Black hole entropy from soft hair

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We start by looking at why we believe that black holes have entropy. According to Boltzmann, the entropy is a measure of the number of microstates of a system. We suggest here that the entropy arises from a holographic conformal field theory on the black hole horizon. Finally, we discuss some of the implications for the information paradox.

Keywords: Black holes. Thermodynamics, Holography. Information Paradox.

1. Introduction

Black holes were first thought about by John Michell back in 1784.[1] He reasoned that if the escape velocity from an object like a star exceeded the speed of light, then it would give rise to an object that cannot be seen optically but whose gravitational field would betray its existence. These ideas were given substance by Einstein's general theory of relativity and the subsequent discovery of solutions of the field equations that represented stationary black holes.[2–6] One puzzling feature of stationary black holes is that they are completely characterised by just a few parameters; their mass M, angular momentum $\mathbf{J}$ and electric charge Q.[7–11] It is this observation that is the basis for the information paradox.[12]

The thermodynamics of black holes in general relativity has a history starting in 1972. The first relevant discovery, the area theorem, was made by Hawking.[13] He found that the area of a black hole horizon could never decrease provided the null convergence condition $R_{ab}k^a k^b \geq 0$ holds for every null vector k^a. Shortly after this, Jacob Bekenstein suggested[14] that the entropy of a black hole must be proportional to the area of its event horizon. His reasoning was based on three observations. The first was that if a Kerr-Newman black hole increased its mass by an amount dM, then

$$dM = \frac{\kappa dA}{8\pi} + \Phi dQ + \mathbf{\Omega} \cdot d\mathbf{J} \tag{1}$$

where κ is the black hole surface gravity, A the area of the event horizon, Φ the electrostatic potential of the black hole and $\mathbf{\Omega}$ its angular velocity. κ, A, Φ and $\mathbf{\Omega}$ are all determined in terms of M, $\mathbf{J}$ and Q. The second piece of evidence was that he reasoned that a black hole must have some kind of internal structure that resulted from it method of formation. That would give rise to an entropy

$$S = -\sum_n p_n \ln p_n \tag{2}$$

arising from the probability of the occupation of the n^{th}-state being p_n. Finally, he showed that it was necessary for this black hole entropy to be added to the thermodynamic entropy of the rest of the universe in order to have a consistent theory of thermodynamics. This came about because otherwise dropping a box of radiation into a black hole would cause the entropy of the universe to decrease, in contradiction to the second law of thermodynamics.

His ideas were met with a certain amount of scepticism because black holes were thought to have vanishing temperature. Despite that, Bardeen, Carter and Hawking[15] pointed out the similarities between the first law of thermodynamics and (1) and also the second law of thermodynamics and the area theorem.

In 1974, Hawking[16,17] showed that black holes had a temperature T_H of $\hbar\kappa/(2\pi)$. Unlike previous work, his calculation was quantum mechanical in nature. Black holes would emit particles with a thermal spectrum at a temperature given by T_H. By identifying (1) with the first law of thermodynamics, one can immediately infer that the entropy must be given by $A/(4\hbar)$. The area theorem is thereby identified with the second law of thermodynamics. A somewhat different view of entropy was taken in Ref. 18. The idea here was to use the path integral for gravity to derive black hole entropy. Although gravity is unrenormalizable, there is no obstacle to using the path integral to lowest order as the uncontrollable divergences only occur at one loop or beyond. The action for pure gravity is, including the Gibbons-Hawking-York boundary terms[18,19]

$$I[g,h] = \frac{1}{16\pi}\int_{\mathcal{M}} R(g)\sqrt{\parallel g \parallel} d^4x + \frac{1}{8\pi}\int_{\partial\mathcal{M}} K\sqrt{\parallel h \parallel} d^3x + C[h] \tag{3}$$

where now $\mathcal{M}$ is the spacetime manifold with metric g and Ricci scalar $R(g)$. The boundary of $\mathcal{M}$ is $\partial\mathcal{M}$ with metric h and second fundamental form K. $C[h]$ is any functional of h and is designed to make the action of flat spacetime vanish. Suppose one wants to find the partition function for a black hole spacetime. Then one wants to compute $Z = \mathrm{tr}(e^{-\beta\mathcal{H}})$ where β is the inverse temperature and $\mathcal{H}$ the Hamiltonian. This can be done by realising that $e^{i\mathcal{H}t}$ is the time evolution operator and so if t is identified with $t + i\beta$ then Z is given by

$$Z = \int D[g] e^{-I[g,h]/\hbar}\,, \tag{4}$$

where now the integral is over all metrics g of positive definite signature and that approach flat space at infinity and are periodic in imaginary time t with period $\beta = T_H^{-1}$.

