La Rivista del Nuovo Cimento (2024) 47:653–728
https://doi.org/10.1007/s40766-025-00064-5

REVIEW PAPER

Renormalized field theory for non-equilibrium systems

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Received: 6 November 2024 / Accepted: 4 December 2024 / Published online: 31 January 2025

Abstract
Renormalized field theory is a most effective framework to carry out asymptotic analysis of non-equilibrium nearly critical systems, especially in high orders of perturbation theory. Here, we review some subtle, slippery and non-conventional aspects of this approach. We present construction of the field-theoretic representation of certain Langevin-type stochastic equations with additive and multiplicative random sources as well as master equations of various birth–death processes. Application of the field-theoretic renormalization group combined with the short-distance operator-product expansion to the analysis of asymptotic scaling behavior is reviewed for passive scalar fields advected by various velocity ensembles, including Kraichnan's rapid-change model and the stochastic Navier–Stokes equation. Infinite sets of anomalous exponents were calculated within regular expansions up to third order. Effects of anisotropy, finite correlation time and compressibility are discussed. The representation of the Kolmogorov constant and the skewness factor suitable for perturbative renormalization-group calculation and the second-order results are presented in a reasonable agreement with experiments in fully developed hydrodynamic turbulence. The recent third-order results for the critical exponents for the directed percolation process are presented; paradigmatic models for irreversible reaction–diffusion processes are discussed with the account of advection in various random velocity fields.

Keywords Non-equilibrium systems · Critical behavior · Renormalized field theory · Functional integral · Multiplicative noise · Dynamic action functional · Renormalization group · Operator-product expansion · Turbulence · Directed percolation · Reaction–diffusion systems

1 Introduction

Over hundreds of years, Humankind tried to understand the powers that move the Earth, the Stars and the Universe. Now this goal is more or less achieved, and the fundamental physics turned to the behavior of sand piles, soap films, raindrops on

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windshields, forms of snowflakes, crowds of birds, colonies of insects and so on. It turns out that these issues are also extremely interesting and highly complicated. They exhibit complex patterns and regularities and require sophisticated tools to describe and understand their behavior.

At the same time, the understanding of origin of large-scale correlations, universal scaling behavior, formation of structures and dynamic emergence of asymptotic symmetries brings us back to the original fundamental problems and throws new light on them.

Numerous specific models of cooperative many-body systems were formulated for the purpose of describing earthquakes, social disturbances, propagation of forest fires, epidemic deceases and tumors—nothing pleasant, to say the least.

Any model of this kind necessarily involves (infinitely) many strongly interacting degrees of freedom of very different scales. The very formulation of such models is an extremely demanding task, and the same is true for their effort- and time-consuming quantitative and qualitative analysis.

Theoretical analysis of such problems can proceed in several ways. One guiding example is provided by equilibrium many-body systems. The very concept of equilibrium state is rather vague [1]: *If a system is very weakly coupled to a thermal bath at a given "temperature," if the coupling is indefinite or not known precisely, if the coupling has been on for a long time, and if all the "fast" things have happened and all the "slow" things not, the system is said to be in "thermal equilibrium."* Nevertheless, thermodynamics and equilibrium statistical physics provide successful universal description of such systems with little connection to their specific "material" nature.

Some general approaches to non-equilibrium systems were proposed with relatively restricted appreciation (N.N. Bogolyubov, I. Prigogine, R. Balescu, G. Gallavotti and others). The open dissipative systems (R.K.P. Zia and others) and systems with self-organized criticality (P. Bak and others) can be viewed as certain separate classes with common general properties.[1]

At the same time, experience with equilibrium *nearly critical* states shows that principal universal properties of many-body cooperative systems can be adequately and nearly exhaustively described by phenomenological or simple microscopic *models*, designed on the base of considerations of simplicity, dimensions, and symmetries. Here guiding examples are provided by various mean-field theories and by simple models of second-order phase transitions in liquid–vapor or magnetic systems (like the Ising and Heisenberg models and some others). Let us briefly recall the most relevant moments in the history of the subject. The most relevant references will be appearing later throughout the text in the course of exposition.

The critical point and intriguing anomalous features of the critical state were discovered in a series of studies by Charles Cagniard de Latour (1822), Michael Faraday (experiments with gas liquefaction), Dmitriy Mendeleev in his experiments with $Si\,Cl_4$ (1860) and Thomas Andrews with $C\,O_2$ (1869) with the work of the latter being the most celebrated one.

[1] Here and below, we do not attempt to give a detailed historical overview and refer the reader to the monographs and reviews [2–16] for the details and references.

Such systems share a characteristic feature which is that the correlation length diverges near critical point despite the microscopic interactions being short-range. It means that the precise nature of the systems (whether they correspond to liquids or to solid states such as magnets and superconductors) becomes unimportant as their behavior is governed by highly universal power laws for correlation functions. Universality here refers to sole dependence on the "global" features (such as the systems' symmetries and dimensions). This makes it possible to develop the critical state theory for all those different systems instead of creating a new particular theory for every example of critical phenomena (which is, of course, a very important undertaking in itself).

The first systematic mean-field theory of the liquid–vapor transition was proposed by Van der Waals (1873). The common key features of numerous mean-field models were summarized by Landau (1937). Those models have clear physical background, predict the transition, but give inaccurate values for the critical exponents. This was made especially clear by Guggenheim (1943), who showed experimentally that the exponent β is closer (but not equal) to $1/3$ than to the mean-field value $1/2$. The ultimate evidence came from Onsager's exact solution of the two-dimensional Ising model in zero magnetic field (1942–1944).

The next step was related to the scaling hypothesis, most clearly formulated by Domb and Hunter, Widom, Kadanoff, Patashinski and Pokrovski. The formulation of Patashinski and Pokrovski, known as the fluctuation theory of critical phenomena, provided the field-theoretic formulation of the problem [12, 13].

One of the most important guiding phenomenological idea was that of universality: that the essential characteristics of critical behavior (such as critical exponents, normalized scaling functions and amplitude ratios) depend only on global characteristics of the system (such as symmetry and spatial dimension), but not on the "details" like material constants. This is expressed in the idea of "universality classes."

Leo Kadanoff provided a witty and illuminating interpretation of the scaling and universality hypotheses in terms of a certain block-spin construction [15]. It was employed by Kenneth Wilson in his formulation of the renormalization group, based on certain recursion relations [2]. An important step forward was made by Wilson and Fisher, who identified a formal small parameter $\varepsilon = 4 - d$, the difference between the upper critical dimension 4 (above which the mean-field theory works) and the physical spatial dimension d, see Ref. [17]. On the base of field-theoretic diagrammatic expansions, they showed that ε can serve as a small parameter in a regular systematic perturbation scheme [18].

Soon after that, J.C. Le Guillou, E. Brézin and J. Zinn-Justin revealed a close resemblance between the Wilsonian renormalization group and the one developed within the framework of quantum-field theory (QFT) designed to meet the needs of relativistic particle physics [19–21].

That allowed one to apply the well-developed techniques of QFT: functional methods (including various Schwinger–Dyson equations, Ward identities, Legendre transformations of generating functionals), various methods of calculation of higher order Feynman diagrams, renormalization theory and renormalization group (including composite operators), short-distance operator-product expansions, instan-

ton calculus and resummations of asymptotic series; see Refs. [4–7] for the details and the references.

As a result of this long history, the modern critical state theory relies on the field-theoretic renormalization group (RG). The most typical ϕ^4 model is written for an n-vector order parameter with $O(n)$-symmetry and describes the universality class of the most common equilibrium phase transitions. The number of components n and the space dimension d are the only global parameters, so that universal features of the critical behavior (critical exponents and universal scaling functions) depend only on them. The latter are calculated via perturbation theory where the role of the small parameter is played by $\varepsilon = 4 - d$, i.e., the deviation of the space dimension from its "logarithmic" value $d = 4$, when the coupling constant ("charge") becomes dimensionless. Above this value, the interaction becomes irrelevant and the critical exponents are given by the mean-field theory.

Near $d = 4$, the problem of finding the infrared (IR) asymptotic behavior is related to the problem of eliminating of ultraviolet (UV) divergences; see Sec. 1.23 in Ref. [7]. The latter is solved by the well-developed machinery of UV renormalization [19–21]. As a by-product, the RG equations are derived; the IR behavior is governed by its IR attractive fixed points.

In general, the choice of such small parameter depends on the model. In particular, for the ϕ^m interaction, the deviation from the logarithmic dimension $d = 2m/(m-2)$ plays that role. For many other models (turbulence, advection, non-local interactions) the logarithmicity is not achieved by changing d, and the analog of ε has different physical nature and is not necessarily related to d. Since the term "ε expansion" has become an idiomatic expression, we will use it in what follows for any expansion of this kind.

For certain numerous models, there can be several such parameters, which leads to generalized "multiple ε expansions"; see, e.g., Refs. [22–32] and references therein. In such models, the UV divergences manifest themselves as singularities in a set of those parameters; the corresponding renormalization procedure is based on the analytic regularization provided by E.R. Speer; see Refs. [4, 5, 21, 33].

In this connection, it is interesting to note that the quantum physics and the physics of critical phenomena were born and then evolved nearly simultaneously, and often the same key researchers were involved. The Mendeleev's periodic law, the striking evidence of the quantum nature of the World, was announced in 1869. Yang and Lee's work on the P-parity violation was published in 1955, soon after their paper on the "YL edge" that explained how critical points may arise in equilibrium statistical physics. The interest in the RG among the QFT community arose anew in the early 70s after the rediscovery of the asymptotic freedom in QCD. G. 't Hooft and M. Veltman proposed the dimensional regularization (1972). In a more general context, the application of functional methods and K. Symanzik's concept of Euclidean field theory revealed a close formal resemblance between the QFT and statistical physics and strongly influenced both the subjects. On the one hand, it gave a deeper mathematical understanding of QFT and provided some rigorous results (M. Kac, B. Simon, J. Glimm and A. Jaffe, K. Osterwalder and R. Schrader, T. Spencer, E. Nelson, J. Fröhlich, K. Gawędzki and A. Kupiainen and others). On the other hand, ideas of spontaneous symmetry breaking (Y. Nambu, B. Goldstone, G. Jona Lasinio), mass generation due to violation of

local gauge symmetry (J. Schwinger, P. Anderson, P. Higgs and others), analytic regularization and continuously varying spatial dimension (E.R. Speer, G. 't Hooft and M. Veltman) became cornerstone conceptions of the newborn theory of classical and quantum many-body strongly interacting particle systems.

By the early 1980s, the RG approach was largely complete and became the common language of physicists when discussing phase transitions and critical phenomena. In subsequent years, the development of the theory came out on a straight road: the previously obtained results were refined, new models were proposed, and new wide areas of its applicability were discovered, in particular, equilibrium critical dynamics [7, 34–36], turbulence and turbulent transport [37, 38], driven diffusive systems [39, 40], percolation [41], roughening of fluctuating surfaces, growth processes and propagation of fronts [42–46], strongly non-equilibrium phase transitions, diffusion-limited chemical reactions [47–51], random walks and anomalous diffusion in random and inhomogeneous media [52–59], systems with self-organized criticality [60–68]; not an exhaustive and constantly expanding list. A general discussion of these and other (e.g., quantum) problems and more references are given in the monograph [36].

In most semi-phenomenological formulations, these phenomena are described by stochastic differential equations for continuous (smoothened, coarse-grained) fields with multiplicative and/or additive random sources (noises, forces, velocities). The key point for the applicability of the field-theoretic RG approach is the general statement that such stochastic equations can be reformulated as certain field-theoretic models for extended sets of fields; see the original references [69–75] and Chapter 5 in Ref. [7] for the review, proof and discussion. This fact opens the possibility to apply to these models the aforementioned field-theoretic methods.

Although the renormalization theory was initially developed for local relativistic quantum-field models and their Euclidean analogs [19–21], no doubt remains about its applicability to non-relativistic models of such kind: it is supported by numerous higher order multi-loop calculations in models of equilibrium critical dynamics, turbulence and turbulent transport, growth processes, polymers, etc.

Of course, it is impossible to cover the whole subject in a single review paper. Here, we selected several topics for the reason that they are essentially different from the ideal paradigmatic example of RG theory, the $O(n)$-ϕ^4_d model in $d = 4 - \varepsilon$ and require additional methods and ideas. On the other hand, these issues reflect our personal interests, opinions and contributions.

In Sect. 2, we explore the meaning of the functional integral in field-theoretic models originated from stochastic processes. The notion of functional integral has rigorous definition only in some special cases, like the Feynman–Kac formula for a particle in imaginary time; see, e.g., Ref. [76], Sec. 5.1.8.

In real QFT models, the only reliable interpretation (and hence the definition) of the functional integral is due to the perturbation expansion around the Gaussian (free) theory, with subsequent renormalization of the coefficients, establishing the asymptotic nature of the resulting series and Borel-type resummation of the latter.

However, even for the relativistic QFT models, the formulation of the original symbol of functional integral and the choice of the action functional is a matter of careful discussion, especially for interactions with derivatives and gauge theories; see, e.g., Refs. [7, 21] and the references therein.

In the functional-integral representation of stochastic differential equations, numerous additional subtleties arise that require special analysis. In Sect. 2 of the present paper, we discuss detailed derivation of the functional integral based on the mathematically consistent integral representation of the Langevin equation with multiplicative noise. Relation between the two most popular interpretations due to Itô and Stratonovich is reviewed and the corresponding action functionals are presented for the case of multiplicative white noise.

Sections 3 and 4 are devoted to the fully developed hydrodynamic turbulence and turbulent advection. In comparison with the RG theory of critical behavior, the RG theory of turbulence has a longer, and probably for this reason, less happy history. The celebrated Kolmogorov scaling was established in 1941, while the first serious studies within the RG approach appeared only at the end of 1970s. The RG was applied to the problem in a variety of formalisms which hinders mutual understanding of specialists working in the area.

A detailed systematic exposition of the RG approach in terms of the standard field-theoretic RG was given in the monograph [37], including advection of scalar fields, magnetohydrodynamic turbulence, effects of anisotropy and Langmuir turbulence of plasma. It was also argued that analysis of some specific features (sweeping effects, infrared singularities, anomalous multiscaling) requires to combine the "default RG" with more advanced methods: renormalization of composite fields (composite operators in quantum-field terminology), short-distance operator-product expansion, and IR perturbation theory.

In Sect. 3, we address the issue of anomalous scaling in turbulence, not touched upon in our elder book [37]. This phenomenon should not be confused with the anomalous scaling in critical behavior. There, the IR asymptotic form of various correlation functions is usually determined by just two principal exponents η and ν. Here, the inertial-range behavior of higher order structure functions is described by infinite set of independent exponents, hence the term "multiscaling." This distinction was sometimes opposed to equilibrium scaling and considered as an argument against applicability of the RG to the turbulence on the whole. Indeed, our study showed that the RG taken in the narrow sense of the word is not sufficient to the analysis of the problem [77, 78].

However, augmented with the operator-product expansion, it justifies the anomalous scaling in the related problem of passive scalar advection with various kinds of velocity ensembles, from the Kraichnan's rapid-change model to the full-scale stochastic Navier–Stokes equation. The anomalous exponents are determined by *negative* scaling dimensions of certain "dangerous" composite operators, which allow one to calculate them in a systematic perturbative expansions, analogous to ε expansions in the RG theory of critical state; see Refs. [78–82] and the references in Sect. 3.

Section 4 shows how the RG approach can be used for practical calculations of such representative quantities as the Kolmogorov constant and the skewness factor. To eliminate the uncertainty in calculating the Kolmogorov constant associated with the choice of external noise in the stochastic Navier–Stokes equation that ensures the stationarity of the turbulent system, some universal relation between it and the skewness factor is used. In addition, using the results of RG analysis of two-point and three-point correlation functions in the energy balance equation allow us to find the

kinetic energy spectrum in the energy-containing interval of decaying turbulence and, thus, to calculate the Kolmogorov constant differently.

Section 5 is devoted to specific non-equilibrium models commonly known as reaction–diffusion processes. We briefly discuss Doi–Peliti approach that allows a straightforward derivation of field-theoretic action starting from microscopic considerations. In Sect. 5.1 as a concrete example, we consider well-known directed percolation process. Furthermore, in Sect. 5.2, we discuss particular irreversible reaction schemes that displays intriguing scaling behavior.

Limited by the length of this paper, we could not include discussion of many interesting problems and related nontrivial effects; we tried to remedy that in the Conclusion. Renormalizable models with infinitely many couplings, violation and asymptotic emergence of symmetries, dimensional transmutation, magneto-hydromagnetic turbulence and turbulent dynamo, and induced nonlinear interaction are among those issues we briefly touched upon there.

2 Functional formulation of Langevin equation with multiplicative noise

The most popular description of near-equilibrium and non-equilibrium dynamics in the realm of classical physics is based on the Langevin equation

$$\partial_t \varphi = -K\varphi + U(\varphi) + B(\varphi) f. \tag{1}$$

where φ stands for a generic field (scalar, vector, tensor; a condensed notation will be used, in which indices and sums over them are implied unless required to avoid misunderstanding). On the right side of (1) K usually is a second-order spatial differential operator acting on φ and $U(\varphi)$, $B(\varphi)$ are polynomial time-local (depending on φ at a single time instant) functionals of φ. Function f is a representative of a random process with zero-mean Gaussian distribution. In most cases B is independent of φ (additive noise). However, in the present review we consider models, in which the coefficient B is a functional of φ (multiplicative noise). Therefore, functional formulation for the stochastic differential equation (SDE) (1) will be reviewed in the generic case.

In case of multiplicative white noise SDE (1) is not well defined and it is not obvious that the customary construction [7, 73] of functional formulation in the continuum form is valid. Therefore, we consider the Langevin equation with multiplicative noise in the integral form

$$\varphi(t) = \varphi(0) + \int_0^t [-K\varphi + U(\varphi)]\,\mathrm{d}u + \int_0^t B(\varphi)\mathrm{d}F,$$

defined by the measure in the stochastic integral (last term of right side). In the white-noise case the external field related to the measure dF is the increment of a representative of a Wiener process, which is a continuous function but not differentiable anywhere. Introducing discrete-time variable we obtain a Volterra-type integral equation with a stochastic integral sum:

$$\varphi_N = \varphi_0 + \sum_{n=1}^{N} \left\{ \left[-K\overline{\varphi}_n + U\left(\overline{\varphi}_n\right) \right] \Delta t_n + B\left(\overline{\varphi}_n\right) \Delta F_n \right\}, \tag{2}$$

where $\Delta t_n = t_n - t_{n-1}$, $\varphi_n \equiv \varphi(t_n)$, $\overline{\varphi}_n = \vartheta\varphi_n + (1-\vartheta)\varphi_{n-1}$, $0 \le \vartheta \le 1$. The measure of the stochastic integral is $\Delta F_n = W_n - W_{n-1}$ in case of a Wiener process and $\Delta F_n = f_n \Delta t_n$ in case of a process with less wild variations.

