SCALAR-TENSOR THEORY AND SCALAR CHARGE

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For a general class of scalar-tensor theories of gravitation the properties of exact static spherically-symmetric vacuum and electrovac solution are discussed. On the other hand, post-Newtonian perfect fluid metric for this class of theories is found and compared with the exact vacuum solution. This comparison shows that the post-Newtonian approximation employs a special value of the scalar charge-mass ratio. This limitation is removed if a term involving scalar charge density is introduced explicitly into the initial Lagrangian. Observational consequences are discussed.

1. *Introduction*

Modern gravitational experiments concern mainly weak fields which faintly differ from Newtonian ones. Thus it is of interest to analyse various theories of gravitation and the ways of their comparison with experiment on post-Newtonian (PN) level.

In the papers by Thorne and Will [1, 2] the most general form of PN metic (parametrized PN, or PPN approximation) is proposed, containing nine arbitrary parameters varying from theory to theory. In the PPN formalism a theory of gravitation is supposed to be a metric one, *i. e.* the matter equations of motion are of the form

$$\nabla_\alpha T^\alpha_\mu = 0, \tag{1}$$

where ∇_α denotes a covariant derivative and T^ν_μ is the matter energy-momentum tensor including all non-gravitational fields.

On the other hand, there exists a method of finding exact static spherically-symmetric solutions for a broad class of scalar-tensor theories. In this paper a comparison of an exact vacuum solution for this class of theories with the PN metric is carried out. It will be shown that the PN approximation requires a special value for the source scalar charge-mass ratio, whereas in the exact solution the scalar charge C is an independent arbitrary constant. It is clear that specific scalar charge (in respect to mass) different from the standard one, may be discovered in nature. To describe this possibility in a consistent way one

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should introduce the scalar charge explicitly into the theory. We discuss some variants and consequences of this innovation.

Possible behaviour of scalar-tensor vacuum and electrovac spherically-symmetric solutions is discussed. To describe their properties in a compact way, we propose a classification of arbitrary spherically-symmetric metrics.

2. Field equations

Consider a class of scalar-tensor theories with the Lagrangian density

$$L = A(\varphi)R + B(\varphi)g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - 2\Lambda(\varphi) + 2\kappa_0 L_m(\Psi(\varphi)g^{\alpha\beta}, \ldots), \tag{2}$$

where φ is the scalar field which we assume to be real (see Appendix 1); L_m is the matter Lagrangian; $\kappa_0 = 8\pi c^{-4}G_0$ is an initial gravitational constant, generally speaking, a non-Einsteinian one; A, B, Λ and Ψ are arbitrary functions; $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature and $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ the Ricci tensor. The Riemann tensor $R^{\mu}_{\nu\varrho\sigma}$ is defined by the formula

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A_\sigma = R^{\alpha}_{\sigma\mu\nu}A_\alpha$$

with an arbitrary vector A_μ. The metric tensor $g_{\mu\nu}$ has the signature $(+\ -\ -\ -)$. Greek indices range from 0 to 3, Latin ones from 1 to 3; $\varphi_{,\alpha} \equiv \partial\varphi/\partial x^\alpha$.

Suppose that at spatial infinity the metric is flat and the φ field tends to some constant value φ_0. It is useful to divide Lagrangian (2) by the (nonzero) constant $A_0 = \underset{0}{A(\varphi)}$. Denote

$$\bar{A}(\varphi) = AA_0^{-1}, \quad \bar{B}(\varphi) = BA_0^{-1}, \quad \bar{\kappa} = \kappa_0 A_0^{-1} = 8\pi\bar{G}c^{-4}. \tag{3}$$

Obviously $\underset{0}{\bar{A}(\varphi)} = 1$.

The transformation [3, 4]

$$g_{\mu\nu} = F(\psi)\tilde{g}_{\mu\nu}, \quad F(\psi) = [\bar{A}(\varphi)]^{-1}; \tag{4}$$

$$\frac{d\varphi}{d\psi} = \bar{A}|\bar{A}\bar{B} + \tfrac{3}{2}\bar{A}_\varphi^2|^{-\frac{1}{2}}, \quad \bar{A}_\varphi \equiv \frac{d\bar{A}}{d\varphi}; \quad \operatorname{sign}(AB + \tfrac{3}{2}A_\varphi^2) = n \tag{5}$$

reduces equation (2) to the form

$$\tilde{L} = \tilde{R} + n\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - 2F^2(\psi)\Lambda + 2\bar{\kappa}F^2(\psi)L_m(F^{-1}\Psi\tilde{g}^{\mu\nu}, \ldots) \tag{6}$$

where a tilde marks quantities obtained with the help of $\tilde{g}_{\mu\nu}$. Noting that formula (5) admits addition of an arbitrary constant to ψ, we put

$$\underset{0}{\psi(\varphi)} = \underset{0}{\psi} = 0. \tag{7}$$

Then according to (4)

$$F(0) = 1. \tag{8}$$

As the tensor $T_{\mu\nu}$ is yielded by the variation

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int L_m \sqrt{-g}\, d^4x, \qquad d^4x \equiv dx^0 dx^1 dx^2 dx^3, \tag{9}$$

in theory (2) the equations of motion have the form (1) only if $\Psi \equiv$ const. A conformal transformation of the type (4) allows us, however, to achieve this for any theory (2) [5]. Besides, here we are dealing with distances small on cosmological scale and so it is reasonable to put $\Lambda \equiv 0$. Thus we consider theory (2) under assumptions

