + \sum_{b=1}^{p+1}\tau^{2p_b}(\mathrm{d}z^b)^2\,, \tag{4.3.11}$$

where

$$\begin{aligned} p_0 &= \frac{\alpha v^{\varphi}}{\sqrt{a}} + (D-2)\,,\\ p_a &= \frac{\alpha v^{\varphi}}{\sqrt{a}} + 1\,,\\ p_b &= \frac{\beta v^{\varphi}}{\sqrt{a}} + \frac{\beta_{bI}v^I}{2\sqrt{a}}\,, \end{aligned} \tag{4.3.12}$$

and a is given by (4.1.3). These numbers p obey the constraints

$$p_0 + 1 = \sum_{i=1}^{D+p} p_i\,, \quad (p_0+1)^2 = \sum_{i=1}^{D+p} p_i^2\,. \tag{4.3.13}$$

• If we consider a curved FLRW-space ($k \neq 0$) in the lower dimension then the uplift gives a vacuum solution with a bit more complicated metric. The uplift gives

$$\mathrm{d}s^2_{D+p+1} = W^u \Big[-\frac{\mathrm{d}t^2}{\mathrm{e}^{2\omega}t^{-2(D-2)} - k} + t^2\,\mathrm{d}\Sigma_k^2\Big] + \sum_{b=1}^{p+1} W^{w_b}(\mathrm{d}z^b)^2\,, \tag{4.3.14}$$

with

$$W = \mathrm{e}^{\omega}t^{2-D} + \sqrt{\mathrm{e}^{2\omega}t^{2(2-D)} - k}\,, \tag{4.3.15}$$

$$u = \tfrac{-2v^{\varphi}}{(D-2)||v||}\sqrt{\tfrac{(p+1)(D-1)}{D+p-1}}\,, \tag{4.3.16}$$

$$w_b = \frac{2v^{\varphi}}{||v||}\sqrt{\tfrac{D-1}{(D+p-1)(p+1)}} - \sqrt{\frac{2(D-1)}{||v||(D-2)}}\beta_{bI}v^I\,, \tag{4.3.17}$$

and ω is left arbitrary.

If we choose $v^I = 0$ and $k = -1$ the solution is the fluxless S-brane of [8, 65, 72]. For $k = 0, +1$ the solution is strictly not called an S-brane since there is no Lorentzian symmetry group $\mathrm{SO}(D-p-2, 1)$.

4.3.2 Dilaton-Gravity

Now we complicate matters by considering a non-zero dilaton ϕ. In D-dimensions the solution is

$$\varphi = v^{\varphi}h + c^{\varphi}\,, \qquad \phi = v^{\phi}h + c^{\phi}\,, \qquad \phi^I = v^I h\,. \tag{4.3.18}$$

The uplift to a fluxless Sp-brane gives a metric of the form (4.3.14) but now there is a non-constant dilaton $\phi(t)$ and $||v||$ gets an extra contribution

$$\phi(t) = -\tfrac{v^{\phi}}{||v||}\sqrt{\tfrac{2(D-1)}{(D-2)}}\log W + c^{\phi}\,, \tag{4.3.19}$$

$$||v||^2 = (v^{\varphi})^2 + (v^{\phi})^2 + \sum_{I=1}^{p}(v^I)^2\,. \tag{4.3.20}$$

When we put the dilaton to constant via $v^{\phi} = 0$ we end up with the pure gravitational solution (4.3.14-4.3.17).

4.3.3 . . . with Non-Trivial Flux

The uplift of the general solution with χ possibly non-zero requires the uplift of all $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ geodesics. But as explained before they can be obtained by an

SL(2, IR)-transformation on the straight line solution trough the origin $\phi = v^\phi h$. Thus we can obtain the general solution by transforming the fluxless solution (4.3.19) with $c^\phi = 0$. The metric reads

$$\mathrm{d}s^2_{D+p+1} = W^u\Big(\zeta'^2 W^z + \eta^2\Big)^x \Big[-\frac{\mathrm{d}t^2}{\mathrm{e}^{2\omega}t^{-2(D-2)} - k} + t^2\,\mathrm{d}\Sigma_k^2\Big] + \tag{4.3.21}$$

$$\sum_{b=1}^{p+1} W^{w_b}\Big(\zeta'^2 W^z + \eta^2\Big)^y (\mathrm{d}z^b)^2\,, \tag{4.3.22}$$

with

$$x = \tfrac{4(p+1)}{b^2(D+p-1)+2(D-2)(p+1)}\,, \quad y = -\tfrac{D-2}{p+1}x\,, \tag{4.3.23}$$

$$u = \frac{-2v^\varphi}{||v||(D-2)}\sqrt{\frac{(p+1)(D-1)}{D+p-1}}\,, \tag{4.3.24}$$

$$z = \frac{bv^\phi + 2(D-2)\alpha v^\varphi}{||v||}\sqrt{\frac{2(D-1)}{(D-2)}}\,. \tag{4.3.25}$$

The function W is defined in (4.3.15), w_b is given by (4.3.17) and $||v||$ by (4.3.20). The dilaton and the form field strength are given by

$$\phi(t) = \tfrac{2(D+p-1)b}{b^2(D+p-1)+2(D-2)(p+1)}\log\Big(\zeta'^2 W^z + \eta^2\Big) - \tfrac{v^\phi}{||v||}\sqrt{\tfrac{2(D-1)}{(D-2)}}\log W\,,$$

$$F_{ti_1\ldots i_{p+1}} = \Big(2(D-2)\sqrt{a}\,\zeta\eta\psi\Big)\times \tag{4.3.26}$$

$$\Big(\frac{t^{1-D}\sqrt{\mathrm{e}^{2\omega}t^{2(2-D)} - k} + \mathrm{e}^\omega t^{3-2D}}{W^{c\psi+1}[\zeta'^2 W^{-c\psi} + \eta^2]^2\sqrt{\mathrm{e}^{2\omega}t^{2(2-D)} - k}}\Big)\,\varepsilon_{i_1\ldots i_{p+1}}\,. \tag{4.3.27}$$

Let us explain the various integration constants. As before ω is left arbitrary and a and c are defined as before. The parameters ζ' and ψ are given by

$$\zeta' = \zeta\frac{\sqrt{a}}{e^\omega}\,, \tag{4.3.28}$$

$$\psi = -\tfrac{1}{c||v||}\sqrt{\tfrac{2(D-1)}{D-2}}\Big(b\,v^\phi + 2(D-1)\sqrt{\tfrac{p+1}{2(D+p-1)(D-2)}}v^\varphi\Big)\,, \tag{4.3.29}$$

where ζ, η come from the SL(2, IR)-transformation

$$\Omega = \begin{pmatrix}\gamma & \delta\\ \zeta & \eta\end{pmatrix}\,, \qquad \gamma\eta - \delta\zeta = 1\,. \tag{4.3.30}$$

The numbers v^φ and v^ϕ are the "velocities" of respectively φ and ϕ in the fluxless solution. One readily checks that the choice $\Omega = \mathbb{1}$ indeed reproduces the fluxless solution given in subsection 4.3.2 with $c^\phi = 0$. If we restrict to $D = 10$, $b = 2$, $p = -1$ and $v^\varphi = 0$ we have the S(-1)-brane of type IIB in a different coordinate frame as discussed in section 2.4.3.

In this section the have written down the most general Sp-brane with a deformed worldvolume.

S0-brane

As an illustration we consider the four-dimensional S0-brane considered in [8]. We do this for three reasons. First of all to show that we indeed reproduce known S-branes. Secondly, the parameters labelling the solutions do not yet have a physical meaning and finally to show that from a higher-dimensional point of view not all parameters are independent.

The four-dimensional non-dilatonic S0-brane belongs to the action (4.3.1) if we take $D = 4$, $p = 0$ and $k = -1$

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big(\mathcal{R} - \tfrac{1}{4}F_2^2\Big)\,. \tag{4.3.31}$$

There is no dilaton present, so we need to take $b = v^\phi = 0$. We then have four remaining parameters ω, v^φ, ζ and η.

The S0-brane follows from (4.3.21) if we require

$$(v^\varphi)^2 \to \frac{2Q\tau_0}{\zeta^2}\,, \quad \omega \to \frac{1}{2}\log(\tau_0^2)\,, \quad \eta^2 \to \frac{Q}{2\tau_0}\,, \tag{4.3.32}$$

together with the new time coordinate τ defined as

$$t^2 = \tau^2 - \tau_0^2\,. \tag{4.3.33}$$

From these relations we derive the metric

$$\mathrm{d}s^2 = -\frac{Q^2}{\tau_0^2}\frac{\tau^2}{\tau^2-\tau_0^2}\mathrm{d}\tau^2 + \frac{\tau_0^2}{Q^2}\frac{\tau^2-\tau_0^2}{\tau^2}\mathrm{d}z^2 + \frac{Q^2}{\tau_0^2}\tau^2\mathrm{d}\mathbb{H}_2^2\,. \tag{4.3.34}$$

This is the S0-brane as given in [8].

Not all parameters we started from are independent parameters from the four-dimensional view point. From the explicit metric and field strength expressions we can see that v^φ only appears in the field strength via a. Furthermore, ζ always appears in the combination $\zeta\sqrt{a}$. So ζ and v^φ are *not* independent variables. This agrees with the first relation in (4.3.32). We can choose $\zeta = 1$ without loss of generality. Similarly one can show that there is one relation between the other three parameters.

Actually, we can also remove the parameter τ_0 from (4.3.34) via re-scaling the coordinates as follows

$$t = \frac{Q}{\tau_0}\tau\,, \quad r = \frac{\tau_0}{Q}z\,. \tag{4.3.35}$$

If we do this we end up with

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{1-\frac{Q^2}{t^2}} + (1-\frac{Q^2}{t^2})\mathrm{d}r^2 + t^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,, \tag{4.3.36}$$

from which it follows that Q is related to the electric charge. The symmetry of the metric (4.3.34) is $\mathrm{SO}(2,1)\times\mathbb{R}$. Here the $\mathrm{SO}(2,1)$ is the symmetry transverse to the brane worldvolume and is referred to as the R-symmetry.

The S0-brane (4.3.34) is a singular solution [8], this follows for example from considering the invariant $\mathcal{R}_{\mu\nu\rho\eta}\mathcal{R}^{\mu\nu\rho\eta}$ where $\mathcal{R}_{\mu\nu\rho\eta}$ is the Riemann tensor. There are two ways that the singularity might disappear in the full theory. Namely the singularity might be smoothed out by stringy effects which are non-perturbative in α' or g_s [8]. All the original isotropic S-branes have singularities similar to the S0-brane.

In 2004 non-singular S-branes were found via a different way. Let us illustrate this with the examples given in [39, 73]. The original S-branes are homogenous, isotropic and time dependent. These solutions can be derived from known isotropic p-brane solutions via analytic continuations. For example, the S0-brane we described above follows from an analytically continuated Reissner-Nordström black hole with mass m and charge Q. To be specific, if we apply the following set of analytic continuations [73]

$$t \to ir\,, \quad r \to it\,, \quad \theta \to i\theta\,, \quad m \to im\,, \tag{4.3.37}$$

to the Reissner-Nordström metric we find

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{1-\frac{2m}{t}-\frac{Q^2}{t^2}} + (1-\frac{2m}{t}-\frac{Q^2}{t^2})\mathrm{d}r^2 + t^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,. \tag{4.3.38}$$

If we consider the massless limit this becomes (4.3.36) [74]. Similarly, the other S-branes can be related to analytic continuations of isotropic p-branes.

A new class of S-branes can be found by deforming the $\mathrm{SO}(D-p-2,1)$ R-symmetry of the S-branes. The easiest way is to consider the analytic continuation of known non-isotropic branes. In [75] is was shown that an analytic continuation of *rotating* p-branes leads to non-singular S-branes. In general, the rotating p-branes have singularities for large angular momenta. However, after the analytic continuation the resulting S-branes are regular everywhere. These branes are called twisted S-branes. Reducing the R-symmetry is thus one way to cure the singularity problem of the original S-branes.

4.4 Discussion

In this chapter we first introduced the concept of a generating solution. A generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group G generates all other geodesics from the generating solution. We then presented the theorem that for maximally non-compact cosets G/H, with H its maximal compact subgroup, the generating solution can be constructed from the Cartan subalgebra only.

We illustrated this technique for Sp-branes. That is we studied a Lagrangian containing gravity, a dilaton and a $(p+1)$-form potential. Reducing this over the worldvolume of the Sp-brane gives the coset $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. Using the above mentioned theorem we presented the generating solution for the S(-1)-brane belonging to this coset. Acting with the $\mathrm{SL}(2,\mathbb{R})$-group on this solution and oxidizing to the higher-dimensional theory we obtained the most general Sp-brane solution with a deformed worldvolume. This is the S(-1)-/Sp-brane map.

However, we made various simplifications. First we did not dualize any forms to scalars in the reduced theory which would add magnetic flux to the above Sp-brane solutions. Secondly we did not consider intersections of S-branes which are carried by multiple forms with different degrees. Nonetheless with our approach they can be found with some extra effort. These extensions would just add extra axions to the lower-dimensional Lagrangian which extend the coset to the cosets in the left column of table 3.4.1. All the geodesics on these cosets must correspond to specific time-dependent S-brane solutions. Since the generating geodesics for the cosets on the left column of table 3.4.1 are the dilatonic straight lines, it must be that all S-brane type solutions can be rotated to pure gravitational solutions in 11 dimensions or to dilaton–Einstein solutions in type II supergravity. For example, if we take $\zeta = 0$ and $\eta = 1$ in (4.3.27) the flux becomes zero. In this way we see that the $SL(2,\mathbb{R})$-subgroup mixes physically distinct solutions in the higher-dimensional theory.

If we reduce to three dimensions a symmetry-enhancement of the coset takes place. The dualisation of the three-dimensional Kaluza–Klein vectors generate the coset $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+2)$ instead of the expected $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. However the generating solution of the $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+2)$-coset has only non-trivial dilatons and is therefore the same as the generating solution of $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. Nonetheless, there is an important difference with the time-dependent solutions from $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. In that case a solution-generating transformation $\in \mathrm{GL}(p+1,\mathbb{R})$ can be interpreted as a coordinate transformation in $D+p+1$ dimensions and therefore maps the vacuum solution to the same vacuum solution in different coordinates. In the case of symmetry enhancement to $\mathrm{SL}(p+2,\mathbb{R})$ a solution-generating transformation is not necessarily a coordinate transformation in $D+p+1$ dimensions. Instead the time-dependent vacuum solution transforms into a "twisted" vacuum solution. Where the twist indicates off-diagonal terms that cannot be redefined away. Such twisted solutions with $k = -1$ have received considerable interest since they can be regular [38, 39].

The solution-generating technique presented here should be considered complementary to the "compensator method" developed by Fré et al in [62]. There the straight line also serves as a generating solution but instead of rigid G-transformations one uses local H-transformations that preserve the solvable gauge to generate new non-trivial solutions. This technique is an illustration of the integrability of the second–order geodesic equations of motion [76].

Chapter 5

Massive Time-Dependent Solutions

In this chapter we extend the time-dependent analysis to Lagrangians with a potential

$$\mathcal{L} = \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j - V(\phi)\Big)\,. \tag{5.0.1}$$

We mentioned in section 3.5 that we can regard the scalar potential V as a 0-form field strength. This can couple magnetically to $(D-2)$-branes, i.e. domain-walls (timelike) and cosmologies (spacelike). In this chapter we will focus mainly on the latter, although many results apply to domain-walls as well.

We begin this chapter with a brief introduction to cosmologies. We will focus on a specific cosmological model, namely the generalization of the multi-exponential potential we introduced in section 3.3. We will not look for the most general solution, but restrict ourself to a critical point analysis. It turns out that these critical points are so-called scaling solutions. The surprising thing is that these scaling solutions are *still* geodesics of the scalar manifold, the presence of the potential does not upset this. In this chapter we are going to state the condition when for a potential V the scaling solution is still a geodesic of the scalar manifold. For a general discussion about cosmologies we refer to e.g. [77, 78].

The work of this chapter is based on collaborations with W. Chemissany, J. Hartong, T. Van Riet and D. Westra [68, 79].

5.1 Cosmologies

Due to cosmological observations we know now that our universe is both homogeneous and isotropic on the large scale. This means that our place in the universe is not special (homogeneous) and that the universe looks the same in all directions (isotropic). For this reason cosmological space-times are described by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 g_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{5.1.1}$$

where g_{ij} is the three-dimensional spatial metric and $a(\tau)$ is called the scale factor. Due to the observation that our universe is both homogeneous and isotropic g_{ij} can written as

$$\mathrm{d}s^2 = \frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{n-1} \,. \tag{5.1.2}$$

The Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G\, T_{\mu\nu} \,, \tag{5.1.3}$$

relates the space-time metric to the matter distribution in space-time. The latter is encoded in the energy-momentum tensor $T_{\mu\nu}$. For the FLRW-metric (5.1.1) we derive that this must have the form

$$T_{\mu\nu} = \mathrm{diag}(\rho, p g_{ij}) \,. \tag{5.1.4}$$

Here $\rho(\tau)$ is the energy density and $p(\tau)$ the pressure. The matter distribution is called the cosmological fluid. After rewriting the Einstein equations we derive the Friedmann equations

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \,, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(3p+\rho) \,. \end{aligned} \tag{5.1.5}$$

Here the function $H = \dot{a}/a$ is called the Hubble parameter and the dot is with respect to τ. Due to the conservation of energy we have for the cosmological fluid the continuity equations

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} + 3H(\rho + p) = 0 \,. \tag{5.1.6}$$

Finally, the relation between the energy density and the pressure is given by the equation of state parameter ω

$$p = \omega\rho \,. \tag{5.1.7}$$

For 'ordinary' matter such as radiation or dust $-1/3 < \omega < 1$. All cosmological fluids can be grouped in two different classes. Namely, those that respect the strong energy condition (SEC) and those that violated it, see e.g. [80]. The SEC is a specific condition on the energy momentum tensor. For the cosmological fluid this means that the matter has to obey $\omega > -1/3$.

For a flat universe we derive from the Friedmann equations (5.1.5) that they imply

$$\ddot{a} > 0 \longleftrightarrow (3\omega + 1)\rho < 0\,. \tag{5.1.8}$$

In other words, if matter does not obey the SEC we have that $\omega < -1/3$ and we see that we have accelerated expansion ($\rho > 0$).

When ω is constant we find from the (5.1.6) that

$$\rho \propto \frac{1}{a^{3(1+\omega)}}\,. \tag{5.1.9}$$

Using this together with the Friedmann equations we can solve for the scale factor, in case $k = 0$ we find

$$a(t) \propto \tau^{\frac{2}{3(\omega+1)}}\,. \tag{5.1.10}$$

Such a scale factor is called a power-law. When $k \neq 0$ the scale factor can also be solved but is more complicated.

5.1.1 Multi-Field Cosmology

Let us assume that we have a system consisting of N scalar fields ϕ^i with scalar manifold metric $G_{ij} = \delta_{ij}$ and a potential $V(\vec{\phi})$

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\Big[\frac{1}{2\kappa^2}\mathcal{R} - \tfrac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} - V(\vec{\phi})\Big]\,, \tag{5.1.11}$$

where $\kappa^2 = 8\pi G$ with G Newton's constant and

$$\partial\vec{\phi}\cdot\partial\vec{\phi} = \sum_{i=1}^{N}\delta_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{5.1.12}$$

The Ansatz for the metric is that of a flat ($k = 0$) FLRW-universe (5.1.1, 5.1.2) and accordingly the scalars only depend on cosmic time τ. The equations of motion are

$$H^2 = \tfrac{\kappa^2}{3}\Big[T + V(\vec{\phi})\Big]\,, \tag{5.1.13}$$

$$\dot{H} = -\kappa^2 T\,, \tag{5.1.14}$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \partial_i V(\vec{\phi}) = 0\,, \tag{5.1.15}$$

where T stands for the kinetic energy

$$T = \frac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi}\,. \tag{5.1.16}$$

Equation (5.1.15) is referred to as the Klein–Gordon equation. In terms of the fluid language we have

$$\rho = T + V\,, \quad p = T - V\,, \quad \omega = \frac{T - V}{T + V}\,. \tag{5.1.17}$$

The above equations of motion are coupled second order differential equations. Therefore the most general solution is hard to find. Instead we will focus on the late-time behaviour of the cosmological solution. For this we need to find the asymptotic behaviour of a solution. It often happens that this behaviour is determined by simpler equations that the ones above. To see why this is so we show that we can rewrite the equations of motion as an autonomous system.

Autonomous systems

Assume that we have a set of variables $x^i(t)$ that obey a first order equation that can be cast into the following form

$$\dot{x}^i(t) = f^i(x)\,. \tag{5.1.18}$$

Here $f^i(x)$ depends only on the variables x and do not contain the evolution parameter t explicitly. We see that we can consider $\vec{f}$ as the velocity field belonging to the curve $\vec{x}$. Assume now that this vector field $\vec{f}$ has a zero at some point $\vec{x}_0$. This simplifies (5.1.18) to

$$\vec{f}(\vec{x}_0) = \vec{0} \Longrightarrow \dot{x}^i(t) = 0\,. \tag{5.1.19}$$

Such a point $\vec{x}_0$ is called a fixed or critical point. Such a critical point is a trivial solutions since we can easily integrate (5.1.19).

What has this to do with late-time cosmology? A critical point can either be a stable or an unstable solution. The stability follows from checking whether a perturbation δ^i of a fixed point x_0^i vanishes or not. For this we have to plug the perturbation $x_0^i + \delta^i$ into the equations of motion (5.1.18) and only keep terms linear in δ^i. This leads to the set of first order equations

$$\dot{\delta}^i = (\partial_j f^i)|_{x=x_0}\delta^j\,. \tag{5.1.20}$$

The general solutions for the perturbations δ^i are given by

$$\delta^i = \sum_j C^i_j e^{\lambda_j t}\,, \tag{5.1.21}$$

where λ_j are the eigenvalues of the matrix $(\partial_j f^i)|_{x=x_0}$ and C^i_j are real constants. If it happens that all λ_j are negative, we see from (5.1.21) that the perturbations decay exponentially. Such a critical point is called an attractor or sink.

If on the other hand some of the λ_j are positive these perturbations grow exponentially. Such a critical point is called a saddle point. When all eigenvalues are positive the point is called a repeller or source.

Let us now answer the question we posed above. Although we cannot solve for a general solution, it will generically be a curve in phase space interpolating between two critical points. For example, the curve can start at a repeller and will asymptotically reach the attractor. Thus we see that critical points, determined by the properties of the vector field $\vec{f}$, are important in understanding the general interpolating solution. In particular, the late-time cosmology is determined by the attractor critical point. Due to the exponential behaviour the attractor will not be reached in finite time.

We will now illustrate this with a specific potential.

5.1.2 Generalized Assisted Inflation

In section 3.3 we found the multi-exponential potential (3.3.21). Such exponential potentials also arise in models motivated by string theory such as supergravities obtained from dimensional reduction (see for instance [46,81–84] and references therein), descriptions of brane interactions [85–87], nonperturbative effects and the effective description of string gas cosmology (see for instance [88]). Also, these models allow to find exact solutions, which correspond to critical points in an autonomous system[1].

Let us generalize (3.3.21) to a sum of M exponential terms

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa \langle \alpha_a \,, \phi \rangle], \tag{5.1.22}$$

where $\langle \alpha_a \,, \phi \rangle = \sum_{i=1}^{N} \alpha_{ai}\phi_i$. There are M vectors α_a with N components α_{ai}. The indices $i, j, \ldots$ run from 1 to N and denote the components of the vectors ϕ and α_a. The indices $a, b, \ldots$ run from 1 to M and label the different vectors α_a and the constants Λ_a.

Let us now make use of linear field redefinitions. If the scalars transform linearly as $\phi \longrightarrow \phi' = S\phi$, where S is an element of $\mathrm{GL}(\mathbb{R}, N)$ then the vectors α_a transform in the dual representation $\alpha_a \longrightarrow \alpha'_a = S^{-T}\alpha_a$. This can be seen from the definition of α'_a

$$\langle \alpha_a \,, \phi \rangle \equiv \langle \alpha'_a \,, \phi' \rangle \,. \tag{5.1.23}$$

Field redefinitions that shift the scalar fields leave the α_a invariant, but change the Λ_a.

From the action we can deduce some properties of this system by looking at transformations in scalar space. The kinetic term is invariant under constant shifts and $O(N)$-rotations of the scalars. These transformations map the multiple exponential

[1]For a review on dynamical systems in cosmology see [89].

potential to another multiple exponential potential but with different Λ_a and α_a. Such redefinitions do not alter the physics, they just rewrite the equations. Therefore qualitative features only depend on $O(N)$-invariant combinations of the α_a-vectors (for example $\langle\alpha_a\,,\,\alpha_b\rangle$). By shifting the scalars we can always re-scale R of the Λ_a to be ± 1, where R is the number of independent α-vectors

$$R = \text{Rank}[\alpha_{ai}]\,. \tag{5.1.24}$$

The number of linearly independent vectors α_a is denoted by R. If $R < N$ one can rotate the scalars such that $\phi_{R+1}, \ldots, \phi_N$ no longer appear in the potential ($\alpha_{ai} = 0$ for $i > R$). These scalars are then said to be decoupled or free.

Let us illustrate this for the single exponential potential

$$V = \Lambda \mathrm{e}^{\vec{\alpha}\cdot\vec{\phi}}\,. \tag{5.1.25}$$

According to the above we should be able to remove all but one scalar field from the potential (5.1.25). To achieve this consider the orthogonal field redefinition $\vec{\phi} \to \vec{\phi}'$

$$\phi'_1 = \frac{1}{||\vec{\alpha}||}\vec{\alpha}\cdot\vec{\phi}\,, \tag{5.1.26}$$

and the ϕ'_i ($i > 1$) are constructed orthogonal to this direction via a Gramm-Schmidt procedure. This procedure is guaranteed to preserve the kinetic term, but the potential now contains only one scalar field as claimed

$$V = \Lambda \mathrm{e}^{||\vec{\alpha}||\phi'_1}\,. \tag{5.1.27}$$

We end up with one massive field $\vec{\phi}'_1$ and $N-1$ massless fields ϕ'_i ($i > 1$). Although (5.1.25) looks like an interaction term, we find that it can be removed via field redefinitions. The resulting theory has only one self-interaction term and $N-1$ free fields.

One can rewrite equations (5.1.13-5.1.15) as an autonomous system. For this we first note that (5.1.14) is not an independent equation and can therefore be ignored. Secondly, we define the following dimensionless variables

$$X_i = \frac{\kappa\dot{\phi}_i}{\sqrt{6}\,H}, \quad Y_a = \frac{\kappa^2}{3\,H^2}\,\Lambda_a \exp[-\kappa\,\langle\alpha_a\,,\,\phi\rangle]\,. \tag{5.1.28}$$

If we write $X^2 = \sum_i X_i^2$ and $Y = \sum_a Y_a$ the equations of motion become

$$X^2 + Y - 1 = 0\,, \tag{5.1.29}$$

$$X'_i = 3X_i\Big(-1 + X^2\Big) + \sqrt{\tfrac{3}{2}}\sum_a \alpha_{ai}Y_a\,, \tag{5.1.30}$$

$$Y'_a = Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 6X^2\Big)\,, \tag{5.1.31}$$

where the prime denotes differentiation with respect to $\log(a)$ [2]. It can be shown that if $X^2 + Y - 1 = 0$ initially then equations (5.1.30-5.1.31) guarantee that it is satisfied at all times. Hence, given the correct initial conditions the dynamics is described by equations (5.1.30-5.1.31).

In the next section we will construct all critical point solutions of the autonomous system (5.1.30-5.1.31). For that purpose it is useful to consider the matrix A

$$A_{ab} = \langle \alpha_a \, , \, \alpha_b \rangle \, . \qquad (5.1.32)$$

The models are divided into two classes. The first class is defined by $R = M$ and the second class by $R < M$. Algebraically the two differ in the following way

$$1. \;\; R = M \quad \Longleftrightarrow \quad \mathrm{Det} A > 0 \, , \qquad (5.1.33)$$
$$2. \;\; R < M \quad \Longleftrightarrow \quad \mathrm{Det} A = 0 . \qquad (5.1.34)$$

The first possibility, where A is invertible, is called generalized assisted inflation. We will not discuss the second class, for this we refer to [79].

Assisted inflation

Assisted inflation is the subclass where A is diagonal. This implies that in assisted inflation the α_a are perpendicular to each other and that one can choose an orthonormal basis in which $\alpha_{ai} = \alpha_a \, \delta_{ai}$. In that basis the potential becomes

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa \, \alpha_a \, \phi_a] \, . \qquad (5.1.35)$$

It is this particular form of the potential that is referred to as assisted inflation in the literature. We want to emphasize that the latter definition is basis-dependent. Potentials different from (5.1.35) but with a diagonal matrix A can be brought to the form (5.1.35) through an $O(N)$-rotation of the scalar fields. An $O(N)$-invariant definition of assisted inflation is that A is a diagonal matrix.

In any system with multiple fields but a single exponential such as studied in [91], the matrix A is trivially diagonal. One can perform a rotation on the scalars such that only one scalar appears in the potential and all the others are decoupled. In order to have a system whose scalars are mutually interacting one needs at least two exponential terms both containing more than one scalar.

[2] The use of $\log(a)$ as a time coordinate fails when $\dot{a} = 0$. This is no problem for studying critical points and their stability because in the neighborhood around a critical point there always exists a region where the coordinate is well defined. In reference [90] an explicit example is given where $\dot{a}$ becomes zero at some point.

5.1.3 The Critical Points

Critical points are defined as solutions of the autonomous system for which $X_i' = Y_a' = 0$. From the acceleration equation,

$$\dot{H} = -\tfrac{\kappa^2}{2}\partial\vec{\phi}\cdot\partial\vec{\phi}\,, \tag{5.1.36}$$

it follows that in a critical point $\dot{H}/H^2$ is constant. If the constant differs from zero we put $\dot{H}/H^2 \equiv -1/p$ and then the scale factor becomes

$$a(\tau) = a_0\,\big(\frac{\tau}{\tau_0}\big)^p\,. \tag{5.1.37}$$

In terms of the dimensionless variables p can be expressed as

$$p = \frac{1}{3X^2}\,. \tag{5.1.38}$$

When $\dot{H}/H^2 = 0$ the scale factor is

$$a(\tau) = a_0\,\mathrm{e}^{H\tau}\,, \tag{5.1.39}$$

and space-time is de Sitter.

The requirement that $Y_a' = 0$ can be satisfied in two ways as can be seen from

$$Y_a' = Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 6X^2\Big)\,. \tag{5.1.40}$$

Either $Y_a = 0$ or the second factor on the right-hand side equals zero. If we put $Y_a = 0$ by hand and then solve for the X_i and the remaining Y_a, the critical point is called a *nonproper critical point.* If we put the second factor to zero by hand and then solve for X_i and Y_a, the critical point is called a *proper critical point.* The name nonproper is given since a critical point with some $Y_a = 0$ has ∞-valued scalar fields. Therefore these critical points are no proper solutions to the equations of motion, they are asymptotic descriptions of solutions. The proper critical points generically have non-zero Y_a and therefore are proper solutions to the equations of motion. But in some cases one finds that $Y_a = 0$ for proper critical points, although one did not put those Y_a to zero by hand. We will not discuss the non-proper solutions, see [79].

Regardless of whether critical points are proper solutions to the equations of motion, they are all equally important in providing information about the orbits. That is, they are either repellers, attractors or saddle points.

Proper critical points

To construct the proper critical points we demand that $X_i' = Y_a' = 0$, we have the algebraic constraints

$$\boxed{\begin{aligned} 3X_i\Big(-1+X^2\Big)+\sqrt{\tfrac{3}{2}}\sum_a \alpha_{ai}Y_a = 0\,,\\ Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle+6X^2\Big)=0\,,\end{aligned}} \tag{5.1.41}$$

To solve these constraints we multiply the first equation above with α_{bi} and sum over i and use (5.1.38, 5.1.41) to find

$$Y_a = 2\frac{3p-1}{3p^2}\sum_b [A^{-1}]_{ab}\,, \quad X_i = \frac{1}{p}\sqrt{\tfrac{2}{3}}\sum_{ab}\alpha_{ai}[A^{-1}]_{ab}\,. \tag{5.1.42}$$

The value of p is found by combining equations (5.1.29, 5.1.38, 5.1.42)

$$p = 2\sum_{ab}[A^{-1}]_{ab}\,. \tag{5.1.43}$$

In terms of the scalar fields the solutions read

$$\boxed{\phi_i = \frac{\sqrt{6}X_i\,p}{\kappa}\log|\tau| + c_i\,.} \tag{5.1.44}$$

The critical point constructed above does not always exist. It is clear from the definition of the Y_a-variables that they must have the same sign as the Λ_a, i.e. [3]

$$(\frac{1}{p}-3)\sum_b [A^{-1}]_{ab} \gtrless 0 \quad \text{for} \quad \Lambda_a \gtrless 0\,. \tag{5.1.45}$$

Let us briefly discuss what happens if we couple the system we described so far to a barotropic fluid ρ which represents the matter in our universe and allow the curvature to be non-zero ($k = \pm 1$) [79]. It turns out that we can classify the solutions as *curvature*, *matter-scaling* or *scalar-dominated* scaling solutions. The curvature scaling solutions have the property that $k \neq 0$ while $\rho = 0$ and $p = 1$ in (5.1.38). For the matter-scaling solutions we have that $k = 0$ while $\rho \neq 0$ and finally for scalar dominated solutions we have $k = \rho = 0$. Especially the matter-scaling solution are of interest to cosmologists, since these solutions have the property that they have a non-zero constant ratio between the energy density of the scalar fields and that of

[3] In [82] it is shown that critical points that violate the existence conditions can still play a role in understanding the late time behaviour of general solutions.

the barotropic fluid. If the matter-scaling solution are attractors they could explain why today we see the same order of energy densities for matter and dark energy. The dark energy is related to the scalar fields. In [79] it is assumed that there is no direct coupling between the scalar fields and the barotropic fluid. The only interaction is via gravity. In such cases one can derive that the power-law is given by $p = \frac{2}{3(\omega+1)}$. Here ω is the relation between the energy density and pressure of the barotropic fluid (5.1.7). Surprisingly, this is the same power-law as that of only a barotropic fluid. In this way we see that the scalar field mimics the barotropic fluid.

