


Article

Entropy Bounds: New Insights

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Abstract: In this paper we review the fundamental concepts of entropy bounds put forward by Bousso and its relation to the holographic principle. We relate covariant entropy with logarithmic distance of separation of nearby geodesics. We also give sufficient arguments to show that the origin of entropy bounds is not indeed thermodynamic, but statistical.

Keywords: holographic principle; generalized covariant entropy bound; light sheets



Citation: Horwitz, L.P.; Namboothiri, V.S.; Varma K, G.; Yahalom, A.; Strauss, Y.; Levitan, J. Entropy Bounds: New Insights. *Symmetry* **2022**, *14*, 126. <https://doi.org/10.3390/sym14010126>

Academic Editors: Andrea Lavagno and Luca Salasnick

Received: 17 November 2021

Accepted: 4 January 2022

Published: 10 January 2022

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1. Introduction

In recent years many theories have emerged with quite a lot of success such as string theory, black hole theory etc as a consequence of which many independent universal entropy bounds that holds for arbitrary systems were postulated. It seems natural that such a paradigm will have implications for cosmology [1–3]. Most entropy bounds that were discussed before could not hold well in cosmology. Thus, a radical modification of standard concepts is required by the theory which is called the holographic principle. The holographic principle requires that the degrees of freedom of a spacial region reside not in the interior as is the case in ordinary quantum field theory but of the surface of the region. Further more the number of degrees of freedom per unit area should never be greater than one per Planck units. In this paper we shall share some preliminary thoughts on how the holographic principle can be formulated in cosmology and discuss a few of its consequences. The Copernican principle implies symmetry of space and time. It turns out to be pretty straight forward and consistent with observation. The universe is spatially homogenous and isotropic (but evolves in time). In general-relativity this translates into the statement that the universe can be foliated into space-like slices in such a way that each of them are maximally symmetric. We shall now discuss about the Covariant Entropy Bound and the Space-like bound. We further establish the space time as $T \times R$, where T represents the Riemannian manifold. We shall now see differences between the covariant bound and the space like bound. The covariant bound is “connected” to the spatial volume which defines a boundary $B = dV$ with an area A being an upper bound on (S_V) , without exceeding the area of B . The space-like bound proceeds in the opposite direction. A co-dimension 2-surface B serves as the starting point for the construction of a co-dimension 1-region L which is called a light-sheet. It is constructed by the use of light rays that emanate from the surface B , as long as they are not expanding. We must remember that we always at least have two suitable directions away from B (which we soon will learn by looking at the two cones F_1 and F_2). When light rays self intersect, they expand, which in turn

implies that the light sheets will terminate at focal points. The covariant entropy bound does not have a fundamental derivation. A variety of entropy bounds have been presented in recent years, some of them with short life time as counter examples invalidate them, some of them based on conjectures, and a few of them with almost rigorous proofs to back up. We shall in the following mention a few of them. We shall begin with the space-like bound. The space like entropy bound is among the early versions of entropy bounds and includes one of the most obvious scenarios for an entropy bound. It states that the entropy contained in any spatial region does not exceed the area of the region's boundary

$$S(V) < \frac{A[B(V)]}{4} \quad (1)$$

B is the boundary of the compact portion of a hyper surface of equal time, A is the area of the boundary of V and $S(V)$ the entropy of all matter systems in V . Supposing that the Universe is a 3-dimensional flat space expanding in time one may choose a 2-sphere in time which implies that the geometry is Euclidian R^3 . It can be shown that the matter entropy

$$S_{matter}(V) = \sigma V = \frac{\sigma A^{3/2}}{6\sqrt{\pi}} \quad (2)$$

For $R > \frac{3}{4\sigma}$, one can find a volume for which the entropy bound is violated [4] The entropy content is characterized by an “average entropy density” [5].

2. The Generalised Covariant Entropy Bound

One of the open problems in cosmology is the definition of the entropy. In a previous study we have defined the entropy [6] as the logarithm of the distance between two neighboring geodesics and explored formal mathematical consequences of such a definition, which has its origin in the geodesic deviation equation that describes the nature of a one-parameter family of nearby geodesics which in the present form is constrained to systems of upto two dimensions. Systems of higher dimensions need more specific tools in order to explain the nature of a bundle of geodesics called congruence. Raychaudhuri equation is a good choice to explain congruence. In a forthcoming study, we shall elaborate on an example of an entropy bound called the covariant entropy bound (added to the mean deviation of geodesics in a congruence.)

The covariant entropy bound [7] states that matter entropy, S_L , on any such light sheet L satisfies the inequality

$$S_L < \frac{A}{4Gh} \quad (3)$$

where G is Newton's constant and A is the area of a space like 2-surface called B , from which non expanding light rays emanate but are terminated before beginning to expand. We shall however study a version of the covariant entropy bound called the generalized covariant entropy bound [8].