The form of the integral sum in (2) is chosen to remind the stochastic integral [83] both in the form due to Itô [84] ($\vartheta = 0$) and due to Stratonovich [85] ($\vartheta = 1/2$). Both choices of discretization have been used in construction of approximations to functional integral [72, 86]. In case of Stratonovich integral, the functional $B(\varphi)$ is assumed to have a continuous derivative with respect to φ [85], which is always the case in renormalized field theories. The stochastic integral with Wiener-process increments depends on the choice of ϑ [87]. Formal manipulations are much simpler in case of Itô integral than that of Stratonovich and—more importantly—lead to approximate generating function of solutions automatically reproducing just the iteration solutions. Therefore, we will carry out derivation with the choice $\vartheta = 0$ and comment on the case $\vartheta = 1/2$ separately.

It is convenient to replace Eq. (2) by the equivalent system ($n = 1, \ldots, N$)

$$\varphi_n = \varphi_{n-1} + \left[-K\varphi_{n-1} + U\left(\varphi_{n-1}\right) \right] \Delta t_n + B\left(\varphi_{n-1}\right) \Delta F_n. \tag{3}$$

This set of equations can be solved by iterations and the result conveniently expressed in terms of Feynman diagrams. Inverse of the operator acting on φ in the free-field part

$$M_{0,nm}(\boldsymbol{x} - \boldsymbol{x}') = \left[\delta_{n\,m} - \delta_{n-1\,m}(1 - K\,\Delta t_n) \right] \delta(\boldsymbol{x} - \boldsymbol{x}')$$

is the propagator and the nonlinear terms on right side of (3) give rise to vertex factors [7, 76]. The iteration solution consists of connected diagrams without loops. In quantum-field theory, a similar tree-graph sum gives the leading contribution in calculation of functional integral in loop expansion [7, 88]. Assuming spatial translation invariance with the use of the Fourier transform the propagator can be expressed as

$$\Delta_{12,mn}(\boldsymbol{x}, \boldsymbol{x}') = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} \exp\left[-i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}') \right] \left\{ \sum_{l=1}^{N} \delta_{m\,n+l-1} \left[1 - K(\boldsymbol{k})\Delta t \right]^{l-1} \right\}, \tag{4}$$

where all time increments are put equal and $K(\boldsymbol{k})$ stands for the operator of the free-field equation in the wave-vector space. Note that all main diagonal elements are equal to the unity and elements above the main diagonal are equal to zero.

We cast the generating function of solutions $\varphi_n[F]$ of the system (3) in functional form. To this end we introduce a multifold integral into the definition using the standard trick [7, 73]

$$G(A,F) = \exp\left(\sum_{n=1}^{N} A_n \varphi_n[F]\Delta t_n\right)$$
$$= \int \prod_{n=1}^{N} \mathrm{d}\varphi_n \, \delta\left(\varphi_n - \varphi_n[F]\right) \exp\left(A_n \varphi_n \Delta t_n\right). \tag{5}$$

A change of variables in the (ordinary with respect to time) δ functions brings about the kinetic equation explicitly in the integrand:

$$G(A,F) = \int \prod_{n=1}^{N} \frac{\mathrm{d}\varphi_n \mathrm{d}\tilde{\varphi}_n}{2\pi i} \,|\det M| \exp\Big(\tilde{\varphi}_n\Big\{-\varphi_n + \varphi_{n-1} + \big[-K\varphi_{n-1} + U\left(\varphi_{n-1}\right)\big]\Delta t_n$$
$$+B\left(\varphi_{n-1}\right)\Delta F_n\Big\} + A_n\varphi_n\Delta t_n\Big). \tag{6}$$

Here, the Fourier representation of the δ function has been introduced (with imaginary $\tilde{\varphi}_n$). Due to the prepoint choice of the time instants in the integral sum (2) the determinant of the corresponding Jacobi matrix

$$M_{nm} = \delta_{n\,m} - \delta_{n-1\,m}\Big\{1 + \big[-K + U'(\varphi_{n-1})\big]\Delta t_n + B'\left(\varphi_{n-1}\right)\Delta F_n\Big\}$$

is equal to the unity. There is no consistent generic definition of functional integral of fields except (renormalized) perturbation theory [88]. Therefore, we use another representation of generating functional (6), which gives rise to the perturbation expansion with well-defined building blocks: the S-matrix functional [76]:

$$G(A,F) = \exp\left(\sum_{m,n=1}^{N} \frac{\delta}{\delta\varphi_m}\Delta_{12,mn}\frac{\delta}{\delta\tilde{\varphi}_n}\right)\exp\sum_{n=1}^{N}\Big\{\tilde{\varphi}_n\big[U\left(\varphi_{n-1}\right)\Delta t_n$$
$$+B\left(\varphi_{n-1}\right)\Delta F_n\big] + A_n\varphi_n\Delta t_n\Big\}\Big|_0, \tag{7}$$

where $|_0$ stands for $|_{\tilde{\varphi}=\varphi=0}$ and this notation will be used henceforth. In (7), the functional differential operator (*reduction operator* in Refs. [7, 76])

$$\mathcal{P} = \exp\left(\sum_{m,n=1}^{N} \frac{\delta}{\delta\varphi_m}\Delta_{12,mn}\frac{\delta}{\delta\tilde{\varphi}_n}\right) \tag{8}$$

spans propagator lines $\Delta_{12,mn}$ between vertex factors, which are functional derivatives of the interaction functional $\sum_{n=1}^{N}\tilde{\varphi}_n\left[U\left(\varphi_{n-1}\right)\Delta t_n + B\left(\varphi_{n-1}\right)\Delta F_n\right]$ with respect to φ_m and $\tilde{\varphi}_n$. In relation (8), contractions are shown explicitly over the discrete-time variable only.

In the S-matrix functional (7), all building blocks are mathematically well defined and relation (7) in fact is the mathematically consistent definition of the functional

integral (6). In general, reduction operator (8) spans propagator lines not only between different vertex factors, which in case of (7) gives—among other things—rise to closed loops of several propagators, but also on single vertex factors giving rise to self-contracted propagators [76]. No such elements are present in the tree graphs of the iteration solution of (3). However, because of the choice of prepoint time instant in the integral sum in the interaction functional in (7) there is a time shift between the auxiliary field $\tilde{\varphi}_n$ and the basic field φ_{n-1}. As a consequence, closed loops of propagators (4) in fact vanish. For instance, for the self-contracted propagator, we obtain expression

$$\sum_{m,n=1}^{N} \frac{\delta}{\delta\varphi_m} \Delta_{12,mn} \frac{\delta}{\delta\tilde{\varphi}_n} \sum_{l=1}^{N} \tilde{\varphi}_l U\left(\varphi_{l-1}\right) \Delta t_l = \sum_{n=1}^{N} \Delta_{12,n-1\,n} \frac{\delta U\left(\varphi_{n-1}\right)}{\delta\varphi_{n-1}} \Delta t_n = 0, \tag{9}$$

since all matrix elements of the triangular propagator (4) above the main diagonal are equal to zero. A closed loop contraction of propagators with the "retarded" interaction gives rise to a nilpotent matrix with vanishing diagonal elements as well. Moreover, from analysis of diagrams of perturbation theory it is known that $\ln G(A, F)$ consists of connected graphs of $G(A, F)$ [76]. Due to definition (5) the coefficient of A in $\ln G(A, F)$ is the graphical solution of equations (3), which thus consists of connected graphs. Therefore, the S matrix functional (7) yields the generating functional in the form of a sum of connected tree graphs—the same as in the iteration solution [50].

In the continuum limit, expression (7) gives rise to the S-matrix functional corresponding to the solution of integral equation (2). However, expression for the functional integral in the continuum limit is not just a replica—up to notation—of that in the discrete approximation. In the discrete-time approximation diagrams with propagator loops are absent. In the ordinary integral sums of the diagrams the continuum limit does not create new graphs or make vanish the existing ones. Therefore, interaction functional in the continuum limit must be such that it does not generate graphs with closed loops of propagators.

In the operator formalism of quantum-field theory (QFT), the Hamilton operator is usually written in the normal-product form, which implies absence of self-contracted propagators in perturbation theory [19]. In presentation of critical dynamics and stochastic field theory, this approach of discarding graphs with self-contracted propagators has been adopted by Vasil'ev [7]. In the functional formulation of QFT to this corresponds the use of the reduced interaction functional (also called the normal form of the interaction functional) in which all self-contractions are explicitly taken into account and, therefore, only contractions between reduced interaction functionals appear in graphs of perturbation theory [76]. Self-contracted propagators are produced by the (continuum) reduction operator (8), therefore, the reduced interaction functional obeys $\mathcal{P}F = F_{\mathrm{red}}$ and in terms of unrestricted functionals is expressed as

$$F(\varphi, \tilde{\varphi}) = \exp\left(-\frac{\delta}{\delta\varphi} \Delta_{12} \frac{\delta}{\delta\tilde{\varphi}}\right) F_{\mathrm{red}}(\varphi, \tilde{\varphi}). \tag{10}$$

Thus, in terms of unrestricted functionals, the counterpart of (7) in the continuum limit assumes the form [50]

$$G(A,F)=\exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left\{\int\left[\tilde{\varphi}U\left(\varphi\right)-\Delta_{12}(0)U'(\varphi)+A\varphi\right]\mathrm{d}t\right.$$
$$\left.+\int\left[\tilde{\varphi}B\left(\varphi\right)-\Delta_{12}(0)B'(\varphi)\right]\mathrm{d}F\right\}\Big|_0, \tag{11}$$

Here, we have used the shorthand notation

$$\int\Delta_{12}(0)U'(\varphi)\mathrm{d}t=\int\mathrm{d}t\iint\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\Delta_{12}(t,\boldsymbol{x};t,\boldsymbol{y})\frac{\delta U(\boldsymbol{x},\varphi)}{\delta\varphi(t,\boldsymbol{y})},$$
$$\int\Delta_{12}(0)B'\left(\varphi\right)\mathrm{d}F=\iiint\mathrm{d}\boldsymbol{x}d\boldsymbol{y}\mathrm{d}z\int\Delta_{12}(t,\boldsymbol{x};t,\boldsymbol{y})\frac{\delta B(\boldsymbol{y},z,\varphi)}{\delta\varphi(t,\boldsymbol{x})}\mathrm{d}F(t,z). \tag{12}$$

Since $\Delta_{12}(t,\boldsymbol{x},t,\boldsymbol{x}')\propto\delta(\boldsymbol{x}-\boldsymbol{x}')$, for local in space functionals $U(\varphi)$ or $B(\varphi)$ the right side of relations (12) is ill defined. This has brought about doubts that the dynamic theory is not determined by perturbation theory completely [6]. As shown here, the terms with $\Delta_{12}(0)$ apparent in (11) are actually absent—with their UV divergences—in perturbation theory.

In (11), the interaction functional is strictly time local and the propagator matrix (4) is replaced by the retarded Green function of the diffusion operator ∂_t+K. Therefore, all Feynman diagrams with closed loops of two or more propagators vanish in the subsequent perturbation expansion, Thus, the S-matrix functional (11) faithfully reproduces results of iteration solution. In particular, there are no self-contracted loops in perturbation theory generated by it. That the diagonal value of the propagator $\Delta_{12}(0)$ appears explicitly in (11) is an artifact of the continuum functional representation, which—contrary to the prepoint discrete approximation (7)—does not automatically reflect properties of the iteration solution of the kinetic equation. Originally the apparent ambiguity in (11) was fixed by condition $\Delta_{12}(0,\boldsymbol{x}-\boldsymbol{x}')=\frac{1}{2}\delta(\boldsymbol{x}-\boldsymbol{x}')$ [70, 73]. This approach has been adopted in presentation of dynamic critical phenomena by Zinn-Justin [6] as well. The continuum functional formulation can be simplified by the prescription $\Delta_{12}(0)=0$ [37, 75]. Both cases have sometimes been described as the choice of the value of the temporal step function at the origin, although in the iteration solution there is no such ambiguity.

2.1 Multiplicative Wiener-process noise

Dependence of the S-matrix functional (7) on the elements of the multiplicative stochastic process F is contained in the exponential only (in case of $\vartheta>0$ the Jacobi determinant would have included them as well making calculation much more elaborate). In case of the Wiener process with the Itô integral sum, the expectation value of the factor in (7) containing elements of the stochastic process is

$$\left\langle \exp\left[\sum_{n=1}^{N}\tilde{\varphi}_n B\left(\varphi_{n-1}\right) S\left(W_n - W_{n-1}\right)\right]\right\rangle_W = \exp\left[\frac{1}{2}\sum_{n=1}^{N}\tilde{\varphi}_n B\left(\varphi_{n-1}\right) DB^{\intercal}\left(\varphi_{n-1}\right)\tilde{\varphi}_n \Delta t_n\right], \tag{13}$$

where the Wiener process is normalized as

$$\left\langle\left[W_n(\boldsymbol{x}) - W_{n-1}(\boldsymbol{x})\right]\left[W_n(\boldsymbol{x}') - W_{n-1}(\boldsymbol{x}')\right]\right\rangle_W = (t_n - t_{n-1})\delta(\boldsymbol{x} - \boldsymbol{x}').$$

The possible nontrivial spatial correlation function $D(\boldsymbol{x} - \boldsymbol{x}')$ of noise is expressed through the factor S: $D = SS^{\intercal}$. Relation (13) gives rise to representation

$$G(A) = \langle G(A, SW)\rangle_W = \exp\left(\sum_{m,n=1}^{N}\frac{\delta}{\delta\varphi_m}\Delta_{12,mn}\frac{\delta}{\delta\tilde{\varphi}_n}\right)\exp\sum_{n=1}^{N}\left\{\tilde{\varphi}_n U\left(\varphi_{n-1}\right) + \frac{1}{2}\tilde{\varphi}_n B\left(\varphi_{n-1}\right) DB^{\intercal}\left(\varphi_{n-1}\right)\tilde{\varphi}_n + A_n\varphi_n\right\}\Delta t_n\Big|_0. \tag{14}$$

Calculation similar to (9) reveals that the new interaction term $\tilde{\varphi}_n B\left(\varphi_{n-1}\right) DB^{\intercal}\left(\varphi_{n-1}\right)\tilde{\varphi}_n$ does not generate self-contracted loops or closed loops of several propagators, so that these are absent in perturbation theory brought about by generating functional (14) as well. Feynman diagrams of perturbation theory generated by (14) have unambiguous continuum limit and define the continuum perturbation theory of the model. The formal continuum limit of expression on the right side of (14) generates diagrams of this perturbation expansion without closed loops of several propagators because of the retarded propagator. Absence of self-contracted propagators requires casting the interaction functional to the normal form according to relation (10). This has to be carried out separately, because averaging over noise has changed the interaction functional. The result has significantly more complicated than in equation (11) apparent dependence of the S-matrix functional on the ambiguous quantity $\Delta_{12}(0)$, therefore we do not quote it here, but indicate the absence of self-contracted propagators in perturbation theory either in the reduction operator of the S-matrix functional

$$G(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta'_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\int \mathrm{d}t\left[\tilde{\varphi}\,U(\varphi) + \frac{1}{2}\tilde{\varphi}B\left(\varphi\right) DB^{\intercal}\left(\varphi\right)\tilde{\varphi} + A\varphi\right]\Big|_0, \tag{15}$$

where the prime has the meaning of discarding all diagrams with self-contracted propagators, or in the dynamic action of the corresponding functional integral

$$G(A) = \iint \mathcal{D}\varphi\mathcal{D}\tilde{\varphi}\exp\int \mathrm{d}t\left\{-\tilde{\varphi}\left(\partial_t\varphi + K\varphi\right) + \left[\tilde{\varphi}U\left(\varphi\right) + \frac{1}{2}\tilde{\varphi}B\left(\varphi\right) DB^{\intercal}\left(\varphi\right)\tilde{\varphi}\right]_{\mathrm{red}} + A\varphi\right\}, \tag{16}$$

where the subscript "red" reminds that there are no graphs with self-contracted propagators at the vertex of the interaction term. It should be noted that in the literature the appearance of the reduced interaction in functional integral (16) is usually not mentioned or substituted by the rule $\Delta_{12}(0) = 0$. The choice of retarded propagator can be included in the definition of space of integration for Gaussian integrals of perturbation theory [76].

The interaction terms brought about by averaging over noise are local in time, but they can be non-local in space. Basic tools of UV renormalization are designed for local interactions [7, 20] so that further transformations of the dynamic action might be useful to carry out the renormalization. In that case applicability of the implementation rule $\Delta_{12}(0) = 0$ must be checked separately with the use of the discrete approximation [89].

2.2 Multiplicative colored noise

In case of colored noise with realizations of bounded variation, the stochastic integral can be understood as ordinary integral. It is, however, convenient to use the prepoint integral sum in this case as well to maintain the correspondence between iteration solution and perturbation expansion. Instead of Eq. (14), the expectation value of the generating functional is obtained in the form

$$G(A) = \exp\left(\sum_{m,n=1}^{N} \frac{\delta}{\delta\varphi_m}\Delta_{12,mn}\frac{\delta}{\delta\tilde{\varphi}_n}\right)\exp\left\{\sum_{n=1}^{N}\left[\tilde{\varphi}_n U\left(\varphi_{n-1}\right) + A_n\varphi_n\right]\Delta t_n + \frac{1}{2}\sum_{m,n=1}^{N}\tilde{\varphi}_m B\left(\varphi_{m-1}\right)\overline{D}_{mn}B^{\intercal}\left(\varphi_{n-1}\right)\tilde{\varphi}_n\Delta t^2\right\}\Big|_0, \tag{17}$$

where $\overline{D}$ is the correlation function of the colored noise. As in the case of Wiener process, due to the incremental time shift between the fields in factors $\tilde{\varphi}_n B\left(\varphi_{n-1}\right)$ of the new interaction term all diagrams with closed loops of propagators containing these vertex factors vanish—including self-contracted propagators—in the perturbation expansion brought about by discrete functional (17). In the continuum limit this property disappears and vanishing of closed loops of propagators takes place due to the emerging retarded propagator.