$$\Lambda(\varphi) \equiv 0; \;\; \Psi(\varphi) \equiv 1. \tag{10}$$

As in (2) a transformation of φ field is possible

$$\varphi \to \tilde{\varphi}, \qquad \varphi = \varphi(\tilde{\varphi}), \tag{11}$$

where $\varphi(\tilde{\varphi})$ is an arbitrary function, for a given concrete choice of gravitation theory the form of the coefficients A and B is not unique. Under assumptions (10) such a choice should be set by means of one function, *e. g.* $\omega(\varphi)$ in Nordtvedt notation [6]

$$A(\varphi) = \varphi, \; B(\varphi) = \omega(\varphi)/\varphi. \tag{12}$$

It is often more convenient to set a theory by means of the function $F(\psi)$ which is invariant under transformation (11).

Taking into account (10) we get the field equations from transformed the Lagrangian (6):

$$\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \tfrac{1}{2}\, \tilde{g}_{\mu\nu}\tilde{R} = -\bar{\kappa} F(\psi) T_{\mu\nu} - n\tilde{S}_{\mu\nu}(\psi); \tag{13}$$

$$2\tilde{\Box}\psi + nF \frac{dF}{d\psi}\bar{\kappa} T = 0, \tag{14}$$

where $\tilde{\Box} = \tilde{g}_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta$ and the tensor

$$\tilde{S}_{\mu\nu}(\psi) = \psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}\, \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}. \tag{15}$$

The matter tensor $T_{\mu\nu}$ is here untransformed and $T = g^{\alpha\beta}T_{\alpha\beta}$.

3. The post-Newtonian approximation

The PN metric is known to be a solution of the field equations in the form of an expansion in powers of reciprocal velocity of light c. In this expansion the components g_{00}, g_{0i} and g_{ik} should be found with an accuracy of $O(c^{-4})$, $O(c^{-3})$ and $O(c^{-2})$ respectively.

Let us find such a solution of equations (13), (14). Following Chandrasekhar [7] and Will [2], assume that the tensor $T_{\mu\nu}$ describes a perfect fluid:

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - g_{\mu\nu}p \tag{16}$$

where p is the pressure, u_μ is the unity-normalized fourvelocity and ε is the energy density from which the rest mass density ϱ is distinguished:

$$\varepsilon = \varrho(c^2 + \Pi). \tag{17}$$

Now the quantities in equations (13) and (14) are presented as series

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}; \quad h_{\mu\nu} = \underset{1}{h_{\mu\nu}}+\underset{2}{h_{\mu\nu}}+\dots,$$

$$\psi = \underset{1}{\psi}+\underset{2}{\psi}+\dots;$$

$$F(\psi) = 1+F'_0\psi+\tfrac{1}{2}F''_0\psi^2+\dots; \quad F'_0 = \frac{dF}{d\psi}(0); \quad F''_0 = \frac{d^2F}{d\psi^2}(0); \quad \dots \tag{18}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is the Minkowski metric and the indices below denote orders in c^{-1}.

The metric to be found $g_{\mu\nu}$ will be determined with the required accuracy if one finds $\tilde{g}_{\mu\nu}$ with the same accuracy and also ψ to within $O(c^{-4})$.

As $\bar{\kappa} = O(c^{-4})$ and

$$T_{00} = \underset{-2}{T_{00}}+\underset{0}{T_{00}}+\dots;$$

$$T_{0i} = \underset{-1}{T_{0i}}+\dots;$$

$$T_{ik} = \underset{0}{T_{ik}}+\dots, \tag{19}$$

in equation (14) $\psi = \underset{2}{\psi}+\underset{4}{\psi}+\dots$ Hence the tensor $\tilde{S}_{\mu\nu}(\psi)$ begins at the fourth order. Together with (19) this gives right to put

$$h_{00} = \underset{2}{h_{00}}+\underset{4}{h_{00}}+\dots;$$

$$h_{0i} = \underset{3}{h_{0i}}+\dots;$$

$$h_{ik} = -\delta_{ik}(1-\underset{2}{h_{00}})+\dots. \tag{20}$$

The first two orders of equation (14) are of the form

$$\partial_{ii}\underset{2}{\psi} = 4\pi\bar{G}nF'_0\varrho c^{-2}; \tag{21}$$

$$\partial_{ii}\underset{4}{\psi}-\partial_{00}\underset{2}{\psi}+\underset{2}{h_{00}}\partial_{ii}\underset{2}{\psi} = 4\pi\bar{G}n\varrho[c^{-4}F'_0(\Pi-3p/\varrho)+c^{-2}(F'^2_0+F''_0)\underset{2}{\psi}], \tag{22}$$

where $\partial_\alpha = \partial/\partial x^\alpha$. (Note that $\partial_0 = \partial/c\partial t$ adds a unity to the order.) Equations (13) are solved similarly to paper [7]. Equation (21) and the ($_{00}$) component of (13) give in the senior order Newton's law, *i. e.*

$$g_{00} = F(\psi)\tilde{g}_{00} = 1-2c^{-2}U = 1-2c^{-2}GI,$$

$$I(\vec{x}, t) = \int d^3x' \frac{\varrho(\vec{x}', t)}{|\vec{x}-\vec{x}'|}, \quad d^3x \equiv dx^1dx^2dx^3, \tag{23}$$