The Schwarzschild metric is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2\,, \tag{5}$$

where M is the mass of the static black hole with horizon at $r = 2M$ and $d\Omega^2$ is the metric on the unit 2-sphere. Taking $t = i\tau$ so that the geometry is as described above gives the Euclidean metric

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{6}$$

Now, $r = 2M$ is a conical singularity that is resolved provided that τ is identified with period $8\pi M$.[20] This periodicity is precisely the same periodicity expected from the Hawking calculation of the black hole temperature. Interpreting the exponential of the action as the partition function, reproduces the black hole entropy. In this calculation, the region $r < 2M$ has been removed from consideration. Implicitly this means that the internal degrees of freedom have been traced over. One is thereby led to believe that the black hole does have some kind of internal structure that cannot be probed by external observers who just look at the classical geometry. The same kind of reasoning can be applied to the Kerr-Newman metric too.

Black holes evaporate. The black hole uniqueness theorems suggest that the only properties that a stationary black hole has are just the mass, charge and spin. As a consequence, there is a tension with the ideas of quantum mechanics. If a black hole completely disappears, then the final state should be unitarily equivalent to the initial state. Obviously, there are enormous number of ways in which the black hole could form. The black hole, once it has settled down to a more or less equilibrium state, is described by just those three parameters. The Hawking radiation is thermal and characterised by the Hawking temperature. Such a final state consisting of Hawking radiation will not be unitarily related to the initial state that gave rise to the black hole. This is the information paradox. It might be that quantum mechanical information really is lost in gravitational collapse. But then, the whole edifice of quantum mechanics would need to be rethought. The incredible success of quantum mechanics would seem to discourage such a viewpoint. Alternatively, there might be something wrong with the uniqueness theorems. It is this latter possibility that we will investigate here.

In what follows, we will use covariant phase space methods[21–27] to understand the nature of charges in general relativity and the consequences for the physics of black holes. The reason for using the covariant phase space method is to preserve as much as possible of the covariance of theory. Had we picked the more conventional canonical methods, we would be forced to pick a particular time coordinate which would obscure matters. Furthermore, it would it impossible to understand what happens on null surfaces such as the event horizon. Our aim now is to try to understand something about the microscopic origin of black hole entropy. The hope is that this will aid a resolution of the information paradox.

In pure general relativity, one can start with the Einstein-Hilbert action I given by

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} R(g)\sqrt{\| g \|} d^4x. \tag{7}$$

This action omits the boundary terms, but these are not germane to the discussion that follows. One finds the Einstein equation by performing a variation of the action induced by a variation of the metric $g_{ab} \to g_{ab} + h_{ab}$. This results in the variation δI given by

$$\delta I = \int_{\mathcal{M}} \; (R_{ab} - \frac{1}{2} R\, g_{ab})\; h^{ab} \sqrt{\| \, g \, \|} d^4x + \int_{\partial\mathcal{M}} \theta \tag{8}$$

The three-form on the boundary $\theta(g,h)$ is known as the presymplectic potential and has components

$$(*\theta)_a = \frac{1}{16\pi} \left(\nabla_b h^b_a - \nabla_a h\right) \tag{9}$$

where $h = h_{ab} g^{ab}$. In canonical general relativity, the boundary term would be thought of

$$\int_{\partial\mathcal{M}} \; \sum_i \; p_i \delta q^i \tag{10}$$

where q^i are the generalised coordinates, p_i are the generalised momenta and i represent the tensor indices of these fields.

The presymplectic density $\omega(g;h,h')$ is defined by a second variation $g_{ab} \to g_{ab} + h'_{ab}$

$$\omega(g;h,h') = \delta\theta(g,h') - \delta'\theta(g,h) \tag{11}$$

Finally, the symplectic form for general relativity is

$$\Omega_\Sigma = \int_\Sigma \omega \tag{12}$$

where Σ is any partial Cauchy surface in the spacetime. In the language of the canonical theory, Ω_Σ would be

$$\int_\Sigma \; \sum_i \; \delta p_i \wedge \delta q^i. \tag{13}$$

One property that ω has is that if the background metric g_{ab} obeys the Einstein equation and both h_{ab} and h'_{ab} obey the linearised Einstein equations, then ω is closed. Thus Ω_Σ is constant under variations of Σ as long as the boundaries of Σ are fixed.