It was noticed in [70, 79, 92] that for the scaling solutions that we discussed above there exist a field redefinitions $\phi^i \to \varphi^i$ such that the potential can be written as

$$V(\varphi) = \mathrm{e}^{c\varphi_1} U(\varphi_2, \ldots, \varphi_N)\,. \tag{5.1.46}$$

To prove this [70] we first note that if the $\vec{\alpha}_a$ are linearly independent there exist a vector $\vec{E}$ such that

$$\vec{\alpha}_a \cdot \vec{E} = c\,, \tag{5.1.47}$$

with c a number. The above can be proven by noting that the $R \times R$ matrix

$$B_{ij} = \sum_{a=1}^{M} \alpha_{ai}\alpha_{aj}\,, \tag{5.1.48}$$

is invertible since the $\vec{\alpha}_a$ are linearly independent. If we now multiply (5.1.47) with α_{aj} and summing over a we see that

$$\sum_i B_{ij} E^i = c \sum_a \alpha_{aj}\,. \tag{5.1.49}$$

Due to the existence of the inverse of B we can find E^i. The above mentioned field redefinition is given by

$$\vec{\phi} = \varphi_1 \vec{E} + \vec{\varphi}_\perp\,. \tag{5.1.50}$$

If we substitute this in (5.1.22) we see that $\alpha_{a1} = c$ and hence we have derive that (5.1.46) holds.

Scaling solutions

The solution (5.1.44) is called a *scaling solution*. The name can be understood as follows. If we calculate the Hubble factor and the kinetic and potential energy we note that they have the same scaling behaviour

$$H^2 \propto V \propto T\,. \tag{5.1.51}$$

We take (5.1.51) as the definition of a scaling solution. These relations imply that the scale factor is a power-law. This follows from the Friedmann equation (5.1.14)

$$\dot{H} \propto H^2 \,. \tag{5.1.52}$$

From this we derive that for scaling solutions we have $H^2 \propto \tau^{-2}$ and as a result $a(\tau)$ must be a power-law. For the scaling solution of the previous section we see from (5.1.37) that this is indeed the case[4].

Interestingly, scaling solutions correspond to the FLRW-geometries that possess a timelike conformal vector field ξ coming from the transformation

$$\tau \to \mathrm{e}^{\lambda}\tau \,, \qquad x^a \to \mathrm{e}^{(1-p)\lambda} x^a \,, \tag{5.1.53}$$

where x^a are the spacelike Cartesian coordinates[5]. In the forthcoming we reserve the indices $a, b, \ldots$ to denote spacelike coordinates when we consider cosmological space-times.

Scaling cosmologies have received a great deal of attention in the dark-energy literature, see [93] for a review and references. Apart from the intriguing cosmological properties of scaling solutions they are also interesting for understanding the dynamics of a general cosmological solution since scaling solutions are often critical points of an autonomous system of differential equations as we explained in subsection 5.1.1. Scaling cosmologies often appear in supergravity theories (see for instance [70, 94]), but remarkably they also appear by spatially averaging inhomogeneous cosmologies in classical general relativity [95].

Let us finish this section by making one surprising observation. When we look at the solution for the scalar fields (5.1.44) we see that this is still a geodesic of the scalar manifold since $G_{ij} = \delta_{ij}$! Apparently the presence of the (complicated) potential does not upset this. A natural question is under what condition does a solution remain a geodesic of the scalar manifold in the presence of a potential. This will be the subject of the next section.

5.2 First Order Formalism

In what follows we consider scalar fields ϕ^i that parameterize a Riemannian manifold with metric G_{ij} coupled to gravity through the action

$$S = \int \mathrm{d}^D x \sqrt{-g} \Big\{ \mathcal{R} - \tfrac{1}{2} G_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - V(\phi) \Big\} \,. \tag{5.2.1}$$

[4]In the case of curved FLRW-universes we also demand that $H \sim k/a^2$, which is only possible for $p = 1$. But in what follows we will not consider the case $k \neq 0$.

[5]For curved FLRW-space-times the spacelike coordinates are invariant.

We restrict to solutions with the following D-dimensional space-time metric

$$\mathrm{d}s_D^2 = g(r)^2 \mathrm{d}s_{D-1}^2 + \eta f(r)^2 \mathrm{d}r^2\,, \quad \mathrm{d}s_{D-1}^2 = (\delta_\eta)_{ab}\mathrm{d}x^a \mathrm{d}x^b\,, \tag{5.2.2}$$

where $\eta = \pm 1$ and $\delta_\eta = \mathrm{diag}(-\eta, 1, \ldots, 1)$. The case $\eta = -1$ describes a flat FLRW-space-time and $\eta = +1$ a Minkowski-sliced domain-wall (DW) space-time. The scalar fields that source these space-times can only depend on the r-coordinate $\phi^i = \phi^i(r)$. The function f corresponds to the gauge freedom of re-parameterizing the r-coordinate.

We will use two coordinate frames to describe scaling cosmologies

$$\tau - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p}\,\mathrm{d}s_{D-1}^2\,, \tag{5.2.3}$$

$$t - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{e}^{2t}\,\mathrm{d}t^2 + \mathrm{e}^{2pt}\mathrm{d}s_{D-1}^2\,. \tag{5.2.4}$$

The first is the usual FLRW-coordinate system and the second can be obtained via the substitution $t = \log\tau$.

If the scalar potential $V(\phi)$ can be written in terms of another function $W(\phi)$ as follows

$$V = \eta\left\{\tfrac{1}{2}G^{ij}\partial_i W \partial_j W - \tfrac{D-1}{4(D-2)}W^2\right\}, \tag{5.2.5}$$

then the action can be written as "a sum of squares" plus a boundary term when reduced to one dimension:

$$\begin{aligned} S =& \eta \int \mathrm{d}r\, f g^{D-1}\Big\{ \tfrac{(D-1)}{4(D-2)}\Big[W - 2(D-2)\frac{\dot{g}}{fg}\Big]^2 - \tfrac{1}{2}||\frac{\dot{\phi}^i}{f} + G^{ij}\partial_j W||^2\Big\} \\ &+ \eta \int \mathrm{d}\Big\{g^{D-1}W - 2(D-1)\dot{g}g^{D-2}f^{-1}\Big\}\,, \end{aligned} \tag{5.2.6}$$

where a dot denotes a derivative w.r.t. r. The term $||\dot{\phi}^i/f + G^{ij}\partial_j W||^2$ is a shorthand notation and the square involves a contraction with the field metric G_{ij}. It is clear that the action is stationary under variations if the terms within brackets are zero[6], leading to the following *first-order* equations of motion

$$\boxed{W = 2(D-2)\frac{\dot{g}}{fg}\,, \qquad \frac{\dot{\phi}^i}{f} + G^{ij}\partial_j W = 0\,.} \tag{5.2.7}$$

For $\eta = +1$ these equations are the standard Bogomol'nyi-Prasad-Sommerfield (BPS) equations for domain-walls that arise from demanding the supersymmetry-variation (susy) of the fermions to vanish, which guarantees that the domain-wall preserves a fraction of the total supersymmetry of the theory. The function W is then the

[6]For completeness we should have added the Gibbons-Hawking term [96] in the action which deletes that part of the above boundary term that contains $\dot{g}$.

superpotential that appears in the susy-variation rules and equation (5.2.5) with $\eta = +1$ is natural for supergravity theories. It is clear that for every W that obeys (5.2.5) we can find a corresponding domain-wall solution, and if W is not related to the susy-variations we call the solutions fake supersymmetric [97].

For $\eta = -1$ these equations are the generalization to cosmologies for arbitrary space-time dimension D and field metric G_{ij}. We refer to these first-order equations as pseudo-BPS equations and W is named the pseudo-superpotential because of the immediate analogy with BPS domain-walls in supergravity [67, 98]. For the case of cosmologies there is no natural choice for W as cosmologies cannot be found by demanding vanishing susy-variations of the fermions. The cosmological solutions is therefore called *pseudo-supersymmetric*.

If we can solve (5.2.7) for a domain-wall we immediately have a cosmological solution by construction. This is called the domain-wall / cosmology correspondence [67, 98]. In the next chapter we will discuss this correspondence in more detail.

In [98] it is proven that for all single-scalar cosmologies (and domain-walls) a pseudo-superpotential W exists such that the cosmology is pseudo-BPS and that one can give a fermionic interpretation of the pseudo-BPS flow in terms of so-called pseudo-Killing spinors. This does not necessarily carry over to multi-scalar solutions as was shown in [99]. Nonetheless, a multi-field solution can locally be seen as a single-field solution [100] because locally we can redefine the scalar coordinates such that the curve $\phi(r)$ is aligned with a scalar axis and all other scalars are constant on this solution. A necessary condition for the single-field pseudo-BPS flow to carry over (locally) to the multi-field system is that the truncation down to a single scalar is consistent (this means that apart from the solution one can put the other scalars always to zero) [99].

5.3 Multi-Field Scaling Cosmologies

Let us turn to scaling solutions in the framework of these first order equations and see how the geodesic motion arises that we found at the end of section 5.1. First we consider the rather trivial case with vanishing scalar potential V and after that we add a scalar potential V. Pseudo-supersymmetry is only discussed in the case of non-vanishing V.

5.3.1 Pure Kinetic Solutions

If there is no scalar potential the solutions trace out geodesics as we learned in section 3.5. The affine velocity $G_{ij}\partial_h\phi^i\partial_h\phi^j = ||v||^2$ is positive and for the metric Ansatz (5.2.2) we derive the Einstein equations

$$\mathcal{R}_{rr} = \tfrac{1}{2}G_{ij}\dot{\phi}^i\dot{\phi}^j = \frac{||v||^2}{2}g^{2-2D}f^2\,, \qquad \mathcal{R}_{ab} = 0\,. \tag{5.3.1}$$

In the gauge $f = 1$ the solution is given by $g = \mathrm{e}^{C_2}(r + C_1)^{\frac{1}{D-1}}$, with C_1 and C_2 arbitrary integration constants, but with a shift of r we can always put $C_1 = 0$ and C_2 can always be put to zero by re-scaling the spacelike coordinates. In the case of a four-dimensional cosmology the geometry is a power-law FLRW-solution with $p = 1/3$.

5.3.2 Potential-Kinetic Scaling Solutions

In a recent paper of Tolley and Wesley an interesting interpretation was given to scaling solutions [101], which we repeat here. The finite transformation (5.1.53) leaves the equations of motion invariant if the action S scales with a constant factor, which is exactly what happens for scaling solutions since all terms in the Lagrangian scale like τ^{-2}. Under (5.1.53) the metric scales like $\mathrm{e}^{2\lambda} g_{\mu\nu}$ and in order for the action to scale as a whole we must have

$$V \to \mathrm{e}^{-2\lambda} V\,, \qquad T = \tfrac{1}{2} g^{\tau\tau} G_{ij} \dot{\phi}^i \dot{\phi}^j \to \mathrm{e}^{-2\lambda} T\,. \tag{5.3.2}$$

Equations (5.3.2) imply that $G_{ij}\dot{\phi}^i\dot{\phi}^j$ remains invariant from which one deduces that $\frac{d\phi^i}{d\lambda} = \xi^i$ must be a Killing vector. The curve that describes a scaling solution follows an isometry of the scalar manifold. It depends on the parametrization whether the tangent vector $\dot{\phi}$ itself is Killing. This happens for the parametrization in terms of $t = \log\tau$ since

$$\xi^i = \frac{\mathrm{d}\phi^i}{\mathrm{d}\lambda} = \lim_{\lambda\to 0} \frac{\phi^i(\mathrm{e}^\lambda \tau) - \phi^i(\tau)}{\lambda} = \frac{\mathrm{d}\phi^i}{\mathrm{d}\log\tau}\,. \tag{5.3.3}$$

Thus a scaling solution is associated with an invariance of the equations of motion for a re-scaling of cosmic time and is therefore associated with a conformal Killing vector on space-time and a Killing vector on the scalar manifold.

Pseudo-supersymmetry comes into play when we check the geodesic equation of motion

$$\nabla_{\dot{\phi}} \dot{\phi}_i = \dot{\phi}^j \nabla_j \dot{\phi}_i = \dot{\phi}^j \left\{ \nabla_{(j} \dot{\phi}_{i)} + \nabla_{[j} \dot{\phi}_{i]} \right\}, \tag{5.3.4}$$

where we denote $\dot{\phi}_i = G_{ik}\dot{\phi}^k$. Now we have that the symmetric part is zero if we parameterize the curve with $t = \log\tau$ since scaling makes $\dot{\phi}$ a Killing vector. We also have that $\nabla_{[j}\dot{\phi}_{i]} = 0$ since the pseudo-BPS condition makes $\dot{\phi}$ a curl-free flow $\dot{\phi}_i = -f\partial_i W$. To check that the curl is indeed zero (when $f \neq 1$) one has to notice that in the parametrization of the curve in terms of $t = \log\tau$ the gauge is such that $\dot{g}/g$ is constant and that $f \sim W^{-1}$. Since the curl is also zero we notice that the curve is a geodesic with $\log\tau$ as affine parametrization[7]

$$\nabla_{\dot\phi}\dot\phi^i = 0 = \ddot\phi^i + \Gamma^i_{jk}\dot\phi^j\dot\phi^k\,. \tag{5.3.5}$$

The link between scaling and geodesics was discovered by Karthauser and Saffin in [102], but no conditions on the Lagrangian were given in [102] such that the relation scaling-geodesic holds. An example of a scaling solution that is not a geodesic was given by Sonner and Townsend in [103].

A more intuitive understanding of the origin of the geodesic motion for some scaling cosmologies comes from the on-shell substitution $V = (3p-1)\,T$ in the Lagrangian to get a new Lagrangian describing seemingly massless fields. Although this is rarely a consistent procedure we believe that this is nonetheless related to the existence of geodesic scaling solutions.

Single field

For single-field models the potential must be exponential $V = \Lambda e^{\alpha\phi}$ in order to have scaling solutions. The simplest pseudo-superpotential belonging to an exponential potential is itself exponential

$$W = \pm\sqrt{\tfrac{8\Lambda}{3-\alpha^2}}\;e^{\frac{\alpha\phi}{2}}\,. \tag{5.3.6}$$

If we choose the plus sign the solution to the pseudo-BPS equation is

$$\phi(\tau) = -\tfrac{2}{\alpha}\log\tau + \tfrac{1}{\alpha}\log[\tfrac{6-2\alpha^2}{\alpha^4\Lambda}]\,,\qquad g(\tau)\sim\tau^{\frac{1}{\alpha^2}}\,. \tag{5.3.7}$$

The minus sign corresponds to the time reversed solution.

Multiple fields

For a general multi-field model a scaling solution with power-law scale factor τ^p obeys $V = (3p-1)T$ from which we derive the **on-shell** relation

$$G^{ij}\partial_i W\partial_j W = \frac{W^2}{4p}\quad\Rightarrow\quad W = \pm\sqrt{\frac{8\,p\,V}{3p-1}}\,. \tag{5.3.8}$$

In general the above expression for the superpotential $W\sim\sqrt{V}$ does not hold off-shell, unless the potential is a function of a specific kind:

$$\frac{1}{p} = \frac{G^{ij}\partial_i V\partial_j V}{V^2}\,. \tag{5.3.9}$$

[7] One could wonder whether the results works in two ways. Imagine that a scaling solution is a geodesic. This then implies that $\nabla_{[j}\dot\phi_{i]} = 0$ and therefore the flow is locally a gradient flow $\dot\phi_i = \partial_i \log W \sim f\partial_i W$.

Scalar potentials that obey (5.3.9) with the extra condition that $p \gtrless \frac{1}{3} \leftrightarrow V \gtrless 0$ allow for multi-field scaling solutions. For a given scalar potential that obeys (5.3.9) there probably exist many pseudo-superpotentials W compatible with V but if we make the specific choice $W = \sqrt{8\,p\,V/(3p-1)}$ then all pseudo-BPS solutions must be scaling and hence geodesic. As a consistency check we substitute the first-order pseudo-BPS equations into the right-hand side of the following second-order equations of motion

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^k\dot{\phi}^j = -f^2 G^{ij}\partial_j V - \Big[3(\dot{\log g}) - (\dot{\log f})\Big]\dot{\phi}^i\,, \tag{5.3.10}$$

and choose a gauge for which

$$\frac{\dot{f}}{f^2} = \frac{1}{4p}W\,, \tag{5.3.11}$$

then we indeed find an affine geodesic motion since the right-hand side of (5.3.10) vanishes.

For some systems one first needs to perform a truncation in order to find the above relation (5.3.9). A good example is the multi-field potential appearing in Assisted Inflation [104]

$$V(\phi^1,\ldots,\phi^n) = \sum_i^n \Lambda_i\,\mathrm{e}^{\alpha_i\phi^i}\,, \qquad G_{ij} = \delta_{ij}\,. \tag{5.3.12}$$

The scaling solution of this system was proven to be the same as the single-exponential scaling [92]. The reason is that we can perform the orthogonal transformation in field space that we discussed below (5.1.46). As a result, the form of the kinetic term is preserved but the scalar potential is given by

$$V = \mathrm{e}^{\alpha\varphi}\,U(\phi^1,\ldots,\phi^{n-1})\,, \quad \frac{1}{\alpha^2} = \sum_i \frac{1}{\alpha_i^2}\,. \tag{5.3.13}$$

The scaling solution is such that $\phi_1,\ldots,\phi_{n-1}$ are frozen in a stationary point of U. This follows from the Klein–Gordon equation for the new fields and making use of the fact that this is a critical point solution. Therefore the system is truncated to a single-field system that obeys (5.3.9). The same proof holds for generalized assisted inflation discussed in section 5.1.2 [79]. As shown in (5.1.44) the scaling solution reads $\phi^i = A^i\log\tau + B^i$, which is clearly a straight line and thus a geodesic.

The scaling solutions of [99,103] were constructed for an axion-dilaton system with an exponential potential for the dilaton

$$S = \int \mathrm{d}^D x\sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mathrm{e}^{\mu\phi}(\partial\chi)^2 - \Lambda\mathrm{e}^{\alpha\phi}\Big\}\,. \tag{5.3.14}$$

Clearly this two-field system obeys (5.3.9) and (one of) the pseudo-superpotential(s) is given by (5.3.6). The pseudo-BPS scaling solution therefore has constant axion and is effectively described by the dilaton in an exponential potential. Note that this solution indeed describes a geodesic on $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ with $\log\tau$ as affine parameter. All examples of scaling solutions in the literature seem to occur for exponential potentials, however by performing a $\mathrm{SL}(2,\mathbb{R})$-transformation on the Lagrangian (5.3.14) the kinetic term is unchanged and the potential becomes a more complicated function of the axion and the dilaton. The same scaling solution then trivially still exists (and (5.3.9) still holds) but the axion is not constant in the new frame and instead the solution follows a more complicated geodesic on $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$.

However another scaling solution is given in [103] that is not geodesic and with varying axion in the frame of the above action (5.3.14). This is an illustration of the above, since the solution is not geodesic we know that there does not exists any other pseudo-superpotential for which the varying axion solution is pseudo-BPS, consistent with what is shown in [99] for that particular solution.

5.4 Uplifts

In this section we illustrate with an example that the cosmologies we worked out in subsection 5.1.3 can be uplifted over their transverse space by using the reduction explained in section 3.3 [45, 64, 70]. We only consider the simple example of a single scalar field and we illustrate that for a given cosmology we find a domain-wall belonging to a model with minus the potential.

In section 3.3 we showed that the reduction of a fluxless brane over its maximally symmetric transverse space with $k = -1$ leads to a Lagrangian with exponential potential (3.3.6)

$$V(\varphi) = n(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi}\,. \tag{5.4.1}$$

Here n is the dimension of the internal manifold and α, β are given in (3.2.7). The four-dimensional $k = 0$ solution with $\kappa^2 = 1/2$ is found from (5.1.43–5.1.44) to be

$$a(\tau) = \tau^{\frac{n}{n+2}}\,, \quad \varphi(\tau) = -\frac{2}{c}\log(\tau) - \frac{2}{c}\log(\frac{n+2}{2})\,, \tag{5.4.2}$$

with $c^2 = (2(\alpha-\beta))^2 = 1 + 2/n$. The constant c_1 in (5.1.44) gets fixed due to the Friedmann equations.

To uplift we plug these solutions in (3.3.1) and derive

$$\mathrm{d}s^2_{4+n} = -\Big(\frac{2}{2+n}\Big)^{\frac{2n}{2+n}}\tau^{-\frac{2n}{2+n}}\mathrm{d}\tau^2 + \Big(\frac{2}{2+n}\Big)^{\frac{2n}{2+n}}\mathrm{d}\vec{x}^2_3 + \Big(\frac{2}{2+n}\Big)^{-\frac{4}{2+n}}\tau^{\frac{4}{2+n}}\mathrm{d}\mathbb{H}^2_n\,. \tag{5.4.3}$$

To identify the solution we apply the following coordinate transformations

$$t = \Big(\frac{2}{n+2}\Big)^{-\frac{2}{2+n}}\tau^{\frac{2}{2+n}}\,, \quad y^i = \Big(\frac{2}{2+n}\Big)^{\frac{n}{2+n}}x^i\,, \tag{5.4.4}$$

and we find

$$\mathrm{d}s^2_{4+n} = \mathrm{d}\vec{y}_3^2 - \mathrm{d}t^2 + t^2 \mathrm{d}\mathbb{H}^2_n \,. \tag{5.4.5}$$

This metric describes $\mathbb{R}^3 \times \text{Milne}_{n+1}$, the latter is a patch of Minkowski space-time in unconventional coordinates. The uplifted solution describes thus a flat space solution.

The extension to the multi-exponential potential given in (3.3.21) does not lead to any qualitative chance. The reason is that the attractor solution is such that only one scalar field is turned on [45]. To find genuine S-branes we need to take flux into consideration. This requires the uplift of a solution belonging to (3.3.12) [45, 105].

According to (5.2.7) the four-dimensional cosmology should give rise to a domain-wall with minus the potential as given in (5.4.1). From (5.2.2) and $\eta = 1$ we indeed find the solution

$$\mathrm{d}s^2_4 = \mathrm{d}r^2 + a(r)^2\Big(-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2\Big)\,, \tag{5.4.6}$$

with power-law and φ given by

$$a(r) = r^{\frac{n}{n+2}}\,, \quad \varphi(r) = -\frac{2}{c}\log(r) - \frac{2}{c}\log(\frac{n+2}{2})\,. \tag{5.4.7}$$

Due to the minus sign in the potential this domain-wall lifts up to a spherical transverse space and we find after appropriate coordinate transformations

$$\mathrm{d}s^2_{4+n} = -\mathrm{d}t^2 + \mathrm{d}\vec{y}_2^2 + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2_n\,, \tag{5.4.8}$$

where $\mathrm{d}\Omega^2_n$ is the metric of a n-dimensional sphere. This metric describes three-dimensional Minkowski space-time $\times\ \mathbb{R}^{n+1}$.

5.5 Discussion

In the first part of this chapter we gave a brief introduction to cosmology and focussed on generalized assisted inflation models. These models have the characteristic that they all have a multi-exponential potential. The analysis was restricted to critical points via re-writing the equations of motion as an autonomous system. These critical points turn out to be scaling solutions and we noted that they are still geodesics of the scalar manifold.

In the second half of this chapter we explained under what condition we have geodesic motion in the presence of a potential. For this we have studied multi-field scaling solutions using a first-order formalism for scalar cosmologies. We derived these first-order equations via a Bogomolnyi-like method that was known to work for domain-wall solutions as was first shown in [106–108] and we showed that it trivially extends to cosmological solutions. This first-order formalism allows a better understanding of the geodesic motion that comes with a specific class of scaling solutions. One of the main results of this chapter is a proof that shows that *all pseudo-BPS* *cosmologies that are scaling solutions must be geodesic.* This complements to the discussion in [99] where the first example of a non-geodesic scaling cosmology was shown to be non-pseudo-BPS. Moreover we gave constraints on multi-field Lagrangians for which the pseudo-BPS cosmologies are geodesic scaling solutions.

By now the first order formalism has been extended to branes of arbitrary dimensions, both space- and timelike. This has been initiated by [109] where it was shown that the *non-extremal* Reissner-Nordström black hole solution of Einstein-Maxwell theory can be found from first-order equations by rewriting the action as a sum of squares à la Bogomol'nyi. In the recent paper [74] this construction of BPS-type equations is extended to branes of arbitrary dimension and to time-dependent solutions. The authors presented the fake- and pseudo-BPS equations for all stationary branes (timelike branes) and all time-dependent branes (spacelike branes) of an Einstein-dilaton-p-form system in arbitrary dimensions[8]. As mentioned before, the word fake refers to time-independent solutions where the superpotential W used in the derivation of the first order equations has no relation to the superpotential appearing in the supersymmetry transformation. In case of time-dependent cosmological solutions the word pseudo-BPS is used for the first order equations governing the dynamics of cosmologies.

[8] They did not included branes with co-dimensions less than three. When the co-dimension is one, the stationary branes are domain walls and the time-dependent branes are cosmologies. The case of branes with co-dimension two is not included as these solutions depend on one complex coordinate rather than on one real coordinate.

Chapter 6

Domain-Wall / Cosmology Correspondence

In the previous chapter we showed that cosmologies and domain-walls satisfy first order equations. Both type of solutions couple to a zero-form field strength given by a potential, although there is an overall minus sign difference due to the relation between the potential and superpotential. Finally, the metric Ansätze are of similar form.

Due to the first order equations (5.2.7) we know that for a given domain-wall a cosmology exist. This is called the domain-wall / cosmology correspondence [98]. In the first section we give a summary of this correspondence as given in [67, 110].

If a domain-wall can be embedded in a supergravity it can preserve (some fraction of) supersymmetry. Due to the explicit time-dependence, cosmologies break all supersymmetry. On the other hand the correspondence tells us that for a given domain-wall there is a corresponding cosmology. In section 2 we present a discussion of the correspondence in a supergravity setting. It turns out that for this to work the cosmologies need to be embedded into the star supergravities of [58]. These cosmologies then turn out to be also (fake) supersymmetric.

This work is done in collaboration with E. A. Bergshoeff, J. Hartong, J. Rosseel and D. Van Den Bleeken [111].

6.1 The Domain-Wall / Cosmology Correspondence

In 1994 it was already noticed that there is a link between domain-walls and cosmologies [66]. This has been worked out in the papers [67, 98, 110, 112–114] and is called the domain-wall / cosmology correspondence. In [110] it was noticed that the correspondence can be extended to instantons in a trivial way. For this reason we will repeat the arguments given in that paper.

For simplicity we consider the single-scalar field model of (3.5.2), that is

$$\mathcal{L} = \sqrt{\epsilon g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,. \tag{6.1.1}$$

Here $\epsilon = 1$ refers to a Euclidean signature while for $\epsilon = -1$ we have a Lorentzian signature.

Since we are initially interested in domain-walls and cosmologies we require a metric Ansatz that has an one-dimensional transverse space. Furthermore, we allow for all three possible choices of $k = 0, \pm 1$. The D-dimensional metric Ansatz is given by

$$\mathrm{d}s^2 = -\epsilon\eta(\mathrm{e}^{\alpha\varphi}f)^2\mathrm{d}z^2 + \mathrm{e}^{2\beta\varphi}\Big(-\eta\frac{\mathrm{d}r^2}{1+\eta k r^2} + r^2\mathrm{d}\Omega_\eta^2\Big)\,. \tag{6.1.2}$$

When $\eta = -1$, $\mathrm{d}\Omega_\eta^2$ describes the SO($D-1$)-invariant metric on the unit radius (D-2)-sphere and for $\eta = 1$ it describes the SO($D-2,1$)-invariant metric on the unit radius ($D-2$)-hyperboloid. The functions φ and f depend only on z while α and β are given by

$$\alpha^2 = \frac{D-1}{2(D-2)}\,,\quad \beta^2 = \frac{1}{2(D-1)(D-2)}\,. \tag{6.1.3}$$

To describe a cosmology we take $\epsilon = -1$ and the choice $\eta = -1$ yields the metric of a homogeneous and isotropic cosmology, describing a universe that is closed if $k = 1$, open if $k = -1$ and flat if $k = 0$. The coordinate z is the time coordinate. For domain-walls we take $\epsilon = -1$ but now with $\eta = 1$. The worldvolume geometry of the domain-wall is anti-de Sitter if $k = -1$, de Sitter if $k = 1$ and Minkowski if $k = 0$. In this case z describes the distance from the wall, while r is the time coordinate. The instanton is described by a Euclidean metric, hence we take $\epsilon = -\eta = 1$. The scalar field ϕ can only depend on the coordinate z.

Let us now see how the correspondence comes about. Since we have maintained the re-parametrization of z due to the inclusion of f we can substitute the metric and scalar field Ansätze into the action (6.1.1). If we do this we find the effective one-dimensional Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}f^{-1}(\dot\phi^2 - \dot\varphi^2) - \epsilon\eta f \mathrm{e}^{2\alpha\varphi}V_{\text{eff}}\,. \tag{6.1.4}$$

Here a dot is a derivative with respect to z and the effective potential is given by

$$V_{\text{eff}}(\phi,\varphi) = V(\phi) - \frac{k}{2\beta^2}\mathrm{e}^{-2\beta\varphi}\,. \tag{6.1.5}$$

From (6.1.4) we see that only the product $\epsilon\eta$ appears in the effective Lagrangian. Even more, we see that (6.1.4) is invariant if we let $\epsilon\eta \to -\epsilon\eta$ together with $V \to -V$ and $k \to -k$.

The domain-wall / cosmology correspondence can be found by considering $\epsilon = -1$. We observe that for every domain-wall solution of a model with potential V there is a cosmology of the model with potential $-V$ and with opposite sign for k if it is non-zero, and *vice-versa*.

Of course, the above also follows from what we derived in section 5.2 for the case $k = 0$. There the sign difference in the potential follows from the parameter η in the relation between the potential and superpotential (5.2.5). For $\eta = 1$ we have that the potential is V, while for $\eta = -1$ we have $-V$. The analysis done in section 5.2 can be extended to include $k \neq 0$ as well.

The extension to instantons is now straightforward. For instantons we require a Euclidean metric so that $\epsilon = -\eta = 1$ or $\epsilon\eta = -1$. This is however the same condition as holds for domain-walls. We see that for a given potential V we find both a Lorentzian domain-wall and a Euclidean solution. The latter can be interpreted as an instanton, but of a model with potential $-V$ because instanton solutions of a mechanical model are precisely solutions with a flipped sign of the potential, see for example [115]. The extended correspondence of [110] can then be summarized as follows. For every domain-wall solution of a model with potential V there corresponds both a cosmology and an instanton of the model with potential $-V$ (although the latter is actually found from the effective Lagrangian with potential V).

Let us illustrate this with the four-dimensional example we worked out in section 5.4. There we showed that a cosmology coupled to a potential V indeed gives rise to a domain-wall coupled to $-V$. According to the above we should also find an instanton solution for this model with potential $-V$. Indeed we find the following Euclidean solution

$$\mathrm{d}s_4^2 = \mathrm{d}r^2 + a(r)^2\Big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big)\,, \tag{6.1.6}$$

with power-law and scalar fields given by (5.4.7). After appropriate coordinate transformations we find the uplifted solution to be

$$\mathrm{d}s_{4+n}^2 = \mathrm{d}\vec{y}_3^{\,2} + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega_n^2\,, \tag{6.1.7}$$

where $\mathrm{d}\Omega_n^2$ is the metric of a n-dimensional sphere. This metric describes $\mathbb{R}^3 \times \mathbb{R}^{n+1}$. So indeed we see that for a given domain-wall we find both a cosmology and an instanton.

The effective Lagrangian (6.1.4) gives rise to second order differential equations for ϕ and φ. On the other hand, in the previous chapter we showed that both domain-walls and cosmologies satisfy the first order equations (5.2.7). In [67] it is shown that one can introduces a function Z which depends on the scalar field ϕ such that one can derive first order equations which automatically satisfy the second order equations that follow from (6.1.4). Since the proof is rather involved, we refer to [67] for this. For $k = 0$ these first order equations agree with (5.2.7).

The analysis so far includes bosonic fields only. It is natural to ask if there is an explanation as to why cosmologies and domain-walls satisfy first order equations such as (5.2.7). Let us comment on these two issues for the case $k = 0$ [67].

The domain-wall / cosmology correspondence is based on the fact that the existence of a domain-wall solution of the effective Lagrangian (6.1.4) with potential V automatically implies the existence of a cosmological solution corresponding to $-V$. The domain-wall solutions generically are 'fake supersymmetric' [67, 98, 112]. This implies that one can write the potential V in terms of a real superpotential W. For the one scalar case this relation schematically looks like

$$V = 2\left((W')^2 - \alpha^2 W^2\right) , \tag{6.1.8}$$

where $W' = \frac{\delta W}{\delta \phi}$ and α is given in (6.1.3). This is, up to a re-scaling of W, the single scalar field version of (5.2.5).