where U is the Newtonian gravitational potential. The Newtonian constant of gravitations is

$$G = \overline{G}(1+\eta), \qquad \eta = \tfrac{1}{2} nF_0'^2. \tag{24}$$

Solving then equation (22) and the ($_{0i}$) and ($_{00}$) components of (13) in the third and fourth or derrespectively and using the gauge condition determining the choice of coordinates in the third order

$$\partial_i h_{0i} = \tfrac{1}{2}(3-\eta)\partial_0 h_{00}, \tag{25}$$

we get finally

$$g_{00} = 1-2c^{-2}U+c^{-4}(2\beta_0 U^2-4G\Phi),$$
$$g_{0i} = Gc^{-3}(\tfrac{7}{2}\Delta_1 V_i+\tfrac{1}{2}\Delta_2 W_i),$$
$$g_{ik} = -\delta_{ik}(1+2c^{-2}\gamma_0 U), \tag{26}$$

where

$$V_i(\vec{x}, t) = \int d^3x' \frac{\varrho(\vec{x}', t)v^i(\vec{x}', t)}{|\vec{x}-\vec{x}'|};$$

$$W_i(\vec{x}, t) = \int d^3x' \frac{\varrho(\vec{x}', t)v^k(\vec{x}', t)(x^i-x^{i'})(x^k-x^{k'})}{|\vec{x}-\vec{x}'|^3};$$

$$\Phi(\vec{x}, t) = \int d^3x' \frac{\varrho(\vec{x}', t)\chi(\vec{x}', t)}{|\vec{x}-\vec{x}'|};$$

$$\chi = \beta_1 v^2+\beta_2 U+\tfrac{1}{2}\beta_3\Pi+\tfrac{3}{2}\beta_4 p/\varrho;$$

$v^2 = v^i v^i$; v^i is the fluid three-velocity. The constants

$$\beta_0 = \frac{1+2\eta+\lambda}{(1+\eta)^2}; \quad \beta_2 = \frac{1-\eta-\eta^2-\lambda}{(1+\eta)^2}; \quad \beta_3 = \Delta_2 = 1;$$

$$\beta_1 = \frac{1}{1+\eta}; \quad \Delta_1 = \frac{7-\eta}{7(1+\eta)}; \quad \beta_4 = \gamma_0 = \frac{1-\eta}{1+\eta}, \tag{27}$$

with

$$\eta = \tfrac{1}{2} nF_0'^2; \quad \lambda = \tfrac{1}{2} n\eta F_0'' \tag{28}$$

are the concrete values of Will's PPN parameters [2] for the theories with Lagrangian (2) (with changes in notation: $\gamma \to \gamma_0$, $\beta \to \beta_0$). Two parameters from [2], Σ and ζ here equal zero due to gauge (25). Transformation formulae to another gauge are given in [2].

Note that for any values of η and λ parameters (27) satisfy all the constraints [8] characterizing an asymptotically Lorentz-invariant theory containing PN conservation laws for energy, momentum and angular momentum. Thus the theory (2) possesses all these properties.

The expressions for η and λ in terms of the initial functions $A(\varphi)$ and $B(\varphi)$ and their derivatives values at $\varphi = \underset{0}{\varphi}$ are:

$$\eta = A_\varphi^2(2AB+3A_\varphi^2)^{-1};$$

$$\lambda = \eta^3 A_\varphi^{-4}[3A_\varphi^2(AB+A_\varphi^2)+A^2(A_\varphi B_\varphi - 2BA_{\varphi\varphi})]. \tag{29}$$

These formulae are insensitive to multiplying (2) by a constant and hence are valid as well without the assumption $A(\underset{0}{\varphi}) = 1$.

In Nordtvedt's notation (12) formula (29) look especially simple:

$$\eta = (2\omega+3)^{-1};$$

$$\lambda = (2\omega+3)^{-3}(2\omega+3+\omega_\varphi \underset{0}{\varphi}). \tag{30}$$

Expressions (27) generalize the result of paper [6] where the PN metric for point masses is obtained.

4. An exact spherically-symmetric solution

In the Lagrangian (2) we take a concrete L_m:

$$L_m = -\tfrac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \tag{31}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. Again we make use of transformation (4), (5). Due to conformal inwariance of (31) the transformed equations (13) and (14) take the form

$$\tilde{G}_\mu^\nu = -\kappa[-\tilde{F}^{\nu\alpha}F_{\mu\alpha} + \tfrac{1}{4}\delta_\mu^\nu \tilde{F}^{\alpha\beta}F_{\alpha\beta}] - n\tilde{S}_\mu^\nu(\psi), \tag{32}$$

$$\tilde{\Box}\psi = 0, \tag{33}$$

where $\tilde{F}^{\mu\nu} = \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}F_{\alpha\beta}$. To these one should add the Maxwell equations

$$\tilde{\nabla}_\alpha \tilde{F}^{\alpha\mu} = 0. \tag{34}$$

The set of equations (32)–(34) is solved in Ref. [9] and more completely in [10, 11] under the assumption that the field is spherically-symmetric and static,

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = e^{\tilde{\gamma}(z)}(dx^0)^2 - e^{\tilde{\alpha}(z)}dz^2 - e^{\tilde{\beta}(z)}d\Omega^2; \tag{35}$$

$$F_{01} = F_{01}(z) = -F_{10}; \text{ the rest } F_{\mu\nu} = 0;$$

$$\psi = \psi(z),$$

where $d\Omega^2 = (dx^2)^2 + \sin^2 x^2 (dx^3)^2$. Let us use the simplest method [11]. So we choose the coordinate z so that

$$\tilde{\alpha}(z) = 2\tilde{\beta}(z) + \tilde{\gamma}(z). \tag{36}$$

Then the solution of equations (33) and (34) has the form

$$\psi(z) = Cz;$$
$$\tilde{F}^{10} = qe^{-\tilde{\alpha}(z)}, \tag{37}$$

where we have put $z = 0$ at spatial infinity and taken into account the condition (7). The integration constant q has the meaning of electric charge and C may be similarly called "scalar charge".

