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Generalized double affine Hecke algebra for double torus

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Abstract

We propose a generalization of the double affine Hecke algebra of type-$C^\vee C_1$ at specific parameters by introducing a "Heegaard dual" of the Hecke operators. Shown is a relationship with the skein algebra on double torus. We give automorphisms of the algebra associated with the Dehn twists on the double torus.

Keywords Double affine Hecke algebra · Skein algebra · Askey–Wilson polynomial

Mathematics Subject Classification 57M27 · 57M25 · 20C08 · 33D52 · 57M60 · 81R12

1 Introduction

The double affine Hecke algebra (DAHA) was introduced by Cherednik for studies on the Knizhnik–Zamolodchikov equation and its applications to orthogonal symmetric polynomials via the Dunkl–Cherednik operators [8]. It is a fundamental modern tool in mathematics and physics. One of applications of DAHA is the skein algebra, which receives renewed interests from a viewpoint of the quantization of the character varieties. Although, the DAHA is so far only applicable for the skein algebra on the once-punctured torus $\Sigma_{1,1}$ and the 4-punctured sphere $\Sigma_{0,4}$. The former is the DAHA of type-A_1, and the latter is of type-$C^\vee C_1$ [5, 16, 22, 24]. The superpolynomials for links on $\Sigma_{1,1}$ and $\Sigma_{0,4}$ were constructed by use of the automorphism of DAHA [9, 10].

A generalization of DAHA for the skein algebra on higher-genus surface was initiated in [2, 3]. Constructed are q-difference operators for generators of the skein algebra on $\Sigma_{2,0}$ as a $\mathbb{Z}_3$ generalization of the DAHA of type-A_1. Though the Iwahori–Hecke algebraic structure seems to be missing, shown was that it is isomorphic to the skein algebra on $\Sigma_{2,0}$ [12].

To the memory of my father.

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Another representation of the skein algebra on $\Sigma_{2,0}$ was given in our previous paper [16]. Combining the DAHAs of type-A_1 and $C^\vee C_1$, we constructed the q-difference operators for generators of the skein algebra [2, 3]. Using the automorphisms of A_1 DAHA, we computed the DAHA polynomial for double twist knots, and observed a relationship with the colored Jones polynomials. The present paper relies on [16], but we rather aim to generalize the $C^\vee C_1$ DAHA at specific parameters by introducing "Heegaard dual" operators of $C^\vee C_1$ DAHA. Thus our generalization is different from [13] where the generalized DAHA was associated with a 2-dimensional crystallographic group.

This paper is organized as follows. In Sect. 2, we recall the skein algebra on surface and DAHA. We review an isomorphism between the skein algebra on the 4-punctured sphere and DAHA of type-$C^\vee C_1$. In Sect. 3, we pay attention to $C^\vee C_1$ DAHA at $\mathbf{t}_\star$. By introducing Heegaard dual operators, we propose a generalized DAHA. Using the automorphisms we study a relationship with the skein algebra on $\Sigma_{2,0}$.

Throughout this article we use

$$\mathrm{ch}(x) = x + x^{-1}, \qquad \mathrm{sh}(x) = x - x^{-1}. \tag{1.1}$$

2 Preliminaries on DAHA of $C^\vee C_1$-type and skein algebra

2.1 DAHA of $C^\vee C_1$-type

We recall the double affine Hecke algebra $H_{q,\mathbf{t}}$ of type-$C^\vee C_1$ with 4 parameters $\mathbf{t} = (t_0, t_1, t_2, t_3)$ [23] (see also [8, 21]). The DAHA $H_{q,\mathbf{t}}$ is generated by $\mathsf{T}_0^{\pm 1}$, $\mathsf{T}_1^{\pm 1}$, $(\mathsf{T}_0^\vee)^{\pm 1}$, and $(\mathsf{T}_1^\vee)^{\pm 1}$ satisfying

$$\begin{aligned} &\mathsf{T}_0 + t_0 - t_0^{-1} = \mathsf{T}_0^{-1}, && \mathsf{T}_1 + t_1 - t_1^{-1} = \mathsf{T}_1^{-1}, \\ &\mathsf{T}_0^\vee + t_2 - t_2^{-1} = \left(\mathsf{T}_0^\vee\right)^{-1}, && \mathsf{T}_1^\vee + t_3 - t_3^{-1} = \left(\mathsf{T}_1^\vee\right)^{-1}, \end{aligned} \tag{2.1}$$

and

$$\mathsf{T}_1^\vee \mathsf{T}_1 \mathsf{T}_0 \mathsf{T}_0^\vee = q^{-\frac{1}{2}}. \tag{2.2}$$

Here and hereafter we use

$$\mathsf{X} = \left(\mathsf{T}_1^\vee \mathsf{T}_1\right)^{-1} = q^{\frac{1}{2}} \mathsf{T}_0 \mathsf{T}_0^\vee, \tag{2.3}$$

$$\mathsf{Y} = \mathsf{T}_1 \mathsf{T}_0. \tag{2.4}$$

The spherical DAHA is defined by

$$SH_{q,\mathbf{t}} = \mathsf{e}\, H_{q,\mathbf{t}}\, \mathsf{e}, \tag{2.5}$$

where the idempotent e is

$$\mathsf{e} = \frac{1}{t_1 + t_1^{-1}} \left(t_1 + \mathsf{T}_1\right). \tag{2.6}$$

The polynomial representation is given as [23]

$$
\begin{aligned}
&\mathsf{T}_0 \mapsto t_0{}^{-1}\,\mathsf{s}\,\eth - \frac{q^{-1}\left(t_0{}^{-1}-t_0\right)x^2 + q^{-\frac{1}{2}}\left(t_2{}^{-1}-t_2\right)x}{1-q^{-1}x^2}\,(1-\mathsf{s}\,\eth)\,,\\
&\mathsf{T}_1 \mapsto t_1{}^{-1}\mathsf{s} + \frac{\left(t_1{}^{-1}-t_1\right)+\left(t_3{}^{-1}-t_3\right)x}{x^2-1}\,(\mathsf{s}-1)\,,\\
&\mathsf{T}_0^\vee \mapsto q^{-\frac{1}{2}}\mathsf{T}_0{}^{-1}x,\\
&\mathsf{T}_1^\vee \mapsto x^{-1}\mathsf{T}_1{}^{-1}.
\end{aligned}
\tag{2.7}
$$

Here we mean

$$
(\mathsf{s}\,f)\,(x) = f(x^{-1}), \qquad (\eth\,f)\,(x) = f(q\,x). \tag{2.8}
$$

Then the idempotent (2.6) is a map to the symmetric Laurent polynomials, e : $\mathbb{C}[x^{\pm 1}] \to \mathbb{C}[x+x^{-1}]$. The Askey–Wilson operator is given from (2.4) as

$$
\mathrm{ch}(\mathsf{Y})|_{\mathrm{sym}} \mapsto W(x;\mathbf{t})\,(\eth-1) + W(x^{-1};\mathbf{t})\left(\eth^{-1}-1\right) + t_0 t_1 + (t_0\,t_1)^{-1}\,, \tag{2.9}
$$

where sym denotes an action on the symmetric Laurent polynomials, and

$$
W(x;\mathbf{t}) = t_0\,t_1\,\frac{\left(1-\frac{1}{t_1 t_3}x\right)\left(1+\frac{t_3}{t_1}x\right)\left(1-q^{\frac{1}{2}}\frac{1}{t_0 t_2}x\right)\left(1+q^{\frac{1}{2}}\frac{t_2}{t_0}x\right)}{\left(1-x^2\right)\left(1-q\,x^2\right)}. \tag{2.10}
$$