Absence of self-contracted propagators in the continuum limit is secured by the reduced interaction functional. In terms of unrestricted functionals the expression for the S-matrix functional is (lengthy spatial contractions are implied in the order of factors, time integrals over the arguments of fields and the correlation function are indicated explicitly) [50, 90]

$$G(A) = \exp\left(\frac{\delta}{\delta\varphi}\Delta_{12}\frac{\delta}{\delta\tilde{\varphi}}\right)\exp\left(\int \mathrm{d}t\left\{\left[\tilde{\varphi}U\left(\varphi\right) - \Delta_{12}(0)U'(\varphi) + A\varphi\right] + \int \mathrm{d}t\int \mathrm{d}t'\left[\frac{1}{2}\tilde{\varphi}B\overline{D}B^{\intercal}\tilde{\varphi} - \Delta_{12}(0)B'\overline{D}B^{\intercal}\tilde{\varphi}\right]\right\}\right)\Big|_0. \tag{18}$$

It should be noted that the expression for the interaction functional is simplified due to vanishing of closed loops of two retarded propagators; in case of Wiener noise, this does not happen. In terms of reduced interaction functional, the corresponding functional integral is

$$G(A) = \int\!\!\int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \exp\Bigg(\int \mathrm{d}t \Big\{ -\tilde{\varphi}\left(\partial_t \varphi + K\varphi\right) + \left[\tilde{\varphi} U\left(\varphi\right)\right]_{\mathrm{red}} + A\varphi \Big\} + \int \mathrm{d}t \int \mathrm{d}t' \left[\frac{1}{2} \tilde{\varphi} B \overline{D} B^{\mathsf{T}} \tilde{\varphi} + \tilde{\varphi} B' \overline{D} \Delta_{12} B^{\mathsf{T}} \right]_{\mathrm{red}} \Bigg). \tag{19}$$

Note the additional (in comparison with (16)) reduced interaction term in the action functional.

Expression (19) readily yields functional-integral representation for the generating function of the Stratonovich SDE with the use of the mathematical theorem that the solution of the SDE in the Stratonovich interpretation is obtained in the white-noise limit of colored noise [91]. To this end, let us introduce a set of correlation functions consisting of a δ sequence in time, i.e.,

$$\langle f(t,\boldsymbol{x}) f(t',\boldsymbol{x}') \rangle = \overline{D}(t,\boldsymbol{x};t',\boldsymbol{x}') \to \delta(t-t') D(\boldsymbol{x},\boldsymbol{x}') \tag{20}$$

and pass to the white-noise limit in (19). In this limit, the contraction of the propagator and correlation function behaves as $\overline{D}\Delta_{12} \to \frac{1}{2} D$ and the functional integral for the generating functional of solutions of the Stratonovich SDE is

$$G(A) = \int\!\!\int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \exp \int \mathrm{d}t \Big\{ -\tilde{\varphi}\left(\partial_t \varphi + K\varphi\right) + A\varphi + \left[\frac{1}{2} \tilde{\varphi} B D B^{\mathsf{T}} \tilde{\varphi} + \tilde{\varphi} U\left(\varphi\right) + \frac{1}{2} \tilde{\varphi} B' D B^{\mathsf{T}} \right]_{\mathrm{red}} \Big\} \tag{21}$$

with a time-local reduced action functional [50] different from that (16) of the Itô SDE, when $B' \neq 0$. The difference between dynamic actions (16) and (21), of course, corresponds to that between the two Fokker–Planck equations [87].

3 Fully developed turbulence, turbulent advection and anomalous multiscaling behavior

Turbulence has famously been called the last unsolved problem of classical physics, thus, it is notable that fully developed hydrodynamic turbulence (in its homogeneous, isotropic, inertial-range form) can be studied with the RG analysis [92–94].

The most typical objects of study are the equal-time structure functions:

$$S_n(r) = \langle [\theta(t,\boldsymbol{x}) - \theta(t,\boldsymbol{x}')]^n \rangle = (\bar{\mathcal{E}} r)^{n/3} f_n(r/l, r/L), \quad r \equiv |\boldsymbol{x} - \boldsymbol{x}'|. \tag{22}$$

The field $\theta = \theta(t, \boldsymbol{x})$ stands for a relevant field: velocity vector, temperature scalar, impurity concentration scalar, etc. The brackets $\langle \dots \rangle$ are for the ensemble averaging. We also assume dependence on time for the fields. The second relation is immediately given by dimensions; $\bar{\mathcal{E}}$ stands for the mean energy dissipation rate, l is the internal (Kolmogorov, dissipation, UV) length, L is the external (outer, integral, IR) turbulence scale and f_n are certain scaling functions.

We are primarily interested in the structure functions (22) behavior in the inertial range $l \ll r \ll L$, where details of the external energy pumping and its following dissipation are irrelevant. Celebrated Kolmogorov–Obukhov (K41) phenomenological theory [92–94] states that the behavior is independent of both the external L and internal l scales (the first and the second Kolmogorov hypotheses, respectively) and solely determined by the parameter $\bar{\mathcal{E}}$. Dimensions analysis then implies

$$S_n(r) \simeq (\bar{\mathcal{E}} r)^{n/3}. \tag{23}$$

In particular, the energy conservation law yields relation $S_3(r) = -(4/5)(\bar{\mathcal{E}} r)$, a firmly established theoretical result.

However, these predictions do not take into account the intermittency (strong fluctuations of the energy flux $\mathcal{E}$); the corrections are referred as "anomalous" scaling, a singular dependence of the correlation functions on the integral scale L:

$$S_n(r) \simeq (\bar{\mathcal{E}} r)^{n/3} \, (r/L)^{q_n}, \tag{24}$$

where the anomalous exponents q_n are certain nontrivial functions of n ($q_3 = 0$).

Various semi-heuristic models associate these exponents with statistical properties of the local dissipation rate, nontrivial topology, etc.; see Refs. [92–94]. However, the models often lack strong hydrodynamic foundation and involve arbitrary adjusting parameters which leaves much to be desired; see e.g., the remark in Ref. [95]. Thus, the question of the anomalous exponents universality and deviations from the K41 theory remains open.

In Refs. [37, 77], the second Kolmogorov hypothesis was justified via the RG approach that established the representation $S_n(r) \simeq (\bar{\mathcal{E}} r)^n \, f_n(r/L)$ with the caveat that explicit form of scaling functions $f_n(r/L)$ is left out of reach. But to deal with the first Kolmogorov hypothesis and the anomalous scaling (24), the standard RG technique has to be broadened to include composite fields or "operators" and short-distance operator-product expansions (OPE). This augmentation results in the following representation:

$$f_n(r/L) = \sum_F C_{nF} \, (r/L)^{-\Delta_F}, \tag{25}$$

where C_{nF} are the coefficients that do no have singularities in (r/L), summation over all possible composite operators F allowed by the symmetry is implied, and Δ_F are their scaling dimensions. It is composite operators with negative scaling dimensions, $\Delta_F < 0$, that play the most important role, thus, they were coined "dangerous" in

Refs. [37, 77]. Conventional models of critical behavior, e.g., ϕ^4, are usually devoid of operators like that.[2]

When developed turbulence is studied with the RG approach, equations of stochastic hydrodynamics are constructed for velocity/temperature/density fields where the source of stochasticity is random stirring forces. Exponents in the correlation functions of the forces become small parameters of the perturbation theory, a part played by $\varepsilon = 4 - d$ in the ϕ^4 model; see Refs. [37, 77]. The corresponding expansions, however, cannot reveal dangerous composite operators for the Navier–Stokes field theory because they arise only for finite values of ε. Complicating the matters further, if these operators exist, they arise in the OPE as infinite families characterized by spectrum of scaling dimensions unbounded from below (for the proof, see, e.g., Appendix A in Ref. [96]). To obtain the small-(r/L) behavior in the form of representations like (25), their contributions must be summed.

Therefore, when studying Galilean-invariant quantities (e.g., equal-time structure functions (22)), scaling dimensions of all Galilean-invariant composite operators must be found by going beyond the ε series and then their contributions have to be summed. This highly complicated problem requires significant development of the current methods.

Nevertheless, for different-time correlation functions, this problem was successfully solved. In the work [77], the expression of the most singular contributions in representations (25) via the powers of the velocity field were found, and the scaling dimensions were obtained exactly through the Ward identities for Galilean symmetry. Therefore, it was possible to establish dependence of the correlation functions on L for $(r/L) \to 0$. In terms of physics, this is connected to the well-known phenomenon of sweeping of the small-scale vortices by the large-scale ones [97, 98].

Therefore, contrary to suspicions that RG approach does not take this effect into account due to its inadequacy [99, 100], the field-theoretic perturbative RG together with OPE and exact relations (derived from Schwinger–Dyson and Ward equations) are sufficient in that regard.

Moreover, models of passively advected scalar or vector fields are successfully studied with that approach because relations between the corresponding anomalous exponents in expressions similar to (24) and dimensions of individual operators can be established; see more on that below.

3.1 Scalar fields: turbulent mixing

On the one hand, numerical simulations and experiments show that the deviation from the K41 theory is more dramatic for passively advected scalar fields than for the velocity field; see, e.g., Refs. [101–105]. On the other hand, theoretical analysis of passive advection is more "profitable" because even simplified models based on "synthetic" velocity fields with prescribed statistics produce anomalous features of turbulent transport.

[2] They also should not be confused with "relevant operators in the sense of Wilson" [2] and with "dangerously irrelevant variables," see, e.g., Ref. [7], sec. 1.16 and Ref.[36], sec. 1.4.

The Kraichnan's rapid-change model can be considered one of the most important developments in the field that can be credited to Batchelor, Obukhov, Kraichnan and Kazantsev; it is a simplified model of the fully developed turbulence written for scalar quantity passively advected by the Gaussian velocity field. It is defined by the pair correlation function $\langle v_i v_j \rangle \simeq \delta(t-t')k^{-d-\xi}$, where k is the wave number and ξ is an arbitrary parameter. With this model, it was possible to establish the existence of anomalous scaling [106] and to obtain the corresponding anomalous exponents within controlled approximations [107, 108] and a regular perturbation scheme [78]. In terms of the inertial-range structure functions, it translates to the following anomalous scaling law:

$$S_{2n}(r) \simeq r^{n(2-\xi)}\,(r/L)^{\Delta_n}, \tag{26}$$

with negative anomalous exponents Δ_n:

$$\Delta_n = -2n(n-1)\xi/(d+2) + O(\xi^2) = -2n(n-1)\xi/d + O(1/d^2), \tag{27}$$

written here up to the first terms of the expansion in ξ [107] and $1/d$ [108], respectively. The relation $\Delta_1 = 0$ is exact.

The equal-time correlation functions for the local dissipation rate of scalar fluctuations, $E(x) = \kappa_0 \partial_i \theta(x) \partial_i \theta(x)$ (here κ_0 is the diffusivity coefficient), in the inertial range behave as [107, 108]

$$\langle E^n(x) E^p(x') \rangle \simeq r^{-\Delta_n - \Delta_p} (mr)^{\Delta_{n+p}}, \tag{28}$$

where the exponents Δ_n are those of (27).

Originally, these results were derived within the so-called "zero-mode" approach [107, 108]; see Ref. [105] for discussion. Another methodical approach combines the field theoretic RG and operator-product expansion (OPE) [78]. It proceeds in two stages. First, the field theory built for the model is analyzed in order to establish its renormalizability which allows to derive differential RG equations for the correlation functions. IR-attractive fixed points of the equations determine asymptotic behavior of the correlation functions in the IR range (UV argument (r/l) for $r \gg l$ and any fixed (r/L)). "Scaling functions" that depend on IR argument (r/L) remain undefined by the RG equations. Second, behavior at $r \ll L$ is given by the OPE and analysis of the RG equations general solution. Critical dimensions of various composite fields (or composite operators in the language of QFT) allow to determine whether there is an infinite family of independent scaling exponents (i.e., multiscaling).

The RG study of critical behavior has long since utilized this approach to study scaling functions at small-(r/L); see, e.g., Refs. [6, 7]. However, appearance of so-called "dangerous" operators with negative critical dimensions is unique to turbulence problems. The term is inspired by the fact that contributions of dangerous operators to the OPE diverge at $(r/L) \to 0$.

The application of OPE to the problems of stochastic hydrodynamics and the concept of dangerous operators were introduced in Ref. [77]; see Refs. [37, 38] and Chapter 6 in Ref. [7] for the NS case.

Analysis performed in Ref. [78] proved appearance of dangerous operators in the Obukhov–Kraichnan model; dimensions of the operators were calculated by constructing regular perturbation theory in the same way critical exponents are obtained through the ε expansion.

Since the original stochastic equations are linear in the passive field, a number of possible dangerous operators affecting a structure function is finite, so one can identify anomalous exponents with critical dimensions of composite operators.

The two-loop and three-loop calculations can be found in Refs. [78] and [79], respectively, where number of loop refers to orders in ξ. The effects of compressibility and anisotropy were studied in Refs. [80–82]; the vector advected field was considered in Refs. [109, 110].

The zero-mode approach focuses on second-order correlation functions and produces exact solutions [108, 111, 112], while the RG approach (OPE and composite operators) opens the door for systematic perturbative calculations [78–81]. The latter is more suitable for the cases with high difficulty of calculations such as anisotropic velocity field, passive advection of vector fields, and advection of extended objects [82, 109, 110].

This is what makes the RG approach particularly attractive as a tool to study more realistic models that account for finite correlation time and non-Gaussianity; see, e.g., Refs. [25, 113, 114]

3.2 Obukhov–Kraichnan ensemble: anomalous scaling

Passive scalar field $\theta(x) \equiv \theta(t, \boldsymbol{x})$ subject to turbulent advection can be modeled by stochastic equation

$$\nabla_t \theta = \kappa_0 \partial^2 \theta + f, \quad \nabla_t \equiv \partial_t + v_i \partial_i, \tag{29}$$

where $\partial_t \equiv \partial/\partial_t$, $\partial_i \equiv \partial/\partial_{x_i}$, $\partial^2 = \partial_i \partial_i$ (summation is implied), κ_0 is the molecular diffusivity coefficient. Velocity field $\boldsymbol{v}(x) \equiv v_i(x)$ is transverse: $\partial_i v_i = 0$. Gaussian random noise $f \equiv f(x)$ is defined by its correlation function

$$\langle f(x) f(x') \rangle = \delta(t - t') C(r/L), \quad r = |\boldsymbol{r}|, \quad \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}', \tag{30}$$

and assumption that its mean is equal to zero. Certain function $C(r/L)$ is finite at $(r/L) \to 0$, where L is integral scale; we can be put $C(0) = 1$ with no loss of generality.

Instead of using Navier–Stokes equation to describe evolution of $\boldsymbol{v}(x)$, we ascribe to the latter the following statistics:

$$\langle v_i(x) v_j(x') \rangle = D_0 \delta(t - t') \int_{k>m} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} P_{ij}(\boldsymbol{k}) \frac{1}{k^{d+\xi}} \exp[i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')] \tag{31}$$

with another assumption of with zero mean. Here $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |\boldsymbol{k}|$, $D_0 > 0$ is a positive amplitude, d is the x-space dimension,

and $0 < \xi < 2$ is an exponent with the most realistic ("Kolmogorov") value being $\xi = 4/3$.

There is an infrared (IR) regularization by the means of the cutoff at $k = m$ in (31), where $m \equiv 1/L$. Anomalous exponents are unaffected by choice of IR regularization which allows to use the cutoff that simplifies calculations.

Coupling constant g_0

$$D_0/\kappa_0 = g_0 = \Lambda^{\xi} \tag{32}$$

serves as the expansion parameter in the perturbation theory; here $\Lambda \equiv 1/l$ is the characteristic ultraviolet (UV) momentum scale.

3.2.1 Field-theory, UV singularities and renormalization

From the general point of view, the velocity field in (29) acts as a multiplicative random noise (see Sect. 2). From the physics viewpoints, the function (31) is an idealized model of a pair correlation function with a narrow width and high maximum, symmetric in t and t'. Thus, the stochastic differential equation (29) must be understood in the Stratonovich sense.

Then the stochastic problem (29)–(31) can be described by the field theory for the fields $\Phi = \{\theta', \theta, \boldsymbol{v}\}$ with the action functional

$$\mathcal{S}_\theta(\Phi) = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t \theta - (v_i \partial_i)\theta + \kappa_0 \partial^2 \theta\right] - \boldsymbol{v} D_v^{-1} \boldsymbol{v}/2. \tag{33}$$

The last term serves as the Gaussian averaging over $\boldsymbol{v}$ with correlation function (31) for De Dominicis–Janssen-type action [69] constructed for equations (29) and (30) at fixed $\boldsymbol{v}$. Here D_θ and D_v stand for the correlation functions (30) and (31); all necessary integrations over $x = (t, \boldsymbol{x})$ are implied and omitted for brevity.

Equivalence of field theory and stochastic problem implies that correlation and response functions of the latter (in the present case, (29)–(31)) can be identified with the Green functions of the former (here, (33)) produced by functional averages with the weight $\exp S(\Phi)$.

Feynman diagrammatic technique can be constructed for the theory (33); it consists of the triple vertex $-\theta'(v_i \partial_i)\theta$ and bare propagators

$$\begin{aligned} \langle \theta \theta' \rangle_0 &= \langle \theta' \theta \rangle_0^* = (-i\omega + \kappa_0 k^2)^{-1}, \quad \langle \theta' \theta' \rangle_0 = 0, \\ \langle \theta \theta \rangle_0 &= C(k)(\omega^2 + \kappa_0^2 k^4)^{-1}, \end{aligned} \tag{34}$$

where $C(k)$ is the Fourier transform of the function $C(r/L)$ in (30); the bare propagator $\langle \boldsymbol{v}\boldsymbol{v} \rangle_0$ is defined by Eq. (31).