Let the light sheet L have generators terminated before they reach caustics (not all generators are terminated before reaching caustics). The endpoints of the generators, which do not reach the caustics, form a second space like 2-surface B' and let A' be the area of B' . The generalized covariant entropy bound is given by

$$S_L < \frac{A - A'}{4Gh} \quad (4)$$

We argue that the above inequality may imply a stronger bound on the matter entropy than the former. It is obvious that this bound is reduced to the covariant entropy bound in case of $A' = 0$. We choose the Robertson-Walker metric for a flat 3-dimensional euclidian space evolving adiabatically in time. Let A_1 be the area of a circular space-like 2-surface, B_1 , and let A_2 , be the area of an elliptic space like 2-surface, B_2 , with the same metric as the

Robertson-Walker metric above. We define the light sheet L_1 as having non expanding light rays emanating from B_1 with area A_1 and endpoints of the generators forming a second space like 2-surface B'_1 with area A'_1 . We further define a second light sheet L_2 with A_2 being the area of an elliptical space-like 2-surface B_2 and with A_1 and A_2 having equally sized areas. L_2 has non expanding light rays emanating from B_2 and endpoints at the generators forming the space-like 2-surface B'_2 with area A'_2 . If A'_2 is larger than A'_1 one has $A_1 = A_2$ and $A'_1 < A'_2$ which implies that $A_1 - A'_1 > A_2 - A'_2$ implying that the entropy bound $\frac{A_2 - A'_2}{4Gh}$ can be bounded from above by $\frac{A_1 - A'_1}{4Gh}$. We shall in a forthcoming study show examples of different light sheets in common universes and with equal sized bases A_1 and A_2 . We have recently shown that the average distance between uniformly distributed pairs of independent points inside a circle is smaller than in any ellipse of the same area. The average value $E(d)$ between any two points in a 2D region of area π can be calculated as [6]

$$E(d) = \frac{1}{\pi^2} |x - y| dA_x dA_y \quad (5)$$

3. Covariant Entropy and Geometrical Entropy Are Related

Let there be a cone with basis area A (our cross sectional area) and let there be given a square based pyramid with the basis area equal to that of the cone. If the covariant entropy bound is applied to the cone and to the pyramid identical bounds are obtained, due to the fact that the covariant entropy bound is exclusively determined by the area A . The generalized covariant entropy bound applied to the two examples also results in identical bounds (for identical time). Bousso has stated that the subtle ingredient of the covariant entropy bound lies in the concept of light-sheets. Given a surface, a light-sheet defines an adjacent region of space-time whose entropy is to be considered. A light-sheet is a way of enclosing a matter system, and the entropy on the light-sheet is given by the entropy of the matter system. The inequality (1) may give the impression, that that the matter entropy only depends on the area of the space-like 2-surface B . But in the example with a cone and a pyramid the basis area A of the two space-like 2-surfaces are identical. However, the areas of the two light sheets are different, and the two light sheets are differently shaped.

The evolution from an infinitesimal circular cross-section to an elliptical one of the same area orthogonal to the flow-lines is brought about by shear. The number of geodesics does not change by the motion of cross section along the flow. However, there is a tendency for geodesics that move along the flow to diverge due to the increase in the mean-distance between the geodesics. This implies that evolution in the presence of shear causes an increase in entropy [6,9].

The covariant entropy bound can be applied to any surface regardless of shape, topology and location [10]. We shall later utilize this characteristic to get a sharper entropy bound for the generalized covariant entropy bound by choosing a spacelike 2-surface B with a circular shape and area A generating a light sheet (a null hyper-surface) and choosing another space-like surface L with the shape of an ellipse area A and also generating a light sheet. It can be proved, that the geodesic deviation equation of Jacobi is unitarily equivalent to that of a harmonic oscillator. The expansion parameter (θ) represents the rate of growth of the cross-sectional area which is orthogonal to the congruence. Increase or decrease of this area is the direct measure of the divergence or convergence of the geodesics. The mean growth of the cross-sectional area is compatible with the mean geodesic deviation. From the paper "simple sufficient conditions for the generalized covariant entropy bound" by Bousso [11], it is obvious that A' approaching A would imply a sharper bound. The question is how it can be realized. One possibility would be if the geodesics tend to keep their mutual distances (not being strongly converging, rather on the way to be diverging) by the above mentioned transformation from a circular to an elliptical cross section resulting in the diverging tendency of the geodesics moving along the flow). It fits unto Bekensteins' bound on the entropy S of any weakly gravitating isolated matter system of mass M which is traversed by a light sheet L that has initially vanishing expansion. It also fits into

Boussos' generalized covariant entropy bound (GCEB) and with Bousso's proof it implies a generalized covariant version of the Bekenstein's bound.