Under the assumptions (35) among the equations (32) there are two independent ones. Using (37) they may be written in the form

$$e^{\tilde{\alpha}}(\tilde{G}_1^1+\tilde{G}_2^2) = \tfrac{1}{2}(\tilde{\beta}''+\tilde{\gamma}'')-e^{\tilde{\beta}+\tilde{\gamma}} = 0;$$
$$e^{\tilde{\alpha}}\tilde{G}_1^1 = -e^{\tilde{\beta}+\tilde{\gamma}}+\tfrac{1}{4}[(\tilde{\beta}'+\tilde{\gamma}')^2-\tilde{\gamma}'^2] = \tfrac{1}{2}nC^2-Q^2e^{\tilde{\gamma}}, \tag{38}$$

where a prime denotes d/dz and $Q^2 = \kappa q^2/2$.

Integrating (38), we finally get for the metric $g_{\mu\nu}$:

$$ds^2 = e^{\gamma(z)}(dx^0)^2-e^{\alpha(z)}dz^2-e^{\beta(z)}d\Omega^2 = \tag{39}$$

$$= F(\psi)\left\{Q^{-2}\frac{(dx^0)^2}{s^2(h, z+z_1)}-Q^2\frac{s^2(h, z+z_1)}{s^2(k, z)}\left[\frac{dz^2}{s^2(k, z)}+d\Omega^2\right]\right\}, \tag{40}$$

where the function

$$S(a, x) = \begin{cases} a^{-1}\sinh ax & \text{for } a > 0, \\ x & \text{for } a = 0, \\ a^{-1}\sin ax & \text{for } a < 0; \end{cases} \tag{41}$$

and the integration constants h and k are connected by the correlation

$$k^2 \operatorname{sign} k = h^2 \operatorname{sign} h+\tfrac{1}{2}n\, C^2. \tag{42}$$

For an arbitrary metric (39) the boundary condition that the space-time is flat at infinity is formulated as

$$e^\gamma \to 1; \quad e^\beta \to \infty; \quad \tfrac{1}{4}\beta'^2e^{\beta-\alpha} \to 1 \tag{43}$$

when z tends to some z_∞. For the metric (40) $z_\infty = 0$ and the condition (43) is fulfilled completely if

$$s^2(h, z_1) = Q^{-2} \tag{44}$$

(remember that $F(0) = 1$). With no loss of generality we may regard the coordinate z definition domain to be

$$0 < z < z_{\max} \tag{45}$$

where $z_{\max}$ depends on which of the functions $F(\psi)$, $s(h, z+z_1)$ or $s(k, z)$ will be the first to go to zero or infinity or, symbolically,

$$z_{\max} = \min\{\underset{a}{\infty};\ \underset{b}{\infty(F)};\ \underset{c}{\text{zero}(F)};\ \underset{d}{\text{zero}[s(k, z)]};\ \underset{e}{\text{zero}[s(h, z+z_1)]}\}. \tag{46}$$

The letters a, b, c, d, e mark the corresponding variants of the solution. Evidently for special values of the integration constants the quantities in the curly brackets in (46) may coincide. For these variants we use double or triple notations, *e. g.* *cd* or *cde*.

In the case $q = 0$ (vacuum) the equations (38) under the condition (43) yield

$$ds^2 = F(\psi)\left\{e^{-2hz}(dx^0)^2 - \frac{e^{2hz}}{s^2(k,z)}\left[\frac{dz^2}{s^2(k,z)} + d\Omega^2\right]\right\}. \tag{47}$$

The definition domain for z is again given by (45) and (46) but obviously in (46) the "e" possibility is absent.

Solution (47) contains three arbitrary constants h, C and $\underset{0}{\varphi}$. The latter is involved in $F(\psi)$ in an indirect form.

In order to have a convenient description of possible properties of the metrics (40) and (47), we introduce a classification or arbitrary spherically-symmetric metrics basing on the behaviour of the functions $e^{\beta(z)}$ and $e^{\gamma(z)}$; e^{β} has an invariant meaning of the area of the sphere $z = \text{const}$ (divided by 4π) and e^{γ} is an invariant (within the given physically preferable reference frame) time slowing-down factor in respect to distant points where the space-time may be treated as flat. The convergence or divergence of the integral

$$\int e^{(\alpha-\gamma)/2} dz \tag{48}$$

when the radial coordinate z tends to the end $\bar{z}$ of its definition domain, is also of interest. If the integral (48) converges, then (in the evident meaning) the points at $z = \bar{z}$ are observable.