The eigen-polynomial of (2.9) is the Askey–Wilson polynomial $P_m(x;q,\mathbf{t})$,

$$
P_m(x;q,\mathbf{t}) = \frac{(ab,ac,ad;q)_m}{a^m(abcdq^{m-1};q)_m}\,{}_4\phi_3\left[\begin{matrix} q^{-m},\,q^{m-1}abcd,\,ax,\,ax^{-1} \\ ab,\,ac,\,ad \end{matrix};q,q\right], \tag{2.11}
$$

satisfying

$$
\left(\mathsf{Y}+\mathsf{Y}^{-1}\right)P_m(x;q,\mathbf{t}) = \mathrm{ch}\left(t_0 t_1 q^{-m}\right)\,P_m(x;q,\mathbf{t}), \tag{2.12}
$$

where

$$
a = \frac{1}{t_1 t_3}, \qquad b = -\frac{t_3}{t_1}, \qquad c = \frac{q^{\frac{1}{2}}}{t_0 t_2}, \qquad d = -\frac{q^{\frac{1}{2}} t_2}{t_0}.
$$

See [4, 15] for properties of the Askey–Wilson polynomials.

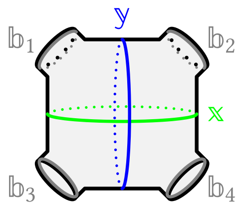
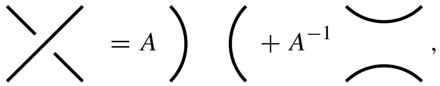


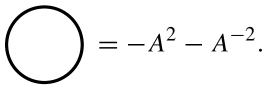
Fig. 1 Simple closed curves on the 4-punctured sphere $\Sigma_{0,4}$

2.2 Skein algebra on $\Sigma_{0,4}$

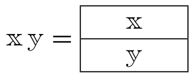
The skein algebra $\mathrm{Sk}_A(\Sigma)$ on surface Σ is generated by isotopy classes of framed links in $\Sigma \times [0, 1]$ satisfying the skein relation



(2.13)



A multiplication $\mathbb{x}\,\mathbb{y}$ of links $\mathbb{x}$ and $\mathbb{y}$ means that $\mathbb{x}$ is vertically above $\mathbb{y}$,



When two simple closed curves $\mathbb{x}$ and $\mathbb{y}$ on Σ intersect exactly once, we have

$$\frac{1}{A^{\pm 2} - A^{\mp 2}} \left(A^{\pm 1} \mathbb{x}\,\mathbb{y} - A^{\mp 1} \mathbb{y}\,\mathbb{x} \right) = \mathscr{D}_{\mathbb{x}}^{\mp 1}(\mathbb{y}), \tag{2.14}$$

where $\mathscr{D}_{\mathbb{x}}$ denotes the left Dehn twist along $\mathbb{x}$. It is noted that

$$\mathscr{D}_{\mathbb{y}}(\mathbb{x}) = \mathscr{D}_{\mathbb{x}}^{-1}(\mathbb{y}). \tag{2.15}$$

A finite set of Dehn twists along non-separating simple closed curves generates the mapping class group $\mathrm{Mod}(\Sigma_{g,0})$ of a surface $\Sigma_{g,0}$. See, e.g., [6, 14].

In the case of the 4-punctured sphere $\Sigma_{0,4}$, the skein algebra is generated by $\mathbb{x}$, $\mathbb{y}$, and $\mathbb{b}_i$ in Fig. 1. It is known that the type-$C^\vee C_1$ DAHA is isomorphic to the skein algebra on the 4-punctured sphere [5, 16, 24]. See also [19, 20, 25, 28] from a point of view of the algebraic structure of the Askey–Wilson polynomials. The Askey–Wilson operator $\mathrm{ch}\,\mathsf{Y}$ and $\mathrm{ch}\,\mathsf{X}$ respectively corresponds to the curves $\mathbb{y}$ and $\mathbb{x}$, while the 4 parameters (t_0, t_1, t_2, t_3) denote the boundary curves $(\mathbb{b}_1, \mathbb{b}_3, \mathbb{b}_2, \mathbb{b}_4)$. See [16] for

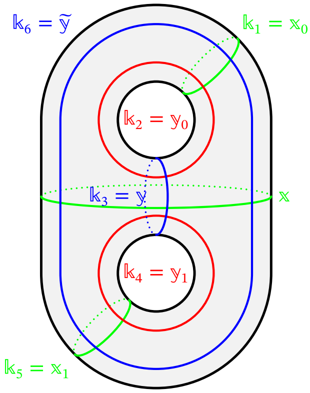


Fig. 2 Simple closed curves on the double torus $\Sigma_{2,0}$

detail. In addition, the generators of the $SL(2;\mathbb{Z})$ action [8] on $H_{q,\mathbf{t}}$

$$\sigma_R = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) : \begin{pmatrix}\mathsf{T}_0\\ \mathsf{T}_1\\ \mathsf{X}\end{pmatrix} \mapsto \begin{pmatrix}q^{-\frac{1}{2}}\mathsf{X}\mathsf{T}_0^{-1}\\ \mathsf{T}_1\\ \mathsf{X}\end{pmatrix}, \tag{2.16}$$

$$\sigma_L = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) : \begin{pmatrix}\mathsf{T}_0\\ \mathsf{T}_1\\ \mathsf{X}\end{pmatrix} \mapsto \begin{pmatrix}\mathsf{T}_0\\ \mathsf{T}_1\\ q^{\frac{1}{2}}\mathsf{T}_0\mathsf{X}^{-1}\mathsf{T}_1^{-1}\end{pmatrix}, \tag{2.17}$$

can be interpreted as the half Dehn twists along $\mathbb{x}$ and $\mathbb{y}$ respectively.

3 DAHA on double torus and skein algebra

3.1 Skein algebra and mapping class group on $\Sigma_{2,0}$

The skein algebra $\mathrm{Sk}_A(\Sigma_{2,0})$ is generated by $\mathbb{k}_1 = \mathbb{x}_0$, $\mathbb{k}_2 = \mathbb{y}_0$, $\mathbb{k}_3 = \mathbb{y}$, $\mathbb{k}_4 = \mathbb{y}_1$, and $\mathbb{k}_5 = \mathbb{x}_1$, where we label each simple closed curve on $\Sigma_{2,0}$ as in Fig. 2 following [16]. The Humphries generators of the mapping class group are $\mathscr{D}_i = \mathscr{D}_{\mathbb{k}_i}$ for $1 \le i \le 5$, and the mapping class group is (see e.g. [6, 26, 27])

$$\mathrm{Mod}(\Sigma_{2,0}) = \left\langle \mathscr{D}_1, \ldots, \mathscr{D}_5 \;\middle|\; \begin{array}{c} \mathscr{D}_{i,i+1,i} = \mathscr{D}_{i+1,i,i+1} \text{ for } 1 \le i \le 4 \\ \mathscr{D}_{i,j} = \mathscr{D}_{j,i} \text{ for } |i-j| > 1 \\ \left(\mathscr{D}_{1,2,3,4,5}\right)^6 = 1 \\ \left(\mathscr{D}_{5,4,3,2,1,1,2,3,4,5}\right)^2 = 1 \end{array} \right\rangle, \tag{3.1}$$

