TN-65-85
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A UNIFORM-FIELD AIR-CORE BENDING MAGNET

"The Electron Pipe"

For particular application to pulsed bending magnets which are to be used relatively infrequently, it is economically reasonable to consider air-core coils. Such a coil system will be considered here as an extension of the theory developed by Helm[1] for the degaussing system. The resulting magnet is uniform to terms involving $(x/r)^6$.

Applying the complex notation used by Helm, we have

$$\tilde{B} = \frac{\mu_o}{2\pi} \frac{dI}{z - c} \qquad (1)$$

for the magnetic induction from current element dI where $z = x + iy$ is the position of the beam and $c = a + ib$ is the position of the current element $dI = jdA$.

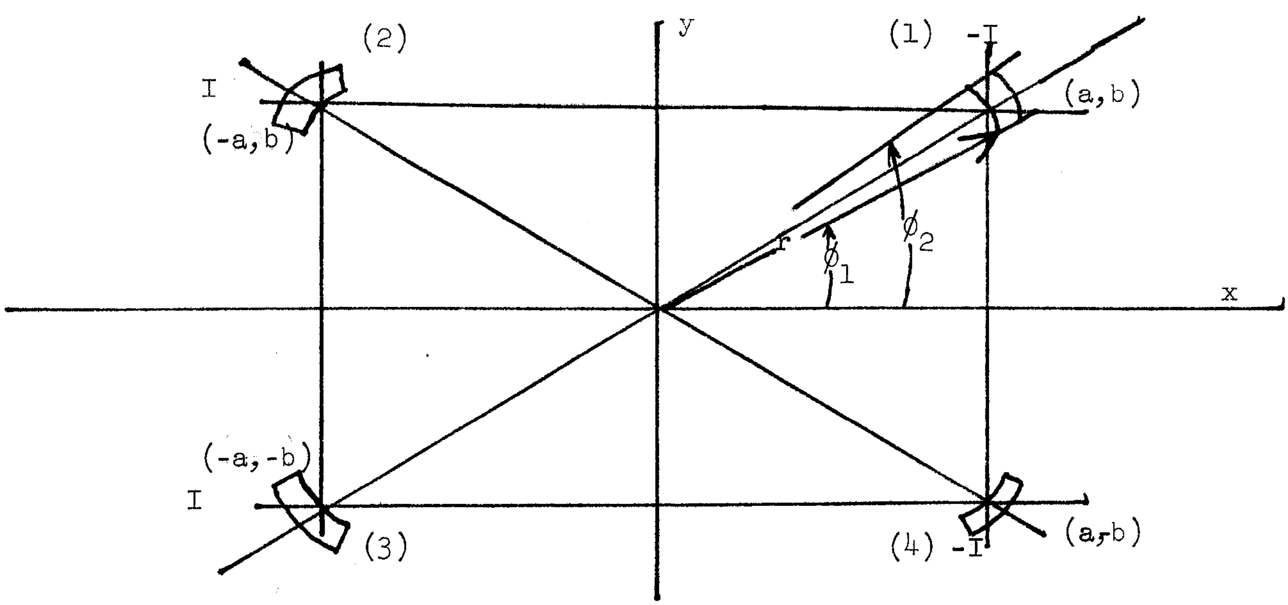

Fig. 1 Four-wire Current Elements
Illustrating the Coordinate Systems

R. H. Helm, "Design Principles for a Degaussing System," TN-61-24

If we consider the four symmetric current elements of Fig. 1, and ignore error terms,

$$\tilde{B} = \frac{\mu o}{2\pi} \int_{\substack{\text{one} \\ \text{wire}}} \left(\frac{1}{c - z} + \frac{1}{c^{*} + z} + \frac{1}{c + z} + \frac{1}{c^{*} - z} \right) dI \qquad (2)$$

where the terms are in the sequence of the numbers in Fig. 1 and $c^{*} = a - ib$. Expanding in powers of z yields

$$\tilde{B} = \frac{\mu o}{\pi} \int_{\substack{\text{one} \\ \text{wire}}} \left\{ \left(\frac{1}{c} + \frac{1}{c^{*}} \right) + z^{2}\left(\frac{1}{c^{3}} + \frac{1}{c^{*3}} \right) + z^{4}\left(\frac{1}{c^{5}} + \frac{1}{c^{*5}} \right) + z^{6}\left(\frac{1}{c^{7}} + \frac{1}{c^{*7}} \right) \right\} dI$$
$$+ \text{ higher order terms } + \text{ error terms} \qquad (3)$$

The odd terms are eliminated by symmetry.

To simplify the integration we will choose for the element of area $dA = r\, dr\, d\phi$. Then by letting $c = r(\cos\phi + i \sin\phi) = re^{i\phi}$ the terms in the expansion are:

$$B_{o} = \frac{2\mu o j}{\pi} \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \int_{r_1}^{r_2} dr = \frac{2\mu o j \Delta r}{\pi} (\sin\phi_2 - \sin\phi_1) \qquad (4)$$

$$B_{2} = z^{2}\left(\frac{\mu o j}{\pi} \int \frac{r dr}{r^{3}} \frac{2}{3} \right) \int_{\phi_1}^{\phi_2} \cos 3\phi \, d3\phi \doteq z^{2} (\quad)(\sin 3\phi_2 - \sin 3\phi_1)$$

$$= z^{2} \left(\frac{4}{3\pi} \mu o j \int \frac{dr}{r^{2}} \right) \cos \frac{3}{2} (\phi_2 + \phi_1) \sin \frac{3}{2} (\phi_2 - \phi_1) \qquad (5)$$

$$B_4 = z^4 \left(\frac{4}{5\pi} \mu o j \int \frac{dr}{r^4} \right) \cos \frac{5}{2} (\phi_2 + \phi_1) \sin \frac{5}{2} (\phi_2 - \phi_1) \tag{6}$$

$$B_6 = z^6 \left(\frac{4}{7\pi} \mu o j \int \frac{dr}{r^6} \right) \cos \frac{7}{2} (\phi_2 + \phi_1) \sin \frac{7}{2} (\phi_2 - \phi_1) \tag{7}$$

To make the second order term zero, we must have $\cos \frac{3}{2}(\phi_2 + \phi_1) = 0$ or $\frac{3}{2} (\phi_2 + \phi_1) = 90^\circ$. If the conductors are single wires, $\phi_1 = \phi_2$ and $\phi = 30^\circ$ which is the result found by Helm.