The domain-walls allow for the existence of a Killing spinor ϵ obeying a Killing spinor equation that can be written in terms of the superpotential W as follows:

$$(D_\mu - W\Gamma_\mu)\epsilon = 0 \,. \tag{6.1.9}$$

In case the Lagrangian (6.1.1) can be obtained as a truncation of a supergravity theory the equations (6.1.8, 6.1.9) can be understood as arising from the structure of the underlying supergravity theory. In particular, the Killing spinor equation could in that case be obtained by putting the supersymmetry transformations of the fermions equal to zero. In [98] it is shown that the first order-equations for domain-walls follow from (6.1.9). The authors also showed that almost all flat ($k = 0$) and AdS-sliced ($k = -1$) domain-walls preserve half of their supersymmetries. The proof of this is based on the fact that for a given domain-wall one can construct out of this solution a superpotential W such that the (fake) Killing spinor is non-zero. This superpotential W is related to the function Z we mentioned earlier. For the exact constraints we refer to [67]. In this sense the first order equations are BPS equations that guarantee the existence of a Killing spinor. For $k = 1$ we can only have a dS-foliation of either Minkowski or AdS space.

However, fake supergravity[1] is much more general and the Lagrangian (6.1.1) can be completely general and does not need to be related to any supergravity theory. The mapping between domain-walls and cosmologies implies that cosmologies also obey a property that looks very much like fake supersymmetry. In this case, it turns out that the cosmology obeys similar equations (6.1.8, 6.1.9) as its corresponding domain-wall solution, with the caveat that now the superpotential W is no longer real but is

[1] In fake supergravity one allows for a superpotential W that is not part of a genuine supergravity. This W is often called an "adapted" superpotential [100].

instead purely imaginary. Redefining $W = i\tilde{W}$, equations (6.1.8, 6.1.9) become

$$V = -2\left((\tilde{W}')^2 - \alpha^2\tilde{W}^2\right) , \tag{6.1.10}$$

$$(D_\mu - i\tilde{W}\Gamma_\mu)\epsilon = 0 \,. \tag{6.1.11}$$

Note the change of sign in (6.1.10), which indeed corresponds to $-V$ in (6.1.1). The structure (6.1.10, 6.1.11) for cosmological solutions was called *pseudo-supersymmetry* [67, 98, 112]. The structure underlying the existence of the first-order equations can be understood from Hamilton-Jacobi theory [110, 112, 114].

From a supergravity point of view, this correspondence is rather odd. Supersymmetric domain-wall solutions can be found rather generically in supergravity theories. For supersymmetric cosmological solutions this is not true. Furthermore, the correspondence involves a sign change in the potential that spoils the supersymmetry of the supergravity theory under consideration. Finally, in fake supergravity theories, one is usually not concerned with the reality properties of the (Killing) spinors and one works with arbitrary Dirac spinors. In real supergravity theories, reality conditions on the spinors have to be imposed in order to account for the correct number of degrees of freedom. In this respect, one no longer has the freedom to take W purely imaginary without upsetting the reality properties of the supersymmetry rules.

A natural question is whether one can give a meaning to pseudo-supersym- metry in a real supergravity context. The fact that the corresponding domain-wall and cosmological solutions differ in the reality properties of the superpotential suggests that, if one can give an embedding of the correspondence in supergravity, one should look for theories in which the spinors obey different reality properties. A priori, it is possible that there are two different theories in the same signature (namely $(1, 9)$) that mainly differ in the reality properties of the spinors. This can then account for a difference in reality properties of the superpotential and for the sign flip in the potential. We present an example of this is in the type II and type II* theories in signature $(1, 9)$. Starting from a supersymmetric domain-wall in type IIA, the corresponding cosmological solution then turns out to be a supersymmetric solution of the type IIA* theory. Pseudo-supersymmetry in this context corresponds to supersymmetry in a star theory.

6.2 ... in a Supergravity Setting

In the coming sections we are going to answer the question posed in the previous section, namely whether one can give a meaning to pseudo-supersymmetry in a real supergravity context. Let us begin by making two remarks.

In the coming sections we will present a complex formulation of 10- and 11-dimensional supergravity theories. One of the reasons to do this stems from the so-called "variant supergravities" in 10 and 11 dimensions, whose existence has been discussed first in [26, 59, 116]. It was argued that upon applying T-dualities along timelike directions new supergravities are found. In particular, timelike T-duality on the usual type IIA theory does not lead to the usual type IIB theory, but instead leads to a different theory, called the type IIB* theory. Similarly, the type IIA* theory is found as the timelike T-dual of the usual type IIB supergravity. Note that both type II and type II* theories share the same space-time signature $(1, 9)$. A crucial difference between type II and type II* is that in the *-theories the RR-forms are ghosts, i.e. they have wrong-sign kinetic terms. Upon applying more general dualities, one is also led to type II supergravities in different signatures. Similarly, it was argued that one should also consider eleven-dimensional supergravity in different signatures. For instance, it was shown that the type IIA* theory could be obtained by dimensional reduction over a timelike direction of 11d supergravity in signature $(2, 9)$.

In the next sections we derive the explicit actions and supersymmetry variations of these variant supergravities. For earlier work on the construction of these theories in the IIA and M-theory case, see [117, 118]. We will adopt a different approach for constructing the actions and furthermore include the IIB case. The strategy we will follow in obtaining actions and supersymmetry transformation rules for these supergravities, is based on the observations made in [119]. There, it was shown that the superalgebras underlying these variant supergravities correspond to different parameterizations of the unique real form of the superalgebra OSp(1|32). Our work can be viewed as a continuation of [119], where now we construct the complex field theory corresponding to the complex algebra presented there. More precisely, starting from the complex algebra, one can impose different reality conditions on the generators. Each choice of reality conditions gives a real superalgebra underlying one of the variant supergravities in a specific signature. Similarly we will start from a single complex action and by imposing different reality conditions obtain the different variant supergravities.

6.3 Type II Actions

In this section we will show how one can obtain supergravity actions for different signatures as different real slices of a single complex action. Sometimes this leads to different supergravity theories with the same signature.

The starting point of our construction will be a complex action that then can be reduced to different real actions. In this thesis we will not address the question of how one can in general construct sensible complex actions or investigate what a general complex action invariant under some complexified symmetry group looks like. Instead we will take a more pragmatic approach. The idea is to start from a known action in terms of some real fields[2] that is invariant under some real symmetry group. The first step is to construct a complexified version of this action that is invariant under the complexified symmetry group. We require that the real action we started from can be obtained from this complexified action by imposing certain reality conditions and similarly for the symmetries. At this point one faces the natural question: are there different real slices leading to other theories? As it will turn out, theories in different signatures are found by taking different reality conditions for a single complex action. In the case one has extended supersymmetry it can even happen that one finds multiple real theories in one signature. It is these issues that we will work out in detail for IIA and IIB supergravity in this section.

This general scheme of finding different real actions as consistent real slices of a given complex action can be applied quite generally. For the interested reader we refer to [111] for the same analysis for M-theory. One would expect the general procedure presented below to hold for all kinds of theories in various dimensions although subtleties can arise and some particular details might change from case to case.

6.3.1 The Complex Type II Action

To start we will deal with the first of the two questions posed above. We will show how one can find complex actions that can respectively be restricted to the known actions of IIA and IIB by reality conditions, and that are furthermore invariant under the complexified super Poincaré group. How the different formulations of the real 10d super Poincaré algebra can be found from the unique ten-dimensional complex OSp(1|32) algebra was described in detail in [119].

In complexifying an action it is crucial that all fields appear holomorphically in the complex action. In other words we replace fields that take values in $\mathbb{R}$ by fields that take values in $\mathbb{C}$ in such a way that no complex conjugates appear. If one does the same complexification on the symmetry transformations, the complexified action is guaranteed to be invariant under these complex transformations as checking the invariance is a pure algebraic computation that nowhere assumes reality of the involved parameters[3].

This procedure of 'holomorphic complexification' is rather straightforward and only requires some more consideration in case of the spinors. Usually spinors appear in the action through bilinears written in terms of the Dirac conjugate $\bar{\chi}^D = \chi^\dagger A$, see appendix B for our conventions and notations regarding spinors. In this form there appears a complex conjugation and as such the action is not holomorphic in

[2] By a real field, we mean a field that satisfies a reality condition, for instance a Majorana fermion.

[3] One might think that complexifying the supersymmetries in a maximal supergravity theory leads to a supergravity with 64 supercharges. This is however not the case. One should view the complexified action as a mathematical tool and not as a new theory describing new physical degrees of freedom.

the spinor χ. There is an easy way around this as using the reality condition on the spinors the original real action can equivalently be written in terms of the Majorana conjugate $\bar{\chi} = \chi^T \mathcal{C}$. In this form spinors appear holomorphically and complexification now amounts to ignoring the reality condition on the spinors.

We will now illustrate this general principle in case of the ten-dimensional type II theories. For our notations we refer to appendix B .

As a starting point we will take the actions of type IIA and type IIB as given in [120]. These actions have the following field content

$$\begin{aligned} \text{IIA} \quad &: \quad \Big\{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(1)}_{\mu}, C^{(3)}_{\mu\nu\rho}, \psi_\mu, \lambda\Big\} \,, \\ \text{IIB} \quad &: \quad \Big\{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\cdots\rho}, \psi_\mu, \lambda\Big\} \,. \end{aligned} \tag{6.3.1}$$

A combined form of the actions is given by (ignoring four fermion terms)

$$\begin{aligned} S \;=\; & -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\Big\{\mathrm{e}^{-2\phi}\Big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2}H\cdot H \\ & -2\partial^\mu\phi\chi^{(1)}_\mu + H\cdot\chi^{(3)} + 2\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - 2\bar{\lambda}\Gamma^\mu\nabla_\mu\lambda + 4\bar{\lambda}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\Big] \\ & + \sum_{n=0,1/2}^{3/2,2}\Big(\tfrac{1}{2}G^{(2n)}\cdot G^{(2n)} + G^{(2n)}\cdot\Psi^{(2n)}\Big) \\ & + \tfrac{1}{4}G^{(5)}\cdot G^{(5)} + \frac{1}{2}G^{(5)}\cdot\Psi^{(5)} - e^{-1}\mathcal{L}_{\mathrm{CS}}\Big\}\,. \end{aligned} \tag{6.3.2}$$

It is understood that the summation in the above action is over integers ($n = 0, 1, 2$) in the IIA case and over half-integers ($n = 1/2, 3/2$) in the IIB case. In the summation range we first write the lowest value for the IIA case, before the one for the IIB case. Remember furthermore that $G^{(5)}$ only appears in IIB and satisfies an additional self-duality constraint $G^{(5)} = \star G^{(5)}$ that does not follow from the field equations. In the IIA case, the massive theory contains an additional mass parameter $G^{(0)} = m$. The Chern-Simons terms are respectively

$$\begin{aligned} \mathcal{L}_{\mathrm{CS}} \;=\; & -\varepsilon^{\mu_1\cdots\mu_{10}}\Big(\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}} \\ & + \tfrac{1}{2\cdot 24^2}\,G^{(0)}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,B^3_{\mu_5\cdots\mu_{10}} + \tfrac{1}{5\cdot 16^2}\,G^{(0)2}\,B^5_{\mu_1\cdots\mu_{10}}\Big)\ (\text{IIA})\,, \end{aligned} \tag{6.3.3}$$

$$\mathcal{L}_{\mathrm{CS}} \;=\; -\frac{1}{3\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}C^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_5}C^{(2)}_{\mu_6\mu_7}\partial_{\mu_8}B_{\mu_9\mu_{10}} \qquad (\text{IIB})\,. \tag{6.3.4}$$

The bosonic fields couple to the fermions via the bilinears $\chi^{(1,3)}$ and $\Psi^{(2n)}$, which read

$$\begin{aligned}
\chi_\mu^{(1)} =& -2\bar\psi_\nu\Gamma^\nu\psi_\mu - 2\bar\lambda\Gamma^\nu\Gamma_\mu\psi_\nu\,,\\
\chi_{\mu\nu\rho}^{(3)} =& \tfrac12\bar\psi_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\mathcal{P}\psi_\beta + \bar\lambda\Gamma_{\mu\nu\rho}{}^\beta\mathcal{P}\psi_\beta - \tfrac12\bar\lambda\mathcal{P}\Gamma_{\mu\nu\rho}\lambda\,,\\
\Psi^{(2n)}_{\mu_1\cdots\mu_{2n}} =& \tfrac12 \mathrm{e}^{-\phi}\bar\psi_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}\mathcal{P}_n\psi_\beta + \tfrac12\mathrm{e}^{-\phi}\bar\lambda\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta\mathcal{P}_n\psi_\beta + \\
& -\tfrac14\mathrm{e}^{-\phi}\bar\lambda\Gamma_{[\mu_1\cdots\mu_{2n-1}}\mathcal{P}_n\Gamma_{\mu_{2n}]}\lambda\,.
\end{aligned} \tag{6.3.5}$$

The supersymmetry rules read (here given modulo cubic fermion terms)

$$\begin{aligned}
\delta_\epsilon e_\mu{}^a =& \bar\epsilon\Gamma^a\psi_\mu\,,\\
\delta_\epsilon\psi_\mu =& \Big(\partial_\mu + \tfrac14\,\not\omega_\mu + \tfrac18\mathcal{P}\,\not H_\mu\Big)\epsilon + \tfrac18\mathrm{e}^\phi\sum_{n=0,1/2}^{3/2,2}\frac{1}{(2n)!}\,\not G^{(2n)}\Gamma_\mu\mathcal{P}_n\epsilon\\
& + \tfrac1{16}\mathrm{e}^\phi\frac{1}{5!}\,\not G^{(5)}\Gamma_\mu\mathcal{P}_{5/2}\epsilon\,,\\
\delta_\epsilon B_{\mu\nu} =& -2\,\bar\epsilon\Gamma_{[\mu}\mathcal{P}\psi_{\nu]}\,,\\
\delta_\epsilon C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& -\mathrm{e}^{-\phi}\,\bar\epsilon\,\Gamma_{[\mu_1\cdots\mu_{2n-2}}\,\mathcal{P}_n\,\Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac12\Gamma_{\mu_{2n-1}]}\lambda\Big)\\
& + (n-1)(2n-1)\,C^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\,\delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]}\,,\\
\delta_\epsilon\lambda =& \Big(\not\partial\phi + \tfrac1{12}\,\not H\mathcal{P}\Big)\epsilon + \tfrac14\mathrm{e}^\phi\sum_{n=0,1/2}^{2,5/2}(-)^{2n}\frac{5-2n}{(2n)!}\,\not G^{(2n)}\mathcal{P}_n\epsilon\,,\\
\delta_\epsilon\phi =& \tfrac12\,\bar\epsilon\lambda\,.
\end{aligned} \tag{6.3.6}$$

Note that for the IIB case $\Gamma_*\epsilon = \epsilon$, $\Gamma_*\psi_\mu = \psi_\mu$ and $\Gamma_*\lambda = -\lambda$, with Γ_* given by (B.1.3).

As explained in appendix B, we work both in IIA and IIB with an implicit doublet notation for the spinors. We use the conventions that symmetrization and anti-symmetrization are with weight one, slashes are short notation for $\not H = H^{\mu\nu\rho}\Gamma_{\mu\nu\rho}$ and $\not H_\mu = H_{\mu\nu\rho}\Gamma^{\nu\rho}$ and the form notations used are

$$\begin{aligned}
A^{(p)}\cdot B^{(p)} =& \frac{1}{p!}A^{(p)}_{\mu_1\cdots\mu_p}B^{(p)\,\mu_1\cdots\mu_p}\,,\\
A^{(p)}\wedge B^{(q)} =& \frac{1}{p!q!}A^{(p)}_{\mu_1\cdots\mu_p}B^{(q)}_{\mu_{p+1}\cdots\mu_{p+q}}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{p+q}}\,,\\
A^{(p)n} =& A^{(p)}\wedge\ldots\wedge A^{(p)} \qquad (n \text{ times})\,,
\end{aligned} \tag{6.3.7}$$

where the label (p) refers to the order of the p-form. The other form conventions are given in appendix A. For notational convenience we group all potentials and field strengths in the formal sums

$$\mathbf{G} = \sum_{n=0,1/2}^{2,5/2} G^{(2n)} \,, \qquad \mathbf{C} = \sum_{n=1,1/2}^{2,5/2} C^{(2n-1)} \,. \tag{6.3.8}$$

The bosonic field strengths are given by

$$H = \mathrm{d}B \,, \qquad \mathbf{G} = \mathrm{d}\mathbf{C} - \mathrm{d}B{\scriptstyle\wedge}\mathbf{C} + G^{(0)}\mathbf{e}^{B} \,, \tag{6.3.9}$$

where it is understood that each equation involves only one term from the formal sums (6.3.8) (only the relevant combinations are extracted). Also we will use the following abbreviation:

$$\mathbf{e}^{\pm B} \equiv \pm B + \tfrac{1}{2} B{\scriptstyle\wedge}B \pm \tfrac{1}{3!} B{\scriptstyle\wedge}B{\scriptstyle\wedge}B + \ldots \tag{6.3.10}$$

In writing down type II actions, we use the following definitions

$$\mathcal{P} \quad = \Gamma_{11} \otimes \mathbb{1}_2 = \quad \mathbb{1}_{32} \otimes \ \sigma_3 \ \text{(IIA)} \ \text{ or } \ - \mathbb{1}_{32} \otimes \sigma_3 \ \text{(IIB)} \,, \tag{6.3.11}$$

and

$$\mathcal{P}_n = (\Gamma_{11} \otimes \mathbb{1}_2)^n \ \text{ (IIA) } \ \text{or} \ \ \mathbb{1}_{32} \otimes \sigma^1 \ (\text{n} + 1/2 \text{ even}), \ \mathbb{1}_{32} \otimes i\sigma^2 \ (\text{n} + 1/2 \text{ odd}) \quad \text{(IIB)} \,.$$

Up till now we have just written down the action of the type IIA/B in (1,9) signature in a standard form. We will now interpret the action (6.3.2) in a different way, as a complex action. All fields are now assumed to be complex, both bosonic and fermionic. For the fermions this means that they are arbitrary Dirac spinors, as stated before they only appear holomorphically in the action through their Majorana conjugate $\bar{\chi} = \chi^{T}\mathcal{C}$. The gamma-matrices with flat indices remain the standard gamma-matrices of (1,9) Minkowski space. As we now allow the vielbein to be complex, the curved gamma-matrices will be part of the complexified Clifford algebra, see section 6.4 for more details. The supersymmetry transformations (6.3.6) are understood to be complex in the same way as the action (6.3.2). The complexified action remains invariant under the complexified supersymmetry transformations as basic manipulations like symmetry properties of bilinears, gamma-matrix algebra and Fierz identities are insensitive to this complexification. In the same way the complex action is invariant under the complexified Lorentz-group $\mathrm{SO}(10, \mathbb{C})$.

6.3.2 Back to Reality

Starting from the complex action and supersymmetry transformations of the previous section we will now explain how one can construct different real actions by taking different real slices. In this subsection, we will do a general analysis determining all variant supergravities. The result is summarized in table 6.3.1. In the next subsection, we will illustrate the method with some specific examples.

Let us start by explaining what we mean by taking a real slice. A reality condition on the fields cannot be chosen at will, but has to satisfy certain consistency conditions. First of all, one can only impose a limited number of reality conditions on the fermions. As is explained in appendix B this leads to the following general reality conditions on the fermionic fields (see (B.2.7))

$$\begin{aligned} \epsilon^* &= -\varepsilon\eta^t\alpha_\epsilon \mathcal{C}A\rho\epsilon\,, \\ \psi^*_\mu &= -\varepsilon\eta^t\alpha_\psi \mathcal{C}A\rho\psi_\mu\,, \\ \lambda^* &= -\varepsilon\eta^t\alpha_\lambda \mathcal{C}A\rho\lambda\,, \end{aligned} \tag{6.3.12}$$

where the α_χ represents a phase factor that can differ from field to field. On the bosonic fields, a general reality condition is given by[4]:

$$\begin{aligned} e_\mu{}^{a*} &= e_\mu{}^a\,, \\ \phi^* &= \phi\,, \\ B^*_{\mu\nu} &= \alpha_B B_{\mu\nu}\,, \\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= \alpha_n C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,, \end{aligned} \tag{6.3.13}$$

where again the α-factors represent phases. Note that we have already taken the dilaton to be real, as this is the only condition consistent with reality of the action. We also choose to work with real vielbeine. This amounts to using the flat gamma-matrices that are appropriate to a specific signature. The complex action is written in terms of fixed flat Γ-matrices in signature (1,9). In principle one could keep these fixed during the whole procedure and allow for purely imaginary vielbein components. Simultaneously redefining the vielbeine and flat gamma-matrices then brings one back to the case where the vielbeine are real and the Clifford algebra has the appropriate signature. For a more thorough and technical discussion of this point, see section 6.4. This reasoning also reveals a subtlety concerning the Chern-Simons terms. Supersymmetry of the action (6.3.2) is established thanks to the relation

$$\Gamma_{a_1\ldots a_n} = -\frac{1}{(10-n)!}\varepsilon_{a_1\ldots a_{10}}\Gamma_{11}\Gamma^{a_{10}\ldots a_{n+1}}\,. \tag{6.3.14}$$

This relation is however only valid for the Clifford algebra with signature $(1,9)$. As explained above, we choose to work with the Clifford algebra that has the same signature as space-time. For this Clifford algebra, the relation (6.3.14) is changed to

$$\Gamma_{a_1\ldots a_n} = \frac{1}{(10-n)!}\varepsilon_{a_1\ldots a_{10}} i^{t+1}\Gamma_{11}\Gamma^{a_{10}\ldots a_{n+1}}\,. \tag{6.3.15}$$

[4]To have a uniform notation the reality condition for $G^{(0)}$ is given in terms of some formal $C^{(-1)}$. This is just a shorthand implying $G^{(0)*} = \alpha_0 G^{(0)}$.

Effectively, rewriting (6.3.15) to (6.3.14) corresponds to replacing $\varepsilon_{0\ldots9}$ by

$$\begin{array}{rcl} \varepsilon_{0\ldots9} & \to & -(-i)^{t+1}\varepsilon_{0\ldots9}\,, \\ \varepsilon^{0\ldots9} & \to & -(i)^{t+1}\varepsilon^{0\ldots9}\,. \end{array} \tag{6.3.16}$$

When going to a real action of a given signature, one has to replace the $\varepsilon_{0\ldots9}$ in the complex Chern-Simons term via the above rule to assure invariance under supersymmetry.

The α-factors appearing in the reality conditions on the bosons and the fermions are not independent. Demanding a real action and consistency with supersymmetry relates them. The latter means that both sides of the supersymmetry rules should have the same behaviour under complex conjugation. In this way, the reality conditions on the fermions determine those of the bosons. Analyzing this in detail leads to the relations

$$\begin{array}{rcl} \alpha_\epsilon & = & \alpha_\psi\,, \\ \alpha_\lambda^2 & = & \alpha_\psi^2 = (-\eta)^{t+1}\,, \\ \alpha_\lambda & = & (-)^{t+1}\eta\rho^T\sigma\rho\sigma\,\alpha_\psi\,, \\ \alpha_H & = & \rho^T\sigma^{t+1}\sigma_3\sigma^{t+1}\rho\sigma_3\,, \\ \alpha_n & = & (-)^{(2n+1)t}(-\eta)^{(2n+1)}\rho^T\sigma^t\mathcal{P}_n\sigma^{t+1}\rho\mathcal{P}_n^{-1}\sigma\,. \end{array} \tag{6.3.17}$$

The possible solutions of these equations lead to consistent reality conditions on all fields. They are summarized in table 6.3.1. Every possible reality condition corresponds to a unique real supergravity theory that has (6.3.2) as complexified action.

Given the data in table 6.3.1, the actions and supersymmetry rules of these variant supergravities can be explicitly written down. These actions are the complex action (6.3.2), where the fields now obey the reality properties (6.3.12,6.3.13), with the α-factors the ones mentioned in table 6.3.1. One notices that in this form some fields might be purely imaginary. In this case, it is more natural to redefine the fields in terms of real fields. This leads to a change in sign of e.g. the kinetic terms of these fields. In order to write the actions in a more conventional form involving Dirac conjugates, one can use the following formula equivalent to (6.3.12) if (6.3.17) is satisfied:

$$\bar\chi = \alpha_\psi\alpha_\lambda\alpha_\chi^*\bar\chi^D\rho\,. \tag{6.3.18}$$

This allows one to rewrite Majorana conjugates appearing in (6.3.2) in terms of Dirac conjugates. As explained above in certain signatures one has to multiply the Chern-Simons term by an additional factor, this factor is given in the last row of table 6.3.1, this same factor also appears in the (anti) self-duality condition of IIB. The procedure described here will be illustrated in more detail for some specific examples in subsection 6.3.3.

	A				B			
t mod 4	0	1		2	1			3
type	*M$^+$	MW	*MW	M$^+$	MW	*MW	′MW	SMW
$\varepsilon = \eta$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
ρ	σ_3	$\mathbb{1}$	σ_3	$\mathbb{1}$	$\mathbb{1}$	σ_3	σ_1	$i\sigma_2$
$\alpha_\epsilon = \alpha_\psi$	i	1	1	i	1	1	1	1
α_λ	i	1	-1	$-i$	1	1	1	1
α_B	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$
$\alpha_0 = \alpha_2,\ \alpha_{1/2} = \alpha_{5/2}$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
$\alpha_1,\ \alpha_{3/2}$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$
$-(i)^{t+1}$	$-i$	1	1	i	1	1	1	-1

Table 6.3.1: *Possible reality conditions on the fields of type II supergravities. t is the number of timelike directions in space-time. The notation concerning the type of fermionic reality condition is explained in appendix B. Every set of reality conditions (column) corresponds to a different variant supergravity theory. The last row refers to the additional factor for the Chern-Simons terms. From this table the actions and supersymmetry transformations of all 10d variant supergravities can be constructed.*

Finally let us give a short overview of the variant theories classified by table 6.3.1. Type IIA supergravity exists in three types of signatures. Note that only in signature $t = 1$ mod 4 there are two different real theories[5]. For IIB the situation is similar. Although table 6.3.1 seems to suggest that there are three different theories in (1,9), IIB* and IIB$'$ are related by a field redefinition that can be interpreted as an S-duality. Note that IIB theories only exist in those signatures where a consistent self-duality condition can be imposed. In our conventions the five form is self-dual in signatures with $t = 1$ mod 4 and anti self-dual when $t = 3$ mod 4, this is due to the subtleties concerning the appearance of $\varepsilon_{0\ldots9}$ explained above.

6.3.3 Examples

In this subsection we will illustrate the previously discussed method of real slices for type II theories in signature (1,9). We will show how to write down the explicit form of the actions starting from table 6.3.1. To illustrate how to write down the Chern-Simons terms in case the real slice involves an additional factor multiplying $\varepsilon_{0\ldots9}$ we discuss this term in signature (0,10) in detail.

[5]The results of table 6.3.1 almost completely agree with those found in [117] for IIA, with the exception that only in signature (1,9) we find two inequivalent theories. In [117] additional IIA theories for $t = 0$ or 2 mod 4 are presented, which we are not able to reproduce in our framework.

IIA

Our first example is how one can recover the usual type IIA theory in signature (1,9). The reality conditions appropriate for this theory are summarized in the second column of table 6.3.1, leading to:

$$\begin{aligned} \epsilon^* &= -\mathcal{C}A\epsilon\,, \\ \psi_\mu^* &= -\mathcal{C}A\psi_\mu\,, \\ \lambda^* &= -\mathcal{C}A\lambda \\ B_{\mu\nu}^* &= B_{\mu\nu}\,, \\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,, \quad (n=0,1,2)\,. \end{aligned} \tag{6.3.19}$$

The real action for this theory is the complex action given above (6.3.2) but restricted to the subspace given by these reality conditions. The Majorana conditions for the spinors (6.3.19) are equivalent to

$$\begin{aligned} \bar\epsilon &= \bar\epsilon^{D}\,, \\ \bar\psi_\mu &= \bar\psi_\mu^{D}\,, \\ \bar\lambda &= \bar\lambda^{D}\,, \end{aligned} \tag{6.3.20}$$

and using these we can write the action (6.3.2) in a standard real form involving Dirac conjugates. Plugging (6.3.19 - 6.3.20) into the action (6.3.2) gives

$$\begin{aligned} S_{\rm IIA} =& -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\Big\{\mathrm{e}^{-2\phi}\big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2}H\cdot H - 2\partial^\mu\phi\chi^{(1)}_\mu + H\cdot\chi^{(3)} \\ &+ 2\bar\psi^{D}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - 2\bar\lambda^{D}\Gamma^\mu\nabla_\mu\lambda + 4\bar\lambda^{D}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\big] + \sum_{n=0,1,2}\tfrac{1}{2}G^{(2n)}\cdot G^{(2n)} \\ &+ G^{(2n)}\cdot\Psi^{(2n)} + e^{-1}\varepsilon^{\mu_1\cdots\mu_{10}}\big[\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}} \\ &+ \tfrac{1}{2\cdot 24^2}\,G^{(0)}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,B^3_{\mu_5\ldots\mu_{10}} + \tfrac{1}{5\cdot 16^2}\,G^{(0)2}\,B^5_{\mu_1\ldots\mu_{10}}\big]\Big\}\,, \end{aligned} \tag{6.3.21}$$

where

$$\begin{aligned} \chi^{(1)}_\mu &= -2\bar\psi^{D}_\nu\Gamma^\nu\psi_\mu - 2\bar\lambda^{D}\Gamma^\nu\Gamma_\mu\psi_\nu\,, \\ \chi^{(3)}_{\mu\nu\rho} &= \tfrac{1}{2}\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\Gamma_{11}\psi_\beta + \bar\lambda^{D}\Gamma_{\mu\nu\rho}{}^\beta\Gamma_{11}\psi_\beta - \tfrac{1}{2}\bar\lambda^{D}\Gamma_{11}\Gamma_{\mu\nu\rho}\lambda\,, \\ \Psi^{(2n)}_{\mu_1\cdots\mu_{2n}} &= \tfrac{1}{2}\mathrm{e}^{-\phi}\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}(\Gamma_{11})^n\psi_\beta + \tfrac{1}{2}\mathrm{e}^{-\phi}\bar\lambda^{D}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta(\Gamma_{11})^n\psi_\beta \\ &\quad -\tfrac{1}{4}\mathrm{e}^{-\phi}\bar\lambda^{D}\Gamma_{[\mu_1\cdots\mu_{2n-1}}(\Gamma_{11})^n\Gamma_{\mu_{2n}]}\lambda\,. \end{aligned} \tag{6.3.22}$$

The action (6.3.21) is invariant under the following supersymmetries

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^a =& \bar\epsilon^D \Gamma^a \psi_\mu \,, \\
\delta_\epsilon \psi_\mu =& \Big(\partial_\mu + \tfrac14 \not\omega_\mu + \tfrac18 \Gamma_{11} \not H_\mu\Big)\epsilon + \tfrac18 \mathrm{e}^{\phi} \sum_{n=0,1,2} \frac{1}{(2n)!} \not G^{(2n)} \Gamma_\mu (\Gamma_{11})^n \, \epsilon \,, \\
\delta_\epsilon B_{\mu\nu} =& -2\, \bar\epsilon^D \Gamma_{[\mu} \Gamma_{11} \psi_{\nu]} \,, \\
\delta_\epsilon C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& -\mathrm{e}^{-\phi}\, \bar\epsilon^D\, \Gamma_{[\mu_1\cdots\mu_{2n-2}} (\Gamma_{11})^n \Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac12 \Gamma_{\mu_{2n-1}]} \lambda\Big) \\
& + (n-1)(2n-1)\, C^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\, \delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]} \,, \\
\delta_\epsilon \lambda =& \Big(\not\partial\phi + \tfrac{1}{12} \not H \Gamma_{11}\Big)\epsilon + \tfrac14 \mathrm{e}^{\phi} \sum_{n=0,1,2} \frac{5-2n}{(2n)!} \not G^{(2n)} (\Gamma_{11})^n \epsilon \,, \\
\delta_\epsilon \phi =& \tfrac12\, \bar\epsilon^D \lambda \,.
\end{aligned}
\qquad (6.3.23)
$$

As the (1,9) IIA supergravity theory was the theory we started from before complexifying, taking the real slice was rather straightforward. Things will become more interesting in case some fields are purely imaginary. We illustrate this in the following example.