In general relativity, the symmetry group is the group of diffeomorphisms. An infinitesimal coordinate transformation is specified by a vector field ζ^a. This induces a variation in any tensor field given by the Lie derivative of that field. Thus, for example, the variation of the metric is given by

$$\delta g_{ab} = \mathcal{L}_\zeta g_{ab} = g_{ab} + \nabla_a \zeta_b + \nabla_b \zeta_a. \tag{14}$$

The bulk term in the variation of the action δI is invariant under such a transformation but the boundary term is not. The infinitesimal co-ordinate transformations obey an algebra whose composition law is

$$\mathcal{L}_\zeta \mathcal{L}_\eta - \mathcal{L}_\eta \mathcal{L}_\zeta = \mathcal{L}_{[\zeta,\eta]} \tag{15}$$

where ζ and η are two (smooth) vector fields and

$$[\zeta, \eta] = \mathcal{L}_\zeta \eta = -\mathcal{L}_\eta \zeta \tag{16}$$

Suppose now that in Ω one makes h'_{ab} a gauge transformation given by the vector field ζ. Then Ω can be written as a boundary integral. Explicitly,

$$Q_\zeta = \frac{1}{16\pi} \int_{\partial\Sigma} F_{ab}\, dS^{ab} \tag{17}$$

with

$$F_{ab} = -2\zeta_{[a}\nabla_{b]}h + 2\zeta_{[a}\nabla^c h_{b]c} - 2\zeta^c \nabla_{[a}h_{b]c} - h\nabla_{[a}\zeta_{b]} + 2h_{c[a}\nabla^c \zeta_{b]}. \tag{18}$$

Let $\partial\Sigma$ is a closed 2-surface S, for example the celestial sphere or a black hole event horizon. One would like to interpret Q_ζ as the variation in the Noether charge conjugate to ζ that is enclosed in the interior of S as one moves between the metric g_{ab} and $g_{ab} + h_{ab}$. There is a complication with this idea because in such a change, there might be a flux of charge crossing S. To take account of this possibility, one needs to examine Q_ζ and identify such terms and subtract them out. In more mathematical language, one we want Q_ζ to be a function of state. As such it must be a 1-form on the infinite-dimensional phase space of the theory. This 1-form needs to be exact so that if one goes along a path Γ between g_{ab} and $g_{ab} + h_{ab}$, then Q_ζ is independent of the path Γ, and therefore dependent only the end-points of that path. The definition on Q_ζ thus needs to be modified by the ddition of a suitable counterterm $Q_\zeta \to Q_\zeta + Q^{ct}_\zeta$. Finding Q^{ct}_ζ needs to be done on a case by case basis as has been elegantly explained in detail by Wald and Zoupas.[26]

In the case that ζ were a time translation, then Q_ζ would be the quasi-local mass enclosed in S.[28] If it were a spatial translation then the momentum. If ζ were a Killing vector, then Q_ζ would be the same as the Komar integral.[29] If ζ were a supertranslation or super-rotation at null infinity, then Q_ζ would be the corresponding supertranslation or super-rotation charge. Equally, one can define charges on the black hole horizon and these are the soft charges or soft black hole hair.[30]

Diffeomorphism invariance of general relativity means that the charges Q_ζ lie in some representation of group of coordinate transformations. Thus

$$\delta_\zeta Q_\eta - \delta_\eta Q_\zeta = Q_{[\zeta,\eta]} \tag{19}$$

Were this relation not to hold, general coordinate invariance would be violated, in gross contradiction to our expectations of what should be true in physics. However, what we find is that this relationship does not hold for charges on black hole event horizons. Instead, we find

$$\delta_\zeta Q_\eta - \delta_\eta Q_\zeta = Q_{[\zeta,\eta]} + K(\zeta, \eta) \tag{20}$$

where $K(\zeta, \eta)$ is a central extension of this algebra.[31] We will now explore a particular example and move on to its interpretation.

We start from the Kerr metric in Boyer-Lindquist coordinates.

$$ds^2 = \rho^2(\frac{dr^2}{\Delta} + d\theta^2) + (r^2 + a^2)\sin^2\theta d\phi^2 - dt^2 + \frac{2Mr}{\rho^2}(a\sin^2\theta d\phi - dt)^2 \tag{21}$$

where

$$\Delta = r^2 - 2Mr + a^2 \tag{22}$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{23}$$

M is the mass of the black hole and $J = Ma$ is its angular momentum. $\Delta = 0$ at $r_\pm$ with r_+ being the location of the outer horizon, r_- the location of the inner horizon and

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{24}$$

Now we will define "conformal" coordinates[32] and assume that the black hole is not extreme so that $m^2 > a^2$.