where we mean $\mathscr{D}_{i,\ldots,j,k} = \mathscr{D}_i \ldots \mathscr{D}_j \mathscr{D}_k$. We note that the 2- and 3-chain relations [14] respectively give

$$\left(\mathscr{D}_{1,2}\right)^6 = \mathscr{D}_{\mathbb{x}}, \tag{3.2}$$

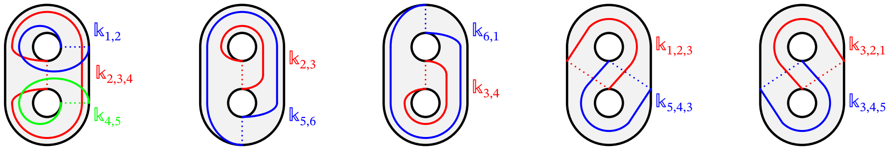


Fig. 3 Several simple closed curves on $\Sigma_{2,0}$ generated by the Dehn twists $\mathscr{D}_j$ from $\mathbb{k}_i$

$$\left(\mathscr{D}_{1,2,3}\right)^4 = \mathscr{D}_5^2. \tag{3.3}$$

See Fig. 3 for several simple closed curves generated by the Dehn twists $\mathscr{D}_i$. We mean for simplicity $\mathbb{k}_{i,\pm j,\dots,\pm k} = \left(\mathscr{D}_k^{\pm 1} \dots \mathscr{D}_j^{\pm 1}\right)(\mathbb{k}_i)$. These were used in [1] in studies of the character variety.

3.2 *q*-difference operators

In [16] studied was the map

$$\mathscr{A} : \mathrm{Sk}_{A=q^{-\frac{1}{4}}}(\Sigma_{2,0}) \to \mathrm{End}\,\mathbb{C}\left(q^{\frac{1}{4}}, x_0, x_1\right)\left[x + x^{-1}\right]. \tag{3.4}$$

Therein given are for the curves in Fig. 2 as

$$\mathscr{A}(\mathbb{x}_b) = \mathrm{ch}(x_b), \tag{3.5}$$

$$\mathscr{A}(\mathbb{y}_b) = \mathrm{i}\, q^{-\frac{1}{4}} G_0(x_b, x), \tag{3.6}$$

$$\mathscr{A}(\mathbb{x}) = \mathrm{ch}(x), \tag{3.7}$$

$$\mathscr{A}(\mathbb{y}) = \sum_{\epsilon=\pm} \omega(x^\epsilon)\left\{-x^{-\epsilon}\left(x_0 + \frac{q^{\frac{1}{2}} x^\epsilon}{x_0}\right)\left(x_1 + \frac{q^{\frac{1}{2}} x^\epsilon}{x_1}\right)\eth^\epsilon + q^{\frac{1}{2}}\,\mathrm{ch}(x_0)\,\mathrm{ch}(x_1)\right\}, \tag{3.8}$$

$$\mathscr{A}(\widetilde{\mathbb{y}}) = \sum_{\epsilon=\pm} \omega(x^\epsilon)\left\{K_0(x_0, x^\epsilon)\, K_0(x_1, x^\epsilon)\,\eth^\epsilon - G_0(x_0, x)\, G_0(x_1, x)\right\}, \tag{3.9}$$

where $b = 0, 1$. Here we have used

$$\omega(x) = \frac{x\left(1 + q^{\frac{1}{2}} x\right)}{q^{\frac{1}{2}}\left(1 - x^2\right)\left(1 - q^{\frac{1}{2}} x\right)}, \tag{3.10}$$

and the q-difference operators $K_n(x_b, x)$ and $G_n(x_b, x)$ for $n \in \mathbb{Z}$ are defined by

$$K_n(x_b, x) = \frac{-x_b^{-n}}{1 - x_b^2}\,\eth_b + \frac{x_b^n\left(q^{\frac{1}{2}} x + x_b^2\right)\left(q^{\frac{3}{2}} x + x_b^2\right)}{q\, x\left(1 - x_b^2\right)}\,\eth_b^{-1}, \tag{3.11}$$

$$G_n(x_b,x) = \frac{-x_b^{-n}}{1-x_b^2}\,\eth_b + \frac{x_b^n\left(q^{\frac{1}{2}}x + x_b^2\right)\left(q^{\frac{1}{2}} + x\,x_b^2\right)}{q^{\frac{1}{2}}\,x\left(1-x_b^2\right)}\,\eth_b^{-1}, \tag{3.12}$$

where the q-shift operators $\eth_b$ for $b = 0, 1$ are

$$(\eth_0 f)(x, x_0, x_1) = f(x, q^{\frac{1}{2}}x_0, x_1), \qquad (\eth_1 f)(x, x_0, x_1) = f(x, x_0, q^{\frac{1}{2}}x_1). \tag{3.13}$$

Note that the q-difference operators $K_0(x_b, x)$, $K_0(x_b, x^{-1})$, and $G_0(x_b, x)$ are respectively related to the raising operator, lowering operator, and the eigen-operator for the type-A_1 Macdonald polynomials a.k.a. the Rogers ultra-spherical polynomials [16]. We see that they fulfill the following;

$$G_n(x_b, x^{-1}) = G_n(x_b, x), \tag{3.14}$$

$$K_n(x_b, x)\,G_n(x_b, q\,x) = G_n(x_b, x)\,K_n(x_b, x), \tag{3.15}$$

$$K_n(x_b, x^{-1})\,K_n(x_b, q^{-1}x) - [G_n(x_b, x)]^2 = -q^{\frac{n}{2}}x^{-1}\left(q^{\frac{1}{2}} - x\right)^2, \tag{3.16}$$

$$\begin{aligned} &q^{\frac{1}{2}}x\,(1-x^2)\left(G_n(x_b, x)\,G_n(x_b, q^{-1}x) - K_n(x_b, x)\,K_n(x_b, x^{-1})\right) \\ &\quad + (1-q)\,x^2\left(K_n(x_b, x) - K_n(x_b, x^{-1})\right)G_n(x_b, q^{-1}x) \\ &\quad = q^{\frac{n}{2}}\left(q^{\frac{1}{2}} - x\right)\left(1 - q^{\frac{1}{2}}x\right)\left(1 - x^2\right)\frac{q - x_b^2}{1 - x_b^2}, \end{aligned} \tag{3.17}$$

$$\begin{pmatrix} K_{n+1}(x_b, x^{-1}) \\ G_{n+1}(x_b, q^{-1}x) \end{pmatrix} = \frac{1}{x - q^{\frac{1}{2}}}\begin{pmatrix} x\,\mathrm{ch}(x_b) & -\frac{q^{\frac{1}{2}} + x\,x_b^2}{x_b} \\ \frac{x + q^{\frac{1}{2}}x_b^2}{x_b} & -q^{\frac{1}{2}}\,\mathrm{ch}(x_b) \end{pmatrix}\begin{pmatrix} K_n(x_b, x^{-1}) \\ G_n(x_b, q^{-1}x) \end{pmatrix}. \tag{3.18}$$