IIA*

The action of the IIA* theory in (1,9) can be constructed by using the third column of table 6.3.1, which leads to the following reality conditions:

$$
\begin{aligned}
\epsilon^* &= -\mathcal{C}A\Gamma_{11}\epsilon \,, \\
\psi^*_\mu &= -\mathcal{C}A\Gamma_{11}\psi_\mu \,, \\
\lambda^* &= \mathcal{C}A\Gamma_{11}\lambda \,, \\
B^*_{\mu\nu} &= B_{\mu\nu} \,, \\
C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= -C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} \,, \quad (n = 0,1,2) \,.
\end{aligned}
\qquad (6.3.24)
$$

Note that now the reality condition for the Ramond-Ramond fields implies that they are purely imaginary. It is therefore natural to make a redefinition to real fields. We also prefer to have the same reality condition for all the fermionic fields. Thus we make the field redefinitions

$$
\begin{aligned}
\zeta &= -i\lambda \,, \\
A^{(2n-1)} &= -iC^{(2n-1)} \,, \\
F^{(2n)} &= -iG^{(2n)} \,.
\end{aligned}
\qquad (6.3.25)
$$

In this case the relation between Majorana and Dirac conjugate of the spinors is

$$\begin{aligned} \bar{\epsilon} &= -\bar{\epsilon}^{D}\Gamma_{11}\,, \\ \bar{\psi}_\mu &= -\bar{\psi}^{D}_\mu\Gamma_{11}\,, \\ \bar{\zeta} &= -\bar{\zeta}^{D}\Gamma_{11}\,. \end{aligned} \tag{6.3.26}$$

Similarly to the IIA case one can obtain a manifestly real action, which now reads

$$\begin{aligned} S_{\mathrm{IIA}^*} = &-\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\Big\{\mathrm{e}^{-2\phi}\big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2}H\cdot H - 2\partial^\mu\phi\xi^{(1)}_\mu + H\cdot\xi^{(3)} + \\ &- 2\bar{\psi}^{D}_\mu\Gamma_{11}\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - 2\bar{\zeta}^{D}\Gamma_{11}\Gamma^\mu\nabla_\mu\zeta - 4i\bar{\zeta}^{D}\Gamma_{11}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\big] - \sum_{n=0,1,2}\tfrac{1}{2}F^{2n}\cdot F^{2n} \\ &+ F^{2n}\cdot\Delta^{(2n)} - e^{-1}\varepsilon^{\mu_1\cdots\mu_{10}}\big[\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}A^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}A^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}} \\ &+ \tfrac{1}{2\cdot 24^2}\,F^{(0)}\,\partial_{\mu_1}A^{(3)}_{\mu_2\mu_3\mu_4}\,B^3_{\mu_5\cdots\mu_{10}} + \tfrac{1}{5\cdot 16^2}\,F^{(0)2}\,B^5_{\mu_1\cdots\mu_{10}}\big]\Big\}\,, \end{aligned} \tag{6.3.27}$$

where

$$\begin{aligned} \xi^{(1)}_\mu &= -2\bar{\psi}^{D}_\nu\Gamma^\nu\Gamma_{11}\psi_\mu + 2i\bar{\zeta}^{D}\Gamma^\nu\Gamma_\mu\Gamma_{11}\psi_\nu\,, \\ \xi^{(3)}_{\mu\nu\rho} &= \tfrac{1}{2}\bar{\psi}^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\psi_\beta - i\bar{\zeta}^{D}\Gamma_{\mu\nu\rho}{}^\beta\psi_\beta - \tfrac{1}{2}\bar{\zeta}^{D}\Gamma_{\mu\nu\rho}\zeta\,, \\ \Delta^{(2n)}_{\mu_1\cdots\mu_{2n}} &= \tfrac{i}{2}\mathrm{e}^{-\phi}\bar{\psi}^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}(\Gamma_{11})^{n+1}\psi_\beta + \tfrac{1}{2}\mathrm{e}^{-\phi}\bar{\zeta}^{D}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta(\Gamma_{11})^{n+1}\psi_\beta \\ &\quad -\tfrac{i}{4}\mathrm{e}^{-\phi}\bar{\zeta}^{D}\Gamma_{[\mu_1\cdots\mu_{2n-1}}(\Gamma_{11})^{n+1}\Gamma_{\mu_{2n}]}\zeta\,. \end{aligned} \tag{6.3.28}$$

The action (6.3.27) is invariant under the supersymmetries

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= \bar{\epsilon}^{D}\Gamma^a\Gamma_{11}\psi_\mu\,, \\ \delta_\epsilon\psi_\mu &= \Big(\partial_\mu + \tfrac{1}{4}\,\not{\omega}_\mu + \tfrac{1}{8}\Gamma_{11}\,\not{H}_\mu\Big)\epsilon + \tfrac{i}{8}\mathrm{e}^\phi\sum_{n=0,1,2}\frac{1}{(2n)!}\,\not{F}^{(2n)}\Gamma_\mu(\Gamma_{11})^n\,\epsilon\,, \\ \delta_\epsilon B_{\mu\nu} &= -\,2\,\bar{\epsilon}^{D}\Gamma_{[\mu}\psi_{\nu]}\,, \\ \delta_\epsilon A^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} &= -\,\mathrm{e}^{-\phi}\,\bar{\epsilon}^{D}\,\Gamma_{[\mu_1\cdots\mu_{2n-2}}\,(\Gamma_{11})^{n+1}\,\Big(i(2n-1)\psi_{\mu_{2n-1}]} + \tfrac{1}{2}\Gamma_{\mu_{2n-1}]}\zeta\Big) \\ &\quad + (n-1)(2n-1)\,A^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\,\delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]}\,, \\ \delta_\epsilon\zeta &= -\,i\Big(\not{\partial}\phi + \tfrac{1}{12}\,\not{H}\Gamma_{11}\Big)\epsilon + \tfrac{1}{4}\mathrm{e}^\phi\sum_{n=0,1,2}\frac{5-2n}{(2n)!}\,\not{F}^{(2n)}(\Gamma_{11})^n\epsilon\,, \\ \delta_\epsilon\phi &= -\,\tfrac{i}{2}\,\bar{\epsilon}^{D}\Gamma_{11}\zeta\,. \end{aligned} \tag{6.3.29}$$

Note that indeed in this real form the Ramond-Ramond fields have wrong sign kinetic terms. Furthermore, there are additional factors of i appearing in the supersymmetry transformations with respect to standard IIA. This is similar to the i's appearing in the pseudo-supersymmetry of [67, 98], we will elaborate on this in section 6.5.1. Another difference is the appearance of the chirality matrix Γ_{11} in various spinor bilinears. They appear for example in the variation of the dilaton, leading to a different transformation of this field under parity[6].

Chern-Simons terms

As explained above there are some subtleties concerning the Chern-Simons terms in certain signatures. Here we will briefly illustrate how the Chern-Simons term of IIA in (0,10) signature can be obtained, the other cases proceed analogously. Of the fields appearing in the IIA Chern-Simons term, B becomes purely imaginary while the others are real, as can be read from the first column of table 6.3.1. We thus make the redefinition

$$\tilde{B}_{\mu\nu} = -iB_{\mu\nu}\,. \tag{6.3.30}$$

Substituting this in the complex Chern-Simons term (6.3.3) and multiplying with the appropriate factor i (see table 6.3.1) gives the following real topological terms:

$$\begin{aligned}-\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\varepsilon^{\mu_1\cdots\mu_{10}}\big[&\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,\tilde{B}_{\mu_9\mu_{10}}\\ &-\tfrac{1}{2\cdot 24^2}\,G^{(0)}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\tilde{B}^3_{\mu_5\cdots\mu_{10}}+\tfrac{1}{5\cdot 16^2}\,G^{(0)2}\,\tilde{B}^5_{\mu_1\cdots\mu_{10}}\big]\,.\end{aligned} \tag{6.3.31}$$

Note that apart from the changes in the Chern-Simons term also the relation between the real potentials and field strengths gets modified, e.g.

$$G^{(4)} = \mathrm{d}C^{(3)} + \mathrm{d}\tilde{B}\wedge A^{(1)} - G^{(0)}\tilde{B}^2\,, \tag{6.3.32}$$

instead of the standard relation (6.3.9).

Similar to this example one can find the Chern-Simons terms and field strengths in other signatures.

IIB*

As our final example we derive the action and supersymmetry equations of IIB*, the alternate real IIB theory in signature (1,9). The reality conditions are:

$$\begin{aligned}\epsilon^* &= \mathcal{C}A\mathcal{P}\epsilon\,,\\ \psi^*_\mu &= \mathcal{C}A\mathcal{P}\psi_\mu\,,\\ \lambda^* &= \mathcal{C}A\mathcal{P}\lambda\,,\\ B^*_{\mu\nu} &= B_{\mu\nu}\,,\\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= -C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,,\quad (n=1/2,3/2,5/2)\,.\end{aligned} \tag{6.3.33}$$

[6]Under parity we understand the transformation $x^i \to -x^i$ that reverses the sign of all 9 spacelike directions.

We redefine the imaginary fields in term of real fields as follows:

$$\begin{aligned} A^{(2n-1)} &= -iC^{(2n-1)}\,, \\ F^{(2n)} &= -iG^{(2n)}\,. \end{aligned} \tag{6.3.34}$$

The reality conditions for the spinors are equivalent to the conditions

$$\begin{aligned} \bar{\epsilon} &= \bar{\epsilon}^{D}\mathcal{P}\,, \\ \bar{\psi}_\mu &= \bar{\psi}_\mu^{D}\mathcal{P}\,, \\ \bar{\lambda} &= \bar{\lambda}^{D}\mathcal{P}\,. \end{aligned} \tag{6.3.35}$$

Substituting this into the complex IIB action (6.3.2) leads to

$$\begin{aligned} S_{IIB*} =& -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x e\Big\{e^{-2\phi}\Big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2}H\cdot H + \\ & -2\partial^\mu\phi\zeta_\mu^{(1)} + H\cdot\zeta^{(3)} + 2\bar{\psi}_\mu^D\mathcal{P}\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - 2\bar{\lambda}^D\mathcal{P}\Gamma^\mu\nabla_\mu\lambda + 4\bar{\lambda}^D\mathcal{P}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\Big] \\ & -\sum_{n=1/2}^{3/2}\left(\frac{1}{2}F^{(2n)}\cdot F^{(2n)} + F^{(2n)}\cdot\Delta^{(2n)}\right) - \frac{1}{4}F^{(5)}\cdot F^{(5)} - \frac{1}{2}F^{(5)}\cdot\Delta^{(5)} \\ & - e^{-1}\frac{1}{3\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}A^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_5}A^{(2)}_{\mu_6\mu_7}\partial_{\mu_8}B_{\mu_9\mu_{10}}\Big\}\,. \end{aligned} \tag{6.3.36}$$

This action needs to be supplemented with the usual self-duality for the RR-five form $F^{(5)}$. The bosonic fields couple to the fermions via the bilinears

$$\begin{aligned} \zeta_\mu^{(1)} =& -2\bar{\psi}_\nu^D\mathcal{P}\Gamma^\nu\psi_\mu - 2\bar{\lambda}^D\mathcal{P}\Gamma^\nu\Gamma_\mu\psi_\nu\,, \\ \zeta_{\mu\nu\rho}^{(3)} =& \tfrac{1}{2}\bar{\psi}_\alpha^D\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\psi_\beta + \bar{\lambda}^D\ \Gamma_{\mu\nu\rho}{}^\beta\psi_\beta - \tfrac{1}{2}\bar{\lambda}^D\Gamma_{\mu\nu\rho}\lambda\,, \\ \Delta^{(2n)}_{\mu_1\cdots\mu_{2n}} =& -i\Big(\tfrac{1}{2}\mathrm{e}^{-\phi}\bar{\psi}_\alpha^D\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}\mathcal{P}\mathcal{P}_n\psi_\beta + \tfrac{1}{2}\mathrm{e}^{-\phi}\bar{\lambda}^D\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta\mathcal{P}\mathcal{P}_n\psi_\beta \\ & - \tfrac{1}{4}\mathrm{e}^{-\phi}\bar{\lambda}^D\Gamma_{[\mu_1\cdots\mu_{2n-1}}\mathcal{P}\mathcal{P}_n\Gamma_{\mu_{2n}]}\lambda\Big)\,. \end{aligned} \tag{6.3.37}$$

The supersymmetry rules are

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^a =& \bar\epsilon^D \mathcal{P} \Gamma^a \psi_\mu \,, \\
\delta_\epsilon \psi_\mu =& \Big(\partial_\mu + \tfrac14 \not{\omega}_\mu + \tfrac18 \mathcal{P} \not{H}_\mu\Big)\epsilon + \tfrac{i}{8} \mathrm{e}^\phi \sum_{n=1/2}^{3/2} \frac{1}{(2n)!} \not{F}^{(2n)} \Gamma_\mu \mathcal{P}_n \epsilon \\
& + \tfrac{i}{16} \mathrm{e}^\phi \frac{1}{5!} \not{F}^{(5)} \Gamma_\mu \mathcal{P}_{5/2} \epsilon \,, \\
\delta_\epsilon B_{\mu\nu} =& -2\,\bar\epsilon^D \Gamma_{[\mu} \psi_{\nu]} \,, \\
\delta_\epsilon A^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& i \mathrm{e}^{-\phi}\, \bar\epsilon^D\, \Gamma_{[\mu_1\cdots\mu_{2n-2}}\, \mathcal{P}\mathcal{P}_n \Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac12 \Gamma_{\mu_{2n-1}]}\lambda\Big) \\
& + (n-1)(2n-1)\, A^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\, \delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]} \,, \\
\delta_\epsilon \lambda =& \Big(\not{\partial}\phi + \tfrac{1}{12} \not{H}\mathcal{P}\Big)\epsilon + \tfrac{i}{4} \mathrm{e}^\phi \sum_{n=1/2}^{5/2} (-)^{2n} \frac{5-2n}{(2n)!} \not{F}^{(2n)} \mathcal{P}_n \epsilon \,, \\
\delta_\epsilon \phi =& \tfrac12\, \bar\epsilon^D \mathcal{P} \lambda \,.
\end{aligned}
\tag{6.3.38}
$$

Note that in contrast to the standard IIB action the IIB* action is no longer invariant under the full S-duality group, but gets mapped to the IIB$'$ theory. Another viewpoint is thus that IIB$'$ is nothing else than a field redefinition of IIB*. As such we will not construct its action and supersymmetry transformations here. They can be obtained either from performing an S-duality or taking a real slice with the appropriate reality conditions in table 6.3.1.

6.3.4 Extended vs. Unextended Supersymmetry

It might be remarkable that in certain signatures different real slices exist while in others only one real theory is consistent. This is related to the number of independent supersymmetries. Although we always discussed theories with 32 real supercharges, this does not necessarily mean that their supersymmetry is extended. Depending from signature to signature the dimension of a real irreducible spinor is 16 or 32. Only in the signatures in which it is 16, and thus the 32 supercharges imply extended supersymmetry, different real slices can occur. This can be understood as in this case different reality conditions can be imposed on the two independent 16-dimensional spinors.

This suggests that in any signature only one real slice of the complex 10d $\mathcal{N}=1$ supergravities exists. These $\mathcal{N}=1$ supergravities can be seen as truncations of the type II theories by a $\mathbb{Z}_2$ truncation. Thus one would expect both the standard theories and their star versions to truncate to the same theory. We will now show that this is indeed the case in IIA. The truncation is made by only keeping the fields invariant under the following fermion number symmetry [120]:

$$\begin{aligned}\{\phi, g_{\mu\nu}, B_{\mu\nu}\} &\to \{\phi, g_{\mu\nu}, B_{\mu\nu}\}\,,\\ \{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\} &\to -\{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\}\,,\\ \{\psi_\mu, \lambda, \epsilon\} &\to \Gamma_{11}\{\psi_\mu, -\lambda, \epsilon\}\,.\end{aligned} \tag{6.3.39}$$

One can see that both IIA and IIA* project to the same theory under identification by this symmetry as this identification is equivalent to demanding the reality conditions (6.3.19) and (6.3.24) to be identical. The other IIA truncation that is given in [120] is no longer consistent.

The situation is similar in IIB. For IIB in (1,9) signature both truncations given in [120] lead to the same result, for IIB* only one truncation is consistent with the reality properties of the spinors while the other identification is the only consistent one for IIB$'$. In the end all possible truncations lead to the same $\mathcal{N} = 1$ theory.

6.4 Reality of the Vielbeine

6.4.1 Imaginary Vielbeine and Signature Change

In this section we will give some more details on the equivalence between choosing to work with on the one hand fixed flat gamma-matrices of signature (1,9) and possibly imaginary vielbein or on the other hand gamma-matrices of the appropriate signature and a real vielbein.

It is important to stress that the flat gamma-matrices appearing in the complex action (6.3.2) are elements of the Clifford algebra of signature (1,9) obeying the standard reality condition[7]

$$\Gamma^{a*} = -\mathcal{C}\Gamma_0\Gamma^a\Gamma_0\mathcal{C}^{-1}. \tag{6.4.1}$$

The curved gamma-matrices $\Gamma_\mu = \Gamma_a e_\mu{}^a$ no longer obey a reality condition as $e_\mu{}^a$ (and the other fields) are complex. Because the vielbein, and thus the metric as well, is complex there is no longer a concept of space-time signature. Note that the complex metric is defined as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta^{(1,9)}_{ab}$, where $\eta^{(1,9)}_{ab} = \mathrm{diag}(- + \ldots +)$.

When we impose reality conditions on the fields appearing in the action we recover a real theory in a signature that can differ from the (1,9) signature we started from. This can happen as some components of the vielbein can be purely imaginary such that $g_{\mu\nu}$ is real but has a signature different from that of $\eta^{(1,9)}_{ab}$.

As explained a choice of reality conditions for the fermions determines the reality properties of all the bosonic fields as well. For the vielbein this happens through the supersymmetry transformation

$$\delta_\epsilon e_\mu{}^a = \bar{\epsilon}\Gamma^a\psi_\mu\,. \tag{6.4.2}$$

[7] In this section we will make the choice $\epsilon = \eta = 1$.

As explained in appendix B in the reality conditions for the spinors (B.2.7) the operator A appears. This A is the product of the timelike gamma-matrices. So by choosing A in the reality conditions for the complex fermions one decides in which space-time signature the real fermions will be consistent. As we will see below consistency of the above supersymmetry variation (6.4.2) implies that also the real metric given by these reality conditions has that signature. If one for example makes a real slice to a theory in signature (t, s), $t + s = 10$, fermions satisfy the following reality conditions

$$\begin{aligned} \epsilon^* &= -\varepsilon\eta^t\alpha_\epsilon \mathcal{C}A\rho\epsilon\,, \\ \psi_\mu^* &= -\varepsilon\eta^t\alpha_\psi \mathcal{C}A\rho\psi_\mu\,, \end{aligned} \tag{6.4.3}$$

with

$$A = (\Gamma_0)(i\Gamma_1)\dots(i\Gamma_{t-1})\,, \tag{6.4.4}$$

where Γ_a are elements of the (1,9) Clifford algebra, i.e. those appearing in the complex action and (6.4.2). We propose the following reality conditions for the vielbeine:

$$(e_\mu{}^a)^* = \alpha_\mu^a e_\mu{}^a\,. \tag{6.4.5}$$

Using this definition and (6.4.3), one can calculate that[8]

$$\alpha_\mu^a = (-)^t A^{-1}\Gamma_0\Gamma^a\Gamma_0 A(\Gamma^a)^{-1}\,, \tag{6.4.6}$$

by taking the complex conjugate of (6.4.2). In the case we take A of the form (6.4.4) and divide the index a as $i = 1 \dots t-1$, $j = t \dots 9$ this implies

$$\alpha_\mu^0 = 1\,,\quad \alpha_\mu^i = -1\,,\quad \alpha_\mu^j = 1\,. \tag{6.4.7}$$

So parts of the vielbein are imaginary and indeed this exactly implies the metric $g_{\mu\nu}$ now has the signature (t, s).

Although everything works perfectly in this way it is rather odd to work with vielbeine that have imaginary components. This is why in the previous section we preferred to work in a formulation where the vielbein is always completely real. This can be accomplished by simultaneously redefining the appropriate components $e_\mu^{\;i} = i\tilde{e}_\mu^{\;i}$ and $\Gamma^i{=}i\tilde{\Gamma}^i$. It is clear that this redefinition changes the signature of the flat metric η_{ab} as the Clifford algebra now has signature (t, s). Furthermore, in all supersymmetry transformations and the action the vielbeine and Γ's appear in pairs of the form $e^\mu{}_a\Gamma^a$ or $e_\mu{}^a\Gamma_a$ and as such always in a combination where one of the redefined variables appears through its inverse. This means that we can put tildes everywhere without changing the form of the expressions or having to add i's or minus signs. One should read the previous section with this redefinition in mind although we did not explicitly write the tildes, i.e. in section 6.3 flat gamma-matrices appearing in real actions and supersymmetry transformations are always elements of the Clifford algebra that has the same signature as space-time and all vielbeine are real.

[8]One has to use that $\alpha_\epsilon\alpha_\psi = (-)^{t+1}$, which follows from analysing the other supersymmetry variations.

6.4.2 Imaginary Vielbeine without Signature Change

The discussion above brings about another point. One can also take some of the components of the vielbein imaginary and still obtain signature (1,9) for the curved metric. This can be achieved by taking for instance the following matrix A:

$$A = i\Gamma_9 \,. \tag{6.4.8}$$

This still leads to a consistent reality condition for the fermions. As explained before A determines what is space and what is time in the real slice. The choice (6.4.8) corresponds to

$$\alpha^0_\mu = -1 \,, \quad \alpha^i_\mu = 1 \,, \quad \alpha^9_\mu = -1 \,, \qquad (i = 1 \ldots 8) \,. \tag{6.4.9}$$

The naturally redefined $\tilde{\eta}^{(1,9)}_{ab}$ now has $\tilde{\eta}^{(1,9)}_{00} = \tilde{\eta}^{(1,9)}_{ii} = 1$ and $\tilde{\eta}^{(1,9)}_{99} = -1$ while the original $\eta^{(1,9)}_{ab}$ from the complex theory had $\eta^{(1,9)}_{00} = -1$ and $\eta^{(1,9)}_{ii} = 1 = \eta^{(1,9)}_{99}$.

This choice for the vielbein does not lead to new real actions. Changing the role of different coordinates from timelike to spacelike and vice versa, but keeping the signature fixed, amounts to no more than a relabelling of the coordinates. The action and supersymmetry variations are not affected by this permutation of coordinates.

This is not true however for solutions of its equations of motion. A generic solution is not invariant under exchange of a timelike and spacelike coordinate. For a complexified version of such a solution, interchanging coordinates again is equivalent to a relabelling that does not lead to a different complex solution, as there is no notion of space or time anymore. So given a real solution, if we complexify it and then go back to a real form by imposing different reality conditions it can happen that two coordinates interchange their space- and timelike character. To keep track of this effect when taking real slices of a complex solution it is most practical to work with imaginary vielbeine. In this way one can see explicitly which coordinates will be timelike and which spacelike in a different real form. One can see this explicitly at work in section 6.5.1.

6.5 Domain-Walls and Cosmologies

In this section we will apply the previously discussed method of complex actions and real slices to construct and relate different real solutions. We discuss two examples. Our first example is in massive IIA, where we find a realisation of the domain-wall/cosmology correspondence of [67, 98, 112] in a supersymmetric theory. After that we look at 9d gauged maximal supergravity. This as an example to show that our method works in different dimensions. Note that our examples are all in theories with extended supersymmetry, as this seems necessary to be able to take different real slices in the same signature with our method.

In this section we work in the Einstein frame (E), related to the string frame (S) via $g^{(S)}_{\mu\nu} = \mathrm{e}^{\phi/2} g^{(E)}_{\mu\nu}$.

6.5.1 10d Massive IIA/A*

We truncate the complex massive IIA theory (6.3.2) to the following action

$$S_{\mathrm{mIIA}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x\, e \Big(\mathcal{R} - \frac{1}{2}\big(\partial\phi\big)^2 - \tfrac{1}{2}\mathrm{e}^{5\phi/2} m^2\Big)\,, \tag{6.5.1}$$

where $m = G^{(0)}$ is the Romans' mass parameter. The fermionic part of the truncated supersymmetry transformations is

$$\begin{aligned} \delta\psi_\mu =& \Big(\nabla_\mu - \frac{1}{32} W \Gamma_\mu\Big)\epsilon\,, \\ \delta_\epsilon\lambda =& \Big(\partial\!\!\!/\phi + \frac{\delta W}{\delta\phi}\Big)\epsilon\,. \end{aligned} \tag{6.5.2}$$

For our theory (6.5.1) the scalar potential and superpotential are respectively

$$V = \frac{1}{2}\big(\frac{\delta W}{\delta\phi}\big)^2 - \frac{9}{32}W^2 = \frac{1}{2}\mathrm{e}^{5\phi/2}m^2\,, \quad W = \mathrm{e}^{5\phi/4}m\,. \tag{6.5.3}$$

The complex equations of motion are given by

$$\begin{aligned} 0 &= \frac{1}{e}\partial_\mu\Big(e g^{\mu\nu}\partial_\nu\phi\Big) - \frac{5}{4}\mathrm{e}^{5\phi/2}m^2\,, \\ G_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\eta\rho}\partial_\eta\phi\partial_\rho\phi) - \frac{1}{2}g_{\mu\nu}\mathrm{e}^{5\phi/2}m^2\Big)\,. \end{aligned} \tag{6.5.4}$$

We propose the following complex Ansatz for a supersymmetric solution. As we will show, it can be seen as the complexification of both a domain-wall and a cosmology:

$$\begin{aligned} e_\mu^0 &= a_0 H^{1/16}\delta_\mu^0\,, \\ e_\mu^i &= a_i H^{1/16}\delta_\mu^i \qquad (i = 1\ldots 8)\,, \\ e_\mu^9 &= a_9 H^{9/16}\delta_\mu^9\,, \\ \phi &= -\frac{5}{4}\log H\,, \end{aligned} \tag{6.5.5}$$

here a_a are some constant complex numbers and H is a complex function depending only on the coordinate x^9. The complex metric is given by $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta^{(1,9)}_{ab}$, as in the previous section 6.3. For this Ansatz the equations of motion (6.5.4) and the supersymmetry condition from (6.5.2) reduce to:

$$\partial_9 H = a_9 m\,. \tag{6.5.6}$$

So we find the following complex solution to the complexified massive theory:

$$\begin{aligned} \mathrm{d}s^2 &= H^{1/8}\Big(-a_0^2(\mathrm{d}x^0)^2 + (a_i)^2(\mathrm{d}x^i)^2\Big) + H^{9/8}a_9^2(\mathrm{d}x^9)^2\,, \\ \mathrm{e}^{\phi} &= H^{-5/4}\,, \quad \text{with } H = 1 + a_9 m x^9\,. \end{aligned} \tag{6.5.7}$$

It is invariant under the following complex supersymmetries:

$$\Gamma_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,, \tag{6.5.8}$$

where ϵ_0 is a constant Dirac spinor. In section 6.3 we explained how the complex action (6.5.1) can give rise to several different real theories by taking different real slices. If we now apply these reality conditions on the bosonic fields to our complex solution we will find different real solutions. The different inequivalent reality properties consistent with section 6.3 and (6.5.6) are given for some signatures in table 6.5.1, note that here we allow for imaginary vielbeine. Let us illustrate how the com-

t	0	1		2
type	mIIA*	mIIA	mIIA*	mIIA
α_m	$+$	$+$	$-$	$-$
α_ϕ	$+$	$+$	$+$	$+$
α_μ^0	$-$	$+$	$-$	$+$
α_μ^1	$+$	$+$	$+$	$+$
α_μ^i	$+$	$+$	$+$	$+$
α_μ^9	$+$	$+$	$-$	$-$
A	$\mathbb{1}$	Γ_0	$i\Gamma_9$	$i\Gamma_0\Gamma_9$

Table 6.5.1: *Possible reality conditions on the fields of the truncated massive IIA supergravity (mIIA), consistent with the equations of motion. t is the number of timelike directions in space-time. When dealing with solutions it is preferable to allow for imaginary vielbeine as explained in section 6.4. A is the product of all Γ's that are timelike in the real theory and it appears in e.g. the reality condition for the fermions.*

plex Ansatz reduces to a supersymmetric domain-wall in massive IIA (mIIA) and a supersymmetric cosmology in mIIA* by imposing the reality conditions.

Domain-wall in mIIA The standard reality conditions lead to mIIA and can be found in the second column of table 6.5.1. As all the fields are real, the action coincides with (6.5.1). The complex solution becomes the well known domain-wall or D8-brane of mIIA:

$$\begin{aligned} ds^2 &= H^{1/8}\Big(-(dx^0)^2 + (d\vec{x})^2\Big) + H^{9/8}(dx^9)^2\,, \\ e^{\phi} &= H^{-5/4}\,, \quad \text{with} \;\; H = 1 + mx^9\,. \end{aligned} \tag{6.5.9}$$

The complex supersymmetry variations (6.5.2) become

$$\begin{aligned} \delta\psi_\mu =& \Big(\nabla_\mu - \frac{1}{32}W\Gamma_\mu\Big)\epsilon\,, \\ \delta_\epsilon\lambda =& \Big(\partial\!\!\!/\phi + \frac{\delta W}{\delta\phi}\Big)\epsilon\,, \end{aligned} \tag{6.5.10}$$

where W is given by

$$W = e^{5\phi/4}m\,, \quad \text{with}\; m \;\in\; \mathbb{R}\,. \tag{6.5.11}$$

It is not difficult to verify that the domain-wall (6.5.9) has the following unbroken real supersymmetries

$$\Gamma_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,, \tag{6.5.12}$$

where ϵ_0 now is a constant Majorana spinor.

Cosmology in mIIA* Alternatively, we can apply the reality conditions of mIIA* to the complex solution (6.5.7). As can be read from table 6.5.1 in this case m is purely imaginary, as are two components of the vielbein: ${e_\mu}^0$ and ${e_\mu}^9$. This implies that in this case $a_9 = a_0 = i$. We redefine $m = i\tilde{m}$, ${e_\mu}^0 = i{\tilde{e}_\mu}{}^0$, ${e_\mu}^9 = i{\tilde{e}_\mu}{}^9$, $\Gamma^0 = i\tilde{\Gamma}^0$ and $\Gamma^9 = i\tilde{\Gamma}^9$. Substituting all this in the complex solution (6.5.7) gives us a supersymmetric cosmological solution of mIIA*:

$$\begin{aligned} ds^2 &= H^{1/8}\Big((dx^0)^2 + (d\vec{x})^2\Big) - H^{9/8}(dx^9)^2\,, \\ e^{\phi} &= H^{-5/4}\,, \quad \text{with} \;\; H = 1 - \tilde{m}x^9\,, \end{aligned} \tag{6.5.13}$$

where $\tilde{m} = F^{(0)}$. Note that this is the E9-brane of [26]. Note that also the real action of mIIA* is different than that of mIIA. It is given in terms of real fields by

$$S_{\text{mIIA}^*} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\, e\,\Big(\mathcal{R} - \frac{1}{2}\big(\partial\phi\big)^2 + \tfrac{1}{2}e^{5\phi/2}\tilde{m}^2\Big)\,, \tag{6.5.14}$$

with the corresponding supersymmetry variations

$$\begin{aligned} \delta\psi_\mu =& \Big(\nabla_\mu - \frac{i}{32}\tilde{W}\Gamma_\mu\Big)\epsilon\,, \\ \delta\zeta =& \Big(-i\partial\!\!\!/\phi + \frac{\delta\tilde{W}}{\delta\phi}\Big)\epsilon\,. \end{aligned} \tag{6.5.15}$$

The superpotential $\tilde{W}$ is real and given by

$$\tilde{W} = \mathrm{e}^{5\phi/4}\tilde{m}\,, \quad \text{with } \tilde{m} \in \mathbb{R}\,. \tag{6.5.16}$$

As was the case for the domain-wall it is easy to check that the cosmology (6.5.13) preserves the following supersymmetries

$$i\tilde{\Gamma}_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,. \tag{6.5.17}$$

Note that now ϵ is not a standard Majorana spinor but satisfies a *MW reality condition instead.

The domain-wall and cosmology presented above are a particular example of the domain-wall / cosmology correspondence of [67, 98, 112], where an embedding in extended supergravity is possible. Indeed the truncated mIIA theory is exactly of the gravity-scalar form as proposed there, as is its domain-wall solution. Furthermore the truncated mIIA* theory is equal to the truncated mIIA theory up to a relative sign in front of the scalar potential. This example places the domain-wall / cosmology correspondence in a supersymmetric context. This means that the Killing spinor of the solution generates a supersymmetry of the theory. Furthermore we see that what was called a pseudo-Killing spinor in [67, 98, 112] now is a generator of a genuine supersymmetry, but in a star theory. In this example pseudo-supersymmetry in an extended supersymmetric theory coincides with the supersymmetry of a superalgebra obeying star reality conditions.

Instanton in Euclidean mIIA* In section 6.1 we noticed that for every domain-wall of a model with potential V there corresponds, besides the cosmology with $-V$, an instanton [113]. The first column of table 6.5.1 agrees with this. The α_m is the same for Euclidean mIIA* in (0,10) and mIIA in (1,9) space-time dimensions, hence they have the same potential. Also do we see that $\alpha^0_\mu = -1$ and hence the corresponding vielbein is imaginary. As a result we have a Euclidean theory.

6.5.2 Maximal Gauged Supergravity in 9d

In the previous example the only scalar field that played a role was the dilaton. In subsection 5.3.2 pseudo-supersymmetry was studied in systems with explicit multiple scalar fields. As mentioned there, in [99, 103] it was shown that for the domain-wall / cosmology correspondence subtleties can appear when axions are included. In light of this we consider a more general example with multiple scalar fields including an axion.