By calculating canonical dimensions, one can identify UV divergences in Green's functions; see, e.g., Refs. [6, 7]. For the theory (33), it is appropriate to introduce two scales that determine canonical dimension of a quantity F: the momentum dimension d_F^k and the frequency dimension d_F^ω so that $[F] \sim [L]^{-d_F^k}[T]^{-d_F^\omega}$; here L is the length scale and T is the time scale. There are normalization conditions $d_k^k = -d_x^k = 1$, $d_\omega^k =$

Table 1 Canonical dimensions of the fields and the parameters in the model (33)

F	θ	θ'	$\boldsymbol{v}$	κ, κ_0	m, μ, Λ	g_0	g
d_F^k	0	d	-1	-2	1	ξ	0
d_F^ω	$-1/2$	$1/2$	1	1	0	0	0
d_F	-1	$d+1$	1	0	1	ξ	0

$d_x^\omega = 0, d_\omega^k = d_t^k = 0, d_\omega^\omega = -d_t^\omega = 1$. Action functional should be dimensionless with respect to each scale which allows to find canonical dimensions of all fields and parameters. Total canonical dimension is then defined as $d_F = d_F^k + 2d_F^\omega$ (factor 2 comes from free theory where $\partial_t \sim \partial^2$).

Table 1 contains canonical dimensions for model (32); renormalized parameters to be introduced later (without the subscript $_0$) are also there. Thus, the model is logarithmic (which makes coupling constant g_0 dimensionless) and the correlation functions diverge as poles in ξ at UV limit.

The total canonical dimension of a 1-irreducible Green function $\Gamma = \langle \Phi \dots \Phi \rangle_{1\text{-ir}}$ is $d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\boldsymbol{v}}\}$ are numbers of fields consisting the function. Formal index of UV divergence is given by the total dimension d_Γ taken at logarithmicity. If d_Γ is a non-negative integer then there are superficial UV divergences in the function Γ that renormalization procedure removes with counterterms.

Additional considerations should be taken into account; first, the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any non-zero function due to the form of the vertex and the bare propagators of model (33). Second, diagrams with closed circuits of retarded propagators $\langle \theta\theta' \rangle_0$ should be put to zero. Finally, the real index of divergence d'_Γ is less than d_Γ if powers of external momenta appear as a factor in every diagram.

The last consideration is relevant because the vertex $\theta'(v_i\partial_i)\theta$ involves derivative that can also be moved to the field θ' by integration by parts (we recall that $\partial_i v_i = 0$). Thus, $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'}$. Moreover, the fields θ and θ' are featured in the counterterms only under derivatives.

Taking all of that into account, we arrive at $d_\Gamma = d + 2 - N_{\boldsymbol{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)(1 - N_{\theta'}) - N_{\boldsymbol{v}}$ and conclusion that, for any d, superficial divergences are present only in the sole 1-irreducible function $\langle \theta'\theta \rangle$; precise form of the counterterm is $\theta'\partial^2\theta$. Renormalized action reads

$$S_R(\Phi) = \theta' D\theta'/2 + \theta'[-\partial\theta - (v_i\partial_i)\theta + \kappa Z_\kappa \partial^2\theta] - \boldsymbol{v} D_v^{-1} \boldsymbol{v}/2 \tag{35}$$

with renormalized parameters related to their bare counterparts via single renormalization constant Z_κ:

$$\kappa_0 = \kappa Z_\kappa, \quad g_0 = g\mu^\xi Z_g, \quad Z_g = Z_\kappa^{-1}. \tag{36}$$

Here, μ is the reference scale (mass) in the minimal subtraction (MS) scheme. Note that the fields and the cutoff value m are not renormalized, i.e., their corresponding renormalization constants are trivial $Z = 1$.

3.2.2 RG equations and functions, fixed points, and invariant variables

Let us derive RG equation:

$$D_{RG} W^R(g, \kappa, \mu, ...) = 0, \quad D_{RG} \equiv D_\mu + \beta(g)\partial_g - \gamma_\kappa(g) D_\kappa, \tag{37}$$

where W^R is renormalized Green's function that coincides with its unrenormalized counterpart (thus, the zero on r.h.s. of (37)) due to the fields not being renormalized; the dots stand for other arguments such as coordinates and momenta. Operators D_F are defined as $D_F \equiv F\partial_F$ for any variable F, while D_{RG} is operator $\widetilde{D}_\mu$ (or $\mu\partial_\mu$ taken at fixed bare parameters g_0, κ_0) expressed in renormalized variables. RG functions (β function and anomalous dimension γ_κ) are defined as

$$\gamma_\kappa(g) \equiv \widetilde{D}_\mu \ln Z_\kappa, \quad \beta(g) \equiv \tilde{D}_\mu g = g[-\xi + \gamma_\kappa]. \tag{38}$$

Anomalous dimension γ_κ appears in β function due to the last relation in (36).

The constant Z_κ eliminates divergence from the exact response function $G \equiv \langle \theta\theta' \rangle$ at $\xi = 0$ so it can be obtained from the Dyson equation

$$G^{-1}(\omega, k) = -i\omega + \kappa_0 k^2 - \Sigma_{\theta'\theta}(\omega, \boldsymbol{k}), \tag{39}$$

where $\Sigma_{\theta'\theta}$ is the self-energy operator.

Due to the form of the vertex and the bare propagators (31), (34), any multi-loop diagram for the function $\Sigma_{\theta'\theta}$ contains a closed circuit of retarded propagators $\langle \theta\theta' \rangle_0$ (this is facilitated by the propagator $\langle \boldsymbol{v}\boldsymbol{v} \rangle_0$ in (31) containing δ function in time as a factor). Thus, one-loop approximation gives the whole function $\Sigma_{\theta'\theta}$ and resulting expression for Z_κ is exact, i.e., has no corrections g^2, etc:

$$Z_\kappa = 1 - \frac{g(d-1)C_d}{2d\xi}. \tag{40}$$

Here, $C_d \equiv S_d/(2\pi)^d$ and $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in the d-dimensional space.

The exact expressions for the RG functions are

$$\gamma_\kappa(g) = \frac{g(d-1)C_d}{2d}, \quad \beta(g) = g\left[-\xi + \frac{g(d-1)C_d}{2d}\right]. \tag{41}$$

An IR-attractive fixed point of the RG equations is a point that turns β function to zero and its first derivative to positive number; here, it is

$$g_* = \frac{2d\xi}{C_d(d-1)} \tag{42}$$

with $\beta(g_*) = 0, \beta'(g_*) = \xi > 0$. Anomalous dimension $\gamma_\kappa(g)$ is known exactly at the fixed point:

$$\gamma_\kappa^* \equiv \gamma_\kappa(g_*) = \xi, \tag{43}$$

without corrections of orders ξ^2, etc.

Consider the even different-time structure functions as an example of the solution of the RG equation:

$$S_{2n}(\boldsymbol{r}, \tau) \equiv \langle [\theta(t, \boldsymbol{x}) - \theta(t', \boldsymbol{x}')]^{2n} \rangle, \quad r = |\boldsymbol{x} - \boldsymbol{x}'|, \quad \tau \equiv t - t'. \tag{44}$$

The structure functions satisfy the RG equation $D_{RG} S_{2n} = 0$ with the operator D_{RG} from Eq. (37) (note that the structure functions involve composite operators θ^n, so the RG equation requires additional justification that will be given later).

Dimensional considerations lead to the following representation:

$$S_{2n}(\boldsymbol{r}, \tau) = \kappa^{-n} r^{2n} R_{2n}(\mu r, \tau\kappa/r^2, mr, g), \tag{45}$$

where R_{2n} is a function of dimensionless arguments. The RG equation leads to the representation

$$S_{2n}(\boldsymbol{r}, \tau) = (\bar{\kappa})^{-n} r^{2n} R_{2n}(1, \tau\bar{\kappa}/r^2, mr, \bar{g},), \tag{46}$$

where the we introduced so-called invariant variables $\bar{e} = \bar{e}(\mu r, e)$ that satisfy the equation $D_{RG}\bar{e} = 0$ and normalization conditions $\bar{e} = e$ at $\mu r = 1$ (here $e \equiv \kappa, g, m$ stands for the full set of renormalized parameters). Representation (46) is accurate because both sides satisfy the RG equation and coincide with each other for $\mu r = 1$ due to the normalization. The bare variables are related to the invariant ones as

$$\kappa_0 = \bar{\kappa} Z_\kappa(\bar{g}), \quad g_0 = \bar{g} r^{-\xi} Z_g(\bar{g}). \tag{47}$$

5 Thus, the invariant variables are functions of the bare parameters. As $\mu r \to \infty$, the invariant coupling constant approaches the IR-attractive fixed point: $\bar{g} \to g_*$. Similar expression can be derived for the invariant diffusivity $\bar{\kappa}$: $\bar{\kappa} = D_0 r^\xi/\bar{g} \to D_0 r^\xi/g_*$ (we recall that $D_0 = g_0\kappa_0$). For $\mu r \to \infty$ and any fixed mr, it translates to

$$S_{2n}(\boldsymbol{r}, \tau) = (D_0/g_*)^{-n} r^{n(2-\xi)} \zeta_{2n}(\tau D_0 r^{\Delta_t}, mr), \tag{48}$$

where

$$\zeta_{2n}(D_0\tau r^{\Delta_t}, mr) \equiv R_{2n}(1, D_0\tau r^{\Delta_t}, mr, g_*) \tag{49}$$

and $\Delta_t \equiv -2 + \gamma_\kappa^* = -2 + \xi$ is the critical time dimension. Note that the dependence of the scaling function ζ_{2n} on its arguments cannot be found using the RG equation (37). In the equal-time structure function (22), ζ_{2n} has only the second argument:

$$S_{2n}(r) = (D_0/g_*)^{-n} r^{n(2-\xi)} \zeta_{2n}(mr). \tag{50}$$

Equations (48)–(50) show that the structure functions do not depend on the diffusivity coefficient in the IR range (large μr and arbitrary mr), moreover, the parameters g_0 and κ_0 enter Eq. (48) only as part of $D_0 = g_0\kappa_0$. Similar result rooted in the second Kolmogorov hypothesis was obtained for the stirred Navier–Stokes equation [37, 38, 115].

In the general case, if $F(r, \tau)$ is a multiplicatively renormalized quantity with canonical dimensions d_F^ω, d_F, its behavior in the IR range ($\Lambda r \sim \mu r$ large, mr arbitrary) is

$$F(\boldsymbol{r}, \tau) \simeq \text{const}\, \Lambda^{-\gamma_F^*} D_0^{d_F^\omega} r^{-\Delta[F]} \zeta_F(D_0\tau r, mr), \tag{51}$$

where

$$\Delta[F] \equiv \Delta_F = d_F^k - \Delta_t d_F^\omega + \gamma_F^*, \quad \Delta_t = -2 + \xi, \tag{52}$$

is the critical dimension of the function F. For nontrivial γ_F^*, the function F still depends on $\Lambda = 1/l$ or, equivalently, on κ_0.

Representation (50) describes the behavior of the structure functions for $\mu r \gg 1$ and any fixed value of $u \equiv mr$; the inertial range is coded by the further condition $u \ll 1$. Taking into account derivations above, one can obtain expansion for the scaling function in ξ:

$$\zeta(u) = \sum_{k=0}^{\infty} \xi^k \zeta^{(k)}(u). \tag{53}$$

However, it can be demonstrated that, for any arbitrarily small value of ξ, there are diagrams diverging at $m \sim u \to 0$. Thus, the coefficients $\zeta^{(k)}$ involve IR singularities $u^p \ln^q u$ termed "large IR logarithms" that compensate for the smallness of ξ so that the actual expansion parameter is $\xi \ln u$. This is why the expansion (53) cannot be used to analyze small-u behavior of $\zeta(u)$.

The goal is to sum up the expansion (53) while assuming that ξ is small and that $\xi \ln u \sim 1$. Following example set by the theory of critical behavior [6, 7], the goal can be achieved using the operator-product expansion; see Refs. [37, 38].

3.2.3 Composite fields: renormalization and scaling dimensions; operator-product expansion and anomalous scaling

Composite fields or composite operators are made of the field $\theta(x)$ and its spatial derivatives. Since the field arguments coincide in correlation functions with an operator F, additional UV divergences appear and are removed by renormalization procedure. Thus, the critical dimension Δ_F is not in general given by the simple sum of critical dimensions of the fields and derivatives entering into F. Composite operators "mix" during renormalization resulting, for example, in a situation when an UV finite renormalized operator is a linear combination of unrenormalized ones.

To find counterterms for an operator F, one needs to consider all possible 1-irreducible Green functions with operator F and arbitrary number of primary fields: $\Gamma = \langle F(x)\Phi(x_1)\ldots\Phi(x_n)\rangle_{1-ir}$. The total canonical dimension for the latter is

$$d_\Gamma = d_f - \sum_\Phi N_\Phi d_\Phi. \tag{54}$$

For superficially divergent diagrams, d_Γ is a non-negative integer.

Let us start with the operators $\theta^n(x)$ from structure functions (22); the corresponding canonical dimension is $d_F = -n$. Using Table 1, we get $d_\Gamma = -n + N_\theta - N_{\boldsymbol{v}} - (d+1)N_{\theta'}$; the form of the diagrams shows the total number of the fields θ in the function Γ cannot exceed the number of the fields θ in the operator θ^n itself: $N_\theta \le n$. Thus, the condition for divergence is $N_{\boldsymbol{v}} = N_{\theta'} = 0$ and arbitrary value of $n = N_\theta$.

But because at least one of external "tails" of the field θ is connected to the vertex $\theta'(v_i\partial_i)\theta$, at least one power of external momentum appears making the real index of divergence d'_Γ negative.

Therefore, the operator θ^n is actually UV finite, $\theta^n = Z[\theta^n]^R$ with $Z = 1$, so the critical dimension of $\theta^n(x)$ can be found using expression (52):

$$\Delta[\theta^n] = n\Delta[\theta] = n(-1+\xi/2). \tag{55}$$

This justifies the RG equation (37) for the correlation functions with the operators $\theta^n(x)$, in particular, the structure functions (22).

Let us now consider the operators

$$F_n \equiv [\partial_i\theta\partial_i\theta]^n \tag{56}$$

with $d_F = 0, d_F^\omega = -n$. They are in the left-hand side of Eq. (28) and it is their critical dimensions that affect the anomalous exponents in (26) and (28).

Using Table 1, we obtain $d_\Gamma = N_\theta - N_{\boldsymbol{v}} - (d+1)N_{\theta'}$ (the necessary condition from the diagrams being $N_\theta \le 2n$). The fields θ, θ' can enter the corresponding counterterms only in the form of derivatives $\partial\theta$, $\partial\theta'$, resulting in the real index of divergence of the form $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'} = -N_{\boldsymbol{v}} - (d+2)N_{\theta'}$. Thus, superficial divergences can only appear in the Green's functions with $N_{\boldsymbol{v}} = N_{\theta'} = 0$ and any $N_\theta \le 2n$ reducing the counterterms to the form F_k with $k \le n$. This is why the operators F_n mix exclusively with each other and the corresponding infinite renormalization matrix $Z_F = Z_{nk}$ is triangular, $Z_{nk} = 0$ for $k > n$. The critical dimensions are given by the diagonal elements $Z_n \equiv Z_{nn}$.

The constants Z_n make the 1-irreducible function

$$\Gamma_n = \langle F_n^R(x)\theta(x_1)\ldots\theta(x_n)\rangle = Z_n^{-1}\langle F_n(x)\theta(x_1)\ldots\theta(x_n)\rangle \tag{57}$$

UV finite. In terms of loops, $\Gamma_n = F_n + \sum_{l=1}^\infty \Gamma^{(l)}$, where $\Gamma^{(l)}$ is the sum of the diagrams with l loops.

For the scaling dimension Δ_n in (52), one obtains

$$\Delta_n = n\xi + \gamma_n^*, \quad \gamma_n = \widetilde{D}_\mu \ln Z_n. \tag{58}$$

Once the dimensions (58) are identified with the anomalous exponents in (26), a perturbation expansion (a series in ξ) can be constructed:

$$\Delta_n = \sum_{k=1}^{\infty} \Delta_n^{(k)} \xi^k. \tag{59}$$

Then, diagrammatic technique can be used to calculate coefficients $\Delta_n^{(k)}$.

The one-loop result is $\Delta_n^{(1)} = -2n(n-1)/(d+2)$ which is in agreement with (27) obtained in Ref. [107] using the zero-mode approach. The result $\Delta_1 = 0$ is exact (valid to all orders in ξ) being an implication of Schwinger equation (the energy conservation law) [78].

The calculation necessary to obtain the next orders is increasingly complicated; for the Kraichnan model, the second-loop result was achieved in Ref. [78] and the third-loop one in Ref. [79].

These findings are in agreement with the $O(1/d)$ result for Δ_n obtained in Ref. [108]; in particular, it could be seen that the $O(1/d)$ contribution is completely contained in the $O(\xi)$ term, while the higher order terms $O(\xi^k)$ with $k \geq 2$ for large d behave as $O(1/d^2)$.

Knowing the three terms of the "ε expansion" in the model (29)–(31), one can analyze its convergence properties and compare predictions for finite ξ with non-perturbative results such as analytical solution of the zero-mode equations for $n = 2$ [108], numerical solutions for $n = 3$ [116] and numerical experiments for $n = 4$ [117] and $n = 6$ [118].

In the OPE approach, the infinite sum

$$[\theta(t, \boldsymbol{x}) - \theta(t, \boldsymbol{x}')]^n = \sum_F C_F(\boldsymbol{r}) F\left(t, \frac{\boldsymbol{x} + \boldsymbol{x}'}{2}\right), \tag{60}$$

describes behavior of the quantities in the right-hand side of Eq. (22) for $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}' \to 0$ and fixed $\boldsymbol{x} + \boldsymbol{x}'$. Here C_F are coefficients (without irregularities in m^2) while F are entire range of renormalized local composite operators allowed by the symmetry. In other words, naive Taylor expansion gives operators entering the OPE and all those admixing to them during renormalization procedure.

We assume that these operators are irreducible tensors such as scalars, vectors and traceless tensors. Another assumption is that they have definite critical dimensions Δ_F (that is, they are "basis operators" in terminology of Ref. [7]; see Section 6.7).

By averaging Eq. (60) with the weight $\exp S_R$ from (35), one obtains the structure functions (22) with the mean values $\langle F \rangle$ appearing on the right-hand side. The RG equations determine their behavior for $m \to 0$: $\langle F \rangle \sim m^{\Delta_F}$.

Thus, the power-law (50) and the OPE (60), yield for the structure function in the inertial range ($\Lambda r \gg 1$, $mr \ll 1$):

$$S_n(\boldsymbol{r}) = D_0^{-n/2} r^{n(1-\xi/2)} \sum_F A_F(mr)(mr)^{\Delta_F}, \tag{61}$$

with factors A_F regular in $(mr)^2$.