We note that

$$\mathscr{A}(\Bbbk_{2,1^n}) = \mathrm{i}\,q^{-\frac{n+1}{4}}\,G_n(x_0, x), \tag{3.19}$$

which follows from the skein algebra

$$\Bbbk_1\,\Bbbk_{2,1^n} = A\,\Bbbk_{2,1^{n-1}} + A^{-1}\,\Bbbk_{2,1^{n+1}}, \quad \Bbbk_{2,1^n} = \begin{cases} \Bbbk_{2,\underbrace{1,\dots,1}_{n}}, & \text{for } n \geq 0, \\ \Bbbk_{2,\underbrace{-1,\dots,-1}_{|n|}}, & \text{for } n \leq 0. \end{cases} \tag{3.20}$$

3.3 Specialization of type-$C^\vee C_1$ DAHA

Hereafter we fix the parameters of the $C^\vee C_1$ DAHA as $\mathbf{t} = \mathbf{t}_\star$,

$$\mathbf{t}_\star = \left(\mathrm{i}\,x_0, \mathrm{i}\,q^{-\frac{1}{2}}x_1, \mathrm{i}\,x_0, \mathrm{i}\,x_1\right), \tag{3.21}$$

to identify the curve $\mathbb{b}_1$ (resp. $\mathbb{b}_3$) with $\mathbb{b}_2$ (resp. $\mathbb{b}_4$) in Fig. 1. At $\mathbf{t}_\star$ (3.21) the Hecke operators (2.7) are read as

$$\begin{aligned}
\mathsf{T}_0 &\mapsto \mathrm{i}\,\frac{x}{q^{\frac{1}{2}}-x}\left(-\frac{q^{\frac{1}{2}}+x\,x_0^2}{x\,x_0}\,\mathsf{s}\,\eth + x_0 + x_0^{-1}\right),\\
\mathsf{T}_1 &\mapsto \mathrm{i}\left(\frac{1+q^{\frac{1}{2}}x}{q^{\frac{1}{2}}\left(1-x^2\right)}\,\frac{q^{\frac{1}{2}}x+x_1^2}{x_1}\,(\mathsf{s}-1) - q^{\frac{1}{2}}x_1^{-1}\right),\\
\mathsf{T}_0^\vee &\mapsto q^{-\frac{1}{2}}\mathsf{T}_0{}^{-1}x,\\
\mathsf{T}_1^\vee &\mapsto x^{-1}\mathsf{T}_1{}^{-1},
\end{aligned} \tag{3.22}$$

which satisfy the Hecke relations

$$\begin{aligned}
\mathsf{T}_0 - \mathsf{T}_0^{-1} &= -\mathrm{i}\,\mathrm{ch}\,(x_0)\,, & \mathsf{T}_0^\vee - \left(\mathsf{T}_0^\vee\right)^{-1} &= -\mathrm{i}\,\mathrm{ch}(x_0),\\
\mathsf{T}_1 - \mathsf{T}_1^{-1} &= -\mathrm{i}\,\mathrm{ch}\left(q^{-\frac{1}{2}}\,x_1\right), & \mathsf{T}_1^\vee - \left(\mathsf{T}_1^\vee\right)^{-1} &= -\mathrm{i}\,\mathrm{ch}(x_1).
\end{aligned} \tag{3.23}$$

The idempotent (2.6) becomes

$$\mathsf{e} \mapsto (1+\mathsf{s})\,\frac{\left(q^{\frac{1}{2}}+x\right)\left(q^{\frac{1}{2}}+x\,x_1^2\right)}{\left(1-x^2\right)\left(q-x_1^2\right)}. \tag{3.24}$$

Note that

$$\mathsf{T}_1\,\mathsf{e} = -\mathrm{i}\,q^{\frac{1}{2}}\,x_1^{-1}\mathsf{e} = \mathsf{e}\,\mathsf{T}_1, \qquad \mathsf{T}_1^{-1}\,\mathsf{e} = \mathrm{i}\,q^{-\frac{1}{2}}\,x_1 = \mathsf{e}\,\mathsf{T}_1^{-1}. \tag{3.25}$$

and

$$\mathsf{X}\mathsf{T}_1\left(1+q^{\frac{1}{2}}\mathsf{X}\right)\mathsf{e} = q^{\frac{1}{2}}\left(1+q^{\frac{1}{2}}\mathsf{X}\right)\mathsf{T}_1^{-1}\,\mathsf{e}. \tag{3.26}$$

The DAHA at $\mathbf{t}_\star$ (3.21) was employed so that the Askey–Wilson operator gives (3.8) as

$$\mathcal{A}(\mathbb{y})\,\mathsf{e} = \mathrm{ch}(\mathsf{T}_1\mathsf{T}_0)\,\mathsf{e}. \tag{3.27}$$

Namely the Askey–Wilson polynomial (2.11) is the eigen-polynomial of (3.8),

$$\mathcal{A}(\mathbb{y})\;P_m(x;q,\mathbf{t}_\star) = -\,\mathrm{ch}\left(\frac{q^{m+\frac{1}{2}}}{x_0x_1}\right)\;P_m(x;q,\mathbf{t}_\star).$$

It should be remarked that the operator $\mathcal{A}(\widetilde{\mathbb{y}})$, commuting with the Askey–Wilson operator as $\mathcal{A}(\mathbb{y})\,\mathcal{A}(\widetilde{\mathbb{y}}) = \mathcal{A}(\widetilde{\mathbb{y}})\,\mathcal{A}(\mathbb{y})$, satisfies [18]

$$\mathcal{A}(\widetilde{\mathbb{y}})\, P_m(x;q,\mathbf{t}_\star) = \frac{\left(q^{m+2}-x_0^2x_1^2\right)^2}{q^{m+\frac{5}{2}}\left(1-x_0^2\right)\left(1-x_1^2\right)} P_{m+1}(x;q,\mathbf{t}_\star)$$
$$-\frac{\left(q^{m+1}-x_0^2\right)^2+\left(q^{m+1}-x_1^2\right)^2}{q^{m+\frac{3}{2}}\left(1-x_0^2\right)\left(1-x_1^2\right)} P_m(x;q,\mathbf{t}_\star) + \frac{(1-q^m)^2}{q^{m+\frac{1}{2}}\left(1-x_0^2\right)\left(1-x_1^2\right)} P_{m-1}(x;q,\mathbf{t}_\star).$$

3.4 Heegaard dual of Hecke operators

Our purpose is to rewrite the map (3.4) in terms of the Iwahori–Hecke operators. The motivation is based on that S^3 has a Heegaard splitting $S^3 = H_1 \cup_{\Sigma_{2,0}} H_2$, where H_i is a 2-handlebody and $\Sigma_{2,0} = \partial H_i$. In gluing, the meridians $\mathbb{x}_b$ on H_1 are mapped to the longitudes $\mathbb{y}_b$ on H_2, and $\mathbb{x}$ and $\mathbb{y}$ are to $\mathbb{x}$ and $\widetilde{\mathbb{y}}$ respectively. The fact that $\mathbb{y}$ corresponds to the Askey–Wilson operator (3.27) suggests that there may exist a "Heegaard dual" U_0 and U_1 of the Hecke operators T_0 and T_1 (3.22) for $\widetilde{\mathbb{y}}$.