The theory we will be working with is given by the following truncated $\mathcal{N} = 2$, d=9 gauged supergravity Lagrangian

$$\mathcal{L}_{9\mathrm{d}} = \tfrac{1}{2}\, e\left[\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mathrm{e}^{2\phi}(\partial l)^2 - \tfrac{1}{2}(\partial\varphi)^2 - V(\phi, l, \varphi)\right], \tag{6.5.18}$$

with V given by

$$V = \frac{1}{2}\mathrm{e}^{-2\phi+4\varphi/\sqrt{7}}\Big(q_1^2 + 2\mathrm{e}^{2\phi}q_1(-q_2+q_1l^2) + \mathrm{e}^{4\phi}(q_2+q_1l^2)^2\Big)\,. \tag{6.5.19}$$

The details of the reduction from IIB are given in [121, 122]. The scalar fields are given by ϕ, φ and l. The constants q_1 and q_2 specify the gauging. We group the 9d, 16-component $\mathcal{N}=2$ spinors χ_i in doublets

$$\chi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}\,. \tag{6.5.20}$$

From table B.1.1 we see that $\epsilon = \eta = -1$. In this section we will use $\mathcal{C} = \gamma_0$. The (1,8) gamma-matrices γ_μ are then purely imaginary. In this notation the supersymmetry transformations of the fermions are

$$\begin{aligned}
\delta_\epsilon\psi_\mu &= \big[\partial_\mu + \omega_\mu - (\tfrac{1}{4}\mathrm{e}^{\phi}\partial_\mu l - \tfrac{i}{28}\gamma_\mu W)i\sigma_2\big]\,\epsilon\,,\\
\delta_\epsilon\lambda &= (i\,\partial\!\!\!/\phi - \mathrm{e}^{-\phi}\delta_l W)\sigma_3\,\epsilon + \mathrm{e}^{\phi}(i\,\partial\!\!\!/ l + \mathrm{e}^{-\phi}\delta_\phi W)\sigma_1\,\epsilon\,,\\
\delta_\epsilon\tilde{\lambda} &= i\,\partial\!\!\!/\varphi\sigma_3\epsilon + \delta_\varphi W\sigma_1\,\epsilon\,.
\end{aligned} \tag{6.5.21}$$

The superpotential W is given by

$$W = \mathrm{e}^{2\varphi/\sqrt{7}}\Big(\mathrm{e}^{-\phi}q_1 + \mathrm{e}^{\phi}(q_2+q_1l^2)\Big)\,. \tag{6.5.22}$$

The above action and supersymmetry rules can be made complex via the method of section 2. It turns out that there are two real slices for signature (1,8), see table 6.5.2. We denote the star version of the truncated $\mathcal{N}$=2, d=9 theory by 9d*. Inspired by

	9d	9d*
$\varepsilon = \eta$	$-$	$-$
ρ	$\mathbb{1}$	σ_3
α_l	$+$	$-$
$\alpha_\epsilon = \alpha_\lambda = \alpha_{\tilde{\lambda}}$	$+$	$+$
$q_1 = q_2$	$+$	$-$

Table 6.5.2: *The two sets of reality conditions appearing in the truncated $\mathcal{N}$=2, d=9 gauged supergravity Lagrangian leading to signature (1,8).*

the domain-walls of [122], we propose the following complex Ansätze:

$$\begin{aligned} e_\mu^0 &= a_0 h^{1/28}\delta_\mu^0\,,\\ e_\mu^i &= h^{1/28}\delta_\mu^i \qquad (i=1\dots7)\,,\\ e_\mu^8 &= a_8 h^{-3/14}\delta_\mu^8\,,\\ \mathrm{e}^\phi &= h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\quad l=c_1{h_1}^{-1}\,,\\ h &= h_1h_2-c_1^2\,, \end{aligned} \tag{6.5.23}$$

where a_a and c_1^2 are some arbitrary complex constants and h_1 and h_2 are functions of x^8 only. This Ansatz is a supersymmetric complex solution if

$$\begin{aligned} \partial_8 h_1 &= 2a_8 q_1\,,\\ \partial_8 h_2 &= 2a_8 q_2\,. \end{aligned} \tag{6.5.24}$$

In this case it has a complex Killing spinor of the form

$$\epsilon = h^{1/56}(\cos f\,\mathbb{1}_2 - i\sin f\,\sigma_2)\epsilon_0\,, \tag{6.5.25}$$

with

$$f=\frac{1}{4}\arctan\left(\frac{2c_1q_1h^{1/2}}{q_2h_1^2-q_1h+q_1c_1^2}\right)\,, \tag{6.5.26}$$

and ϵ_0 is a doublet of constant Dirac spinors that satisfies

$$\gamma^8\sigma_2\epsilon_0=\epsilon_0\,. \tag{6.5.27}$$

As in the previous subsection one can now take two different real slices leading to a pair of real solutions in signature (1,8). Taking all the fields in (6.5.23) real leads back to the familiar domain-walls of [122]:

$$\begin{aligned} ds^2 &= h^{1/14}(-(\mathrm{d}x^0)^2+\mathrm{d}\vec{x}^2)+h^{-3/7}(\mathrm{d}x^8)^2\,,\\ \mathrm{e}^\phi &= h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\qquad l=c_1{h_1}^{-1}\,,\\ h_1 &= 2q_1x^8+k_1^2,\ h_2=2q_2x^8+k_2^2 \quad \text{and} \quad h=h_1h_2-c_1^2\,, \end{aligned} \tag{6.5.28}$$

where k_i are integration constants. As noted above there is another set of consistent reality conditions. This second real slice of (6.5.23) will give a cosmological solution. From table 6.5.2 one can see which fields become purely imaginary. To write everything in terms of real fields we redefine $q_i = i\tilde{q}_i$, $c_1 = i\tilde{c}_1$ and $l = i\tilde{l} = i\tilde{c}_1h_1^{-1}$. Consistency with the equations of motion also requires ${e_\mu}^0$ and ${e_\mu}^8$ to be imaginary, i.e. $a_8 = a_0 = i$. The Ansatz (6.5.23) written in terms of these real fields gives the following cosmological solution:

$$\begin{aligned} ds^2 &= h^{1/14}((\mathrm{d}x^0)^2+\mathrm{d}\vec{x}^2)-h^{-3/7}(\mathrm{d}x^8)^2\,,\\ \mathrm{e}^\phi &= h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\qquad \tilde{l}=\tilde{c}_1{h_1}^{-1}\,, \end{aligned} \tag{6.5.29}$$

where

$$h = h_1 h_2 + \tilde{c}_1^2\,, \qquad h_1 = 2\tilde{q}_1 x^8 + k_1^2\,, \qquad h_2 = 2\tilde{q}_2 x^8 + k_2^2\,. \tag{6.5.30}$$

This is a real cosmological solution of the star version of the 9d theory, of which the action can easily be constructed along the lines of section 6.3.

Again we find a natural relation between domain-wall solutions and cosmological solutions as different real slices of a single complex solution. In this case the relation between the two real theories is slightly more involved than just reversing the overall sign of the scalar potential. It is not difficult to see that the scalar potential in the 9d* theory is now

$$V = -\frac{1}{2}\mathrm{e}^{-2\phi+4\varphi/\sqrt{7}}\Big(\tilde{q}_1^2 - 2\mathrm{e}^{2\phi}\tilde{q}_1(\tilde{q}_2 + \tilde{q}_1\tilde{l}^2) + \mathrm{e}^{4\phi}(\tilde{q}_2 - \tilde{q}_1\tilde{l}^2)^2\Big)\,. \tag{6.5.31}$$

So apart from an overall change in sign with respect to (6.5.19) there are also relative sign changes between the different terms in the potential. This goes together with a signature change of the scalar manifold in the 9d* case as the axion $\tilde{l}$ has wrong sign kinetic term. Note that if the theory is truncated by setting the axion to zero we again find an example where one can embed the correspondence of [67, 98, 112] in a supersymmetric theory. Also in this case the pseudo-supersymmetry of the cosmology can be interpreted as the vanishing of the fermionic supersymmetry transformations of a theory with star reality conditions.

6.5.3 E-branes

In the previous sections we have seen that star supergravities have a non-Riemannian scalar coset. In section 6.5.1 we gave an example of a time-dependent half supersymmetric solution of mIIA*. This solution is called an E8-brane or E*p*-brane in general. The E stands for the Euclidean worldvolume and the *p* reflects that we have a (*p*+1)-dimensional worldvolume[9]. Note that S*p*-branes also have a (*p*+1)-dimensional worldvolume but they are brane solutions of standard type II theories instead of type II* theories.

In [26, 59] the full BPS-brane analysis for the star supergravities was done. Just like S*p*-branes they are time-dependent. The difference is that E-branes are BPS-solutions of star supergravities, while S*p*-branes are not supersymmetric solutions of standard type II supergravities. At the end of section 2.4 we mentioned that if we demand that an extremal S-brane satisfies the extremality condition (2.4.20) the field strength turns out to be imaginary. Such a solution can be embedded in a star supergravity. This shows that in a star supergravity we can have solutions satisfying (2.4.20).

[9] In the original papers an E*p*-brane means a brane with a *p*-dimensional Euclidean worldvolume. We however prefer to use the same notation as that for *p*- and S*p*-branes.

Let us show this explicitly for the S(−1)-brane given in section 2.4. We need to find the extremal version of this solution. The only deformation parameter of the metric (2.4.32) is $||v||$, we expect that the limit of $||v|| \to 0$ should give us the extremal solution. For this we need to restrict to a $k = -1$ slicing since only this describes Minkowski space-time in Milne coordinates. However, from (2.4.33) we see that the limit $||v|| \to 0$ gives us real and constant scalar fields. We therefore have to re-scale the constants in a specific way.

Inspired by the extremal limit given in [123] we consider the following series of limits

$$c_2 \to c_2 + \frac{1}{2}\pi i\,, \quad c_1 \to \frac{c_1}{i||v||}\,, \quad c_2 \to \frac{g_s||v||}{c_1}\,, \quad c_3 \to ic_3\,, \quad ||v|| \to 0\,. \tag{6.5.32}$$

A direct calculation shows that this leads to the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^2\mathrm{d}\mathbb{H}^2_{D-1}\,, \tag{6.5.33}$$

and the axion and dilaton are given by

$$\mathrm{e}^{\phi} = h\,, \quad \chi = i(\pm h^{-1} + c_3)\,. \tag{6.5.34}$$

Note that the axion is imaginary as required. The harmonic function h is given by

$$h = g_s + \frac{c_1}{(D-2)t^{D-2}}\,. \tag{6.5.35}$$

This is the solution as given in [41]. We see that the extremal limit of the S(−1)-brane indeed leads to an imaginary solution which can be embedded in IIB* supergravity. We have derived the extremal E(−1)-brane as given in [26].

6.6 Discussion

In the first section we gave an overview of the domain-wall / cosmology correspondence. A natural question that arose is whether one can give a meaning to pseudo-supersymmetry in a real supergravity context.

To achieve this we complexified the type II supergravities and their supersymmetry rules. These complex actions do not describe physical theories but are a useful mathematical tool that allows to write down the actions for all variant supergravities as real slices of the complex action. We illustrated the method in detail for the standard type II theories and their corresponding star versions in signature (1,9). Although we restricted our analysis to 10 dimensions one can generalize it to lower dimensions, for some related results see [124–132]. In this chapter we gave an additional example for $\mathcal{N} = 2$ in 9 dimensions.

In the last part of this chapter, we have looked at solutions of these complex theories and shown that one can obtain solutions of the different real theories by taking real slices. In particular, we have seen that in this way supersymmetric domain-walls and (pseudo-) supersymmetric cosmologies can arise as different slices of one complex solution. The domain-walls are solutions in an ordinary supergravity, while the cosmologies arise as solutions of the star version. In this sense the pseudo-supersymmetry of cosmologies corresponds to supersymmetry in the star theory. We presented a ten-dimensional example where the domain-wall / cosmology correspondence of [67,98,112] can be embedded into an extended supergravity context. We also noticed that it can be extended to include instantons as well [110]. In another example in 9 dimensions we again constructed a domain-wall and corresponding cosmology. A noteworthy feature of this last example is that the potential no longer gets an overall sign flip, but only certain terms in the potential change sign. Furthermore the scalar manifold changes signature. This might hint that also in a fake supergravity context more general changes in the potential could appear under the map of a domain-wall to a cosmology.

In the last subsection we pointed out a relation between extremal Sp- and Ep-branes.

Chapter 7

Instantons

So far we have considered Riemannian scalar cosets. In the previous chapter we showed that the domain-wall / cosmology correspondence can be extended to include instantons as well. It is therefore not surprising that the analysis we did for Riemannian scalar cosets can be extended to non-Riemannian scalar cosets. Actually, we know already from section 3.5 that if there is no potential present the instantons also describe geodesic motion on the scalar manifold, with the complication that the affine velocity $||v||^2$ is no longer strictly positive due to the fact that we now have to deal with a non-Riemannian scalar manifold. This leads to different classes of instantons, labeled by the sign of $||v||^2$.

The prototype Lorentzian scalar coset is $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ of Euclidean type IIB supergravity. The extremal solution belonging to this coset is the D(-1)-instanton [133,134]. In our language this is a lightlike geodesic. The extension to non-extremal D-instantons was considered in [123,135]. These are related to space- and timelike geodesics. In this chapter we are going to consider two extensions of the scalar coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$.

First we begin with considering more general non-Riemannian cosets. In section 3.2.1 we noticed that reducing pure gravity over a Lorentzian torus gives the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$. In the next section we will derive the generating solution for this coset and extend it to $\mathrm{GL}(p+q,\mathbb{R})/\mathrm{SO}(p,q)$.

The second extension will be the inclusion of a potential. This extends the results of section 5.2 to non-Riemannian cosets. After that, we restrict ourselves to a potential given by a cosmological constant Λ. We will see that this potential is special in that it never upsets the geodesic motion of the scalar fields.

The work in this chapter is done together with E. A. Bergshoeff, W. Chemissany, A. Collinucci, T. Van Riet, M. Trigiante and S. Vandoren [40,69].

7.1 Instanton Geometries

From section 3.5 we know that the geometry of the (-1)-brane or instanton entirely depends on the character of the geodesic curve (spacelike, lightlike or timelike), independently of the scalar manifold coset. The metric is given by (3.5.3) where g can be found by solving (3.5.10) with $\epsilon = +1$

$$\dot{g}^2 = \frac{||v||^2}{2(D-2)(D-1)} f^2 g^{4-2D} + k f^2 \,. \tag{7.1.1}$$

Some of these solutions have appeared in the literature before [40, 56, 123, 135, 136].

- $||v||^2 > 0$

  For this class of instantons we will be using the gauge $f = g$. In the table below we present the conformal factor f that determines the metric and the radial harmonic function h. Note that for all three values of k the solutions have singularities.

$k=-1$	$f(r) = (\frac{\lvert\lvert v\rvert\rvert^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \cos^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{8(D-1)}{(D-2)\lvert\lvert v\rvert\rvert^2}}\, \text{arctanh}[\tan(\frac{D-2}{2} r)] + b$
$k=0$	$f(r) = \left(\sqrt{\frac{(D-2)\lvert\lvert v\rvert\rvert^2}{2(D-1)}}\, r\right)^{\frac{1}{D-2}}$ $h(r) = \sqrt{\frac{2(D-1)}{(D-2)\lvert\lvert v\rvert\rvert^2}} \log r + b$
$k=+1$	$f(r) = (\frac{\lvert\lvert v\rvert\rvert^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \sinh^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{2(D-1)}{\lvert\lvert v\rvert\rvert^2(D-2)}}\, \log[\tanh(\frac{D-2}{2} r)] + b$

Table 7.1.1: *The Euclidean geometries with $||v^2|| > 0$ in the gauge $f = g$. The real number b is an integration constant.*

- $||v||^2 = 0$

  We will be using the Euclidean "FLRW gauge" for which $f = 1$. It is clear from (3.5.10) that for $k = -1$ we do not find a solution and that for $k = 0$ we find flat space in Cartesian coordinates ($g = 1$) and for $k = +1$ we find flat space in spherical coordinates ($g = r$). This makes sense since a lightlike geodesic motion comes with zero "energy-momentum"[1]. The harmonic function is

$$\begin{aligned} k=0: &\qquad h(r) = a\,r + b\,, \\ k=1: &\qquad h(r) = \frac{a}{r^{D-2}} + b\,. \end{aligned} \tag{7.1.2}$$

In Euclidean IIB supergravity the axion-dilaton parameterize $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ and for $||v||^2 = 0$ and $k=1$ we have the standard half-supersymmetric D-instanton [133, 134].

- $||v||^2 < 0$

We will present the solutions in the conformal gauge $g = fr$, which has the advantage that the coordinates cover the whole space. For $k=0$ and $k=-1$ we clearly have no solutions since the right-hand side of (3.5.10) is always negative. For $k=+1$ a solution does exist, and in the conformal gauge it is given by

$$f(r) = \Big(1 - \tfrac{||v||^2}{8(D-1)(D-2)} r^{-2(D-2)}\Big)^{\frac{1}{D-2}}\,, \tag{7.1.3}$$

where indeed only $||v||^2 < 0$ is valid. This geometry is smooth everywhere and describes a wormhole since there is an inversion-symmetry which interchanges the two asymptotically flat regions, see figure 7.1.1. This symmetry acts as follows [123]

$$r^{D-2} \to \tfrac{-||v||^2}{8(D-1)(D-2)} r^{-(D-2)}\,. \tag{7.1.4}$$

The neck of the wormhole is located at the self-dual radius defined by $r_s^{2(D-2)} = \frac{-||v||^2}{8(D-1)(D-2)}$. The two asymptotic regions are connected via a neck with a minimal physical radius at the self-dual radius r_s. This physical radius can be calculated in the FLRW-gauge

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + a(\rho)^2 \mathrm{d}\Omega_{D-1}^2\,. \tag{7.1.5}$$

From this we find that the physical radius is given by $a(\rho_s)^{D-2} = r^{D-2} f(r_s)$.

The harmonic function is given by

$$h(r) = \sqrt{-\tfrac{8(D-1)}{(D-2)||v||^2}}\,\arctan\Big(\sqrt{\tfrac{-||v||^2}{8(D-1)(D-2)}}\; r^{-(D-2)}\Big) + b\,. \tag{7.1.6}$$

[1] The fact that the $k=-1$ solution does not exists reflects that there does not exist a hyperbolic slicing of the Euclidean plane.

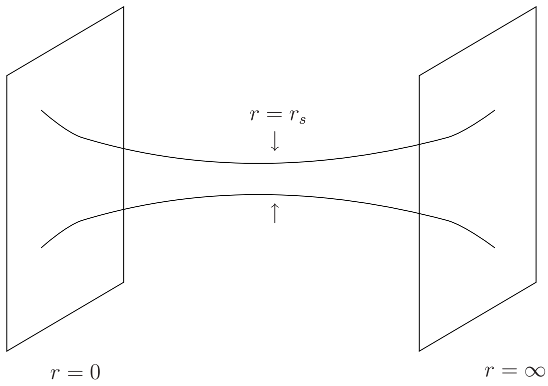


Figure 7.1.1: For the class $||v||^2 < 0$ the space is a wormhole with the neck of the wormhole at the self-dual radius r_s. The two asymptotically flat regions at $r = 0$ and $r = \infty$ are connected via a neck with a minimal physical radius at the self-dual radius r_s.

7.2 Solutions of Kaluza–Klein Theory

We know that the scalar fields are geodesics, but these are not described by the Cartan subalgebra only since $||v||^2$ can have any sign. Let us therefore focus on the geodesic motion that comes about. The approach that we will follow allows us to re-derive the geodesics of $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{SO}(n)$ coset but also allows for a generalization to $\mathrm{GL}(p + q = n, \mathbb{R})/\,\mathrm{SO}(p, q)$.

7.2.1 The Geodesic Curves

In the following we will not make use of a coordinate system on the coset, instead we will work directly on the level of $\mathcal{M}$, see section 3.4. So that we do not need to be bothered with subtleties regarding the Borel gauge.

The action for the geodesic curves can be compactly written in terms of the symmetric coset matrix $\hat{\mathcal{M}}$

$$S = \int \mathrm{Tr}[\partial\hat{\mathcal{M}}\partial\hat{\mathcal{M}}^{-1}]\,, \qquad \hat{\mathcal{M}} = \hat{L}\eta\hat{L}^T\,, \quad \eta = (-\mathbb{1}_p, \mathbb{1}_q)\,. \tag{7.2.1}$$

Here we have included the breathing mode φ in $\mathcal{M}$, see (3.2.12), to make $\hat{\mathcal{M}}$. Here $\hat{L}$ is a representative of $\mathrm{GL}(p + q = n, \mathbb{R})/\,\mathrm{SO}(p, q)$. The relation between $\hat{\mathcal{M}}$ and the moduli φ and $\mathcal{M}$ is as follows

$$\hat{\mathcal{M}} = (|\det\hat{\mathcal{M}}|)^{\frac{1}{n}}\mathcal{M}\,, \qquad |\det\hat{\mathcal{M}}| = \exp\sqrt{2n}\varphi\,. \tag{7.2.2}$$

The equations of motion can compactly be written as

$$[\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}']' = 0\,, \tag{7.2.3}$$

where a prime is a derivative with respect to an affine parameter. This implies that $\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}' = K$ with K a constant matrix, which can be seen as the matrix of Noether charges. The affine velocity squared of the geodesic curve is $||v||^2 = \frac{1}{2}\mathrm{Tr}[K^2]$. Since $\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}' = K$ the problem is integrable and a general solution is given by

$$\hat{\mathcal{M}}(h) = \hat{\mathcal{M}}(0)\mathrm{e}^{Kh}\,. \tag{7.2.4}$$

The isometry group $\mathrm{GL}(n,\mathrm{I\!R})$ has a transitive action on the coset space which implies that we can restrict ourselves to geodesics that go trough the origin. Since we have the freedom of affine re-parameterization of h we can assume that $\hat{\mathcal{M}}(0) = \eta$. The matrix of Noether charges K is not completely arbitrary and the only constraint it fulfills can be derived from the properties of $\hat{\mathcal{M}}$

$$\eta K = K^T\eta\,, \tag{7.2.5}$$

where the signature of η depends on the isotropy group $\mathrm{SO}(p,q)$ we are considering, see (7.2.1).

K is an element of the Lie algebra of $\mathrm{GL}(n,\mathrm{I\!R})$ and accordingly it transforms in the adjoint of $\mathrm{GL}(n,\mathrm{I\!R})$

$$K \to \Omega\, K\, \Omega^{-1}\,, \tag{7.2.6}$$

under which the n Casimirs $\mathrm{Tr}K$, $\mathrm{Tr}K^2$,..., $\mathrm{Tr}K^n$ are invariant. The constraints given in (7.2.5) are not invariant under the total isometry group $\mathrm{GL}(n,\mathrm{I\!R})$ but only under the smaller isotropy group $\mathrm{SO}(p,q)$.

In the following sections we derive the generating geodesics for the three possible cases $\mathrm{GL}(n,\mathrm{I\!R})/\,\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathrm{I\!R})/\,\mathrm{SO}(n-1,1)$ and $\mathrm{GL}(n+1,\mathrm{I\!R})/\,\mathrm{SO}(n-1,2)$, although it can easily be extended to $\mathrm{GL}(p+q,\mathrm{I\!R})/\,\mathrm{SO}(p,q)$. As explained in subsection 4.3.1, for pure Kaluza–Klein theory in $D > 3$ all geodesics that are related through a $\mathrm{SL}(n)$-transformation lift up to exactly the same vacuum solution in $D+n$ dimensions since the $\mathrm{SL}(n)$ corresponds to rigid coordinate transformations from a $(D+n)$-dimensional point of view. Here the $\mathrm{SL}(n,\mathrm{I\!R})$ follows from $\mathrm{GL}(n,\mathrm{I\!R}) = \mathbb{R}\times\mathrm{SL}(n,\mathrm{I\!R})$ with $\mathbb{R}$ related to the breathing mode. So, in this sense it is absolutely necessary to understand the generating geodesic since it classifies higher-dimensional solutions modulo coordinate transformations. Of course, this is not true for $D = 3$ where $\mathrm{SL}(n+1)$ maps higher-dimensional solutions to each other that are not necessarily related by coordinate transformations.

7.2.2 Normal Form of $\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$

Consider $K \in \mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$. By definition K obeys (7.2.5)

$$\eta K = K^T\eta\,, \quad \text{with} \quad \eta = (-\mathbb{1}_p, +\mathbb{1}_q)\,. \tag{7.2.7}$$

Two eigenvectors of K, $\mathbf{v_1}$ and $\mathbf{v_2}$, that belong to different eigenvalues λ_1 and λ_2 are necessarily orthogonal with respect to the inner product (,) defined with the bilinear form η, because $(\mathbf{v_2}, K\mathbf{v_1}) = (K\mathbf{v_2}, \mathbf{v_1})$ and thus $\lambda_1(\mathbf{v_1}, \mathbf{v_2}) = \lambda_2(\mathbf{v_1}, \mathbf{v_2})$. Now if $\lambda_1 \neq \lambda_2$ then this is only consistent if $(\mathbf{v_1}, \mathbf{v_2}) = 0$. If two eigenvectors have the same eigenvalue we can always perform a pseudo Gramm–Schmidt procedure such that they become orthogonal with respect to η, which we refer to as pseudo-orthogonal.

Assume we have a complex eigenvalue $\lambda \neq \bar{\lambda}$ with corresponding eigenvectors $\mathbf{v}$ and $\bar{\mathbf{v}}$. If we write $\mathbf{v} = \mathbf{v_1} + \mathbf{i}\mathbf{v_2}$ and $\lambda = \lambda_1 + i\lambda_2$ then this means that

$$K\mathbf{v_1} = \lambda_1\mathbf{v_1} - \lambda_2\mathbf{v_2}\,, \qquad K\mathbf{v_2} = \lambda_2\mathbf{v_1} + \lambda_1\mathbf{v_2}\,, \tag{7.2.8}$$

(pseudo)-orthogonality between $\mathbf{v}$ and $\bar{\mathbf{v}}$ implies $(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2})$.

In what follows we will construct a normal form for K using the eigenvectors as a basis.

$\mathbf{min}(p, q) = 0$

In this case we see from (7.2.7) that K is a symmetric matrix. With the help of $\mathrm{SO}(n)$ we can diagonalize K to a real matrix, written in terms of its orthogonal basis of eigenvectors as

$$K_N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}. \tag{7.2.9}$$

Note that this result is compatible with the result given in (4.2.2). The fact that K_N is diagonal reflects that the generating solution can be rotated to the Cartan subalgebra only.

If we instead consider $\mathrm{SO}(p, q)$ we will have in general complex eigenvalues (and its conjugates). Since $\hat{\mathcal{M}}$, see eq. (7.2.4), contains the scalar fields it should always be real. One complex eigenvalue and its conjugate can always be written as a 2×2 real block. So for each complex eigenvalue the matrix K_N will have a 2×2 block. Therefore knowing the maximal number of complex eigenvalues leads us to the normal form. In the following we will derive the maximal number of complex eigenvalues one can have for the coset $\mathrm{GL}(p+q, \mathbb{R})/\,\mathrm{SO}(p, q)$. We will show that this number is given by $\min(p, q)$.

$\mathbf{min}(p, q) = 1$

Assume there are at least two complex eigenvalues λ, σ, that correspond to respectively $\mathbf{v} = \mathbf{v_1} + \mathbf{i}\mathbf{v_2}$ and $\mathbf{w} = \mathbf{w_1} + \mathbf{i}\mathbf{w_2}$, and that they are not related through conjugation ($\lambda \neq \bar{\sigma}$). Then the four vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}$ obey (from pseudo-orthogonality constraints between the vectors themselves and the vectors with the conjugated vectors)

$$(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2}), \qquad (\mathbf{w_1}, \mathbf{w_1}) = -(\mathbf{w_2}, \mathbf{w_2}), \qquad (\mathbf{w_i}, \mathbf{v_j}) = 0, \quad \forall\, i, j = 1, 2. \tag{7.2.10}$$

From here on we write a vector $\mathbf{v}$ as $\mathbf{v} = (v, \vec{V})$. Here v is the time component and $\vec{V}$ are the spatial components of $\mathbf{v}$. Now assume $\mathbf{v}_i$ as $\mathbf{w}_j$ are all lightlike. Lightlike vectors then obey $v^2 = \vec{V} \cdot \vec{V}$. Pseudo-orthogonality between the lightlike vectors $\mathbf{v_1}$ and $\mathbf{w_1}$ implies $v_1 w_1 = \vec{V_1} \cdot \vec{W_1}$. Therefore $(\vec{V_1} \cdot \vec{W_1})^2 = v_1^2 w_1^2 = (\vec{V_1} \cdot \vec{V_1})(\vec{W_1} \cdot \vec{W_1})$, which, according to Cauchy–Schwarz, is only possible when $\vec{V_1}$ and $\vec{W_1}$ are parallel. But in that case we find $\mathbf{v_1} \sim \mathbf{w_1}$. Similarly we find that $\mathbf{v_1} \sim \mathbf{w_2}$ and $\mathbf{v_2} \sim \mathbf{w_2}$ which implies $\mathbf{v_1} \sim \mathbf{v_2}$ which is impossible for complex eigenvectors. Therefore our assumption was wrong.

Assume that one couple of vectors is lightlike, say $\mathbf{v_1}, \mathbf{v_2}$. Then $\mathbf{w_1}$ is spacelike and $\mathbf{w_2}$ timelike (or vice versa). We can always find a frame in which $\mathbf{w_2}$ is given by $(w_2, \vec{0})$. Now it is straightforward that there cannot exist a lightlike vector (like $\mathbf{v_1}$) pseudo-orthogonal to $\mathbf{w_2}$. Therefore our assumption was wrong.

Assume that no couple is lightlike. Take $\mathbf{v_2}$ and $\mathbf{w_2}$ timelike. In a frame in which $\mathbf{w_2}$ is given by $(w_2, \vec{0})$ it is clear that $\mathbf{v_2}$ cannot exist as there does not exist a timelike vector pseudo-orthogonal to $\mathbf{w_2}$.

Therefore, having at least two different complex eigenvalues, not related through complex conjugation is a false assumption and we conclude that there can be at maximum one complex eigenvalue (and its conjugated one).

From this we can find the normal form K_N. For that we write the normal form in terms of a pseudo-orthogonal basis made from the (real and imaginary) parts of the eigenvectors.

Assume $\mathbf{v_1}$ is timelike and normalized, $(\mathbf{v_1}, \mathbf{v_1}) = -1$. Because $\mathbf{v_1}$ is timelike all vectors orthogonal to it are spacelike. In the following we assume that the $\mathbf{v_i}$ are normalized $(\mathbf{v_i}, \mathbf{v_i}) = +1$ for $i > 1$. We define the orthonormal basis $(\mathbf{u_i}, i = 1, \ldots n)$ where the unit vectors $\mathbf{u_i}$, $i > 2$ are orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$ via the Gramm-Schmidt procedure

$$\mathbf{u_1} = \mathbf{v_1}, \tag{7.2.11}$$

$$\mathbf{u_2} = \sin\alpha\, \mathbf{v_1} + \cos\alpha\, \mathbf{v_2}, \tag{7.2.12}$$

where $\tan\alpha = (\mathbf{v_1}, \mathbf{v_2})$. Using (7.2.8) we find that

$$K\mathbf{u_1} = (\lambda_1 + \lambda_2 \tan\alpha)\mathbf{u_1} - \lambda_2 \cos^{-1}\alpha\, \mathbf{u_2}, \; K\mathbf{u_2} = \lambda_2 \cos^{-1}\alpha\, \mathbf{u_1} + (-\lambda_2 \tan\alpha + \lambda_1)\mathbf{u_2}. \tag{7.2.13}$$

From this we easily read off the components of K_N in the new basis

$$K_N = \begin{pmatrix} \lambda_1 + \lambda_2 \tan\alpha & -\lambda_2 \cos^{-1}\alpha & 0 & \dots & 0 \\ \lambda_2 \cos^{-1}\alpha & -\lambda_2 \tan\alpha + \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & \dots & \dots & \lambda_n \end{pmatrix} . \tag{7.2.14}$$

If $\mathbf{v_1}$ is spacelike, then the above normal form is still valid if we interchange $\mathbf{v_1}$ with $\mathbf{v_2}$ in the definition of the orthonormal basis.

Assume on the other hand that $\mathbf{v_1}$ and $\mathbf{v_2}$ are lightlike. Like before it is easy to understand that all other eigenvectors $\mathbf{v_{i>2}}$ must be spacelike $(\mathbf{v_{i>2}}, \mathbf{v_{i>2}}) = +1$. We define an orthonormal basis $(\mathbf{u_i}, i = 1, \dots n)$, again the unit vectors $\mathbf{u_i}$, $i > 2$ are orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$ via the Gramm-Schmidt procedure

$$\mathbf{u_1} = \frac{\mathbf{v_1} \pm \mathbf{v_2}}{\sqrt{2|(\mathbf{v_1}, \mathbf{v_2})|}} , \tag{7.2.15}$$

$$\mathbf{u_2} = \frac{\mathbf{v_1} \mp \mathbf{v_2}}{\sqrt{2|(\mathbf{v_1}, \mathbf{v_2})|}} . \tag{7.2.16}$$

Here the upper sign must be chosen when $(\mathbf{v_1}, \mathbf{v_2}) < 0$ and vice versa. The normal form is given by

$$K_N = \begin{pmatrix} \lambda_1 & \pm\lambda_2 & 0 & \dots & 0 \\ \mp\lambda_2 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & \dots & \dots & \lambda_n \end{pmatrix} . \tag{7.2.17}$$

Note that the 2×2 block in both (7.2.14) and (7.2.17) correspond to at most one complex eigenvalue (and its conjugated one).

$\mathbf{min}(p, q) = 2$

Now, assume there exist at least three complex eigenvalues (which is possible if $p+q \geq 6$). This implies the existence of six vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}, \mathbf{x_1}, \mathbf{x_2}$ that obey

$$(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2}) , \quad (\mathbf{w_1}, \mathbf{w_1}) = -(\mathbf{w_2}, \mathbf{w_2}) , \quad (\mathbf{x_1}, \mathbf{x_1}) = -(\mathbf{x_2}, \mathbf{x_2}) , \tag{7.2.18}$$

$$(\mathbf{v_i}, \mathbf{w_j}) = (\mathbf{v_i}, \mathbf{x_j}) = (\mathbf{w_i}, \mathbf{x_j}) = \mathbf{0} , \qquad \forall \quad \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2} . \tag{7.2.19}$$

Let us first assume all vectors are lightlike. There always exists a frame for which $\mathbf{v_1} = (v_1, 0, \vec{V_1})$, $\mathbf{v_2} = (0, v_2, \vec{V_2})$. Then write $\mathbf{w_1} = (a, b, \vec{W_1})$. We have that $a^2 + b^2 = \vec{W_1} \cdot \vec{W_1}$. But orthogonality implies $a = ||\vec{V_1}||^{-1} \vec{V_1} \cdot \vec{W_1}$ and $b = ||\vec{V_2}||^{-1} \vec{V_2} \cdot \vec{W_1}$. Also

$\vec{W}_1$ is fixed and it equals $a\vec{V}_1 + b\vec{V}_2$. Therefore all other lightlike vectors are the same and thus parallel, which is not what we want.

