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Linearized renormalization

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Abstract Using an infinitesimal approach, this work addresses the renormalization problem to deal with the ultraviolet divergences arising in quantum field theory. Under the assumption that the action has already been renormalized to yield an ultraviolet-finite effective action that satisfies a certain set of renormalization conditions, we analyze how the action must be adjusted to reproduce a first-order change in these renormalization conditions. The analysis then provides the change that is induced on the correlation functions of the theory. This program is successfully carried out in the case of super-renormalizable theories, namely, a scalar field with cubic interaction in four space-time dimensions and with quartic interaction in three space-time dimensions. Relying on existing results in the theory of perturbative renormalization, we derive explicit renormalized expressions for these theories, each of which involves only a finite number of graphs constructed with full propagators and full n-point vertices. The renormalizable case is analyzed as well; the derived expressions are ultraviolet finite as the regulator is removed but cannot be written without a regulator. In this sense, the renormalization is not fully explicit in the renormalizable case. Nevertheless, a perturbative solution of the equations starting from the free theory provides the renormalized Feynman graphs, similar to the BPHZ program. For compatibility with the preservation of the renormalization conditions, a projective renormalization scheme, as opposed to a minimal one, is also introduced. The ideas developed are extended to the study of the renormalization of composite operators and the Schwinger–Dyson equations.

Contents

1 Introduction and conclusions

Quantum field theory is an extremely beautiful and successful tool in theoretical physics for describing the interactions between particles and in other areas such as condensed matter [1,2]. It accommodates the Standard Model of Particles which currently is being actively tested [3,4]. Perturbative calculations using Feynman diagrams typically display divergences associated with virtual particles in loops in the asymptotic region of large loop momenta. The same

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ultraviolet (UV) divergences appear in the correlation functions for fields at short distances. Such divergences are not pure mathematical artifacts, rather, they signal the failure of the model to include the proper degrees of freedom relevant at very short distances [5]. In general, a field theory is designed to work at scales below a certain momentum cutoff Λ and becomes inconsistent above it, developing unphysically large contributions. Through the process of renormalization, the divergences can be removed by enlarging the action through the addition of suitable counterterms, in general involving operators of increasingly larger dimension. Renormalizable and super-renormalizable theories are those that only require a finite number of operators in the action to produce finite correlation functions. While a renormalizable theory such as Quantum Chromodynamics needs not, and in fact does not, describe the physics at arbitrarily short scales, it remains an internally consistent theory at all scales. Quantum electrodynamics is also renormalizable at the perturbative level, although it develops a non-perturbative Landau pole at extremely high energies [6]. Due to the difficult renormalization problem, only a few quantum field theories currently exist from a rigorous mathematical point of view, at the non-perturbative level [7]. These theories exist in lower-dimensional spaces where ultraviolet divergences are less severe. Certainly, no four-dimensional renormalizable gauge theory has yet been constructed in a mathematical sense.

The physics of the non-perturbative renormalization and its naturalness were greatly clarified after the analysis by Wilson and others. Nevertheless, even if the perturbative analysis does not cover the whole subject of renormalization, it is still an indispensable and insightful tool in any theory. The systematics of the renormalization at the perturbative level has been the subject of intense mathematical work in the past. In particular, the work of Bogoliubov and Parasiuk [8] and Hepp [9] allowed to organize in a systematic way the renormalization of the momentum space Feynman diagrams of any theory and the identification of the counterterms. Epstein and Glaser [10] developed an equivalent procedure in space-time space (see [11] for a modern presentation). The forest formula by Zimmermann provided an explicit solution to Bogoliubov's recursion that, as a by-product, allowed to yield manifestly renormalized ultraviolet-finite Feynman graphs for massive theories [12,13]. The same subtraction rules work regardless of the regularization and apply to the massless case and gauge theories, where dimensional regularization becomes the method of choice [14].

A field theory is solved when its correlation functions are known. In perturbation theory, this corresponds to summing the relevant Feynman graphs. The effective action functional is the classical action, or tree-level action, that reproduces the correlation functions of a theory, while the full propagator and full vertices contain all loop effects [15]. In [16], the following problem was posed and answered: If a perturbation is added to the action, must the Feynman graphs be recomputed, including both the old and new vertices, to obtain the new effective action? Such procedure corresponds to a scheme '$S_{\text{free}} + S_I + \delta S$', where $S = S_{\text{free}} + S_I$ is the original action, and δS is the perturbation. In fact, it is not mandatory to follow such route. The new effective action can also be obtained using the scheme '$\Gamma + \delta S$', where Γ is the effective action of the original theory. In this scheme, standard Feynman graphs are used with vertices of δS, as well as full vertices and full propagators of the effective action, but include only graphs without unperturbed loops. This means that every loop must include a vertex of δS. If the vertices of δS are deleted in one such graph, what remains is a tree graph of Γ.

In this work, we explore how the above result can be exploited in the context of renormalization theory. In a renormalizable theory, every correlation function remains finite (barring coincident points) when the parameters in the action are given a suitable dependence on the UV regulator, say Λ, as the regulator is removed, i.e., for large Λ. In this way, the theory becomes a renormalized one. In a theory that is already renormalized, one can introduce a first-order perturbation in the action. In the '$\Gamma + \delta S$' scheme, all the UV divergences from the loops inside Γ have been canceled, but the perturbation δS introduces new loops and thus new divergences that need a regulator. These new divergences must be canceled through a suitable dependence on the regulator of the parameters in δS. As usual, the cancellation of divergences leaves finite ambiguities in the parameters. Rather than applying a minimal subtraction scheme, the approach we adopt here is to enforce some renormalization conditions on the correlation functions by defining some renormalized parameters. Correspondingly, the subtractions appearing in, e.g., the forest formula, follow a projective renormalization scheme. The procedure then yields the change in the correlation functions induced by the first-order variation in the renormalized parameters.

This infinitesimal approach, which is implemented through a form of Schwinger's principle, has an obvious limitation, namely, it does not directly provide the renormalized effective action, only its variation as the renormalized parameters change. On the other hand, the renormalization takes place at the linear level, hence it is structurally simpler and the divergences are softer. Another virtue is that the expressions involve the full propagator and vertices, then the equations found are non-perturbative. This means that each equation involves only a finite number of graphs constructed with the propagator and vertices of the effective action, rather than an infinite number of graphs of the action, much as the Schwinger–Dyson equations do. Of course, a different question is that some of the theories only exist, in principle, in a perturbative sense, i.e., as a formal power series in the coupling or in $\hbar$. The fact that the equations do not directly depend on perturbation theory is an interesting feature, which could potentially be adapted for a non-perturbative construction of the theory. This is not attempted in this work.

The linearized approach explored here is applied to super-renormalizable theories, namely, scalar ϕ^3_4 and ϕ^4_3. The fact that in these theories the UV divergences are rather mild allows us to obtain explicit and fully renormalized non-perturbative equations for the variation of the effective with respect to the renormalized parameters. No regulator is needed. Not unexpectedly, the renormalizable case, e.g. ϕ^3_6 or ϕ^4_4, is not so well behaved since the UV divergences are stronger there. In the renormalizable case, expressions are found that still involve only a finite number of graphs constructed with the propagator and vertices of the effective action, however, the momentum integrals need to be regulated. The renormalizability of the theory guarantees that there is a finite limit after removing the regulator, but the limit itself cannot be expressed in closed form. In the renormalizable case, the equations display a finite number of explicit loops, but the divergences involved are not accounted by a naive expectation from these loops only. Ultimately, the reason why the limit cannot be taken explicitly is that the composite operators in the perturbation δS activate subdivergences involving vertices from δS and propagator lines (of the action) hidden within the effective action even if the latter has been renormalized. Nevertheless, the enforcement of the renormalization conditions, which is built into the formulation, ensures that when the equations are solved perturbatively as a formal series in the renormalized coupling, each contribution is a subtracted and UV-finite Feynman graph so that the regulator can be removed there.

The subject of perturbing the action to first order is, of course, closely related to the study of composite operators and their renormalization. In fact, most of the tools developed in one case immediately apply to the other, so this topic is also discussed.

The linearized renormalization is also closely related to the subject of Schwinger–Dyson equations and their renormalization. In both cases, the equations follow from graphs involving vertices and lines from the effective action and from the action itself (in the case of the Schwinger–Dyson equations) or from δS in the linearized renormalization approach. The two sets of equations are related and fulfill consistency conditions between them.

The paper is organized as follows: Sect. 2 revisits the effective action machinery. The concept of $\langle A\rangle^\phi$ and some related diagrammatic properties are discussed. Our form of the Schwinger principle is also introduced along with the definition of the coefficients H and G. Section 3 details the application of the linearized approach to the super-renormalizable scalar field theories ϕ^3_4 and ϕ^4_3. The result is an infinite hierarchy of non-perturbative equations that are free from UV divergences and express the variation of the effective action with respect to the renormalized parameters. The renormalizable case is addressed in Sect. 4. First, the projective renormalization scheme is introduced. This is presented within the framework of a Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) approach, by specifying the operation T to be employed in the forest formula. Then, the linearized equations are constructed. As previously mentioned, these equations are less explicit than those of the super-renormalizable case since the regulator cannot be lifted in their non-perturbative form. However, the regulator can be removed in a perturbative solution of the equations and this yields subtracted UV-finite Feynman graphs. Section 5 deals with the renormalization of composite operators, building on the formalism previously developed for the linearized renormalization. In the super-renormalizable case this approach produces UV-finite non-perturbative expressions for the matrix elements of the operators. The renormalization approach is adapted to the Schwinger–Dyson equations in Sect. 6. The renormalized Schwinger–Dyson equations follow the same pattern as the linearized renormalization and the composite operators; they are manifestly UV-finite in the super-renormalizable case but not in the renormalizable one. Some properties of the effective action exposed in Sect. 1, including the Schwinger principle, are proven in Appendix A. Appendix B illustrates properties of the projective renormalization scheme developed in Sect. 4.1. Appendix C elucidates further the concept of anti-canonical pattern discussed in Sect. 4.2 by analyzing a simple subcase of the scalar theory ϕ^3_6. It also illustrates the consistency of other results obtained in the main text. Finally Appendix D discusses a technical aspect of the linearized approach, namely, that the renormalized results do not depend on the assumption that the effective action has already been renormalized prior to its perturbation.

2 The Schwinger action principle

For simplicity, let us consider a theory ϕ^κ_d, a single scalar field $\varphi(x)$ with action

$$S[\varphi] = \int d^d x \left(\frac{1}{2} Z (\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{\kappa!} g \varphi^\kappa \right). \tag{2.1}$$

Here, a Euclidean signature-like notation is used. For convenience, an explicit wave function renormalization factor Z has also been introduced.

The generating functional $Z[J]$ of the Green or correlation functions is then

$$Z[J] = \int D\varphi \, e^{\int d^d x (\mathcal{L}(x) + J(x)\varphi(x))} := e^{W[J]} \tag{2.2}$$

so that

$$\langle T\varphi(x_1)\cdots\varphi(x_n)\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0},$$
$$\langle T\varphi(x_1)\cdots\varphi(x_n)\rangle_c = \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}W[J]\Big|_{J=0}. \quad (2.3)$$

Here and in what follows, we will use a convention with Boltzmann weight e^{+S} instead of e^{-S}. This implies that Z, m^2, g, $S[\varphi]$, $W[J]$, $\Gamma[\phi]$, etc, have non-standard signs. In particular, the free propagator in momentum space becomes

$$D_0(k) = -(Zk^2+m^2)^{-1}. \quad (2.4)$$

While this convention may be inconvenient for performing detailed calculations, it enjoys the crucial advantage that there are no minus signs between Green functions contributions and their Feynman graphs, nor in the Feynman rule of the vertices.

2.1 The effective action functional

Using the DeWitt notation $\varphi(x)\to\varphi^i$ (with φ^i real and i including all labels present in the field) a general action takes the form

$$S[\varphi] = c + h_i\varphi^i + \frac{1}{2}m_{ij}\varphi^i\varphi^j + \sum_{n\geq 3}\frac{1}{n!}g_{i_1\ldots i_n}\varphi^{i_1}\cdots\varphi^{i_n} \quad (2.5)$$

or simply

$$S[\varphi] = \sum_{n\geq 0}\frac{1}{n!}g_{i_1\ldots i_n}\varphi^{i_1}\cdots\varphi^{i_n}. \quad (2.6)$$

Here m_{ij} and $g_{i_1\ldots i_n}$ are completely symmetric covariant tensors with respect to linear transformations of the φ^i, which are contravariant. The (free connected) propagator is

$$s^{ij} := (-m^{-1})^{ij}, \quad s^{ij}m_{jk} = -\delta^i_k. \quad (2.7)$$

In this notation, the generating function is then

$$Z[J] = e^{W[J]} = \int D\varphi\, e^{S[\varphi]+J_i\varphi^i} \quad (2.8)$$

and

$$\langle\varphi^{i_1}\cdots\varphi^{i_n}\rangle_c = \partial^{i_1}\cdots\partial^{i_n}W[J]\Big|_{J=0}, \quad \partial^i := \partial/\partial J_i. \quad (2.9)$$

The effective action $\Gamma[\phi]$ will be used extensively. It is defined as the Legendre transformation of the connected generating functional,

$$\Gamma[\phi] = W[J] - J_i\phi^i, \quad \phi^i[J] = \partial^i W[J] = \langle\varphi^i\rangle^J,$$
$$J_i[\phi] = -\partial_i\Gamma[\phi], \quad \partial_i := \partial/\partial\phi^i. \quad (2.10)$$

ϕ^i is known as the classical field. The expansion

$$\Gamma[\phi] = C + H_i\phi^i + \frac{1}{2}M_{ij}\phi^i\phi^j + \sum_{n\geq 3}\frac{1}{n!}\Gamma_{i_1\ldots i_n}\phi^{i_1}\cdots\phi^{i_n}$$
$$= \sum_{n\geq 0}\frac{1}{n!}\Gamma_{i_1\ldots i_n}\phi^{i_1}\cdots\phi^{i_n}, \quad (2.11)$$

provides the full propagator (the connected two-point function)

$$D^{ij} := (-M^{-1})^{ij} = \langle\varphi^i\varphi^j\rangle_c, \quad D^{ij}M_{jk} = -\delta^i_k, \quad (2.12)$$

and the vertex functions $\Gamma_{i_1\ldots i_n}$, which are the connected, irreducible amputated graphs of the action S.[1] In turn, the Green functions of S are obtained by using the standard Feynman rules with propagator D^{ij} and vertices $\Gamma_{i_1\ldots i_n}$ (for $n\neq 2$) *but including only tree graphs*. Hence Γ is the action that produces classically (i.e. at tree level) the correlation functions generated by S quantum-mechanically (i.e. allowing loops),

$$Z[J] = e^{\Gamma[\phi]+J_i\phi^i}. \quad (2.13)$$

2.2 Schwinger's action principle

If the current J is not set to zero in the functional integral, the expectation value of an observable $A[\varphi]$ becomes

$$\langle A\rangle^J = \frac{1}{Z[J]}\int D\varphi\, e^{S[\varphi]+J_i\varphi^i}A[\varphi]. \quad (2.14)$$

Let us introduce the notation

$$\langle A\rangle^\phi := \langle A\rangle^{J=J[\phi]}. \quad (2.15)$$

That is, $\langle A\rangle^\phi$ denotes the expectation value in the presence of the current J such that $\langle\varphi\rangle^J = \phi$. In particular

$$\langle\varphi^i\rangle^\phi = \phi^i. \quad (2.16)$$

Under a first-order variation of the action $S\to S+\delta S$, the generator of the connected Green functions is modified as

$$\delta W[J] = \langle\delta S\rangle^J. \quad (2.17)$$

As a consequence, for the effective action one obtains immediately (see Appendix A) that

$$\delta\Gamma[\phi] = \langle\delta S\rangle^\phi, \quad (2.18)$$

which is likely a version of Schwinger's quantum action principle. This identity will play a central role in this work.

The first-order variation in Eq. (2.18) is consistent (in the technical sense $[\delta_1,\delta_2]=0$); therefore, if one wants to compute variations of higher order it suffices to use the identity recursively. Alternatively, the theorem proven in [16] applies:

[1] Exceptionally, for $n=2$ these graphs produce the selfenergy Σ_{ij} and $M_{ij} = m_{ij}+\Sigma_{ij}$.

If S is perturbed to $S + S_I$ the new correlation functions follow from using the Feynman rules of '$\Gamma + S_I$' (namely, the propagator and vertices of Γ plus vertices of S_I) but retaining only graphs such that any loop must have at least one vertex of S_I.

2.3 The coefficients H

Clearly, when the current J is not removed, the expectation values $\langle A\rangle^J$ can be computed with the same Feynman rules of S but using $h_i + J_i$ as the new 1-point vertex. A related result that will be needed is as follows:

Theorem. For $n \geq 2$, the expectation values $\langle\varphi^{i_1}\cdots\varphi^{i_n}\rangle^\phi$ are obtained from the Feynman rules at the tree level but using the following propagator (line) and vertices:

$$\hat{D}^{ij}[\phi] := ((-\partial^2\Gamma[\phi])^{-1})^{ij},$$
$$\hat{\Gamma}_{i_1\ldots i_n}[\phi] := \begin{cases} \partial_{i_1}\cdots\partial_{i_n}\Gamma[\phi] & n \geq 3 \\ 0 & n \leq 2 \end{cases}. \tag{2.19}$$

This statement is proven in Appendix A. Obviously when ϕ is set to 0, $\hat{D}^{ij}[\phi]$ and $\hat{\Gamma}_{i_1\ldots i_n}[\phi]$ become D^{ij} and $\Gamma_{i_1\ldots i_n}$ introduced in (2.12) and (2.11).

Applying the Theorem, for $n = 1, 2, 3$ one obtains[2]

$$\begin{aligned} \langle\varphi^i\rangle^\phi &= \phi^i, \\ \langle\varphi^i\varphi^j\rangle_c^\phi &= \hat{D}^{ij}[\phi], \\ \langle\varphi^i\varphi^j\varphi^k\rangle_c^\phi &= \hat{D}^{ia}[\phi]\hat{D}^{jb}[\phi]\hat{D}^{kc}[\phi]\hat{\Gamma}_{abc}[\phi]. \end{aligned} \tag{2.20}$$

Schematically,

$$\begin{aligned} \langle\varphi\rangle^\phi &= \phi, \\ \langle\varphi^2\rangle_c^\phi &= \hat{D}, \\ \langle\varphi^3\rangle_c^\phi &= \hat{D}^3\hat{\Gamma}_3, \\ \langle\varphi^4\rangle_c^\phi &= \hat{D}^4\hat{\Gamma}_4 + 3\times\hat{D}^2\hat{\Gamma}_3\hat{D}\hat{\Gamma}_3\hat{D}^2. \end{aligned} \tag{2.21}$$

The rule is that each $\partial^i = \partial/\partial J_i$ generates a new leg from either a vertex ($\hat{\Gamma}_n$) or a line ($\hat{D}$). Likewise $\partial_j = \partial/\partial\phi^j$ generates an amputated leg because $\partial^i = \hat{D}^{ij}[\phi]\partial_j$. So, for instance

$$\begin{aligned} \langle\varphi^i\varphi^j\varphi^k\rangle_c^\phi &= \partial^i\langle\varphi^j\varphi^k\rangle_c^\phi = \hat{D}^{ia}\partial_a\hat{D}^{jk} \\ &= \hat{D}^{ia}\hat{D}^{jb}\hat{D}^{kc}\hat{\Gamma}_{abc}. \end{aligned} \tag{2.22}$$

Let us introduce the family of functionals $\hat{H}$ from

$$\langle\varphi^{i_1}\cdots\varphi^{i_\ell}\rangle^\phi = \phi^{i_1}\cdots\phi^{i_\ell} + \hat{H}^{i_1\ldots i_\ell}[\phi]. \tag{2.23}$$

Note that these expectation values are not connected. Also, for convenience, the completely disconnected term has been removed from the definition of $\hat{H}$. Hence, in particular $\hat{H}^{i_1\ldots i_\ell}[\phi]$ vanishes for $\ell = 0, 1$. Explicitly for $\ell = 2, 3$ one has

$$\begin{aligned} \hat{H}^{i_1i_2}[\phi] &= \hat{D}^{i_1i_2}[\phi] \\ \hat{H}^{i_1i_2i_3}[\phi] &= \phi^{i_1}\hat{D}^{i_2i_3}[\phi] + \phi^{i_2}\hat{D}^{i_1i_3}[\phi] + \phi^{i_3}\hat{D}^{i_1i_2}[\phi] \\ &\quad + \hat{D}^{i_1a}[\phi]\hat{D}^{i_2b}[\phi]\hat{D}^{i_3c}[\phi]\hat{\Gamma}_{abc}[\phi]. \end{aligned} \tag{2.24}$$

Schematically

$$\begin{aligned} \hat{H}^2 &= \hat{D}, \\ \hat{H}^3 &= 3\times\phi\hat{D} + \hat{D}^3\hat{\Gamma}_3, \\ \hat{H}^4 &= 6\times\phi^2\hat{D} + 4\times\phi\hat{D}^3\hat{\Gamma}_3 + 3\times\hat{D}^2 \\ &\quad + 3\times\hat{D}^2\hat{\Gamma}_3\hat{D}\hat{\Gamma}_3\hat{D}^2 + \hat{D}^4\hat{\Gamma}_4. \end{aligned} \tag{2.25}$$

Furthermore, one can define the coefficients H from

$$\hat{H}^{i_1\ldots i_\ell}[\phi] = \sum_{n=0}^\infty \frac{1}{n!} H^{i_1\ldots i_\ell}_{j_1\ldots j_n}\phi^{j_1}\cdots\phi^{j_n}. \tag{2.26}$$

These coefficients $H^{i_1\ldots i_\ell}_{j_1\ldots j_n}$ represent tree level Γ-graphs (in general disconnected) with legs $i_1, \ldots, i_\ell$ (contravariant) plus the amputated legs $j_1, \ldots, j_n$ (covariant), constructed with the effective action propagator D^{ab} and vertices $\Gamma_{a_1\ldots a_k}$ (hence Γ-graphs). They are fully symmetric tensors with respect to the covariant and contravariant indices separately.

Since (2.26) represents a Taylor expansion in ϕ, the coefficients H are easily constructed by applying ∂_j on $\hat{H}[\phi]$ to produce the amputated legs $j_1, \ldots, j_n$, and then setting $\phi = 0$. Diagrammatically, this corresponds to recursively extracting the amputated legs in all possible ways from the initial graph of $\hat{H}^{i_1\ldots i_\ell}[\phi]$. The legs are extracted from explicit ϕ (once), from unamputated legs and from vertices, but not from amputated legs. At the end ϕ must be set to zero for each coefficient, but not of course during the recursion.

For $\ell = 2$ one obtains

$$\begin{aligned} H^{i_1i_2} &= D^{i_1i_2}, \\ H^{i_1i_2}_{j_1} &= D^{i_1a}D^{i_2b}\Gamma_{j_1ab}, \\ H^{i_1i_2}_{j_1j_2} &= D^{i_1a}D^{i_2b}\Gamma_{j_1j_2ab} + D^{i_1a}D^{i_2b}D^{cd}\Gamma_{j_1ac}\Gamma_{j_2db} \\ &\quad + D^{i_1a}D^{i_2b}D^{cd}\Gamma_{j_2ac}\Gamma_{j_1db}, \end{aligned} \tag{2.27}$$

and similarly for higher values of n.

Likewise, for $\ell = 3$ the extraction of the amputated legs $j_1, \ldots, j_n$ in all possible ways yields

$$\begin{aligned} H^{i_1i_2i_3} &= D^{i_1a}D^{i_2b}D^{i_3c}\Gamma_{abc}, \\ H^{i_1i_2i_3}_{j_1} &= \delta^{i_1}_{j_1}D^{i_2i_3} + \delta^{i_2}_{j_1}D^{i_1i_3} + \delta^{i_3}_{j_1}D^{i_1i_2} \\ &\quad + D^{i_1d}\Gamma_{j_1de}D^{ea}D^{i_2b}D^{i_3c}\Gamma_{abc} \\ &\quad + D^{i_1a}D^{i_2d}\Gamma_{j_1de}D^{eb}D^{i_3c}\Gamma_{abc} \\ &\quad + D^{i_1a}D^{i_2b}D^{i_3d}\Gamma_{j_1de}D^{ec}\Gamma_{abc} \\ &\quad + D^{i_1a}D^{i_2b}D^{i_3c}\Gamma_{j_1abc}. \end{aligned} \tag{2.28}$$

[2] The case $n = 1$ is included for completeness; it is not derived from the above rules.

For convenience, let us also introduce the notation

$$\hat{G}^{i_1\dots i_\ell}[\phi] = \langle \varphi^{i_1}\cdots\varphi^{i_\ell}\rangle^\phi, \tag{2.29}$$

that is, as $\hat{H}^{i_1\dots i_\ell}[\phi]$ but including the completely disconnected term. These functionals can be collected into generating functionals, namely,[3]

$$\begin{aligned} G[J,\phi] = \langle e^{J_i\varphi^i}\rangle^\phi = \frac{Z[J+J[\phi]]}{Z[J[\phi]]} \\ =: e^{J_i\phi^i} + H[J,\phi], \end{aligned} \tag{2.30}$$

so that

$$\begin{aligned} G[J,\phi] &= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}J_{i_1}\cdots J_{i_\ell}\hat{G}^{i_1\dots i_\ell}[\phi] \\ &= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\ell!}\frac{1}{n!}G^{i_1\dots i_\ell}_{j_1\dots j_n}J_{i_1}\cdots J_{i_\ell}\phi^{j_1}\cdots\phi^{j_n}. \end{aligned} \tag{2.31}$$

In addition, for an observable $A[\varphi]$,

$$A[\varphi] = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}A_{i_1\dots i_\ell}\varphi^{i_1}\cdots\varphi^{i_\ell}, \tag{2.32}$$

one can define

$$G^A[\phi] := A[\phi] + H^A[\phi] := \langle A[\varphi]\rangle^\phi. \tag{2.33}$$

This functional can be expanded as

$$G^A[\phi] = \sum_{n=0}^{\infty}\frac{1}{n!}G^A_{j_1\dots j_n}\phi^{j_1}\cdots\phi^{j_n}, \tag{2.34}$$

where

$$G^A_{j_1\dots j_n} = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}A_{i_1\dots i_\ell}G^{i_1\dots i_\ell}_{j_1\dots j_n}. \tag{2.35}$$

From Schwinger's principle, it follows that if $S[\phi]$ is perturbed by adding a term $\delta S[\phi] = \delta\lambda A[\phi]$,

$$\delta\Gamma[\phi] = \delta\lambda\langle A[\varphi]\rangle^\phi = \delta\lambda G^A[\phi]. \tag{2.36}$$

Hence, diagrammatically, $G^A_{j_1\dots j_n}$ is given by connected and irreducible graphs with amputated legs $j_1\dots j_n$, constructed with the propagator and the vertices of the action plus exactly one vertex of the composite operator $A[\phi]$.[4] We will refer to the $G^A_{j_1\dots j_n}$ as the *amputated matrix elements* of A, since the standard matrix elements of A, $\langle A\varphi^{i_1}\cdots\varphi^{i_m}\rangle$, can be recovered from them, to wit, by adding tree graphs of Γ (in general several trees) in all possible ways saturating all amputated legs j_r and fields i_s, provided that each tree contains at most one end of type j_r (so that no new loops are generated).

[3] Here J is an independent variable, not $J[\phi]$.

[4] The Feynman rules of these vertices are $A_{i_1\dots i_\ell}$ and diagrammatically are represented by a single point.