We shall denote the behaviour of the metric (39) by means of two figures, the first one corresponding to the behaviour of e^{γ} when $z \to \bar{z}$ and the second one to that of e^{β}. Namely we write 1, 2 or 3 if the function tends to zero, infinity or a finite value, respectively. Besides, we denote the convergence or divergence of integral (48) by a plus or minus sign. *E. g.* the Schwarzschild metric belongs to class 13_- because when the spherical radius r tends to the gravitational radius r_g, $e^{\gamma(z)} \to 0$, $e^{\beta(z)} = r^2 \to \text{const.}$ and the sphere $r = r_g$ is invisible for an observer at rest.

Let us give a brief characteristic of the obtained classes.

11, 21, 31. The three-space includes the center. Moreover, in class 31 there are singularity-free metrics. This is so if the local euclidity conditions are fulfilled: for $z \to \bar{z}$

$$e^{\gamma} \to \text{const.}, \; \gamma' \to 0, \; (z-\bar{z})^2 e^{\alpha-\beta} \to 1. \tag{49}$$

12, 22, 32. The three-space has a "neck" that means that $e^{\beta(z)}$ has a minimum. In case 32 $z = \bar{z}$ corresponds to another spatial infinity (not always flat) rather than to a singularity. A two-dimensional analog of such geometry is a hyperboloid of one sheet.

13, 23. To study the geometry completely it is necessary to convert to another reference frame perhaps allowing to continue the metric further than $z = \bar{z}$.

33. The coordinate z should be changed to a more licky one ($z \to z'(z)$) allowing to penetrate further than $z = \bar{z}$ by analytic continuation. One will naturally find one of the rest eight classes.

The proposed classification allows us to describe the solutions (40) and (47) in a compact way, see Table I. The table shows that classes 13 and 23 containing Schwarzschild-type singularities may appear only for special values of the constants, whatever the function $F(\psi)$ is.

Of certain interest are the solutions of class 32_- which occur for general values of the constants only for $n = -1$. It is easily assured that at $z \to z_{max}$ the 32_- metrics (40) and (47) become flat but the time has in general another rate than at $z = 0$. One may try to apply such metrics to describe topologies of the kind of Wheeler handles [12] but, unlike the familiar Schwarzschild and Reissner-Nordström metrics, these ones are singularity-free. It may be shown that under natural assumptions on the orders of the quantities involved in (40) and (47), the "necks" dimensions, *i. e.* $e^{\beta_{min}/2}$, are of the order of gravitational radii $2GM\,c^{-2}$ for corresponding masses.

Note that for the solution (40) the complete electromagnetic field energy in the outer region of an arbitrary sphere $z = z^*$ may be found:

$$\int T_0^0 \sqrt{-g}\, d^3x = \frac{4\pi}{\kappa} \int_0^{z^*} \frac{dz}{s^2(h, z+z_1)} = \frac{4\pi}{\kappa} \left[\frac{s'}{s}(h, z_1) - \frac{s'}{s}(h, z^*+z_1) \right]. \tag{50}$$

The integral (50) taken over the whole space ($z^* = z_{max}$) is finite for all variants of z_{max} given in (46) but those containing e.

TABLE I

Possible behaviour of the metrics (40) and (47)

Variants by (46)	Behaviour	Solution (40)				Solution (47)	
		$n=1$, $h \geq 0$	$n=1$, $h<0$	$n=-1$, $h \geq 0$	$n=-1$, $h<0$	$n=1$ or $n=-1$, $\lvert h\rvert \geq \lvert C\rvert/\sqrt{2}$	$n=-1$, $\lvert h\rvert < \lvert C\rvert/\sqrt{2}$
a	(*)	+	—	+	—	+	—
b	22_+	+	+	+	+	+	+
c	11_+	+	+	+	+	+	+
d	32_-	—	—	—	+	—	+
e	21_+	+	+	+	+	—	—
bd	22_-	—	—	—	+	—	+
cd	$(**)_-$	—	—	—	+	—	+
be	$(^*_*)_+$	+	+	+	+	—	—
ce	$(***)_+$	+	+	+	+	—	—
de	23_+	—	—	—	+	—	—
bde	22_+	—	—	—	+	—	—
cde	$(***)_+$	—	—	—	+	—	—

Comments: In the six right-hand columns the "+" sign means that there exist $F(\psi)$ for which the metric behaves correspondingly. Otherwise the "—" sign is written.

(*) is any of the nine classes depending on the integration constants values and the form of $F(\psi)$. Classes 13, 23, 31, 32 may occur only for special choices of the constants and class 33, moreover, for special $F(\psi)$.

(**) means classes 11, 12 or 13
($^*_*$) classes 21, 22 or 23
(***) classes 11, 21 or 31

} depending on the function $F(\psi)$.