So, at maximum two couples can be lightlike, say $(\mathbf{v_1}, \mathbf{v_2})$ and $(\mathbf{w_1}, \mathbf{w_2})$. Then say $\mathbf{x_1}$ is timelike and thus $\mathbf{x_2}$ spacelike. We can find a frame for which $\mathbf{x_1} = (x, 0, \vec{0})$. So all lightlike vectors perpendicular to these must have the form $\mathbf{L_i} = (0, l_i, \vec{L}_i)$. And if we want the lightlike vectors to be mutually perpendicular (with respect to η) we find again that they are all parallel which gives rise to contradictions.

Now, maximally one couple of the vectors can be lightlike, say the couple $(\mathbf{v_1}, \mathbf{v_2})$ and the rest not lightlike. There must exist two timelike vectors, say $\mathbf{w_1}$ and $\mathbf{x_1}$. As before we can always fix a frame in which $\mathbf{w_1} = (0, w, \vec{0})$ and $\mathbf{x_1} = (x, 0, \vec{0})$. But clearly we cannot find a lightlike vector orthogonal to them.

Finally assume that none of them are lightlike. Then we have three spacelike and three lightlike vectors. This however is impossible because they must be mutually orthogonal with respect to η. To show this assume $\mathbf{v_1}, \mathbf{w_1}, \mathbf{x_1}$ are timelike. After a boost there always exists a frame in which $\mathbf{v_1} = (v, 0, \vec{0})$. There is still a SO$(1,n)$-boost $\in$ SO$(2,n)$ to bring $\mathbf{w_1}$ to the form $\mathbf{w_1} = (0, w, \vec{0})$. Clearly there does not exist a $\mathbf{x_1} = (x, y, \vec{X}_1)$ since the orthogonality condition implies $x = y = 0$ and thus $\mathbf{x_1}$ is not timelike, contrary to the assumption.

Therefore, having at least three different complex eigenvalues is a false assumption and we conclude that there can be at most two complex eigenvalues (and the conjugated ones). Similar to eq. (7.2.17), we now have in general two 2×2 blocks in K_N and $n-4$ number of diagonal elements.

$\mathbf{min}(p, q) > 2$

In case $(p, q) > 2$ a similar analysis applies. Now we need at least $\min(p, q) + 1$ complex eigenvalues to find a contradiction. We can therefore have at most $\min(p, q)$ complex eigenvalues. The normal form of K_N will in general have $\min(p, q)$ number of 2×2 blocks and $n - 2\min(p, q)$ number of diagonal elements.

7.2.3 Uplift to Vacuum Solutions

In order to uplift the solutions from $D > 3$ dimensions to $D + n$ dimensions one uses the Kaluza–Klein Ansatz (4.3.2) with Kaluza–Klein vectors put to zero

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}\mathrm{d}z^m\mathrm{d}z^n \,. \tag{7.2.20}$$

Consider the symmetric coset matrix $\hat{\mathcal{M}}(h) = \eta \exp K_N h$ with K_N the normal form of K that generates all other geodesics and h the harmonic function given in section 7.1. The relation between $\hat{\mathcal{M}}$ and the moduli φ and $\mathcal{M}$ is given in (7.2.2).

Time-dependent solutions from $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n)$

Although the method we use here is different from that we used in chapter 4, the vacuum solutions are of course the same. We refer to [69] to see the vacuum solutions in this approach explicitly.

Stationary solutions from $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n-1,1)$

For K_N we use the normal form $K_N =$

$$\begin{pmatrix} \lambda_1 & \omega & 0 & \dots & 0 \\ -\omega & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_a & \omega & 0 & \dots & 0 \\ -\omega & -\lambda_a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_b & 0 & 0 & \dots & 0 \\ 0 & \lambda_b & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{7.2.21}$$

We exponentiate this to

$$\hat{\mathcal{M}}(h(r)) = \eta\,\mathrm{e}^{K_N h(r)} = \begin{pmatrix} a(r) & b(r) & 0 & \dots & 0 \\ b(r) & c(r) & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 h} & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{e}^{\lambda_n h} \end{pmatrix}, \tag{7.2.22}$$

with

$$\begin{aligned} a(r) &= -\mathrm{e}^{\lambda_b h(r)}\Big(\cosh(\Lambda h(r)) + \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda}\Big)\,, \\ b(r) &= -\omega \mathrm{e}^{\lambda_b h(r)} \frac{\sinh(\Lambda h(r))}{\Lambda}\,, \\ c(r) &= \mathrm{e}^{\lambda_b h(r)}\Big(\cosh(\Lambda h(r)) - \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda}\Big)\,, \end{aligned} \tag{7.2.23}$$

and we define the $\mathrm{SO}(1,1)$ invariant quantity Λ as

$$\Lambda = \sqrt{\lambda_a^2 - \omega^2}\,. \tag{7.2.24}$$

There exist three distinctive cases depending on the character of Λ. If Λ is real the above expression does not need rewriting but we can put ω to zero using a $\mathrm{SO}(1,1)$-boost and then the generating solution is just the straight line solution. If $\Lambda = i\tilde{\Lambda}$ with $\tilde{\Lambda}$ real then the terms with $\cosh(\Lambda h)$ become $\cos\tilde{\Lambda}$ and $\Lambda^{-1}\sinh\Lambda h$ become $\tilde{\Lambda}^{-1}\sin\tilde{\Lambda}h$. Finally, if $\Lambda = 0$ then the term $\Lambda^{-1}\sinh\Lambda h$ becomes just h and the term with $\cosh\Lambda h$ becomes equal to one.

To discuss the zoo of solutions one should make a classification in terms of the different signs for k, $||v||^2$ and Λ^2. The solutions with spherical symmetry have the more interesting properties that they lift up to vacuum solutions that can be asymptotically flat. Moreover, the brane solutions in supergravity always have $k = +1$. Let us briefly discuss them.

- $||v||^2 > 0$: There are three metric solutions, depending on the sign of Λ^2. Only in case $\Lambda^2 > 0$ we can diagonalize it to a straight line via a SO(1,1) boost. In the other two cases there will be a cross-term.
- $||v||^2 = 0$: There are only two solutions, namely $\Lambda^2 = 0$ and $\Lambda^2 < 0$. Both will have a cross-term.
- $||v||^2 < 0$: This implies that $\Lambda^2 < 0$ and hence there will be a cross term.

As an example consider $||v||^2 < 0\,, \Lambda^2 < 0$. The metric solution is given by

$$
\begin{aligned}
\mathrm{d}s^2 =& \mathrm{e}^{p_0 h(r)} f(r)^2 \Big(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{D-1}\Big) \\
&+ \mathrm{e}^{p_1 h(r)} \Big(-\tilde{a}(r)\mathrm{d}t^2 + 2\tilde{b}(r)\mathrm{d}t\mathrm{d}x + \tilde{c}(r)\mathrm{d}x^2\Big) + \mathrm{e}^{2p_{i-1}}\mathrm{d}z^i \mathrm{d}z_i\,,
\end{aligned}
\tag{7.2.25}
$$

where $f(r)$ can be found in (7.1.3) and $h(r)$ is defined in equation (7.1.6). The numbers p_i are

$$
p_0 = (2\lambda_b + \sum_{i=3}^{n} \lambda_i)\sqrt{\tfrac{1}{(D+n-2)(D-2)}}\,, \tag{7.2.26}
$$

$$
p_1 = -\frac{(D-2)p_0 - \lambda_b(n-2) + \sum_{i=3}^n \lambda_i}{n}\,, \tag{7.2.27}
$$

$$
p_{i-1} = -\frac{(D-2)p_0 + 2\lambda_b + \sum_{j=3}^n \lambda_j - \lambda_i}{n}\,, \tag{7.2.28}
$$

and the functions $\tilde{a}(r), \tilde{b}(r), \tilde{c}(r)$ are given by

$$
\tilde{a}(r) = \cos(|\Lambda| h(r)) + \lambda_a \frac{\sin(|\Lambda| h(r))}{|\Lambda|}\,, \tag{7.2.29}
$$

$$
\tilde{b}(r) = \sqrt{\lambda_a^2 - \Lambda^2}\frac{\sin(|\Lambda h(r)|)}{|\Lambda|}\,, \tag{7.2.30}
$$

$$
\tilde{c}(r) = \cos(|\Lambda| h(r)) - \lambda_a \frac{\sin(|\lambda| h(r))}{|\Lambda|}\,. \tag{7.2.31}
$$

In [69] all the different $k = +1$ metrics are given.

7.3 Massive Instantons

Just as we did for time-dependent solutions, we now investigate the effect of adding a potential to our non-Riemannian scalar coset. The analysis we did in section 5.2 can be extended to the instanton case as well, we only consider the case $k = 0$.

Consider the following action

$$S = \int \mathrm{d}^D x \sqrt{\epsilon g} \Big\{ \mathcal{R} - \tfrac{1}{2} G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) \Big\} . \tag{7.3.1}$$

For $\epsilon = 1$ we have a Euclidean theory, while for $\epsilon = -1$ we have a Lorentzian theory. The metric Ansatz is

$$\mathrm{d}s_D^2 = -\eta\epsilon f(r)^2 \mathrm{d}r^2 + g(r)^2(-\eta\, \mathrm{d}\rho^2 + \mathrm{d}\vec{x}^2_{D-2}) , \tag{7.3.2}$$

and we extend (5.2.5) to

$$V = -\eta\epsilon \left\{ \tfrac{1}{2} G^{ij} \partial_i W \partial_j W - \tfrac{D-1}{4(D-2)} W^2 \right\} . \tag{7.3.3}$$

If $\epsilon = -1$ we rediscover the analysis of section 5.2. That is for $\eta = 1$ we have a domain-wall and for $\eta = -1$ a cosmology. When $\epsilon = 1$ and $\eta = -1$ we have a Euclidean metric. We see that both the domain-wall and instanton have $\epsilon\eta = -1$, just as we found in section 6.1. For instantons the same first order equations (5.2.7) hold, since we derive the same effective action as (5.2.6). The only difference is that the overall factor η has been replaced with $-\eta\epsilon$. For the Euclidean case the scalar metric G_{ij} is in general non-compact as we saw in the previous section.

There is therefore one complicating issue. The above analysis shows that for a given domain-wall we find a Euclidean solution belonging to the *same* scalar manifold metric G_{ij}. We know however from the previous section that in general a Euclidean theory contains ghost fields, while domain-walls are considered to belong to theories without ghosts. In case we have only a dilaton this problem does not occur, see for example the instanton of massive IIA* we discussed in subsection 6.5.1. However, let us consider the multi-scalar case (5.3.14) with $\alpha = 0$ for simplicity. Below we show that the Euclidean version of this action requires the axion χ to become a ghost. So we have a different scalar coset. That means that for a given domain-wall we do not have an instanton belonging to the Euclidean version of (5.3.14), instead the correspondence gives us a Euclidean solution of the same scalar coset without a ghost.

Note that we found a similar thing in section 6.5.2 when we discussed the 9d $\mathcal{N} = 2$ domain-wall / cosmology example. When we include the ghost axions the supergravity embedding of the correspondence does not simply require V replaced by $-V$.

Cosmological constant

We now restrict to the case that the potential is a cosmological constant

$$S=\int \mathrm{d}^D x\sqrt{g}\Big(\mathcal{R}+\frac{1}{4}\mathrm{Tr}(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1})-\Lambda\Big)\,. \qquad (7.3.4)$$

In chapter 5 we have seen that the solution of a theory with a potential is only under certain conditions still a geodesic of the scalar manifold, namely when the cosmology is pseudo-BPS. The potential that we consider here is a negative cosmological constant. There is no coupling of the scalar fields in $\mathcal{M}$ to Λ. The equations of motion for $\mathcal{M}$ are still given by

$$\partial_\mu\Big(\sqrt{g}\mathcal{M}^{-1}\partial^\mu\mathcal{M}\Big)=0\,. \qquad (7.3.5)$$

That means that if we introduce the harmonic function h as the new parameter they can still be geodesics. Indeed if we define the Noether charge as $\mathcal{M}^{-1}\partial_h\mathcal{M}=K$ and the affine velocity as $||v||^2=1/2\mathrm{Tr}(K^2)$, the metric can be solved again independently. The presence of the cosmological constant will only change the shape of the metric. This is no longer true if we allow for a direct coupling of the scalars to Λ.

Let us now turn to the solutions. For all the cosets we considered in the previous sections $\mathcal{M}$ is unchanged, since we still have geodesics on the scalar manifold. The metric can be solved independently of $\mathcal{M}$, a similar analysis as for the case $\Lambda=0$ gives

$$\mathrm{d}s^2=\frac{\mathrm{d}r^2}{\frac{||v||^2}{2(D-2)(D-1)}r^{-2(D-2)}-\frac{\Lambda}{(D-2)(D-1)}r^2+k}+r^2\Big(\frac{\mathrm{d}\rho^2}{1-k\rho^2}+\rho^2\mathrm{d}\Omega^2_{D-2}\Big)\,. \qquad (7.3.6)$$

As a concrete example consider the Euclidean version of the action (5.3.14) with $\alpha=0$ and $\mu=2$

$$S=\int \mathrm{d}^D x\sqrt{-g}\Big\{\mathcal{R}-\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2-\Lambda\Big\}\,. \qquad (7.3.7)$$

For a proper derivation as to why we need to replace in the Euclidean theory $(\partial\chi)^2$ by $-(\partial\chi)^2$ we refer to e.g. [137] and references therein. The scalar field solutions are given in table 7.3.1 [2].

The harmonic function h satisfies the differential equation

$$\partial_r h(r)=\frac{1}{\sqrt{\Big(\frac{||v||^2}{2(D-2)(D-1)}r^{-2(D-2)}-\frac{\Lambda}{(D-2)(D-1)}r^2+k\Big)r^{2(D-1)}}}\,. \qquad (7.3.8)$$

[2] The $||v||^2>0$ geodesic is related to $||v||^2<0$ via the continuations $||v||\to i||\tilde v||\,,\quad c_2\to ic_2\,,\quad c_1\to\frac{c_1}{i}$. Similarly the $||v||^2=0$ geodesics follows from $||v||^2>0$ if we apply $c_1\to\frac{c_1}{||v||}\,,\quad c_2\to\frac{g_s||v||}{c_1}$.

$\|\|v\|\|^2 > 0$	$\phi = -\log\|c_1 \sinh(\|\|v\|\|h + c_2)\|$	$\chi = \pm\frac{1}{c_1}\coth(\|\|v\|\|h + c_2) + c_3$
$\|\|v\|\|^2 = 0$	$\phi = -\log\|c_1 h + g_s\|$	$\chi = \pm(c_1 h + g_S)^{-1} + c_3$
$\|\|v\|\|^2 < 0$	$\phi = -\log\|c_1 \sin(\|\|\tilde{v}\|\|h + c_2)\|$	$\chi = \pm\frac{1}{c_1}\cot(\|\|\tilde{v}\|\|h + c_2) + c_3$

Table 7.3.1: The scalar fields belonging to the action (7.3.7) for each sign of $||v||^2$. For clarity we have defined $v = i\tilde{v}$ such that $||\tilde{v}||^2 > 0$.

In general the harmonic function h can no longer be solved explicitly with the exception $D = 3$ [40]. In case $||v|| = 0$ we can solve (7.3.8), but since the solution is rather involved we do not write it down.

There is a close connection to the S(−1)-brane of subsection 2.4.3 and the E(−1)-brane we discussed in subsection 6.5.3. For simplicity we consider only those solutions that are related to type IIB supergravities and have a string theory interpretation.

Let us first consider the link to the type IIB $k = -1$ S(−1)-brane. For this we consider the $||v||^2 > 0$ instanton with $k = 1$ and $\Lambda = 0$. Consider the analytically continuation $r \to it$. We see from the g_{rr}-component of (7.3.6) that $||v||^2 r^{-2(D-2)} \to ||v||^2(-1)^{D-2}t^{-2(D-2)}$. The latter should be positive for S(−1)-branes, so in general we have to make a difference between even and odd dimensions. If we restrict to $D = 10$ and apply the following analytical continuations on the instanton

$$r \to it\,, \quad \rho \to i\tilde{\rho}\,, \quad c_2 \to c_2 + i\frac{\pi}{2}\,, \quad c_1 \to -ic_1\,, \quad c_3 \to ic_3\,, \tag{7.3.9}$$

we find the S(−1)-brane solution (2.4.32–2.4.34) with $k = -1$.

To find the non-extremal version of the type IIB* E(−1)-brane it is sufficient to consider the Wick rotations $r \to it$ and $\rho \to i\tilde{\rho}$ applied to the $||v||^2 > 0$ and $k = 1$ instanton. We derive the following non-extremal type IIB* E(−1)-brane

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\mathrm{d}t^2}{\frac{||v||^2}{144}t^{-16} + 1} + t^2\Big(\frac{\mathrm{d}\tilde{\rho}^2}{1+\tilde{\rho}^2} + \tilde{\rho}^2\mathrm{d}\Omega_8^2\Big)\,,\\
\phi &= \log\Big[c_1\sinh\Big(||v||h + c_2\Big)\Big]\,,\\
\chi &= \pm\frac{1}{c_1}\coth\Big(||v||h + c_2\Big) + c_3\,,\\
h(t) &= \frac{3}{2||v||}\mathrm{arctanh}\Big(\frac{||v||}{\sqrt{144t^{16} + v^2}}\Big)\,,
\end{aligned} \tag{7.3.10}$$

which after appropriate re-scalings can be linked to (6.5.34). Note that the spatial part of the metric describes a nine-dimensional hyperboloid as expected and that we can extend both (−1)-branes to theories with $\Lambda \neq 0$.

Let us finish by relating our general solution to other known type IIB solutions. For this we restrict to $k = +1$. For $D = 10$ and $\Lambda = 0$ the solutions are the extremal [133,134] and non-extremal D-instanton [123,135] of type IIB for respectively $||v||^2 = 0$ and $||v||^2 \neq 0$. For $2 < D < 10$ and $\Lambda = 0$ it can be considered as a consistent truncation of type IIB reduced over a torus. In case we have $D = 5$ and consider a negative cosmological constant, the action is a compactification of type IIB over S^5. This is the natural setting for the AdS/CFT correspondence [40, 138–140].

7.4 Discussion

In this chapter we have extended the analysis of the generating solution to non-Riemannian cosets. In particular we focussed on $\mathrm{GL}(p+q,\mathbb{R})/\,\mathrm{SO}(p,q)$ and showed that the number of complex eigenvalues is at most $\min(p,q)$. For $\min(p,q) = 1$ we discussed the oxidation of the various generating solutions to vacuum solutions.

In the last section we looked at what happens if we consider massive instantons. We showed that we can find similar first order equations as we derived for domain-walls and cosmologies in section 5.2.7. The main difference is that the scalar metric is no longer positive definite. We then focused on the special case that the potential is a cosmological constant. The scalar fields still describe geodesic motion. At the end of the last section we showed a link between non-extremal D(−1)-instantons, S(−1)-branes and non-extremal E(−1)-branes.

Chapter 8

Conclusions and Future Research

8.1 Summary

In the first chapter we pointed out what we wanted to achieve in this thesis. First we wanted to show that p- and Sp-branes can be linked to lower-dimensional actions whose solutions are respectively given by instantons or S(-1)-branes if we reduce over the worldvolume of the brane. And similarly, if we reduce a p- or Sp-brane over all but one of its transverse directions we find a domain-wall or a cosmology. See also figure 3.5.1. The main goal was to derive the generating solution for the massless geodesics that we obtain in the lower dimension. For the massive theory the focus was on re-writing the second order differential equations as first order equations and explaining why we sometimes still have geodesic motion.

The first goal was explained in section 3.5. Let us summarize the situation for massless theories first. In section 3.5 we showed that the scalar part of the action for massless theories, obtained after a worldvolume reduction, leads to geodesic motion. The gravity part gets decoupled and can be solved independently. This applies both to instantons and S(-1)-branes.

To find the geodesics for both types of (-1)-branes we introduced the concept of a generating solution. By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group G generates all other geodesics from the generating solution. The isometry group G is the symmetry group of the lower-dimensional equations of motion. This way we found in chapter 4 the most general Sp-brane with deformed worldvolume via a reduction over a Euclidean torus.

In case we reduce over a Lorentzian torus, the scalar manifold becomes non-

Riemannian. As a result, there are different classes of generating solutions labeled by the sign of the affine velocity. We tackled this problem in chapter 7. We showed how to derive the generating solution for the coset $\mathrm{GL}(p+q,\mathbb{R})/\,\mathrm{SO}(p,q)$.

If we instead do a maximally symmetric compactification, the lower-dimensional theory can contain a potential. This leads to both domain-walls (stationary) and cosmologies (time-dependent). Although we also considered some examples of an instanton with a potential. In general, the presence of the potential upsets the geodesic motion. In chapter 5 we first showed that if we can rewrite the potential in terms of a superpotential in a specific way the equations of motion become first order equations. These cosmologies are called pseudo-BPS. Furthermore, we found examples of scaling solutions that did turn out to be geodesics on the scalar manifold. To explain this we showed in chapter 5 that all pseudo-BPS cosmologies that are scaling solutions must be geodesic. In case the potential is a cosmological constant, the geodesic motion is always preserved.

The resemblance between domain-walls and cosmologies is explained by the domain-wall / cosmology correspondence. A natural question is what pseudo-supersymmetry (pseudo-BPS) means in a real supergravity context. We answered this question in chapter 6. The domain-walls are solutions of an ordinary supergravity, while the cosmologies arise as solutions of the corresponding star supergravity. In this sense the pseudo-supersymmetry of cosmologies corresponds to supersymmetry in the star theory.

8.2 Future Research

Let us give some research problems that are natural to consider next.

- In case of the p-brane we restricted to pure gravity in the higher-dimensional theory. A natural extension would be to extend this with a $(p+1)$-form potential as well. This would allow us to write down the most general p-brane with deformed worldvolume.

- In the non-Riemannian case we restricted the analysis of the generating solution to the coset $\mathrm{GL}(p+q,\mathbb{R})/\,\mathrm{SO}(p,q)$. A natural extension is to look for the generating solutions of the maximally non-compact supergravities given in the right column of table 3.4.1 [69].

- The embedding of the domain-wall / cosmology correspondence was restricted to $\mathcal{N}=2$ supergravities in nine and ten dimensions. Note however that when one continues to lower the dimension, more possibilities could arise since one can then have extended supersymmetry with $\mathcal{N}>2$. This allows for more general reality conditions on the fermions than considered in chapter 6. The matrix ρ appearing in these reality conditions will in general be a $\mathcal{N}\times\mathcal{N}$ matrix. It might be interesting to find out if for $\mathcal{N} > 2$ there can be more than two inequivalent real slices in certain signatures.

Appendix A

Differential Geometry: Formulae and Conventions

In this appendix we fix the conventions used in the main text. We also give a brief introduction to the Einstein-Hilbert action.

A.1 Conventions

We take the following metric signature convention,

$$\eta = \mathrm{diag}(- \cdots -, + \cdots +)\,, \tag{A.1.1}$$

writing first the timelike directions and then the spacelike ones. Symmetrization of a tensor $T_{\mu_1 \ldots \mu_p}$ is given by

$$T_{(\mu_1 \ldots \mu_p)} = \frac{1}{p!}\Big(T_{\mu_1 \ldots \mu_p} + \text{even permutations} + \text{odd permutations}\Big)\,. \tag{A.1.2}$$

Whereas the anti-symmetrization of $T_{\mu_1 \ldots \mu_p}$ is given by

$$T_{[\mu_1 \ldots \mu_p]} = \frac{1}{p!}\Big(T_{\mu_1 \ldots \mu_p} + \text{even permutations} - \text{odd permutations}\Big)\,. \tag{A.1.3}$$

A.2 General Relativity

In general relativity space-time is a D-dimensional (pseudo-)Riemannian manifold (M, g). This means that it is a manifold M endowed with a bilinear form $g_{\mu\nu}$ with signature $(-\cdots-,+\cdots+)$, writing first the timelike directions and then the spacelike ones. In components we write this as

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\,, \qquad \mu,\nu = 1,\dots D\,. \tag{A.2.1}$$

To shorten the notation, we will omit the $\otimes$. For a given metric $g_{\mu\nu}$ we use the following Levi-Civita connection

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\Big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\Big)\,, \tag{A.2.2}$$

from which we construct the Riemann tensor

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\gamma_{\nu\sigma}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\sigma}\,. \tag{A.2.3}$$

From this tensor we can define the Ricci tensors $\mathcal{R}_{\nu\sigma}$ and the Ricci scalar $\mathcal{R}$ via the contractions of the Riemann tensor

$$\mathcal{R}_{\ \nu\sigma} \equiv \mathcal{R}^\mu_{\nu\mu\sigma}\,, \quad \mathcal{R} \equiv \mathcal{R}_\nu{}^\nu\,. \tag{A.2.4}$$

The Einstein tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}\,. \tag{A.2.5}$$

The Ricci tensor of a conformal re-scaled metric $\tilde{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu}$ is

$$\tilde{\mathcal{R}}(\tilde{g})_{\mu\nu} = \mathcal{R}_{\mu\nu} - (D-2)\alpha^2(\partial\phi)^2 g_{\mu\nu} + (D-2)\alpha^2\partial_\mu\phi\partial_\nu\phi - (D-2)\alpha\nabla_\mu\partial_\nu\phi - \alpha g_{\mu\nu}\Box\phi\,. \tag{A.2.6}$$

All the tensors appearing on the right-hand side are defined with respect to $g_{\mu\nu}$. The action of the covariant derivative ∇_η on a tensor $T^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p}$ is defined as

$$\begin{aligned}\nabla_\eta T^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p} =& \partial_\eta T^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p} - \Gamma^\rho_{\eta\mu_1} T^{\nu_1\dots\nu_q}_{\rho\mu_2\dots\mu_p} \dots - \Gamma^\rho_{\eta\mu_p} T^{\nu_1\dots\nu_q}_{\mu_1\mu_2\dots\mu_{p-1}\rho} \\ &+ \Gamma^{\nu_1}_{\rho\eta} T^{\rho\nu_2\dots\nu_q}_{\mu_1\dots\mu_p} + \dots + \Gamma^{\nu_q}_{\rho\eta} T^{\nu_1\dots\nu_{q-1}\rho}_{\mu_1\dots\mu_p}\,.\end{aligned} \tag{A.2.7}$$

Finally, the $\Box$-operator is defined as

$$\Box\phi \equiv \nabla_\mu\partial^\mu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\Big)\,, \tag{A.2.8}$$

where g stands for the determinant of $g_{\mu\nu}$.

For the cosmologies and instantons we often work in the gauge where the metric is given by

$$\mathrm{d}s_D^2 = \epsilon f(r)^2\mathrm{d}r^2 + g(r)^2 g^{D-1}_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{A.2.9}$$

For $\epsilon = -1$ we have a cosmology, while for $\epsilon = +1$ this describes an instanton geometry. The function f corresponds to the gauge freedom of reparameterizing the r-coordinate. In case $f = 1$ and $\epsilon = -1$ we have a cosmology in the FLRW-gauge. For the metric Ansatz (A.2.9) the Ricci tensor is given by

$$\mathcal{R}_{ij} = -\epsilon\Big\{\frac{\mathrm{d}}{\mathrm{d}r}[\frac{g\dot{g}}{f^2}]+\frac{g\dot{g}\dot{f}}{f^3}+(D-3)\frac{\dot{g}^2}{f^2}\Big\}g_{ij}^{D-1}+\mathcal{R}_{ij}^{D-1}\,, \qquad \mathcal{R}_{rr} = (D-1)\Big\{-(\frac{\ddot{g}}{g})+\frac{\dot{g}\dot{f}}{gf}\Big\}\,, \tag{A.2.10}$$

where a dot refers to a derivative with respect to r.

A homeomorphism $f : M \to M$ is an *isometry* if it preserves the metric, in components this is the statement

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p)\,, \tag{A.2.11}$$

where x and y are the coordinates of p and $f(p)$ respectively. If a displacement ϵX, ϵ being infinitesimal, generates an isometry, the vector field X is called a *Killing vector field*. The coordinates x^μ of a point $p \in M$ change to $x^\mu + \epsilon X^\mu(p)$ under this displacement. If $f : x^\mu \to x^\mu + \epsilon X^\mu$ is an isometry, it satisfies (A.2.11). From this we can derive that $g_{\mu\nu}$ and X^μ satisfy the *Killing equation*

$$X^\zeta\partial_\zeta g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0\,. \tag{A.2.12}$$

If the right-hand side is non-zero and given by $\psi g_{\mu\nu}$ with ψ a function then X is called a conformal Killing vector field. The metric gets re-scaled by an overall factor related to ψ.

The Killing vector fields represent the direction of the symmetry of a manifold. In D-dimensional Minkowski space-time ($D \geq 2$) there are $D(D+1)/2$ Killing vector fields, D of which generate translations, $(D-1)$ boosts and $(D-1)(D-2)/2$ space rotations. Those space-times which admit $D(D+1)/2$ Killing vector fields are called *maximally symmetric spaces*. One can prove that the Riemann tensor is then given by

$$\mathcal{R}_{\rho\sigma\mu\nu} = \alpha(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})\,, \tag{A.2.13}$$

with α a constant.

In the metric (A.2.9), g_{ij} often describes a Euclidean maximally symmetric space. That is we have the sphere S^n (k=1) or the hyperboloid $\mathbb{H}^n$ (k=-1) or flat space $\mathbb{E}^n$ (k=0). The metrics read

$$\mathrm{d}s^2 = \frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n-1}^2\,. \tag{A.2.14}$$

Also, $\mathrm{d}\Omega_m^2$ is the metric on the S^m sphere

$$\mathrm{d}\Omega_m^2 = \mathrm{d}\theta_1^2 + \sin^2(\theta_1)\mathrm{d}\theta_2^2 + \ldots + \sin^2(\theta_1)\ldots\sin^2(\theta_{m-1})\mathrm{d}\theta_m^2\,. \tag{A.2.15}$$

Via the coordinate redefinition

$$\frac{1}{1-kr^2}\mathrm{d}r^2 = \mathrm{d}\psi^2 \,, \tag{A.2.16}$$

we find the metric

$$\begin{aligned} k=-1: &\quad \mathrm{d}s^2 = \mathrm{d}\psi^2 + \sinh^2\psi\,\mathrm{d}\Omega_{n-1}^2\,, \\ k=0\quad: &\quad \mathrm{d}s^2 = \mathrm{d}\psi^2 + \psi^2\,\mathrm{d}\Omega_{n-1}^2\,, \\ k=+1: &\quad \mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\Omega_{n-1}^2\,. \end{aligned} \tag{A.2.17}$$

The convention is such that $\alpha = k$ for these metrics. The Ricci scalar is

$$\mathcal{R}_n = kn(n-1)\,. \tag{A.2.18}$$

A.2.1 Vielbeine

Instead of writing the metric g in terms of coordinate one-forms $\mathrm{d}x^\mu$, we can use vielbein one-forms $e^a = e^a_\mu \mathrm{d}x^\mu$ for which the metric is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}e^a \otimes e^b\,, \quad a,b = 1,\dots D\,, \tag{A.2.19}$$

where η is given by (A.1.1) and we used $\otimes$ for clarity. We use Greek indices $\mu,\nu,\rho\dots$ to denote space-time coordinates and Latin indices $a,b,c\dots$ represent the so-called tangent directions, which are raised and lowered with η. The determinant of the vielbein is denoted by e.

We define the spin connection $\omega^a_b = \omega_\mu{}^a{}_b \mathrm{d}x^\mu$ via

$$\mathrm{d}e^a = -w^a_b \wedge e^b\,, \quad \omega_{ab} = -\omega_{ba}\,. \tag{A.2.20}$$

The spin connection can be expressed in terms of the vielbeine

$$\omega_{abc} = \omega_{\mu bc}e^\mu{}_a = \frac{1}{2}\Big(-\Omega_{abc} + \Omega_{bca} + \Omega_{cab}\Big)\,, \quad \Omega^a_{bc} = 2\partial_{[\mu}e^a_{\nu]}e^\mu_b e^\nu{}_c\,. \tag{A.2.21}$$

Here Ω_{abc} is called the object of anholonomicity. In terms of the spin connection the two-form curvature is given by

$$R^a_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,, \tag{A.2.22}$$

from which we find the Ricci tensor

$$\mathcal{R}_{\mu\nu} = (R^a_b)_{\rho\nu}e^b{}_\mu e^\rho{}_a\,. \tag{A.2.23}$$

The covariant derivative with respect to local Lorentz transformations is denoted by ∇_μ, acting on spinors χ as

$$\begin{aligned}\nabla_\mu \chi &= \partial_\mu \chi + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\chi\,,\\ \nabla_\mu \chi^\nu &= \partial_\mu \chi^\nu + \Gamma^\nu_{\mu\rho}\chi^\rho + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\chi^\nu\,,\end{aligned} \tag{A.2.24}$$

where in general we denote $\Gamma_{a_1\,\ldots\,a_n} = \Gamma_{[a}\ldots\Gamma_{a_n]}$ and Γ_a is an element of the Clifford algebra, see appendix B.

