Hindawi
Advances in Mathematical Physics
Volume 2019, Article ID 6765827, 10 pages
https://doi.org/10.1155/2019/6765827

Research Article

Yukawa Potential Orbital Energy: Its Relation to Orbital Mean Motion as well to the Graviton Mediating the Interaction in Celestial Bodies

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Received 25 September 2018; Revised 25 November 2018; Accepted 4 December 2018; Published 1 January 2019

Academic Editor: Eugen Radu

Research on gravitational theories involves several contemporary modified models that predict the existence of a non-Newtonian Yukawa-type correction to the classical gravitational potential. In this paper we consider a Yukawa potential and we calculate the time rate of change of the orbital energy as a function of the orbital mean motion for circular and elliptical orbits. In both cases we find that there is a logarithmic dependence of the orbital energy on the mean motion. Using that, we derive an expression for the mean motion as a function of the Yukawa orbital energy, as well as specific Yukawa potential parameters. Furthermore, various special cases are examined. Lastly, expressions for the Yukawa range λ and coupling constant α are also derived. Finally, an expression for the mass of the graviton m_{gr} mediating the interaction is calculated using the expression its Compton wavelength (i.e., the potential range λ). Numerical estimates for the mass of the graviton mediating the interaction are finally obtained at various eccentricity values and in particular at the perihelion and aphelion points of Mercury's orbit around the sun.

1. Introduction

Any scientist will agree that Einstein's general relativity theory (GR) is one of the most mathematically elegant theories invented in the human history. Even though the theory explains many physical phenomena, it is unable to shed light on the problem of the observed accelerating universe. To do that, GR introduces a cosmological constant lambda Λ as well as the so-called dark energy. Various gravitational theories exist today which try to explain the observed acceleration of the universe [1]. These gravitational theories are nonsymmetric, scalar-tensor, quantum gravitational, or $F(R)$ theories of gravity, etc. As a common denominator in the weak limit, all theories result in a Yukawa type of gravitational potential. In a paper by Chan [2], the authors put forward the idea that observational evidence for the existence of cold dark matter particles in the cores of dwarf galaxies could be explained through the interaction of a Yukawa potential. Therefore, the experimental and observational search for such deviation might result in new type of physics [3].

In a recent paper by Haranas et al. (2018), the authors examine dust particle orbits under the influence of Poynting-Robertson effect in which Newtonian gravity has been modified by a Yukawa term. Similarly, in Mukherjee and Sounda [4], the authors investigate the orbits resulting from various coupling constants α (alpha) of a Yukawa correction to the Newtonian potential. Quite often in celestial mechanics, a Yukawa-type potential is proposed to modify the Newtonian gravity [5–9] (Iorio 2007), and its effect on various gravitational, astrophysical, and orbital scenarios is examined. In this contribution, we examine the time rate of the Yukawa orbital energy of circular and elliptical orbits, and from that we derive relations for the orbital energy as a function of mean motion n of the orbiting body. To relate α

and λ to the orbital parameters of the secondary, expressions are derived that relate them to various orbital parameters. In particular, considering the expression for λ, we obtain a Lambert function that relates the mass of the graviton along the orbit of the secondary to the Yukawa parameters, eccentric anomaly, orbital energy, and eccentricity. This is done using the already derived expression for lambda and substituting it into the corresponding equation for the range of graviton λ and then solving for its mass m_{gr}. It is well known that if gravitation is propagated by a massive field, the velocity of the gravitational waves (gravitons) will depend upon their frequency, and the effective Newtonian potential will have a Yukawa form, i.e., $V(r) \propto r^{-1}e^{-r/\lambda_{gr}}$, where $\lambda_{gr} = \hbar/m_{gr}c$ is the graviton Compton wavelength. Today's research for the mass of the graviton includes both theoretical and observational work. For example, in Stavridis and Will [10] the authors try to bound the graviton mass using gravitational effects and their effect in the spin precessions of massive hole binaries. Similarly, in Mureika and Mann [11], the authors use an entropic gravity approach to estimate a bound for the mass of the graviton. Finally, in Zacharov et al. [12], the authors consider Yukawa gravity interactions of S2 star orbits near the galactic plane to improve expectations for graviton mass bounds. At this point we must say that in today's gravity research various methods have been employed in the determination of the graviton mass. Our motivation for paper emanates from the fact that this work can serve as another possible observational test in setting solar system as well as binary system bounds on graviton mass m_{gr}, where the bound depends on the mass of the source, which in this case is a sun like type of star.

2. The Yukawa Potential

Let us consider a two-body problem, where a secondary body of mass m orbits under the influence of a primary body of mass M. With the potential being central, the two-body problem can be reduced to a central-force problem, and the motion of the secondary body can be examined. The effects of gravity on the secondary in the presence of a Yukawa correction can be described by the modified potential energy per unit mass [13].

$$V(r) = -\frac{GM}{r}\left(1 + \alpha e^{-r/\lambda}\right), \tag{1}$$

In (1), r denotes the distance between the centers of the two bodies, G is the Newtonian gravitational constant, $\alpha = kK/GmM$, where k and K are the coupling constants of the new force to the bodies relative to the gravitational one, and λ is the range of this interaction (ibid. 2016). Next, let us now write down an expression of the orbital energy by making use of the virial theorem which states that for a bounded unperturbed system the following relation holds [14]:

$$\langle T\rangle = -\frac{\langle V\rangle}{2} \tag{2}$$

where T is the time average on the system's kinetic energy and V the time average of the system's potential energy. Therefore, conservation of energy implies that the total energy is equal to (ibid. 2002)

$$\mathscr{E} = \langle T\rangle + \langle V\rangle = \frac{\langle V\rangle}{2}. \tag{3}$$

Anticipating small perturbations, we can add the average value of the perturbing term to the average of the Newtonian potential to that of the perturbative term making also use that the semimajor axis a is the average value of the radial distance r along the orbit; we can further write that as

$$\mathscr{E} \cong -\frac{GMm}{2a}\left(1 + \alpha e^{-r/\lambda}\right) \tag{4}$$