2.4 Momentum representation conventions

In momentum space we use the following conventions:

$$\Gamma[\phi] = \sum_{n\geq 0}\frac{1}{n!}\int\prod_{i=1}^{n}\frac{d^dp_i}{(2\pi)^d}\tilde{\Gamma}_n(p_1,\dots,p_n)\tilde{\phi}_1(p_1)\cdots\tilde{\phi}_n(p_n) \tag{2.37}$$

with $\tilde{\phi}(p) = \int d^dx\, e^{ipx}\phi(x)$ and

$$\tilde{\Gamma}_n(p_1,\dots,p_n) = (2\pi)^d\delta(p_1+\cdots+p_n)\Gamma_n(p_1,\dots,p_{n-1}) \tag{2.38}$$

due to invariance under translations.[5]

Likewise, we introduce the momentum version of the coefficients $H^{i_1\dots i_\ell}_{j_1\dots j_n}$, denoted

$$\begin{aligned} &\tilde{H}^\ell_n(q_1,\dots,q_\ell;p_1,\dots,p_n) \\ &= (2\pi)^d\delta\left(\sum_{i=1}^{\ell}q_i+\sum_{j=1}^{n}p_j\right)H^\ell_n(q_1,\dots,q_\ell;p_1,\dots,p_n) \end{aligned} \tag{2.39}$$

or simply $(2\pi)^d\delta(\sum q+\sum p)H^\ell_n(q;p)$. Explicitly, for $n=\ell=2$,

$$\begin{aligned} &H^2_2(q_1,q_2;p_1,p_2) \\ &= D(q_1)D(q_2)D(q_1+p_1)\Gamma_3(p_1,q_1,q_1+p_1) \\ &\quad + D(q_1)D(q_2)D(q_1+p_2)\Gamma_3(p_2,q_1,q_1+p_2) \\ &\quad + D(q_1)D(q_2)\Gamma_4(p_1,p_2,q_1,q_2), \end{aligned} \tag{2.40}$$

where

$$D(p) := (-\Gamma_2(p))^{-1} \tag{2.41}$$

denotes the full propagator (connected 2-point function).

Similarly $H^A_{j_1\dots j_n}$ corresponds to $\tilde{H}^A_n(p)$, or $H^A_n(p)$ after extracting $(2\pi)^d\delta(\sum p)$ if $A[\phi]$ is translationally invariant, and the same goes for $G^A_n(p)$.

In what follows, different types of Feynman graphs will appear:

(i) S-graphs. They are constructed using the free propagator $D_0(k)$ and the vertex of the action g, equivalently, s^{ij} and $g_{i_1\dots i_n}$.
(ii) Γ-graphs. They are constructed using the full propagator $D(k)$ and the full vertices $\Gamma_n(p)$, equivalently, D^{ij} and $\Gamma_{i_1\dots i_n}$.
(iii) $\hat{\Gamma}$-graphs. These are those of the Theorem, using $\hat{D}^{ij}[\phi]$ as line and $\hat{\Gamma}_{i_1\dots i_n}[\phi]$ as vertices and ϕ is not set to zero.

[5] The functions $\Gamma_n(p_1,\dots,p_n)$ or simply $\Gamma_n(p)$, are only defined in the subspace $\sum p = 0$; therefore, depending on the context, we may write them as $\Gamma_n(p_1,\dots,p_{n-1})$ for convenience. The same notational liberty will be taken for similar functions, e.g. $H_n(q;p)$ below.

Here, and in what follows, we use a Euclidean signature.

3 Linearized renormalization of super-renormalizable theories

We want to investigate the idea of using the Schwinger principle (2.18) to perform the renormalization of a renormalizable theory. In this Section we start with the more amenable case of a super-renormalizable theory.

A (first-order) infinitesimal variation of the bare action S, in DeWitt notation

$$\delta S[\phi] = \sum_{n\geq 0} \frac{1}{n!}\delta g_{i_1\ldots i_n}\phi^{i_1}\cdots\phi^{i_n}, \tag{3.1}$$

will induce an infinitesimal variation in the effective action

$$\begin{aligned}\delta\Gamma[\phi] &= \langle\delta S[\varphi]\rangle^{\phi}\\ &= \delta S[\phi] + \sum_{\ell\geq 2}\frac{1}{\ell!}\delta g_{i_1\ldots i_\ell}\hat{H}^{i_1\ldots i_\ell}[\phi],\end{aligned} \tag{3.2}$$

and

$$\delta\Gamma_{j_1\ldots j_n} = \delta g_{j_1\ldots j_n} + \sum_{\ell\geq 2}\frac{1}{\ell!}\delta g_{i_1\ldots i_\ell} H^{i_1\ldots i_\ell}_{j_1\ldots j_n}. \tag{3.3}$$

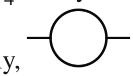
In the spirit of [16], our point of view is that $\Gamma[\phi]$ is already renormalized and the correlation functions are free from ultraviolet (UV) divergences, and the same goes for the functionals $\hat{H}$ and the coefficients H, which are finite combinations of the propagators and vertices of $\Gamma[\phi]$. However, new UV divergences arise in $\delta\Gamma[\phi]$ since $\langle\delta S[\phi]\rangle$ requires the expectation value of the composite local operators present in the action. In Eq. (3.3) the divergences are introduced by the sum over the indices $i_1\ldots i_\ell$, or equivalently due to loops in a momentum space representation. To render $\delta\Gamma[\phi]$ finite, such divergences have to be canceled by divergences in the bare parameters of the action, $\delta g_{i_1\ldots i_\ell}$.

3.1 The ϕ_4^3 theory

In order to analyze this matter, let us consider the theory ϕ_4^3 (i.e., $\kappa = 3$ and $d = 4$ in (2.1)) which is super-renormalizable. In this theory, the bare coupling and the bare wavefunction renormalization factor are finite and need not be renormalized; they will be denoted g and Z (rather than g_0 and Z_0). Only the bare mass, m_0, requires renormalization, so we only introduce one renormalization condition,

$$m_R^2 = \Gamma_2(p)\Big|_{p=0}. \tag{3.4}$$

The parameters Z and g are those of the action but they can also be recovered from the effective action in the large momentum limit (Eq. (3.25) below).

We will use a Euclidean cutoff Λ to regularize the UV divergences. In the ϕ_4^3 theory, there is a single primitive divergent S-graph, namely, , which is canceled by using as the bare mass

$$m_0^2 = m_R^2 + m_{\rm ct}^2 \tag{3.5}$$

with

$$m_{\rm ct}^2(\Lambda) = -\frac{1}{2}\frac{g^2}{Z^2}\Omega_4 L_\Lambda + \text{s.d.t.}, \tag{3.6}$$

where "s.d.t." stands for subdominant terms. Also, we have introduced the notations

$$L_\Lambda := \log(\Lambda/\mu) \tag{3.7}$$

and

$$\int \frac{d^dq}{(2\pi)^d} f(q^2) = \Omega_d \int_0^\infty dt\, t^{d-1} f(t^2). \tag{3.8}$$

The value of the scale μ is not relevant as the renormalization condition eliminates its dependence. Using the cutoff action with bare mass $m_0(\Lambda)$ in the Feynman rules and then removing the cutoff produces a finite $\Gamma[\phi]$. This is the standard approach.

Now, our point of view will be that the effective action $\Gamma[\phi]$ of the ϕ_4^3 theory is already renormalized and so it no longer depends on any regulators. Such an effective action provides finite values for the correlation functions.[6] The functional $\Gamma[\phi]$ is completely determined by the values of the parameters m_R^2, Z, and g. We consider a first-order variation in the action

$$\delta S = \delta m_0^2\, O^m + \delta Z\, O^Z + \delta g\, O^g. \tag{3.9}$$

Here all three operators include cutoffs through profile factors F, which we assume to be equal for simplicity:

$$\begin{aligned} O^m &:= \int d^dx \frac{1}{2}\hat{\phi}^2(x),\\ O^Z &:= \int d^dx \frac{1}{2}(\partial_\mu\hat{\phi}(x))^2,\\ O^g &:= \int d^dx \frac{1}{\kappa!}\hat{\phi}^\kappa(x).\end{aligned} \tag{3.10}$$

with $d = 4$ and $\kappa = 3$, and

$$\hat{\phi}(x) := F(-\partial^2/\Lambda^2)\phi(x). \tag{3.11}$$

Here $F(x)$ is a real decreasing smooth function rapidly approaching 1 as $x \to 0$, and 0 as $x \to \infty$. Valid convergence conditions are

$$\forall\alpha\in\mathbb{R} \quad \lim_{x\to 0^+} x^\alpha(F(x)-1) = \lim_{x\to+\infty} x^\alpha F(x) = 0. \tag{3.12}$$

[6] Of course, new divergences arise as two or more fields are located at ever closer points to produce a composite operator.

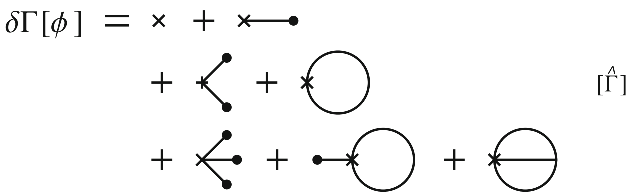


Fig. 1 Graphs contributing to $\delta\Gamma[\phi]$ (Eq. (3.16)) for a variation $\delta S[\phi]$ with two- and three-point vertices (Eq. (3.15)). These are $\hat{\Gamma}$-graphs (indicated by the tag $[\hat{\Gamma}]$ in the figure), that is, the lines and (uncrossed) vertices in the graphs are $\hat{D}^{ij}[\phi]$ and $\hat{\Gamma}_{i_1\ldots i_n}[\phi]$ (without setting the field ϕ to zero). The vertices represented by a cross are those of δS, i.e., $\delta g_{i_1\ldots i_\ell}$ (for $\ell = 2, 3$). In the figure, the contributions from the vertices of δS of 0- and 1-points – omitted in the formulas – have been included by completeness

For convenience, we assume that $F(x) = 1$ for $0 \le x \le 1$ and $F(x) = 0$ for $x \ge 2$; otherwise F is smooth and decreasing. The effect of the regulator F is to suppress the interaction at momenta well above Λ. For regularization in coordinate space see [17,18].

At the level of the Feynman rules in momentum space, the interaction vertex of δS are

$$(\delta m_0^2 + p^2 \delta Z) F^2(p^2/\Lambda^2) \tag{3.13}$$

for $\delta m_0^2\, O^m + \delta Z\, O^Z$ and

$$\delta g \prod_{j=1}^{\kappa} F(p_j^2/\Lambda^2) \tag{3.14}$$

for $\delta g\, O^g$, where the p_j denote the momenta at the κ-point vertex. In the presence of Λ, that is, before taking the limit $\Lambda \to \infty$, all divergences from the loops are regulated in $\langle \delta S[\phi]\rangle$.

In (3.9) 2- and 3-point operators are present. Terms corresponding to 0- and 1-point vertices should also be included in $\delta S[\phi]$ to renormalize the 0- and 1-point vertex functions in $\delta\Gamma[\phi]$, but we will often omit them as they will not be relevant for the discussion (they do not induce divergences on m_0^2).

For a ϕ^3 theory

$$\delta S = \frac{1}{2}\delta m_{ij}\phi^i\phi^j + \frac{1}{3!}\delta g_{ijk}\phi^i\phi^j\phi^k \tag{3.15}$$

Equation (3.2) yields

$$\delta\Gamma[\phi] = \delta S[\phi] + \frac{1}{2}\delta m_{ij}\hat{H}^{ij}[\phi] + \frac{1}{3!}\delta g_{ijk}\hat{H}^{ijk}[\phi]. \tag{3.16}$$

The graphs are displayed in Fig. 1. A key point here is that the number of $\hat{\Gamma}$-graphs involved is finite. Upon expansion

$$\begin{aligned}\delta\Gamma_{j_1\ldots j_n} &= \delta m_{j_1 j_2}\delta_{n,2} + \delta g_{j_1 j_2 j_3}\delta_{n,3}\\ &\quad + \frac{1}{2}\delta m_{ij} H^{ij}_{j_1\ldots j_n} + \frac{1}{3!}\delta g_{ijk} H^{ijk}_{j_1\ldots j_n}.\end{aligned} \tag{3.17}$$

The number of Γ-graphs is also finite for each given n. They correspond to an infinite number of S-graphs.

In a momentum representation, (3.15) becomes

$$\delta S_n(q) = \left(\delta m_0^2 + \delta Z q_1^2\right) F_{q_1}^2 \delta_{n,2} + \delta g F_{q_1} F_{q_2} F_{q_3}\delta_{n,3}, \tag{3.18}$$

where $F_q := F(q^2/\Lambda^2)$. In turn, (3.17) becomes

$$\begin{aligned}\delta\Gamma_n(p) &= \delta m_0^2\left(\delta_{n,2} + H_n^m(p)\right)\\ &\quad + \delta Z\left(\delta_{n,2}p^2 + H_n^Z(p)\right)\\ &\quad + \delta g\left(\delta_{n,3} + H_n^g(p)\right).\end{aligned} \tag{3.19}$$

Here, we use p^2 to refer to p_1^2 (or p_2^2), but more importantly, in writing this expression, we have dropped explicit factors F_p (they are implicit). The form factor F_q is relevant when q can be large, but does not differ from 1 for momenta that are never in the UV region, so we adopt this convention to have simpler expressions.

The contributions from the loops have (relevant) form factors in the internal lines. Explicitly,

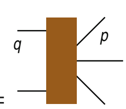
$$\begin{aligned}H_n^m(p) &= \int \frac{d^dq}{(2\pi)^d} F_q^2 A_n(q;p)\\ H_n^Z(p) &= \int \frac{d^dq}{(2\pi)^d} F_q^2 q^2 A_n(q;p)\\ H_n^g(p) &= \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d} F_{q_1}F_{q_2}F_{q_1+q_2} B_n(q;p)\end{aligned} \tag{3.20}$$

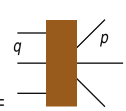
with $d = 4$ and

$$\begin{aligned}A_n(q;p) &:= \frac{1}{2}H_n^2(q,-q;p),\\ B_n(q;p) &:= \frac{1}{3!}H_n^3(q_1,q_2,-q_1-q_2;p).\end{aligned} \tag{3.21}$$

The functions $H_n^\alpha(p)$ (where the label α refers to m, Z or g) as well as A_n and B_n, are only defined in the subspaces $\sum p = 0$ and $\sum q = 0$.

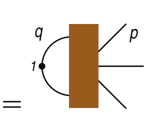
The functions $A_n(q;p)$ and $B_n(q;p)$ depend only on the effective action and not on the regularization details nor the cutoff. They can be represented diagrammatically as

$A_n(q;p) =$ 

$B_n(q;p) =$ 

(3.22)

It should be recalled that in these functions, the n lines p are amputated, but not the lines q. Likewise $H_n^m(p)$, $H_n^Z(p)$ and $H_n^g(p)$ can be represented as

$H_n^m(p) =$ 

(3.23)

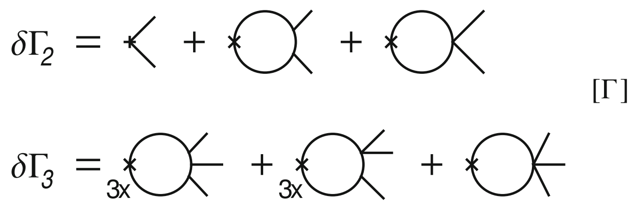
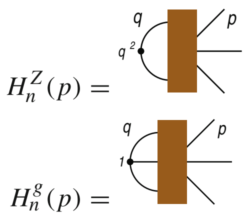


Fig. 2 For the theory ϕ_4^3, graphs contributing to $\delta\Gamma_2(p)$ and $\delta\Gamma_3(p)$, Eq. (3.19) for $\delta g = 0$. The lines and (uncrossed) vertices in the graphs are those of $\Gamma[\phi]$, i.e., $D(p)$ and $\Gamma_n(p)$, hence these are Γ-graphs. This is indicated by the tag $[\Gamma]$ in the figure. The vertices represented by a cross are those of δS, with Feynman rule $F_q^2(\delta m_0^2 + q^2\delta Z)$. The p-legs are distinguishable; therefore, the first two graphs of $\delta\Gamma_3(p)$ have three versions



3.1.1 Renormalization from δZ

To simplify the discussion, we will start with the case $\delta g = 0$. In the theory ϕ_4^3 the mass term δm_0^2 is divergent but does not itself introduce divergences; instead the divergences are induced on δm_0^2 by δZ and δg, which are finite.[7] Therefore, while the mass term δm_0^2 cannot be assumed to vanish if either δZ or δg are present, the choices $\delta g = 0$ or $\delta Z = 0$ are consistent. This is no longer true in the renormalizable case (e.g., the theory ϕ_6^3).

The contributions to $\Gamma_2(p)$

$$\delta\Gamma_2(p) = \delta m_0^2\big(1 + H_2^m(p)\big) + \delta Z\big(p^2 + H_2^Z(p)\big) \tag{3.24}$$

are displayed in the first row of Fig. 2.

The asymptotic UV behavior of S-graphs is described by Weinberg's theorem [12,19]. In the ϕ_4^3 theory, the effect of the quantum fluctuations is suppressed in the asymptotic region; for large momenta q with fixed p and k,[8]

$$\begin{aligned} D(q) &= D_0(q) + O(\log(q^2)/q^4) \\ \Gamma_3(q, k-q, p) &= g + O(\log(q^2)/q^2) \\ \Gamma_n(q, k-q, p) &= O(1/q^2) \quad n \geq 4, \end{aligned} \tag{3.25}$$

[7] This is already clear from the form of m_{ct}^2 in (3.6), which depends on Z and g but not on the mass.

[8] For a quantity $f(x)$ defined as a formal power series, $f(x) = \sum_k f_k(x) g^k$, the statement $f(x) = O(x^n)$ should be understood as $f_k(x) = O(x^n)\ \forall k$.

where

$$D_0(q) := -\frac{1}{Zq^2 + m_R^2}. \tag{3.26}$$

The first Eq. (3.25) would be equally correct using any other mass scale instead of m_R. The choice adopted might be preferable to connect with perturbation theory and the BPHZ renormalization scheme [12].

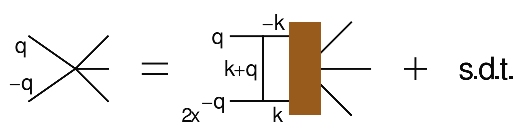
Due to this asymptotic behavior and attending to the graphs in Fig. 2, $H_2^m(p)$ is UV-finite as $\Lambda \to \infty$, while $H_2^Z(p)$ has a logarithmic divergence. In (3.24) the divergence from $\delta Z H_2^Z(p)$ must be canceled by a logarithmically divergent $\delta m_0^2(\Lambda)$, which cannot depend on p. This requires that the *divergent component* of $H_2^Z(p)$ should be proportional to $1 + H_2^m(p)$ as p changes. Moreover (from (3.19)) the same (divergent) proportionality constant must hold between the divergent component of $H_n^Z(p)$ and $H_n^m(p)$ (which is finite) for $n > 2$.

To see how this works, we will analyze the asymptotic behavior of $A_n(q; p)$ in the regime of large q with fixed p. To this end, let us first establish the relation

$$\Gamma_{n+2}(q, -q, p) = 2g^2 D(q) H_n^m(p) + O^+(1/q^3) \quad n \geq 2. \tag{3.27}$$

Here, we have introduced the notation $f(x) = O^+(x^n)$ when $x \to \infty$ to denote $f(x) = O(x^{n+\eta})$ for all $\eta > 0$. In particular $O(x^n \log^k(x)) \subseteq O^+(x^n)$.

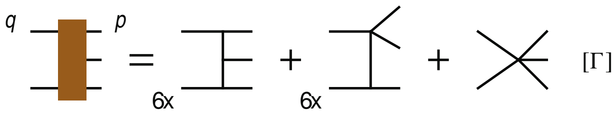
Asymptotically the leading terms are those with a minimum number of propagator-lines traversed by the momentum flow of q. In the S-graphs of $\Gamma_{n+2}(q, -q, p)$ with $n \geq 2$, the two external lines q and $-q$ must go to two different vertices, because the vertex functions are one-particle irreducible. The leading terms are those with just one internal line carrying a large momentum $q+k$ connecting the two vertices (see (3.28)). The two outgoing lines attached to those two vertices must be internal (otherwise the graph would be reducible) and recombine to produce the final n p-legs; the recombination involves precisely the amplitude $A_n(k; p)$. Diagrammatically,



(3.28)

The left-hand side (LHS) represents $\Gamma_{n+2}(q, -q, p)$; all legs are amputated. There are two q-legs and n p-legs. The box is $A_n(k; p)$. The legs are distinguishable, so there are two versions of the graph. Furthermore, for large q the internal propagator $D(k+q) = D(q) + O(1/q^3)$; in the leading term, that with $D(k+q) \to D(q)$, the integration over k can be carried out and it produces $H_n^m(p)$. This proves (3.27). The factor 2 compensates for the symmetry factor $1/2$ present in $H_n^m(p)$ but not in $\Gamma_{n+2}(q,-q,p)$.

Let us now analyze the asymptotic behavior of $A_n(q;p)$. The cases $n=2$ and $n\geq 3$ are discussed separately. Let us consider $A_n(q;p)$ for $n=3$,



(3.29)

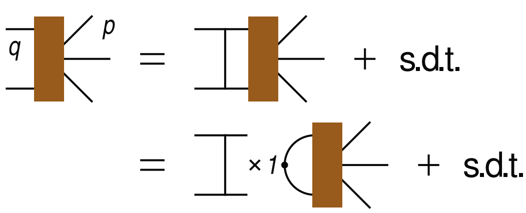
Recall that in $A_n(q;p)$ the p-lines are amputated and the q-lines are not. The first Γ-graph in the right-hand side (RHS) is $O(1/q^8)$ from the four explicit propagators (two external and two internal) carrying the large momentum q. The second graph is also $O(1/q^8)$ from the three explicit propagators and the vertex $\Gamma_4=O(1/q^2)$. The third graph, with two explicit external propagators and a vertex $\Gamma_5=O(1/q^2)$, dominates the asymptotic behavior. More generally, for $n\geq 3$ the leading term of $A_n(q;p)$ comes from the graph with a vertex Γ_{n+2} and

$$A_n(q;p)=\frac{1}{2}D^2(q)\Gamma_{n+2}(q,-q,p)+O(q^{-8})\quad n\geq 3. \tag{3.30}$$

Therefore, using (3.27),

$$A_n(q;p)=g^2D^3(q)H_n^m(p)+O^+(q^{-7})\quad n\geq 3. \tag{3.31}$$

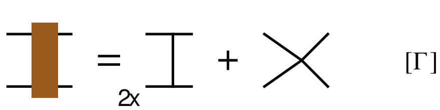
Diagrammatically, for $A_n(q;p)$ and $n\geq 3$,



(3.32)

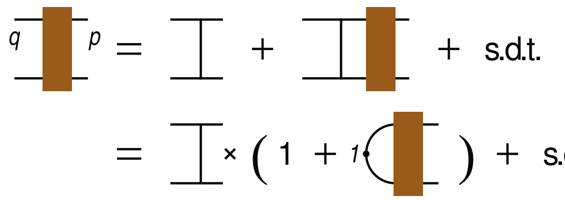
Here, we have chosen to use the full propagator $D(q)$ in the three lines carrying the momentum q. $D_0(q)$ would be also a valid choice. The choice affects the precise definition of the "finite component" of $H_n^Z(p)$ but this ambiguity will be removed by the renormalization condition.

The case $n=2$ is special because the two Γ-graphs in $A_2(q;p)$ are of the same order



(3.33)

The graph with Γ_4 does not dominate; hence, there is an additional term in $A_2(q;p)$:



(3.34)

In summary,

$$\begin{aligned}&A_n(q;p)=g^2D^3(q)(\delta_{n,2}+H_n^m(p))+\hat{A}_n(q;p),\\&\hat{A}_n(q;p)=O^+(1/q^7).\end{aligned} \tag{3.35}$$

We can use this result in (3.20). Since $A_n(q;p)=O(q^{-6})$, the integral over q in $H_n^m(p)$ is UV-finite, while $H_n^Z(p)$ diverges logarithmically,

$$H_n^Z(p)=C_Z(\Lambda)(\delta_{n,2}+H_n^m(p))+\hat{H}_n^Z(p), \tag{3.36}$$

where the remainder $\hat{H}_n^Z(p)$ is UV-finite and

$$C_Z(\Lambda)=\int\frac{d^4q}{(2\pi)^d}F_q^2q^2g^2D^3(q)=-\Omega_4\frac{g^2}{Z^3}L_\Lambda+\text{s.d.t.} \tag{3.37}$$

The structure of (3.36) introduced in (3.19) allows to write

$$\begin{aligned}\delta\Gamma_n(p)=\delta m^2\left(\delta_{n,2}+H_n^m(p)\right)+\delta Z\left(\delta_{n,2}p^2+\hat{H}_n^Z(p)\right)\\+\delta g\left(\delta_{n,3}+\hat{H}_n^g(p)\right)\end{aligned} \tag{3.38}$$

where

$$\delta m^2:=\delta m_0^2+\delta Z\,C_Z+\delta g\,C_g, \tag{3.39}$$

and we have already allowed for the presence of a δg contribution, which will be discussed subsequently.

The crucial observation is that the bare-mass parameter $\delta m_0^2(\Lambda)$ can be chosen so that δm^2 is finite, and this renders all $\delta\Gamma_n(p)$ also finite.

In (3.38) it can be assumed that the limit $\Lambda\to\infty$ has been taken, so the result no longer depends on the value of the cut-off or the regularization procedure. However $\hat{H}_n^Z(p)$, $\hat{H}_n^g(p)$ and δm^2 still have a dependence on the conventions adopted regarding how to split some functions into their leading and subleading components. For instance, in (3.35) the leading term was chosen with $D^3(q)$ instead of $D^2(q)D_0(q)$. In this sense, these quantities have no direct physical meaning as they are not determined solely from Γ and $\delta\Gamma$.

The ambiguities are fixed by enforcing the renormalization condition in (3.4),

$$\begin{aligned}\delta m_R^2&=\delta\Gamma_2(0)\\&=\delta m^2(1+H_2^m(0))+\delta Z\,\hat{H}_2^Z(0)+\delta g\,\hat{H}_2^g(0).\end{aligned} \tag{3.40}$$

Using this relation to eliminate δm^2 in favor of δm_R^2 and substituting in (3.38), yields

$$\begin{aligned}\delta\Gamma_n(p) = \delta m_R^2(\delta_{n,2} + H_{R,n}^m(p)) \\ + \delta Z(\delta_{n,2}p^2 + H_{R,n}^Z(p)) \\ + \delta g(\delta_{n,3} + H_{R,n}^g(p))\end{aligned} \tag{3.41}$$

with

$$\begin{aligned}H_{R,n}^m(p) &= \frac{H_n^m(p) - \delta_{n,2}H_2^m(0)}{1 + H_2^m(0)}, \\ H_{R,n}^Z(p) &= \hat{H}_n^Z(p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)}\hat{H}_2^Z(0), \\ H_{R,n}^g(p) &= \hat{H}_n^g(p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)}\hat{H}_2^g(0).\end{aligned} \tag{3.42}$$

By construction, $H_{R,2}^m(0) = H_{R,2}^Z(0) = H_{R,2}^g(0) = 0$, so that $\delta m_R^2 = \delta\Gamma_2(0)$, regardless of the values of δZ and δg.

Note that the same Eqs. (3.42) hold using H_n^Z and H_n^g instead of $\hat{H}_n^Z$ and $\hat{H}_n^g$,[9] although in that case each term would be divergent separately.