There is a number of observations to mention. First, operators that are total derivatives can be omitted due to the translational invariance: $\langle \partial F(x) \rangle = \partial \langle F(x) \rangle = \partial \times \text{const} = 0$. Second, mean values of operators involving an odd number of fields θ also are equal to zero in the model (29)–(31). Their contributions disappear similarly to the odd structure functions. Finally, because the model is isotropic, all contributions but those of the scalar operators vanish in (61). Since we consider the case of $SO(d)$-symmetry, tensor indices of a tensor operator's mean value must be the ones of Kronecker delta symbols. But the latter are not enough to construct traceless tensor. This is why coefficients A_F and functions S_{2n} depend solely on $r = |\boldsymbol{r}|$.

Let us recall that is 'dangerous' composite operators with negative critical dimensions that distinguish turbulence models [37, 38]. Contributions of those operators to the OPE are behind anomalous scaling of the scaling functions.

Due to Δ_F being given by $\Delta_F = d_F + O(\xi)$, operators whose Δ_F are minimal include maximum number of fields θ and minimum number of derivatives (under assumption of small ξ). Since symmetry $\theta \to \theta + \text{const}$ governs both problem (29)–(31) and the quantities (22), operators in the expansion (60) must obey the symmetry too and, thus, be made out of the gradients of θ.

Now let us consider the naive Taylor expansion and operator mixing. The leading term of the former for S_n is the second rank operator $(\partial\theta)^n$ with lower ranked operators resulting from its decomposition in irreducible tensors. If there is stirring force, operators $(\partial\theta)^k$ with $k < n$ admix with them during renormalization procedure and have to be accounted for in the OPE (similar to solutions of inhomogeneous equations).

Since problem (29) is linear, operators with $k > n$ and the Taylor expansion terms for S_n do not mix. Canonical dimensions of the operators are $d_F = 0$ (see Table 1) which are minimal possible; the operators set the leading terms of the $mr \to 0$ behavior when $j \le 2n$. Correction terms for $mr \to 0$ are given by operators with more derivatives than fields θ because their canonical dimensions are $d_F = 1, 2$, etc.

Thus, we obtained representations (26) using the RG approach and OPE; we also matched anomalous exponents Δ_n and scaling dimensions of the composite fields $F_n \equiv [\partial_i\theta\partial_i\theta]^n$.

It is possible to derive similar representations for other quantities, e.g., equal-time pair correlation functions of the operators $E^n(x) = \kappa_0^n F_n(x)$ with F_n from (56):

$$\begin{aligned} \langle E^n(x) E^p(x') \rangle &= (\Lambda r)^{-\Delta_n - \Delta_p} f_{n,p}(mr), \\ f_{n,p}(mr) &= \text{const}(mr)^{\Delta_{n+p}}, \end{aligned} \tag{62}$$

where Δ_n are from (26). RG equation provides the first expression which holds for $(\Lambda r) \gg 1$ and arbitrary fixed (mr). OPE with its leading term set by the operator

F_{n+p} gives the second one which holds for $(mr) \ll 1$. See Refs. [78, 108] for other examples.

It is worth noting that when $n = p = 1$, the correlation function (62) does not depend on UV scale Λ, a result reminiscent of the second Kolmogorov hypothesis. Moreover, it can be said that operators $E^n \sim F_n$ are closed with respect to the fusion which means that the leading term in the OPE for the pair correlation function (62) is determined by the operator F_{n+m}. Coupled with inequality $\Delta_n + \Delta_m > \Delta_{n+m}$ (that follows from the explicit forms of Δ_n), this observation implies multifractality of the local dissipation rate correlations in the model (29)–(31) [119, 120]. Finally, Hölder inequality for the structure functions (26) follows from inequality $\Delta_n + \Delta_m > \Delta_{n+m}$.

3.3 Universality facets: memory, anisotropy and compressibility

Undoubtedly, the Kraichnan model (29)–(31) is based on many assumptions that simplify turbulence phenomenon. Gaussianity, isotropy, incompressibility and vanishing correlation time are a natural starting point for a model of turbulence but the next steps should turn to the cases of anisotropic compressible field correlated in time etc. For example, addition of finite correlation time [121, 122] or compressibility [123–125] strongly affect the results.

Investigation of a generalized model where eddy turnover was affecting temporal correlation of the advecting field was performed in Ref. [126]. It argued for the dependence of anomalous exponents on a description of the velocity statistics that was broader than just the exponents. The argument was supported by other studies [121, 127]. Small correlation time case for the Kraichnan model was considered in Ref. [121] where one-loop approximation of anomalous exponents was obtained; the exponents turned out to be non-universal as they depended on the correlation time. See also Refs. [128, 129] for effects of finite correlation time on transport in random Gaussian self-affine velocity fields.

Now let us consider large-scale anisotropy (often introduced via the correlation function of the random noise) and how it affects passively advected fields [102, 112, 116, 125, 130] and the velocity field [131].

The local isotropy restoration hypothesis [93] posits that large-scale anisotropy disappears as the energy flows to smaller scales due to the cascade mechanism. Both the zero-mode approach and the RG analysis specify that process by establishing the hierarchy of the restoration: the coefficient functions display scaling with non-universal amplitudes and universal exponents in the inertial range. It is the amplitudes that "encode" the degree of anisotropy. The leading contribution to the scaling is given by the isotropic term, its corresponding exponent matching the one for the purely isotropic model (30). The anisotropic contributions, on the other hand, produce only corrections.

For the model (29) and (31), some of the exponents were found exactly using the zero-mode approach [108]. For the case of the finite correlation time, one-loop approximation for all of the exponents was obtained with the RG and OPE approach [25]; the two-loop result was later achieved in Ref. [113].

Study of passive magnetic fields [109] revealed that while the hierarchy remains for all $\alpha > 0$, the corrections grow relative to leading terms as α increases. The hierarchy seems rather general because numerical simulations and experiments in the atmosphere indicate a compatible structure [131]. All the same, the anisotropy persists in the inertial range even if it appears solely in odd correlation functions [45, 50, 103].

Compressibility seems to strengthen large-scale anisotropy. Similar effect was established for the model of the passively advected magnetic field [109]. Another investigation of the effects of finite correlation time and large-scale anisotropy can be found in Ref. [81].

Now let us turn to effects of "pure" compressibility. Passive advection of a density field (e.g., impurity) can be modeled by the equation

$$\partial_t \theta + \partial_i (v_i \theta) = \kappa_0 \partial^2 \theta + f \tag{63}$$

while advection of a tracer field (temperature, impurity concentration) can be modeled by

$$\partial_t \theta + (v_i \partial_i)\theta = \kappa_0 \partial^2 \theta + f. \tag{64}$$

In the Kraichnan model, statistics of f are given by correlation function (30); velocity field statistics, on the other hand, require substitution

$$D_0 P_{ij}(\boldsymbol{k}) \to D_0 P_{ij}(\boldsymbol{k}) + D_0' Q_{ij}(\boldsymbol{k}) = D_0 (P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})) \tag{65}$$

in (31); here $Q_{ij}(\boldsymbol{k}) = k_i k_j / k^2$ and $D_0' > 0$. The case of $\alpha = 0$ describes a transverse velocity field while the case of $\alpha \to \infty$ at fixed $D_0' = D_0 \alpha$ does the same for a potential velocity. Degree of compressibility can be measured by parameter $\rho = \alpha/(d-1+\alpha)$ so that $0 \le \rho \le 1$.

Investigation in Ref. [125] of the compressible case showed that the anomalous scaling regime vanishes for sufficiently big ξ and α when $\rho \equiv \alpha/(d-1+\alpha) > d/\xi^2$); instead there is a regime of inverse energy cascade. Nevertheless, the anomalous scaling is still present for small ξ or $1/d$.

RG analysis of the case $\alpha \neq 0$ proves its multiplicative renormalizability; there is an IR-attractive fixed point with a coordinate

$$g_* = \frac{2d\xi}{C_d(d-1+\alpha)}, \tag{66}$$

where C_d is from (40). This point appears in both cases: the density field case and the tracer field case. In the RG approach, α is not an expansion parameter but it is dimensionless and as such serves as another coupling constant. However, equation $\beta_\alpha = 0$ brings no new information.

Since the tracer field in Eq. (64) is under derivative, the symmetry $\theta \to \theta +$ const is unbroken for all α. This is why the analysis and OPE discussed in Sect. 3.2.3 is applicable for all α too. Dimensions of operators θ^n do not dependent on α and are

given by Eq. (55); inertial-range behavior of the structure functions is described by expressions (26) with the anomalous exponents Δ_n that now depend on α:

$$\Delta_n = -2n(n-1)\xi(1+2\rho)/(d+2) + O(\xi^2) \tag{67}$$

where $\rho = \alpha/(d-1+\alpha)$. This approximation is a result of the zero-mode technique [125] while the derivation of the $O(\xi^2)$ correction was obtained with the RG approach and OPE [81].

Now let us discuss the density field case [80]. The composite operators θ^n are now renormalized in a nontrivial way: $\theta^n = \overline{Z}_n[\theta^n]^R$ where $\overline{Z}_n \neq 1$ so that dimensions $\overline{\Delta}_n$ are nontrivial [80]:

$$\begin{aligned}\overline{\Delta}_n = n(-1+\xi/2) - \frac{\alpha n(n-1)d\xi}{2(d-1+\alpha)} + \frac{\alpha(\alpha-1)n(n-1)(d-1)\xi^2}{2(d-1+\alpha)^2} \\ + \frac{\alpha^2 n(n-1)(n-2)dh(d)\xi^2}{4(d-1+\alpha)^2} + O(\xi^3),\end{aligned} \tag{68}$$

where $h(d) \equiv F(1,1;d/2+1;1/4)$.

Therefore, it is the composite fields θ^n pair correlation functions that display anomalous scaling now rather than the structure functions:

$$\begin{aligned}\langle\theta^n(x)\theta^p(x')\rangle &= (\kappa_0\Lambda^2)^{-(n+p)/2}(\Lambda r)^{-\overline{\Delta}_n-\overline{\Delta}_p} f_{n,p}(mr), \\ f_{n,p}(mr) &= \text{const}\,(mr)^{\overline{\Delta}_{n+p}}.\end{aligned} \tag{69}$$

Due to the restricted size of this paper, we will not discuss numerous recent interesting generalizations and higher order computations for the problems mentioned in this Section. The reader can be addressed to the papers [132, 133] that focus on the effects of helicity, compressibility, anisotropy and non-Gaussianity, with a detailed bibliography.

4 Navier–Stokes equation, Kolmogorov constant

4.1 RG approach to the stochastic Navier–Stokes equation

The RG approach to the stirred NS equation pioneered in Ref. [134] was applied to the problem of developed turbulence in Ref. [74]; it was essentially developed further since then. Let us briefly go through the necessary steps of the analysis based on the standard field-theoretic RG and ε expansion; for more details, see Refs. [37, 38].

NS equation for transverse ($\partial_i v_i = 0$) velocity field is

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \wp + f_i, \tag{70}$$

where $\nabla_t = \partial_t + v_i\partial_i$ is Lagrangian derivative, $\wp$ is pressure, f_i is transverse random force per unit mass. The latter is defined by Gaussian distribution with zero mean and

correlation function of the form

$$\langle f_i(x) f_j(x') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int_{k \geq m} d\boldsymbol{k} P_{ij}(\boldsymbol{k}) d_f(k) \exp[i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')], \tag{71}$$

where $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2$, $k \equiv |\boldsymbol{k}|$, and d is the spatial dimension. There is a sharp cutoff $m = 1/L$ for IR regularization (L is the integral scale).

The problem (70) and (71) corresponds to renormalizable field theory if function $d_f(k)$ has a following power form:

$$d_f(k) = D_0 k^{4-d-y}, \tag{72}$$

where $D_0 > 0$, and the exponent $0 < y \leq 4$ serves as a small parameter of the RG expansion (instead of ξ that we used for the Kraichnan model). The value $y = 4$ has the most significant physical interpretation because it turns $d_f(k)$ into the delta function: $d_f(k) \sim \delta(\boldsymbol{k})$. The latter models injection of energy to the system that occurs due to interaction with the largest turbulent eddies; see Refs. [37, 38] for more details. It should be noted that RG analysis of the model (70)–(72) is sound for small y, however, the question of whether it can be extrapolated to $y = 4$ remains open.

The Galilean symmetry holds for the problem (70)–(72) as well as for its fuller version that includes Eqs. (29) and (30) for the case of passively advected scalar field. Thus, Ward identities for correlation functions originated from it are satisfied in every order of perturbation theory. The RG procedure does not interfere with the symmetry, so covariant structures remain so at all stages. This is why the Galilean-invariant quantities such as the equal-time structure functions (22) are indifferent to the sweeping (contrary to the synthetic Gaussian models like the Kraichnan model).

After constructing field theory corresponding to the problem (70)–(72), its multiplicative renormalizability can be established; RG functions can be calculated (as series in the coupling constant) and IR-attractive fixed points can be found (their coordinates series in y). The action reads

$$S_v(\boldsymbol{v}', \boldsymbol{v}) = v' D_v v'/2 + v'(-\nabla_t + \nu_0 \partial^2) v, \tag{73}$$

where D_v stands for the correlation function (71); all the integrations over necessary arguments and summations over the tensor indices are implied. The auxiliary vector field v_i' is transverse so it serves as the transverse projector which justifies "disappearance" of the pressure term.

The coupling constant is $g_0 \equiv D_0 \nu_0^3 \sim \Lambda^y$; Λ is the characteristic UV momentum scale. Field v_i shares its canonical dimensions with those of the Kraichnan model field v_i, given in Table 1; the dimensions of v_i' are $d_{v'}^k = (d+1)$, $d_{v'}^\omega = -1$, $d_{v'} = (d-1)$. Analysis of UV divergences reveals that only counterterm $v' \partial^2 v$ is needed for renormalization (we assume that $d > 2$ as the case of $d = 2$ is special). Therefore, only one independent renormalization constant appears in the renormalized action:

$$S_{vR}(\boldsymbol{v}', \boldsymbol{v}) = v' D_v v'/2 + v'(-\nabla_t + \nu Z_1 \partial^2) v. \tag{74}$$

Bare parameters are related to their renormalized counterparts as

$$D_0 = g_0 \nu_0^3 = g\mu^y \nu^3, \quad g_0 = g\mu^y Z_g, \quad \nu_0 = \nu Z_\nu, \quad Z_\nu = Z_1, \quad Z_g = Z_1^{-3}. \tag{75}$$

As usual, μ is the reference mass scale. The fields and cutoff $m = 1/L$ are not renormalized. One-loop approximation for Z_ν is well known (see, e.g., Ref. [37]) and allows to find an IR-stable fixed point.

Since canonical dimensions of the fields v_i, v_i' are positive and due to (52), for small y, critical dimensions of nontrivial composite operators made out of these fields and their derivatives are similarly positive. The leading contribution in OPE, then, is due to the operator $F = 1$ with $\Delta_F = 0$ (finite for $(mr) = 0$); nontrivial operators only produce corrections that disappear as $(mr) \to 0$. The correlation functions do not display anomalous scaling for small y.

Numerical simulations [135], however, predict appearance of anomalous scaling for structure functions with increase in y. In terms of RG procedure, it implies that there are Galilean-invariant operators whose critical dimensions become negative for y close to the value $y = 4$. At the same time, the RG approach cannot identify those operators because y is no longer small. For thorough discussion of this issue, see Refs. [37, 38] and papers cited therein. Another analytical method is needed to shed light on this problem.

4.2 Stochastic Navier–Stokes equation near two dimensions: double expansion

Coefficient functions in the perturbation solution of the stochastic problem (70), (71), and (72) exhibit singular behavior in the limit $d \to 2$. To account for their effect to the solution in a consistent manner, it is convenient to start with the modified kernel function [136–138]

$$d_f(k) = D_{10} k^{4-d-y} + D_{20} k^2 = g_{10} \nu_0^3 \, k^{4-d-y} + g_{20} \nu_0^3 \, k^2, \tag{76}$$

where the deviation of the space dimension d from its critical value two $d - 2$ is assumed small. The two parameters y and $d - 2$ are the UV-regulators customarily used in the analysis of this and other double-expansion problems with the aid of a combination of the analytic and dimensional regularization schemes [21]. Divergences in the Green functions show in the form of poles in definite linear combinations of the parameters y and $d - 2$ regarded as complex numbers. The term $\propto k^2$ in (76) is generated by renormalization and becomes relevant at two dimensions. It must be added to the forcing correlation function at the outset to keep the model multiplicatively renormalizable. Physically, this terms describes the effect of thermal fluctuations [139]. The corresponding field-theoretic model is logarithmic, i.e., $d_{g10} = d_{g20} = 0$, when $y = 0$ and $d = 2$. Due to the Galilean invariance, only diagrams of the one-irreducible Green functions $\langle v' v \rangle_{1-\mathrm{ir}}$ and $\langle v' v' \rangle_{1-\mathrm{ir}}$ yield divergent contributions which lead to the renormalization of the parameters ν_0 and D_{20}. Renormalization of the model (76) with

the two regulators gives rise to a double expansion in y and $d-2$ of the renormalized Green functions [136].

RG calculations with two (or even more) small parameters which serve as regulators in dimensional or analytic renormalization have been widely used in the analysis of static critical phenomena, dynamic critical phenomena, diffusion in random environment, interface growth and in stochastic hydrodynamics (see Refs. [50, 140] for representative references). Critical exponents and other relevant quantities may be expressed in a double expansion in these parameters. The small parameters may both be regulators of analytic renormalization or one of them is the regulator of dimensional renormalization. In the following, this pair of parameters will be denoted ε and Δ. It has been almost invariably assumed (implicitly or explicitly) that both parameters are of the same order of magnitude. This assumption effectively restores the dimensional renormalization with a single small parameter and the effective scheme on minimal subtractions (MS) [20] can be used—at least formally.

Typically, there are at least two charges (i.e., expansion parameters of perturbation theory) in these models and therefore a rather generic case of two charges and a single anomalous dimension γ (corresponding to a field renormalization) will be analyzed here. When there are several coupling constants, it is customary to classify the order of perturbation theory by the number of loops. In multi-charge problems, there are coupling constants, which should be calculated in closed form at each such order of perturbation theory (e.g., ratios of coefficients of viscosity, diffusion and thermal conductivity). We do not discuss such coupling constants here. Two different structures of β functions are met.