Now rewriting (1) only the Yukawa orbital energy we have is

$$\mathscr{E} = -\frac{GMm}{2r}\left(\alpha e^{-r/\lambda}\right). \tag{5}$$

Next differentiating (5) with respect to time we obtain that the total time rate of the energy of the Yukawa correction is

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{GMm}{2r^2}\left[\alpha\left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}\right]\frac{dr}{dt}. \tag{6}$$

3. Circular Orbits

To examine the case of circular orbits, namely, we let the eccentricity $e_0 = 0$ and $r = a$, (6) becomes

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{GMm}{2a^2}\left[\alpha\left(1 + \frac{a}{\lambda}\right)e^{-a/\lambda}\right]\frac{da}{dt}. \tag{7}$$

Next, following Haranas et al. [9], we consider the unperturbed relative orbit of the secondary body, in this case a Keplerian ellipse. If a is the semimajor axis, e_0 is its eccentricity and n is its mean motion. On the unperturbed Keplerian ellipse, this law is expressed as [15]

$$GM = n^2a^3. \tag{8}$$

Differentiating (8) w.r.t. time, the rate of change of the mean motion can then be expressed as

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{3n}{2a}\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{9}$$

from which we obtain that

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2a}{3n}\frac{\mathrm{d}n}{\mathrm{d}t}. \tag{10}$$

Therefore, substituting (10) in (7) and simplifying, we obtain

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -\frac{GMm}{3an}\left[\alpha\left(1 + \frac{a}{\lambda}\right)e^{-a/\lambda}\right]. \tag{11}$$

Next defining the potential energy at semimajor a to be

$$\frac{1}{3}V_N(a) = -\frac{1}{3}\left(\frac{GMm}{a}\right), \tag{12}$$

the Newtonian part of the potential, (11) can written as follows:

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -\frac{V_N(a)}{3n}\left[\alpha\left(1 + \frac{a}{\lambda}\right)e^{-a/\lambda}\right]. \tag{13}$$

Integrating, we obtain the orbital energy dependence on the mean motion n to be

$$\mathscr{E} = \mathscr{E}_0 - \frac{\alpha V_N(a)}{3}\left(1+\frac{a}{\lambda}\right)e^{-a/\lambda}\ln\left(\frac{n}{n_0}\right). \quad (14)$$

On the other hand, to obtain the dependence of the mean motion on the Yukawa parameters and orbital energy $\mathscr{E}$, we use (14) and solving for n we obtain

$$n(\mathscr{E}) = n_0 e^{-3e^{a/\lambda}(\mathscr{E}-\mathscr{E}_0)/\alpha V_N(a)(1+a/\lambda)}. \quad (15)$$

In the case where the range of the potential equals the semimajor axis λ = a, (15) becomes

$$n(\mathscr{E}) = n_0 e^{-3e(\mathscr{E}-\mathscr{E}_0)/2\alpha V_N(a)}. \quad (16)$$

At this point we can see that value of the mean motion n depends on the energy difference $(\mathscr{E}-\mathscr{E}_0)$. If $(\mathscr{E} > \mathscr{E}_0)$ then $(\mathscr{E}-\mathscr{E}_0) > 0$ and therefore the exponent remains negative and the mean motion reduces exponentially with the energy difference. On the other hand, if $(\mathscr{E} < \mathscr{E}_0)$ and $(\mathscr{E}-\mathscr{E}_0) < 0$, the exponent is positive and therefore the mean motion increases. Finally, if $(\mathscr{E}-\mathscr{E}_0) = V_N(a)$, then (15) for the mean motion becomes

$$n(\mathscr{E}) = n_0 e^{-3e^{a/\lambda}/\alpha(1+a/\lambda)}, \quad (17)$$

decreasing for $\alpha > 0$ and increasing for $\alpha < 0$. Moreover, solving for the range of the potential λ we find that

$$\lambda = -\frac{a}{1+W\left[3(\mathscr{E}-\mathscr{E}_0)/e\alpha V(a)\ln(n/n_0)\right]}, \quad (18)$$

where W is the Lambert function of the indicated argument. Similarly, solving for the coupling constant α we obtain

$$\alpha = -\frac{3e^{a/\lambda}(\mathscr{E}-\mathscr{E}_0)}{(1+a/\lambda)V(a)\ln(n/n_0)}. \quad (19)$$

In the case where $(\mathscr{E}-\mathscr{E}_0) = V(a)$ we find that the range lambda λ of the Yukawa potential is equal to

$$\lambda = -\frac{a}{1+W\left[3/e\alpha\ln(n/n_0)\right]}, \quad (20)$$

and the coupling contract α becomes

$$\alpha = -\frac{3e^{a/\lambda}}{(1+a/\lambda)\ln(n/n_0)}. \quad (21)$$

The semimajor axis a is equal to the range of potential λ and according to (20) must satisfy the following equation:

$$1+W\left[\frac{3(\mathscr{E}-\mathscr{E}_0)}{e\alpha V(a)\ln(n/n_0)}\right] = -1, \quad (22)$$

from which we find that the gravitation satisfies the following equation:

$$V(a) = -\frac{3e(\mathscr{E}-\mathscr{E}_0)}{2a\ln(n/n_0)}. \quad (23)$$

Consequently, the energy difference satisfies the following equation:

$$(\mathscr{E}-\mathscr{E}_0) = -\frac{2a}{3e}V(a)\ln\left(\frac{n}{n_0}\right). \quad (24)$$

Using (24) we find that the mean motion n for $a = \lambda$ must independently satisfy the equation

$$n(\mathscr{E}) = n_0 e^{-3e(\mathscr{E}-\mathscr{E}_0)/2aV(a)}. \quad (25)$$

In particular if $(\mathscr{E}-\mathscr{E}_0) = V(a)$, (25) satisfies the equation

$$n(\mathscr{E}) = n_0 e^{-3e/2a}. \quad (26)$$

Furthermore, using (19) if $(\mathscr{E}-\mathscr{E}_0) = -\alpha V(a)e^{-a/\lambda}/3$ we find that

$$\lambda = \frac{\ln(n/n_0)}{(1-\ln(n/n_0))}a. \quad (27)$$

Using, (19) it is impossible to find an expression for α since the resulting equation takes the form

$$\left(1+\frac{a}{\lambda}\right)\ln\left(\frac{n}{n_0}\right) = 1 \quad (28)$$

Moreover, if in (28) we impose the condition λ = a, we obtain that

$$n = n_0\sqrt{e}. \quad (29)$$

4. Elliptical Orbits

Similarly, for elliptical orbits we have the following expression:

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{GMm}{2r^2}\left[\alpha\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right]\frac{dr}{dt} \quad (30)$$

where r is the orbital radial vector. Using (8) we find that

$$3n^2r^2\frac{dr}{dt} + 2nr^3\frac{dn}{dt} = 0. \quad (31)$$

The resulting simplified equation is expressed in terms of the eccentric anomaly, i.e., $r = a(1-e_0\cos E)$ where e_0 is the orbital eccentricity of the secondary and E is the eccentric anomaly; equation (31) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2r}{3n}\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{2a(1-e_0\cos E)}{3n}\frac{\mathrm{d}n}{\mathrm{d}t} \quad (32)$$

Next substituting (33) in (30) we find that

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = -\frac{GMm}{3an(1-e_0\cos E)}\left[\alpha\left(1+\frac{a(1-e_0\cos E)}{\lambda}\right)\cdot e^{-a(1-e_0\cos E)/\lambda}\right]\frac{\mathrm{d}n}{\mathrm{d}t}, \quad (33)$$

from which we obtain the following equation for the change of orbital energy w.r.t. the mean motion n:

$$\frac{d\mathscr{E}}{dn} = -\frac{GMm}{3an\left(1-e_0\cos E\right)}\left[\alpha\left(1+\frac{a\left(1-e_0\cos E\right)}{\lambda}\right) \cdot e^{-a(1-e_0\cos E)/\lambda}\right]. \tag{34}$$

Integrating and rearranging (34) with respect to n, we obtain that

$$\ln\left(\frac{n}{n_0}\right) = \frac{3\left(\mathscr{E}-\mathscr{E}_0\right)\left(1-e_0\cos E\right)e^{a(1-e_0\cos E)/\lambda}}{\alpha V_N\left(a\right)\left(1+\left(a/\lambda\right)\left(1-e_0\cos E\right)\right)}. \tag{35}$$

Solving (35) for the mean motion n we find that

$$n\left(\mathscr{E}\right) = n_0 e^{((3(1-e_0\cos E)e^{a(1-e_0\cos E)/\lambda}/\alpha V_N(a)(1+(a/\lambda)(1-e_0\cos E)))(\mathscr{E}-\mathscr{E}_0))}. \tag{36}$$

As a check of (36), let us note that if $e=0$ we can easily recover (15) for the circular orbits. In the case where $\lambda = a$, (36) simplifies in the following way:

$$n\left(\mathscr{E}\right) = n_0 e^{((3(1-e_0\cos E)/\alpha(2-e_0\cos E))((\mathscr{E}-\mathscr{E}_0)/V_N(a))e^{-(1-e_0\cos E)})}. \tag{37}$$

At this point we find that value of the mean motion n again depends on the energy difference $(\mathscr{E}-\mathscr{E}_0)$. If again $(\mathscr{E}>\mathscr{E}_0)$ then $(\mathscr{E}-\mathscr{E}_0)>0$ implies that the mean motion reduces exponentially to the energy difference. On the other hand, if $(\mathscr{E}<\mathscr{E}_0)$ and $(\mathscr{E}-\mathscr{E}_0)<0$ the motion increases exponentially. Finally, if $(\mathscr{E}-\mathscr{E}_0)=V(a)$, (36) for the mean motion becomes

$$n\left(\mathscr{E}\right) = n_0 e^{(3(1-e_0\cos E)e^{a(1-e\cos E)/\lambda}/\alpha(1+a(1-e_0\cos E)/\lambda))}. \tag{38}$$

Next, if simultaneously $(\mathscr{E}-\mathscr{E}_0)=V(a)$ and $\lambda = a$, we find that the mean motion of the orbiting body takes the following form:

$$n\left(E\right) = n_0 e^{((3(1-e_0\cos E)/\alpha(2-e_0\cos E))e^{(1-e_0\cos E)})}. \tag{39}$$

Furthermore, if $\lambda = a$ and $(\mathscr{E}-\mathscr{E}_0)=(1/3)V_N(r)\alpha e^{-r/\lambda}$ and $r=a(1-e\cos E)$ solving for the mean motion n we find that

$$n\left(E\right) = n_0 e^{(((1-e_0\cos E)/(2-e_0\cos E))e^{(1-a)(1-e_0\cos E)})}. \tag{40}$$

Finally, if $(\mathscr{E}-\mathscr{E}_0)=(1/3)V_N(r)\alpha e^{-r/\lambda}$ then we find that

$$n\left(E\right) = n_0 e^{((1-e_0\cos E)e^{(a/\lambda)(1-\lambda(e+e_0)\cos E)}/(1+(a/\lambda)(1-e_0\cos E)))}. \tag{41}$$

Going back to (36) which is the most general equation and solving for α and λ, respectively, we obtain that

$$\lambda = -\frac{a\left(1-e_0\cos E\right)}{1+W\left[-3\left(\mathscr{E}-\mathscr{E}_0\right)\left(1-e_0\cos E\right)/\alpha e V_N\left(a\right)\ln\left(n/n_0\right)\right]}. \tag{42}$$

Similarly, for α we find that

$$\alpha = \frac{3\lambda\left(\mathscr{E}-\mathscr{E}_0\right)\left(1-e_0\cos E\right)e^{a(1-e_0\cos E)/\lambda}}{V_N\left(a\right)\left(\lambda+a\left(1+e_0\cos E\right)\right)\ln\left(n/n_0\right)}. \tag{43}$$