Definition 3.1 We define the representation of U_0 and U_1 by

$$\mathsf{U}_0 \mapsto \frac{q^{-\frac{1}{4}}x}{q^{\frac{1}{2}}-x} K_0(x_0,x^{-1})\,\mathsf{s}\,\eth - \frac{q^{-\frac{1}{4}}x}{q^{\frac{1}{2}}-x} G_0(x_0,x), \tag{3.28}$$
$$\mathsf{U}_1 \mapsto -\frac{x\left(1+q^{\frac{1}{2}}x\right)}{q^{\frac{1}{4}}\left(1-x^2\right)} K_0(x_1,x)\,(\mathsf{s}-1) + \frac{q^{\frac{1}{4}}}{1-q^{\frac{1}{2}}x}\left(G_0(x_1,x)-q^{\frac{1}{2}}x\,K_0(x_1,x)\right), \tag{3.29}$$
$$\mathsf{U}_0^\vee \mapsto q^{-\frac{1}{2}}\mathsf{U}_0^{-1}x, \tag{3.30}$$
$$\mathsf{U}_1^\vee \mapsto x^{-1}\mathsf{U}_1^{-1}, \tag{3.31}$$

where $K_n(x_b,x)$ and $G_n(x_b,x)$ are given in (3.11) and (3.12).

The invertibilities of U_0 and U_1 can be checked using (3.14)–(3.17) by

$$\mathsf{U}_0^{-1} \mapsto \frac{q^{-\frac{1}{4}}x}{q^{\frac{1}{2}}-x} K_0(x_0,x^{-1})\,\mathsf{s}\,\eth - \frac{q^{\frac{1}{4}}}{q^{\frac{1}{2}}-x} G_0(x_0,x), \tag{3.32}$$
$$\mathsf{U}_1^{-1} \mapsto \left\{(\mathsf{s}+1)\frac{q^{\frac{1}{2}}+x}{q^{\frac{1}{4}}(1-x^2)} K_0(x_1,x^{-1}) + \frac{q^{\frac{1}{4}}}{q^{\frac{1}{2}}-x}\left(x\,G_0(x_1,q^{-1}x)-q^{\frac{1}{2}}\,K_0(x_1,x^{-1})\right)\right\} \times \frac{1-x_1^2}{q-x_1^2}. \tag{3.33}$$

By construction, we have an analogue of (2.2)

$$\mathsf{U}_1^\vee \mathsf{U}_1 \mathsf{U}_0 \mathsf{U}_0^\vee = q^{-\frac{1}{2}}. \tag{3.34}$$

Moreover we get the followings.

Lemma 3.2 *(1)*

$$\mathsf{U}_0\mathsf{T}_0\mathsf{U}_0^\vee = -q^{\frac{1}{2}}\,\mathsf{T}_0, \tag{3.35}$$

(2)

$$\mathsf{U}_1\mathsf{T}_1^{-1}\mathsf{U}_1^\vee\mathsf{T}_1 \mapsto -q - \frac{(1-q)\left(1+q^{\frac{1}{2}}x\right)\left(q^{\frac{1}{2}}x+x_1^2\right)}{\left(1-x^2\right)\left(q-x_1^2\right)}\,(\mathsf{s}-1)\,. \tag{3.36}$$

The representations in Definition 3.1 and the inverses show that the Heegaard dual operators satisfy the Hecke-type relations. We have for U_0 and $\mathsf{U}_0^\vee$

$$\begin{aligned}\mathsf{U}_0 - \mathsf{U}_0^{-1} &= q^{-\frac{1}{4}}\,G_0(x_0,x),\\ \mathsf{U}_0^\vee - \left(\mathsf{U}_0^\vee\right)^{-1} &= q^{-\frac{1}{4}}\,G_0(x_0,x).\end{aligned} \tag{3.37}$$

For U_1 and $\mathsf{U}_1^\vee$, we have preferable expressions with the symmetrizer e (3.24)

$$\begin{aligned}\left(q^{-\frac{1}{2}}\mathsf{U}_1 - q^{\frac{1}{2}}\mathsf{U}_1^{-1}\right)\mathsf{e} &= q^{-\frac{1}{4}}G_0(x_1,x)\,\mathsf{e},\\ \left(\mathsf{U}_1^\vee - \left(\mathsf{U}_1^\vee\right)^{-1}\right)\mathsf{e} &= q^{-\frac{1}{4}}G_0(x_1,x)\,\mathsf{e},\end{aligned} \tag{3.38}$$

which can be seen by use of an analogous identity to (3.26),

$$\mathsf{X}\mathsf{U}_1\left(1+q^{\frac{1}{2}}\mathsf{X}\right)\mathsf{e} = q^{\frac{1}{2}}\left(1+q^{\frac{1}{2}}\mathsf{X}\right)\mathsf{U}_1^{-1}\,\mathsf{e}. \tag{3.39}$$

Furthermore we can prove

$$\mathcal{A}(\widetilde{\mathbb{y}})\,\mathsf{e} = \operatorname{ch}\left(\mathsf{U}_1\mathsf{U}_0\right)\mathsf{e}. \tag{3.40}$$

In summary, all the generators $\mathbb{k}_i$ for the skein algebra $\mathrm{Sk}_{A=q^{-\frac{1}{4}}}(\Sigma_{2,0})$ given in (3.5)–(3.9) can be written as follows.

Proposition 3.3

$$\begin{aligned}
\mathcal{A}(\mathbb{k}_1) &= \mathcal{A}(\mathbb{x}_0) = \operatorname{ch}\left(\mathrm{i}\,\mathsf{T}_0\right) = \operatorname{ch}\left(\mathrm{i}\,\mathsf{T}_0^\vee\right),\\
\mathcal{A}(\mathbb{k}_2) &= \mathcal{A}(\mathbb{y}_0) = \operatorname{ch}\left(\mathrm{i}\,\mathsf{U}_0\right) = \operatorname{ch}\left(\mathrm{i}\,\mathsf{U}_0^\vee\right),\\
\mathcal{A}(\mathbb{k}_3)\,\mathsf{e} &= \mathcal{A}(\mathbb{y})\,\mathsf{e} = \operatorname{ch}(\mathsf{T}_1\mathsf{T}_0)\,\mathsf{e} = \operatorname{ch}(\mathsf{T}_0\mathsf{T}_1)\,\mathsf{e},\\
\mathcal{A}(\mathbb{k}_4)\,\mathsf{e} &= \mathcal{A}(\mathbb{y}_1)\,\mathsf{e} = \operatorname{ch}\left(\mathrm{i}\,q^{-\frac{1}{2}}\mathsf{U}_1\right)\mathsf{e} = \operatorname{ch}\left(\mathrm{i}\,\mathsf{U}_1^\vee\right)\mathsf{e},\\
\mathcal{A}(\mathbb{k}_5)\,\mathsf{e} &= \mathcal{A}(\mathbb{x}_1)\,\mathsf{e} = \operatorname{ch}\left(\mathrm{i}\,q^{-\frac{1}{2}}\mathsf{T}_1\right)\mathsf{e} = \operatorname{ch}\left(\mathrm{i}\,\mathsf{T}_1^\vee\right)\mathsf{e},\\
\mathcal{A}(\mathbb{k}_6)\,\mathsf{e} &= \mathcal{A}(\widetilde{\mathbb{y}})\,\mathsf{e} = \operatorname{ch}\left(\mathsf{U}_1\mathsf{U}_0\right)\mathsf{e}.
\end{aligned} \tag{3.41}$$