First, two (or more) random sources with different powerlike falloff of correlation functions are often introduced. Two analytic regulators appear: deviations of exponents of powerlike correlation functions from their critical values. Regulators are put proportional to each other and renormalization is treated in the framework of the usual dimensional renormalization, although formally in terms of a multiple expansion. In these models the β functions of charges are similar to that of the single-charge case (38), i.e., a renormalized coupling constant is factorized and the γ functions are regular functions of the charges (regular multi-charge case [140]):

$$\beta_1(g_1, g_2) = g_1\left[-\varepsilon - \gamma_1(g_1, g_2)\right], \quad \beta_2(g_1, g_2) = g_2\left[-\Delta - \gamma_2(g_1, g_2)\right]. \quad (77)$$

In the framework of analytic renormalization, coefficients of the RG functions are regular functions of the two parameters ε and Δ at the origin by construction of the renormalization scheme. Due to the analytic properties and structure of the RG functions, nontrivial fixed points and the RG functions themselves can be found in the form of a regular double expansion in ε and Δ.

Second, dimensional regularization is amended by analytic regularization (only one analytic regulator will be considered here, although several have been introduced) [140]. Either in propagators or in interactions the wave-number dependence includes the combination $a + bk^{-2\Delta}$, in which $\Delta > 0$ (in propagators usually multiplied by the factor k^2). For small Δ, a nontrivial problem of renormalization of such structures arises since in the limit $\Delta \to 0$ the terms in $a + bk^{-2\Delta}$ become indistinguishable and it is not clear, which of them should be renormalized. The problem is solved

by the prescription of the counter terms to renormalization of the local (analytic in k^2) contribution [136, 137], The point is that renormalization produces only local counterterms. Renormalization constants are constructed in a regularized model with clearly distinguishable local and non-local terms. The counterterms are local and renormalize a similar term. In the field-theoretic approach, such "generation terms" are added to the original model at the outset to deal with a multiplicatively renormalizable model [7].

In practical calculations, the ray scheme is often used, in which the regulators are put proportional to each other and the procedure of dimensional renormalization is used with an additional finite and fixed parameter $\zeta = \Delta/\varepsilon$ [137]. At one-loop order, the RG functions are analytic functions of regulators in this scheme. At higher orders, divergent subdiagrams produce contributions with denominators of the form $m\varepsilon + n\Delta$ with integer m and n. In the ray scheme, the common factor ε is pulled out to yield denominators of the form $m + n\zeta$ considered finite. Such factors are not removed by the MS procedure in the ray scheme and appear in RG functions. This feature of the ray scheme has been demonstrated and analyzed in detail in Ref. [137].

Contributions to renormalization of a generation term produced by other charges change the structure of the corresponding β function. For instance, if in (77), g_2 were the charge of a generation term, then the corresponding $\beta_2(g_1, g_2)$—apart from a regular function $\gamma_2(g_1, g_2)$—would contain a singular term of the form $(g_1/g_2)\gamma_2'(g_1, g_2)$ (with regular γ_2') leading to nontrivial fixed points singular in ε, Δ. Contrary to the regular multi-charge case, the fixed-point values of charges in the analytic renormalization are not necessarily regular functions of the regulators (although the RG functions are). Therefore, critical exponents may not be regular functions of regulators either. Another feature of this class of models is that the very number of the fixed points becomes scheme dependent (see Ref. [140] for details).

4.3 Passive advection by the Navier–Stokes velocity ensemble

As was observed in Ref. [78] and studied in depth in Ref. [114], there is anomalous scaling of scalar field for small values of y when the field is advected by velocity field described by the NS equation (70)–(72) (if statistics are out of the range for intermittency). RG approach and OPE can be used to obtain the anomalous exponents with y from (72) serving again as expansion parameter. Two-loop result was achieved in Ref. [114]; see Ref. [113] for finite correlation time case.

The field theory that can be considered in place of the full problem (29), (30), and (70)–(72) is

$$S(\Phi) = S_v(\boldsymbol{v}', \boldsymbol{v}) + \theta' D_\theta \theta'/2 + \theta'(-\nabla_t + u_0 \nu_0 \partial^2)\theta, \tag{78}$$

where ratio $u_0 = \kappa_0/\nu_0$ is the inverse Prandtl number, D_θ is the correlation function (30) for the noise f in (29), and S_v is the action (73).

Canonical dimensions analysis shows that there are two counterterms: $v'\partial^2 v$ and $\theta'\partial^2\theta$; the renormalized action reads

$$S_R(\Phi) = S_{vR}(\boldsymbol{v}', \boldsymbol{v}) + \theta' D_\theta \theta'/2 + \theta'(-\nabla_t + u\nu Z_2 \partial^2)\theta \tag{79}$$

with S_{vR} from (74). Since we assume that field θ is passive (it does not affect dynamics of the velocity field), there is relation $Z_1 = Z_\nu$ for renormalization constants in S_{vR}; Z_2 is a new constant. Relations (75) are augmented by

$$u_0 = u Z_u, \quad Z_u = Z_2 Z_1^{-1}. \tag{80}$$

It is important here that correlation function D_θ does not enter function $\langle \theta' \theta \rangle_{1-ir}$ whose divergence defines renormalization constant Z_2. This is why Z_2 and $\beta_u = \widetilde{D}_\mu u$ in its turn match the ones of the model that does not involve random noise f in equation (29). If we assume that $g > 0$ and $u > 0$, we will find a single IR-attractive fixed point.

Additionally, and even more importantly, the noise correlation function can be substituted outright with its large-scale form (30). The canonical dimensions of the fields θ and θ' are then given by Table 1, which means that one of them is negative: $d_\theta = -1$. This makes composite fields built of the derivatives of θ dangerous even for small values of ε. One-loop expressions for the dimensions Δ_n shows that $\Delta_n < 0$. Thus, anomalous scaling relations for the structure functions (24) can be established for the model (29), (30), and (70)–(72) using the RG approach and OPE. Dimensions Δ_n and Δ_{nl} were obtained in two-loop approximation in Ref. [65].

RG analysis of the model (29), (30), and (70)–(72) has two implications.

First, critical dimensions of composite operators (56) and, thus, the anomalous exponents do not depend on the specific choice of the correlator (30). Therefore, if the noise in (29) is substituted with a constant gradient, for example, as has been done in Ref. [103], they still will not change. The random noise supports steady state of the system and so ensures non-zero amplitudes in the power-law.

This means that the concept of zero modes (and related statistical conservation laws) can be extended to the case of passive scalar advected by the NS described velocity field; indeed, passive scalar advected by Gaussian velocity ensemble (31) with vanishing correlation time behaves similarly while the RG and OPE procedure and results match exactly for both models.

Second, the exponents depend on y and on the spatial dimension d. In the case of Gaussian velocity ensembles with finite correlation time, there was additional dependence on the dimensionless ratio of the correlation times. In the present case, there is dimensionless parameter $u_0 \equiv \kappa_0/\nu_0$, the (inverse) Prandtl number. However, during the RG procedure, the respective invariant variable takes place of u_0, a variable that can be interpreted as the ratio of correlation times, see Ref. [113] for details. The coordinate u_* (to which that variable tends) depends only on d and y so the critical dimensions are also independent of u_0.

Nonuniversality of the exponents in the Gaussian model (their dependence on u_0) is a direct result the IR-stable fixed point being degenerate. In the case of the NS advection, the fixed point is unique so the exponents are universal. It is possible that nonuniversality in synthetic case is caused by the Gaussianity, while more realistic models yield universality.

For further details and discussion, see Ref. [114]. Studies of more realistic models based on the NS equation with compressibility and large-scale anisotropy taken into account can be found in Refs. [29–31, 96, 141–147].

In passive advection by stochastic Navier–Stokes drift new universal quantities appear on top of critical dimensions of fields and composite operators, the most important being the Prandtl number Pr (ratio of kinematic viscosity and thermal diffusivity) [93] or—in the setup of the Obukhov–Kraichnan model (29)—the Schmidt number Sc (ratio of kinematic viscosity and mass diffusivity). It was calculated in the RG and the y expansion at one-loop order in Refs. [148, 149]. Due to the effective replacement of the molecular coefficients of viscosity and thermal diffusivity by their turbulent analogs it is necessary to define turbulent transport coefficients, i.e., the Prandtl number (or Schmidt number). In the two-loop calculation initiated in Ref. [150], the following approach was put forward: in Dyson equations for the response functions

$$G^{-1}_{vv'}(k,\omega) \equiv \Gamma_{vv'}(k,\omega) = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{v'v}(k,\omega)\,,$$
$$G^{-1}_{\theta\theta'}(k,\omega) \equiv \Gamma_{\theta\theta'}(k,\omega) = -\mathrm{i}\omega + \kappa_0 k^2 - \Sigma_{\theta'\theta}(k,\omega)\,,$$

where Σ are self-energy operators, the right-side expressions at $\omega = 0$ yield effective transport coefficients. Therefore, the inverse effective Prandtl number u_{eff} is defined by the relation

$$u_{eff} \equiv \frac{\Gamma_{\theta\theta'}(k,\omega=0)}{\Gamma_{vv'}(k,\omega=0)} = \frac{u R_\theta(k/\mu, g, u)}{R_v(k/\mu, g)}, \tag{81}$$

where the dimensionless functions R_v and R_θ of dimensionless arguments are

$$R_v(k/\mu, g) = Z_v - \frac{\Sigma_{v'v}(k,\omega=0)}{\nu k^2}, \quad R_\theta(k/\mu, g, u) = Z_\kappa - \frac{\Sigma_{\theta'\theta}(k,\omega=0)}{u\nu k^2}\,.$$

The effective inverse Prandtl number (81) in the inertial range predicted by the RG fixed point g_*, u_* is

$$u_{eff} = u_* \frac{R_\psi(1,\, g_*,\, u_*)}{R_\varphi(1,\, g_*)}. \tag{82}$$

This quantity is universal: the result of its calculation in the inertial range with the aid of relation (82) does not depend on the renormalization scheme and has an unambiguous y expansion. The result of the two-loop calculation (i.e., to order $O(y)$) [150–152] at the physical value $y = 4$ for the turbulent Prandtl number Pr_t is

$$\mathrm{Pr}_t^{(0)} = \frac{2}{\sqrt{43/3} - 1} \simeq 0.7179, \qquad \mathrm{Pr}_t \simeq 0.7040$$

in one-loop and two-loop accuracy, respectively. The correction term is strikingly small. Even at $y = 4$, it is only 2% of the leading contribution.

4.4 Kolmogorov constant and skewness factor

Experimental data and the results of calculation of scaling functions in toy models of turbulence reveal that it is reasonable to assume that the scaling exponent $2/3$ in the second-order (longitudinal velocity) structure function (24) is exact. This will be assumed in what follows.

In addition to the universal scaling exponents, important quantities of physical interest are the amplitudes of structure functions and their ratios. Among them the paramount role is played by the Kolmogorov constant—a dimensionless numerical parameter entering the amplitude of the second-order structure function $S_2(r)$ (or kinetic energy spectrum) and the skewness factor $\mathcal{S} \equiv S_3/S_2^{3/2}$, [92, 93]. Their values can be reliably experimentally measured (see Tables I–IV in Refs. [153, 154]), the measurement accuracy has been steadily improving and the influence of the Reynolds number, the integral scale L and other parameters on their values are being investigated up to now [155, 156]. Starting from the second half of the 1980s, attempts have been made to analytically determine the Kolmogorov constant [77, 157–159] accompanied later by numerical calculations requiring large computing power [160–162]. As in field-theoretic models of phase transitions, amplitudes are not universal: they depend on details of the model and perturbative scheme. Although the earliest calculations of the Kolmogorov constant were performed in a straightforward manner in the lowest approximation of the models, the results were in surprisingly good agreement with the experimental values [77, 157–159, 163–167]. Certain ratios of amplitudes (like the skewness factor), however, can be considered universal, because parameters of the non-universal details actually drop out in relations arising in the perturbative calculation. The crucial step is to utilize the relationship between the Kolmogorov constant (non-universal) and the universal skewness factor. Calculating the skewness factor using perturbative methods we immediately obtain the value of the Kolmogorov constant, thus bypassing the difficulties associated with its ambiguous definition [168].

The Kolmogorov constant C_K is the amplitude in the second-order structure function $S_2(r)$ in the inertial range:

$$S_2(r) = C_K \mathcal{E}^{2/3} r^{2/3}, \tag{83}$$

where $\mathcal{E}$ is the average energy dissipation rate per unit mass (in the steady state, it coincides with the mean energy injection rate $\overline{\mathcal{E}}$). The y expansion of the Kolmogorov constant in the model with the power-law forcing (72) is not unambiguous [169], therefore, attempts to calculate C_K have produced significant variation in the predicted numerical values.

Only quantities independent of the model parameter D_0 have unambiguous dependence on y, e.g., the skewness factor $\mathcal{S} \equiv S_3/S_2^{3/2}$, connected with the Kolmogorov constant through the exact relation [92, 93]

$$\mathcal{S} = -\frac{4}{5} C_K^{-3/2}, \tag{84}$$

on which an unambiguous calculation of C_K might be based [137, 138, 168, 169]. However, in the unphysical region $y < 4$, the structure function $S_2(r)$ in the model with the power-law forcing correlation $d_f(k) \sim k^{4-d-y}$ contains—at $y < 3$—an independent of r UV-divergent additive term $\sim \Lambda^{2-2y/3}$. Therefore, a generalization of the skewness factor $\mathcal{S} = S_3/S_2^{3/2}$ to the region $y < 3$ is pointless, because the powers of r do not cancel due to the constant term in $S_2(r)$. A suitable quantity related to the skewness factor independent of D_0 and independent of r in the interval $0 < y < 4$ is [169]

$$Q(y,d) \equiv \frac{r\partial S_2(r)/\partial r}{|S_3(r)|^{2/3}} = \frac{r\partial S_2(r)/\partial r}{(-S_3(r))^{2/3}}. \tag{85}$$

In terms of $Q(y,d)$, the physical values of $\mathcal{S}$ and C_K are

$$\mathcal{S} = -\left[\frac{2}{3Q(4,d)}\right]^{3/2}, \quad C_K = \frac{3}{2}\left[\frac{12}{d(d+2)}\right]^{2/3} Q(4,d). \tag{86}$$

$S_3(r)$ is found exactly from the spectral energy balance for all $y < 4$

$$S_3(r) = -\frac{3(d-1)\,\Gamma(2-y/2)\,(r/2)^{y-3} D_0}{(4\pi)^{d/2}\,\Gamma(d/2+y/2)}.$$

Therefore, in the limit $y \to 4$, the pumping parameter must behave as $D_0 \propto (4-y)$. The relationship between the energy dissipation rate $\bar{\mathcal{E}}$ and the noise d_f follows directly from the energy balance equation [77]

$$\bar{\mathcal{E}} = \frac{d-1}{2(2\pi)^d} \int d\boldsymbol{k}\, d_f(k), \tag{87}$$

which models the energy injection necessary to maintain steady turbulent state. In order to apply the field-theoretic RG, the power-law form of the correlator of the noise (72) is most convenient. Substitution in (87) with the ultraviolet cutoff Λ yields

$$D_0 = \frac{2(4-y)\Lambda^{y-4}}{C_d(d-1)}\bar{\mathcal{E}}. \tag{88}$$

It is readily seen that the power-law representation (72) of the noise correlator with the amplitude (88) in the limit $y \to 4$ tends to $\delta(\boldsymbol{k})$ and thus represents energy injection from the region of infinite-size eddies [77].

The ultraviolet cutoff Λ in Eq. (88) and the inverse dissipation length l^{-1} are of the same order of magnitude. Further, Λ is related to the viscosity coefficient ν_0 and the mean energy dissipation rate as $\Lambda = (\bar{\mathcal{E}}/\nu_0^3)^{1/4}$. Obviously, in Eq. (88), Λ can be replaced with a rescaled counterpart $a\Lambda$ with an arbitrary dimensionless parameter a of the order of unity and in the limit $y \to 4$, the physical injection will not change. This means that D_0 is ambiguously defined in the range $0 < y < 4$, i.e., it can be multiplied by an arbitrary function $f(y)$ providing $f(4) = 1$. Thus, amplitudes in the structure

functions are ambiguous as well. In pioneering calculations of the Kolmogorov constant [77, 157–159, 163–167], different functions $f(y)$ were introduced with slightly different results. The main conclusion is that within the framework of the y expansion, it is not possible to unambiguously determine the value of the Kolmogorov constant due to the explicit dependence of the second order structure function on the ambiguous value D_0.

The logarithmic derivative of S_2 in (85) is given by an integral which converges at all values $0 < y < 4$

$$r\partial_r S_2(r) = 2\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\, G(k)\left[1 - (\boldsymbol{k}\cdot\boldsymbol{r})^2/(kr)^2\right](\boldsymbol{k}\cdot\boldsymbol{r})\,\sin(\boldsymbol{k}\cdot\boldsymbol{r}).$$

Here, $G(k)$ is the Fourier transform of the scalar part of the equal-time velocity correlation function $G_{ij}(\boldsymbol{x}-\boldsymbol{x}') = \langle v_i(t,\boldsymbol{x})v_j(t,\boldsymbol{x}')\rangle$ defined by relation $G_{ij}(\boldsymbol{k}) = P_{ij}(\boldsymbol{k})G(k)$, where $P_{ij}(\boldsymbol{k})$ is the transverse projection operator. Large-scale asymptotics of $G(k)$ determined by the fixed point g_{1*} are

$$G(k) = \nu^2 k^{-d+2} R(\mu/k, g) \sim (D_0/g_*)^{2/3} k^{2-d-2y/3} R(1, g_*), \qquad \mu/k \to \infty,$$

where R is a scaling function of dimensionless arguments. It follows that [168]

$$Q(y,d) = \left[\frac{4(d-1)g_*}{9}\right]^{1/3} \frac{\Gamma(2-y/3)\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+y/2)}{\Gamma(d/2+y/3)\Gamma^{2/3}(2-y/2)}\, R(1,g_*).$$

Quantity $Q(y,d)$ can also be represented as a power series in the parameter y

$$Q(y,d) = y^{1/3}\sum_{k=0}^{\infty} Q_k(d)y^k \tag{89}$$

with coefficients $Q_k(d)$ depending on the space dimension d. Two-loop calculations corresponding to y precision according to Eq. (86) led to following values of the skewness factor and the Kolmogorov constant, respectively,

$$\mathcal{S} = -0.15, \qquad C_K = 3.02.$$

A detailed analysis of the field-theoretic model of turbulence (70), (71) and (72) shows that the integrals in the Fourier representation of correlation functions of the velocity field, in addition to poles at $y = 0$, also display additional ultraviolet divergences at $d = 2$ [37]. Account of these divergences in a renormalization scheme combining analytic renormalization used in (70), (71) and (72) with dimensional renormalization [136] gives rise to a double expansion in parameters y and $d-2$ and leads to significant improvement of the numerical performance of the field-theoretic model [137, 138, 169].