After removing the cutoff, the function $H_{R,n}^m(p)$ is well-defined and is determined solely and completely from the effective action. The same statement holds for $H_{R,n}^Z(p)$. In fact, whatever the concrete choice of the leading term of $A_n(q;p)$ used in the definition of the remainder $\hat{A}_n(q;p)$, it is erased in $H_{R,n}^Z(p)$, as this function can be expressed as

$$\begin{aligned}H_{R,n}^Z(p) &= \int \frac{d^4q}{(2\pi)^4} q^2 A_{R,n}(q;p), \\ A_{R,n}(q;p) &:= A_n(q;p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)}A_2(q;0).\end{aligned} \tag{3.43}$$

By construction, $A_{R,2}(q;0) = 0$. The integrand $A_{R,n}(q;p) = O^+(q^{-7})$, hence no regulator is required.

As a technical aside, a function $F(q;p)$ *factorizes for large* q *and fixed* p when there exist functions $A(q)$ and $B(p)$ such that $\lim_{q\to\infty} F(q,p)/A(q) = B(p)$. Then $F(q;p) = A(q)B(p) + \hat{F}(q;p)$ where $\hat{F}(q;p)$ is a subleading component. For a given $F(q;p)$ there is an ambiguity in $A(q)$ since it is always possible to use instead $A'(q) = \lambda A(q) + R(q)$, where $\lambda \neq 0$, and $R(q)$ is subleading. However, the subtracted function

$$\begin{aligned}F_R(q;p) &:= F(q;p) - \frac{B(p)}{B(p_0)}F(q;p_0) \\ &= \hat{F}(q;p) - \frac{B(p)}{B(p_0)}\hat{F}(q;p_0)\end{aligned} \tag{3.44}$$

is free from such ambiguity and is more convergent than the original F. In turn $F_R(q;p)$ depends on the choice of p_0 such that $F_R(q;p_0) = 0$.

[9] Such equations are obtained directly from (3.19) by eliminating δm_0^2 in favor of δm_R^2.

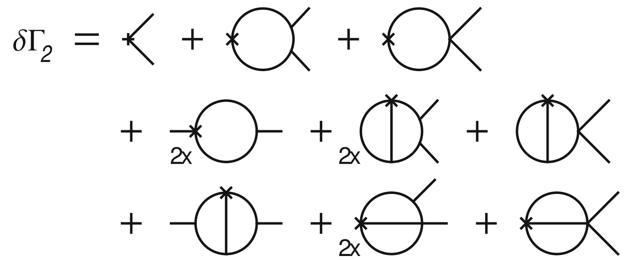


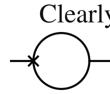
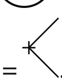
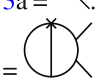
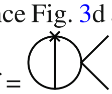
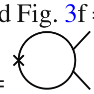
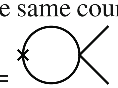
Fig. 3 For the ϕ_4^3 theory, Γ-graphs contributing to $\delta\Gamma_2(p)$. The lines and (uncrossed) vertices in the graphs are $D(p)$ and $\Gamma_n(p)$. The vertices represented by a cross are those of δS, with Feynman rule $F_q^2(\delta m_0^2 + q^2\delta Z)$ for the two-point vertex and $+F_{q_1}F_{q_2}F_{q_1+q_2}\delta g$ for the three-point vertex

3.1.2 Renormalization from δg

Let us include the effect of a non-vanishing δg. Since the infinitesimal contributions are additive, we can set $\delta Z = 0$ in this discussion. We start with the case $n = 2$.

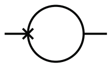
$$\delta\Gamma_2(p) = \delta m_0^2\big(1 + H_2^m(p)\big) + \delta g\, H_2^g(p). \tag{3.45}$$

The corresponding Γ-graphs are displayed in Fig. 3. The graphs may contain divergences coming from i) loops and ii) from parameters (namely, δm_0^2). Because $\delta Z = 0$, the divergences from the loops are present exclusively in the three graphs in the second row, and the divergences from the parameters are present exclusively in the three graphs of the first row. The three graphs in the third row are fully finite when the regulator in δS is removed.

Clearly, the logarithmic loop-divergence in Fig. 3d =  can be can removed by a mass counterterm in Fig. 3a = . In turn, since Fig. 3d appears as subgraph in Fig. 3e =  and Fig. 3f = , the same counterterm acting in Fig. 3b =  and Fig. 3c =  will cancel the subdivergences in Fig. 3e, f, respectively.

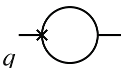
The three graphs in the first row of Fig. 3 correspond to the $\delta m_0^2\big(1 + H_2^m(p)\big)$ terms depending on $A_2(q;p)$ already discussed. The remaining six graphs correspond to $\delta g\, H_2^g(p)$, as defined in (3.20). It will be convenient to split this function into two components (d) and (f) (for *divergent* and *finite*) corresponding respectively to the second and third row graphs of Fig. 3:

$$H_2^g(p) = H_2^{g(\mathrm{d})}(p) + H_2^{g(\mathrm{f})}(p). \tag{3.46}$$

$H_2^{g(\mathrm{f})}(p)$ is already UV-finite, while $H_2^{g(\mathrm{d})}(p)$ contains the UV-divergent graph Fig. 3d, and the same graph as a subgraph in Fig. 3e, f. When  is contracted to a point, the three graphs of $H_2^{g(\mathrm{d})}(p)$ turn into the three graphs of $H_2^m(p)$. Algebraically, let

$$K(q) := \int \frac{d^4k}{(2\pi)^4} F_k F_{k+q} B(k;q) \\ B(k;q) := D(k)D(k+q)\Gamma_3(k,q) \tag{3.47}$$

so that

q 

$$= \frac{1}{2}\delta g\, F_q K(q) \tag{3.48}$$

and

$$H_2^{g(\mathrm{d})}(p) = K(p) + \int \frac{d^4q}{(2\pi)^4} F_q K(q)\, A_2(q;p). \tag{3.49}$$

If K is replaced by 1 (and the regulator is removed), this expression becomes $1 + H_2^m(p)$.

$K(p)$ is logarithmically divergent. To disentangle the (sub)divergence, let us extract the leading term. A possible choice is

$$B(q;p) = gD^2(q) + \hat{B}(q;p), \quad \hat{B}(q;p) = O^+(q^{-5}), \tag{3.50}$$

so that

$$K(p) = C_g + \hat{K}(p) \tag{3.51}$$

with $\hat{K}(p)$ UV-finite and[10]

$$C_g(\Lambda) = \int \frac{d^4q}{(2\pi)^4} F_q^2\, g D^2(q) = \Omega_4 \frac{g}{Z^2} L_\Lambda + \text{s.d.t.} \tag{3.52}$$

Correspondingly, we can define

$$\hat{H}_2^{g(\mathrm{d})}(p) := \hat{K}(p) + \int \frac{d^4q}{(2\pi)^4} \hat{K}(q)\, A_2(q;p), \tag{3.53}$$

which is UV-finite since $\hat{K}(q) = O(\log(q^2))$ and $A_2(q;p) = O(q^{-6})$.

As a consequence

$$H_2^g(p) = \hat{H}_2^g(p) + C_g\left(1 + \int \frac{d^4q}{(2\pi)^4} F_q\, A_2(q;p)\right), \tag{3.54}$$

with

$$\hat{H}_2^g(p) := \hat{H}_2^{g(\mathrm{d})}(p) + H_2^{g(\mathrm{f})}(p), \tag{3.55}$$

which is also UV-finite.

The integral in the second term of (3.54) is very similar to $H_2^m(p)$, but not identical: In (3.20) the form factor appears as F_q^2 while in (3.54) it appears as F_q. The difference $F_q(1-F_q)$ vanishes outside the range $\Lambda^2 < q^2 < 2\Lambda^2$, hence the integral in (3.54) differs from $H_2^m(p)$ by an $O(\Lambda^{-2})$ contribution, which vanishes as the cutoff is removed. Note that the form factor of the 3-point vertex needs not coincide with that of the 2-point vertex (or even the form factors of δm_0^2 and δZ could be different). The $O(\Lambda^{-2})$ dependence does not rely on the details of the form factors, provided they are sufficiently well-behaved.

It follows that (up to terms irrelevant after removing the cutoff)

$$H_2^g(p) = \hat{H}_2^g(p) + C_g(\Lambda)(1 + H_2^m(p)). \tag{3.56}$$

When this expression is introduced in (3.45), the Eqs. (3.38) (for $n = 2$) and (3.39) are verified. Not surprisingly, the values of C_Z and C_g are such that $\delta m^2 - \delta m_0^2 = \delta Z C_Z + \delta g C_g$ matches $\delta(-m_{\mathrm{ct}}^2)$, with m_{ct}^2 introduced in (3.5),

$$-\delta Z \Omega_4 \frac{g^2}{Z^3} L_\Lambda + \delta g \Omega_4 \frac{g}{Z^2} L_\Lambda = \delta\left(\frac{1}{2}\frac{g^2}{Z^2}\Omega_4 L_\Lambda\right). \tag{3.57}$$

As before, δm^2 can be traded for δm_R^2 by imposing the renormalization condition $\delta m_R^2 = \delta\Gamma_2(0)$, and in this case (3.41) (for $n = 2$) and (3.42) are also fulfilled. The concrete choice of the leading and subleading components in $B(q;p)$ in (3.50) is not relevant. In fact $H_{R,2}^g(p)$ can be rewritten as

$$H_{R,2}^g(p) = H_{R,2}^{g(\mathrm{d})}(p) + H_{R,2}^{g(\mathrm{f})}(p) \tag{3.58}$$

with

$$H_{R,2}^{g(\mathrm{d})}(p) = K_R(p) + \int \frac{d^4q}{(2\pi)^4} K_R(q)\, A_{R,2}(q;p), \\ H_{R,2}^{g(\mathrm{f})}(p) = H_2^{g(\mathrm{f})}(p) - \frac{1 + H_2^m(p)}{1 + H_2^m(0)} H_2^{g(\mathrm{f})}(0), \tag{3.59}$$

and

$$K_R(p) := \int \frac{d^4q}{(2\pi)^4} (B(q;p) - B(q;0)). \tag{3.60}$$

$H_{R,2}^{g(\mathrm{d})}(0) = H_{R,2}^{g(\mathrm{f})}(0) = 0$, so that the renormalization condition is preserved.

So $H_{R,2}^g(p)$ is expressed entirely in terms of the propagator and vertices of $\Gamma[\phi]$ with UV convergent integrals without a regulator, just like $H_{R,n}^m(p)$ and $H_{R,n}^Z(p)$.

The previous formulas can be extended to $n > 2$. The Γ-graphs in Fig. 3, for $n = 2$, were obtained from the $\hat{\Gamma}$-graphs in Fig. 1 by extracting two (amputated) p-legs in all possible ways, either from the lines or the vertices. The same procedure applies for $n \geq 2$. The result for $n = 3$ is displayed

[10] Actually the factor in (3.52) would be $F_q F_{q+p}$ instead of F_q^2, however, in the UV limit, the difference yields a constant term which is absorbed in subleading terms.

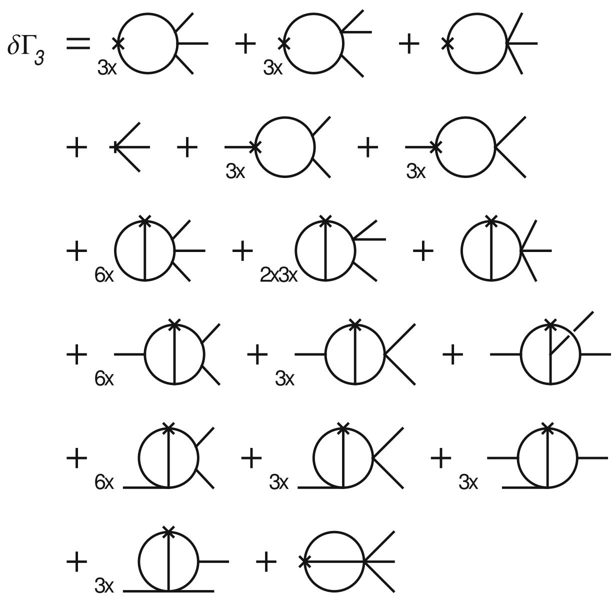
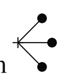


Fig. 4 Γ-graphs for $\delta\Gamma_3(p)$ in the ϕ_4^3 theory

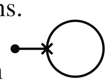
in Fig. 4. Algebraically

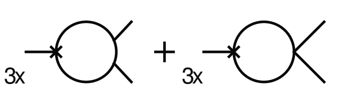
$$\delta\Gamma_n(p) = \delta m_0^2\big(\delta_{n,2} + H_n^m(p)\big) + \delta g\big(\delta_{n,3} + H_n^g(p)\big). \tag{3.61}$$

In (3.61) the term $g\delta_{n,3}$ of course comes from  of Fig. 1. The term $gH_n^g(p)$ comes from extracting n p-legs from the graphs with loops in the third line of Fig. 1. The different contributions are eventually classified into two types

$$H_n^g(p) = H_n^{g(\mathrm{d})}(p) + H_n^{g(\mathrm{f})}(p), \tag{3.62}$$

which are divergent and finite, respectively. Let us analyze these contributions.

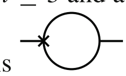
In the $\hat{\Gamma}$-graph , necessarily one of the p-legs is extracted from the crossed vertex, and $n-1$ legs are extracted from the line. For instance, for $n = 3$, the Γ-graphs produced are

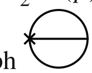


(3.63)

This type of terms yield a contribution

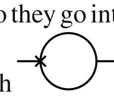
$$\sum_{j=1}^{n}\int \frac{d^4q}{(2\pi)^4} F_q^2 A_{n-1}(q;\, p) = \sum_{j=1}^{n} H_{n-1}^m(p_1, \ldots, \widehat{p}_j, \ldots, p_n), \tag{3.64}$$

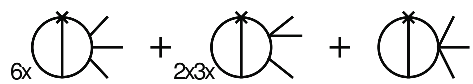
which is convergent without a regulator for $n \geq 3$ and adds to $H_n^{g(\mathrm{f})}(p)$. For $n = 2$ such a contribution is ; it is just $K(p)$, which is logarithmically divergent; therefore, it was included in $H_2^{g(\mathrm{d})}(p)$ in (3.49).

In the $\hat{\Gamma}$-graph  all p-legs are to be extracted from the three lines and the uncrossed vertex. Two types of Γ-graphs may be distinguished:

(i) Those where all n legs are extracted from the same line.
(ii) Otherwise.

Type (ii) graphs are UV-finite and so they go into $H_n^{g(\mathrm{f})}(p)$.

Type (i) graphs contain the subgraph  and in fact, this is the best way to characterize type (i) graphs. Hence, they are divergent and go into $H_n^{g(\mathrm{d})}(p)$. Explicitly, for $n = 3$, these are



(3.65)

The assembling of the n legs is such that they are organized to form the amplitude $A_n(q;\, p)$, thus (including now the $n = 2$ exceptional term)

$$H_n^{g(\mathrm{d})}(p) = \delta_{n,2}K(p) + \int \frac{d^4q}{(2\pi)^4} F_q K(q) A_n(q;\, p). \tag{3.66}$$

Once again, this leads to

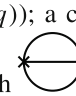
$$H_n^g(p) = \hat{H}_n^g(p) + C_g(\delta_{n,2} + H_n^m(p)), \tag{3.67}$$

that introduced in (3.61) comply with (3.38) and (3.39). Elimination of δm^2 in favor of the renormalization condition then produces (3.42). Finally, $H_{R,n}^g(p)$ can be arranged as

$$H_{R,n}^g(p) = H_{R,n}^{g(\mathrm{d})}(p) + H_{R,n}^{g(\mathrm{f})}(p) \tag{3.68}$$

with

$$\begin{aligned} H_{R,n}^{g(\mathrm{d})}(p) &= \delta_{n,2}K_R(p) + \int \frac{d^4q}{(2\pi)^4} K_R(q)\, A_{R,n}(q;\, p), \\ H_{R,n}^{g(\mathrm{f})}(p) &= H_n^{g(\mathrm{f})}(p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)} H_2^{g(\mathrm{f})}(0), \end{aligned} \tag{3.69}$$

where everything is convergent without the UV regulator. Unlike the case of $H_{R,n}^Z(p)$, $H_{R,n}^g(p)$ requires two subtractions (one in $A_{R,n}(q;\, p)$ and another in $K_R(q)$); a consequence of the two-loop structure of the $\hat{\Gamma}$-graph .

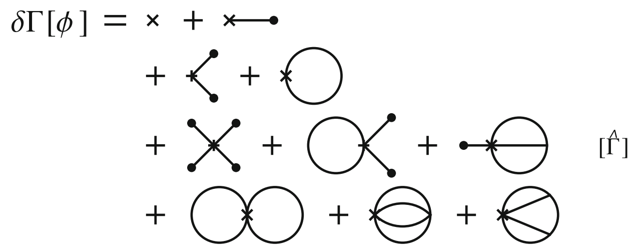
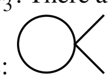
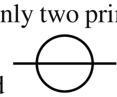


Fig. 5 $\hat{\Gamma}$-graphs contributing to $\delta\Gamma[\phi]$ for a variation $\delta S[\phi]$ with two- and four-point vertices. The contributions from the vertices of δS of 0- and 1-points have been included by completeness

3.2 The ϕ_3^4 theory

The theory ϕ_3^4 is also super-renormalizable, and in fact, it is one of the few quantum field theories for which there exists a mathematically rigorous non-perturbative treatment. Unlike ϕ_4^3, the Hamiltonian of ϕ_3^4 is bounded from below and the renormalized correlation functions exist beyond perturbation theory [7,20].

Despite that, both theories have substantial similarities and most of the treatment just developed for the renormalization of ϕ_4^3 carries over to ϕ_3^4. There are only two primitive divergent Feynman S-graphs:  and . The first one diverges linearly and the second one logarithmically. They are canceled by a counterterm mass term

$$m_{\rm ct}^2 = \frac{1}{2}\frac{g}{Z}\Omega_3\Lambda + \frac{1}{3!}\frac{g^2}{Z^3}\frac{1}{8}\Omega_3 L_\Lambda + \text{s.d.t.} \tag{3.70}$$

Once again, the parameters Z and g need no renormalization.

In this theory there is symmetry under $\phi \to -\phi$, and we will assume that all vertex functions of odd order vanish.

Fig. 5 displays the contributions to $\delta\Gamma[\phi]$ induced by δS with two- and four-point vertices by applying Schwinger's principle.[11] The various vertex functions $\delta\Gamma_n(p)$ are then obtained by extracting the n p-legs in all possible ways. The Γ-graphs corresponding to the contributions to $\delta\Gamma_2(p)$ are displayed in Fig. 6.

In this theory

$$\delta S_n(q) = \left(\delta m_0^2 + \delta Z q^2\right) F_q^2 \delta_{n,2} + \delta g F_{q_1} F_{q_2} F_{q_3} F_{q_4} \delta_{n,4}, \tag{3.71}$$

and

$$\begin{aligned}\delta\Gamma_n(p) = \delta m_0^2\big(\delta_{n,2} + H_n^m(p)\big)\\ + \delta Z\big(\delta_{n,2}p^2 + H_n^Z(p)\big)\\ + \delta g\big(\delta_{n,4} + H_n^g(p)\big).\end{aligned} \tag{3.72}$$

The expressions of $H_n^m(p)$ and $H_n^Z(p)$, as well as $A_n(q;p)$, are as in (3.20) and (3.21), while $H_n^g(p)$ and $B_n(q;p)$

[11] Note that $\Gamma_n(p)$ vanishes for odd n but not so $\hat{\Gamma}_n[\phi]$.

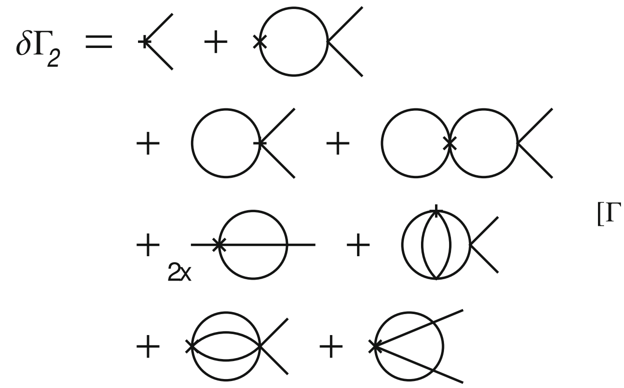


Fig. 6 Γ-graphs for the ϕ_3^4 theory contributing to $\delta\Gamma_2(p)$, assuming $\Gamma_n(p) = 0$ for odd n. The vertices represented by a cross are those of δS, with Feynman rule $F_q^2(\delta m_0^2 + q^2\delta Z)$ for the two-point vertex and $+F_{q_1}F_{q_2}F_{q_3}F_{q_1+q_2+q_3}\delta g$ for the four-point vertex

become

$$\begin{aligned}H_n^g(p) = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\\ F_{q_1}F_{q_2}F_{q_3}F_{q_1+q_2+q_3}B_n(q;p)\\ B_n(q;p) := \frac{1}{4!}H_n^4(q_1,q_2,q_3,-q_1-q_2-q_3;p),\end{aligned} \tag{3.73}$$

with $d = 3$.

As happened with ϕ_4^3, here a finite variation of the mass does not introduce UV divergences while δZ and δg do induce divergences to be absorbed by the mass counterterm.

3.2.1 *Renormalization from δZ*

Let us start by analyzing the effect of δZ, with $\delta g = 0$. Our aim is to establish a relationship as in (3.36). Let us consider the case $n = 2$. $\delta\Gamma_2(p)$ only receives contributions from the first two graphs in the RHS of Fig. 6. Explicitly,

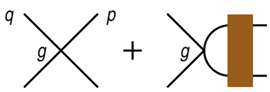
$$A_2(q;p) = \frac{1}{2}D^2(q)\Gamma_4(q,-q,p). \tag{3.74}$$

Since, in this theory,

$$D(q) = O(1/q^2), \quad \Gamma_n(q,-q,p) = O(1) \quad n \geq 4, \tag{3.75}$$

$H_2^m(p)$ is UV-finite while $H_2^Z(p)$ has a linear divergence.

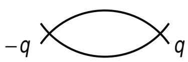
In more detail, when the integration over q in (3.20) is carried out to yield $H_2^Z(p)$, a linear divergence is produced from terms $O(1)$ in $\Gamma_4(q,-q,p)$ and a logarithmic divergence is produced from terms $O(1/q)$; more convergent terms do not produce UV divergences. An analysis similar to that performed for ϕ_4^3 can be carried out for ϕ_3^4. By inspection of the S-graphs of the theory, it can be seen that the $O(1)$ terms are in fact q-independent: they come from graphs where the two external lines q and $-q$ go to the same vertex (with Feynman rule $+g$). The two outgoing lines may not interact again or else they may interact producing the amplitude $A_2(k; p)$. Diagrammatically, the q-independent terms of $\Gamma_4(q, -q, p)$ are

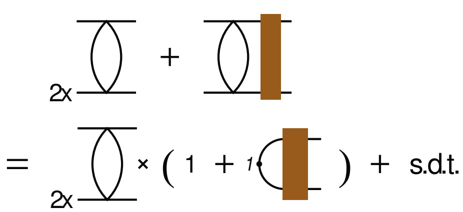


(3.76)

and algebraically

$$\Gamma_4(q, -q, p)\Big|_{q\text{-indep.}} = g(1 + H_2^m(p)). \tag{3.77}$$

On the other hand, the terms $O(1/q)$ in $\Gamma_4(q, -q, p)$ are produced when the large momentum q flows through the graph  which behaves as a $\frac{1}{2}\frac{g^2}{Z^2}\frac{1}{8}\frac{1}{q} + O^+(1/q^2)$. Once again, the two outgoing legs may or may not interact again. Diagrammatically, the $O(1/q)$ contributions to $\Gamma_4(q, -q, p)$ are



(3.78)

corresponding to

$$\Gamma_4(q, -q, p)\Big|_{O(1/q)} = \frac{g^2}{Z^2}\frac{1}{8}\frac{1}{q}(1 + H_2^m(p)) + O^+(1/q^2). \tag{3.79}$$

Combining both results

$$\Gamma_4(q, -q, p) = \tilde{g}(q)(1 + H_2^m(p)) + O^+(1/q^2) \tag{3.80}$$

where

$$\tilde{g}(q) := g + \frac{g^2}{Z^2}\frac{1}{8}\frac{1}{q}. \tag{3.81}$$

Thus

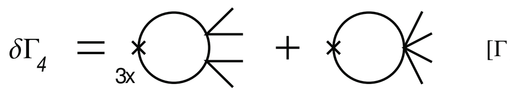
$$\begin{aligned} A_2(q; p) &= \frac{1}{2}\tilde{g}(q)D^2(q)(1 + H_2^m(p)) + \hat{A}_2(q; p), \\ \hat{A}_2(q; p) &= O^+(1/q^6). \end{aligned} \tag{3.82}$$

Then (3.36) is reproduced for $n = 2$ with

$$\begin{aligned} C_Z(\Lambda) &= \frac{1}{2}\Omega_3\left(\frac{g}{Z^2}\Lambda + \frac{1}{8}\frac{g^2}{Z^4}L_\Lambda + \text{s.d.t.}\right) \\ &= -\frac{\partial m_{\text{ct}}^2(\Lambda)}{\partial Z}. \end{aligned} \tag{3.83}$$

This guarantees that a suitable $\delta m_0^2(\Lambda)$ cancels the divergences induced by δZ on $\delta\Gamma_2(p)$.

The analysis for $n \geq 4$ and still $\delta g = 0$ is similar to that done for ϕ_4^3. For $\delta\Gamma_4(p)$, the Γ-graphs are



(3.84)

The divergence comes only from the second graph, with a vertex Γ_6, as the first one has three explicit internal propagators with large momentum q. More generally, for $n \geq 4$, the leading terms in $\delta\Gamma_n(p)$ come from $\Gamma_{n+2}(q, -q, p)$, that fulfills, using the arguments already presented for $\Gamma_4(q, -q, p)$,

$$\begin{aligned} \Gamma_{n+2}(q, -q, p) &= \tilde{g}(q)(\delta_{n,2} + H_n^m(p)) \\ &\quad + O^+(1/q^2) \quad n \geq 2. \end{aligned} \tag{3.85}$$

Therefore

$$\begin{aligned} A_n(q; p) &= \frac{1}{2}\tilde{g}(q)D^2(q)(\delta_{n,2} + H_n^m(p)) + \hat{A}_n(q; p), \\ \hat{A}_n(q; p) &= O^+(1/q^6). \end{aligned} \tag{3.86}$$

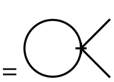
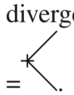
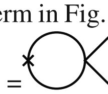
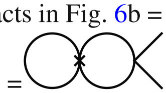
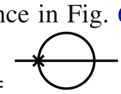
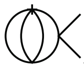
From this point on, the equations already derived for the theory ϕ_4^3 regarding δZ follow (some with obvious modifications) including (3.36), (3.38)–(3.43). Thus $H_n^Z(p)$ is expressed through integrals which are UV-finite without a regulator.

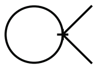
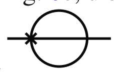
3.2.2 *Renormalization from δg*

Let us now study the renormalization of $H_n^g(p)$, thus we assume $\delta Z = 0$. Let us first consider the case $n = 2$. The Γ-graphs corresponding to $\delta\Gamma_2(p)$ are displayed in Fig. 6. The two graphs in the first row are $\delta m_0^2(1 + H_2^m(p))$ already discussed and $H_2^m(p)$ is finite. The remaining graphs are $\delta g H_2^g(p)$. Again this function can be split into UV-divergent and UV-finite components:

$$H_2^g(p) = H_2^{g(\text{d})}(p) + H_2^{g(\text{f})}(p). \tag{3.87}$$

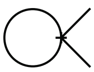
$H_2^{g(\text{f})}(p)$ corresponds to the two graphs in the last row, and $H_2^{g(d)}(p)$ to the four graphs in the second and third rows of Fig. 6.