$$w^+ = \sqrt{\frac{r - r_+}{r - r_-}}\, e^{2\pi T_R \phi} \tag{25}$$

$$w^- = \sqrt{\frac{r - r_+}{r - r_-}}\, e^{2\pi T_L \phi - t/2M} \tag{26}$$

$$y = \sqrt{\frac{r_+ - r_-}{r - r_-}}\, e^{\pi(T_L + T_R)\phi - t/4M} \tag{27}$$

where

$$T_L = \frac{r_+ + r_-}{4\pi a} \quad \text{and} \quad T_R = \frac{r_+ - r_-}{4\pi a} \tag{28}$$

The future outer horizon is $w^- = 0$ and the past outer horizon is $w^+ = 0$. The azimuthal coordinate ϕ is identified with period 2π and this induces an identification on w^+, w^- and y as

$$w^+ \to e^{4\pi^2 T_R} w^+, \quad w^- \to e^{4\pi^2 I_L} w^-, \quad y \to e^{2\pi^2(T_L + T_R)} y. \tag{29}$$

The line element close to the horizon bifurcation surface $w^+ = w^- = 0$ is[33]

$$ds^2 = \frac{4\rho_+^2}{y^2} dw^+ dw^- + \frac{16M^2a^2\sin^2\theta}{\rho_+^2 y^2} dy^2 + \rho_+^2 d\theta^2 + O(w^+, w^-) \tag{30}$$

where $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. If one looks at the $w^+, w^-, y-$ plane by setting θ to be a constant, then this line element is that of AdS_3/Γ with Γ being some discrete group. So close to the horizon bifurcation surface, the geometry of spacetime is some kind of warped product of the line segment $\theta \in [0, \pi]$ and a deformed portion of three-dimensional anti-de Sitter space. The classic work of Brown and Henneaux[31] shows that in spacetimes that are asymptotic to anti-de Sitter spacetime, the diffeomorphism algebra has anomalies. One might therefore suspect that something similar happens in the case of the Kerr black hole.

Consider the diffeomorphism given by the vector field ζ_n

$$\zeta_n = \epsilon_n(w^+)\partial_+ + \frac{1}{2}\epsilon'(w^+)y\partial_y \tag{31}$$

with

$$\epsilon_n(w^+) = 2\pi T_R (w^+)^{(1+\frac{in}{2\pi T_R})} \tag{32}$$

and n being any integer. It should be noted that under the identifications of either $w^+ \to w^+ e^{4\pi^2 T_R}$ or $y \to y e^{2\pi^2(T_L+T_R)}$ that ζ_n is invariant. This vector field is well-defined on the future horizon. These vector fields obey the Witt (or centerless Virasoro) algebra with the commutator

$$[\zeta_n, \zeta_m] = i(m-n)\zeta_{n+m} \tag{33}$$

Similarly, one can find a second vector field $\tilde{\zeta}_n$ given by

$$\tilde{\zeta}_n = \tilde{\epsilon}_n(w^-)\partial_- + \tfrac{1}{2}\tilde{\epsilon}'(w^-)y\partial_y \tag{34}$$

and $\tilde{\epsilon}$ being given

$$\tilde{\epsilon}_n(w^-) = 2\pi T_L (w^-)^{(1+\frac{in}{2\pi T_L})} \tag{35}$$

again with n being any integer. This vector field is well-defined on the past horizon. Again, it is invariant under the identifications $w^- \to w^- e^{4\pi^2 T_L}$ or $y \to y e^{2\pi^2(T_L+T_R)}$. It too obeys the Virasoro algebra

$$[\tilde{\zeta}_n, \tilde{\zeta}_m] = i(m-n)\tilde{\zeta}_{n+m}. \tag{36}$$

Both of these vector fields are well-defined on the bifurcation surface and commute with each other there

$$[\zeta_n, \tilde{\zeta}_m] = 0. \tag{37}$$

These vector fields can be used to generate charges on the bifurcation surface. To do this we need to introduce an appropriate counterterm. This is given by

$$-\frac{1}{8\pi}\int dS^{ab}\ \nabla_a(\zeta^c h^d_b)N_{cd} \tag{38}$$

where N_{ab} are the components of the volume form on the normal bundle to the horizon. There is a precisely similar expression for the fields $\tilde{\zeta}_n$. One then finds that the charges on the bifurcation surface obey the algebra

$$[Q_n, Q_m] = i(n-m)Q_{n+m} + in^3 J\delta_{n,-m} \tag{39}$$

for the right-handed algebra and

$$[\tilde{Q}_n, \tilde{Q}_m] = i(n-m)\tilde{Q}_{n+m} + in^3 J\delta_{n,-m} \tag{40}$$

for the left-handed algebra. Finally, the left and right algebras commute with each other

$$[Q_n, \tilde{Q}_m] = 0. \tag{41}$$

In both cases, the central terms shown here correspond to the conventionally normalised Virasoro algebra with central charges given by $c_L = c_R = 12J$. Thus the diffeomorphism algebra has an anomaly.