We can thus regard the map (3.4) as

$$\mathcal{A} : \mathrm{Sk}_{A=q^{-\frac{1}{4}}}(\Sigma_{2,0}) \to SH_{q,\mathbf{t}_\star}^{gen} \tag{3.42}$$

where $SH_{q,\mathbf{t}_\star}^{gen}$ is a spherical subalgebra of our generalized DAHA,

$$H_{q,\mathbf{t}_\star}^{gen} = \left\langle \mathsf{T}_0^{\pm 1}, \mathsf{T}_1^{\pm 1}, \mathsf{X}^{\pm 1}, \mathsf{U}_0^{\pm 1}, \mathsf{U}_1^{\pm 1} \;\middle|\; \begin{array}{c} \text{the Hecke relations (3.23), (3.37), (3.38)} \\ \mathsf{T}_0\mathsf{U}_0^{-1}\mathsf{X}\mathsf{T}_0^{-1}\mathsf{U}_0 = -q \\ \mathsf{U}_1\mathsf{T}_1^{-1}\mathsf{X}^{-1}\mathsf{U}_1^{-1}\mathsf{T}_1\,\mathsf{e} = -q\,\mathsf{e} \end{array} \right\rangle. \tag{3.43}$$

The conditions are from Lemma 3.2.

3.5 Automorphisms

By definition of $H_{q,\mathbf{t}_\star}^{gen}$ (3.43), we find the automorphisms $\mathscr{T}_i = \mathscr{T}_{\Bbbk_i}$ as follows.

Proposition 3.4 *We have the automorphisms of $H_{q,\boldsymbol{t}_\star}^{gen}$ (3.43);*

$$\mathscr{T}_1 = \mathscr{T}_{\mathbb{x}_0} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ -\mathrm{i}\, q^{\frac{1}{4}} \mathsf{U}_0\mathsf{T}_0^{-1} \\ \mathsf{U}_1 \end{pmatrix}, \tag{3.44}$$

$$\mathscr{T}_2 = \mathscr{T}_{\mathbb{y}_0} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{i}\, q^{-\frac{1}{4}} \mathsf{U}_0\mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix}, \tag{3.45}$$

$$\mathscr{T}_3 = \mathscr{T}_{\mathbb{y}} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ (\mathsf{T}_0\mathsf{T}_1)^{-1}\,\mathsf{X}\mathsf{T}_1\mathsf{T}_0 \\ q^{-\frac{1}{4}}\,(\mathsf{T}_0\mathsf{T}_1)^{-1}\,\mathsf{U}_0 \\ q^{\frac{1}{4}}\mathsf{U}_1\mathsf{T}_0\mathsf{T}_1 \end{pmatrix}, \tag{3.46}$$

$$\mathscr{T}_4 = \mathscr{T}_{\mathbb{y}_1} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_0 \\ \mathrm{i}\, q^{-\frac{1}{4}}\,(\mathsf{U}_1\mathsf{X})^{-1}\,\mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix}, \tag{3.47}$$

$$\mathscr{T}_5 = \mathscr{T}_{\mathbb{x}_1} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ -\mathrm{i}\, q^{\frac{1}{4}}\mathsf{U}_1\mathsf{X}\mathsf{T}_1 \end{pmatrix}. \tag{3.48}$$

We note that the map $\mathscr{T}_3$ (3.46) originates from the Dehn twist σ_L^{-2} on $\Sigma_{0,4}$ (2.17). Our claim is as follows.

Proposition 3.5 *We have a commutative diagram;*

$$\begin{array}{ccc} \mathrm{Sk}_{A=q^{-\frac{1}{4}}}(\Sigma_{2,0}) & \xrightarrow{\mathscr{D}_i} & \mathrm{Sk}_{A=q^{-\frac{1}{4}}}(\Sigma_{2,0}) \\ \mathscr{A}\downarrow & & \downarrow\mathscr{A} \\ SH^{gen}_{q,\boldsymbol{t}_\star} & \xrightarrow[\mathscr{T}_i]{} & SH^{gen}_{q,\boldsymbol{t}_\star} \end{array}$$

We shall check the relations in (3.1). From the definitions (3.44)–(3.48) it is straightforward to see case-by-case that both the braid relations and the commutativities hold;

$$\mathscr{T}_{i,i+1,i} = \mathscr{T}_{i+1,i,i+1}, \qquad \text{for } 1 \leq i \leq 4, \tag{3.49}$$

$$\mathscr{T}_{i,j} = \mathscr{T}_{j,i}, \qquad \text{for } |i-j| > 1, \tag{3.50}$$

where we mean $\mathscr{T}_{i,\ldots,j,k} = \mathscr{T}_i \ldots \mathscr{T}_j \mathscr{T}_k$. We can also find that

$$\mathscr{T}_{1,2,3,4,5} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{U}_0 \\ \mathrm{i}\, q^{-\frac{1}{2}} \mathsf{U}_0^{-1} \mathsf{T}_1^{-1} \mathsf{X}^{-1} \mathsf{U}_1^{-1} \mathsf{T}_1 \\ \mathsf{T}_1^{-1} \mathsf{U}_0^{-1} \mathsf{X} \mathsf{T}_1 \mathsf{U}_0 \\ -\mathrm{i}\, \mathsf{T}_1^{-1} \mathsf{T}_0^{-1} \\ \mathsf{T}_1 \end{pmatrix},$$

$$\mathscr{T}_{5,4,3,2,1} : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{i}\, q \mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \mathsf{U}_0 \\ -q^{-\frac{1}{2}} \mathsf{X}^{-1} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right)^{-1} \\ \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right) \mathsf{X} \mathsf{T}_0^{-1} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right)^{-1} \mathsf{T}_0 \\ q^{-\frac{1}{2}} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right) \mathsf{X} \mathsf{T}_0^{-1} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right)^{-1} \\ \mathrm{i}\, \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \mathsf{T}_0 \mathsf{X}^{-1} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \right)^{-1} \end{pmatrix},$$

which result in

$$\left(\mathscr{T}_{1,2,3,4,5}\right)^6 = \left(\mathscr{T}_{5,4,3,2,1,1,2,3,4,5}\right)^2 : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \mathsf{U}_1^{-1} \right) \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \left(\mathsf{T}_1^{-1} \mathsf{U}_1 \mathsf{X} \mathsf{T}_1 \mathsf{U}_1^{-1} \right)^{-1}. \tag{3.51}$$

As seen from (3.36), the operator $\mathsf{U}_1 \mathsf{T}_1^{-1} \mathsf{X}^{-1} \mathsf{U}_1^{-1} \mathsf{T}_1$ acts as a scalar on the symmetric Laurent polynomials, and the maps (3.51) are identities on $\mathbb{C}(q^{\frac{1}{4}}, x_0, x_1)[x + x^{-1}]$.