In the Navier–Stokes problem generated by the random force (76), additional information obtained by renormalization of divergences near two dimensions has been used

to improve numerical performance of predictions about physically relevant quantities such as the skewness factor and Kolmogorov constant [137, 138, 169]. There are two charges in the model (76), therefore, large-scale asymptotics of $G(k)$ are determined by the fixed point g_{1*}, g_{2*} as

$$\begin{aligned} G(k) &= \nu^2 k^{-d+2} R(\mu/k, g_1, g_2) \\ &\sim (D_{10}/g_{1*})^{2/3} k^{2-d-2y/3} R(1, g_{1*}, g_{2*}), \qquad \mu/k \to \infty, \end{aligned}$$

with the scaling function R. In this case [138],

$$Q(y,d) = \left[\frac{4(d-1)}{9u_{1*}^2}\right]^{1/3} \frac{\Gamma(2-y/3)\Gamma^{1/3}(d/2)\Gamma^{2/3}(d/2+y/2)}{\Gamma(d/2+y/3)\Gamma^{2/3}(2-y/2)} R(1, u_{1*}, u_{2*}).$$

The y expansion of $Q(y,d)$ at $d > 2$ contains poles at $d = 2$ [168] and the coefficients of $Q_k(d)$ in (89) take the form of a Laurent series

$$Q_k(d) = \sum_{l=0}^{\infty} q_{kl}\,(d-2)^{l-k}. \tag{90}$$

Thus, the y expansion of Q assumes the form

$$Q(y,d) = y^{1/3} \sum_{k=0}^{\infty} Q_k(d) y^k = y^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\frac{y}{d-2}\right)^k q_{kl}\,(d-2)^l. \tag{91}$$

On the other hand, in the double expansion in the ray scheme, the ratio $y/(d-2) = \zeta$ is kept fixed and the function Q acquires an alternative y expansion

$$Q(y,d) = y^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\frac{y}{d-2}\right)^k q_{kl}\,(d-2)^l = y^{1/3} \sum_{k=0}^{\infty} b_k(\zeta) y^k. \tag{92}$$

Here, in the y expansion on the right side, a subsequence of the double sum of (91) is summed which is different from the y expansion of (89). Complementary information in the n-loop approximation of the two expansions

$$Q_{y,d}^{(n)} \equiv y^{1/3} \sum_{k=0}^{n-1} Q_k(d) y^k, \qquad Q_{y,\zeta}^{(n)} \equiv y^{1/3} \sum_{k=0}^{n-1} b_k(\zeta) y^k,$$

can be used to improve numerical predictions for $Q(y,d)$ by adding the two and subtracting the common part to avoid double counting [137, 169] (see Fig. 1):

$$Q_{eff}^{(n)} = Q_{y,d}^{(n)} + Q_{y,\zeta}^{(n)} - \delta Q^{(n)}, \quad \delta Q^{(n)} = y^{1/3} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \zeta^k q_{kl}\,(d-2)^l. \tag{93}$$

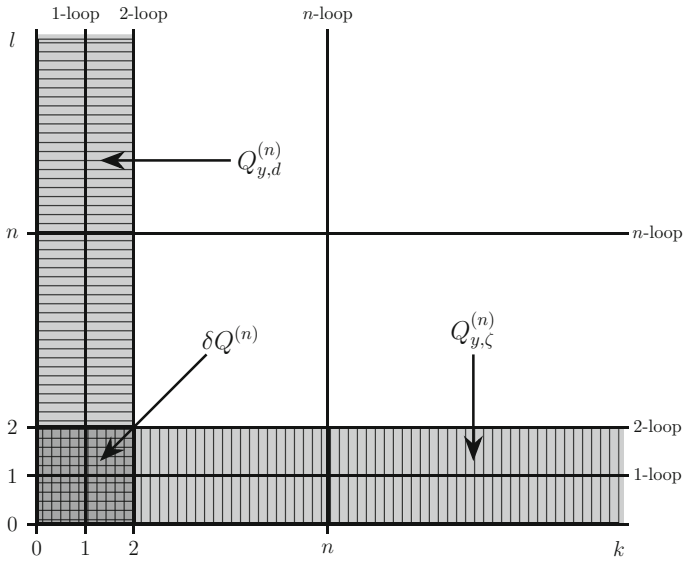

Fig. 1 Subsequences of the double sum (91) summed in the improved expansion (93). Terms in (93) included in $Q^{(n)}_{y,\zeta}$ from (92) and $Q^{(n)}_{y,d}$ from the usual y expansion (89), are depicted as the light-grey horizontal and vertical stripes, respectively. The correction term $\delta Q^{(n)}$ corresponds to the sum over the dark-grey square. Data and figure reproduced with kind permission of The European Physical Journal (EPJ) from L. Ts. Adzhemyan, M. Hnatich, and J. Honkonen, Eur. Phys. J. B **73**, 275 (2010); DOI: 10.1140/epjb/e2009-00432-8

This approach yields a significant improvement in the agreement of the calculated values of the Kolmogorov constant and the skewness factor with experimental values.

In the two-loop approximation, the results are [137, 169]

$$\mathcal{S} = -0.22, \qquad C_K = 2.37.$$

Moreover, due to the specific structure of the model, the double expansion at the physically most interesting fixed point is reduced to an expansion in the single parameter y [140]. Due to potential problems in the ray scheme, the normalization-point scheme was used in Ref. [138] with conditions on one-irreducible functions

$$\Gamma_{v'v} = \frac{\langle v'_i v_i \rangle_{1-\mathrm{ir}}|_{\omega=0}}{\nu k^2 (1-d)}\bigg|_{\substack{k=0 \\ \mu=m}} = 1, \quad \Gamma_{v'v'} = \frac{\langle v'_i v'_i \rangle_{1-\mathrm{ir}}|_{\omega=0}}{\nu^3 k^2 (d-1)} - g_1(\mu/k)^{d-2+y} - g_2\bigg|_{\substack{k=0 \\ \mu=m}} = 0.$$

The one-loop RG functions are ($u_1 = 2g_1/(4\pi)^{d/2}\Gamma(d/2), u_2 = 2g_2/(4\pi)^{d/2}\Gamma(d/2)$)

$$\gamma_1 = -\frac{3(d-1)}{4(d+2)}(u_1+u_2), \quad \gamma_2 = \frac{(d^2-2)(u_1+u_2)^2}{4d(d+2)\,u_2} - \frac{3(d-1)}{4(d+2)}(u_1+u_2).$$

Coordinates of the nontrivial fixed point $u_{1*} > 0$, $u_{2*} > 0$ are found as the solution of the equations $\beta_1(u_*) = 0$, $\beta_2(u_*) = 0$:

$$u_{1*} + u_{2*} = \frac{4y(d+2)}{3(d-1)}, \quad u_{2*} = \frac{y^2}{y+d-2}\,\frac{4(d^2-2)(d+2)}{9d(d-1)^2}. \tag{94}$$

Any relevant quantity is calculated as expansion in u_1 and u_2. Physical results are obtained at the fixed point of the RG. Fixed point values (94) are small when y is small regardless of d. In particular, for the physical value $d = 3$, the parameter $d - 2$ is not small and (94) indicate that $u_{1*} = O(y)$ and $u_{2*} = O(y^2)$ giving rise to an improved y expansion, which takes into account additional divergences near $d = 2$ to all orders in the deviation $d - 2$. These results can be extrapolated to values with a finite deviation from $d = 2$, since $d - 2$ is not needed as an expansion parameter.

There is an important difference between this procedure and the extrapolation of results of the $\varepsilon = 4 - d$ expansion to $d = 3$ in the theory of critical phenomena. In the latter extrapolation goes to values of d at which there is no regular way to directly take into account the IR divergences (below $d_c = 4$). In the Navier–Stokes problem, the extrapolation is made to values of d at which IR divergences are less severe than in the logarithmic theory [140].

It is of natural interest to determine the Kolmogorov constant and the skewness factor in higher orders of perturbation theory. However, already three-loop calculations pose a nontrivial numerical problem due to the extremely time-consuming integration procedures stemming from the large number of Feynman diagrams. The structure of the diagrams allows to perform calculations in the $1/d$ expansion of inverse space dimension d, which significantly reduces the number of relevant diagrams that have to be taken into account. In the three-loop approximation, only 80 diagrams survive out of 8160 calculated diagrams in the limit $1/d \to \infty$. This is a significant simplification of the computational process. In addition to general theoretical interest, such approach allows us to estimate how far the values of the Kolmogorov constant and the skewness factor deviate from their values for the physical dimension $d = 3$. The value of $Q(y, d)$ in leading order in $1/d$ with the precision y^2 is [170]

$$Q(y,d) = \frac{(2yd)^{1/3}}{3}\left\{1 + \frac{y}{36} + \left(\frac{49}{162} - \frac{\pi^2}{27}\right)\frac{y^2}{4} + O(y^3)\right\}. \tag{95}$$

Inserting formula (95) into (86) at $d = 3$ and $y = 4$, we get the following values in the three-loop approximation

$$\mathcal{S} = -0.73, \qquad C_K = 1.50.$$

Note that for the calculation of all physical quantities (including the Kolmogorov constant and the skewness factor) using field-theoretic methods an essential step was to employ the energy balance equation for stationary turbulence as an auxiliary tool, from which a relation was inferred between the stochastic noise amplitude and a physically relevant quantity—the energy dissipation rate $\bar{\mathcal{E}}$. It is possible to proceed in the opposite direction: viz the results of renormalization-group analysis of two-point

and three-point correlation functions can be used as an Ansatz for finding the three-dimensional kinetic energy spectrum $E(k,t)$, and hence the Kolmogorov constant, from the energy balance equation for decaying turbulence:

$$\partial_t E(k,t) = T(k,t) - 2\nu_0 k^2 E(k,t), \tag{96}$$

where the spectral kinetic energy transfer function $T(k,t)$ is defined through S_3.

It has been proposed to use (96) having turned off the energy pumping at time intervals, when the Reynolds number significantly exceeds critical values [171], i.e., when the steady solutions for two-point and three-point correlation functions in the RG approach retain their form, but the amplitudes (e.g., $\bar{\mathcal{E}}$) and the turbulent length scale L in begin to depend on time:

$$E(k,t) = C'_K [\bar{\mathcal{E}}(t)]^{2/3} k^{-5/3} F(kL(t)). \tag{97}$$

Here, the Kolmogorov constant C'_K for the three-dimensional kinetic energy spectrum is related to the Kolmogorov constant in S_2 (83) as

$$C_K = \frac{3 \times 2^{1/3}\,\Gamma(2/3)\,\Gamma(d/2)}{(d+2/3)\,\Gamma(d/2+1/3)}\, C'_K. \tag{98}$$

Directly at a fixed point, the transfer function $T(k,t)$ can be represented as a twofold integral over a particular region of the wave-vector space of a quadratic form of the forcing function $F(kL(t))$ in (97) [172]—an approximation resembling the EDQNM approximation [94]. Inserting the Ansätze for $E(k,t)$ and $T(k,t)$ into the energy balance equation (96), we get an integro-differential nonlinear equation for the function $F(kL(t))$ in the energy-containing interval. Its form can be estimated if we require analyticity in the zero scale limit and asymptotic power behavior governed by the exponent of the Kolmogorov value $-5/3$ in the inertial interval. Parametrization with four parameters b_1, b_2, b_3, and b_h in the form

$$F(x) = \frac{x^{11/3}}{(x^4 + 2b_2^2 x^2 + b_1^4)^{11/12}[1 + b_h(b_3^2 + x^2)^{-1/3}]}, \qquad x \equiv kL(t) \tag{99}$$

allows us to calculate values of the parameters with high accuracy by numerically solving the integro-differential equation. Expression

$$C'_K = \frac{d}{2}\left(\int_0^\infty dx\, x^{-5/3} F(x)\right)^{-1}$$

determines the value of the Kolmogorov constant in the three-dimensional kinetic energy spectrum for the function F (99): $C'_K = 1.578$ or $C_K = 2.075$. The solution (99) allows to obtain a theoretical curve for the longitudinal turbulence spectrum [93] and compare it with the experimental data in the energy-containing interval. Figure 2

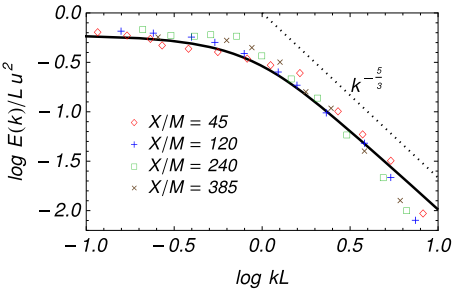

Fig. 2 Comparison of the theoretically predicted the longitudinal kinetic energy spectrum $E(k)$ normalized by RMS velocity u and the experimental data in the range $0.1/L < k < 10/L$. The experimental points correspond to data taken from Ref. [173]. They were chosen at different past grid distances X related to decay time t; the dimensionless ratio X/M represents the relative distance measured in mesh length units M. Data and figure reproduced with permission from L. Ts. Adzhemyan, M. Hnatich, D. Horváth, and M. Stehlik, Phys. Rev. E **58**, 4511 (1998); DOI: 10.1103/PhysRevE.58.4511; copyright (1998) by the American Physical Society

demonstrates a good agreement between the theoretical curve and data from different experiments behind the grid [171].

Table 2 contains values of the Kolmogorov constant C_K in S_2 (83) and C'_K in the energy spectrum (97) obtained analytically or by numerical simulation. To complete columns, quoted results for C_K are recalculated to C'_K and vice versa by (98). Dependence of calculations on the amplitude of forcing is labeled "NU". For general picture, values of the skewness factor are calculated by (86) or (84). For completeness, we also quote results for skewness factor obtained by authors themselves, and corresponding works are denoted with asterisk in Table 2. For comparison with the experimental data, we quote the averaged value of C'_K by Sreenivasan [153], based on the analysis of experimental data for one-dimensional spectra in the past grid turbulence, laboratory shear flows and geophysical flows. We also quote experimental data [155] with investigation of dependence of C_K on large-scale fluctuations of energy injection.

5 Functional approach to master equation

In previous sections, we have analyzed particular turbulence problems, which are probably one of the most famous far from equilibrium systems. As we have seen their theoretical investigation is based on a formulation via stochastic partial differential equation of the Langevin type. The construction of governing dynamic equations follows from symmetry considerations and a deep understanding of the physical nature of the problem. The properties of random noise are postulated ad hoc without reference to microscopic degrees of freedom. However, systems exist for which such an approach is not feasible, and alternative methods are called for. From a theoretical point of view, a particularly interesting class is formed by reaction–diffusion systems [36, 47, 178]. Their applications can be found in various research areas such as population dynamics, the creation of fractal structures, epidemic modeling, sociology, etc. In practical terms, we can interpret reaction–diffusion systems as interacting classical stochastic

Table 2 The C'_K is notation of 3-dimensional Kolmogorov constant

Author (year), [Ref.]	C'_K	C_K	S	Remarks
Yakhot et al. (1986) [157]	1.62	2.13	-0.49^*	RG, 1-loop, NU
Adzhemyan et al. (1989) [77]	2.99	3.93	-0.10	RG, 1-loop, NU
Zhou et al. (1989) [159]	1.44	1.89	-0.31	RG
Carati (1990) [164]	1.54	2.03	-0.28	RG, 1-loop, NU
Smith et al. (1992) [174]	1.62	2.13	-0.59^*	RG, 1-loop, NU
Giles (1994) [166]	1.44	1.89	-0.31	RG, 1-loop, NU
Sreenivasan (1995) [153]	1.62 ± 0.17	2.13 ± 0.22	-0.26 ± 0.03	Review article
Yeung et al. (1997) [160]	1.62	2.13	-0.26	DNS $R_\lambda = 240$
Adzhemyan et al. (1998) [171]	1.58	2.08	-0.27	
Gotoh et al. (2002) [161]	1.64 ± 0.04	2.16 ± 0.06	$-0.55 \pm 0.01^*$	DNS $R_\lambda = 460$
Adzhemyan et al. (2003) [168]	1.12	1.47	-0.45	1-loop
	2.29	3.02	-0.15	2-loop
	1.33	1.75	-0.58	1-loop, $d \to \infty$
	1.48	1.94	-0.49	2-loop, $d \to \infty$
Kaneda et al. (2003) [162]	(1.6, 1.7)	(2.10, 2.23)	$(-0.24, -0.26)$	DNS $R_\lambda = 1201$
Adzhemyan et al. (2003,2005) [137, 169]	1.36	1.79	-0.33	RG 1-loop
	1.80	2.37	-0.22	2-loop
Adzhemyan et al. (2008) [170]	1.14	1.50	-0.73	3-loop, $d \to \infty$
Donzis et al. (2010) [175]	1.58	2.08	-0.27	DNS
Adzhemyan et al. (2010) [138]	1.44	1.89	-0.31	RG, 1-loop
Ni et al. (2013) [154]	1.68	2.21	-0.24	Experiment
Chien et al. (2013) [155]	(1.23, 1.56)	(1.62, 2.05)	$(-0.27, -0.39)$	Experiment
Ishihara et al. (2016) [176]	1.8 ± 0.1	2.37 ± 0.02	-0.22 ± 0.01	DNS $R_\lambda = 1423$
Zhou (2021) [177]	(1.5, 1.7)	(1.97, 2.24)	$(-0.24, -0.29)$	Review article

DNS—direct numerical simulation and R_λ is Taylor microscale Reynolds number ($R_\lambda \approx \sqrt{Re}$). The values are rounded to two decimal places

systems in which the microscopic degrees of freedom evolve according to prescribed probabilistic rules. Remarkably, despite the notorious difficulty of non-equilibrium models, they might display a substantial simplification under specific circumstances and can be, in fact, analytically tractable. We anticipate such behavior to occur when a studied system undergoes a continuous phase transition akin to equilibrium second-order phase transitions [47, 48]. At criticality, underlying degrees of freedom behave coherently over many spatiotemporal scales, which gives rise to self-similar scaling behavior. A particular hallmark is a large correlation length with respect to both time and spatial directions. In the critical region, it is then permissible [6] to model a system using the continuous (mesoscopic) field theory. The ensuing model allows us to investigate universal quantities that correspond to the gross properties of the system. Moreover, the properties of a noise variable naturally arise during the derivation process.