5. The Yukawa Potential Range and the Particle Mediating the Interaction

At this point I would like to bring the readers' attention to an issue that might have some of them concerned. This is the issue of closed bounded orbits under the light of Bertrand's theorem which proves that only a Newtonian as well as a Hook type of potential results in closed bounded orbits. This is an issue that the authors of this paper will extensively deal with in a paper to come soon. As a preliminary result we say that all the applications of the Yukawa potential in our research are related to the solar system and within the boundaries of the solar system dimensions where the following condition, namely, $\lambda \gg r$, is true [16]. The parameter λ is the Compton wavelength of the exchange particle, which in the present case is a graviton. In other words, the range of the Yukawa potential is always larger than any distance r away from the primary star the sun in our case that the secondary body orbits. Our preliminary analysis of nearly circular orbits has shown that the ratio of a harmonic oscillator with frequency ω oscillating around a point situated at r to that of the orbital frequency $\dot{\theta}$ at r is given the ratio $\omega/\dot{\theta} \cong 1-(r/\sqrt{2}\lambda)^2$. In the case of the solar system with sun as the primary gravitating body producing the interaction, and since $\lambda \gg r$, we obtain to a second order approximation that $\omega/\dot{\theta} \cong 1$ and therefore the orbits are bounded and closed. As a support of our findings we refer to a paper by Mukherjee and Sounda [4]. In this paper the authors find closed and bounded orbits for various values of the coupling constant α of the Yukawa potential.

One of the primary motivations for today's interest in non-Newtonian gravity emanates from our interest to probe and understand long-range forces. This interest addresses the following question: Why a certain class of theories predict the existence of gravity - like interactions. At this point we must say that many various models in which gravity is unified with other forces have been studied in the recent years. These models are formulated along the consideration that the product μ,

$$\mu \equiv \left(\frac{f^2}{\hbar c}\right)m_N \cong 10^{-10}\,\mathrm{eV}/c^2, \tag{44}$$

determines the mass of the new field [17]. The Compton wavelength (or "range") λ associated with the field characterises the distance at which the corresponding field acts and is given by the equation below:

$$\lambda = \frac{\hbar}{mc}. \tag{45}$$

In Haranas et al. [13] and Moffat and Toth [16], the strength α of the potential near the sun in a point source scenario is given by the following equation:

$$\alpha = \frac{M_{sun}}{\left(\sqrt{M_{sun1}} + C_1'\right)^2}\left(\frac{G_\infty}{G_N} - 1\right), \quad (46)$$

where $G_\infty \cong 20G_N$ and $C_1' = 25000\sqrt{M_{sun}}$. Similarly, the range λ can be written as follows:

$$\lambda = \frac{\sqrt{M_{sun}}}{C_2'}, \quad (47)$$

and $C_2' = 6250\sqrt{M_{sun}}\,\mathrm{kpc}^{-1}$. Therefore equating (42) and (45) we obtain that the mass of the mediating of the interaction graviton is

$$m_{gr} = -\frac{\hbar}{ca(1 - e\cos E)}\left[1 + W\left(-\frac{3(1 - e_0\cos E)}{\alpha e V_N(a)\ln(n/n_0)}(\mathcal{E} - \mathcal{E}_0)\right)\right], \quad (48)$$

and finally using (46) the equation takes the form

$$m_{gr} = -\frac{\hbar}{ca(1 - e\cos E)}\left[1 + W\left(-\frac{3G_N\left(\sqrt{M_{sun}} + C_1'\right)^2(\mathcal{E} - \mathcal{E}_0)(1 - e_0\cos E)}{M_{sun}(G_\infty - G_N)V_N(a)\ln(n/n_0)}\right)\right]. \quad (49)$$

In the case of circular motion, i.e., $e = 0$, (49) simplifies to

$$m_{gr} = -\frac{\hbar}{ca}\left[1 + W\left(-\frac{3G_N\left(\sqrt{M_{sun}} + C_1'\right)^2(\mathcal{E} - \mathcal{E}_0)}{M_{sun}(G_\infty - G_N)V_N(a)\ln(n/n_0)}\right)\right]. \quad (50)$$

Similarly, in the case of circular motion and if $(\mathcal{E} - \mathcal{E}_0) = V(a)$, (50) becomes

$$m_{gr} = -\frac{\hbar}{ca}\left[1 + W\left(-\frac{3G_N\left(\sqrt{M_{sun}} + C_1'\right)^2}{M_{sun}(G_\infty - G_N)\ln(n/n_0)}\right)\right]. \quad (51)$$

Equations (48) to (51) give the mass of the graviton field mediating the interaction, i.e., a massive graviton field. According to (48) we see that the mass of graviton continuously varies along elliptical orbit with a maximum value at the perihelion/periastron point (binary star scenario). Its mass also depends on the ratio of $(\mathcal{E}-\mathcal{E}_0)/V_N(a)$ as well as on ratio of the initial to final mean orbital motion $n/n(0)$ and finally on the mass of the primary body producing the Yukawa field, Moffat and Toth [16].

6. The Extended Source Case

In the case of an extended mass distribution following Haranas et al. [13], we say that the equation of Yukawa potential must be modified. For an extended matter distribution and in relation to the MOG field, there are not presently any solutions. In this case, authors treat the problem phenomenologically, seeking to find an effective mass distribution $M(x, \boldsymbol{r})$ that could be used in (46) and (47). What this function does simply determines an "effective mass" which determines α and λ. This way, the gravitational influence of matter located at a point of distance x on the test particle located at r is calculated. Thus, the point source function can simply yield a mass proportional to the volume if the distribution is constant. Therefore, following Haranas et al. [13], we can write

$$M(\boldsymbol{x}, \boldsymbol{r}) = \int_V \rho(\overline{\boldsymbol{x}})\, e^{(-\xi|(\overline{x}-x)/(\overline{x}-x)|)}\,\mathrm{d}^3\overline{x}, \quad (52)$$

where the coefficient ξ is to be determined by observation, for example, comparison with Bullet Cluster Data. For a mass density function, $\rho(x) = \rho_0$; then we will have that $M(\boldsymbol{r}, x) \propto |\boldsymbol{r} - \boldsymbol{x}|^3$. Furthermore, in the case of a nonconstant density ρ, the coupling constant α and range of the potential λ become (ibid. 2016)

$$\alpha = \frac{M(\boldsymbol{x}, \boldsymbol{r})}{\left(\sqrt{M(\boldsymbol{x}, \boldsymbol{r})} + C_1'\right)^2}\left(\frac{G_\infty}{G_N} - 1\right), \quad (53)$$