A.3 Forms

Instead of working with the index notation, it is often highly preferable to work with the language of differential forms. A differential form of order p or a p-form for short reads in components

$$A_p = \frac{1}{p!}A_{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}\,, \tag{A.3.1}$$

where the wedge product $\wedge$ is defined by the totally anti-symmetric tensor product, for example

$$\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2} = \mathrm{d}x^{\mu_1}\otimes\mathrm{d}x^{\mu_2} - \mathrm{d}x^{\mu_2}\otimes\mathrm{d}x^{\mu_1}\,. \tag{A.3.2}$$

We will only give some relevant properties of differential forms, a good textbook on this subject is for example [51], see also [44].

Due to the wedge product, a p- and a q-form obey

$$A_p\wedge B_q = (-)^{pq}B_q\wedge A_p\,. \tag{A.3.3}$$

The action of the exterior derivative d on the p-form (A.3.1) is defined as

$$\mathrm{d}A_p = \frac{1}{p!}\partial_{[\mu}A_{\mu_1\ldots\mu_p]}\mathrm{d}x^\mu\wedge\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}\,, \tag{A.3.4}$$

resulting in a $(p+1)$-form. This exterior derivative obeys the Leibniz rule

$$\mathrm{d}(A_p\wedge B_q) = \mathrm{d}A_p\wedge B_q + (-)^p A_p\wedge\mathrm{d}B_q\,. \tag{A.3.5}$$

In D dimensions we define the epsilon *symbol* $\varepsilon_{\mu_1\ldots\mu_D}$ via

$$\varepsilon_{1\ldots D} = 1\,, \tag{A.3.6}$$

and it is antisymmetric in all its indices. This allows us to define the epsilon tensor $\epsilon_{\mu_1\ldots\mu_D}$ via

$$\epsilon_{\mu_1\ldots\mu_D} = \sqrt{|g|}\varepsilon_{\mu_1\ldots\mu_D}\,. \tag{A.3.7}$$

Sometimes it is also useful to define a totally antisymmetric epsilon symbol with *upstairs* indices, the components are given numerically by

$$\varepsilon^{\mu_1 \cdots \mu_D} = (-)^t \varepsilon_{\mu_1 \cdots \mu_D} \,, \tag{A.3.8}$$

where t is the number of timelike coordinates. Contractions of the epsilon tensor (and symbol) obey the following relation[1]

$$\epsilon_{\mu_1 \cdots \mu_q \mu_{q+1} \cdots \mu_D} \epsilon^{\mu_1 \cdots \mu_q \nu_{q+1} \cdots \nu_D} = (-)^t q!(D-q)! \delta^{[\nu_{q+1} \cdots \nu_D]}_{[\mu_{q+1} \cdots \mu_D]} \,. \tag{A.3.9}$$

The Hodge operator $\star$ is a linear map of a p-form into a $(D-p)$-form, whose action on a p-form is defined by

$$*(\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}) = \frac{1}{(D-p)!} \epsilon_{\nu_1 \ldots \nu_{D-p}}{}^{\mu_1 \ldots \mu_p} \, \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_{D-p}} \,. \tag{A.3.10}$$

As a particular case,

$$*1 = \epsilon = \sqrt{|g|} \mathrm{d}^D x \,, \tag{A.3.11}$$

where we identify $\mathrm{d}^D x$ as $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D$. Three frequently used expressions for arbitrarily p-forms are

$$\begin{aligned} *A_p \wedge B_p &= *B_p \wedge A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} B^{\mu_1 \ldots \mu_p} * 1 \,, \\ * * A_p &= (-)^{p(D-p)+t} A_p \,, \end{aligned} \tag{A.3.12}$$

and

$$*\mathrm{d} * A_p = \frac{(-)^{p(D-p+1)-1+t} \partial_\rho \left(\sqrt{|q|} A^{\rho \mu_1 \ldots \mu_{p-1}} \right)}{(p-1)! \sqrt{|q|}} g_{\mu_1 \nu_1} \ldots g_{\mu_{p-1} \nu_{p-1}} \, \mathrm{d}x^{\nu_1} \wedge \ldots \mathrm{d}x^{\nu_{p-1}} \,. \tag{A.3.13}$$

A.4 Euler-Lagrange Variation

To obtain the Einstein equation, we need the following relations under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \,, \quad & \delta\sqrt{-g} &= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \,, \\ \delta g^{\mu\nu} &= -g^{\mu\kappa} g^{\nu\rho} \delta g_{\kappa\rho} \,, & \delta\sqrt{-g}\mathcal{R} &= -\sqrt{-g} G^{\mu\nu} \delta g_{\mu\nu} \,. \end{aligned} \tag{A.4.1}$$

[1] We define $0! = 1$.

The action of matter coupled to D-dimensional gravity is given by

$$S = \int \mathrm{d}^D x \, \sqrt{|g|} \left(\frac{1}{2\kappa^2} \mathcal{R} + \mathcal{L} \right), \tag{A.4.2}$$

where the first term on the right-hand side is called the Einstein-Hilbert action and $\mathcal{L}$ is the matter Lagrangian density of the theory. Newton's constant G is related to κ, for example in four dimensions we have that $\kappa^2 = 8\pi G$. If the matter part of action is changed under $\delta g_{\mu\nu}$, the energy-momentum tensor is defined by

$$\delta S_{\mathrm{M}} = \frac{1}{2} \int \mathrm{d}^D x \, \sqrt{|g|} \, T^{\mu\nu} \, \delta g_{\mu\nu} \, . \tag{A.4.3}$$

From demanding that the total variation of the action (A.4.2) vanishes under $\delta g_{\mu\nu}$, we obtain the Einstein equation

$$G_{\mu\nu} = \kappa^2 \, T_{\mu\nu} \, . \tag{A.4.4}$$

In particular we take for the matter Lagrangian density a A_{p-1} potential. This is described by the action

$$\mathcal{L} = *\mathcal{R} - \frac{1}{2} \mathrm{e}^{a\phi} * \mathrm{d}A_{p-1} \wedge \mathrm{d}A_{p-1} - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi \, , \tag{A.4.5}$$

where a is a real number. The equations of motion, together with the Bianchi identity, are then given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{p-1}{2(D-2)(p!)} g_{\mu\nu} \mathrm{e}^{a\phi} F_p^2 + \frac{1}{(p-1)!2} \mathrm{e}^{a\phi} (F_p^2)_{\mu\nu} \, , \tag{A.4.6}$$

$$\mathrm{d}\Big(*\mathrm{e}^{a\phi} F_p \Big) = 0 \, , \quad \mathrm{d}F_p = 0 \, , \tag{A.4.7}$$

$$\Box \phi = \frac{a}{p!2} F_p^2 \mathrm{e}^{a\phi} \, , \tag{A.4.8}$$

where for a p-form field strength we have used the definitions

$$\begin{aligned} F_p^2 &= F_{\mu_1 \ldots \mu_p} F_{\nu_1 \ldots \nu_p} g^{\mu_1 \nu_1} \ldots g^{\mu_p \nu_p} \, , \\ (F_p^2)_{\mu\nu} &= F_{\mu \rho_1 \ldots \rho_{p-1}} F_{\nu \nu_1 \ldots \nu_{p-1}} g^{\rho_1 \nu_1} \ldots g^{\rho_{p-1} \nu_{p-1}} \, . \end{aligned} \tag{A.4.9}$$

Appendix B

Spinors and their Reality Properties

In this appendix, we will recall various properties of Clifford algebras and spinors. The purpose of this appendix is two-fold. On the one hand it serves to introduce our conventions and notations regarding spinors. On the other hand, the discussion on the reality conditions on spinors is also rather crucial for the results presented in chapter 6. In the first section of this appendix, we will recall some general properties of Clifford algebras in various dimensions and signatures. In the second section, we will then discuss how appropriate reality conditions can be imposed on the spinors. The latter discussion will be mainly restricted to 10 and 11 dimensions. A good review concerning the matter presented here is offered in [141], whose conventions we will mainly follow.

B.1 Clifford Algebras in Various Dimensions and Signatures

In this section we will consider arbitrary dimensions $d = t + s$, where t is the number of timelike and s the number of spacelike directions. The Clifford algebra is then defined by the following anticommutation relation

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}\,, \tag{B.1.1}$$

where $\eta_{ab} = \mathrm{diag}(- \cdots - + \cdots +)$, writing first the timelike directions and then the spacelike ones.

We will always work with unitary representations of (B.1.1):

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1}\,, \tag{B.1.2}$$

where we define A to be the product of all timelike Γ-matrices : $A = \Gamma_1 \cdots \Gamma_t$. In this way, timelike Γ-matrices are anti-hermitian, while the spacelike ones are hermitian.

In even dimensions, we will define the chirality matrix Γ_* as follows

$$\Gamma_* = (-i)^{d/2+t}\Gamma_1 \cdots \Gamma_d \quad \Rightarrow \quad (\Gamma_*)^2 = \mathbb{1}\,. \tag{B.1.3}$$

When we restrict to 10 dimensions, we will also denote Γ_* by Γ_{11}. Note that in odd dimensions the product of all Γ-matrices is always given by a power of i times the unit matrix.

One can show that there always exists a unitary matrix $\mathcal{C}_\eta$ such that

$$\mathcal{C}_\eta^T = -\varepsilon\mathcal{C}_\eta \qquad \text{and} \qquad \Gamma_a^T = -\eta\mathcal{C}_\eta\Gamma_a\mathcal{C}_\eta^{-1}\,, \tag{B.1.4}$$

where ε, η can be ± 1. In even dimensions, both signs for η are possible, corresponding to the fact that both Γ_a^T and $-\Gamma_a^T$ are representations that are equivalent to Γ_a. The two possibilities for the charge conjugation matrix are then related by

$$\mathcal{C}_+ = \mathcal{C}_-\Gamma_*\,. \tag{B.1.5}$$

In odd dimensions, due to the constraint on the product of all Γ-matrices, only one of the representations Γ_a^T or $-\Gamma_a^T$ is equivalent to Γ_a and hence only one sign for η is possible. Once the sign of η is fixed, the sign of ε can be determined. The possibilities for these signs are summarized in table B.1.1.

d mod 8	0	1	2	3	4	5	6	7
(ϵ, η)	$(-,+)$ $(-,-)$	$(-,-)$	$(-,-)$ $(+,+)$	$(+,+)$	$(+,+)$ $(+,-)$	$(+,-)$	$(+,-)$ $(-,+)$	$(-,+)$

Table B.1.1: *The possible signs for ε and η for all dimensions (modulo 8).*

Defining the following matrix B_η

$$B_\eta = -\varepsilon\eta^t\mathcal{C}_\eta A\,, \tag{B.1.6}$$

equations (B.1.2) and (B.1.4) then imply that

$$\Gamma_a^* = (-)^{t+1}\eta B_\eta\Gamma_a B_\eta^{-1}\,. \tag{B.1.7}$$

As for the $\mathcal{C}_\eta$-matrix, in even dimensions both signs of η are possible, while in odd dimensions only one possibility for η is allowed. Finally, note that the matrix B_η satisfies

$$B_\eta B_\eta^* = -\varepsilon\eta^t(-)^{t(t+1)/2}\mathbb{1}\,. \tag{B.1.8}$$

B.2 Reality Conditions for Spinors

In this thesis we define the Majorana conjugate $\bar{\chi}$ of a spinor χ as

$$\bar{\chi} = \chi^T \mathcal{C}_\eta \,, \tag{B.2.1}$$

whereas the Dirac conjugate $\bar{\chi}^D$ is given by

$$\bar{\chi}^D = \chi^\dagger A \,. \tag{B.2.2}$$

In order to formulate reality conditions in 10 dimensions, we will work with a doublet notation, allowing us to treat type IIA and type IIB theories in a single framework. The 64-component doublets are the following

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} \text{ (type IIA)} \,, \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \text{ (type IIB)} \,, \tag{B.2.3}$$

where $\Gamma_* \chi^\pm = \pm \chi^\pm$. Gamma-matrices and the charge conjugation matrix $\mathcal{C}_\eta$ then act on the doublets by making the following replacements

$$\Gamma_a \quad \rightarrow \quad \Gamma_a \otimes \sigma \,, \tag{B.2.4}$$
$$\mathcal{C}_\eta \quad \rightarrow \quad \mathcal{C}_\eta \otimes \sigma \,, \tag{B.2.5}$$

where σ is given by σ_1 in type IIA and by $\mathbb{1}_2$ in type IIB. Note furthermore that Γ_* can be represented by $\mathbb{1}_{32} \otimes \sigma_3$ in type IIA and by $\mathbb{1}_{32} \otimes \mathbb{1}_2$ in type IIB. In the following and in chapter 6 we will always assume that matrices act on doublets as indicated in (B.2.4), without writing the tensor products explicitly. We use the following three Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{B.2.6}$$

Using this doublet notation, a general reality condition can now be denoted as follows:

$$\chi^* = -\varepsilon \eta^t \alpha_\chi \mathcal{C}_\eta A \rho \chi \,, \tag{B.2.7}$$

where α_χ represents a phase factor. The presence of $-\varepsilon \eta^t \mathcal{C}_\eta A$ is dictated by compatibility with Lorentz transformations. Note that the condition (B.2.7) now contains a 2×2-matrix ρ, that can mix the two components of the doublets (B.2.3); the action of ρ on a doublet should thus be interpreted as $\mathbb{1}_{32} \otimes \rho$. We will take the following possibilities for ρ :

$$\rho \in \{\mathbb{1}_2, \sigma_1, \mathrm{i}\sigma_2, \sigma_3\} \,. \tag{B.2.8}$$

Note that in the type IIA case the matrix ρ is required to be diagonal, since complex conjugation should preserve the chirality of the spinor. In the type IIB case, we do not have to impose this restriction as both parts of the doublet now have the same chirality. Note that upon making a field redefinition, the reality conditions with $\rho = \sigma_1$ and $\rho = \sigma_3$ can be related [1]. We can thus restrict to $\rho \in \mathbb{1}, \mathrm{i}\sigma_2, \sigma_3$ without loss of generality.

The requirement that $\chi^{**} = \chi$ leads to a non-trivial requirement:

$$(\sigma^{t+1}\rho)^2 = -\varepsilon\eta^t(-)^{\frac{t(t+1)}{2}} . \tag{B.2.9}$$

In the IIB case, there is moreover an extra consistency condition, due to the fact that the theory is chiral. Indeed the reality condition (B.2.7) has to respect the chirality, which in 10 dimensions is only possible when t is odd.

The different reality conditions that can be consistently imposed are then summarized in table B.2.1. In this table we always choose $\epsilon = \eta = 1$. This is possible as

t mod 4	0	1	2	3
IIA	*M	MW *MW	M	/
IIB	/	MW *MW	/	SMW

Table B.2.1: *This table gives all the possible ten-dimensional reality conditions of the form (B.2.7) for a doublet of chiral spinors in type IIA and IIB respectively. t denotes the number of timelike dimensions. Here M, *M or SM respectively stand for $\rho = \mathbb{1}_2$, σ_3 or $i\sigma_2$. The addition of W means that the reality condition respects chirality of the spinors.*

$C_- = C_+\Gamma_{11}$, and thus (B.2.7) with the choice $\epsilon = \eta = -1$ can always be rewritten in terms of C_+ and $\eta = \epsilon = 1$ by redefining ρ and α_χ since Γ_{11} can be represented as σ_3 or $\mathbb{1}_2$ in IIA respectively IIB.

Finally a word on notation. Note that in denoting the types of reality conditions on the fermions in table B.2.1, we reserve the * when $\rho = \sigma_3$ in (B.2.7). M, MW and SMW then correspond to what is known in the literature as Majorana, Majorana-Weyl and symplectic Majorana-Weyl (see for instance [141]). Although *M suggests a Majorana condition, this is not true. For instance, what we have called *M in Euclidean type IIA, corresponds to what in the literature is called symplectic Majorana.

[1]Explicitly, this redefinition is given by $\chi'_1 = \chi_1 + \chi_2$ and $\chi'_2 = \chi_1 - \chi_2$. Note that this redefinition involves only real numbers. Furthermore as one can see in table 6.3.1 in the main text, this redefinition corresponds to going from IIB$'$ to IIB*.

Appendix C

Lie Group and Lie Algebra

A *Lie Group* G is a differentiable manifold which is endowed with a group structure such that the two group operations

- $\cdot \ : G \times G \to G\,,\ (g_1, g_2) \to g_1 \cdot g_2\,,$
- ${}^{-1} : G \to G\,,\ g \to g^{-1}\,,$

are differentiable. A Lie group is *abelian* if $a \cdot b = b \cdot a$, $\forall a, b \in G$, else it is called non-abelian. From now on we will write $a \cdot b$ as ab.

The Lie group G can act on a manifold M. The action of G on a point p of the manifold M is a differentiable map $\sigma : G \times M \to M$ which satisfies the conditions

- $\sigma(e, p) = p\,,$
- $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)\,,$

where e is the identity element of G, $g_i \in G$ and $p \in M$. We call the action of σ *transitive* if for any $p_1, p_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, p_1) = p_2$. This means that given any point $p \in M$, the action of G on p allows us to go to all the points of M. Such a manifold is called *homogeneous*. For example, Lie groups act transitively on themselves via the group multiplication. The *isotropy group* $H(p)$ of $p \in M$ is a subgroup of G defined by

$$H(p) = \{g \in G | \sigma(g, p) = p\}\,. \tag{C.0.1}$$

There is a theorem that states that if a Lie group G acts on a manifold M the isotropy group $H(p)$ for any $p \in M$ is a Lie subgroup [51].

There are two ways to define a *Lie algebra*. First, the tangent space of a group G at the identity element e can be identified with the Lie algebra $\mathfrak{G}$ of G. Secondly, there is a more algebraic approach [142]. A Lie algebra is a vector space $\mathfrak{G}$ together with a bilinear operation $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ satisfying

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z] && \text{(bilinearity)}\,, \\ [x, y] &= -[y, x] && \text{(anticommutativity)}\,, \\ 0 &= [x, [y, z]] + [y, [z, x]] + [z, [x, y]] && \text{(Jacobi identity)}\,, \end{aligned} \tag{C.0.2}$$

where $x, y, z \in \mathfrak{G}$ and $a, b \in F$ with F a field over which $\mathfrak{G}$ is a vector space, for example $\mathbb{R}$ or $\mathbb{C}$.

A Lie algebra is specified by its generators t_a and their commutation relations

$$[t_a, t_b] = f_{ab}^{\ \ c} t_c\,, \tag{C.0.3}$$

here $f_{ab}^{\ \ c}$ is called the structure constant. The dimension of a Lie algebra is the dimension of the underlying vector space spanned by the generators t_a.

By a *representation* of an algebra we mean a set of matrices T_a with the same commutation relations as the t_a's given in (C.0.3). This is a mapping of the generators t_a into linear operators T_a, which act on some vector space V. When this vector space is the Lie algebra $\mathfrak{G}$ itself we call this representation the *adjoint* representation. Let $x \in \mathfrak{G}$ and take

$$x \to [t_a, x]\,. \tag{C.0.4}$$

This linear transformation is called the ad t_a.

A *subalgebra* $\mathfrak{h} \subset \mathfrak{G}$ is a subspace of $\mathfrak{G}$ which is closed under the Lie product. An *ideal* is a special kind of subalgebra. Namely, if $\mathfrak{h}$ is an ideal and $x \in \mathfrak{h}$ and y is an element of $\mathfrak{G}$ then $[x, y] \in \mathfrak{h}$. If $\mathfrak{h}$ would have been a subalgebra only, $y \in \mathfrak{h}$ instead of $\mathfrak{G}$. A Lie algebra which has no trivial ideals it is called *simple*. The trivial ideals are the full algebra and the ideal $\{0\}$. An algebra which has no abelian ideals is called *semi-simple*.

The generators of the simple Lie algebra can be chosen so that one subset of them generates a commutative Cartan subalgebra (CSA). We denote these generators by h_I, so that $[h_I, h_J] = 0$. The other remaining generators are eigenvectors of ad h for every $h \in$ CSA. We call these the shift operators and denote them by e_α. Here α is a r-dimensional vector $\alpha = (\alpha_1, \ldots, \alpha_r)$ and r is called the rank of the algebra. The α_I, $I = 1, \ldots, r$, are the eigenvalues of H_I in the adjoint representation, i.e. $[H_I, E_\alpha] = \alpha_I E_\alpha$. The α_I is called a *root* which form the root vector α and E_α is the adjoint representation of e_α.

Let $\alpha_1, \ldots, \alpha_r$ be a fixed basis of roots so that any other root ρ can be written as $\rho = \sum_{i=1}^r c_i \alpha_i$ with c_i some coefficient. We call ρ a *positive* root if the first non-zero $c_i > 0$, else it is called a negative root. A *simple* root is a positive root which cannot be written as the sum of two positive roots. The number of simple roots equals the rank of the Lie algebra.

For a general representation we indicate the basis of the CSA $\{h_I\}$, $I = 1, \dots r$, by matrices H_I and the step operators e_α by E_α. The H_I and E_α act on vectors ϕ^a in some space V. Since the H_I commute we take them diagonal

$$H_I \phi^a = \lambda_I^a \phi^a \,. \tag{C.0.5}$$

The eigenvalue λ_I^a is called a *weight* which form the weight vector λ^a. We see that in case of the adjoint representation the weights are the roots.

The canonical commutation relations can be summarized by

$$[H_I, H_J] = 0\,, \quad [H_I, E_\alpha] = \alpha_I E_\alpha\,, \quad [E_\alpha, E_\beta] = N(\alpha,\beta) E_{\alpha+\beta}\,. \tag{C.0.6}$$

The last line is to be understood as follows. If $\alpha+\beta$ is not a root we have $N(\alpha,\beta) = 0$, else we have $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$.

Appendix D

Publications

- [A] E. A. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren and T. Van Riet, *Non-Extremal D-instantons and the AdS/CFT Correspondence*, JHEP **0601**, 061 (2006) [arXiv:hep-th/0510048].
- [B] J. Hartong, A. Ploegh, T. Van Riet and D. B. Westra, *Dynamics of Generalized Assisted Inflation*, Class. Quant. Grav. **23**, 4593 (2006) [arXiv:gr-qc/0602077].
- [C] W. Chemissany, A. Ploegh and T. Van Riet, *Scaling Cosmologies, Geodesic Motion and Pseudo-Susy*, [arXiv:0704.1653 [hep-th]].
- [D] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel and D. Van den Bleeken, *Pseudo-Supersymmetry and a Tale of Alternate Realities*, JHEP **0707**, 067 (2007) [arXiv:0704.3559 [hep-th]].
- [E] E. A. Bergshoeff, J. Hartong, A. Ploegh and D. Sorokin, *Q-instantons*, [arXiv:0801.4956 [hep-th]].
- [F] E. A. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet *Generating Geodesic Flows and Supergravity Solutions*, to appear.

Bibliography

[1] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, Phys. Rev. Lett. **83** (1999) 3370–3373, hep-ph/9905221

[2] L. Randall and R. Sundrum, *An alternative to compactification*, Phys. Rev. Lett. **83** (1999) 4690–4693, hep-th/9906064

[3] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge, UK: Univ. Pr. (1998) 402 p

[4] M. Green, J. Schwarz and E. Witten, *Superstring Theory. Vol. 1: Introduction*, Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)

[5] E. Kiritsis, *Introduction to superstring theory*, hep-th/9709062

[6] B. Zwiebach, *A first course in string theory*, Cambridge, UK: Univ. Pr. (2004) 558 p

[7] R. J. Szabo, *BUSSTEPP lectures on string theory: An introduction to string theory and D-brane dynamics*, hep-th/0207142

[8] M. Gutperle and A. Strominger, *Spacelike branes*, JHEP **04** (2002) 018, hep-th/0202210

[9] T. Frankel, *The geometry of physics: an introduction*, Cambridge Univ. Pr. (1997)

[10] J. Callan, Curtis G., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in Background Fields*, Nucl. Phys. **B262** (1985) 593

[11] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, Cambridge, UK: Univ. Pr. (1998) 531 p

[12] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. **B76** (1978) 409–412

[13] R. G. Leigh, *Dirac-Born-Infeld Action from Dirichlet Sigma Model*, Mod. Phys. Lett. **A4** (1989) 2767

[14] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, *Stainless super p-branes*, Nucl. Phys. **B456** (1995) 669–698, hep-th/9508042

[15] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213–326, hep-th/9412184

[16] C. Teitelboim, *Monopoles of Higher Rank*, Phys. Lett. **B167** (1986) 69

[17] R. I. Nepomechie, *Magnetic Monopoles from Antisymmetric Tensor Gauge Fields*, Phys. Rev. **D31** (1985) 1921

[18] A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207

[19] L. J. Romans, *Massive N=2a Supergravity in Ten-Dimensions*, Phys. Lett. **B169** (1986) 374

[20] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, hep-th/9711200

[21] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253–291, hep-th/9802150

[22] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. **323** (2000) 183–386, hep-th/9905111

[23] E. B. Bogomolny, *Stability of Classical Solutions*, Sov. J. Nucl. Phys. **24** (1976) 449

[24] M. K. Prasad and C. M. Sommerfield, *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, Phys. Rev. Lett. **35** (1975) 760–762

[25] E. Witten and D. I. Olive, *Supersymmetry Algebras That Include Topological Charges*, Phys. Lett. **B78** (1978) 97

[26] C. M. Hull, *Timelike T-duality, de Sitter space, large N gauge theories and topological field theory*, JHEP **07** (1998) 021, hep-th/9806146

[27] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, *SUPERSTRINGS AND SOLITONS*, Nucl. Phys. **B340** (1990) 33–55

[28] J. X. Lu, *ADM masses for black strings and p-branes*, Phys. Lett. **B313** (1993) 29–34, hep-th/9304159

[29] A. Dabholkar, *Microstates of non-supersymmetric black holes*, Phys. Lett. **B402** (1997) 53–58, hep-th/9702050

[30] H. Lu, S. Mukherji and C. N. Pope, *From p-branes to cosmology*, Int. J. Mod. Phys. **A14** (1999) 4121–4142, hep-th/9612224

[31] G. T. Horowitz and A. Strominger, *Black strings and P-branes*, Nucl. Phys. **B360** (1991) 197–209

[32] M. J. Duff, H. Lu and C. N. Pope, *The black branes of M-theory*, Phys. Lett. **B382** (1996) 73–80, hep-th/9604052

[33] M. Kruczenski, R. C. Myers and A. W. Peet, *Supergravity S-branes*, JHEP **05** (2002) 039, hep-th/0204144

[34] C.-M. Chen, D. V. Gal'tsov and M. Gutperle, *S-brane solutions in supergravity theories*, Phys. Rev. **D66** (2002) 024043, hep-th/0204071

[35] S. Arapoglu, N. S. Deger, A. Kaya, E. Sezgin and P. Sundell, *Multi-spin giants*, Phys. Rev. **D69** (2004) 106006, hep-th/0312191

[36] N. Ohta, *Intersection rules for S-branes*, Phys. Lett. **B558** (2003) 213–220, hep-th/0301095

[37] A. Lukas, B. A. Ovrut and D. Waldram, *String and M-theory cosmological solutions with Ramond forms*, Nucl. Phys. **B495** (1997) 365–399, hep-th/9610238

[38] G. Jones, A. Maloney and A. Strominger, *Non-singular solutions for S-branes*, Phys. Rev. **D69** (2004) 126008, hep-th/0403050

[39] J. E. Wang, *Twisting S-branes*, JHEP **05** (2004) 066, hep-th/0403094

[40] E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren and T. Van Riet, *Non-extremal D-instantons and the AdS/CFT correspondence*, JHEP **01** (2006) 061, hep-th/0510048

[41] S. Bhattacharya and S. Roy, *Time dependent supergravity solutions in arbitrary dimensions*, JHEP **12** (2003) 015, hep-th/0309202

[42] E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109

[43] A. Strominger, *The dS/CFT correspondence*, JHEP **10** (2001) 034, hep-th/0106113

[44] C. Pope, *Kaluza-Klein Theory*, Lecture Notes see: `http://faculty.physics.tamu.edu/pope/ihplec.ps`

[45] T. M. Van Riet, *Cosmic Acceleration in Kaluza-Klein Supergravity.* PhD thesis, Rijksuniversiteit Groningen, 2007.

[46] M. S. Bremer, M. J. Duff, H. Lu, C. N. Pope and K. S. Stelle, *Instanton cosmology and domain walls from M-theory and string theory*, Nucl. Phys. **B543** (1999) 321–364, `hep-th/9807051`

[47] P. G. O. Freund and M. A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97** (1980) 233–235

[48] C.-M. Chen, P.-M. Ho, I. P. Neupane, N. Ohta and J. E. Wang, *Hyperbolic space cosmologies*, JHEP **10** (2003) 058, `hep-th/0306291`

[49] C.-M. Chen, P.-M. Ho, I. P. Neupane and J. E. Wang, *A note on acceleration from product space compactification*, JHEP **07** (2003) 017, `hep-th/0304177`

[50] M. N. R. Wohlfarth, *Inflationary cosmologies from compactification?*, Phys. Rev. **D69** (2004) 066002, `hep-th/0307179`

[51] M. Nakahara, *Geometry, topology and physics*, IOP Publishing (2003)

[52] D. B. Westra and W. Chemissany, *Coset symmetries in dimensionally reduced heterotic supergravity*, JHEP **02** (2006) 004, `hep-th/0510137`

[53] M. Trigiante, *Dualities in supergravity and solvable Lie algebras*, `hep-th/9801144`

[54] M. Caselle and U. Magnea, *Random matrix theory and symmetric spaces*, Phys. Rept. **394** (2004) 41–156, `cond-mat/0304363`

[55] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (1978)

[56] P. Breitenlohner, D. Maison and G. W. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, Commun. Math. Phys. **120** (1988) 295

[57] A. Keurentjes, *Poincare duality and G+++ algebras*, `hep-th/0510212`

[58] C. M. Hull and B. Julia, *Duality and moduli spaces for time-like reductions*, Nucl. Phys. **B534** (1998) 250–260, `hep-th/9803239`

[59] C. M. Hull and R. R. Khuri, *Branes, times and dualities*, Nucl. Phys. **B536** (1998) 219–244, `hep-th/9808069`

[60] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, `hep-th/9510017`

[61] D. V. Gal'tsov and O. A. Rytchkov, *Generating branes via sigma-models*, Phys. Rev. **D58** (1998) 122001, `hep-th/9801160`

[62] P. Fré *et al.*, *Cosmological backgrounds of superstring theory and solvable algebras: Oxidation and branes*, Nucl. Phys. **B685** (2004) 3–64, `hep-th/0309237`

[63] M. Cvetic, S. S. Gubser, H. Lu and C. N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and Coulomb branch of gauge theories*, Phys. Rev. **D62** (2000) 086003, `hep-th/9909121`

[64] P. K. Townsend and M. N. R. Wohlfarth, *Accelerating cosmologies from compactification*, Phys. Rev. Lett. **91** (2003) 061302, `hep-th/0303097`

[65] N. Ohta, *Accelerating cosmologies from S-branes*, Phys. Rev. Lett. **91** (2003) 061303, `hep-th/0303238`

[66] M. Cvetic and H. H. Soleng, *Naked singularities in dilatonic domain wall space times*, Phys. Rev. **D51** (1995) 5768–5784, hep-th/9411170

[67] K. Skenderis and P. K. Townsend, *Pseudo-supersymmetry and the domain-wall / cosmology correspondence*, hep-th/0610253

[68] W. Chemissany, A. Ploegh and T. Van Riet, *A note on scaling cosmologies, geodesic motion and pseudo- susy*, arXiv:0704.1653 [hep-th]

[69] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, *Generating Geodesic Flows and Supergravity Solutions*, To appear

[70] J. Rosseel, T. Van Riet and D. B. Westra, *Scaling cosmologies of $N = 8$ gauged supergravity*, Class. Quant. Grav. **24** (2007) 2139–2152, hep-th/0610143

[71] A. W. Knapp, *Lie groups beyond an introduction*, Birkhäuser, Second Edition (2002)

[72] P. K. Townsend, *Cosmic acceleration and M-theory*, hep-th/0308149

[73] O. Obregon, H. Quevedo and M. P. Ryan, *Regular non-twisting S-branes*, JHEP **07** (2004) 005, hep-th/0406116

[74] B. Janssen, P. Smyth, T. Van Riet and B. Vercnocke, *A first-order formalism for timelike and spacelike brane solutions*, arXiv:0712.2808 [hep-th]

[75] H. Lu and J. F. Vazquez-Poritz, *Non-singular twisted S-branes from rotating branes*, JHEP **07** (2004) 050, hep-th/0403248

[76] P. Fre and A. Sorin, *Integrability of supergravity billiards and the generalized Toda lattice equation*, Nucl. Phys. **B733** (2006) 334–355, hep-th/0510156

[77] U. H. Danielsson, *Lectures on string theory and cosmology*, Class. Quant. Grav. **22** (2005) S1–S40, hep-th/0409274

[78] S. M. Carroll, *TASI lectures: Cosmology for string theorists*, hep-th/0011110

[79] J. Hartong, A. Ploegh, T. Van Riet and D. B. Westra, *Dynamics of generalized assisted inflation*, Class. Quant. Grav. **23** (2006) 4593–4614, gr-qc/0602077

[80] R. Wald, *General Relativity.* The University of Chicago Press, 1984.