4. From Bekensteins Bound to Horwitz Conjecture

We shall in the following study the more "classical" entropy bounds. Bekenstein proposed the existence of a universal bound on the entropy S of any thermodynamic system of a total energy E : $S < (\text{or equal to}) 2\pi RE$, where R is defined as the circumferential radius and $R = \sqrt{\frac{A}{4\pi}}$, with A being the area of the smallest sphere circumscribing the system. L. P. Horwitz has conjectured that it is not the form of the geodesic curves which is a measure of stability, but stability of these curves under small perturbations determines the evolution. The increase in the average distance between nearby geodesics occurs while transforming from a circular to an elliptical cross section. By looking at the effect of shear in the separation of points in a given area, it may be argued that this separation can actually be associated with greater order consistent with the stability measure on the geodesics flow. This is due to the fact that the entropy is really a many body effect in statistical mechanics and by separating the particles such as in a many body collision or particle production process, the final state is more ordered. For example, Oppenheimer thought that a nucleus after collision boils up leading to high entropy that reduces when individual particles come flying out leaving a state of separated particles with little thermodynamics in their description. One would imagine that an unstable system composed of many degrees of freedom would evolve towards equilibrium with an increase in entropy. However the state of the system passed a point where the binding forces become ineffective and the system then falls apart. Thermodynamics will no longer be applicable to the disconnected particles in the final state. This conjecture seems to be consistent with the idea that a decaying particle runs to an increase in entropy leading to decay. However the final state after decay could be almost a pure state with zero entropy. Bousso has underlined that the covariant entropy bound is time-invariant. But Bekenstein's so called spacelike projection theorem is not time invariant. It refers both to past and future explicitly because the second law of thermodynamics enters its derivation and Bekenstein's bound rests on the second law. The former seems to fit into L.P. conjecture, which also hints to the conclusion that the origin of the covariant bound is not thermodynamic, but statistical.

5. Connection to the Holographic Principle

GCEB takes into account a light sheet L some of whose generators are terminate before they reach caustics. The endpoints of these generators form a second 2-surface B' with non-zero area. The GCEB is the conjecture which states that the entropy of matter on such a light sheet satisfies the inequality $S_L < \frac{A-A'}{4Gh}$. This bound reduces to the covariant bound in the special case $A' = 0$. It can be shown that the generalized covariant bound will lead to a generalized version of the Bekenstein bound. We shall study the paper [11] which provides an introduction for the untrained reader of the 'Holographic Principle', that fits very well into Boussos' generalized covariant entropy bound and also the generalized Bekenstein bound. The covariant entropy bound is a hypothesis that connects the area of two-dimensional surfaces to the entropy content of adjacent regions. Non-expanding light-sheets emanating orthogonally from a spacelike 2-surface B of area generate a null hypersurface L known as a light sheet. The light sheet must be terminated before it can begin to expand; this usually happens when neighbouring rays intersect. The matter entropy S_L on any such light sheet L satisfies $S_L < \frac{A-A'}{4Gh}$ where G is Newton's constant. It is obvious that A' approaching A would apply a sharper bound. The question is how could it be "done". One possibility would be if the geodesics could tend to keep their mutual distances (not being strongly converging, rather on the way to be diverging). One could "hope" that the angle of the light-cone will not be an acute angle. It fits into Bekensteins' bound and also the generalized covariant entropy bound (GCEB).

6. Application to Black Holes

We will now discuss the entropy of a black hole. It is obvious that the observable properties of a black hole, such as mass, electric charge, and angular momentum, influence its entropy. It is worth noting that those three parameters only appear in the same order as “which” represents the black hole’s surface area. The fact that the event horizon area of a black hole cannot decrease in most black hole transformations is one way to understand the “area theorem” [12,13]. This increasing behaviour is reminiscent of closed-system thermodynamic entropy. As a result, it stands to reason that the black hole entropy should be proportional to the area. More information on black hole entropy can be found extensively in literature.

7. Conclusions

In this paper, we have reviewed the basic concepts of entropy bounds put forward by Bousso. We have discussed the use of light sheets for studying entropy bounds and as a way to tackle the inconsistencies with space-like entropy bounds. We have discussed the relation between covariant entropy and the geometrical entropy which in turn is the logarithmic distance of separation of nearby geodesics. We have also discussed why the origin of covariant bounds is statistical rather than thermodynamic. We can have examples where there is less thermodynamic description (less entropy) in the final state of the system. In a forthcoming study, we shall include explicit examples that use the formalism mentioned in this paper and their application to cosmology.

Author Contributions: Conceptualization, J.L.; Formal analysis, V.S.N., G.V.K., A.Y. and Y.S.; Investigation, L.P.H.; writing—review and editing, L.P.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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