It should be noted that, for the 3-chain relation (3.3), we have

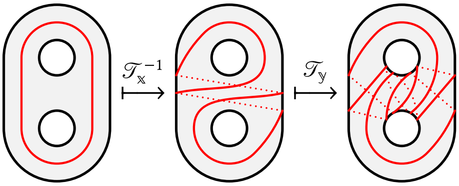
$$(\mathscr{T}_{1,2,3})^4 : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_1^{-1}\mathsf{T}_0\mathsf{T}_1 \\ \mathsf{T}_1 \\ \mathsf{T}_1^{-1}\mathsf{X}\mathsf{T}_1 \\ \mathsf{T}_1^{-1}\mathsf{U}_0\mathsf{T}_1 \\ \mathsf{U}_1\mathsf{X}\mathsf{T}_1^2 \end{pmatrix}, \qquad (\mathscr{T}_5)^2 : \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ \mathsf{U}_1 \end{pmatrix} \mapsto \begin{pmatrix} \mathsf{T}_0 \\ \mathsf{T}_1 \\ \mathsf{X} \\ \mathsf{U}_0 \\ -q^{\frac{1}{2}}\mathsf{U}_1\mathsf{X}\mathsf{T}_1\mathsf{X}\mathsf{T}_1 \end{pmatrix}.$$

Both actions on the generators (3.41) are same.

We explicitly give the map for the curves in Fig. 3. We have checked the consistency with the skein algebra, e.g. (2.14), of the curves.

$$\begin{aligned}
\mathscr{A}(\Bbbk_{1,2}) &= \mathscr{T}_2(\mathscr{A}(\Bbbk_1)) = \mathrm{ch}\left(-q^{-\frac{1}{4}}\mathsf{U}_0\mathsf{T}_0\right) = \mathrm{ch}\left(q^{\frac{1}{4}}\mathsf{X}^{-1}\mathsf{U}_0\mathsf{T}_0\right) \\
&= \mathrm{i}\, G_{-1}(x_0, x), \\
\mathscr{A}(\Bbbk_{2,3})\,\mathsf{e} &= \mathscr{T}_3(\mathscr{A}(\Bbbk_2))\,\mathsf{e} = \mathrm{ch}\left(\mathrm{i}\, q^{-\frac{1}{4}}(\mathsf{T}_0\mathsf{T}_1)^{-1}\mathsf{U}_0\right)\mathsf{e} = \mathrm{ch}\left(-\mathrm{i}\, q^{\frac{1}{4}}(\mathsf{T}_1\mathsf{T}_0)^{-1}\mathsf{X}^{-1}\mathsf{U}_0\right)\mathsf{e} \\
&= \mathrm{i}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-K_1(x_0,x^\epsilon)\,\frac{x_1^2+q^{\frac{1}{2}}x^\epsilon}{x_1}\,\eth^\epsilon + G_1(x_0,x)\;\mathrm{ch}(x_1)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{3,4})\,\mathsf{e} &= \mathscr{T}_3^{-1}(\mathscr{A}(\Bbbk_4))\,\mathsf{e} = \mathrm{ch}\left(\mathrm{i}\, q^{-\frac{3}{4}}\mathsf{U}_1(\mathsf{T}_0\mathsf{T}_1)^{-1}\right)\mathsf{e} = \mathrm{ch}\left(-\mathrm{i}\, q^{-\frac{1}{4}}\mathsf{U}_1\mathsf{X}(\mathsf{T}_1\mathsf{T}_0)^{-1}\right)\mathsf{e} \\
&= \mathrm{i}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-\frac{x^{-\epsilon}x_0^2+q^{\frac{1}{2}}}{x_0}K_{-1}(x_1,x^\epsilon)\,\eth^\epsilon + q^{\frac{1}{2}}\,\mathrm{ch}(x_0)\,G_{-1}(x_1,x)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{4,5})\,\mathsf{e} &= \mathscr{T}_5(\mathscr{A}(\Bbbk_4))\,\mathsf{e} = \mathrm{ch}\left(q^{-\frac{1}{4}}\mathsf{U}_1\mathsf{X}\mathsf{T}_1\right)\mathsf{e} = \mathrm{ch}\left(-q^{\frac{1}{4}}\mathsf{U}_1\mathsf{X}\mathsf{T}_1\mathsf{X}\right)\mathsf{e} \\
&= \mathrm{i}\, q^{-\frac{1}{2}}G_1(x_1,x)\,\mathsf{e}, \\
\mathscr{A}(\Bbbk_{5,6})\,\mathsf{e} &= \mathscr{T}_5^{-1}\,(\mathscr{A}(\Bbbk_6))\,\mathsf{e} = \mathrm{ch}\left(\mathrm{i}\, q^{-\frac{1}{4}}\mathsf{U}_1(\mathsf{X}\mathsf{T}_1)^{-1}\mathsf{U}_0\right)\mathsf{e} \\
&= q^{\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(K_0(x_0,x^\epsilon)\,K_{-1}(x_1,x^\epsilon)\,\eth^\epsilon - G_0(x_0,x)\,G_{-1}(x_1,x)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{6,1})\,\mathsf{e} &= \mathscr{T}_1\,(\mathscr{A}(\Bbbk_6))\,\mathsf{e} = \mathrm{ch}\left(-\mathrm{i}\, q^{\frac{1}{4}}\mathsf{U}_1\mathsf{U}_0\mathsf{T}_0^{-1}\right)\mathsf{e} \\
&= q^{-\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(K_1(x_0,x^\epsilon)\,K_0(x_1,x^\epsilon)\,\eth^\epsilon - G_1(x_0,x)\,G_0(x_1,x)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{1,2,3})\,\mathsf{e} &= \mathscr{T}_3\left(\mathscr{A}(\Bbbk_{1,2})\right)\mathsf{e} = \mathrm{ch}\left(q^{\frac{1}{2}}\mathsf{T}_1^{-1}\mathsf{X}^{-1}\mathsf{U}_0\right)\mathsf{e} = \mathrm{ch}\left((\mathsf{X}\mathsf{T}_1\mathsf{T}_0)^{-1}\mathsf{U}_0\mathsf{T}_0\right)\mathsf{e} \\
&= \mathrm{i}\, q^{\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-K_0(x_0,x^\epsilon)\,\frac{x_1^2+q^{\frac{1}{2}}x^\epsilon}{x_1}\eth^\epsilon + G_0(x_0,x)\;\mathrm{ch}(x_1)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{5,4,3})\,\mathsf{e} &= \left(\mathscr{T}_5^{-1}\mathscr{T}_3\right)(\mathscr{A}(\Bbbk_4))\,\mathsf{e} = \mathrm{ch}(\mathsf{U}_1\mathsf{T}_0)\,\mathsf{e} = \mathrm{ch}\left(-q^{\frac{1}{2}}\mathsf{U}_1\mathsf{T}_1^{-1}\mathsf{X}^{-1}\mathsf{T}_0\mathsf{T}_1\right)\mathsf{e} \\
&= \mathrm{i}\, q^{\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-\frac{x_0^2+q^{\frac{1}{2}}x^\epsilon}{x_0}\,K_0(x_1,x^\epsilon)\,\eth^\epsilon + \mathrm{ch}(x_0)\,G_0(x_1,x)\right)\mathsf{e}, \\
\mathscr{A}(\Bbbk_{2,3,4})\,\mathsf{e} &= \mathscr{T}_2^{-1}\left(\mathscr{A}(\Bbbk_{3,4})\right)\mathsf{e} = \mathrm{ch}\left(-q^{-1}\mathsf{U}_1(\mathsf{T}_0\mathsf{T}_1)^{-1}\mathsf{U}_0\right)\mathsf{e} = \mathrm{ch}\left(-q^{\frac{1}{2}}\mathsf{U}_1\mathsf{U}_0(\mathsf{T}_1\mathsf{T}_0)^{-1}\right)\mathsf{e} \\
&= \sum_{\epsilon=\pm}\omega(x^\epsilon)\left(K_1(x_0,x^\epsilon)\,K_{-1}(x_1,x^\epsilon)\,\eth^\epsilon - G_1(x_0,x)\,G_{-1}(x_1,x)\right)\mathsf{e},
\end{aligned}$$