[81] N. Ohta, *Accelerating cosmologies and inflation from M / superstring theories*, Int. J. Mod. Phys. **A20** (2005) 1–40, hep-th/0411230

[82] E. Bergshoeff, A. Collinucci, U. Gran, M. Nielsen and D. Roest, *Transient quintessence from group manifold reductions or how all roads lead to Rome*, Class. Quant. Grav. **21** (2004) 1947–1970, hep-th/0312102

[83] J. L. P. Karthauser and P. M. Saffin, *The dynamics of coset dimensional reduction*, hep-th/0601230

[84] A. M. Green and J. E. Lidsey, *Assisted dynamics of multi-scalar field cosmologies*, Phys. Rev. **D61** (2000) 067301, astro-ph/9907223

[85] A. Mazumdar, S. Panda and A. Perez-Lorenzana, *Assisted inflation via tachyon condensation*, Nucl. Phys. **B614** (2001) 101–116, hep-ph/0107058

[86] K. Becker, M. Becker and A. Krause, *M-theory inflation from multi M5-brane dynamics*, Nucl. Phys. **B715** (2005) 349–371, hep-th/0501130

[87] J. Ward, *Instantons, assisted inflation and heterotic M-theory*, hep-th/0511079

[88] A. Berndsen, T. Biswas and J. M. Cline, *Moduli stabilization in brane gas cosmology with superpotentials*, JCAP **0508** (2005) 012, hep-th/0505151

[89] A. A. Coley, *Dynamical systems in cosmology*, gr-qc/9910074

[90] I. P. C. Heard and D. Wands, *Cosmology with positive and negative exponential potentials*, Class. Quant. Grav. **19** (2002) 5435–5448, gr-qc/0206085

[91] R. J. van den Hoogen and L. Filion, *Stability analysis of multiple scalar field cosmologies with matter*, Class. Quant. Grav. **17** (2000) 1815–1825

[92] K. A. Malik and D. Wands, *Dynamics of assisted inflation*, Phys. Rev. **D59** (1999) 123501, astro-ph/9812204

[93] E. J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, Int. J. Mod. Phys. **D15** (2006) 1753–1936, hep-th/0603057

[94] M. de Roo, D. B. Westra and S. Panda, *Gauging CSO groups in $N = 4$ supergravity*, JHEP **09** (2006) 011, hep-th/0606282

[95] T. Buchert, J. Larena and J.-M. Alimi, *Correspondence between kinematical backreaction and scalar field cosmologies: The 'morphon field'*, Class. Quant. Grav. **23** (2006) 6379–6408, gr-qc/0606020

[96] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, Phys. Rev. **D15** (1977) 2752–2756

[97] D. Z. Freedman, C. Nunez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, Phys. Rev. **D69** (2004) 104027, hep-th/0312055

[98] K. Skenderis and P. K. Townsend, *Hidden supersymmetry of domain walls and cosmologies*, Phys. Rev. Lett. **96** (2006) 191301, hep-th/0602260

[99] J. Sonner and P. K. Townsend, *Axion-dilaton domain walls and fake supergravity*, hep-th/0703276

[100] A. Celi, A. Ceresole, G. Dall'Agata, A. Van Proeyen and M. Zagermann, *On the fakeness of fake supergravity*, Phys. Rev. **D71** (2005) 045009, hep-th/0410126

[101] A. J. Tolley and D. H. Wesley, *Scale-invariance in expanding and contracting universes from two-field models*, hep-th/0703101

[102] J. L. P. Karthauser and P. M. Saffin, *Scaling solutions and geodesics in moduli space*, Class. Quant. Grav. **23** (2006) 4615–4624, hep-th/0604046

[103] J. Sonner and P. K. Townsend, *Recurrent acceleration in dilaton-axion cosmology*, Phys. Rev. **D74** (2006) 103508, hep-th/0608068

[104] A. R. Liddle, A. Mazumdar and F. E. Schunck, *Assisted inflation*, Phys. Rev. **D58** (1998) 061301, astro-ph/9804177

[105] A. Collinucci, M. Nielsen and T. Van Riet, *Scalar cosmology with multi-exponential potentials*, Class. Quant. Grav. **22** (2005) 1269–1288, hep-th/0407047

[106] I. Bakas and K. Sfetsos, *States and curves of five-dimensional gauged supergravity*, Nucl. Phys. **B573** (2000) 768–810, hep-th/9909041

[107] K. Skenderis and P. K. Townsend, *Gravitational stability and renormalization-group flow*, Phys. Lett. **B468** (1999) 46–51, hep-th/9909070

[108] I. Bakas, A. Brandhuber and K. Sfetsos, *Domain walls of gauged supergravity, M-branes, and algebraic curves*, Adv. Theor. Math. Phys. **3** (1999) 1657–1719, hep-th/9912132

[109] C. M. Miller, K. Schalm and E. J. Weinberg, *Nonextremal black holes are BPS*, Phys. Rev. **D76** (2007) 044001, hep-th/0612308

[110] P. K. Townsend, *From Wave Geometry to Fake Supergravity*, arXiv:0710.5709 [hep-th]

[111] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel and D. Van den Bleeken, *Pseudo-supersymmetry and a tale of alternate realities*, arXiv:0704.3559 [hep-th]

[112] K. Skenderis and P. K. Townsend, *Hamilton-Jacobi for domain walls and cosmologies*, hep-th/0609056

[113] K. Skenderis, P. K. Townsend and A. Van Proeyen, *Domain-wall/Cosmology correspondence in adS/dS supergravity*, JHEP **08** (2007) 036, arXiv:0704.3918 [hep-th]

[114] P. K. Townsend, *Hamilton-Jacobi Mechanics from Pseudo-Supersymmetry*, arXiv:0710.5178 [hep-th]

[115] R. Rajaraman, *SOLITONS AND INSTANTONS. AN INTRODUCTION TO SOLITONS AND INSTANTONS IN QUANTUM FIELD THEORY*, Amsterdam, Netherlands: North-holland (1982) 409p

[116] C. M. Hull, *Duality and the signature of space-time*, JHEP **11** (1998) 017, hep-th/9807127

[117] S. Vaula, *On the construction of variant supergravities in D = 11, D = 10*, JHEP **11** (2002) 024, hep-th/0207080

[118] H. Nishino and J. Gates, S. James, *The *report*, Class. Quant. Grav. **17** (2000) 2139–2148, hep-th/9908136

[119] E. Bergshoeff and A. Van Proeyen, *The many faces of OSp(1—32)*, Class. Quant. Grav. **17** (2000) 3277–3304, hep-th/0003261

[120] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, *New formulations of D = 10 supersymmetry and D8 - O8 domain walls*, Class. Quant. Grav. **18** (2001) 3359–3382, hep-th/0103233

[121] E. Bergshoeff, T. de Wit, U. Gran, R. Linares and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions*, JHEP **10** (2002) 061, hep-th/0209205

[122] D. Roest, *M-theory and gauged supergravities*, Fortsch. Phys. **53** (2005) 119–230, hep-th/0408175

[123] E. Bergshoeff, A. Collinucci, U. Gran, D. Roest and S. Vandoren, *Non-extremal D-instantons*, JHEP **10** (2004) 031, hep-th/0406038

[124] J. Lukierski and A. Nowicki, *All possible de Sitter superalgebras and the presence of ghosts*, Phys. Lett. **B151** (1985) 382

[125] K. Pilch, P. van Nieuwenhuizen and M. F. Sohnius, *de Sitter superalgebras and supergravity*, Commun. Math. Phys. **98** (1985) 105

[126] B. de Wit and A. Zwartkruis, *SU(2,2/1,1) Supergravity and N=2 supersymmetry with arbitrary cosmological constant*, Class. Quant. Grav. **4** (1987) L59

[127] I. Y. Park, C. N. Pope and A. Sadrzadeh, *AdS braneworld Kaluza-Klein reduction*, Class. Quant. Grav. **19** (2002) 6237–6258, hep-th/0110238

[128] K. Behrndt and M. Cvetic, *Time-dependent backgrounds from supergravity with gauged non-compact R-symmetry*, Class. Quant. Grav. **20** (2003) 4177–4194, hep-th/0303266

[129] R. D'Auria and S. Vaula, *D = 6, N = 2, F(4)-supergravity with supersymmetric de Sitter background*, JHEP **09** (2002) 057, hep-th/0203074

[130] H. Lu and J. F. Vazquez-Poritz, *From de Sitter to de Sitter*, JCAP **0402** (2004) 004, hep-th/0305250

[131] J. T. Liu, W. A. Sabra and W. Y. Wen, *Consistent reductions of IIB*/M* theory and de Sitter supergravity*, JHEP **01** (2004) 007, hep-th/0304253

[132] S. Ferrara, *Spinors, superalgebras and the signature of space-time*, hep-th/0101123

[133] M. B. Green and M. Gutperle, *Effects of D-instantons*, Nucl. Phys. **B498** (1997) 195–227, hep-th/9701093

[134] G. W. Gibbons, M. B. Green and M. J. Perry, *Instantons and Seven-Branes in Type IIB Superstring Theory*, Phys. Lett. **B370** (1996) 37–44, hep-th/9511080

[135] M. Gutperle and W. Sabra, *Instantons and wormholes in Minkowski and (A)dS spaces*, Nucl. Phys. **B647** (2002) 344–356, hep-th/0206153

[136] N. Arkani-Hamed, J. Orgera and J. Polchinski, *Euclidean Wormholes in String Theory*, arXiv:0705.2768 [hep-th]

[137] A. Collinucci, *Instantons and Cosmologies in String Theory.* PhD thesis, Rijksuniversiteit Groningen, 2004.

[138] C.-S. Chu, P.-M. Ho and Y.-Y. Wu, *D-instanton in AdS(5) and instanton in SYM(4)*, Nucl. Phys. **B541** (1999) 179–194, hep-th/9806103

[139] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory*, JHEP **08** (1998) 013, hep-th/9807033

[140] T. Banks and M. B. Green, *Non-perturbative effects in AdS(5)* $\times$ S^5 *string theory and* $d = 4$ *SUSY Yang-Mills*, JHEP **05** (1998) 002, hep-th/9804170

[141] A. Van Proeyen, *Tools for supersymmetry*, hep-th/9910030

[142] R. N. Cahn, *Semisimple Lie algebras and their representations*, Menlo Park, Usa: Benjamin/cummings (1984) 158 P. (Frontiers In Physics, 59)

Nederlandse Samenvatting

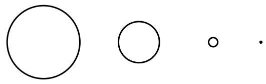
Dit proefschrift heeft als doel een methode te ontwikkelen waarmee braanoplossingen[1] eenvoudiger kunnen worden geformuleerd. Hieronder wordt getracht deze zin uit te leggen op een zo'n simpel mogelijke manier en tevens een beknopt overzicht te geven van wat er in dit proefschrift behandeld wordt. We beginnen met een korte introductie in de snarentheorie en gaan dan over op de aspecten die in dit proefschrift worden behandeld.

In de elementaire deeltjes fysica worden de "bouwstenen"van de materie om ons heen beschreven. Men zoomt als het ware met een sterke microscoop in op een materiaal en kijkt waaruit dat bestaat. Een voorbeeld van een bouwsteen is het elektron. In de natuurkunde is de microscoop een deeltjesversneller. De krachtigste versneller, de LHC, gaat binnenkort van start in CERN nabij Genève. Door de botsingen van deeltjes te bestuderen kan men veel leren over de bouwstenen van onze wereld.

Naast deze bouwstenen zijn er ook krachten. Er zijn op dit moment vier krachten bekend in ons universum. Laten we beginnen met de sterke en zwakke kernkracht en het elektromagnetisme. De eerste twee krachten spelen bijvoorbeeld een rol bij het (in)stabiel zijn van de atoomkern. Elektromagnetisme speelt een rol bij bijvoorbeeld elektriciteit en magnetisme. De bouwstenen en deze drie krachten zijn in de loop van de twintigste eeuw in één theorie samengevat. Dit model heet het standaardmodel. Gerard 't Hooft en Martinus Veltman hebben een belangrijke rol gespeeld bij het consistent maken van dit model. Hiervoor ontvingen zij in 1999 de Nobelprijs voor de natuurkunde.

De vierde en meest bekende kracht is de zwaartekracht. In 1915 gaf Albert Einstein een goede verklaring voor dit verschijnsel via de algemene relativiteitstheorie. Eén van de grote uitdagingen in de moderne natuurkunde is het ontwikkelen van één model dat alle vier krachten plus de elementaire deeltjes samenvat. Tot nu toe is men niet in staat geweest om een succesvolle theorie op te stellen die uitgaat van puntdeeltjes. Hiermee wordt bedoeld dat de bouwstenen van het model geen interne structuur

[1] Het Engelse woord membrane wordt in het Nederlands vertaald als membraan. Vandaar dat hier het woord braan gebruikt wordt voor het Engelse woord brane.

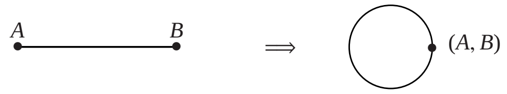


Figuur D.0.1: Hier zien we vier keer dezelfde cirkel met afnemende straal. In het vierde plaatje is het verschil met een punt niet meer te zien.

hebben zoals bij een punt. Nu komt de snarentheorie om de hoek kijken. Zoals de naam al zegt zijn de "bouwstenen"van deze theorie geen puntdeeltjes maar snaren. Waar een bewegend puntdeeltje een lijn in de tijd beschrijft, beschrijft een bewegende snaar een twee-dimensionale oppervlakte. Immers de snaar zelf is één-dimensionaal (denk aan een lijn) maar als de snaar beweegt kun je je dit voorstellen als een twee-dimensionaal oppervlakte in de ruimte-tijd. De ruimte-tijd kun je je voorstellen als een kaart waarbij ook de tijd is aangegeven.

Het idee achter de snarentheorie is, dat alle deeltjes en krachten opgevat kunnen worden als verschillende trillingen van de snaar. Als de snaar op de juiste manier trilt gedraagt het zich bijvoorbeeld als een elektron. Als de snaartjes maar klein genoeg zijn, lijkt het voor ons als waarnemers net alsof het puntdeeltjes zijn. We moeten heel goed inzoomen willen we een snaartje zien in plaats van een puntdeeltje. Denk bijvoorbeeld aan een cirkel, dit is een voorbeeld van een gesloten snaar. Als we de straal van de cirkel heel erg klein kiezen lijkt het een punt, zie figuur D.0.1. In de natuurkunde geldt, dat hoe meer men wil inzoomen des te meer moeite men moet doen, oftewel des te meer energie men in het systeem moet stoppen. Of de LHC in staat is om de snaren te detecteren is maar zeer de vraag. De karakteristieke lengte van een snaar is ongeveer 10^{-35} meter, terwijl de LHC niet verder kan "inzoomen"dan 10^{-19} meter (waar bijvoorbeeld $10^{-1} = 0.1$, $10^{-2} = 0.01$ enzovoort).

Hoe simpel deze uitbreiding op het eerste gezicht ook lijkt, een snaar in plaats van een punt, de fysische consequenties zijn groot. We noemen een aantal in het oogspringende eigenschappen. Wil de snarentheorie werken dan vereist de theorie een tien-dimensionale ruimte! Ter vergelijking: Wij kunnen slechts drie ruimtelijke dimensies zien (denk bijvoorbeeld aan een kubus). Samen met de tijd spreken we van een vier-dimensionale ruimte-tijd. Deze ogenschijnlijke tegenstelling tussen vier en tien kan worden opgelost door te "compactificeren". Laten we dit uitleggen aan de hand van een voorbeeld. We nemen aan dat de overige zes dimensies heel erg klein zijn. Dit kan bijvoorbeeld door aan te nemen dat elk van de zes dimensies de vorm heeft van een cirkel, zie figuur D.0.2. Als we de straal erg klein kiezen dan zullen deze extra dimensies niet te zien zijn voor ons, denk aan figuur D.0.1. De sterkste versneller op dit moment kan lengtes tot 10^{-19} meter detecteren. Willen we geen tegenspraak hebben dan betekent dit dat de grootst mogelijke lengte van de extra



Figuur D.0.2: De linker lijn stelt een extra dimensie voor. Om ervoor te zorgen dat we deze dimensie niet kunnen zien nemen we aan dat deze dimensie de vorm heeft van een (hele kleine) cirkel zoals weer gegeven in het rechter plaatje. We verbinden hierbij de punten A en B met elkaar.

dimensies 10^{-19} meter is[2].

De oplettende lezer zal op dit moment zeggen: "Waarom zou ik stoppen bij een snaar?". We zouden net zo goed als bouwsteen een membraan kunnen gebruiken. Dit is een twee-dimensionaal object, denk bijvoorbeeld aan een vel papier. In een tien-dimensionale wereld kunnen we dit concept snel uitbreiden tot het idee van een braan. Dit zijn hoger-dimensionale versies van de membranen. De algemene naamgeving is p-braan. De p geeft het aantal ruimtelijke dimensies aan van de braan. We nemen aan dat ook de tijd onderdeel is van de braan. De totale oppervlakte van de braan bestaat dus uit $(p+1)$ dimensies[3]. Kortom een p-braan stelt een $(p+1)$-dimensionaal oppervlakte voor. Dit oppervlakte noemen we het *wereldvolume* van de braan. Bijvoorbeeld een 0-braan is een puntdeeltje (de tijd vormt een één-dimensionaal wereldvolume) en een 1-braan is een snaar (samen met de tijd geeft dit een twee-dimensionaal wereldvolume). Het mooie van wiskunde is dat het werken met tien dimensies geen probleem is. Het is niet fundamenteel anders dan werken met "slechts" vier dimensies. Een voorstelling hiervan in je hoofd maken is daarentegen een heel ander verhaal...

Het interessante van de snarentheorie is dat deze hoger-dimensionale branen automatisch verschijnen in de theorie als men deze analyseert. In zekere zin is de naam snarentheorie dus onjuist! De reden dat we beginnen met snaren in plaats van branen is dat men nog niet goed weet hoe om te gaan met branen. Dit is een openstaand probleem in de snarentheorie.

Maar ook de snarentheorie zelf is nog niet volledig ontwikkeld. De zogeheten perturbatieve beschrijving van de theorie heeft zich in de afgelopen twintig jaar flink ontwikkeld. Met het woord perturbatief bedoelen we dat er een kleine koppelingsconstante is. Dit betekent ruwweg dat de interacties tussen de objecten in de theorie zwak zijn. Dankzij de ontwikkelingen van de perturbatieve snarentheorie weten we dat de hierboven beschreven branen onderdeel van de theorie zijn. Deze branen zijn als het

[2]Er bestaan ook andere manieren om extra dimensies te hebben die we niet zien zonder aan te nemen dat ze klein zijn. Dit kan bijvoorbeeld door zogeheten braan-wereld modellen.

[3]We hoeven overigens de tijd niet perse als onderdeel van de braan te zien. In dat geval is de naamgeving nog steeds zo dat een p-braan een $(p+1)$-dimensionaal oppervlakte voorstelt. We noemen zo'n braan een Sp-braan.

ware de niet-perturbatieve kant van de snarentheorie. Als de koppelingsconstante groot is en de interacties dus sterk zijn, spelen branen een belangrijke rol.

Een ander intrigerend aspect is dat in het niet-perturbatieve regime van de theorie een extra dimensie ontstaat. Dit kan men zich het beste voorstellen aan de hand van een cirkel. De koppelingsconstante bepaalt de grootte van de straal. Als de koppelingsconstante groter wordt, wordt de straal langer. Op een gegeven moment kan men dus een extra dimensie "zien". Denk hierbij aan figuur D.0.1 waarbij je nu van rechts naar links kijkt. Door het ontstaan van de extra dimensie hebben we dus niet een tien- maar een elf-dimensionale theorie. Deze elf-dimensionale theorie wordt M-theorie genoemd. Er is weinig bekend over deze theorie[4], alleen dat de bouwstenen geen snaren zijn maar 2- en 5-branen.

Echter het feit dat we de niet-perturbatieve kant van de snarentheorie niet kennen, hoeft niet perse een probleem te zijn. De huidige versnellers zijn bij lange na niet krachtig genoeg om de energieschaal van snaren te benaderen (vergelijk maar de lengtes 10^{-35} meter en 10^{-19} meter). Het is dus belangrijk om de snarentheorie (en M-theorie) bij lage energie te bestuderen. Het blijkt dat er vijf verschillende lage energielimieten zijn van de snarentheorie. Het idee is dat al deze vijf theorieën op een andere manier kijken naar M-theorie. Net zoals bijvoorbeeld de zes kanten van een dobbelsteen allemaal deel uit maken van dezelfde dobbelsteen.

Deze vijf theorieën zijn voorbeelden van de zogeheten superzwaartekracht. Dit is een uitbreiding van de algemene relativiteitstheorie van Einstein met extra deeltjes. De theorie heeft verder een extra symmetrie genaamd supersymmetrie. Dit is een symmetrie die fermionen (zoals bijvoorbeeld het elektron) relateert aan bosonen (bijvoorbeeld het graviton wat de zwaartekracht overbrengt). Alle deeltjes in de natuur zijn of bosonen of fermionen. Dit verklaart ook de naam, het combineert *super*symmetrie met de zwaartekracht. In dit proefschrift zullen we ons beperken tot deze lage energielimieten.

Een ander interessant aspect van snarentheorie is dat het geïntroduceerd wordt als een theorie die de wereld beschrijft op het allerkleinste niveau. Oftewel bij hele hoge energieschalen. Wanneer bevond ons universum zich in een hoge energetisch toestand? Observaties in de sterrenkunde laten zien dat ons heelal steeds groter wordt. Gaan we dus ver terug in de tijd dan is de voor de hand liggende conclusie dat er ooit een moment is geweest waarop alle materie in ons heelal op één punt samenkwam. Dit moment wordt de Big Bang of oerknal genoemd. Het allereerste moment van ons universum moet dus beschreven worden door de snarentheorie! Ook al is de snarentheorie in eerste instantie een theorie die het allerkleinste beschrijft, uiteindelijk heeft het dus ook veel te zeggen over het heelal als geheel. Dit is ook op een andere manier te zien. De snarentheorie tracht alle vier de krachten die wij kennen in één theorie samen te vatten. Zwaartekracht is dan automatisch een onderdeel van de theorie. Op grote afstanden is het juist de zwaartekracht die regeert en dus de

[4] Zelfs waar de letter M voorstaat is niet echt bekend.

evolutie van ons universum bepaalt. Dit wordt bestudeerd in de kosmologie.

In dit proefschrift zal de nadruk liggen op het bestuderen van de branen die voorkomen in de lage energielimieten van de snarentheorie. Het doel is om een methode te ontwikkelen die het vinden van expliciete braanoplossingen eenvoudiger maakt. In principe kunnen we de vergelijkingen, die volgen uit de lage energielimiet van snarentheorie, proberen op te lossen. Echter die vergelijkingen zijn erg moeilijk om exact op te lossen. Zoals vaker in de natuurkunde zullen we daarom gebruik maken van symmetrieën om het probleem te simplificeren.

Laten we dit uitleggen aan de hand van een concreet voorbeeld. Denk aan een vel papier. Dit is een twee-dimensionaal object oftewel een 2-braan in onze taal. Zoals gezegd, dit twee-dimensionale oppervlakte noemen we het wereldvolume van de braan. Als we voor het gemak even de tijd vergeten kunnen we een derde richting loodrecht op dit papier voorstellen. Te samen met het twee-dimensionale wereldvolume hebben we een drie-dimensionale ruimte. Wij gaan op zoek naar braanoplossingen die alleen afhangen van de derde dimensie loodrecht op het papier. De twee dimensies van het wereldvolume spelen hierdoor geen rol. Waar we ons op het wereldvolume ook bevinden de oplossing gedraagt zich hetzelfde, want alleen de derde richting heeft invloed op deze braanoplossing. Als we het vel papier samendrukken tot een klein propje is de situatie dus eigenlijk nog hetzelfde! Zoals gezegd, dit noemen we compactificeren of het *oprollen* van dimensies. Het probleem is hierdoor ineens een stuk eenvoudiger op te lossen. In plaats van na te gaan hoe een vel papier zich gedraagt in de drie-dimensionale ruimte hebben we slechts een punt dat zich in één dimensie kan voort bewegen! Deze aanpak van het oprollen van dimensies van het wereldvolume (het vel papier) is een belangrijk onderdeel van de techniek die in dit proefschrift wordt toegepast. De effectieve beschrijving van de braan (vel papier) is dus weer een puntdeeltje. Dit verklaart de titel van het proefschrift, namelijk “de deeltjes dynamica van branen”. Hetzelfde kunnen we doen met alle andere branen.

Omdat we met deze truc het hele wereldvolume hebben laten verdwijnen spreken we van een (-1)-braan. Immers onze telling was zo dat een p-braan een $(p+1)$-dimensionaal wereldvolume vormt en in ons geval is er geen wereldvolume meer (dat wil zeggen $0 = -1 + 1$ oftewel $p = -1$).

We moeten nu onderscheid maken naar waar de tijd zich bevindt. Als de tijd deel uitmaakt van het oorspronkelijke wereldvolume noemen we het een (-1)-braan of instanton. Merk op dat na het oprollen van het wereldvolume ook de tijd verdwenen is! Hoe raar dit ook mag lijken voor ons, wiskundig is er (zoals zo vaak) niks aan de hand. Als de tijd geen deel uitmaakt van het (oorspronkelijke) wereldvolume noemen we het een S(-1)-braan. Na het oprollen van het wereldvolume is de tijd dus nog wel aanwezig. De oorspronkelijke braan heet een Sp-braan. Het verschil met een p-braan is dus dat de tijd geen deel uitmaakt van het wereldvolume (zie ook voetnoot 3 op pagina 161).

We zullen ook nog een andere vorm van oprollen (compactificeren) beschouwen.

In plaats van het oprollen van het wereldvolume beschouwen we nu (op één na) de richtingen loodrecht op de braan als irrelevant. Hiermee bedoelen we dat de oplossing weer niet afhangt van die loodrechte richtingen. Dit is een stukje moeilijker voor te stellen, maar ook dit kan men wiskundig hard maken. De effectieve theorie bestaat dan na het oprollen uit het wereldvolume en nog één loodrechte richting.

Ook hier moeten we weer onderscheid maken waar de tijd zich bevindt. Als de tijd een onderdeel is van het wereldvolume spreken we na het oprollen van domain-walls (in slecht Nederlands domein-muren). Hierbij kan men ook letterlijk aan een muur denken. Als je in een omgeving van een domain-wall zou leven, zou je, als je van links naar rechts loopt, halverwege een "muur"tegenkomen.

Als daarentegen de tijd de enige overgebleven loodrechte richting is, noemen we de oplossing een kosmologie. Het verschil met S(−1)-branen heeft te maken met een extra term in de vergelijkingen. Deze extra term noemen we de potentiaal en verschijnt standaard bij het oprollen van richtingen loodrecht op het wereldvolume. De aanwezigheid van de potentiaal maakt het probleem een stuk complexer om op te lossen.

Kortom, na het oprollen hebben we dus vier verschillende situaties. Namelijk instantonen (of (−1)-branen), S(−1)-branen, domain-walls en kosmologieën. We zien dus dat alle branen uiteindelijk gerelateerd kunnen worden aan één van deze vier verschillende mogelijkheden[5], zie ook figuur 3.5.1. Elk van deze oplossingen wordt apart geanalyseerd in dit proefschrift. Vooral de eerste twee type oplossingen (instantonen en S(−1)-branen) blijken exact te kunnen worden opgelost met deze truc die we hier hebben beschreven. Als we deze vier oplossingen hebben gevonden gaan we de opgerolde dimensies weer uitrollen en krijgen we een oplossing van de oorspronlijke p-braan!

Het proefschrift bestaat uit de volgende hoofdstukken.

In hoofdstuk 2 geven we een korte introductie in de snarentheorie. Hierbij ligt de focus op het introduceren van de branen.

In hoofdstuk 3 laten we zien hoe branen effectief te beschrijven zijn als deeltjes. Als we over het wereldvolume van een braan reduceren leidt dit tot instantonen en S(−1)-branen. Als we over de richtingen loodrecht op de braan reduceren krijgen we domain-walls of kosmologieën. Op deze manier zien we dat alle branen gerelateerd kunnen worden aan deze vier type branen.

In hoofdstuk 4 beginnen we met het oplossen van de Sp-branen. Hiervoor moeten we de S(−1)-branen oplossen. Dit doen we door middel van een genererende oplossing. Een genererende oplossing is de meest simpele oplossing die toch alle informatie van de S(−1)-braan in zich heeft. Elke andere S(−1)-braanoplossing is hieruit te verkrijgen. Daarna gaan we de reductie ongedaan maken en laten we zien hoe de bijbehorende Sp-braan eruit ziet.

[5]Er is één uitzondering. De zogeheten 7-branen kunnen niet op deze manier beschreven worden om een technische reden.

In hoofdstuk 5 bestuderen we kosmologieën. Dit betekent dat we een braan reduceren over de transversale ruimte en we krijgen dan een potentiaal. We introduceren eerst een bepaald type kosmologie, namelijk die bekend staat onder de naam "generalized assisted inflation". Daarna laten we zien hoe we ondanks de aanwezigheid van de potentiaal toch veel over het systeem te weten kunnen komen.

In hoofdstuk 6 gaan we verder met het bestuderen van de kosmologieën. We laten zien dat er een directe link is met de domain-walls. Dit heet de domain-wall / kosmologie correspondentie. Ruwweg komt dit erop neer dat voor een gegeven kosmologie men direct een domain-wall oplossing kan formuleren en *vice versa*. De nadruk zal liggen op het bestuderen van hoe deze correspondentie in detail werkt binnen superzwaartekrachten.

In hoofdstuk 7 gaan we de stap maken naar instantonen. We beperken ons hier tot de instantonen die behoren tot het zogeheten $\mathrm{SL}(p+q,\mathbb{R})/\,\mathrm{SO}(p,q)$ systeem. We leiden af wat de genererende oplossing is. Daarna bekijken we het effect van het toevoegen van een potentiaal.

In hoofdstuk 8 presenteren we de conclusies en geven we een paar suggesties voor verder onderzoek.

Er zijn ook nog vier appendices toegevoegd. Appendix A geeft veel gebruikte conventies en formules voor de algemene relativiteitstheorie. Appendix B behandelt de spinoren conventies, welke worden gebruikt in hoofdstuk 6. Appendix C geeft een korte introductie in Lie groepen en Lie algebra's. In appendix D tot slot geven we een overzicht van de gepubliceerd artikelen waarop dit proefschrift (deels) is gebaseerd.

Dankwoord

Laat ik beginnen met het bedanken van mijn werkgever, de Stichting voor Fundamenteel Onderzoek der Materie (FOM), voor de mogelijkheid om mijn PhD te volbrengen. In het bijzonder wil ik ze bedanken voor de vele scholingsmogelijkheden en conferenties in binnen- en buitenland. Ik had nooit gedacht dat ik de afgelopen jaren Uppsala, Corfu, Triëst, Napels, Stockholm, Genève, Madrid, Dubna, Valencia, Leuven en Turijn zou bezoeken!

Mijn promotor Eric Bergshoeff ben ik dankbaar voor het mij leren een goed onderzoeker te worden en voor de hulp bij het tot stand brengen van dit proefschrift en vele artikelen. Ik wil Mees de Roo bedanken voor de discussies in de afgelopen vier jaren.

Dit proefschrift zou niet tot stand zijn gekomen zonder de volgende mensen met wie ik plezierig en goed heb samengewerkt. Stefan Vandoren, Thomas Van Riet, Andrés Collinucci, Eric Bergshoeff, Dennis Westra, Jelle Hartong, Dmitri Sorokin, Wissam Chemissany, Dieter Van den Bleeken, Jan Rosseel en Mario Trigiante. Ook de leescommissie, Ana Achúcarro, Joaquim Gomis en Adel Bilal, wil ik bedanken voor het doorlezen van dit proefschrift.

Daarnaast wil ik de AIO/OIO's en de postdocs van de afdeling theoretische hoge-energiefysica te Groningen bedanken. Te weten: Andrés Collinucci, Martijn Eenink, Dennis Westra, Thomas Van Riet, Jelle Hartong, Wissam Chemissany, Teake Nutma, Sven Kerstan en Olaf Hohm. Wissam, thanks for all the Naka(hara) discussions. Thomas bedankt voor het vele samenwerken en discussies over het werk en de vele onderwerpen daar buiten. Dennis bedankt voor de hulp bij het proefschrift en gerelateerde zaken. Ook alle andere mensen van de theorie-gang bedankt voor de leuke discussies, groepsuitjes en borrels. Hetzelfde geldt voor de secretaresses, Ynske, Iris, Annelies en Sietske voor hun administratieve hulp.

Of je nu werkt om te leven of leeft om te werken in beide gevallen wil ik de volgende mensen bedanken voor de erg leuke vier (oké eigenlijk negen jaar...) die we met z'n allen in Groningen hebben doorgebracht! Christiaan "sniffmeister" Boersma bedankt voor alle woensdagavonden om onze (mijn?) kookkunsten up te graden. Niels Bos bedankt voor alle films en muziek die ik de voorbije jaren heb leren kennen. Tevens jullie twee bedankt voor het willen zijn van mijn paranimfen. Daarnaast wil ik bij deze graag Chris Ormel bedanken voor alle (politieke) discussies die we regelmatig hebben gevoerd en voor de hulp met de figuren in dit proefschrift. Verder wil ik Erwin Platen en Anneke Praagman bedanken voor de vele leuke avonden. Ook wil ik graag mijn oom Cees Swart bedanken voor het ontwerpen van het boekomslag en de uitnodigingskaart.

Tot slot wil ik mijn ouders en mijn broer Albert bedanken voor hun steun in de afgelopen jaren. Mam bedankt voor het doorlezen van de Nederlandse gedeeltes. Ik hoop de komende jaren net zoveel steun en liefde te mogen ontvangen van jullie!