The graph Fig. 6c =  is linearly divergent, and the divergence can be canceled by a mass-counterterm in Fig. 6a = . When such a counterterm acts in Fig. 6b =  it cancels the divergence in Fig. 6d = . Likewise the graph Fig. 6e =  diverges logarithmically to be canceled by a mass-counterterm in Fig. 6a. When such counterterm is introduced in Fig. 6b it cancels the divergence in Fig. 6f = .

When  is contracted to a point, Fig. 6c, d become to Fig. 6a, 6b, respectively. Likewise when  is contracted to a point Fig. 6e, f become Fig. 6a, b. Algebraically, let

$$C_{g,a}(\Lambda) := \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} F_k^2 D(k) \tag{3.88}$$

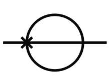
so that

 $= \delta g\, F_q^2 C_{g,a},$ (3.89)

and

$$\begin{aligned} K(q) &:= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} F_{k_1}F_{k_2}F_{k_1+k_2+q} B(k_1,k_2;q) \\ B(k_1,k_2;q) &:= \frac{1}{3} D(k_1)D(k_2)D(k_1+k_2+q)\Gamma_4(k_1,k_2,q) \end{aligned} \tag{3.90}$$

so that

q  $= \frac{1}{2}\delta g\, F_q K(q).$ (3.91)

Then, the four divergent contributions in Fig. 6c–f, can be expressed as

$$\begin{aligned} H_2^{g(\mathrm{d})}(p) &= C_{g,a} + K(p) \\ &\quad + \int \frac{d^3q}{(2\pi)^3}\left(F_q^2 C_{g,a} + F_q K(q)\right) A_2(q;p). \end{aligned} \tag{3.92}$$

If $C_{g,a} + K$ is replaced by 1 (and the regulator is removed), this expression becomes $1 + H_2^m(p)$.

$C_{g,a}$ is a linearly divergent constant,

$$C_{g,a} = -\frac{1}{2}\frac{1}{Z}\Omega_3\Lambda + \text{s.d.t.} \tag{3.93}$$

On the other hand, $K(q)$ has a logarithmic divergence. To isolate it, let us extract a leading term in B,

$$\begin{aligned} B(q_1,q_2;p) &= \frac{1}{3} g D(q_1)D(q_2)D(q_1+q_2) \\ &\quad + \hat{B}(q_1,q_2;p). \end{aligned} \tag{3.94}$$

The vertex $\Gamma_4(q_1,q_2,q_3)$ is $O(1)$ when q_1 and q_2 are large and more specifically it is $g + O(1/q)$ when q_3 is also large. Therefore

$$\hat{B}(q_1,q_2;p) = O^+(q^{-7}) \tag{3.95}$$

except in the zero-measure case $q_1+q_2 =$ finite. (Such exceptional case is precisely that in (3.80), where $q_1 = -q_2 = q$.) We then obtain

$$K(p) = C_{g,b}(\Lambda) + \hat{K}(p) \tag{3.96}$$

with $\hat{K}(p)$ UV-finite and

$$\begin{aligned} C_{g,b}(\Lambda) &= \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3} F_{q_1}F_{q_2}F_{q_1+q_2} \\ &\quad \times \frac{1}{3} g D(q_1)D(q_2)D(q_1+q_2) \\ &= -\frac{1}{3}\frac{g}{Z^3}\frac{1}{8}\Omega_3 L_\Lambda + \text{s.d.t.} \end{aligned} \tag{3.97}$$

We can now define (similar to (3.53))

$$\hat{H}_2^{g(\mathrm{d})}(p) := \hat{K}(p) + \int \frac{d^3q}{(2\pi)^3}\hat{K}(q) A_2(q;p), \tag{3.98}$$

which is UV-finite since $\hat{K}(q) = O(\log^2(q^2))$ and $A_2(q;p) = O(q^{-4})$, and also

$$\hat{H}_2^g(p) := \hat{H}_2^{g(\mathrm{d})}(p) + H_2^{g(\mathrm{f})}(p). \tag{3.99}$$

As a consequence,

$$\begin{aligned} H_2^g(p) &= \hat{H}_2^g(p) + C_{g,a} + C_{g,b} \\ &\quad + \int \frac{d^3q}{(2\pi)^3}(C_{g,a}F_q^2 + C_{g,b}F_q) A_2(q;p). \end{aligned} \tag{3.100}$$

By the same argument given after (3.55), F_q is equivalent to F_q^2 within the integral, up to terms vanishing when the cutoff is removed, and we finally arrive at

$$H_2^g(p) = \hat{H}_2^g(p) + C_g(\Lambda)(1 + H_2^m(p)), \tag{3.101}$$

with

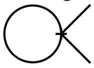
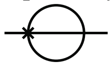
$$C_g := C_{g,a} + C_{g,b} = -\frac{\partial m_{\mathrm{ct}}^2(\Lambda)}{\partial g}. \tag{3.102}$$

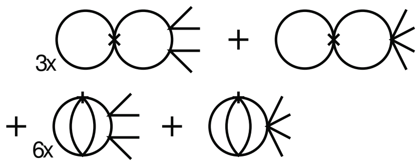
Equation (3.101) is in fact identical to (3.56) of the theory ϕ_4^3. The relations found there regarding $H_2^g(p)$ apply also here (with $d = 3$), in particular, after imposing the renormalization condition, Eqs. (3.58) and (3.59) also hold here with

$$K_R(p) := \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}(B(q_1,q_2;p) - B(q_1,q_2;0)). \tag{3.103}$$

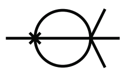
Let us analyze now $H_n^g(p)$ for $n \geq 4$. The discussion is similar to that for ϕ_4^3. The contributions follow by extracting n p-legs in all possible ways from the five $\hat{\Gamma}$-graphs with loops and δg in Fig. 5. The Γ-graphs so obtained are of two types, divergent and finite,

$$H_n^g(p) := H_n^{g(\mathrm{d})}(p) + H_n^{g(\mathrm{f})}(p). \tag{3.104}$$

Divergent are precisely those graphs containing either  or  as a subgraph. All other graphs are finite. Explicitly, for $n = 4$, the divergent graphs are



(3.105)

Note that a graph like  is UV-finite; a divergence would require the three internal lines with large momentum, but in that case $\Gamma_6(q) = O(1/q)$. More generally, $\Gamma_n(q) = O(1/q)$ when $n > 4$ and more than two lines carry a large momentum. This is not in conflict with the fact that $\Gamma_n(q)$ is indeed $O(1)$ for $n \geq 4$ in a kinematic configuration $\Gamma_n(q, -q+k, p)$ with q large and k, p finite.

As already happened for ϕ_4^3, the structure of the divergent graphs is such that the amplitude $A_n(q;\, p)$ is produced. Thus

$$H_n^{g(\mathrm{d})}(p) = \delta_{n,2}(C_{g,a} + K(p)) + \int \frac{d^4q}{(2\pi)^4} \left(F_q^2 C_{g,a} + F_q K(q)\right) A_n(q;\, p). \tag{3.106}$$

This leads to

$$H_n^g(p) = \hat{H}_n^g(p) + C_g(\delta_{n,2} + H_n^m(p)), \tag{3.107}$$

where $\hat{H}_n^g(p)$ is UV-finite. This expression ensures that a single p-independent parameter $\delta m_0^2(\Lambda)$ renormalizes all divergences induced by δg.

After imposing the renormalization conditions, (3.41) (with $\delta_{n,4}$ instead of $\delta_{n,3}$) holds along with (3.68) and (3.69). There, $K_R(p)$ is defined in (3.103) and $A_{R,n}(q;\, p)$ is defined in (3.43). In these expressions all integrals are convergent without a regulator.

3.3 The effective action in the manifold (m_R^2, Z, g)

The previous Sects. 3.1 and 3.2 have dealt with the super-renormalizable theories ϕ_d^κ for $(\kappa, d) = (3, 4)$ and $(4, 3)$. In both cases, the effective action functional is determined by the three parameters (m_R^2, Z, g),[12] and

$$\begin{aligned}
\frac{\partial \Gamma_n(p)}{\partial m_R^2} &= \delta_{n,2} + H_{R,n}^m(p), \\
\frac{\partial \Gamma_n(p)}{\partial Z} &= \delta_{n,2} p^2 + H_{R,n}^Z(p), \\
\frac{\partial \Gamma_n(p)}{\partial g} &= \delta_{n,\kappa} + H_{R,n}^g(p).
\end{aligned} \tag{3.108}$$

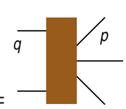
Explicit expressions were obtained for the $H_{R,n}^\alpha(p)$ in terms of Γ-graphs (for $n \geq 2$). Since previously the results themselves necessarily appear mixed with their derivation and with intermediate definitions, for convenience we summarize here the main formulas:

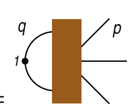
$$\begin{aligned}
H_{R,n}^m(p) &= \frac{H_n^m(p) - \delta_{n,2} H_2^m(0)}{1 + H_2^m(0)} \\
H_{R,n}^Z(p) &= \int \frac{d^dq}{(2\pi)^d} q^2 A_{R,n}(q;\, p) \\
H_{R,n}^g(p) &= H_{R,n}^{g(\mathrm{d})}(p) + H_{R,n}^{g(\mathrm{f})}(p)
\end{aligned} \tag{3.109}$$

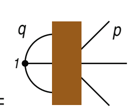
$$\begin{aligned}
A_{R,n}(q;\, p) &= A_n(q;\, p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)} A_2(q;\, 0) \\
H_{R,n}^{g(\mathrm{d})}(p) &= \delta_{n,2} K_R(p) + \int \frac{d^dq}{(2\pi)^d} K_R(q)\, A_{R,n}(q;\, p) \\
H_{R,n}^{g(\mathrm{f})}(p) &= H_n^{g(\mathrm{f})}(p) - \frac{\delta_{n,2} + H_n^m(p)}{1 + H_2^m(0)} H_2^{g(\mathrm{f})}(0)
\end{aligned} \tag{3.110}$$

$$\begin{aligned}
K_R(p) &= \int \frac{d^4q}{(2\pi)^4} (B(q;\, p) - B(q;\, 0)) \quad [\phi_4^3] \\
K_R(p) &= \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} (B(q_1, q_2;\, p) - B(q_1, q_2;\, 0)) \quad [\phi_3^4] \\
B(q;\, p) &= D(q) D(q+p) \Gamma_3(q, p) \\
B(q_1, q_2;\, p) &= \frac{1}{3} D(q_1) D(q_2) D(q_1 + q_2 + p) \Gamma_4(q_1, q_2, p)
\end{aligned} \tag{3.111}$$

Finally,

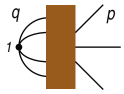
$A_n(q;\, p) =$ 

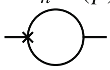
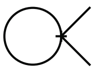
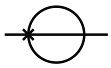
$H_n^m(p) =$ 

$H_n^g(p) =$ 

$[\phi_4^3]$ (3.112)

[12] Actually two further parameters come from the renormalization conditions on $\Gamma_n(p)$ for $n = 0$ and $n = 1$ which can be disregarded in this discussion.

$H_n^g(p) =$  $[\phi_3^4]$

but $H_n^{g(\mathrm{f})}(p)$ only includes the graphs of $H_n^g(p)$ without  as a subgraph for ϕ_4^3, and without subgraphs  or  for ϕ_3^4.

The expressions enjoy several interesting properties:

(i) They are UV-finite and renormalized without resorting to UV regulators. In this sense they are "manifestly" or "explicitly" renormalized.
(ii) The expressions are non-perturbative. They are exact and do not rely on the analysis of the behavior of an infinite number of graphs of S.
(iii) They define an autonomous set of equations. For each value of n, the three derivatives are expressed using a finite number of Γ-graphs, that is, constructed with the full propagator and vertices of the effective action. At most $\kappa - 1$ explicit loops are present.
(iv) They allow to move on the manifold (m_R^2, Z, g). In particular, starting from the free theory $(m_R^2, Z, 0)$, for which $\Gamma[\phi] = S[\phi]$, the equation of $\partial\Gamma[\phi; g]/\partial g$ can be solved in powers of g. Due to the hierarchical structure of the equations, this procedure delivers systematically the perturbative series already renormalized within a BPHZ-like scheme.
(v) The equations are consistent, that is, successive derivatives commute; hence,

$$\frac{\partial H_{R,n}^{\beta}(p)}{\partial g_{R,\alpha}} = \frac{\partial H_{R,n}^{\alpha}(p)}{\partial g_{R,\beta}}, \tag{3.113}$$

where $g_{R,\alpha}$ refers to the coordinates (m_R^2, Z, g). The consistency properties provide identities among the functions $\Gamma_n(p)$.

The constructions performed for ϕ_4^3 and ϕ_3^4 show that, at least in the super-renormalizable case, the renormalization can be achieved by perturbing an already renormalized and finite $\Gamma[\phi]$. The divergences introduced by the perturbation through the presence of loops can be canceled by taking suitable divergent parameters in the perturbing action.

It can be noted that δm_R^2 is of course an exact differential in the manifold (m_R^2, Z, g), as are δZ, δg and $\delta\Gamma[\phi]$, while the auxiliary quantity δm^2 (Eq. (3.39)) is UV-finite but may not be an exact differential. Nevertheless, the coefficients of δm^2, δZ and δg in (3.38) (and the analogous equation for $\kappa = 4$) have the nice property of having a polynomial dependence with respect to the full propagator and vertices, while $H_{R,n}^{\alpha}(p)$ are rational functions of them.

Another observation is that $H_n^m(p)$ is finite without renormalization (i.e., without requiring any subtractions) as is also $H_{R,n}^m(p)$. Therefore, no (UV) renormalization is required when moving between different values of m_R^2 with (Z, g) fixed.

The derivatives $H_{R,n}^{\alpha}(p)$ are consistent with the symmetries of the theory. In particular, under a rescaling of the field $\phi(x)$, the action $S[\lambda\phi]$ has $\Gamma[\lambda\phi] + c(\lambda)$ as effective action. This can be implemented through a rescaling of the parameters of the action,

$$\Gamma[\lambda\phi; m_R^2, Z, g] = \Gamma[\phi; \lambda^2 m_R^2, \lambda^2 Z, \lambda^{\kappa} g] \tag{3.114}$$

(up to a ϕ-independent term) or equivalently,

$$\lambda^n \Gamma_n(p; m_R^2, Z, g) = \Gamma_n(p; \lambda^2 m_R^2, \lambda^2 Z, \lambda^{\kappa} g) \quad n > 0. \tag{3.115}$$

This leads to the identities, such as

$$\begin{aligned} n\Gamma_n(p) = {} & 2m_R^2\big(\delta_{n,2} + H_{R,n}^m(p)\big) + 2Z\big(\delta_{n,2}p^2 + H_{R,n}^Z(p)\big) \\ & + \kappa g\big(\delta_{n,\kappa} + H_{R,n}^g(p)\big) \quad n > 0. \end{aligned} \tag{3.116}$$

This formula allows us to express $H_{R,n}^g(p)$ in terms of $\Gamma_n(p)$ and the derivatives $H_{R,n}^m(p)$ and $H_{R,n}^Z(p)$, which are considerably simpler, as they have just one explicit loop. Naturally, and regrettably, such a simplified form of $H_{R,n}^g(p)$ cannot be used to solve the equation for $\partial\Gamma[\phi; g]/\partial g$ perturbatively, as it is singular at $g = 0$.

Likewise, if $\hbar$ is introduced explicitly in the functional integral through $S[\phi] \to S[\phi]/\hbar$ this is equivalent to a rescaling of the parameters, that is,

$$\frac{1}{\hbar}\Gamma[\phi; m_R^2, Z, g; \hbar] = \Gamma[\phi; \frac{m_R^2}{\hbar}, \frac{Z}{\hbar}, \frac{g}{\hbar}; 1], \tag{3.117}$$

which leads to the identity

$$\left(-1 + \hbar\frac{\partial}{\partial\hbar} + m_R^2\frac{\partial}{\partial m_R^2} + Z\frac{\partial}{\partial Z} + g\frac{\partial}{\partial g}\right)\Gamma[\phi; \hbar] = 0. \tag{3.118}$$

Again, the singularity at $\hbar = 0$ prevents to reconstruct $\Gamma[\phi]$ with loops starting from the tree level effective action $S[\phi]$. Eq. (3.118) simply states that $\Gamma^{(L)}[\phi]$ (the contributions with L loops) is a homogeneous function of the variables (m_R^2, Z, g) of degree $1 - L$.

The theory ϕ_3^4 is stable and admits a mathematically rigorous construction. Perhaps the closed expressions derived for $\partial\Gamma[\phi; g]/\partial g$ allow to provide an alternative construction. The analysis of this theory can be repeated without assuming Γ_n to vanish for odd n. The evolution (with respect to g) starting from the free theory should presumably arrive at the same conclusion without introducing it as an assumption.

4 Linearized renormalization of renormalizable theories

4.1 Projective renormalization scheme

The discussion here will concern the *perturbative* renormalization of renormalizable theories of the type ϕ_d^κ, although it may also apply to the super-renormalizable case. Actually, a theory such as ϕ_4^4 is afflicted by the problem of triviality; unless $g = 0$ it is thought to be non-renormalizable beyond perturbation theory.[13] Likewise ϕ_d^3 has a problem of stability (even for $d = 1$) since $\phi^{\rm odd}$ is not semi-bounded.[14] Here, we refer to the correlation functions at a perturbative level, i.e., as a formal power series in the coupling constant or in $\hbar$. Hence in that mathematical sense, such correlation functions do exist and are well-defined.

Of course for, e.g. ϕ_6^3, the action in Eq. (2.1), which contains fully local operators, would not yield finite and well-defined values for the Feynman graphs due to UV divergences, and a regularization scheme is required. One such scheme is dimensional regularization (DR) [12,14] where the fields propagate in $\bar{d} = d - 2\epsilon$ space-time dimensions, being d the physical dimension. Another is a Euclidean cutoff as in Sect. 3. In what follows, we will adopt the latter scheme for definiteness and briefly comment on DR below.

To be more specific, let us denote by $S_0[\phi]$ the action of the theory. This is the *true action* that (upon removal of the regulator) produces all the physical UV-finite correlation functions of $\phi(x)$. We will call it the *action* or the *bare action*, but warn that in the literature it is often denominated the renormalized action [12]. This functional takes the form

$$S_0 = m_0^2\, O^m + Z_0\, O^Z + g_0\, O^g. \tag{4.1}$$

The operators O^α are those in (3.10), with a profile factor F and a cutoff Λ. Equivalently,

$$S_0[\phi] = g_{0,\alpha}\, O^\alpha[\phi]. \tag{4.2}$$

A sum over α is implicit, $\alpha \in \{m, Z, g\}$ (here m, Z, g are merely labels, not variables). The bare couplings $g_{0,\alpha}$ are m_0^2, Z_0, and g_0 and have a dependence on Λ. This action yields the regulated effective action $\Gamma[\phi; \Lambda]$.

As is well-known, the theory will be perturbatively renormalizable (super-renormalizable) when the coupling constant has zero (positive) mass dimension. The (regulated) effective action itself depends on Λ and is UV-finite. This means that it is possible to choose the Λ dependence in the bare parameters of the action, $g_{0,\alpha}(\Lambda)$, in such a way that the limit $\Lambda \to \infty$ of $\Gamma[\phi; \Lambda]$, denoted $\Gamma[\phi]$, exists and is finite for sufficiently regular configurations ϕ.

[13] See however [21].

[14] Although the functional integral could be rendered convergent by taking a suitable path in the complex plane of ϕ [22].

In the renormalization program developed by Bogoliubov and collaborators [12,23] the dependence of the bare parameters on the regulator is obtained through counterterms. Let the functional S_b,

$$S_b[\phi] = m^2\, O^m + Z\, O^Z + g\, O^g = g_{b,\alpha}\, O^\alpha[\phi] \tag{4.3}$$

be (what we will call) the *basic action*. Here $g_{b,\alpha}$ (that is, m^2, Z, g) are *finite* (Λ-independent) parameters that will serve as coordinates in the manifold of effective actions of the theory and need not have a particular physical meaning. The basic action yields, using the Feynman rules, an effective action $\Gamma_b[\phi; \Lambda]$ which is UV divergent. To remove the divergences and obtain $\Gamma[\phi]$, counterterms $S_{\rm ct}$ are added to the basic action,

$$S_0 = S_b + S_{\rm ct}. \tag{4.4}$$

The graphs in Γ and Γ_b are identical and only differ by the couplings, $g_{0,\alpha}$ and $g_{b,\alpha}$, respectively. The basic parameters are finite and Γ_b diverges from the loops. Γ is finite after the divergences of the loops are compensated by the divergences in the bare parameters.

The counterterms can be determined recursively within a perturbative or loop expansion. At each new order, the new Feynman graphs of that order have the *subdivergences* automatically canceled by lower order counterterms, while the remaining superficial divergences are primitive and give a new contribution to the counterterm action. For a graph with all subdivergences subtracted, some prescription, implemented by a linear operation T, extracts the divergent component while $(1 - T)$ gives a purely finite component. The operation T can be chosen in many ways, but it must fulfill three essential requirements, i) $T^2 = T$ (idempotent), ii) the component $(1 - T)$ must be finite, hence any possible divergences must be isolated by T, and iii) T must always act in the same form when acting on the same graph or subgraph.

Denoting, as usual, $\bar{R}$ the operation of recursively extracting all subdivergences (but not the overall divergence), the counterterms can be summarized in the following formula:

$$S_{\rm ct} = -T\, \bar{R}(\Gamma_b - S_b) \tag{4.5}$$

so

$$S_0 = S_b - T\, \bar{R}(\Gamma_b - S_b). \tag{4.6}$$

$\Gamma_b - S_b$ is the sum of the graphs of Γ_b with loops. Explicitly [13,24] on each graph of Γ_b,

$$\bar{R} = \prod_\gamma{}'(1 - T_\gamma), \tag{4.7}$$

where γ denotes every possible subgraph with loops of effective-action type, that is, amputated, connected, and irreducible. The full graph itself is excluded from the product. The factor $1 - T_\gamma$ removes any possible divergent component of the subgraph γ. The subtractions are to be applied orderly, with smaller subgraphs first. The prime indicates that, upon expansion, monomials with factors of the type $T_{\gamma_1}T_{\gamma_2}$ are to be removed when the subgraphs γ_1 and γ_2 are overlapping, that is, they share at least one line but neither is a subgraph of the other.

Another standard definition is

$$R := (1 - T)\bar{R}. \tag{4.8}$$

The operation R represents the removal of all subdivergences, including the overall divergence. The effective action is obtained after applying the operation R to the graphs of Γ_b with loops; therefore,

$$\Gamma = S_b + R(\Gamma_b - S_b). \tag{4.9}$$

The meaning of this relation is that if one adds all (regulated) effective-action-like graphs produced by S_0, upon removal of the regulator the functional $\Gamma[\phi]$ is obtained on the LHS, and the same result is obtained on the RHS by using instead S_b plus systematic subtraction of the divergences in the graphs with loops.

As said, there is much freedom in the choice of T. The most usual choices are of the minimal type, that is, $T_{\min}$ vanishes if the graph or subgraph has no divergences. This is the case of minimal subtraction (MS) in DR, where $T_{\min}$ selects the principal part in the power series in ϵ, or in BPHZ, where $T_{\min}$ extracts the component of the Taylor series in powers of the external momenta up to a degree equal to the degree of divergence (or nothing if there is no divergence). In minimal schemes

$$T_{\min}\Gamma = T_{\min}S_b = 0, \quad (1 - T_{\min})S_0 = S_b. \tag{4.10}$$

The first equations follow from the fact that Γ and S_b are finite, the latter equation indicates that S_b is the finite component of S_0, because the counterterms are strictly divergent and so $T_{\min}S_{\rm ct} = S_{\rm ct}$.

Here we will discuss a different, non-minimal, scheme, which will be called *projective renormalization scheme*. As already noted in Sect. 3.3, a perturbative solution of the equation for $\partial\Gamma_n(p)/\partial g$ provides something similar to the BPHZ expansion, where each graph is renormalized before integration. However, in BPHZ the subtraction is minimal, while here even finite graphs are subtracted since what is enforced are the *renormalization conditions* (RC).

To see this in more detail, let us consider the renormalizable case, where in principle all the three bare parameters need renormalization. The effective action manifold is coordinated by three renormalized finite parameters $g_{R,\alpha}$, namely, m_R^2, Z_R and g_R, defined through some RC. A typical choice (for the massive case) is that of renormalization conditions at zero momentum, or more generally at a scale $\mu_{\rm RC} \geq 0$,

$$\begin{aligned} \Gamma_2(p)\Big|_{\mu_{\rm RC}} - \mu_{\rm RC}^2 \frac{\partial\Gamma_2(p)}{\partial(p_1^2)}\Big|_{\mu_{\rm RC}} &= m_R^2, \\ \frac{\partial\Gamma_2(p)}{\partial(p_1^2)}\Big|_{\mu_{\rm RC}} &= Z_R, \\ \Gamma_\kappa(p)\Big|_{\mu_{\rm RC}} &= g_R, \end{aligned} \tag{4.11}$$

where m_R^2, Z_R and g_R are three finite Λ-independent parameters. Here $\Gamma_n(p)$ refers to the n-point vertex function, and the kinematic condition $\big|_{\mu_{\rm RC}}$ refers to

$$p_i p_j = \frac{\mu_{\rm RC}^2}{n-1}(n\delta_{ij} - 1). \tag{4.12}$$

If the theory is massless, there are only two parameters: Z_R and g_R. In the *renormalizable* case the theory has no scale and DR does not introduce one, so one can adopt as renormalization conditions

$$\begin{aligned} \frac{1}{\mu_{\rm RC}^2}\Gamma_2(p)\Big|_{\mu_{\rm RC}} &= Z_R, \\ \Gamma_\kappa(p)\Big|_{\mu_{\rm RC}} &= g_R, \end{aligned} \tag{4.13}$$

for $\mu_{\rm RC} > 0$, and the operator O^m is not present in the actions.[15]

In a general renormalization scheme, the basic couplings $g_{b,\alpha}$ together with some subtraction prescription T produce the effective action functional (Eq. (4.9)). These basic couplings, which define a coordinate system in the manifold, are then adjusted to reproduce some RC. The idea of the projective renormalization scheme is to adopt $g_{b,\alpha} = g_{R,\alpha}$ from the beginning, hence,

$$S_b = S_R \tag{4.14}$$

where

$$S_R[\phi] := m_R^2\, O^m + Z_R\, O^Z + g_R\, O^g = g_{R,\alpha} O^\alpha[\phi]. \tag{4.15}$$

In addition, within this scheme, the corresponding effective action Γ_b will be denoted Γ_R.

In order to provide proper definitions, let us introduce the linear space $\mathscr{H}_{\rm C}$ spanned by the functionals $\Gamma[\phi]$ which are translationally invariant, with arbitrary (but sufficiently regular) coefficients $\Gamma_n(p)$.[16] Then the renormalization conditions in Eq. (4.11) or Eq. (4.13) can be cast in the form

$$\hat{T}_m\Gamma = m_R^2, \quad \hat{T}_Z\Gamma = Z_R, \quad \hat{T}_g\Gamma = g_R, \tag{4.16}$$

[15] O^m is needed also in the massless case when the regulator is a cutoff.