Fig. 4 A rational tangle for $2+\frac{1}{2}$ which corresponds to the figure-eight knot 4_1



$$\begin{aligned}
\mathcal{A}(\Bbbk_{3,4,5})\,\mathsf{e} &= \mathscr{T}_5\left(\mathcal{A}(\Bbbk_{3,4})\right)\mathsf{e} = \mathrm{ch}\left(q^{-\frac{1}{2}}\mathsf{U}_1\mathsf{X}\mathsf{T}_0^{-1}\right)\mathsf{e} = \mathrm{ch}\left(-\mathsf{U}_1\mathsf{X}\mathsf{T}_1\mathsf{X}(\mathsf{T}_1\mathsf{T}_0)^{-1}\right)\mathsf{e}\\
&= \mathrm{i}\,q^{\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-\frac{q^{-\frac{1}{2}}x^{-\epsilon}x_0^2+1}{x_0}K_0(x_1,x^\epsilon)\,\eth^\epsilon+\mathrm{ch}(x_0)\,G_0(x_1,x)\right)\mathsf{e},\\
\mathcal{A}(\Bbbk_{3,2,1})\,\mathsf{e} &= (\mathscr{T}_1\mathscr{T}_2)\left(\mathcal{A}(\Bbbk_3)\right)\mathsf{e} = \mathrm{ch}(\mathsf{T}_1\mathsf{U}_0)\,\mathsf{e} = \mathrm{ch}(\mathsf{U}_0\mathsf{T}_1)\,\mathsf{e}\\
&= \mathrm{i}\,q^{\frac{1}{4}}\sum_{\epsilon=\pm}\omega(x^\epsilon)\left(-K_0(x_0,x^\epsilon)\,\frac{q^{-\frac{1}{2}}x^{-\epsilon}x_1^2+1}{x_1}\,\eth^\epsilon+G_0(x_0,x)\,\mathrm{ch}(x_1)\right)\mathsf{e}.
\end{aligned}$$

Here we avoid to use $\mathscr{T}_4$ due to that T_1, used for the idempotent e, is no longer invariant.

3.6 Rational tangles

In [11], Conway introduced tangle operations, and showed that continued fraction can be assigned to a certain family of knots and links. In view from links on the double torus $\Sigma_{2,0}$, the tangle operations correspond to the Dehn twists along $\mathbb{x}$ and $\mathbb{y}$ acting on the curve $\widetilde{\mathbb{y}}$. We can thus construct the rational tangle $\widetilde{\mathbb{y}}_r$ associated with the continued fraction r with even integers. The automorphism $\mathscr{T}_{\mathbb{y}}$ for the Dehn twist along $\mathbb{y}$ is (3.46), and $\mathscr{T}_{\mathbb{x}}$ is given from the 2-chain relation (3.2) as

$$\mathscr{T}_{\mathbb{x}}:\begin{pmatrix}\mathsf{T}_0\\ \mathsf{T}_1\\ \mathsf{X}\\ \mathsf{U}_0\\ \mathsf{U}_1\end{pmatrix}\mapsto\begin{pmatrix}\mathsf{X}\mathsf{T}_0\mathsf{X}^{-1}\\ \mathsf{T}_1\\ \mathsf{X}\\ \mathsf{X}\mathsf{U}_0\mathsf{X}^{-1}\\ \mathsf{U}_1\end{pmatrix}, \tag{3.52}$$

which is consistent with the Dehn twist σ_R^2 on $\Sigma_{0,4}$ (2.16).

We show a few examples. The figure-eight knot 4_1 is a rational tangle with $\frac{5}{2}=2+\frac{1}{2}$ as in Fig. 4. We have

$$\widetilde{\mathbb{y}}_{5/2}=\left(\mathscr{D}_{\mathbb{y}}\mathscr{D}_{\mathbb{x}}^{-1}\right)(\widetilde{\mathbb{y}}), \tag{3.53}$$

which gives

$$\mathcal{A}(\widetilde{\mathbb{y}}_{5/2})=\mathrm{ch}\left(\mathsf{U}_1\mathsf{T}_0\mathsf{T}_1\left(\mathsf{T}_0^\vee\mathsf{T}_1\mathsf{T}_0\right)^{-1}\mathsf{T}_0^{-1}\mathsf{U}_0\mathsf{T}_1\left(\mathsf{T}_0^\vee\mathsf{T}_1\mathsf{T}_0\right)\right). \tag{3.54}$$

The knot 5_2 is $\frac{7}{2}=4+\frac{1}{-2}$, and

$$\widetilde{\mathbb{y}}_{7/2}=\left(\mathscr{D}_{\mathbb{y}}^{-1}\mathscr{D}_{\mathbb{x}}^{-2}\right)(\widetilde{\mathbb{y}}), \tag{3.55}$$

which gives

$$\mathcal{A}(\widetilde{\mathbb{y}}_{7/2}) = \mathrm{ch}\left(\mathsf{U}_1\mathsf{T}_1^{-1}\mathsf{T}_0^{-1}\left(\mathsf{T}_1\mathsf{T}_0\mathsf{T}_1^\vee\mathsf{T}_0^{-1}\right)\mathsf{T}_1\mathsf{T}_0\mathsf{T}_1^\vee\mathsf{T}_1\mathsf{U}_0\left(\mathsf{T}_1\mathsf{T}_0\mathsf{T}_1^\vee\mathsf{T}_0^{-1}\right)^{-2}\right). \tag{3.56}$$

In both cases, the DAHA polynomials $\mathcal{A}(\mathbb{k}_r)(1)$ are too involved to give here. Nonetheless Mathematica shows that the constant terms $\mathfrak{d}^0$ of $\mathcal{A}(\widetilde{\mathbb{y}}_r)$ reduce to

$$\begin{aligned}
\mathrm{Const}(\mathcal{A}(\widetilde{\mathbb{y}}_{5/2}))(1)\Big|_{x_0=x_1=-x=q^{\frac{1}{2}}} &= \frac{q^{-2}-q^{-1}+1-q+q^2}{(1-q)(1-q^2)},\\
\mathrm{Const}(\mathcal{A}(\widetilde{\mathbb{y}}_{7/2}))(1)\Big|_{x_0=x_1=-x=q^{\frac{1}{2}}} &= \frac{q(1-q+2q^2-q^3+q^4-q^5)}{(1-q)(1-q^2)}.
\end{aligned} \tag{3.57}$$

These computations support a relationship with the Jones polynomial as observed in [16].

4 Concluding remarks

We have proposed a generalization of the type-$C^\vee C_1$ DAHA at $\mathbf{t}_\star$ by introducing the Heegaard dual operators. We hope to report on their roles on the (non-symmetric) Askey–Wilson polynomials at $\mathbf{t}_\star$, and also on the generalization to the higher-rank skein algebras. It would be promising to incorporate results from the cluster algebra [7, 17].

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Data availibility The Mathematica code for (3.57) is attached as supplementary material.

Declarations

Conflict of interest The author has no Conflict of interest to declare that are relevant to the content of this article.

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