5. The exact solution and the post-Newtonian metric

Static spherically-symmetric PN metric out of the gravitational field sources, as well as vacuum metric (47) in the asymptotic region, may be presented as a series in inverse powers of the so-called isotropic radius r:

$$ds^2 = \left(1+\frac{\xi_1}{r}+\frac{\xi_2}{r^2}+\dots\right)(dx^0)^2-\left(1-\frac{\eta_1}{r}+\dots\right)(dr^2+r^2d\Omega^2). \tag{51}$$

Moreover, in the PN expansion the factors $\xi_1, \dots, \eta_1, \dots$ are in their turn power series in c^{-1}:

$$\xi_1 = -\frac{2GM_0}{c^2}+O(c^{-4}); \qquad \xi_2 = 2\beta_0\frac{G^2M_0^2}{c^4}+O(c^{-6});$$

$$\eta_1 = 2\gamma_0\frac{GM_0}{c^2}+O(c^{-4}) \tag{52}$$

where

$$M_0 = 4\pi\int\varrho(r')r'^2dr'.$$

For exact solution (47) ξ_1, ξ_2 and η_1 are combinations of the integration constants:

$$\xi_1 = \lim_{z\to z_\infty}[-\varepsilon\gamma' e^{(2\beta-\alpha)/2}] = -2h+CF_0';$$

$$\xi_2 = \tfrac{1}{2}\lim_{z\to z_\infty}\{e^{(3\beta-\alpha)/2}[\varepsilon\gamma'+e^{(\beta-\alpha)/2}(\gamma''+\gamma'^2+\tfrac{1}{2}\beta'\gamma'-\tfrac{1}{2}\alpha'\gamma')]\} =$$

$$= \tfrac{1}{2}[C^2(F_0''-F_0'^2)+(2h-CF_0')^2];$$

$$\eta_1 = \lim_{z\to z_\infty}\{e^{\beta/2}[2-|\beta'|e^{(\beta-\alpha)/2}]\} = 2h+CF_0', \tag{53}$$

where $\varepsilon = \operatorname{sign}\beta'$ at $z\to z_\infty$. After the first equality sign the expressions for ξ_1, ξ_2 and η_1 are given for arbitrary metric (39) satisfying the boundary condition (43).

Comparing (52) and (53), we obtain the constants h and C as expansions in c^{-1}:

$$h = \frac{GM_0}{c^2(1+\eta)}+O(c^{-4});$$

$$C = -nF_0'\frac{GM_0}{c^2(1+\eta)}+O(c^{-4}). \tag{54}$$

In the PN approximation only the senior terms of these series are taken into consideration. (The third integration constant of the solution (47), $\underset{0}{\varphi}$ is the same as in the PN approximation.)

Thus in the solution (47) the constant h is related to the active gravitational mass which in accordance with (54) is mainly determined by the rest mass M_0 if the PN expansion is applicable. As for the independent constant C, it is forcedly connected with the mass in the PN metric.

6. Discussion

The results of Section 5 may be interpreted as imputing a special value to the ratio of the scalar charge density and the rest mass density in theory (2). If this ratio differs from its standard value dictated by (54), it should effect observable phenomena. Thus we conclude that it is desirable to introduce the scalar charge explicitly into the theory, writing in the Lagrangian (2) an additional term of the form, say,

$$S(\varphi)j(x), \tag{55}$$

where $S(\varphi)$ is a certain function and $j(x)$ is independent of φ and plays the role of charge density which naturally should be related to some matter parameters. Such a term describes direct scalar field — matter interaction and its appearance does not look unexpected.

However, if the term (55) is present, theory (2) is no more a metric one. On the right side of equation of motion (1) there emerges and expression proportional to $Sj_{,\mu}$. Nevertheless, if the quantity j is proportional to the energy — momentum tensor trace of to the matter density, then the senior order of c^{-1} expansion similar to Section 3, gives Newton's law for the interaction of point particles. The PN metric and the PN equations of motion are got in a rather cumbersome manner in this case and we will not give them here. The whole situation is rather well illustrated by the following simple formal modification of the theory. In equation (14) for the transformed scalar field ψ in the second term (which plays the role of scalar charge density) write an indefinite factor σ, leaving equations (13) for the metric field unchanged. Then, carrying out the expansion in c^{-1}, it is easily assured that for σ = const., in the senior order Newton's law with the constant

$$G = \overline{G}(1 + \tfrac{1}{2}\sigma n F_0'^2) \tag{56}$$

is valid and the PN metric again takes form (26) with PPN parameters (27) where instead of η and λ there stand

$$\eta^+ = \sigma\eta; \quad \lambda^+ = \sigma^2\lambda. \tag{57}$$

This metric accords with exact solution (47) with arbitrary C and h. (Note that inclusion of terms of the type (55) does not alter the form of the exterior vacuum solution.) Introduction of σ increases the uncertainty in interpreting the results of PN effects measurements. In fact, if such measurements are in agreement with (27), then they establish within this "σ formalism" only the values of η^+ and λ^+ with unknown σ. (For expressions for concrete effects using the PPN parameters see *e.g.* in paper [8].) More details may be obtained only by studying effects out of the PN frames, *e. g.* time variation of the gravitational constant due to the Universe expansion.

In the notation (12) the local constant of gravitation and its variation due to variation of φ are given in the σ formalism by the formulae generalizing those given by Nordtvedt (Ref. [6], Appendix):

$$G = \frac{G_0}{\varphi}\,\frac{2\omega+3+\sigma}{2\omega+3}; \quad \frac{G_{,\alpha}}{G} = \left[1 + \frac{2\sigma\omega_\varphi\,\varphi}{(2\omega+3)(2\omega+3+\sigma)}\right]\frac{\varphi_{,\alpha}}{\varphi}. \tag{58}$$

However one should be careful in comparing this kind of formulae with observations using cosmological solutions for φ because perhaps scalar fields local from the cosmological viewpoint (*e. g.* that of Galaxy) may mask possible changes of the cosmological background.

Note that, besides scalar-tensor theories, variable gravitational coupling was considered by Staniukovich [13].