[16] Here we are using the symbol Γ to denote both a generic effective action-like functional and the actual effective action corresponding to the action S_0. We will let the two uses be distinguished by the context rather than introducing a different new notation.

or

$$\hat{T}_\alpha \Gamma[\phi] = g_{R,\alpha} \quad \alpha \in \{m, Z, g\}, \tag{4.17}$$

with $g_{R,\alpha} = m_R^2, Z_R, g_R$ (or just Z_R, g_R in the massless case). Here the mappings $\hat{T}_\alpha$ are linear forms on $\mathscr{H}_{\mathrm{C}}$ (i.e., $\hat{T}_\alpha \in \mathscr{H}_{\mathrm{C}}^*$). We can further define the subspace $\mathscr{H}_{\mathrm{S}}$ as

$$\mathscr{H}_{\mathrm{S}} := \mathrm{span}(\{O^\alpha\}) \subseteq \mathscr{H}_{\mathrm{C}}, \tag{4.18}$$

where $O^\alpha[\phi]$ are the operators appearing in the action. The 1-forms $\hat{T}_\alpha$ have been defined in such a way that *when restricted to* $\mathscr{H}_{\mathrm{S}}$ they are the dual basis of the O^α, i.e. they fulfill the relation

$$\hat{T}_\alpha O^\beta = \delta_\alpha^\beta, \tag{4.19}$$

and hence also

$$\hat{T}_\alpha S_0 = g_{0,\alpha}. \tag{4.20}$$

This is readily verified: for the functional S_0 in (4.1) and using the definitions in (2.37) and (2.38), one obtains in momentum space (dropping irrelevant form factors)

$$S_{0,2}(p) = m_0^2 + Z_0 p^2, \quad S_{0,3}(p) = g_0, \quad S_{0,n\geq 4}(p) = 0, \tag{4.21}$$

hence $\hat{T}_m S_0 = m_0^2$, $\hat{T}_Z S_0 = Z_0$, and $\hat{T}_g S_0 = g_0$ for the RC in (4.11), or in (4.13) when $m_0^2 = 0$.

The dual basis property of the $\hat{T}_\alpha$ allows us to define the projector operator T as

$$T := O^\alpha \hat{T}_\alpha, \quad T^2 = T. \tag{4.22}$$

Also $\hat{T}_\alpha T = \hat{T}_\alpha$.

T acts on the space $\mathscr{H}_{\mathrm{C}}$ and projects onto $\mathscr{H}_{\mathrm{S}}$. Hence in particular

$$T S_0 = S_0. \tag{4.23}$$

Note that the conditions (4.19) restrict, but do not completely determine, the 1-forms $\hat{T}_\alpha$ (e.g., they depend on the scale μ_{RC}). By the same token, while the subspace $\mathscr{H}_{\mathrm{S}}$ is determined from (the operators in) S_0, the projector T itself depends on how the complementary subspace is chosen in the direct sum decomposition $\mathscr{H}_{\mathrm{C}} = \mathscr{H}_{\mathrm{S}} \oplus \mathscr{H}_{\mathrm{S}}^\perp$, i.e., on the concrete choice of the renormalization conditions. Here $\mathscr{H}_{\mathrm{S}}^\perp$ is just a name for the space $(1-T)\mathscr{H}_{\mathrm{C}}$, no scalar product is introduced in $\mathscr{H}_{\mathrm{C}}$ to define the orthogonality.

The basic action functional $S_b = S_R$ is also in $\mathscr{H}_{\mathrm{S}}$, hence

$$T S_R = S_R. \tag{4.24}$$

From its definition, this action fulfills the identity

$$T\Gamma = S_R, \tag{4.25}$$

which is an alternative form of the RC (4.17), and

$$\hat{T}_\alpha S_R = g_{R,\alpha}. \tag{4.26}$$

By its very definition

$$\bar{R} S_R = S_R, \tag{4.27}$$

since the graphs of S_R contain no subgraphs (with loops). It follows that

$$R S_R = (1-T)\bar{R} S_R = (1-T) S_R = 0. \tag{4.28}$$

Every (effective-action-like) graph defines a functional of ϕ, and the action of T only depends on such a functional. This is unlike the operations $\bar{R}$ and R, which depend on the detailed structure of the graph. The projector T in (4.22) characterizes this renormalization scheme. T is not minimal and acts as an idempotent operator on the space of (translationally invariant) functionals of ϕ, so its action depends on the subgraph and not on where the subgraph is. In this sense T has a geometric nature. In a renormalizable (or super-renormalizable) theory, the superficial divergences have the same form as the action itself, hence $1-T$ selects finite components only. When a cutoff is used, the operators O^α depend on the profile function F and the regulator. Both $\mathscr{H}_{\mathrm{S}}$ and T inherit such dependence since the O^α spans $\mathscr{H}_{\mathrm{S}}$. For the theory ϕ_6^3, or for ϕ_4^4 assuming no breaking of the symmetry $\phi \to -\phi$, the three operators in (3.10) suffice to span $\mathscr{H}_{\mathrm{S}}$. In general all operators required by counterterms need to be included, and the theory is deemed renormalizable when the dimension of $\mathscr{H}_{\mathrm{S}}$ is finite.

The operator T is not of the minimal-subtraction type: the relations (4.10) are very different from those in (4.25) or (4.28). Nevertheless, the two general relations

$$S_0 = S_R - T\bar{R}(\Gamma_R - S_R) \tag{4.29}$$

and

$$\Gamma = S_R + R(\Gamma_R - S_R), \tag{4.30}$$

are fulfilled, where Γ_R denotes the (divergent by loops) effective action produced by S_R. Subtraction of the two equations (noting that $R + T\bar{R} = \bar{R}$) yields

$$\Gamma - S_0 = \bar{R}(\Gamma_R - S_R). \tag{4.31}$$

This formula is remarkable because it contains the other two: (4.29) follows immediately from (4.31) by applying T on both sides, using $TS_0 = S_0$ and $T\Gamma = S_R$. Likewise, (4.30) follows by applying $1-T$.[17]

Note that the relations (4.29) and (4.30) are not specifically related to UV divergences or even to quantum field theory. These identities hold equally well in models where the range of the index i in ϕ^i is discrete or even finite. They are diagrammatic identities that follow from the form of the

[17] Also a relation $\Gamma - S_0 = \bar{R}(\Gamma_b - S_b)$ holds in any scheme, after subtracting (4.6) from (4.9), but in general neither contains the other two.

partition function $Z[J]$ as a sum or integral over the Boltzmann weight in (2.8). Given the action functional $S_0[\phi]$, the functional $\Gamma[\phi]$ is fully determined through the Feynman rules as a formal sum of graphs constructed with the parameters in S_0. The choice of projector T then completely fixes $S_R = T\Gamma$ in terms of S_0. Everything is expressed in terms of S_0 or equivalently in terms of the parameters $g_{0,\alpha}$. One can then decide using instead the variables $g_{R,\alpha}$. In this view, (4.29) is just the change of variables expressing the parameters $g_{0,\alpha}$ in terms of the parameters $g_{R,\alpha}$, that is, S_0 as a function of S_R. Likewise (4.30) expresses Γ in terms of S_R. The $\bar{R}$ and R operations are automatically implemented when the inversion $S_R(S_0) \to S_0(S_R)$ is performed perturbatively. This point is illustrated in Appendix B.

Equation (4.31) is noteworthy because, regarding S_0 as the independent variable, the LHS is independent of the RC, that is, the concrete choice of T or equivalently the choice of the subspace $\mathscr{H}_S^\perp$, hence the equation implies that the dependence on T must also cancel on the RHS (when everything is expressed in terms of S_0). An infinitesimal change in T (e.g. in the scale μ_{RC}) leads to the Callan–Symanzik equations [12,15,25,26] which we do not analyze further here.

Until now we have assumed that the image of the projector T fills up the space $\mathscr{H}_S$ spanned by the action. More generally, one can consider a subspace $\mathscr{H}_T \subseteq \mathscr{H}_S$, such that still $T^2 = T$ but $T\mathscr{H}_C = \mathscr{H}_T$. Now, the operation $1-T$ in $\bar{R}$ and $R = (1-T)\bar{R}$ removes only the component parallel to $\mathscr{H}_T$ leaving the rest of $\mathscr{H}_S$ in place. Accordingly, the definition of the basic action S_R is generalized to a partial projection

$$S_R = T\Gamma + (1-T)S_0. \tag{4.32}$$

Consider for instance a theory

$$S_0 = Z_0 O^Z + g_0 O^g, \tag{4.33}$$

with $T = O^g \hat{T}_g$, in this case

$$S_R = Z_0 O^Z + g_R O^g. \tag{4.34}$$

This amounts to saying that instead of the full change of variables $(Z_0, g_0) \to (Z_R, g_R)$, a partial change of variables $(Z_0, g_0) \to (Z_0, g_R)$ is performed, leaving Z_0 unchanged because the removal operation $(1-T)$ only acts on three-point subgraphs.

Since $1-T$ still acts recursively, the three relations

$$\begin{aligned} \Gamma - S_0 &= \bar{R}(\Gamma_R - S_R), \\ S_0 &= S_R - T\bar{R}(\Gamma_R - S_R), \\ \Gamma &= S_R + R(\Gamma_R - S_R), \end{aligned} \tag{4.35}$$

still hold and are consistent with (4.32).

This partial renormalization may be particularly useful for super-renormalizable theories, such as ϕ_4^3. There only m_0^2 needs to be renormalized as Z_0 and g_0 are UV-finite. There are no UV divergences in the sector spanned by O^Z and O^g, so the action of T can be restricted to the sector spanned by O^m.[18]

The projective scheme has an obvious disadvantage in concrete calculations: it is not minimal, every graph of effective action type with a component along $\mathscr{H}_S$ (i.e., with 2 or κ legs in a ϕ_d^κ theory) gives a contribution to the counterterms, even if it is fully UV-finite. On the other hand, virtues are its geometric and systematic nature, and that by construction $T(\Gamma - S_R) = 0$; the tree level S_R already saturates the RC and they are preserved at every order hence the effective action is expressed in terms of physical coordinates $g_{R,\alpha}$ automatically, rather than in terms of intermediate coordinates $g_{b,\alpha}$. The perturbative solution of $\partial\Gamma_n(p)/\partial g_R$ yields the graphs renormalized under the projective scheme. Let us also remark that the use of the projective scheme is compatible with DR.

4.2 Linearized renormalization

The UV divergences in the renormalizable theories are more severe than those of super-renormalizable ones, correspondingly, the results that we have obtained are also less conclusive.

According to Schwinger's principle, for a theory with bare action (4.2)

$$\delta\Gamma[\phi] = \delta g_{0,\alpha} \langle O^\alpha \rangle^\phi. \tag{4.36}$$

Our point of is be that the unperturbed effective action is already renormalized and only the new divergences need regularization. To do this a cutoff will be assumed. The possibility of using DR is briefly discussed below.

We will use the notation $G^\alpha[\phi] := \langle O^\alpha \rangle^\phi$ already introduced in (2.33) , that is,

$$G^\alpha[\phi] := \frac{\partial\Gamma[\phi]}{\partial g_{0,\alpha}}, \tag{4.37}$$

or equivalently

$$G_n^\alpha(p) := \frac{\partial\Gamma_n(p)}{\partial g_{0,\alpha}}, \tag{4.38}$$

and correspondingly,

$$\begin{aligned} \delta\Gamma_n(p) &= \delta g_{0,\alpha} G_n^\alpha(p) \\ &= \delta m_0^2\, G_n^m(p) + \delta Z_0\, G_n^Z(p) + \delta g_0\, G_n^g(p). \end{aligned} \tag{4.39}$$

We remark that here we are using the notation based on the functions $G_n^\alpha(p)$ while $H_n^\alpha(p)$ were used in Sect. 3 (e.g., in (3.19)). They are related by $G^\alpha[\phi] = O^\alpha[\phi] + H^\alpha[\phi]$,

[18] The formalism allows to consider non-standard partial projections, mixing subgraphs with different number of legs, such as $T = (O^Z + O^g)\frac{1}{2}(\hat{T}_Z + \hat{T}_g) := O\,\hat{T}_O$. It is not clear whether such a possibility could find any useful applications.

that is,

$$G_n^m(p) = \delta_{n,2} + H_n^m(p),$$
$$G_n^Z(p) = \delta_{n,2} p^2 + H_n^Z(p), \tag{4.40}$$
$$G_n^g(p) = \delta_{n,\kappa} + H_n^g(p).$$

The functions $H_n^\alpha(p)$ only include the loop terms while $G_n^\alpha(p)$ contain also tree graphs. Apart from this difference, the formulas given for $H_n^\alpha(p)$ in the super-renormalizable case (e.g. (3.20)) in terms of the propagators and effective vertices also apply in the renormalizable case up to the different dimension in the momentum integrals.

$G^\alpha[\phi]$, or equivalently $G_n^\alpha(p)$, are the quantities that we can obtain explicitly from $\Gamma[\phi]$, and they are divergent by loops. The quantities that are completely finite are, instead,

$$G_R^\beta[\phi] := \frac{\partial \Gamma[\phi]}{\partial g_{R,\beta}}, \tag{4.41}$$

so that

$$\delta\Gamma_n(p) = \delta g_{R,\alpha} G_{R,n}^\alpha(p) = \delta m_R^2 G_{R,n}^m(p) + \delta Z_R G_{R,n}^Z(p) + \delta g_R G_{R,n}^g(p). \tag{4.42}$$

In order to relate both sets of quantities, let us introduce the matrix

$$W^\alpha{}_\beta := \frac{\partial g_{R,\beta}}{\partial g_{0,\alpha}}, \tag{4.43}$$

which depends on the regulator and is divergent. In terms of it

$$G_R^\beta[\phi] = \frac{\partial \Gamma[\phi]}{\partial g_{R,\beta}} = \frac{\partial g_{0,\alpha}}{\partial g_{R,\beta}} \frac{\partial \Gamma[\phi]}{\partial g_{0,\alpha}}, \tag{4.44}$$

therefore

$$G_R^\beta[\phi] = (W^{-1})^\beta{}_\alpha \, G^\alpha[\phi], \tag{4.45}$$

or equivalently

$$G_{R,n}^\beta(p) = (W^{-1})^\beta{}_\alpha \, G_n^\alpha(p). \tag{4.46}$$

The matrix W can be computed from $G^\alpha[\phi]$ by using the RC in (4.17),

$$W^\alpha{}_\beta = \frac{\partial g_{R,\beta}}{\partial g_{0,\alpha}} = \frac{\partial \hat{T}_\beta \Gamma[\phi]}{\partial g_{0,\alpha}} = \hat{T}_\beta \frac{\partial \Gamma[\phi]}{\partial g_{0,\alpha}} \tag{4.47}$$

hence

$$W^\alpha{}_\beta = \hat{T}_\beta G^\alpha[\phi]. \tag{4.48}$$

The consistency condition

$$\hat{T}_\alpha G_R^\beta[\phi] = \delta_\alpha^\beta, \tag{4.49}$$

which follows from the definition of G_R^β and the RC, is also derived from the relations (4.45) and (4.48).

To be more explicit, for the RC in (4.11) with $\mu_{\rm RC} = 0$ the matrix W takes the form

$$W = \begin{pmatrix} G_2^m(0) & \partial_{p^2} G_2^m(0) & G_\kappa^m(0) \\ G_2^Z(0) & \partial_{p^2} G_2^Z(0) & G_\kappa^Z(0) \\ G_2^g(0) & \partial_{p^2} G_2^g(0) & G_\kappa^g(0) \end{pmatrix}, \tag{4.50}$$

while for the massless case RC in (4.13)

$$W = \begin{pmatrix} \mu_{\rm RC}^{-2} G_2^Z(\mu_{\rm RC}) & G_\kappa^Z(\mu_{\rm RC}) \\ \mu_{\rm RC}^{-2} G_2^g(\mu_{\rm RC}) & G_\kappa^g(\mu_{\rm RC}) \end{pmatrix}. \tag{4.51}$$

In the DeWitt notation, from

$$O^\alpha[\phi] = \sum_{\ell \ge 0} \frac{1}{\ell!} O^\alpha_{i_1 \dots i_\ell} \phi^{i_1} \cdots \phi^{i_\ell}, \tag{4.52}$$

it follows that

$$G^\alpha[\phi] = O^\alpha[\phi] + \sum_{\ell \ge 2} \frac{1}{\ell!} O^\alpha_{i_1 \dots i_\ell} \hat{H}^{i_1 \dots i_\ell}[\phi]. \tag{4.53}$$

The main formulas are then i) (4.53) (or its momentum space version (4.54)) expressing $G^\alpha[\phi]$ in terms of the effective action, ii) (4.48) which provides the matrix $W^\alpha{}_\beta$, and iii) (4.45) that yields $G_R^\beta[\phi]$.

The divergent functions $G_n^\alpha(p)$ enjoy two distinct properties, namely, their divergences are *explicit* and *anti-canonical*.

Let us explain the meaning of *explicit* in this context. In (4.53), both the operator coefficients $O^\alpha_{i_1 \dots i_n}$ and the functionals $\hat{H}^{i_1 \dots i_\ell}[\phi]$ are free from divergences (for regular field configurations). The divergence arises from the sum over the indices. In momentum space,[19]

$$G_n^\alpha(p) = O_n^\alpha(p) + \sum_{\ell \ge 2} \frac{1}{\ell!} \int^\Lambda \prod_i^{\ell-1} \frac{d^d q_i}{(2\pi)^d} O_\ell^\alpha(q) H_n^\ell(q; p), \tag{4.54}$$

and $G_n^\alpha(p)$ is expressed as a momentum integral (identical to those found in the super-renormalizable case) where the integrand itself is completely finite and independent (or essentially independent) of the cutoff. The divergence emerges only as the cutoff grows and the integration region increases, and in this sense the divergence is *explicit*, there are no hidden divergences in $G_n^\alpha(p)$ and so they are also absent in $W^\alpha{}_\beta$, due to $W = \hat{T} G$.

The fact that the divergences are explicit in $G_n^\alpha(p)$ or $W^\alpha{}_\beta$ is not to say that they are *superficial* in a technical sense. Let us clarify this point. As is well-known, when an S-graph having only an overall divergence of degree $\gamma \ge 0$ (all sub-divergences removed) is differentiated more than γ times with respect to (external) momenta or masses, it becomes

19 $O_n^\alpha(p)$ is only defined in the subspace $\sum_{i=1}^n p_i = 0$. In the integral $\sum_{i=1}^\ell q_i = \sum_{j=1}^n p_j = 0$. The form factors are included in $O_\ell^\alpha(q)$.

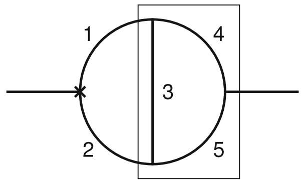
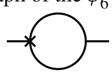


Fig. 7 An S_R-graph of the ϕ_6^3 theory for $G_2^g(p)$. The graph belongs to the Γ-graph class . The divergence of the subgraph (345) has been subtracted, and this is indicated by the thin-line box

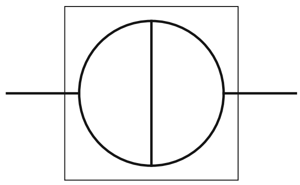


Fig. 8 An S_R-graph of the ϕ_6^3 theory for $\Gamma_2(p)$. The box indicates that all the divergences are subtracted

convergent; consequently, the divergent component of such graph is a polynomial of degree γ of the momenta and the masses [12]. On the other hand, the corresponding (regulated) momentum integral of the graph (that is, excluding coupling constants) has a mass dimension γ. As a mathematical consequence, the dependence of the integral on the cutoff Λ follows a *canonical pattern*:

$$\sum_{k=1}^{\gamma} P_k \Lambda^k + P_0 \log(\Lambda) + f(\Lambda), \tag{4.55}$$

where the P_k are Λ-independent polynomials of degree $\gamma - k$ in momenta and masses, and $f(\Lambda)$ has a finite limit as the cutoff is removed. The presence of subdivergences introduces non-canonical divergent terms, such as $\Lambda^n \log^m(\Lambda)$. Of course, in the renormalizable case, the dependence on the momenta in the canonical divergent terms conforms to the operators in the action, and the divergence in the bare parameters $g_{0\alpha}(\Lambda)$ is canonical because the counterterms collect only contributions from superficial divergences.

The dependence on Λ in $G_n^\alpha(p)$ or the matrix W is explicit but not necessarily canonical due to the presence of subdivergences: even if the functions $H_n^\ell(q; p)$ are finite, new divergences and subdivergences are produced by attaching the vertex $O_\ell^\alpha(q)$ and integrating over q. This is illustrated in Fig. 7, which represents an S_R-graph of the theory ϕ_6^3 contributing to $G_2^g(p)$. The subgraph with lines (345) comes from $\Gamma[\phi]$ and is already subtracted. When the crossed vertex is added, two new divergences arise, one is the subgraph (123), logarithmically divergent, and the other is the full graph (12345), which diverges quadratically. If they are not subtracted, this is a contribution to $G_2^g(p)$. Since the subgraph (123) is not subtracted, the divergence in the full graph is not only superficial and hence $G_2^g(p; \Lambda)$ will contain non-canonical terms.

Using this example, it is interesting to emphasize that if the two new divergences are also subtracted, what is obtained is a contribution to $G_{R,2}^g(p)$. Let us discuss this point in more detail. As already noted after (4.9), Eq. (4.30) states that the same functional Γ is obtained from the S_0-graphs without subtractions or from the S_R-graphs with subtractions. This observation can be applied to $G^\alpha[\phi]$ and $G_R^\alpha[\phi]$. The coefficients $\hat{H}^{i_1 \ldots i_\ell}$ or $H_n^\ell(q, p)$ can be (finitely) expressed through Γ-graphs. In turn, they can be expanded in terms of S_0-graphs or in terms of *subtracted* S_R-graphs. In the latter representation (4.53) expresses $G^\alpha[\phi]$ by means of graphs of the theory '$S_R + O^\alpha$'[20] but including only graphs with exactly one vertex O^α and with subtraction of all subgraphs not involving the vertex O^α. The only remaining divergences are those induced by the composite operator O^α. If one proceeds to subtract these remaining divergences, $G_{R,n}^\alpha(p)$ is obtained instead of $G_n^\alpha(p)$. To see this consider, for instance, $\Gamma_2(p)$ expanded as a sum of subtracted S_R-graphs, as required by (4.30). Figure 8 shows one such S_R-graph. These contributions depend on the variables $g_{R,\alpha}$. If a variation δg_R is applied, the corresponding derivative is $G_{R,2}^g(p)$. The variation of δg_R in the graph of Fig. 8 produces (among other) the graph in Fig. 7 with all divergences subtracted, including the subgraph (123) and the full graph (12345).

Hence $G^\alpha[\phi]$ and $G_R^\alpha[\phi]$ share the same graphs, with the partial and full removal of the divergences, respectively. In this view, Eq. (4.45) is nothing but the linear version of (4.30). Bogoliubov's R and $\bar{R}$ operators act by eliminating divergences on every subgraph; in the linearized version, the divergences irradiate from a single composite operator and their removal is obtained by a simple inversion of the matrix W; G and W share the same divergences and they cancel in the combination $W^{-1}G = G_R$. Of course, the price to pay for the linear simplicity is that only the derivatives of the effective action are obtained rather than the effective action itself.

However, the structure of the divergences in W is by no means arbitrary, on the contrary: *the matrix* W^{-1} *has only canonical divergences*. To see this, let us introduce the matrix $V^\beta{}_\alpha$ through

$$(W^{-1})^\beta{}_\alpha = \delta_\alpha^\beta - V^\beta{}_\alpha. \tag{4.56}$$

Using (4.43), this definition implies the relation

$$\delta g_{0,\alpha} = \delta g_{R,\alpha} - \delta g_{R,\beta} V^\beta{}_\alpha. \tag{4.57}$$

This is the linear version of (4.29). The matrix V collects the contributions of the counterterms to the (variation of the) bare couplings. Such contributions come from the superficially divergent component of the primitively divergent graphs after all subdivergences have been subtracted, and thus, as $g_{0,\alpha}(\Lambda)$ itself, they have only a canonical dependence on Λ.

[20] That is, graphs constructed with lines and vertices of S_R plus the vertex corresponding to O^α.

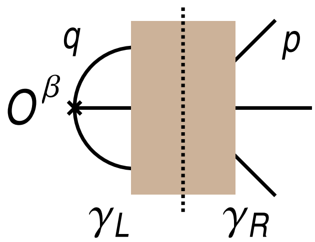


Fig. 9 Schematics of the graphs in $G_n^\beta(p)$, to expose Eq. (4.58). The vertical dotted line separates the left-hand subgraphs containing the vertex O^β from right-hand subgraphs, containing the dependence on p

Additionally, (4.45) also implies the relation

$$G^\beta[\phi] = G_R^\beta[\phi] + V^\beta{}_\alpha G^\alpha[\phi]. \tag{4.58}$$

The interpretation of this formula is as follows. We consider the typical S_R-graphs contributing to $G_n^\beta(p)$, in Fig. 9. They start at the vertex O^β, located on the left in the figure, and propagate to the right, where n (amputated) legs emerge carrying momentum p. All divergences from subgraphs not involving the operator O^β have been subtracted, but there remain the new divergences irradiating from the composite operator. We can consider all possible cuts of each graph, represented by the vertical dotted line in the figure, separating the graph into two subgraphs, γ_L and γ_R, where the former contains the source O^β and the latter does not. To each such cut, we can associate a decomposition $1 = T_{\gamma_L} + (1 - T_{\gamma_L})$. For a renormalizable theory ϕ^κ only cuts cutting at most κ lines need be considered, since otherwise T_{γ_L} vanishes. As the cut is moved from left to right, factors $(1 - T_{\gamma_L})$ accumulate, hence subtracting the subdivergences. The product of the $(1 - T_{\gamma_L})$ factors produces the fully subtracted graphs, and this is $G_R^\beta[\phi]$ on the RHS in (4.58). On the other hand, when a factor T_{γ_L} acts for the first time, and therefore γ_L has all subdivergences subtracted and only displays an overall divergence, the projector onto $\mathscr{H}_\mathrm{S}$ selects operators O^α, with divergent coefficients that are counterterms of the canonical type. These add to $V^\beta{}_\alpha$. Once the operator O^α is formed and all possible graphs γ_R attached to it are added, they reproduce the original structure, this time as $G^\alpha[\phi]$.

Summarizing, in all cases, the divergences present in $\Gamma[\phi]$ have been renormalized (subtracted if expressed in terms of S_R-graphs). $G^\alpha[\phi]$ describes the effect induced by a source O^α, and corresponds to graphs with unsubtracted divergences generated by that source. $W^\alpha{}_\beta$ is the component of $G^\alpha[\phi]$ along O^β. $V^\beta{}_\alpha$ corresponds to graphs with source O^β and effect along O^α with subtractions of all *subdivergences*, but not the overall divergence. $G_R^\beta[\phi]$ is the renormalized effect of a source O^β, it corresponds to graphs with all divergences subtracted.

It is interesting to note that $G^\alpha[\phi]$ does not know about the RC, hence T or $\hat{T}^\beta$ only appears once in the matrix $W = \hat{T} G$. Upon matrix inversion, $W^{-1} = 1 - V$ will produce instances of $\hat{T}$ in several places. At a perturbative level, it can be checked that such instances of T precisely combine to produce the operators $\hat{T}\bar{R}$, so that V contains only superficially divergent graphs, without subdivergences. An explicit perturbative calculation illustrates this point in Appendix C.