and

$$\lambda = \frac{\sqrt{M(\boldsymbol{x}, \boldsymbol{r})}}{C_2'}. \quad (54)$$

In the case of an extended mass distribution, calculation of α and λ will result in better estimates of the two corresponding parameters and therefore a better estimate of the effect. With reference to Haranas et al. [13] in the case of an extended source we can write the gravitational potential in the following way:

$$V(\vec{r}, \lambda) = 2\pi\rho G_\infty \int_0^\infty r'^2 dr' \cdot \int_0^\pi \sin\theta' \left(\frac{e^{-\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}/\lambda}}{\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}}\right) d\theta' \quad (55)$$

which finally integrates to

$$V(\vec{r}, \lambda) = G_\infty M \begin{bmatrix} \dfrac{e^{-r/\lambda}}{r}\Phi\left(\dfrac{R}{\lambda}\right) & r \geq R \\ \dfrac{6r\lambda^2}{2R^3r^2}\left[1 - \left(1 + \dfrac{R}{r}\right)e^{-R/\lambda}\dfrac{\sinh(r/\lambda)}{r/\lambda}\right] & r \leq R \end{bmatrix} \quad (56)$$

where

$$\Phi_s(x) \equiv \frac{3}{x^3}(x\cosh x - \sinh x), \quad (57)$$

and $x = R/\lambda$, and it has the following limits:

$$\Phi_s = 1 + \frac{x^2}{10} + \frac{x^4}{280} + \ldots x \ll 1, \quad (58)$$

$$\Phi_s \cong \frac{3e^x}{2x^2} \quad x \gg 1. \quad (59)$$

In a paper by Branco et al. [18], by examining the g_{tt} component of the metric the authors identify a Newtonian potential plus a Yukawa perturbation of the form (ibid. 2014)

$$V(r) = -\frac{GM_s}{r}\left(1+\left(\frac{1}{3}-4\xi\right)A(m,R_s)e^{-r/\lambda}\right), \quad (60)$$

where $\lambda = 1/m$ is the characteristic length, $a = 1/3 - 4\xi$ is the strength of the Yukawa addition to the field, and $A(m, R_s)$ is the form factor defined as follows [18]:

$$A(m,R_s) = \frac{4\pi}{mM_s}\int_0^{R_s} r\rho(r)\sin(mr)\,\mathrm{d}r, \quad (61)$$

which integrates to

$$A(m,R_s) = 3\left(\frac{mR_s\cosh(mR_s)-\sinh(mR_s)}{(mR_s)^3}\right). \quad (62)$$

Equation (62) admits the following limiting cases (ibid. 2014):

$$A(m,R_s) \cong 1+\frac{(mR_s^2)}{10} \cong 1.0 \quad \text{if } mR_s \ll 1 \quad (63)$$

$$A(m,R_s) \cong \frac{3e^{mR_s}}{2(mR_s)} \quad \text{if } mR_s \gg 1. \quad (64)$$

For a sun like central body (ibid. 2014) the more accurate NASA model has been used for the density of the sun that obeys the following conditions $\rho(R_{sun}) = 0$ and $\mathrm{d}\rho(R_{sun})/\mathrm{d}r = 0$. Thus, the density function is of the form

$$\rho(r) = \rho_0\left[1-5.74\left(\frac{r}{R_s}\right)+11.9\left(\frac{r}{R_s}\right)^2 - 10.5\left(\frac{r}{R_s}\right)^3+3.34\left(\frac{r}{R_s}\right)^4\right]. \quad (65)$$

Integrating (65) the authors obtained the following function for the form factor $A(m, R_S)$ (http://spacemath.gsfc.nasa.gov/(2014)):

$$A(m,R_s) = x^{-7}\left[\left(\begin{array}{c}4.6\times10^4x+2.1\times10^3x^3+x\left(2.7\times10^4+131x^2\right)\cosh x\\ -\left(7.3\times10^4+3.6\times10^3x^2-14.6x^4\right)\sinh x\end{array}\right)\right], \quad (66)$$

with limiting cases given by $x = mR_s$, and $m = 1/\lambda$ with the limiting cases

$$A(m,R_s) \cong 1+6\times10^{-2}(mR_s)^2 \approx 1 \quad \text{if } mR_s \ll 1, \quad (67)$$

$$A(m,R_s) \cong 7.3\frac{e^{mR_s}}{(mR_s)^3} \quad \text{if } mR_s \gg 1. \quad (68)$$

In the case of an extended source we only consider a sun like star. In this case according to Haranas et al. [13] and Moffat and Toth [16] $\lambda \cong 4.937\times10^{15}$ m will imply $mR_s = R_s/\lambda \ll 1$ and therefore $A(m, R_s) \cong 1+6\times10^{-2}(mR_s)^2 \approx 1$. Therefore, in the case of a sta,r our analysis above does not need to involve the $A(m, R_s)$ factor function.

7. Discussion and Numerical Results

To obtain numerical results, we first start with circular orbits. Using (13) and the relation $GM = n^2a^3$ to eliminate n from the equation, we obtain that

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -\frac{1}{3}\alpha m\sqrt{GMa}\left(1+\frac{a}{\lambda}\right)e^{-a/\lambda}. \quad (69)$$

Furthermore, using the fact that $h = \sqrt{GMa(1-e^2)} = \sqrt{GMa}$ or angular momentum per unit mass and for circular orbits, i.e., $e = 0$, we write (69) in the following way:

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -\frac{1}{3}\alpha mh\left(1+\frac{a}{\lambda}\right)e^{-a/\lambda} = -\frac{1}{3}mhf_{Yuk}(a,\lambda). \quad (70)$$

In other words, we find that the rate $\mathrm{d}\mathscr{E}/\mathrm{d}n$ has the units of angular momentum. Similarly, in the elliptic case, (33) results in (71) below:

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -\frac{\alpha mh}{3\sqrt{1-e_0^2}\,(1-e_0\cos E)}\left(1+\frac{a(1-e_0\cos E)}{\lambda}\right)e^{-a(1-e_0\cos E)/\lambda} = -\frac{mh}{3\sqrt{1-e_0^2}}\cdot f_{Yuk}(a,\lambda), \quad (71)$$

where we define as "*Yukawa function*" the term appearing in (71) and (72) that is originally derived from the term $\alpha r^{-1}(1+r/\lambda)e^{-r/\lambda}$. Assuming circular orbits and a semimajor axis equal to the semimajor axis of Mercury $a = 5.79\times10^7$ km and Yukawa coupling constants $\alpha = 3.039\times10^{-8}$ Moffat and Toth [16] and $\alpha = 3.57\times10^{-10}$ and range $\lambda = 4.937\times10^{15}$ m as in [13] in the case where $\lambda > a$ we obtain that

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -1.19\times10^{-10}h. \quad (72)$$

Similarly, if $a = \lambda$ we obtain that

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -8.755\times10^{-11}h. \quad (73)$$

Finally, if $a = \lambda$ and using α as it is given in Moffat and Toth [16], we obtain the following relation:

TABLE 1: Table of the rate $\mathrm{d}\mathscr{E}/\mathrm{d}n$ as a function of eccentric anomaly for an elliptical orbit in a Yukawa potential.

Eccentric Anomaly $E[^\circ]$	$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n}$ $[\mathrm{kgm^2s^{-1}}]$
40	$-4.738\times10^{13}h$
60	$-4.450\times10^{13}h$
80	$-4.141\times10^{13}h$
100	$-3.856\times10^{13}h$
120	$-3.622\times10^{13}h$
140	$-3.451\times10^{13}h$
160	$-3.348\times10^{13}h$
180	$-3.314\times10^{13}h$
200	$-3.348\times10^{13}h$
220	$-3.451\times10^{13}h$
240	$-3.622\times10^{13}h$
260	$-3.856\times10^{13}h$
280	$-4.141\times10^{13}h$
300	$4.450\times10^{13}h$
320	$-4.738\times10^{13}h$
340	$-4.946\times10^{13}h$
360	$-5.023\times10^{13}h$

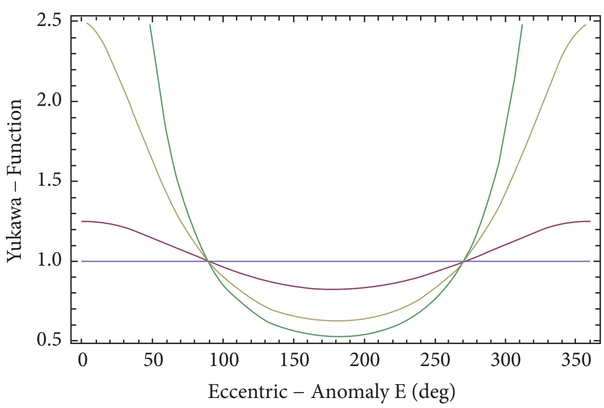

FIGURE 1: Plot of the "Yukawa function" for elliptical orbits of various eccentricities and eccentric anomaly during one full blue $e = 0.01$, red $e = 0.205$, olive green $e = 0.6$, green $e = 0.9$.

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}n} = -1.013 \times 10^{-8} h. \tag{74}$$

Similarly, in Table 1 looking at elliptical orbits, we consider Mercury's actual orbit with semimajor axis $a = 57{,}909{,}050$ km, and eccentricity $e = 0.205$ and thus we have obtained the values shown in Table 1 for the rate $\mathrm{d}\mathscr{E}/\mathrm{d}n$ as a function of eccentric anomaly.

In Figure 1 the Yukawa function is plotted for elliptical orbits of various eccentricities, i.e., blue $e = 0.01$, red $e = 0.205$, olive green $e = 0.6$, and green $e = 0.9$, and for eccentric anomaly values during one full revolution. Similarly, in Figure 2 we plot the Yukawa function for planet Mercury as a function of eccentric anomaly and during one full revolution. Next, in Table 2 we tabulate the parameters $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ and Q as related to the graviton mass at perihelion as is "felt" from an orbiting around a sun like star body. For this numerical calculation for planet Mercury, $\alpha = 3.04 \times 10^{-8}$, $\lambda = 4.937 \times 10^{15}$ m, and $e = 0.205$. Similarly, in Table 3 we tabulate the same parameters as in Table 2 for Mercury using $\alpha = 3.57 \times 10^{-10}$ and $\lambda = 4.937 \times 10^{15}$ m. In Table 4 we tabulate the ratio of the mediating of the interaction graviton mass at perihelion and aphelion points as a function of various eccentricities using $Q_0 = 138.326$. In Figure 3 the mass ratio m_p/m_{ap} of graviton particle mediating the Yukawa interaction as is felt from Mercury at perihelion and aphelion position is plotted, as a function of eccentric anomaly for $\lambda = 5.79 \times 10^{15}$ m and $\alpha = 3.04 \times 10^{-8}$, 3.57×10^{-11}, respectively. Here m_p and m_{ap} indicate the mass of the massive graviton mediating the interaction at the perihelion and aphelion points on the orbit of the secondary. In Table 5, we tabulate the ratio of the graviton mass ratios as a function of orbital eccentricities and

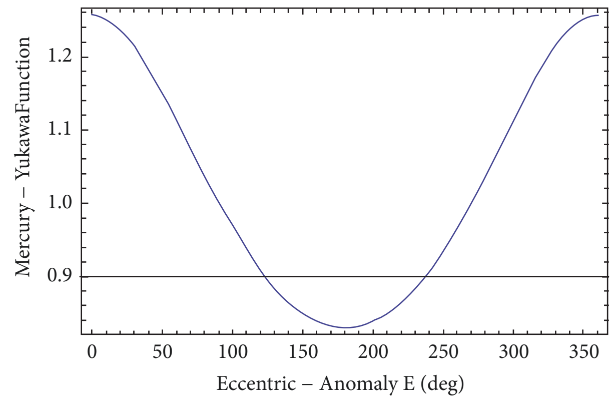

FIGURE 2: Plot of the Yukawa function for planet Mercury as a function of eccentric anomaly during a full revolution.