To make the fourth-order term zero from equation (6), $\sin \frac{5}{2} (\phi_2 - \phi_1) = 0$ and $\frac{5}{2} (\phi_2 - \phi_1) = 180^\circ$. Solving equations (8) and (9) we have $\phi_2 = 66^\circ$ and $\phi_1 = -6^\circ$. (10)

The negative angle means that one current element overlaps with another. The overlap can be accomplished by using a good conductor, i.e., silver or copper, in the 12° overlapped segment and using a poorer conductor like aluminum for the rest of the conductor. See Figure 2.

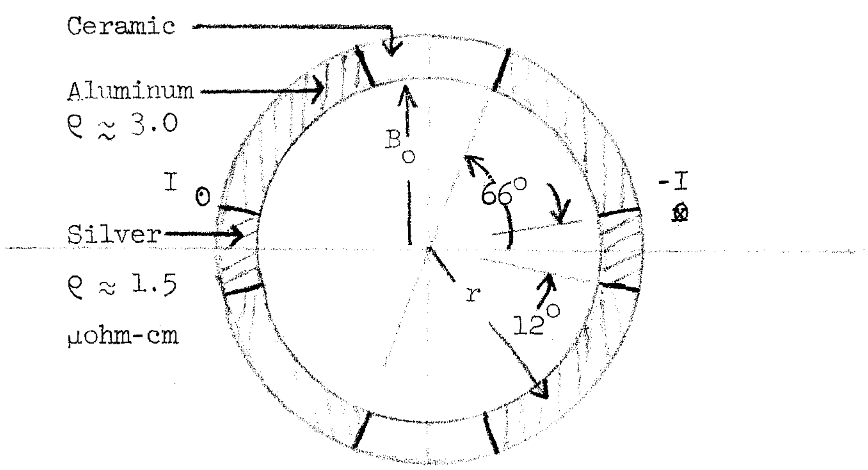

Fig. 2 Possible Cross Section of Electron Pipe

From Eq. (4), by letting $j = \frac{I}{A} = \frac{I}{r\Delta r\ (\phi_2 - \phi_1)}$, we have

$$B_o = \frac{2\mu oI}{\pi r} \ \frac{\sin \phi_2 - \sin \phi_1}{\phi_2 - \phi_1} = \frac{4\mu oI}{\pi r} \cos \frac{1}{2} (\phi_2 + \phi_1) \frac{\sin \frac{1}{2} (\phi_2 - \phi_1)}{(\phi_2 - \phi_1)} \qquad (11)$$

Converting to engineering units this becomes

$$B_o = \frac{0.63\ I}{r} \ \cos \frac{1}{2} (\phi_2 + \phi_1) \frac{\sin \frac{1}{2} (\phi_2 - \phi_1)}{(\phi_2 - \phi_1)} \quad \text{gauss (amperes, inches)}$$

As a check, we can let $\phi_2 = \phi_1 = 30^\circ$, as for a four wire system, which yields

$$B_o = \frac{0.63I}{r} \frac{\sqrt{3}}{4} = \frac{0.272I}{r} \text{ gauss (amperes, inches).}$$

which is the same as the result for the degaussing system. Substituting instead $\phi_2 = 66^\circ$ and $\phi_1 = -6^\circ$ we have the result for the fourth-order coil as

$$B_o = \frac{0.63I}{r} \frac{(\cos 30^\circ) \sin 36^\circ}{(72\pi / 180)} = 0.255\frac{I}{r} \text{ gauss (amperes, inches),}$$

where r is the radius of the median circle. The current is that assigned to just one of four conductors, so that when they are merged, as in Fig. 2, the expression becomes

$I_{total} = 7.88\ B_o r$ amperes (gauss, inches).

For a beam with diameter one centimeter, a one inch radius tube could be clamped to the outside of a ceramic tube using ceramic (or insulated metal) spacers at the top and bottom.

The assembly, which would sort of resemble a pipe for an electron beam, would then be laid in an aligned curved bed. The field strength, radius of curvature and length of the assembly would of course, have to be determined with reference to the rest of the optics of the system.

To evaluate the inhomogeniety requires considering misalignments and tolerances of the construction process. However, the inherent inhomogeniety is due to the sixth-order term of equation 7. Taking a ratio of equation 7 to equation 4 yields

$$\frac{B_6}{B_o} \approx \frac{z^6}{r^6} \frac{1}{7} \frac{\cos\frac{7}{2}(\phi_2+\phi_1)\sin\frac{7}{2}(\phi_2-\phi_1)}{\cos\frac{1}{2}(\phi_2+\phi_1)\sin\frac{1}{2}(\phi_2-\phi_1)} \approx \frac{1}{4}\frac{z^6}{r^6}$$

which for the example of a one centimeter beam is $\frac{1}{4}\left(\frac{1}{5}\right)^6 \approx 1.6 \times 10^{-5}$.

To make good use of such a uniform field requires special attention to details of construction. For example, Dr. Chu points out that end effects could be significant. The end at which the connectors are located must be designed with particular attention to symmetry.