Once again, the consistency conditions

$$\frac{\partial G_R^\beta[\phi]}{\partial g_{R,\alpha}} = \frac{\partial G_R^\alpha[\phi]}{\partial g_{R,\beta}}, \tag{4.59}$$

or equivalently

$$\frac{\partial G^\beta[\phi]}{\partial g_{0,\alpha}} = \frac{\partial G^\alpha[\phi]}{\partial g_{0,\beta}}, \tag{4.60}$$

apply and lead to multiple identities among the correlation functions.

The formulas derived above apply to the super-renormalizable case as well. To this end, for the theories ϕ_4^3 or ϕ_3^4 we define[21]

$$m_R^2 = \Gamma_2(0), \quad Z_R := Z_0, \quad g_R := g_0. \tag{4.61}$$

Besides convenience, these are proper RC, since Z_0 and g_0 can be extracted from $\Gamma_2(p)$ and $\Gamma_\kappa(p)$ (with $\kappa = 3$ or 4) in the large momentum limit. Specifically, these RC correspond to (4.11) with a suitable choice of the scale μ_RC in each case,

$$\begin{aligned} m_R^2 &= \hat{T}_m \Gamma \quad \text{with } \mu_\mathrm{RC} = 0, \\ Z_R^2 &= \hat{T}_Z \Gamma \quad \text{with } \mu_\mathrm{RC} = \infty, \\ g_R^2 &= \hat{T}_g \Gamma \quad \text{with } \mu_\mathrm{RC} = \infty. \end{aligned} \tag{4.62}$$

Therefore, in the super-renormalizable case, the matrix W takes the form

$$W = \begin{pmatrix} G_2^m(0) & 0 & 0 \\ G_2^Z(0) & 1 & 0 \\ G_2^g(0) & 0 & 1 \end{pmatrix}, \tag{4.63}$$

and so

$$V = \begin{pmatrix} 1 - G_2^m(0)^{-1} & 0 & 0 \\ G_2^m(0)^{-1} G_2^Z(0) & 0 & 0 \\ G_2^m(0)^{-1} G_2^g(0) & 0 & 0 \end{pmatrix}. \tag{4.64}$$

$G_2^m(0)$ is finite and $G_2^Z(0)$ and $G_2^g(0)$ have a canonical logarithmic divergence, hence the divergences in V are canonical too (and exceptionally in the present case, also those of W). The renormalized derivatives of the action follow from using

[21] Z_0 and g_0 were denoted Z and g in Sect. 3.

(4.45), and this yields[22]

$$
\begin{aligned}
G_R^m[\phi] &= \frac{G^m[\phi]}{G_2^m(0)},\\
G_R^Z[\phi] &= G^Z[\phi] - \frac{G_2^Z(0)}{G_2^m(0)} G^m[\phi],\\
G_R^g[\phi] &= G^g[\phi] - \frac{G_2^g(0)}{G_2^m(0)} G^m[\phi].
\end{aligned}
\tag{4.65}
$$

These equation are, of course, identical to those in (3.42) when the latter are expressed in terms of $G_n^\alpha(p)$.

In Sect. 3 for ϕ_4^3 and ϕ_3^4 it was possible to rearrange the integrals expressing $G_{R,n}^\alpha(p)$ in such a way that convergence was explicit and an UV regulator was not needed. The divergences in the renormalizable case are more severe than those in the super-renormalizable case. For ϕ_6^3 or ϕ_4^4 all three parameters in the action require renormalization, and in the massive case, the divergences are quadratic instead of logarithmic. It can be conjectured that, in the renormalizable case, the very same rearrangements of the integral made in Sect. 3 (but increasing the dimension and keeping the regulator) while not completely eliminating the divergences, will remove quadratic divergences and leave logarithmic ones only. This is based on the observation that in the super-renormalizable case, the matrix W has the divergence structure

$$
V = \begin{pmatrix} \text{fin} & \text{fin} & \text{fin} \\ \log & \text{fin} & \text{fin} \\ \log & \text{fin} & \text{fin} \end{pmatrix}
\tag{4.66}
$$

while in the renormalizable case

$$
V = \begin{pmatrix} \log & \log & \log \\ \text{quad} & \log & \log \\ \text{quad} & \log & \log \end{pmatrix}.
\tag{4.67}
$$

The relation (4.45) can be expressed more explicitly as

$$
G^\alpha[\phi;\Lambda] = W^\alpha{}_\beta(\Lambda;\mu_{\mathrm{RC}}) \left(G_R^\beta[\phi;\mu_{\mathrm{RC}}] + R^\beta[\phi;\Lambda;\mu_{\mathrm{RC}}] \right),
\tag{4.68}
$$

or

$$
G_n^\alpha(p;\Lambda) = W^\alpha{}_\beta(\Lambda;\mu_{\mathrm{RC}}) \left(G_{R,n}^\beta(p;\mu_{\mathrm{RC}}) + R_n^\beta(p;\Lambda;\mu_{\mathrm{RC}}) \right),
\tag{4.69}
$$

where μ_{RC} indicates the dependence on the RC and $R^\beta(\Lambda;\mu_{\mathrm{RC}})$ vanishes for large Λ. The information encoded here is that asymptotically, in the large cutoff limit, $G^\alpha[\phi;\Lambda]$ is the product of two factors, one that depends on Λ but not on ϕ, and the other conversely. This is the necessary and sufficient condition for the theory to be renormalizable in the linearized version. The ϕ-independent factor W can be extracted from G^α by applying the RC. The quantities $G_n^\alpha(p)$, and hence W, depend only polynomially on the basic elements $D(p)$ and $\Gamma_n(p)$, with regulated momentum integrals inserted in various places. In the combination $G_R = W^{-1}G$ such dependence is no longer polynomial but of the rational type. From this point of view, a probably crucial simplification is that in the super-renormalizable case, the determinant of W (namely $G_2^m(0)$) is UV finite, and this guarantees that the divergences in the matrix W^{-1} have only a polynomial dependence on the basic regulated integrals. This fact makes it easier to devise a rearrangement of the terms to produce explicitly convergent blocks, as done in Sect. 3. Let us note that in the super-renormalizable case, the divergence structure of W is particularly simple and the equation of $G_R^g[\phi]$ in (4.65) would suggest a simple subtraction scheme to cancel the divergences, yet actually two subtractions were needed to achieve manifestly convergent momentum integrals in (3.110).

The situation is much harder in the renormalizable case. While renormalizability guarantees that the combination $W^{-1}G$ has a limit as the regulator is removed, it is far from clear that the momentum integrals can be explicitly rearranged as in the super-renormalizable case to yield manifestly UV convergent blocks. Therefore, in the renormalizable case, one arrives at expressions of G and W with an explicit regulator, and only after the combination $W^{-1}G$ has been constructed the regulator can be removed to yield G_R. By way of illustration, one can imaging two conditionally convergent series; in one case it has been possible to rearrange the series into an absolutely convergent one with the same sum while for the other series no rearrangement has been found or it may not exist. In the super-renormalizable case the theory is manifestly renormalized, while in the renormalizable case, it is renormalized but not in an explicit manner.

To achieve an explicitly convergent expression of G_R in the renormalizable case (if that is at all possible), it might be of help the constraint that the divergences in W^{-1} are not arbitrary, but instead they follow a canonical pattern. Thus W enjoys two non-trivial properties: an *anti-canonical pattern* (meaning that W^{-1} is canonical) and also its divergences are *explicit* (meaning that they come from a single cutoff momentum integral with a cutoff-independent integrand).

The divergences of $G^\alpha[\phi]$ are also explicit and anti-canonical, which we define to mean that they can be canceled by multiplication with a canonically divergent matrix, namely, W^{-1}. Another way to express the anti-canonical divergence-pattern of $G^\alpha[\phi]$ is to take three independent field configurations ϕ_A, $A = 1, 2, 3$, and form a 3×3 matrix $G^\alpha{}_A$ with the three column-vectors $G^\alpha[\phi_A;\Lambda]$, then the inverse

[22] We have neglected the correlation functions with $n \in \{0, 1\}$ throughout. Their inclusion adds terms to these relations that do not affect the sector $n \geq 2$.

matrix $(G^{-1})^A{}_\alpha(\Lambda)$ has only canonical divergences,

$$
\begin{aligned}
&(W^{-1})^\beta{}_\alpha G^\alpha{}_A =: G^\beta{}_{R,A} \\
&(G^{-1})^A{}_\alpha = (G^{-1})^A{}_{R,\beta}(W^{-1})^\beta{}_\alpha
\end{aligned}
\tag{4.70}
$$

Instead of using three field configurations, one can use three sets of momenta p, or in fact any three observables linearly extracted from the functional $G^\alpha[\phi]$, the RC producing $W^\alpha{}_\beta$ being a particular case. It is important to note that $(G^{-1})^A{}_\alpha$ is canonical only with respect to its dependence on Λ. In general, however, its divergent component will not be a polynomial on external momenta and masses (see Appendix C).

Another observation is that while the regulator cannot be fully removed in the explicit non-perturbative expressions of $G^\beta_R[\phi]$, it can be removed in a perturbative treatment. That is, if the equation for $G^g_R[\phi]$ is solved perturbatively in powers of g_R, starting from $m_R^2 O^m + Z_R O^Z$ at zeroth order, what is obtained are the usual (regulated) momentum integrals from the standard Feynman rules of S_R plus systematic subtraction of divergences and subdivergences (with T as operator isolating the divergences) implementing Bogoliubov's R operation on the S_R-graphs. Then the regulator can be removed since the subtracted integrals are already convergent. This is illustrated in Appendix C. In the super-renormalizable case G_R is still obtained from $W^{-1}G$, as explained around (4.64), hence the R operation acts in the standard way. The treatment in Sect. 3 looks different because it uses Γ-graphs and the RC in (4.62) automatically set to zero the action of $\hat{T}$ on most subgraphs.

A technical assumption has not yet been explicitly addressed. We have considered renormalizable theories, at least in a perturbative sense. This means that for finite values g_R (meaning here the various $g_{R,\alpha}$) and some regularization with cutoff Λ, there are parameters $g_0(\Lambda)$ in the action S_0 producing the functional $\Gamma[\phi;\Lambda;g_R]$ which fulfills the RC of g_R for all Λ, and has a finite limit $\Gamma[\phi;g_R]$ as the regulator is removed. Clearly, in this case, for $g_R+\delta g_R$ there will be $g_0(\Lambda)+\delta g_0(\Lambda)$ such that $\Gamma[\phi;\Lambda;g_R+\delta g_R]$ also has a finite limit. This can be denoted the theory '$\Gamma(\Lambda)+\delta S_0(\Lambda)$'. The point here is that Λ appears both in the original (unperturbed) system and in the perturbation, as the regulator is removed. However, our point of view has been slightly different since we have been working with the theory '$\Gamma+\delta S_0(\Lambda)$', i.e., the effective action of the unperturbed system is already renormalized and only the perturbation is regulated. Technically, the validity of such an approach is not guaranteed as an immediate consequence of renormalizability. While the assumption worked well in the super-renormalizable case, the divergences of just renormalizable theories are more untamed. It is argued in Appendix D that the assumption is indeed valid: even if the parameters $\delta g_0(\Lambda)$ required to have a finite limit for '$\Gamma+\delta S_0(\Lambda)$' need not be identical to those from '$\Gamma(\Lambda)+\delta S_0(\Lambda)$', they do exist. This is not surprising since such final-result-independence also holds in other aspects of renormalization. For instance, in the projective scheme T acts according to the overlap of the graph with the space $\mathscr{H}_S$, whether the graph is actually divergent or not, while in the standard BPHZ approach $T_{\min}$ only acts if the graph is divergent; the two approaches produce the same effective action when the cutoff is removed, although the detailed parameters $g_{0,\alpha}(\Lambda)$ are different in both cases.

We conclude our discussion of the renormalizable case by considering the use of DR as the UV regulator. In DR, the perturbative amplitudes are meromorphic functions of ϵ with poles at most at integer values, hence finite for ϵ close but different from 0. The operators are still fully local in $\bar{d}=d-2\epsilon$ dimensions but (slightly) non-local from the point of view of the d dimensional degrees of freedom (i.e., after integrating out the $(\bar{d}-d)$-dimensional degrees of freedom), and in this way ϵ acts as a regulator. A logarithmic mass scale ν is introduced as the measure $\int d^d p/(2\pi)^d$ must be replaced by $\nu^{2\epsilon}\int d^{\bar{d}}p/(2\pi)^{\bar{d}}$ to restore proper dimensions in the Feynman graphs. All the amplitudes are then finite but depend on ϵ. By definition, the fact that a quantity $A(\epsilon)$ is UV-finite means that the limit $\epsilon\to 0$ exists and is finite, equivalently, the residue at $\epsilon=0$ vanishes.

In our discussion a cutoff was applied as a regulator of the momentum integrals. As it turns out, a possibility is open to regulate the UV divergences by means of DR within the linearized approach. A way to do this can be illustrated with $G^m_n(p)$ in (3.20),

$$
G^m_n(p)=\delta_{n,2}+\nu^{2\epsilon}\int\frac{d^{\bar{d}}q}{(2\pi)^{\bar{d}}}\frac{1}{2}H^2_n(q,-q;p). \tag{4.71}
$$

Here, the integrand is expressed in terms of the functions $D(k)$ and $\Gamma_n(k)$. In the spirit of the linearized approach, these $\Gamma_n(k)$ are derived from $\Gamma[\phi]$ at $\epsilon=0$ (i.e., for d spacetime dimensions), but they are needed in $\bar{d}=d-2\epsilon$ dimensions in (4.71). An obvious prescription is to exploit the Lorentz (or Euclidean) invariance to express $\Gamma_n(k)$ in d dimensions in terms of the invariants p_ip_j, which are then identified with the corresponding invariants in $\bar{d}$ dimensions. The implementation of DR is then straightforward if the vertices $\Gamma_n(k)$ are explicitly given as functions of the invariants p_ip_j (this would be the case, for instance, within a perturbative calculation). This prescription does not cover the possibility of pseudoscalar invariants (involving the Levi–Civita symbol), but this is rather an issue of DR itself, related to the presence of anomalies. A more serious obstacle is that a Lorentz invariant function $\Gamma_n(k)$, known only in d dimensions, can be written in many inequivalent ways in terms of the invariants p_ip_j when $n-1>d$, and therefore the extension would not be unique.[23] This ambiguity does not arise in the

[23] For instance, $k_1^2k_2^2-(k_1k_2)^2$ vanishes identically in $d=1$, so a multiple of this function can be added to $\Gamma_3(k)$ in $d=1$ producing an

standard application of DR in perturbation theory because in that case there are explicit formulas in terms of $k_i k_j$ for the functions. In our case the functions $\Gamma_n(k)$ are defined by their values rather than by explicit formulas. A prescription would be required for a "natural" or "minimal" extension of $\Gamma_n(k)$ when $n > d+1$. A compromise solution would be to use only the d-dimensional components of the momenta in the vertex functions with $n > d+1$ and the full $\bar{d}$-dimensional momenta otherwise, or even to make the extension only in the propagator function $D(k)$. On the other hand, the ambiguity is of order $\bar{d}-d$, so its likely effect would be to redefine the bare couplings without changing $\delta\Gamma[\phi]$ as the regulator is removed. This point and the use of DR for $\delta\Gamma$ is illustrated in Appendix D. The possibility of using DR as a regulator is of interest because the formulas of $G_n^\alpha(k)$ are non-perturbative, and a non-perturbative formulation does not exist for DR at present.

5 Composite operators

5.1 General considerations

In the previous section we introduced the quantities

$$G^\alpha[\phi] = \frac{\partial \Gamma[\phi]}{\partial g_{0,\alpha}} = \langle O^\alpha \rangle^\phi. \tag{5.1}$$

They represent the (unrenormalized) amputated matrix elements of the operators O^α (discussed after (2.36) in Sect. 2.3). The renormalized ones can be obtained as

$$G_R^\beta[\phi] = \frac{\partial \Gamma[\phi]}{\partial g_{R,\beta}} =: \langle O^\beta \rangle_R^\phi. \tag{5.2}$$

In terms of S_R-graphs, all divergences not touched by the composite operator are subtracted; hence, they are already renormalized. In $\langle O^\alpha \rangle^\phi$ the new divergences induced by the composite operator remain unsubtracted, while in $\langle O^\beta \rangle_R^\phi$ they are removed through the R operation. In view of the relation $G_R^\beta[\phi] = (W^{-1})^\beta{}_\alpha G^\alpha[\phi]$, one can define the *renormalized operators*

$$O_R^\beta := (W^{-1})^\beta{}_\alpha O^\alpha, \tag{5.3}$$

(hence $\hat{T}_\alpha O_R^\beta = (W^{-1})^\beta{}_\alpha$) such that

$$\langle O_R^\beta \rangle^\phi = \langle O^\beta \rangle_R^\phi. \tag{5.4}$$

The operators $O^\alpha[\phi]$ are functionals that are finite for regular field configurations, and so finite at the tree or classical level; however, they have divergent matrix elements when introduced in the dynamics described by $\Gamma[\phi]$ due to the quantum fluctuations of the fields, i.e., radiative corrections. In turn, the renormalized operators are divergent as functionals but have finite matrix elements.

A first-order variation in the relation $S_0 = g_{0,\alpha} O^\alpha$ implies

$$\delta S_0 = \delta g_{0,\alpha} O^\alpha = \delta g_{R,\beta} O_R^\beta. \tag{5.5}$$

δS_0 diverges at the tree level but has finite matrix elements,

$$\delta\Gamma[\phi] = \langle \delta S_0 \rangle^\phi = \delta g_{0,\alpha} \langle O^\alpha \rangle^\phi = \delta g_{R,\beta} \langle O_R^\beta \rangle^\phi. \tag{5.6}$$

As indicated by these formulas, the amputated matrix elements of the operator O^α can be obtained from the perturbation it produces on the effective action when it is coupled to the action. By the same token, one can obtain the matrix elements of other operators not necessarily present in the original action. For instance[24]

$$O^a[\phi] := \frac{1}{2}\hat{\phi}^2(0), \tag{5.7}$$

can be coupled as $\delta S_0 = \delta\lambda_{0,a} O^a$, and

$$\delta\Gamma[\phi] = \delta\lambda_{0,a} \langle O^a \rangle^\phi. \tag{5.8}$$

As a rule, $\langle O^a \rangle^\phi$ will present UV-divergences, and in general, it will not be possible to cancel them (for all ϕ) by a suitable choice of the cutoff dependence in the bare parameter $\delta\lambda_{0,a}$. The cancellation will require mixing with other composite operators of equal or smaller dimension [12]. For the operator O^a above in the ϕ_6^3 theory, the mixing requires the operators

$$O^b := \hat{\phi}(0), \quad O^c := -(\partial^2\hat{\phi})(0), \quad O^d := 1. \tag{5.9}$$

That is

$$\delta S_0 = \delta\lambda_{0,a} O^a + \delta\lambda_{0,b} O^b + \delta\lambda_{0,c} O^c + \delta\lambda_{0,d} O^d. \tag{5.10}$$

The bare parameters $\delta\lambda_{0,\mu}(\Lambda)$ can be chosen so that

$$\delta\Gamma[\phi] = \delta\lambda_{0,\mu} \langle O^\mu \rangle^\phi \quad \mu \in \{a, b, c, d\}, \tag{5.11}$$

with implicit sum over the label $\mu \in \{a, b, c, d\}$, is UV-finite. Once again, here the unperturbed effective action $\Gamma[\phi]$ (and so its propagator and all its correlations functions) is already renormalized and UV-finite.

Note that also in the renormalization of the operators O^α in S_0 further operators $\propto 1$ and $\propto \phi$ are needed in the mixing to produce a finite $\delta\Gamma_n(p)$ for $n = 0, 1$. Such operators were mostly disregarded in the discussion as they do not affect the renormalization of $\delta\Gamma_n(p)$ for $n \geq 2$. An important difference between the operators in the action and the sporadic composite operators considered here, is that the former act many times while the latter act only once (or at any rate, act a finite number of times). As a consequence, the non-renormalizable operators in the action (i.e., with negative mass-dimension coupling) require the introduction new

ambiguity in its extension to $\bar{d}$. Ultimately, this problem is also related to the Levi–Civita symbol; the scalar quantity $|k_1 \wedge k_2 \wedge \cdots \wedge k_{d+1}|^2$ vanishes in d dimensions.

[24] $\hat{\phi}(x)$ is the field regulated with a cutoff, as defined in (3.11). The operator is at $x = 0$.

counterterm operators in the action of increasingly larger dimensions, while the composite operators require only a finite number of counterterms.

The new divergences introduced by the loops in the amputated matrix elements of the composite operators O^μ are to be canceled by the bare parameters $\delta\lambda_{0,\mu}(\Lambda)$. In order to do this, RC plus a projective renormalization scheme can be employed.

Until now the effective action was parameterized by the three values $g_{R,\alpha}$, so $\Gamma[\phi; g_{R,\alpha}]$, and these values are enforced through RC. One can consider the enlarged action $S_0 + \lambda_{0,\mu} O^\mu$, and correspondingly $\Gamma[\phi; g_{R,\alpha}, \lambda_{R,\mu}]$. The new parameters $\lambda_{R,\mu}$ require additional RC and the following are suitable in the present case,[25]

$$\begin{aligned} \tilde\Gamma_2(0) &= \lambda_{R,a}, \\ \tilde\Gamma_1(0) &= \lambda_{R,b}, \\ \partial_{p^2}\tilde\Gamma_1(0) &= \lambda_{R,c}, \\ \tilde\Gamma_0(0) &= \lambda_{R,d}. \end{aligned} \tag{5.12}$$

Here $\tilde\Gamma_n(p)$ denotes the momentum amplitudes in the expansion of $\Gamma[\phi]$, as defined in (2.37). Because the operators O^μ are not translationally invariant, the total momentum is not conserved and $\sum p$ needs not vanish in $\tilde\Gamma_n(p)$ in the theory with non-vanishing $\lambda_{0,\mu}$.

If only the matrix elements of the operators O^μ are needed, and not more general correlations functions, the conditions can be relaxed to

$$\begin{aligned} \delta\tilde\Gamma_2(0) &= \delta\lambda_{R,a}, \\ \delta\tilde\Gamma_1(0) &= \delta\lambda_{R,b}, \\ \partial_{p^2}\delta\tilde\Gamma_1(0) &= \delta\lambda_{R,c}, \\ \delta\tilde\Gamma_0(0) &= \delta\lambda_{R,d}, \end{aligned} \tag{5.13}$$

as a perturbation around the action S_0. i.e, the point $\lambda_{R,\mu} = 0$. We refer to this point as the point $\lambda = 0$.

To include the operators O^μ it is necessary to extend the space of functionals, as the space $\mathscr{H}_\mathrm{C}$ only includes the translationally invariant ones. The total space will be

$$\mathscr{H} = \mathscr{H}_\mathrm{C} \oplus \mathscr{H}_\mathrm{NC} \tag{5.14}$$

(conserving and non-conserving momentum). A neat way to keep the two spaces well separated and avoid problems, such as evaluating at $p = 0$ when $\delta(p)$ is present, is to introduce a projector P_C so that for a generic functional $F[\phi]$

$$P_C \tilde F_n(p) := \lim_{\eta\to 0^+} h\left((\textstyle\sum p)^2/\eta\right) \tilde F_n(p). \tag{5.15}$$

Here $h(x)$, defined for $x \geq 0$, is a decreasing smooth function with compact support such that $h(0) = 1$, and $h^{(k)}(0) = 0 \ \forall k \geq 1$. Then

$$\mathscr{H}_\mathrm{C} := P_C \mathscr{H}, \quad \mathscr{H}_\mathrm{NC} := (1 - P_C)\mathscr{H}. \tag{5.16}$$

The RC in (5.12) can be cast in the form

$$\hat T_\mu \Gamma[\phi] = \lambda_{R,\mu}, \tag{5.17}$$

where the $\hat T_\mu \in \mathscr{H}^*$. These operators fulfill the duality property[26]

$$\hat T_\nu O^\mu = \delta^\mu_\nu \quad \mu, \nu \in \{a, b, c, d\}. \tag{5.18}$$

Furthermore, due to the separation between $\mathscr{H}_\mathrm{C}$ and $\mathscr{H}_\mathrm{NC}$,

$$\begin{aligned} \hat T_\mu \mathscr{H}_\mathrm{C} &= 0 \quad \mu \in \{a, b, c, d\}, \\ \hat T_\alpha \mathscr{H}_\mathrm{NC} &= 0 \quad \alpha \in \{m, Z, g\}. \end{aligned} \tag{5.19}$$

Since $O^\alpha \in \mathscr{H}_\mathrm{C}$ and $O^\mu \in \mathscr{H}_\mathrm{NC}$,

$$\hat T_\mu O^\alpha = \hat T_\alpha O^\mu = 0. \tag{5.20}$$

The projector

$$T = O^\alpha \hat T_\alpha + O^\mu \hat T_\mu \tag{5.21}$$

is such that

$$T\mathscr{H}_\mathrm{C} = \mathscr{H}_\mathrm{S}, \quad T\mathscr{H}_\mathrm{NC} = \mathscr{H}_\mathrm{NS}, \tag{5.22}$$

where $\mathscr{H}_\mathrm{NS} \subseteq \mathscr{H}_\mathrm{NC}$ is the space spanned by the operators O^μ. The projector T serves to construct the R and $\bar R$ operations to renormalize both the action and the composite operators.

Specifically, following the previous steps, one can define the functionals

$$G^\mu[\phi] := \langle O^\mu\rangle^\phi = \frac{\partial\Gamma[\phi]}{\partial\lambda_{0,\mu}}. \tag{5.23}$$

These functions, or equivalently the functions $\tilde G^\mu_n(p)$, are needed only at S_0 i.e. at the point $\lambda = 0$. We assume that scenario in what follows.

The functions $\tilde G^\mu_n(p)$ are expressed in terms of the vertex $\tilde O^\mu_\ell(q)$ and the functions $H^\ell_n(q; p)$, similar to (4.54), except that O^μ does not conserve momentum, namely,

$$\tilde G^\mu_n(p) = \tilde O^\mu_n(p) + \sum_{\ell\geq 2}\frac{1}{\ell!}\int^\Lambda \prod_i^{\ell-1}\frac{d^d q_i}{(2\pi)^d}\tilde O^\mu_\ell(q) H^\ell_n(q; p). \tag{5.24}$$

Here $\sum p$ is arbitrary but the condition $\sum q + \sum p = 0$ fixes one of the q_i. Likewise,

$$G^\nu_R[\phi] := \langle O^\nu\rangle^\phi_R := \frac{\partial\Gamma[\phi]}{\partial\lambda_{R,\nu}}. \tag{5.25}$$

[25] Actually $\tilde\Gamma_0(p)$ has zero arguments p; we set 0 for uniformity of the notation.

[26] Indeed $\tilde O^a_n(p) = \delta_{n,2}$, $\tilde O^b_n(p) = \delta_{n,1}$, $\tilde O^c_n(p) = p^2\delta_{n,1}$ and $\tilde O^d_n(p) = \delta_{n,0}$, up to regulator-factors $F(p)$, which have no effect at $p = 0$.

The relation

$$W^{\mu}{}_{\nu} := \frac{\partial \lambda_{R,\nu}}{\partial \lambda_{0,\mu}} \tag{5.26}$$

then implies

$$G_R^{\nu}[\phi] = (W^{-1})^{\nu}{}_{\mu}\, G^{\mu}[\phi]. \tag{5.27}$$

Since there is no mixing between the two sectors $\mathscr{H}_{\mathrm{C}}$ and $\mathscr{H}_{\mathrm{NC}}$ the would-be matrix elements $W^{\alpha}{}_{\nu} = W^{\mu}{}_{\beta}$ are zero. The matrix W in the $\mathscr{H}_{\mathrm{NC}}$ sector follows from

$$W^{\mu}{}_{\nu} = \hat{T}_{\nu} G^{\mu}[\phi]. \tag{5.28}$$

A composite operator of a mass-dimension can only produce UV divergences of that very dimension or less; therefore, in minimal schemes $W^{\mu}{}_{\nu}$ would be a triangular matrix, as the matrix element would vanish when the dimension of O^{ν} is larger than that of O^{μ}. In the projective scheme, those matrix elements are UV-finite but not necessarily zero. By assumption, the space $\mathscr{H}_{\mathrm{NS}}$ spanned by the O^{μ} is such that all divergences generated by those operators are also contained in the same space.