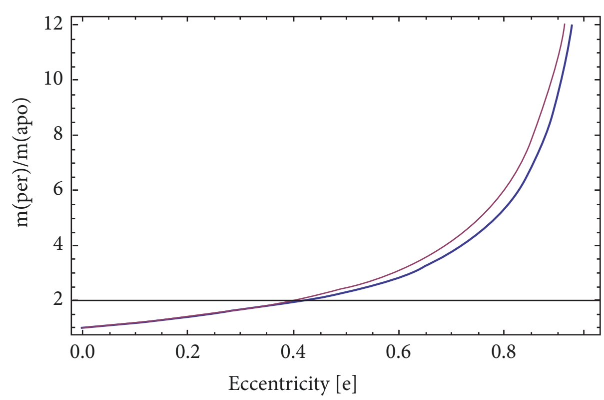

FIGURE 3: Mass ratio of the mediating the Yukawa interaction particle as it is felt by Mercury at perihelion and aphelion is plotted as a function of eccentric anomaly for $\lambda = 5.79 \times 10^{15}$ m and $\alpha = 3.04 \times 10^{-8}$, 3.57×10^{-11}.

Table 2: Tabulation of parameters $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ and Q_0 relating to the mediating graviton mass at perihelion for a sun like star as is "felt" by the secondary i.e. Mercury for $\alpha = 3.04\times10^{-8}$ and $\lambda = 4.937\times10^{15}$ m, e= 0.205.

Coupling constant α	Eccentric Anomaly $E[^\circ]$	$\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ [J]	Q_0
	0.0	-7.708×10^{24}	
3.04×10^{-8}	45	-7.685×10^{24}	13.83408952
	90	-7.631×10^{24}	
	180	-7.555×10^{24}	
	270	-7.593×10^{24}	
	360	-7.708×10^{24}	

Table 3: Tabulation of parameters relating to the ratio of the mediating graviton mass at perihelion for a sun like star as is "felt" from the orbiting secondary i.e. Mercury for $\alpha = 3.57\times10^{-10}$ and $\lambda = 4.937\times10^{15}$ m and for various values of the eccentric anomaly.

Coupling constant α	Eccentric Anomaly $E[^\circ]$	$\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ $[J]$	Q_0
	0.0	-9.052×10^{22}	
3.57×10^{-10}	45	-9.025×10^{22}	138.326
	90	-8.961×10^{22}	
	180	-8.872×10^{22}	
	270	-8.916×10^{22}	
	360	-9.052×10^{22}	

Table 4: Ratio m_p/m_{ap} of the mediating the interaction graviton as a function of various eccentricities as is "felt" by the orbiting secondary body Q_0 = 138.326 between perihelion and aphelion positions.

Eccentricity e_0	m_p/m_{ap}
0.0	1.000000
0.001	1.001660
0.1	1.182141
0.2	1.400340
0.3	1.671480
0.4	2.018540
0.5	2.482622
0.6	3.142260
0.7	4.170730
0.8	6.049680
0.9	10.90870

for Q_0 = 13.83408952. Using (48) we find that at perihelion the mass of the graviton is given by

$$m_p = -\frac{\hbar}{ca(1-e_0)}\left[1+W\left(-\frac{3(\mathscr{E}-\mathscr{E}_0)(1-e_0)}{\alpha e V_N(a)\ln(n/n_0)}\right)\right]. \tag{75}$$

Similarly, at aphelion we find that

$$m_{ap} = -\frac{\hbar}{ca(1+e_0)}\left[1+W\left(-\frac{3(\mathscr{E}-\mathscr{E}_0)(1+e_0)}{\alpha e V_N(a)\ln(n/n_0)}\right)\right]. \tag{76}$$

Moreover, since the graviton mass must be a positive number, (48) implies that the following equation must be satisfied:

$$\frac{1}{(1-e_0\cos E)}\left[1 + W\left(-\frac{3(1-e_0\cos E)}{\alpha e V_N(a)\ln(n/n_0)}(\mathscr{E}-\mathscr{E}_0)\right)\right] = -1. \tag{77}$$

This can be true if the following conditions are met for all the corresponding parameters separately. First from (77) solving for the coupling constant α, we find that (83) is satisfied if

$$\alpha = \frac{3(\mathscr{E}-\mathscr{E}_0)(1-e_0\cos E)\,e^{(2-e_0\cos E)}}{eV_N(a)(2-e_0\cos E)\ln(n/n_0)}, \tag{78}$$

where e is the exponential function. For circular orbits, (78) takes the form

$$\alpha = \frac{3e(\mathscr{E}-\mathscr{E}_0)}{2V_N(a)\ln(n/n_0)}. \tag{79}$$

Similarly, for the mean motion n from Eq. (77) we find that satisfied if:

$$n = n_0 e^{3(\mathscr{E}-\mathscr{E}_0)(1-e_0\cos E)e^{2-e_0\cos E}/e\alpha V_N(a)(2-e_0\cos E)\ln(n/n_0)}, \tag{80}$$

which for circular orbits becomes

$$n = n_0 e^{3(\mathscr{E}-\mathscr{E}_0)e/2\alpha V_N(a)\ln(n/n_0)}. \tag{81}$$

Next, in relation to the final orbital energy, (77) is satisfied if

$$\mathscr{E} = \mathscr{E}_0 + \frac{e\alpha V_N(a)(2-e_0\cos E)\,e^{-(2-e_0\cos E)}}{3(1-e_0\cos E)}\ln\left(\frac{n}{n_0}\right), \tag{82}$$

TABLE 5: Ratio of the mediating the interaction particle as a function of various eccentricities as is "felt" by the orbiting secondary body for $Q_0 = 13.83408952$.