We postulate that this anomaly is cancelled by holographic degrees of freedom on the horizon expressed in terms of a two-dimensional conformal field theory. Consider for a moment the expressions for the absorption probabilities for particles incident on a Kerr black hole. Suppose we look at a particle with energy δE and angular momentum parallel to the black hole spin δJ. Then we observe that the absorption probability obtains a suggestive factor of

$$|\Gamma(1 + \frac{i\omega_L}{2\pi T_L})|^2 \; |\Gamma(1 + \frac{i\omega_R}{2\pi T_R})|^2 \tag{42}$$

where

$$\omega_L = \frac{2M^3}{J}\delta E \quad \omega_R = \frac{2M^3}{J}\delta E - \delta J. \tag{43}$$

This is precisely what is to be expected for a conformal field theory where the left-handed degrees of freedom are at a temperature of T_L and the right-handed degrees of freedom are at a temperature of T_R and one is asking for the absorption probability for particles of energy ω_L in the left-handed sector and energy ω_R in the right-handed sector. We take it that there are no coincidences in nature and therefore we really can attribute our observations to the existence of holographic degrees of freedom on the horizon described by a two-dimensional conformal field theory.

A general property of conformal field theories, provided the central charge is sufficiently large, was first described by Cardy.[34] The entropy for a system with central charges c_L and c_R for the two sectors at temperatures T_L and T_R is given by

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R). \tag{44}$$

Plugging in our expressions for c_L, c_R, T_l and T_R gives

$$S = \frac{1}{4}A. \tag{45}$$

It is hard to believe that this is a coincidence. It appears therefore we have identified the degrees of freedom responsible for black hole entropy.[33]

Subsequent to the conference, it has been shown that the same methods reproduce the entropy for the Kerr-Newman family of black holes[35] and for uncharged black holes in anti-de Sitter spacetime.[36]

A key question is to ask how this affects our view of the information paradox. We have shown how to account for black hole entropy in terms of a holographic two-dimensional conformal field theory living on the black hole horizon. It is however far from clear that the states of such a theory can record all of the quantum mechanical information that is pertinent to black hole formation from ordinary matter. We are therefore left with a collection of problems that need exploration and solution before there can be any claim of solving the information paradox. We conclude this essay with a summary of outstanding issues. Does the horizon conformal field theory contain a complete description of the black hole formation process? How does the Hawking radiation encode this information so as to preserve unitary time evolution? Why is it that the black hole entropy is independent of the spectrum of elementary particles when the number of ways a black hole can be formed is highly dependent on that spectrum. For example, if there were a million different species of electron, the number of ways a black hole could form would be vastly higher than if there a single type of electron. Nevertheless, the Hawking entropy would be same.

Suppose a particle falls into a black hole. Classically, a co-moviong observer sees it pass through the horizon without anything obvious happening. In the case of a Schwarzschild black hole, it will reach the singularity in a finite amount of proper time. The singularity is a boundary of spacetime and so we believe the particle to have disappeared. In the case of rotating black holes, it seems plausible that it will also inevitably reach a singularity as the inner horizon of a Kerr black hole is unstable and is presumed to become singular once any energy-momentum arrives there. However, if the particle is to leave an imprint on the state of the horizon conformal field theory, it appears to have violated the quantum no-cloning theorem. Roughly speaking, the no-cloning theorem says that you cannot duplicate the state of a particle by unitary time evolution. A number of technical assumptions go into this amongst which is a notion of locality, a dubious assumption in the case of gravitation.

Then there are some more challenging issues. What happens to the singularity? It is a classical concept and shows that classical general relativity is an incomplete theory. What happens quantum mechanically? There is no satisfactory answer at present. What are the final stages of black hole evaporation? The picture presented seems to suggest that all symmetries in nature are gauge symmetries and not global symmetries. For example in the standard model, baryon number is a global symmetry, but it is hard to see how this could be encoded in the picture presented here. There is one ambitious theory that predicts that all symmetries are gauge symmetries and that is string theory. Although string theory is successful in resolving the divergence problems of quantum gravity, and potentially geometrizing the spectrum of elementary particles, it is far from being a theory of spacetime. Hopefully, the picture here will provide a guide to the true nature of quantum gravity, but there are immense and exciting challenges to the construction of such a theory. Eventually, we hope that the construction of such a theory will lead to deep insights into the nature of our Universe.

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