The renormalized composite operators are[27]

$$O_R^{\nu} = (W^{-1})^{\nu}{}_{\mu} O^{\mu} \tag{5.29}$$

and they fulfill the relation

$$\langle O_R^{\nu}\rangle^{\phi} = \langle O^{\nu}\rangle_R^{\phi}. \tag{5.30}$$

For the operator O^a in ϕ_6^3 above, the renormalization yields

$$O_R^a = \frac{1}{\tilde{G}_2^q(0)}\left(O^a - \tilde{G}_1^a(0) O^b - \partial_p^2 \tilde{G}_1^a(0) O^c - \tilde{G}_0^a(0) O^d\right) \tag{5.31}$$

where $\tilde{G}_2^q(0)$ and $\partial_p^2 \tilde{G}_1^a(0)$ are $O(L_\Lambda)$, $\tilde{G}_1^a(0) = O(\Lambda^2)$ and $\tilde{G}_0^a(0) = O(\Lambda^4)$. In the present case, the matrix W is particularly easy to invert because, regardless of the interaction, $\tilde{G}_n^A(p) = \tilde{O}_n^A(p)$ whenever A contains at most one field ϕ, hence $\tilde{G}_n^b(p) = \delta_{n,1}$, $\tilde{G}_n^c(p) = p^2\delta_{n,1}$, and $\tilde{G}_n^d(p) = \delta_{n,0}$.

As already happened for the action, also for composite operators the $\tilde{G}_n^{\mu}(p)$ and $W^{\mu}{}_{\nu}$ are explicit in terms of the cutoff momentum integrals and present an anti-canonical divergence-pattern, while

$$(W^{-1})^{\nu}{}_{\mu} = \delta_{\mu}^{\nu} - V^{\nu}{}_{\mu} \tag{5.32}$$

is canonical. Some related ideas are discussed in Appendix C.

[27] If the set of composite operators is enlarged by adding new operators, the previous matrix elements of W do not change, however W^{-1}, and so the combinations O_R^{μ} in terms of the O^{ν}, may change in the unextended sector, unless the new operators are chosen so that the matrix W is triangular.

5.2 Renormalization of $\phi^3(0)$ in the theory ϕ_4^3

In the super-renormalizable theory ϕ_4^3, we consider the renormalization of the composite operator

$$O^a := \frac{1}{3!}\hat{\phi}^3(0). \tag{5.33}$$

Dimensionally, it mixes with the operator

$$O^b := \frac{1}{2}\hat{\phi}^2(0), \tag{5.34}$$

and the operators $\partial^2\hat{\phi}(0)$, $\hat{\phi}(0)$, and 1. We disregard these because they only involve the renormalization of $G_n^{\mu}(p)$ for $n \le 1$.

Note that the relations

$$O^g = \int d^d x\, O^a(x), \quad O^m = \int d^d x\, O^b(x), \tag{5.35}$$

imply

$$\begin{aligned}\tilde{G}_n^a(p) &= G_n^g(p)\\ \tilde{G}_n^b(p) &= G_n^m(p)\end{aligned} \quad \text{if } \sum p = 0. \tag{5.36}$$

For the operators O^a and O^b we will adopt RC similar to those of O^g and O^m in (4.62), that is,

$$\begin{aligned}\lambda_{R,a} &= \hat{T}_a \Gamma := \tilde{\Gamma}_3(p)\Big|_{\mu_{\mathrm{RC}}=\infty},\\ \lambda_{R,b} &= \hat{T}_b \Gamma := \tilde{\Gamma}_2(p)\Big|_{\mu_{\mathrm{RC}}=0}.\end{aligned} \tag{5.37}$$

Because the kinematical conditions (4.12) enforce $\sum p = 0$ for $n \ge 2$, our choice of RC in turn ensures that

$$\begin{aligned}\tilde{G}_{R,n}^a(p) &= G_{R,n}^g(p)\\ \tilde{G}_{R,n}^b(p) &= G_{R,n}^m(p)\end{aligned} \quad \text{if } \sum p = 0. \tag{5.38}$$

Furthermore, the matrix $W^{\mu}{}_{\nu}$ of the composite operators (the basis is ordered first a then b) is

$$\begin{aligned}W^{\mu}{}_{\nu} &= \begin{pmatrix}\tilde{G}_3^a(\infty) & \tilde{G}_2^a(0)\\ \tilde{G}_3^b(\infty) & \tilde{G}_2^b(0)\end{pmatrix} = \begin{pmatrix}G_3^g(\infty) & G_2^g(0)\\ G_3^m(\infty) & G_2^m(0)\end{pmatrix}\\ &= \begin{pmatrix}1 & G_2^g(0)\\ 0 & G_2^m(0)\end{pmatrix} = W^{\alpha}{}_{\beta},\end{aligned} \tag{5.39}$$

That is, it coincides with the matrix $W^{\alpha}{}_{\beta}$ of the operators in the action. As a consequence relations similar to (4.65) apply (for $n \ge 2$)

$$\begin{aligned}\tilde{G}_{R,n}^a(p) &= \tilde{G}_n^a(p) - \frac{G_2^g(0)}{G_2^m(0)}\tilde{G}_n^b(p),\\ \tilde{G}_{R,n}^b(p) &= \frac{1}{G_2^m(0)}\tilde{G}_n^b(p),\end{aligned} \tag{5.40}$$

or equivalently

$$O_R^a = O^a - \frac{G_2^g(0)}{G_2^m(0)} O^b,$$
$$O_R^b = \frac{1}{G_2^m(0)} O^b, \tag{5.41}$$

up to terms proportional to the operators $1, \phi(0)$ and $\partial^2\phi(0)$.

It should be noted that not only the expressions for $\tilde{G}^a_{R,n}(p)$ and $\tilde{G}^b_{R,n}(p)$ are explicit and finite, in fact, they can be arranged so that no regulator is needed, as already happened for the functions $G^\alpha_{R,n}(p)$ in Sect. 3.1. This follows from the fact that the expressions in both cases are very similar, namely (with implicit form factors)

$$\tilde{G}^a_n(p) = \delta_{n,3} + \int^\Lambda \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{3!}H_n^3 \times (q_1, q_2, -q_1 - q_2 - \sum p; p) \tag{5.42}$$
$$\tilde{G}^b_n(p) = \delta_{n,2} + \int^\Lambda \frac{d^dq_1}{(2\pi)^d}\frac{1}{2!}H_n^2(q_1, -q_1 - \sum p; p),$$

only differ from $G^g_n(p)$ and $G^m_n(p)$ in (3.20) and (3.21) by the term $\sum p$. Because $\sum p$ is finite, the asymptotic properties of the functions are identical and the same arrangements carried out in Sect. 3.1 work here with straightforward modifications. Entirely similar considerations apply to the renormalization of the composite operator $\frac{1}{2}(\partial\phi)^2(0)$. Moreover, they also apply to the corresponding operators in the theory ϕ^4_3.

6 Schwinger–Dyson equations

The relation

$$0 = \int D\varphi \partial_i \left(e^{S+J\cdot\varphi}A\right) \tag{6.1}$$

is valid for any sufficiently convergent observable $A[\varphi]$. It immediately produces the Schwinger–Dyson equations (SDE)

$$0 = \langle \partial_i A + A\partial_i S + AJ_i\rangle^J. \tag{6.2}$$

More specifically, the choice $A = 1$ yields $\langle\partial_i S\rangle^J = -J_i$, and hence

$$\partial_i \Gamma[\phi] = \langle\partial_i S\rangle^\phi. \tag{6.3}$$

The same result is also deduced directly from the identity $\delta\Gamma[\phi] = \langle\delta S\rangle^\phi$ by noting that a shift $S[\phi] \to S[\phi+\epsilon]$ is accompanied by $\Gamma[\phi] \to \Gamma[\phi+\epsilon]$. In what follows by SDE we will refer to the relations in (6.3), or equivalent to them.

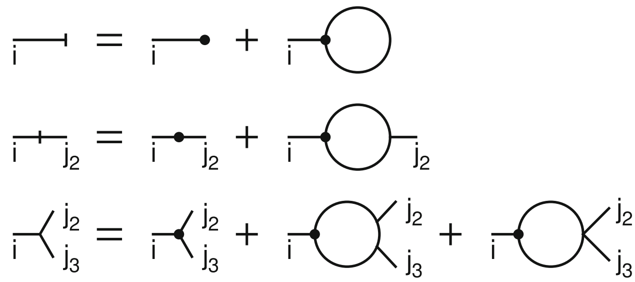


Fig. 10 Graph representation of SDE in (6.7). The dots represent the vertices of S, the other vertices and the lines are those of Γ

The SDE can be made more explicit by expanding both sides of (6.3) in powers of ϕ. This procedure yields

$$\sum_{n\ge1}\frac{1}{(n-1)!}\Gamma_{ij_2\dots j_n}\phi^{j_2}\cdots\phi^{j_n} = \sum_{\ell\ge1}\frac{1}{(\ell-1)!}g_{ii_2\dots i_\ell}\langle\varphi^{i_2}\cdots\varphi^{i_\ell}\rangle^\phi \tag{6.4}$$

and so

$$\Gamma_{ij_2\dots j_n} = g_{ij_2\dots j_n} + \sum_{\ell\ge3}\frac{1}{(\ell-1)!}g_{ii_2\dots i_\ell}H^{i_2\dots i_\ell}_{j_2\dots j_n} \quad n\ge1. \tag{6.5}$$

The LHS is completely symmetric, so the RHS is symmetric too, although not manifestly so. These are the (unsymmetrized) SDE. The coefficients H are tree graphs of Γ, then after joining the i-legs in the vertex g_ℓ of S, graphs with at most $\ell-2$ explicit loops are obtained.

Let us consider in more detail the particular case of a ϕ^3 theory, i.e., $g_{i_1\dots i_n} = 0$ for $n\ge4$

$$\Gamma_i = h_i + \frac{1}{2}g_{ijk}H^{jk}$$
$$\Gamma_{ij_2} = m_{ij_2} + \frac{1}{2}g_{ijk}H^{jk}_{j_2}$$
$$\Gamma_{ij_2j_3} = g_{ij_2j_3} + \frac{1}{2}g_{ijk}H^{jk}_{j_2j_3} \tag{6.6}$$
$$\Gamma_{ij_2\dots j_n} = \frac{1}{2}g_{ijk}H^{jk}_{j_2\dots j_n} \quad n\ge4.$$

More explicitly, for $1\le n\le3$,

$$\Gamma_i = h_i + \frac{1}{2}g_{ijk}D^{jk}$$
$$\Gamma_{ij_2} = m_{ij_2} + \frac{1}{2}g_{ijk}D^{ja}D^{kb}\Gamma_{j_2ab}$$
$$\Gamma_{ij_2j_3} = g_{ij_2j_3} + g_{ijk}D^{ja}D^{kb}D^{cd}\Gamma_{j_2ac}\Gamma_{j_3bd} + \frac{1}{2}g_{ijk}D^{ja}D^{kb}\Gamma_{j_2j_3ab}. \tag{6.7}$$

The diagrammatic representation of these equations is displayed in Fig. 10. This is an infinite hierarchy of exact relations fulfilled by the theory ϕ^3. Furthermore, through systematic iteration (inserting the LHS in the RHS) the hierarchy produces the perturbative series in powers of the coupling g in terms of Feynman graphs (S-graphs). In such a construction the perturbative counting is assigned as follows:

(i) m_{ij}, Γ_{ij} and D^{ij}, as well as h_i and Γ_i, are of $O(g^0)$,
(ii) g_{ijk} and Γ_{ijk} are of $O(g)$, and
(iii) the vertices $\Gamma_{i_1\ldots i_n}$ ($n \geq 4$) are of $O(g^n)$.

In each iteration D^{ij} and $\Gamma_{i_1\ldots i_n}$ are expressed using s^{ij} and g_{ijk} plus terms of higher order. Then, at each step, there are contributions involving only s^{ij} and g_{ijk} plus a remainder that contains the full propagator and vertices. The terms constructed solely with S no longer evolve and these are the standard Feynman graphs of the theory S, while the remainder becomes of higher order at each iteration. Hence, the procedure not only produces the perturbative series but also provides a closed form for the exact remainder corresponding to the truncated series.

Since, as noted above, the RHS of (6.5) is not manifestly symmetric in the indices $i, j_2, \ldots, j_n$, there is an ambiguity in how to expand D^{ij} and $\Gamma_{i_1\ldots i_n}$ present on the RHS using the SDE. A simple prescription is to use a symmetrized version of the SDE, which will also be useful later; namely, the RHS of Eq. (6.5) is symmetrized by hand. This gives rise to the *symmetrized* SDE:

$$\Gamma_{j_1\ldots j_n} = g_{j_1\ldots j_n} + \sum_{\ell\geq 3} \frac{1}{\ell!} g_{i_1\ldots i_\ell} H'^{i_1\ldots i_\ell}_{j_1\ldots j_n} \quad n \geq 1 \tag{6.8}$$

where

$$H'^{i_1\ldots i_\ell}_{j_1\ldots j_n} := \frac{1}{n}\sum_{q=1}^{\ell}\sum_{p=1}^{n} \delta^{i_q}_{j_p} H^{i_1\ldots \widehat{i_q}\ldots i_\ell}_{j_1\ldots \widehat{j_p}\ldots j_n} \quad n \geq 1, \quad \ell \geq 3. \tag{6.9}$$

By construction the coefficients $H'^{i_1\ldots i_\ell}_{j_1\ldots j_n}$ are completely symmetric and vanish for $\ell = 1, 2$. Correspondingly, in the momentum space, there are functions $H'^{\ell}_{n}(q; p)$ in the subspace $\sum q + \sum p = 0$.

In complete analogy with the relations

$$\begin{aligned} \delta\Gamma[\phi] &= \delta S_0[\phi] + \sum_{\ell\geq 2} \frac{1}{\ell!} \delta g_{i_1\ldots i_\ell} \hat{H}^{i_1\ldots i_\ell}[\phi] = \delta g_{0,\alpha} G^\alpha[\phi], \\ G^\alpha[\phi] &= O^\alpha[\phi] + H^\alpha[\phi], \end{aligned} \tag{6.10}$$

the SDE in (6.8) can be cast in the form

$$\begin{aligned} \Gamma[\phi] &=: S_0[\phi] + \sum_{\ell\geq 3} \frac{1}{\ell!} g_{i_1\ldots i_\ell} \hat{H}'^{i_1\ldots i_\ell}[\phi] =: g_{0,\alpha} G'^\alpha[\phi], \\ G'^\alpha[\phi] &=: O^\alpha[\phi] + H'^\alpha[\phi]. \end{aligned} \tag{6.11}$$

Similar to the functionals $G^\alpha[\phi]$, the $G'^\alpha[\phi]$ or the functions $H'^\alpha_n(p)$ are expressed in terms Γ-graphs, a finite number of graphs for each given number n of (amputated) p-legs. As in (3.23), ℓ q-legs are attached to a vertex O^α.

The integral over q is, in general, UV divergent and *explicit* in the sense that a regulator Λ bounds the integration region, but otherwise it is not present in the integrand. As was the case for $\delta\Gamma[\phi]$ through Schwinger's principle in previous Sections, here we assume that the effective action, which determines the lines and vertices to construct the Γ-graphs, is already renormalized. The regulator is needed only for the explicit loops in the Γ-graphs and the divergences are compensated through a suitable cutoff dependence in the bare couplings $g_{0,\alpha}(\Lambda)$ explicit in the formula. The consistency of this assumption is based on the analysis carried out in Appendix D.

Applying the forms $\hat{T}_\beta$ on both sides of (6.11), in order to enforce the RC,

$$\hat{T}_\beta \Gamma[\phi] = g_{R,\beta} = g_{0,\alpha} W'^\alpha{}_\beta \tag{6.12}$$

with

$$W'^\alpha{}_\beta := \hat{T}_\beta G'^\alpha[\phi]. \tag{6.13}$$

Clearly, the matrix V', defined by

$$(W'^{-1})^\beta{}_\alpha =: \delta^\beta_\alpha - V'^\beta{}_\alpha \tag{6.14}$$

fulfills the relation

$$g_{R,\alpha} = g_{0,\alpha} + g_{R,\beta} V'^\beta{}_\alpha. \tag{6.15}$$

This equation conforms to the structure $S_R = S_0 + T\bar{R}(\Gamma_R - S_R)$. Therefore V' has only canonical divergences (those in $g_{0,\alpha}$) and in turn $W'^\alpha{}_\beta$ and $G'^\alpha[\phi]$ are anti-canonical. Specifically, the functionals

$$G'^\beta_R[\phi] := (W'^{-1})^\beta{}_\alpha G'^\alpha[\phi] \tag{6.16}$$

are UV-finite, and

$$\Gamma[\phi] = g_{R,\beta} G'^\beta_R[\phi]. \tag{6.17}$$

However, note that the functionals $G'^\beta_R[\phi]$ are not univocally determined by this equation, they are defined by the very hierarchy of SDE. When (6.16) is expanded perturbatively the pattern $\Gamma = S_R + R(\Gamma_R - S_R)$ is reproduced. The equation paralleling (4.58),

$$G'^\beta[\phi] = G'^\beta_R[\phi] + V'^\beta{}_\alpha G'^\alpha[\phi], \tag{6.18}$$

also holds.

Clearly, the relations fulfilled by the SDE and their renormalization are very similar to those of the linearized renormalization in Sect. 4. Nevertheless, one noteworthy difference between the matrices W and W' (or V and V') arises, namely, while the matrix elements of W are all non-trivial and divergent in the renormalizable case, W' and V' have the structures

$$W' = \begin{pmatrix} I & 0 \\ X & X \end{pmatrix}, \quad V' = \begin{pmatrix} 0 & 0 \\ X & X \end{pmatrix}, \tag{6.19}$$

where I is the identity matrix and the first subset of indices refers to operators O^α with $\ell \leq 2$. This pattern follows from $\hat{H}'^{i_1\ldots i_\ell}[\phi] = 0$ for $\ell \leq 2$. Also, for a ϕ_d^κ theory, the Γ-graphs of the SDE involve $\kappa - 2$ explicit loops, for $\kappa - 1$ loops in the case of $\delta\Gamma$.

The two sets of matrices W and W', or V and V', can be related. Using (4.43) and (6.12),

$$W^\alpha{}_\beta = W'^\alpha{}_\beta + g_{0,\gamma}\frac{\partial W'^\gamma{}_\beta}{\partial g_{0,\alpha}}, \tag{6.20}$$

or equivalently

$$V^\beta{}_\alpha = V'^\beta{}_\alpha + g_{R,\gamma}\frac{\partial V'^\gamma{}_\alpha}{\partial g_{R,\beta}}. \tag{6.21}$$

Furthermore, from (6.17)

$$G_R^\beta[\phi] = G_R'^\beta[\phi] + g_{R,\gamma}\frac{\partial G_R'^\gamma[\phi]}{\partial g_{R,\beta}}. \tag{6.22}$$

The RHS automatically implements the consistency conditions (4.59).

In the renormalizable case, the $G_R'^\beta[\phi]$ are UV-finite but not manifestly so: the regulator appears in $W'^\alpha{}_\beta$ and $G'^\alpha[\phi]$ in such a way that the combination $W'^{-1}G'[\phi]$ has a finite limit. This is exactly the same situation as for $G_R[\phi] = W^{-1}G[\phi]$ in the linearized renormalization of a renormalizable theory. On the other hand, for super-renormalizable theories, the renormalization of the SDE is manifest. For instance, for ϕ_4^3, using the RC in (4.62),

$$\begin{aligned} W'^g{}_m &= \hat{T}_m G'^g = G_2'^g(0), \\ W'^g{}_Z &= \hat{T}_Z G'^g = \partial_{p^2} G_2'^g(\infty) = 0, \\ W'^g{}_g &= \hat{T}_g G'^g = G_3'^g(\infty) = 1. \end{aligned} \tag{6.23}$$

Hence,

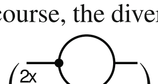
$$W' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ G_2'^g(0) & 0 & 1 \end{pmatrix}, \quad V' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G_2'^g(0) & 0 & 0 \end{pmatrix}. \tag{6.24}$$

In view of the SDE for this theory, Fig. 10,

$$\begin{aligned} G'^m_{R,n}(p) &= G'^m_n(p) = \delta_{n,2}, \\ G'^Z_{R,n}(p) &= G'^Z_n(p) = p^2\delta_{n,2}, \\ G'^g_{R,n}(p) &= G'^g_n(p) - G'^g_2(0)\delta_{n,2}. \end{aligned} \tag{6.25}$$

Hence

$$\Gamma_n(p) = (m_R^2 + Zp^2)\delta_{n,2} + g(G_n'^g(p) - G_2'^g(0)\delta_{n,2}). \tag{6.26}$$

Of course, the divergent component of the quantity $g G_2'^g(0)$ $= \frac{1}{2}\Big($  $\Big)_{p=0}$ is just $-m_{\mathrm{ct}}^2(\Lambda)$ in (3.6). If the iteration discussed after (6.7), eliminating $\Gamma[\phi]$ in favor of $S_R[\phi]$, is systematically applied from (6.26), it produces the subtracted Feynman graphs of the theory.

Using the expression for $V^\beta{}_\alpha$ in (4.64), the identities (6.21) imply the relations

$$\begin{aligned} 1 &= \frac{1}{G_2^m(0)} + g\frac{\partial G_2'^g(0)}{\partial m_R^2} \\ \frac{G_2^Z(0)}{G_2^m(0)} &= g\frac{\partial G_2'^g(0)}{\partial Z} \\ \frac{G_2^g(0)}{G_2^m(0)} &= \frac{\partial(gG_2'^g(0))}{\partial g}. \end{aligned} \tag{6.27}$$

Because the divergent component of $G_2'^g(0)$ is independent of m_R^2 the RHS of the first equation is UV-finite.

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Code Availability Statement This manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

Appendix A: Derivation of some formulas

1. Proof of Eq. (2.18)

Under the first-order variation $S \to S + \delta S$, it follows from (2.8) that $\delta W[J] = \langle \delta S \rangle^J$. Hence, applying the variation δ to $W[J] = \Gamma[\phi] + J_i\phi^i$, with $J = J[\phi]$ and $\delta\phi = 0$, we obtain

$$\delta W[J] + \delta J_i \partial^i W[J] = \delta\Gamma[\phi] + \delta J_i \phi^i. \tag{A1}$$

Since $\phi^i = \partial^i W[J]$, it follows that $\langle \delta S \rangle^J = \delta\Gamma$ which proves Eq. (2.18).

2. Proof of the Theorem around Eq. (2.19)

The action producing the expectation values $\langle\varphi^{i_1}\cdots\varphi^{i_n}\rangle^\phi$ is just $S'[\varphi] = S[\varphi] + J[\phi]\cdot\varphi$, and therefore the corresponding generator of the connected Green functions is $W[J'+J]$. Here the current is J' while ϕ, and so $J = J[\phi]$, are parameters. For $n \geq 2$ such generator may be changed to

$$W'[J'] = W[J+J'] - (J+J')\cdot\phi. \tag{A2}$$

Although this functional would give a vanishing value for $\langle\varphi\rangle^J$, it correctly reproduces the expectation values for $n \geq 2$. The effective action $\Gamma'[\phi']$ that at the tree level produces the same Green functions (for $n \geq 2$) is then obtained as the Legendre transform of $W'[J']$, as follows.

Using the Legendre-transform relations of the type

$$\Gamma[\phi] = \inf_J (W[J] - J\cdot\phi), \quad W[J] = \sup_\phi(\Gamma[\phi] + J\cdot\phi), \tag{A3}$$

we can write

$$\begin{aligned}\Gamma'[\phi'] &= \inf_{J'}(W'[J'] - J'\cdot\phi') \\ &= \inf_{J'}(W[J+J'] - (J+J')\cdot\phi - J'\cdot\phi') \\ &= \inf_{J'}(\sup_{\phi_1}(\Gamma[\phi_1] + (J+J')\cdot\phi_1) - \phi\cdot(J+J') - J'\cdot\phi').\end{aligned} \tag{A4}$$

The extremum with respect to J' requires $\phi_1 = \phi+\phi'$, hence

$$\Gamma'[\phi'] = \Gamma[\phi+\phi'] + J[\phi]\cdot\phi'. \tag{A5}$$

Here ϕ' is the classical field and ϕ is a parameter. This effective action has a vanishing one-point vertex.

The statement of the Theorem in Eq. (2.19) is that the derivatives of $\Gamma[\phi]$ of order two or higher produce the correct propagator and vertices to be used in the Feynman rules at tree level to reproduce $\langle\varphi^{i_1}\cdots\varphi^{i_n}\rangle^\phi$, for $n \geq 2$. Those Feynman rules obviously derive from the effective action in Eq. (A5), hence the statement is proven.

It can be noted that the Legendre transformation of the functional $W[J+J']$, namely, $\Gamma[\phi'] + J\cdot\phi'$, also produces the correct Green functions, albeit with different propagator and vertices; however, it contains a non-vanishing one-point vertex. A resummation of those graphs to eliminate the one-point vertex is directly provided by the effective action in Eq. (A5).

Alternatively, the Theorem can be obtained by recursively applying derivatives, starting from the two-point function:

$$\langle\varphi^i\varphi^j\rangle_c^J = \partial^i\partial^j W[J] = \frac{\partial\phi^j}{\partial J_i}. \tag{A6}$$

By the symmetry $W \leftrightarrow -\Gamma$ of the Legendre transformation, we also have

$$-\partial_i\partial_j\Gamma[\phi] = \frac{\partial J_i}{\partial\phi^j}. \tag{A7}$$

This proves $\langle\varphi^i\varphi^j\rangle_c^\phi = ((-\partial^2\Gamma)^{-1})^{ij}$. Then for $n=3$

$$\langle\varphi^i\varphi^j\varphi^k\rangle_c^J = \partial^i\langle\varphi^j\varphi^k\rangle_c^J = \partial^i\phi^a\partial_a((-\partial^2\Gamma)^{-1})^{ij} \tag{A8}$$

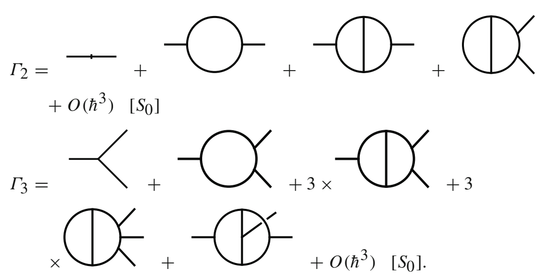
produces the third equation in (2.20). The rule is that each $\partial^a = \partial/\partial J_a$ generates a new leg from either a vertex or a (external or internal) line, while each $\partial_a = \partial/\partial\phi^a$ generates an amputated leg.