Eccentricity e_0	m_p/m_{ap}	$m_g(p)$ [g]	$m_g(ap)$ [g]
0.0	1.00000	1.00000×10^{-62}	1.000×10^{-62}
0.001	1.00155	1.00155×10^{-62}	9.984×10^{-63}
0.1	1.16851	1.16851×10^{-62}	8.558×10^{-63}
0.2	1.36960	1.36959×10^{-62}	7.301×10^{-63}
0.3	1.61576	1.61574×10^{-62}	6.190×10^{-63}
0.4	1.92698	1.92695×10^{-62}	5.190×10^{-63}
0.5	2.33765	2.33761×10^{-62}	4.277×10^{-63}
0.6	2.91303	2.91297×10^{-62}	3.433×10^{-63}
0.7	3.79632	3.79623×10^{-62}	2.634×10^{-63}
0.8	5.38523	5.38508×10^{-62}	1.856×10^{-63}
0.9	9.46986	9.46964×10^{-62}	1.056×10^{-63}

which for circular orbits takes the form

$$\mathscr{E} = \mathscr{E}_0 + \frac{2\alpha V_N(a)}{3e} \ln\left(\frac{n}{n_0}\right). \tag{83}$$

Moreover, solving for the value of $V_N(a)$ satisfying (77) we obtain

$$V_N(a) = \frac{3(\mathscr{E} - \mathscr{E}_0)(1 - e_0)e^{(2-e_0)}}{e\alpha(2 - e_0 \cos E)\ln(n/n_0)}, \tag{84}$$

which for circular orbits becomes

$$V_N(a) = \frac{3e(\mathscr{E} - \mathscr{E}_0)}{2\alpha \ln(n/n_0)}. \tag{85}$$

Next using (75) and (76) we can obtain a relation for the ratio of the graviton mediating the interaction at perihelion and aphelion correspondingly, to be

$$m_p = \left(\frac{1+e_0}{1-e_0}\right)\left[\frac{1 + W(Q_0(1-e_0))}{1 + W(Q_0(1+e_0))}\right] m_{ap}, \tag{86}$$

where the Q_0 is given by

$$Q_0 = -\frac{3(\mathscr{E} - \mathscr{E}_0)}{\alpha e V_N(a)\ln(n/n_0)}. \tag{87}$$

Next calculating we calculate the orbital energy difference $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$, for planet Mercury using $\alpha = 3.04 \times 10^{-8}$ and $\lambda = 4.937 \times 10^{15}$ m and for various values of the eccentric anomaly. We tabulate the results in Tables 2 and 3.

Taking the numerical values of Q_0 into account as in (87) we can write the mass of the graviton at perihelion and aphelion to be

$$m_p = \left(\frac{1+e_0}{1-e_0}\right)\left[\frac{1 + W(13.84208952(1-e_0))}{1 + W(13.84208952(1+e_0))}\right] m_{ap}, \tag{88}$$

and

$$m_p = \left(\frac{1+e_0}{1-e_0}\right)\left[\frac{1 + W(138.326(1-e_0))}{1 + W(13.326(1+e_0))}\right] m_{ap}, \tag{89}$$

where W is the Lambert function of the indicated argument. In Table 4 we calculate and plot the variation of the mass of the mediating particle for the fixed values of the coupling constant alpha and range lambda of the potential as a function of eccentricity e.

With reference to Ohanian and Ruffini [19] looking at (91) and referring to Yukawa gravitational potential according to relativistic quantum theory, we say that the mass of the graviton is inversely proportional to the range λ. Relying on the observational limit of $\lambda > 10^{24}$ cm we find that $m_g < 10^{-62}$ g. Next, assuming the graviton mass to be $m_g = 10^{-62} g$ we tabulate the corresponding mass values of graviton at perihelion and aphelion as "felt" by the orbiting Mercury.

8. Conclusions

In this paper we have derived the rate of change of the energy of an orbiting body with respect to mean motion in the influence of Yukawa potential of coupling constant α and range λ. We have found that the rate of change of the orbital energy of circular and elliptical orbits w.r.t. the mean motion is a logarithmic function of the mean anomaly n. Furthermore, using this expression we have derived expressions for the mean anomaly as a function of the Yukawa parameters (α, λ) and the orbital energy difference which simplifies considerably in the case where that range of the Yukawa potential is equal to the semimajor axis, i.e., $\lambda = a$. Furthermore we have found that in the case where the energy difference $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ is negative the orbital mean motion n increases exponentially. If $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0$ is positive, then mean motion decreases exponentially. In the special case where $\Delta\mathscr{E} = \mathscr{E} - \mathscr{E}_0 = V_N(a)$, the mean motion also decreases exponentially for $\alpha > 0$ and increases for $\alpha < 0$. Similarly, for circular orbits we have derived expressions for the Yukawa parameters α and λ. We have found that λ is given in terms of a Lambert function of argument $W(3(\Delta\varepsilon)/e\alpha V(a)\ln(n/n_0))$. Similarly, for circular orbits we find that $\lambda = -3e^{a/\lambda}\Delta\varepsilon/(1 + a/\lambda)V(a)\ln(n/n_0)$. Analogous relations have been derived for elliptical orbits. Next expressions for the mass of graviton have been derived in terms of the orbital elements of the orbiting secondary for circular and elliptical orbits, from which special conditions have been extracted for a positive graviton mass. We have found that there is no graviton mass difference effect for circular orbits, as to what the mass of the graviton between perihelion and aphelion is. Therefore, the strength of the gravitational interaction along the orbit propagates with the help of a constant mass graviton. As the eccentricity increases, the mass of the massive graviton appears to increase, being larger at perihelion where at aphelion it is always less by at least two orders of magnitude from its assumed mass, i.e., $m_g = 10^{-62}g$. Our motivation for this paper emanates from the hope that this work can add an extra possibility in the estimation of the graviton mass. This will be the addition of one more observational test in setting solar system as well as binary system bounds for the graviton mass m_{gr} in terms of orbital energy mean motion relations. This is in the case where the bound depends on the mass of the source, which in this case is a sun like type of star. Thus near the massive body we would expect the graviton mass to have its higher value where its lowest value will be at the aphelion point, respectively. In other words, the strength of the interaction of gravity will reduce and this is something we already know. The only difference is that now we can have an explanation for the strength of gravitational interaction in terms of a massive graviton mass change as the new theories predict.

Appendix

The Lambert W function is defined as the inverse function of the $x| \longrightarrow xe^x$ mapping and thus solving the equation $ye^y = x$. This solution is given in the form of the Lambert W function, $y = W(x)$ which according to the relation $ye^y = x$ satisfies the following equation: $W(x)e^{W(x)} = x$ [20]. The function is also known as Omega or Product Log function.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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