Up to now we have been using the assumption that in the senior order of the c^{-1} expansion, Newton's law should be fulfilled. However it seems physically more plausible to introduce scalar charge as an elementary particle characteristic, *e. g.* as a function of the particle rest mass. Then one obtains different values of the constant G for different substances. There is experimental evidence that this effect really exists, *e. g.* [14] and this is perhaps the strongest argument in favour of explicit inclusion of the scalar charge into the theory. Further consequences of this hypothesis are under study.

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APPENDIX 1

Complex scalar field

There is no evident reason to consider the scalar field φ in scalar-tensor theories to be real *a priori*. The complexity of φ may play some role in cosmological (see [15]) or other problems where it yields an additional degree of freedom. However we will convince ourselves that changing of $A(\varphi)$, $B(\varphi)$ and $\varphi_{,\alpha}\varphi_{,\beta}$ in the Lagrangian (2) for $A(|\varphi|)$, $B(|\varphi|)$ and $\overset{*}{\varphi}_{,\alpha}\varphi_{,\beta}$ respectively, does not alter the metrics obtained in Sections 3 and 4.

Denote

$$x = |\varphi|; \quad y = \arg \varphi. \tag{A1}$$

The transformation (4), (5) (where instead of φ and ψ we write x and $\tilde{x}$) brings the initial equations to the form

$$\tilde{G}_{\mu\nu} = -\bar{\kappa}F(\tilde{x})T_{\mu\nu} - n\tilde{S}_{\mu\nu}(\tilde{x}) - \bar{B}Fx^2\tilde{S}_{\mu\nu}(y), \tag{A2}$$

$$2n\,\tilde{\Box}\,\tilde{x} + F\frac{dF}{d\tilde{x}}\bar{\kappa}\,T - \frac{d}{d\tilde{x}}(\bar{B}Fx^2)\tilde{g}^{\alpha\beta}y_{,\alpha}y_{,\beta} = 0, \tag{A3}$$

$$\bar{B}Fx^2\,\tilde{\Box}\,y + \frac{d}{d\tilde{x}}(\bar{B}Fx^2)\tilde{g}^{\alpha\beta}\tilde{x}_{,\alpha}y_{,\beta} = 0 \tag{A4}$$

with the notations (3) and (15) under the assumptions (10). From a PPN expansion for the equation (A4) similar to Section 3 it follows:

$$y = \underset{0}{y} = \text{const}. \tag{A5}$$

and hence $\tilde{S}_{\mu\nu}(y) \equiv 0$, and thus y does not enter the further calculations. Thus the formulae (27) for the PPN parameters remain unchanged but in (29) the index φ should be changed for x.

A static spherically-symmetric solution with the electric field may be found using the transformation

$$g_{\mu\nu} = \bar{A}^{-1}\tilde{g}_{\mu\nu}; \qquad \frac{dx}{d\tilde{x}} = \frac{x}{\tilde{x}}\left(\frac{\bar{A}\bar{B}}{\bar{A}\bar{B}+\frac{3}{2}\bar{A}_x^2}\right)^{\frac{1}{2}} \tag{A6}$$

which leads to the field equations

$$\tilde{G}_\mu^\nu = -\bar{\kappa}[-\tilde{F}^{\nu\alpha}F_{\mu\alpha}+\tfrac{1}{4}\,\delta_\mu^\nu\tilde{F}^{\alpha\beta}F_{\alpha\beta}]-nD(\overset{*}{\psi}\psi)\,[\tilde{g}^{\nu\alpha}\overset{*}{\psi}_{,\mu}\psi_{,\alpha}-\tfrac{1}{2}\,\delta_\mu^\nu\tilde{g}^{\alpha\beta}\overset{*}{\psi}_{,\alpha}\psi_{,\beta}], \tag{A7}$$

$$\tilde{\Box}\,\psi+\frac{d}{d\psi}(\ln D)\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} = 0, \tag{A8}$$

with the Maxwell equations (34) added, where

$$\psi = \tilde{x}e^{iy}; \qquad nD(\overset{*}{\psi}\psi) = \bar{B}x^2/(\bar{A}\tilde{x}^2); \qquad n = \text{sign}\,B. \tag{A9}$$

Under the assumptions (35), (36) the equation (A8) is reduced to

$$\psi''+\frac{d}{d\psi}(\ln D)\psi'^2 = 0. \tag{A10}$$

Divinding this equation by ψ' and adding its complex-conjugate, we obtain an exact differential equation which gives:

$$D(\overset{*}{\psi}\psi)\overset{*}{\psi}'\psi' = C^2 = \text{const.}, \tag{A11}$$

and the further calculation repeats exactly that for the real field.

APPENDIX 2

Some particular cases

We point out some particular cases of gravitation theories described by Lagrangian (2) under assumptions (10).

1. General relativity:

$$A(\varphi) \equiv 1;\, B(\varphi) \equiv 0;\, F(\varphi) \equiv 1;\, \eta = \lambda = 0. \tag{A12}$$

2. The Brans-Dicke theory [16]:

$$A(\varphi) = \varphi;\, B(\varphi) = \omega/\varphi;\, \omega = \text{const.} \neq -3/2;$$

$$F(\psi) = \exp\,[-\psi/\sqrt{|\omega+3/2|}];\, \eta = (2\omega+3)^{-1};\, \lambda = (2\omega+3)^{-2}. \tag{A13}$$

The PPN parameters calculated from here using (27) coincide with those given in paper [2].