Appendix B: Renormalization as reparametrization

In this appendix, we want to illustrate that the relation (4.31)

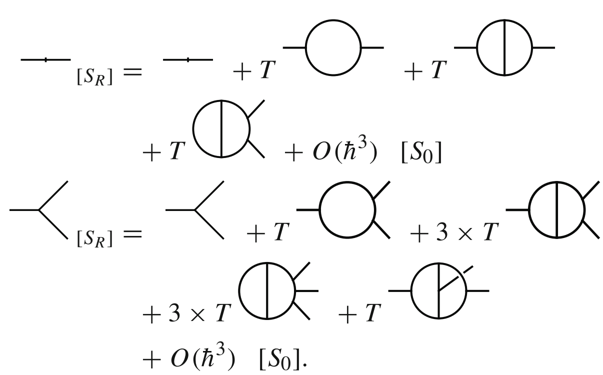
$$\Gamma - S_0 = \bar{R}(\Gamma_R - S_R) \tag{B1}$$

expresses a change of variables from S_0 to S_R, where the latter is defined by $S_R = T\Gamma$. Γ and Γ_R denote the effective actions diagrammatically constructed with S_0 and S_R, respectively. As previously noted, this relation already implies the other two: $S_R - S_0 = T\bar{R}(\Gamma_R - S_R)$ and $\Gamma - S_R = R(\Gamma_R - S_R)$. We will work at low orders of the perturbative expansion in the Γ_n sectors $n = 2, 3$ and assume that S_0 has only operators with $\ell = 2, 3$. The expansion of Γ as one-particle irreducible amputated graphs of S_0, yields



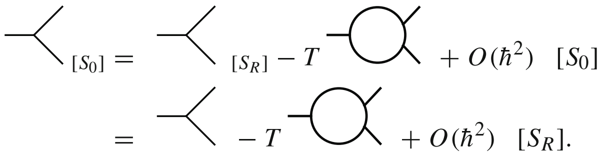
(B2)

Here the external legs are amputated. The label $[S_0]$ on the RHS indicates that the lines and vertices are those of S_0 unless otherwise indicated. The prescription $S_R = T\Gamma$ then implies



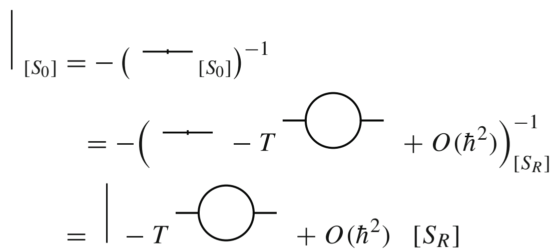
(B3)

We want to express S_0 in terms of S_R. The $n = 3$ relation in (B3) yields



(B4)

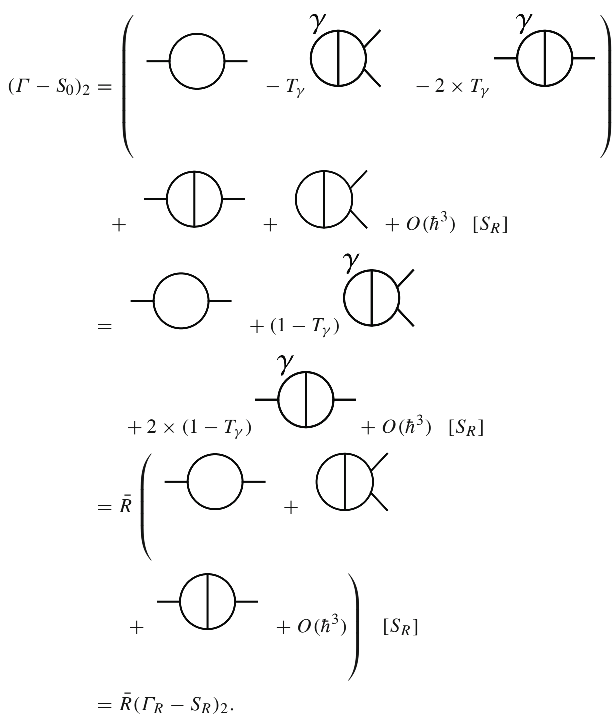
On the other hand, for $n = 2$, the free propagator relations $D_{0,0} = -S_{0,2}^{-1}$ and $D_{R,0} = -S_{R,2}^{-1}$ apply. Upon inversion, one immediately finds, for the bare and renormalized free propagators,



(B5)

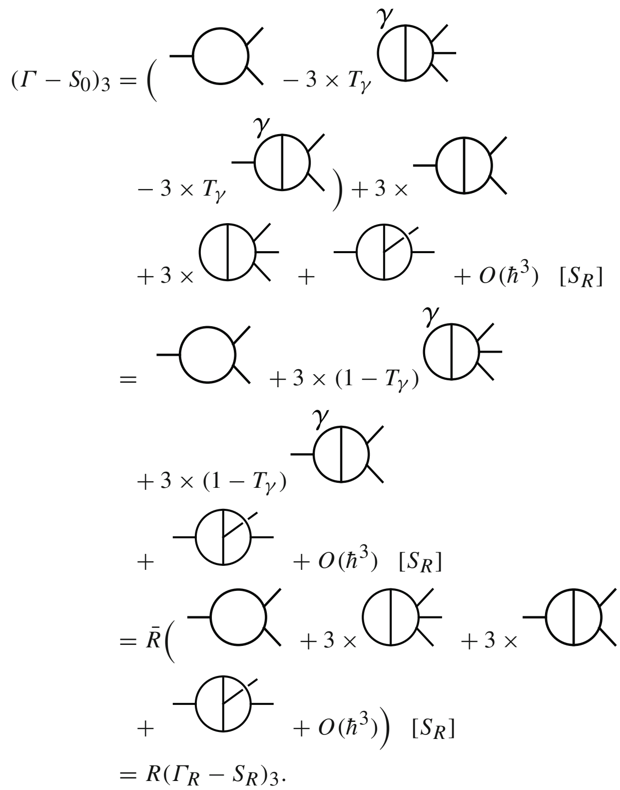
The lines are not amputated here.

Applying (B4) and (B5) one can proceed to the systematic elimination of the line and vertices of S_0 in favor of those of S_R for Γ_n in (B2). This produces, for $n = 2$,



(B6)

Likewise, for $n = 3$,



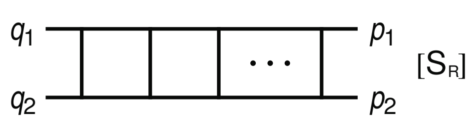
(B7)

Hence, the statement is verified to this order.

Appendix C: Cancellation of divergences and canonical and anti-canonical patterns

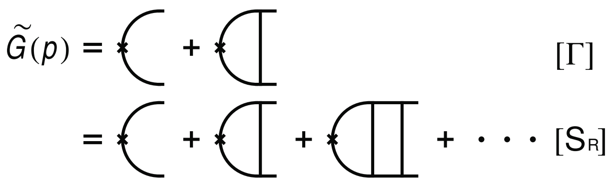
As stated in Sect. 5.1, the matrix $W^\mu{}_\nu$ has an anti-canonical pattern of divergences, while $V^\nu{}_\mu$ is canonical, and similarly for $W^\alpha{}_\beta$ and $V^\beta{}_\alpha$ in Sect. 4.2. In addition, the construction $(W^{-1})^\mu{}_\nu G^\nu[\phi] = G^\mu_R[\phi]$ eliminates the divergences.

These statements can be explicitly verified through a perturbative calculation to any desired order. Nevertheless, to illustrate them in a simple setting, we consider the renormalization of the composite operator $O = \frac{1}{2}\phi^2(0)$ in the theory ϕ^3_6, but including only a restricted set of graphs of S_R. To be more explicit, we consider the class



(C1)

$(q_1 + q_2 + p_1 + p_2 = 0)$ with $k = 0, 1, 2, \ldots$ exchanges in the t-channel, such S_R-graphs are of perturbative order g_R^{2k}. Within this approximation, only the term $\tilde{G}_2(p)$ (i.e., $n = 2$) can be described, so we denote it as $\tilde{G}(p)$:



(C2)

Analytically,

$$\tilde{G}(p) = 1 + \frac{1}{2}\int^{\Lambda} \frac{d^6q}{(2\pi)^6}\, D_q^2\, H(q;p). \tag{C3}$$

In our approximation, there is no mixing with other operators. As the renormalization condition we adopt $\hat{T}G = \tilde{G}(0)$, hence

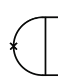
$$W = \tilde{G}(0) = 1 + \frac{1}{2}\int^{\Lambda} \frac{d^6q}{(2\pi)^6}\, D_q^2\, H(q;0), \tag{C4}$$

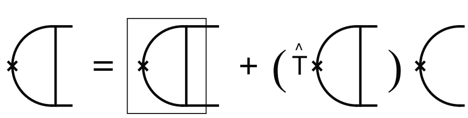
and $G_R^{\mu}[\phi] = (W^{-1})^{\mu}{}_{\nu} G^{\nu}[\phi]$ reduces to

$$\tilde{G}_R(p) = \frac{\tilde{G}(p)}{\tilde{G}(0)}. \tag{C5}$$

In the theory ϕ_6^3 all the box subgraphs in the S_R-graph in (C1) are UV-finite. On the other hand, the composite operator induces divergences with up to k-loops in the graph of order g_R^{2k}.

In (C2), the S_R-graph of order g_R^0, , is a tree graph with the value 1.

The one-loop S_R-graph of order g_R^2 in (C2), , has only a logarithmic superficial divergence, without subdivergences. Applying the identity $1 = (1-T) + T$, the term $1 - T$ gives the subtracted graphs and $T = O\hat{T}$ isolates the divergence in the sector O, hence:



(C6)

The two-loop S_R-graph, as well as the higher-order graphs, can be treated similarly in diagrammatic form; however, we turn to a more convenient algebraic notation. Let γ_k, $k = 0, 1, 2, \ldots$, denote the S_R-graph as in (C2) of order g_R^{2k}, which also appears as a subgraph of γ_ℓ for $\ell \geq k$. Then (C2) becomes

$$\tilde{G}(p) = \gamma_0 + \gamma_1 + \gamma_2 + O(g^6) \tag{C7}$$

and (C6) becomes

$$\gamma_1 = T\gamma_1 + (1-T)\gamma_1 = (\hat{T}\gamma_1)\gamma_0 + R\gamma_1. \tag{C8}$$

As always, R denotes the operation of subtracting all sub- and overall divergences, while $\bar{R}$ subtracts only subdivergences.

Likewise, denoting T_{γ_k} the projection acting on the (sub)graph γ_k,

$$\gamma_2 = T_{\gamma_1}\gamma_2 + (1 - T_{\gamma_1})\gamma_2 \tag{C9}$$

which can be worked out so that any divergence is either subtracted or is a projected overall divergence:

$$\begin{aligned}(1 - T_{\gamma_1})\gamma_2 &= (1 - T_{\gamma_2})(1 - T_{\gamma_1})\gamma_2 + T_{\gamma_2}(1 - T_{\gamma_1})\gamma_2\\ &= R\gamma_2 + T\bar{R}\gamma_2 = R\gamma_2 + (\hat{T}\bar{R}\gamma_2)\gamma_0,\\ T_{\gamma_1}\gamma_2 &= (\hat{T}\gamma_1)\gamma_1 = (\hat{T}\gamma_1)((\hat{T}\gamma_1)\gamma_0 + R\gamma_1),\end{aligned} \tag{C10}$$

that is,

$$\gamma_2 = (\hat{T}\bar{R}\gamma_2 + (\hat{T}\gamma_1)^2)\gamma_0 + (\hat{T}\gamma_1)R\gamma_1 + R\gamma_2. \tag{C11}$$

Collecting the various terms

$$\begin{aligned}\tilde{G}(p) &= (1 + \hat{T}\gamma_1 + (\hat{T}\gamma_1)^2 + \hat{T}\bar{R}\gamma_2)\gamma_0\\ &\quad + (1 + \hat{T}\gamma_1)R\gamma_1 + R\gamma_2 + O(g^6).\end{aligned} \tag{C12}$$

Using $\hat{T}\gamma_0 = 1$ and $\hat{T}R = 0$,

$$W = \hat{T}\tilde{G}(p) = 1 + \hat{T}\gamma_1 + (\hat{T}\gamma_1)^2 + \hat{T}\bar{R}\gamma_2 + O(g^6) \tag{C13}$$

and

$$\tilde{G}(p) = W\tilde{G}_R(p) \tag{C14}$$

with

$$\tilde{G}_R(p) = \gamma_0 + R(\gamma_1 + \gamma_2 + O(g^6)). \tag{C15}$$

Thus $\tilde{G}_R(p)$ is UV-finite and conforms to the expected structure $\Gamma = S_R + R(\Gamma_R - S_R)$. Effectively, in (C12) all divergences and subdivergences present in $\tilde{G}(p)$ have been systematically transferred to a factor W with no dependence on p, leaving a factor $\tilde{G}_R(p)$ with dependence on p but no divergences.

On the other hand, W has an anti-canonical pattern of divergences, because

$$\begin{aligned}V &= 1 - W^{-1} = \hat{T}\gamma_1 + \hat{T}\bar{R}\gamma_2 + O(g^6)\\ &= \hat{T}\bar{R}(\gamma_1 + \gamma_2 + O(g^6))\end{aligned} \tag{C16}$$

is canonical, i.e., $\hat{T}$ acts only after all subdivergences have been subtracted. V provides the counterterms and also conforms to the expected structure $S_0 = S_R - T\bar{R}(\Gamma_R - S_R)$. Let us point out that $\tilde{G}(p)$ does not know about T (i.e., about the RC chosen for the composite operator O) and T appears just once in W; nevertheless, the construction $V = 1 - W^{-1}$ arranges the insertions of T to produce the correct ordered subtractions to select superficial divergences only. Not only that, $\tilde{G}_R(p) = W^{-1}\tilde{G}(p)$ implies that W^{-1} introduces the projectors and subtractions in the graphs in the correct form to subtract all divergences.

In what follows, we present some speculation about the possible use of the anti-canonical pattern to extract useful information. The fact that the expressions are only guaranteed to exist perturbatively is disregarded here. As discussed in Sect. 4.2 $\tilde{G}(p)$ also has an anti-canonical dependence on the regulator. In fact since W^{-1} diverges logarithmically, Eq. (C14) implies a Λ-dependence in $\tilde{G}(p)$ of the form

$$\tilde{G}(p;\Lambda) = (f(p;\Lambda) - h(p)L_\Lambda)^{-1} \tag{C17}$$

for some UV-finite functions $f(p;\Lambda)$ and $h(p)$. The anti-canonical divergence-pattern of $\tilde{G}(p;\Lambda)$ does not mean that $1/\tilde{G}(p;\Lambda)$ is also canonical in its dependence on p; a finite number of derivatives with respect to p does not render this quantity UV-finite.[28]

According to the expression in (C17), $\tilde{G}(p)$ has a finite limit as the cutoff is removed; however, it still needs renormalization since the limit is zero. The renormalized value is

$$\tilde{G}_R(p) = \lim_{\Lambda\to\infty} \frac{\tilde{G}(p;\Lambda)}{\tilde{G}(0;\Lambda)} = \frac{h(0)}{h(p)}. \tag{C18}$$

The limit is approached at a rate $O(1/L_\Lambda)$. Knowledge of the anti-canonical pattern can be exploited to accelerate the rate of convergence. For instance, a single subtraction produces

$$\frac{\tilde{G}^{-1}(p;\Lambda_1) - \tilde{G}^{-1}(p;\Lambda_2)}{L_{\Lambda_1} - L_{\Lambda_2}} = -h(p) + \frac{f(p;\Lambda_1) - f(p;\Lambda_2)}{L_{\Lambda_1} - L_{\Lambda_2}}. \tag{C19}$$

The second term would vanish if the cutoff dependence in f were negligible. Further acceleration should be achieved by applying more subtractions.

In the super-renormalizable case (Secs. 3 or 5.2) it was possible to arrange the momentum integrals to avoid the need for a regulator altogether, and using a finite number of Γ-graphs in each case. Here, such a rearrangement does not seem possible. In fact the bare $\tilde{G}(p)$ in (C7) comes from a series of unsubtracted graphs, and each new loop should produce a new factor $L_q = \log(q^2/\mu^2)/2$ in the integrand, that in turn produces a new factor L_Λ in $\tilde{G}(p;\Lambda)$. This is reflected in the expansion

$$\tilde{G}(p) = f_p^{-1} + f_p^{-2} h_p L_\Lambda + f_p^{-3} h_p^2 L_\Lambda^2 + \cdots \tag{C20}$$

based on the anti-canonical pattern of $\tilde{G}(p)$ in (C17).

[28] Since the RC centered at $p = 0$ are arbitrary, and W^{-1} can be expressed using superficially divergent integrals only, a suitable RC centered at p (instead of $p = 0$) should also express $1/\tilde{G}(p;\Lambda)$ in terms of superficially divergent integrals; however, as the very RC depend on p, it would not follow that the divergences are polynomial in p.

The information encoded in the anti-canonical pattern can be exploited as follows. Let $\bar{H}(q;p)$ be the angular-averaged value of $H(q;p)$ in (C3) over the directions of q. Then

$$\begin{aligned} \tilde{G}(p;\Lambda) &= 1 + \int^\Lambda dq\, Q(q;p) \\ Q(q;p) &:= \frac{1}{2}\Omega_6 q^5 D_q^2\, \bar{H}(q;p). \end{aligned} \tag{C21}$$

Neglecting the dependence of f_p on Λ, which should be a valid assumption for a large cutoff,

$$Q(\Lambda;p) = \frac{\partial \tilde{G}(p;\Lambda)}{\partial \Lambda} = \frac{1}{(f_p - h_p L_\Lambda)^2}\frac{h_p}{\Lambda}. \tag{C22}$$

This relation implies an asymptotic behavior for large q

$$\bar{H}(q;p) = \frac{Z^2}{\Omega_6}\frac{1}{h_p}\frac{1}{q^2 L_q^2}, \tag{C23}$$

which allows us to extract the renormalized value of $\tilde{G}(p)$ directly from $\bar{H}(q;p)$, namely,

$$\tilde{G}_R(p) = \lim_{q\to\infty}\frac{\bar{H}(q;p)}{\bar{H}(q;0)}. \tag{C24}$$

To arrive at this result, we have exploited not only the anti-canonical pattern of $\tilde{G}(p)$, but also its "explicit" property, namely, the dependence on Λ comes only from the boundary of the momentum integral and not from the integrand itself.

Appendix D: '$\Gamma + \delta S_0(\Lambda)$' vs '$\Gamma(\Lambda) + \delta S_0(\Lambda)$'

In this Appendix, we discuss the technical point raised near the end of Sect. 4.2: we argue that using an already renormalized effective action Γ (cutoff removed) to make the perturbation (the '$\Gamma + \delta S_0(\Lambda)$' approach) does not modify the final result $\Gamma + \delta\Gamma$ as compared to the standard approach of using the same regulator everywhere when doing the perturbation and then removing the regulator (the '$\Gamma(\Lambda) + \delta S_0(\Lambda)$' approach).

To show this we will consider the S_R-graph of the ϕ_6^3 theory shown in Fig. 11. Let us denote γ_1 the one-loop subgraph defined by the inner-loop, with internal momentum r, γ_2 the two-loop subgraph with internal momenta r and k, and γ_3 the full three-loop graph, with internal momenta r, k and q and external momentum p. Also, $\gamma_1(k^2;\epsilon)$, $\gamma_2(q^2;\epsilon)$ and $\gamma_3(p^2;\epsilon)$ denote their values. For simplicity, we work in the massless theory with the RC in (4.13) and apply DR.

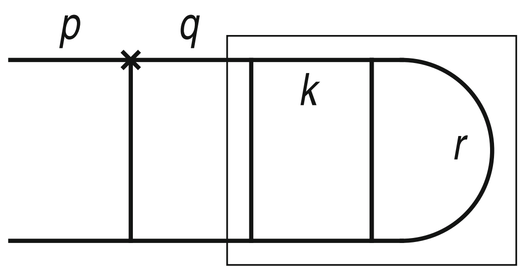
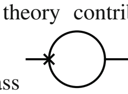


Fig. 11 An S_R-graph of the massless ϕ_6^3 theory contributing to $G_{R,2}^g(p)$. The graph belongs to the Γ-graph class . The two internal loops are part of $\Gamma[\phi]$ and they are already subtracted (indicated by the box). The outer loop is induced by the perturbation and requires a further subtraction

Let us first consider the standard calculation where a non-vanishing ϵ is held until the graph is computed and then set to zero. For the first subgraph (omitting trivial factors g_R, $-Z_R$ and δg_R)

$$\gamma_1(k^2;\epsilon) = \frac{1}{2}\nu^{2\epsilon}\int \frac{d^{\bar{d}}r}{(2\pi)^{\bar{d}}}\frac{1}{r^2}\frac{1}{(k-r)^2}, \tag{D1}$$

with $\bar{d} = 6-2\epsilon$. This and all other required integrals can be explicitly obtained from the basic expression [27]

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2)^\alpha}\frac{1}{((q-p)^2)^\beta} = (p^2)^{d/2-\alpha-\beta}C_2(\alpha,\beta;d) \tag{D2}$$

with

$$C_2(\alpha,\beta;d) := \frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha+\beta)}\frac{B(d/2-\alpha,d/2-\beta)}{B(\alpha,\beta)} \tag{D3}$$

and $B(a,b) \equiv \Gamma(a)\,\Gamma(b)/\Gamma(a+b)$. For instance,

$$\gamma_1(k^2;\epsilon) = \frac{1}{2}\nu^{2\epsilon}(k^2)^{1-\epsilon}C_2(1,1;6-2\epsilon). \tag{D4}$$

The graph γ_1 is quadratically divergent and requires subtractions. The canonical divergences in the ϕ_6^3 massless theory are proportional to k^2. It is clear from the integral in (D1), that three derivatives with respect to k^μ make the integral convergent; thus, the divergence in γ_1 is canonical and is removed by $1-T$,

$$\begin{aligned}\bar{\gamma}_1(k^2;\epsilon) &:= (1-T)\gamma_1(k^2;\epsilon)\\ &= \gamma_1(k^2;\epsilon) - \frac{k^2}{\mu_{\mathrm{RC}}^2}\gamma_1(\mu_{\mathrm{RC}}^2;\epsilon).\end{aligned} \tag{D5}$$

The subtracted integral $\bar{\gamma}_1(k^2;\epsilon)$ has only a regular power series in ϵ. A further feature is that the zeroth-order term,

$$\bar{\gamma}_1(k^2;0) = k^2\frac{\log(k^2/\mu_{\mathrm{RC}}^2)}{768\pi^3}, \tag{D6}$$

is independent of the scale ν. The renormalization condition erases the dependence on ν in the ϵ-independent term. The second subgraph (already using the subtracted γ_1) is

$$\gamma_2(q^2;\epsilon) := \nu^{2\epsilon}\int\frac{d^{\bar{d}}k}{(2\pi)^{\bar{d}}}\frac{1}{(k^2)^2}\frac{1}{(q-k)^2}\bar{\gamma}_1(k^2;\epsilon). \tag{D7}$$

Again, it has only a superficial divergence that is canonical and is canceled by the subtraction

$$\begin{aligned}\bar{\gamma}_2(q^2;\epsilon) &:= (1-T)\gamma_2(q^2;\epsilon)\\ &= \gamma_2(q^2;\epsilon) - \frac{q^2}{\mu_{\mathrm{RC}}^2}\gamma_2(\mu_{\mathrm{RC}}^2;\epsilon),\end{aligned} \tag{D8}$$

and

$$\bar{\gamma}_2(q^2;0) = q^2\frac{\log(q^2/\mu_{\mathrm{RC}}^2)(3\log(q^2/\mu_{\mathrm{RC}}^2)-11)}{1769472\pi^6}. \tag{D9}$$

Likewise

$$\gamma_3(p^2;\epsilon) := \nu^{2\epsilon}\int\frac{d^{\bar{d}}q}{(2\pi)^{\bar{d}}}\frac{1}{(q^2)^2}\frac{1}{(p-q)^2}\bar{\gamma}_2(q^2;\epsilon), \tag{D10}$$

and

$$\begin{aligned}\bar{\gamma}_3(p^2;\epsilon) &:= (1-T)\gamma_3(p^2;\epsilon)\\ &= \gamma_3(p^2;\epsilon) - \frac{q^2}{\mu_{\mathrm{RC}}^2}\gamma_3(\mu_{\mathrm{RC}}^2;\epsilon).\end{aligned} \tag{D11}$$

The final result of the standard calculation is then $\bar{\gamma}_3(p^2;0)$,

$$\begin{aligned}\bar{\gamma}_3(p^2;0) &= p^2\log(p^2/\mu_{\mathrm{RC}}^2)\\ &\quad\times\frac{3\log^2(p^2/\mu_{\mathrm{RC}}^2)-33\log(p^2/\mu_{\mathrm{RC}}^2)+103}{2038431744\pi^9}.\end{aligned} \tag{D12}$$

In the alternative computation, the two inner loops are contributions to $\Gamma[\phi]$, which is already renormalized, hence ϵ is set to zero in $\bar{\gamma}_2(q^2;\epsilon)$. The outer loop comes from the perturbation and there ϵ is non-vanishing when performing the integration over q,

$$\gamma_3'(p^2;\epsilon) := \nu^{2\epsilon}\int\frac{d^{\bar{d}}q}{(2\pi)^{\bar{d}}}\frac{1}{(q^2)^2}\frac{1}{(p-q)^2}\bar{\gamma}_2(q^2;0). \tag{D13}$$

Upon subtraction

$$\bar{\gamma}_3'(p^2;\epsilon) = \gamma_3'(p^2;\epsilon) - \frac{q^2}{\mu_{\mathrm{RC}}^2}\gamma_3'(\mu_{\mathrm{RC}}^2;\epsilon), \tag{D14}$$

and the final result of the alternative calculation is $\bar{\gamma}_3'(p^2;0)$. By direct calculation of the integrals it is easy to verify that the two calculations yield the same result,

$$\bar{\gamma}_3'(p^2;0) = \bar{\gamma}_3(p^2;0). \tag{D15}$$

As already said $\bar{\gamma}_2(q^2;\epsilon)$ has a regular power series in ϵ,

$$\bar{\gamma}_2(q^2;\epsilon) = \bar{\gamma}_2(q^2;0) + \epsilon q^2 R(q^2;\epsilon) \tag{D16}$$

and the function $R(q^2;\epsilon)$ is also regular. The argument why neglecting such remainder $\epsilon q^2 R$ has no effect in $\bar{\gamma}_3(p^2;0)$ is actually simple: when the extra term $q^2 R(q^2;\epsilon)$ is introduced in q-integral (D10) it will produce both finite and divergent contributions. The finite contributions are irrelevant due to the extra factor ϵ. On the other hand, the divergent contributions can compensate for the factor ϵ and yield a net result, however such divergent terms are necessarily canonical and so they are removed by the subtraction $1-T$ in (D11). That the divergences induced by $q^2 R(q^2;\epsilon)$ are necessarily canonical follows from the fact that the function R has only a soft dependence on q^2, namely, of the type $(q^2)^{n\epsilon}$ (for a few small values n). Upon expansion in ϵ, only terms of the type $\log^m(q^2)$ are produced. Such soft dependence cannot overturn the dominant factor $1/(q^2(p-q)^2)$: still taking three derivatives with respect to p^μ makes the integral convergent; hence, the divergence is proportional to p^2 and so is canonical.

In the graph considered above, no subdivergences were induced by adding the composite operator (the crossed vertex), only a superficial divergence. However, this is not relevant to the argument. One can consider instead the graph in Fig. 7. There, the subgraph (345) is already subtracted, but the composite operator induces both a subdivergence and an overall divergence. Again, the subtracted subgraph (345) has only a soft momentum dependence. The remainder after subtracting its value at $\epsilon = 0$ is $O(\epsilon)$ so, after performing the last momentum integration, the finite contributions vanish when ϵ is set to zero, while the divergent contributions are canonical due to the soft momentum dependence. They are then removed by the last subtraction $1-T$.

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