3. The Zaitsev-Kolesnikov theory with a conformally covariant scalar field [3, 10, 17]:

$$A(\varphi) = 1+\tfrac{1}{6}\,\varphi^2; \quad B(\varphi) = -1;$$

$$F(\psi) = \frac{\cos^2[(\psi+\psi_0)/\sqrt{6}]}{\cos^2(\psi_0/\sqrt{6})}; \qquad \psi_0/\sqrt{6} = \arctan(\varphi/\sqrt{6});$$

$$\eta = -\tfrac{1}{18}\,\underset{0}{\varphi}^2; \quad \lambda = -\tfrac{1}{108}\,\underset{0}{\varphi}^2(1-\tfrac{1}{6}\,\underset{0}{\varphi}^2) = \tfrac{1}{6}\,\eta(1+3\eta). \tag{A14}$$

4. This scheme includes also the case of material conformally covariant scalar field in general relativity which has been studied in paper [11]:

$$A(\varphi) = 1-\tfrac{1}{3}\,\kappa_0\varphi^2; \quad B(\varphi) = 2\kappa_0,$$

with Einstein value of the constant κ_0;

$$F(\psi) = \frac{\cosh^2[(\psi+\psi_0)/\sqrt{6}]}{\cosh^2(\psi_0/\sqrt{6})}; \qquad \psi_0/\sqrt{6} = \tanh^{-1}(\sqrt{\kappa_0/3}\;\underset{0}{\varphi}). \tag{A15}$$

For this $F(\psi)$ solutions (40) and (47) completely coincide with the corresponding solutions in [11] if one changes the notation in the following way:

$$\kappa \to \kappa\cosh^2 z_0; \quad \frac{|C|z}{\sqrt{6}} \to z; \quad \frac{\psi_0}{\sqrt{6}} \to z_0; \quad \frac{\sqrt{6}\,h}{|C|} \to h; \quad \frac{\sqrt{6}\,k}{|C|} \to k;$$

$$C^2 \to 2\kappa|C|^2\cosh^4 z_0. \tag{A16}$$

The properties of metrics (40) and (47) for the concrete cases 2, 3 and 4 are given in Table 2. In case 4 at $h = |C|/\sqrt{6}$ these metrics belong to class 33_+ and their further study takes place in the transformed coordinates:

$$y = \coth(|C|z/\sqrt{6}) \quad \text{for } \psi_0 \neq 0; \tag{A17}$$

$$r = (y+y_1)\sqrt{C^2/6} \quad \text{for } \psi_0 = 0,$$

in which the metric takes the form

$$ds^2 = (y+y_0)^2\left\{\frac{(dx^0)^2}{(y+y_1)^2} - \frac{1}{6}\,C^2\,\frac{(y+y_1)}{y^4}\,(dy^2+y^2d\Omega^2)\right\} \quad \text{for } \psi_0 \neq 0,$$

$$ds^2 = (1-r_0/r)^2(dx^0)^2-(1-r_0/r)^{-2}dr^2-r^2d\Omega^2 \quad \text{for } \psi_0 = 0. \tag{A18}$$

where

$$y_0 = \tanh(\psi_0/\sqrt{6}); \quad r_0 = y_1|C|/\sqrt{6}; \qquad y_1 = \begin{cases}\coth(|C|z_1/\sqrt{6}) & \text{for solution (40),}\\ 1 & \text{for solution (47).}\end{cases} \tag{A19}$$

TABLE II

Properties of the solutions (40) and (47) in some particular cases

Example 2. Brans-Dicke theory

Constants	$\omega > -3/2$		$\omega < -3/2$					
	$h < 0$ or (and) $z_1 < 0$	$h \geqslant 0$, $z_1 > 0$	$h < \lvert C \rvert/\sqrt{2}$ or (and) $z_1 < 0$			$h \geqslant \lvert C \rvert/\sqrt{2}$, $z_1 > 0$		
						$h < h_0$	$h = h_0$	$h > h_0$
Variants by (46)	e	a	d	e	de	a	a	a
Behaviour	21_{+}	11_{+}	32_{-}	21_{+}	23_{+}	12_{-}	13_{-}	11_{-}

Example 3. Zaitsev-Kolesnikov theory

Variations by (46)	c	d	e	cd	ce	de	cde
Behaviour	11_{+}	32_{-}	21_{+}	13_{-}	31_{+}	23_{+}	31_{+}

Example 4. Conformally covariant scalar field in general relativity

Constants	$h > \lvert C \rvert/\sqrt{6}$, $z_1 > 0$	$h < 0$ or (and) $z_1 < 0$	$0 \leqslant h < \lvert C \rvert/\sqrt{6}$ $z_1 > 0$	$h = \lvert C \rvert/\sqrt{6}$, $z_1 > 0$		
				$\psi_0 > 0$	$\psi_0 = 0$	$\psi_0 < 0$
Variations by (46)	a	e	a	a	a	a
Behaviour	12_{+}	21_{+}	21_{+}	32_{-}	13_{-}	11_{+}

$$h_0 = \tfrac{1}{4}\lvert C\rvert \left(\lvert\omega+3/2\rvert^{-1/2}+2\lvert\omega+3/2\rvert^{1/2}\right)$$

Comment: Variants of the behaviour of solution (40) are given in the table. To get variants for (47) one should reject those involving e.

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