Arguably the most fundamental description for reaction–diffusion problems starts at the level of the master equation [179, 180]. Since there is a formal similarity to quantum-mechanical problems, it is often possible to employ quantum mechanical methods (such as occupation number representation [36, 181]) to recast master equation in terms of appropriate action functional, which serves as a weighting factor for a specific realization of a stochastic process.

In what follows, we succinctly summarize general principles behind the Doi–Peliti formalism [182, 183] that effectively incorporates these ideas. We show how the master equation can be mapped onto a suitable Fock space, from which we further derive a continuum limit. The main outcome is an action functional amenable to ensuing systematic treatment by powerful field-theoretic methods such as the Feynman diagrammatic technique, renormalization group and operator-product expansion [6, 7]. The general aim is to determine and quantitatively analyze large-scale asymptotic properties of a model.

As a prototypical example, we focus on a continuum model corresponding to directed percolation universality class [41], which serves us for a demonstration of particularities arising in the Doi–Peliti approach. In the end, we briefly summarize results derived through field-theoretic RG techniques for other prominent reaction processes displaying scaling behavior that fall outside the percolation universality class.

Percolation process is probably the most common and famous paradigmatic model of fractal growth. Initially, it was developed [184] to model a spreading of fluid through an irregular porous medium in the presence of some external force, e.g., gravity. In biological terms, DP can be interpreted as a simple model for the epidemic process in which there is no immunization of infected individuals. In the evolution of infectious diseases, sick individuals can spread infection to neighbors. The infected individuals are allowed to recover, but later the individuals can be sick again. The simple model exhibits a critical region between the state without the disease (the absorbing state) and the epidemic (the active state). The existence of an absorbing state is precisely the crucial physical property that determines the non-equilibrium character of directed percolation. In particular, ergodicity together with detailed balance condition are explicitly violated.

Applications of directed percolation have been found in diverse research areas such as high energy physics [47, 185, 186], population dynamics [187], reaction–diffusion problems [188], modeling of forest fires spreading [189], etc. Despite its prominence in theoretical physics, critical exponents were measured only in a relatively few experiments. The directed percolation phase transition was experimentally studied in nematic liquid crystals [190, 191], the transition from laminar to turbulent flow in channel flow [192, 193] and in Couette flow [194, 195]. Actual experimental verifications are surprisingly low, especially as various possible experiments have been suggested in the past. Clearly, the direct experimental investigation of directed percolation constitutes a formidable task itself.

Let us remark that the Doi–Peliti approach is versatile enough to incorporate additional physical effects [49] as multiple reaction schemes, disorder effects, the influence of spatial boundaries, etc. Since the corresponding area is rather huge, for details, we refer the interested readers to the existing literature on the subject [36, 49, 50, 188].

Also the analysis of asymptotic behavior in reaction–diffusion systems is not limited to perturbative RG methods. Significant contributions have been also obtained through non-perturbative functional RG [196–198], Monte-Carlo simulations [47, 199], and mapping to quantum system [200–202]. Since these methods fall outside our present scope, we do not attempt to give an exhausting overview of all related topics.

5.1 The Doi–Peliti approach

We begin with a non-equilibrium system located at a single site (zero-dimensional field theory). Further, we assume that there are countably many states in which the system can be found. We enumerate possible states by non-negative integer index n. Then, a complete statistical description of such a system can be given in terms of evolution equations for probabilities $p_n(t)$, where n corresponds to a given state. Time evolution of probabilities $p_n(t)$ is accounted for by the master equation that in the compact notation takes the following form:

$$\frac{dp_n(t)}{dt} = \sum_m R_{m\to n}\, p_m(t) - \sum_m R_{n\to m}\, p_n(t), \tag{100}$$

where $R_{m\to n}$ is the transition probability rate from the state m to the state n. The first term on the right-hand side accounts for the increase in $p_n(t)$ because of the incoming probability to the state n, whereas the second term accounts for decrease in probability $p_n(t)$ due to outgoing processes.

Our aim is now to recast the master equation (100) in a suitable form of functional integrals, which can be further analyzed by field-theoretic methods. To this end, we follow a well-known route [49, 50, 203]. First, we convert the master equation to a Schrödinger-like equation containing creation and annihilation operators. The associated state space bears a close resemblance to Fock space in quantum mechanics [204, 205]. To perform the required transition, we assign to a state n a ket vector $|n\rangle$ of the Hilbert space, which is composed of a vacuum state $|0\rangle$ with zero "particles" (excitations), along with n particles created by a successive application of the creation operator $a^\dagger$. The creation operator $a^\dagger$ and the annihilation operator a are also introduced according to the standard rules [183, 203, 204]

$$a^\dagger\,|n\rangle = |n+1\rangle\,, \quad a\,|n\rangle = n\,|n-1\rangle\,. \tag{101}$$

From these relations, we can directly infer the expected commutation relation

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \tag{102}$$

The Hilbert space is further endowed with the inner product between two states $\langle m\,|n\rangle = n!\delta_{n,m}$. Among others this implies that the identity can be written as $1 = \sum_n \frac{1}{n!}\,|n\rangle\,\langle n|$ and also that operators a and $a^\dagger$ are Hermitian conjugate to each other [204].

The whole set of probabilities $p_n(t)$ can be compactly encoded into the state vector $|\phi(t)\rangle$ defined as

$$|\phi(t)\rangle = \sum_{n=0}^{\infty} p_n(t)\,|n\rangle\,. \tag{103}$$

Note that such definition leads to substantial differences with the usual relations known from quantum mechanics. Physically relevant quantities cannot be given by a bilinear form in ket vector $|\phi(t)\rangle$.

Using this formalism, we can transfer information about the time evolution of probabilities $p_n(t)$ onto the state vector $|\phi(t)\rangle$. Equation (100) is rewritten into a form of an imaginary time Schrödinger-like equation

$$\frac{d\,|\phi(t)\rangle}{dt} = L\,|\phi(t)\rangle\,, \tag{104}$$

where we have introduced the Liouville operator (also known in the literature as Liouvillian [183] or pseudo-Hamiltonian [36]) L, whose concrete form can be readily obtained once rates $R_{n\to m}$ are specified. By a formal exponentiation of Eq. (104), we get

$$|\phi(t)\rangle = U_{t,0}\,|\phi_0\rangle\,, \qquad U_{t,t_0} = \exp\left(L(t-t_0)\right), \tag{105}$$

where operator U_{t,t_0} plays a role of time evolution operator akin to quantum-mechanical propagator, and $|\phi_0\rangle = |\phi(t=0)\rangle$ is an initial state at time instant $t=0$. Hereinafter, we always set lower bound $t_0 = 0$ in the corresponding expressions.

For the later use of functional integrals, it is useful to utilize Bargmann–Segal representation [183, 206, 207] based on an equivalence between analytic functions $\phi(z)$ of a complex variable z with the aforementioned Hilbert space, i.e., $|\phi\rangle = \sum_n \phi_n\,|n\rangle \leftrightarrow \phi(z) = \sum_n \phi_n z^n$. The scalar product between two states is defined as $\langle\phi\,|\psi\rangle = \sum_n n!\phi_n\psi_n$ and be further conveniently expressed through an integral formula $\langle\phi\,|\psi\rangle = \int\int \frac{dzdz'}{2\pi}\phi(z)\psi(iz')\exp\left(-izz'\right)$, where both integrals are taken over the entire real axis. To a given operator A with matrix elements $A_{m,n} = \langle m|A\,|n\rangle$, we associate the kernel

$$A(z,\xi) = \sum_{m,n} \frac{z^m}{m!}\frac{\xi^n}{n!}A_{m,n}. \tag{106}$$

For our discussion, two important formulas are needed [183, 203]. First, if two states $|\phi_1\rangle$ and $|\phi_2\rangle$ related by $|\phi_1\rangle = A\,|\phi_2\rangle$, then we have a relation for the function $\phi_1(z)$ associated with the state $|\phi_1\rangle$ of the following form:

$$\phi_1(z) = \int \frac{\mathrm{d}\xi\,\mathrm{d}\xi'}{2\pi}A(z,\xi)\phi_2(i\xi')\exp\left(-i\xi\xi'\right). \tag{107}$$

The second formula is related to a kernel of the product of two operators A and B, and can be presented as a double integral [183, 207]

$$AB(z,\xi) = \int \frac{\mathrm{d}\varphi \mathrm{d}\tilde{\varphi}}{2\pi} A(z,\varphi) B(i\tilde{\varphi},\xi) \mathrm{e}^{-i\varphi\tilde{\varphi}}. \tag{108}$$

In this formula, both integration variables φ and $\tilde{\varphi}$ are real, and it is also possible to consider complex valued $\tilde{\varphi}$ [36, 208]. To a given operator A, we can formally assign its normal order form as follows $A \to \mathcal{A} = \sum_{m,n} \mathcal{A}_{m,n} a^{\dagger m} a^n$. The corresponding kernel is then defined in the usual quantum-mechanical sense $\mathcal{A}(z,\xi) = \sum_{m,n} \mathcal{A}_{m,n} z^m \xi^n$, which assumes slightly different normalization than the previously defined kernel (106). It can be shown [207] that kernels $A(z,\xi)$ and $\mathcal{A}(z,\xi)$ are proportional to each other through the relation $A(z,\xi) = \mathrm{e}^{z\xi} \mathcal{A}(z,\xi)$. With the operator L we thus associate its normal-ordered form $\mathcal{L}$. The transition from a given operator expression to a corresponding kernel simply amounts to the replacement $a^\dagger \to i\tilde{\varphi}$ and $a \to \varphi$ [183, 203]. With this machinery at our disposal, it is possible by relatively straightforward algebraic manipulations to derive a functional representation for the time evolution operator $U_{t,0}$ introduced in Eq. (105). The derivation consists of two steps. First, with the operator $U_{t,0}$, we associate the kernel $U_{t,0}(z,\xi)$ employing the definition (106). Second, according to the well-known Trotter product formula [209], we may express $U_{t,0}$ as the infinite product

$$U_{t,0} = \lim_{n\to\infty} \left(1 + \frac{t}{n} L\right)^n = \lim_{\tau\to 0} (1 + \tau L)^{t/\tau}, \tag{109}$$

where in the last relation, we have introduced a small time interval τ defined as $\tau \equiv \frac{t}{n}$. Then, the kernel associated with each of the n factors in Eq. (109) can be expressed through the normal kernel as follows:

$$(1 + \tau L(z,\xi)) = \mathrm{e}^{z\xi} (1 + \tau \mathcal{L}(z,\xi)). \tag{110}$$

Applying this together with the formula for the kernel products (108), we finally get the discretized formula for the kernel of the time evolution operator

$$U_{t,0}(z,\xi) = \lim_{n\to\infty} \int\int \prod_{j=1}^{n-1} \frac{\mathrm{d}\varphi_j \mathrm{d}\tilde{\varphi}_j}{2\pi} \exp\Bigg(- \sum_{k=1}^{n-1} \left[i\tilde{\varphi}_k (\varphi_k - \varphi_{k-1}) - \tau \mathcal{L}(i\tilde{\varphi}_k, \varphi_{k-1}) \right] + \tau \mathcal{L}(z,\varphi_{n-1}) + z\varphi_{n-1} \Bigg), \tag{111}$$

where the standard approximation for the exponential function has been used. The expression in the exponential function takes a slightly different form than common in the literature [183, 203]. The discretization points were chosen as $\tilde{\varphi}_k = \tilde{\varphi}(t_k)$, $\varphi_k = \varphi(t_k)$, where $t_k = k\tau$ and $k = 0, 1, \ldots, n$. The appropriate boundary conditions are $\varphi_0 = \xi$, and $i\tilde{\varphi}_n = z$. It should be noted that the time ordering of the fields in

(111) – which is a consequence of the normal form of the action—leads to perturbation expansion without self-contracted propagators, as in Sect. 2.

Expectation value of many relevant quantities Q takes the form

$$\langle Q(t)\rangle = \sum_{n=0}^{\infty} Q(n)\, p_n(t), \tag{112}$$

where $Q(n)$ is the value of a quantity Q in state n-th state. To represent expectation values (112) in the operator formalism, it is convenient to introduce the projection vector $\langle P|$:

$$\langle P| = \sum_{n=0}^{\infty} \frac{1}{n!}\, \langle n| = \sum_{n=0}^{\infty} \frac{1}{n!}\, \langle 0|\hat{a}^n = \langle 0|\mathrm{e}^{\hat{a}}. \tag{113}$$

The projection vector is a left eigenvector of the creation operator with the eigenvalue equal to unity

$$\langle P|\, a^\dagger = \langle P|. \tag{114}$$

From here, it follows that

$\forall\, n : \langle P|n\rangle = 1.$

Basis states $|n\rangle$ are eigenstates of the number operator $n = a^\dagger a$, therefore, we get $\langle P|(a^\dagger a)^m\, |n\rangle = n^m \langle P|n\rangle = n^m$.

With help of the projection operator, we readily recast (112)

$$\langle Q(t)\rangle = \langle P|Q|\phi(t)\rangle. \tag{115}$$

Further, using aforementioned Bargmann–Segal representation, it is straightforward [183] to derive

$$\langle Q(t)\rangle = \int \frac{\mathrm{d}z\mathrm{d}z'}{2\pi} \int \frac{\mathrm{d}\xi\mathrm{d}\xi'}{2\pi} \mathrm{e}^{z - izz'}\, Q(iz', \xi)\mathrm{e}^{-i\xi\xi'}\phi(i\xi'). \tag{116}$$

If quantity Q is represented in its normal form $Q = \sum_m \mathcal{Q}_{mm} a^{\dagger m} a^n$, the expression (116) takes even simpler form

$$\langle Q(t)\rangle = \int \frac{\mathrm{d}z\mathrm{d}z'}{2\pi} \mathrm{e}^z \sum_m \mathcal{Q}_{mm} z^m \phi(iz'). \tag{117}$$

For practical purposes, here, we choose the stochastic Verhulst (logistic) model of population growth as a typical representative of reaction–diffusion models. Its importance is related to the fact that it exhibits critical behavior that corresponds to the well-known directed percolation universality class [41]. Its microscopic description

can be effectively formulated by means of a particular reaction–diffusion system consisting of three reaction processes for interacting species A

$$A \xrightarrow{K_{10}} \varnothing, \quad A \xrightarrow{K_{12}} 2A, \quad 2A \xrightarrow{K_{21}} A, \tag{118}$$

where $\varnothing$ corresponds to an inert particle not participating in the reaction process anymore. Three processes in (118) can be interpreted [47] as death process (self-destruction), birth process (offspring production), and damping (coagulation) necessary to avert the population explosion, respectively. At the level of the rate equation (mean-field approximation with complete neglect of spatial fluctuations), we get the dynamic equation $\partial_t n(t) = (K_{12} - K_{10})n(t) - K_{21}n(t)^2$ for the total number of infected individuals $n = n(t)$. This dynamic equation predicts the existence of two fixed points: $n_1^* = 0$ and $n_2^* = (K_{12} - K_{10})/K_{21}$. It is easy to see that the former is stable in the limit $t \to \infty$ for $K_{12} < K_{10}$, whereas the latter becomes stable for $K_{12} > K_{10}$. A state corresponding to n_1^* is known as an absorbing state, and n_2^* is an active state. Naively, we can guess that the transition between these two states corresponds to the critical region, which in effect, more sophisticated methods confirm. It is precisely this region that is particularly interesting as it exhibits all the features associated with criticality, i.e., strong fluctuations, universal behavior, power-law scaling, etc.

Indicated factors K in (118) are rate constants for a given reaction process. In terms of transition probabilities $R_{m\to n}$ (see Eq. (100)), the reaction scheme (118) is transposed into the following reaction rate:

$$R_{n\to m} = K_{10} n \delta_{n,m+1} + K_{12} n \delta_{n+1,m} + K_{21} n(n-1)\delta_{n,m+1}. \tag{119}$$

By direct manipulations with creation and annihilation operators, we obtain the corresponding normal-ordered Liouville operator

$$\mathcal{L}_{\text{reac}} = K_{10}\left(1 - a^\dagger\right) a + K_{12}\left(a^\dagger - 1\right) a^\dagger a + K_{21}\left(1 - a^\dagger\right) a^\dagger a^2. \tag{120}$$

To account for spatial fluctuations in infected individuals, it is customary, first, to place reaction schemes on the regular (infinite) lattice and then introduce an unbiased hopping process between neighboring lattice sites. This can be readily modeled by the following Liouville operator [183, 204]:

$$\mathcal{L}_{\text{diff}} = \sum_{\langle i,j \rangle} D_0 \left[a_i^\dagger a_j - a_i^\dagger a_i \right], \tag{121}$$

where creation and annihilation operators are now enumerated with the additional lattice index i, and summation runs over all neighboring sites. Coarse-grained continuum limit in d-dimensional space yields then a contribution of the form

$$i D_0 \int \mathrm{d}s \int \mathrm{d}\boldsymbol{x}\, \partial_j \tilde{\varphi}(s, \boldsymbol{x}) \partial_j \varphi(s, \boldsymbol{x}), \tag{122}$$

where discrete index i (see Eq. (121)) has been replaced by the continuous spatial variable $\boldsymbol{x} \in \mathbb{R}^d$, and summation over index j is implied.

For many physical applications, the continuous limit of the kernel (111) plays a crucial role [49, 183, 210]. In terms of continuous fields $\varphi(t)$ and $\tilde{\varphi}(t)$, it takes the form of the following functional integral:

$$U_{t,0}(z,\xi) = \int\int \mathcal{D}\varphi\mathcal{D}\tilde{\varphi} \exp\left(-\int_0^t \mathrm{d}s \left[i\tilde{\varphi}(s)\partial_s\varphi(s) - \mathcal{L}(i\tilde{\varphi}(s),\varphi(s))\right] + i\tilde{\varphi}(t)\varphi(t)\right), \quad (123)$$

where s plays a role of time-like integration variable, and numerical factor 2π was absorbed in the functional measure. The expression appearing in the argument of the exponential function of Eq. (123)

$$\mathcal{L}_{\mathrm{eff}} = -\int_0^t \mathrm{d}s \left[i\tilde{\varphi}(s)\partial_s\varphi(s) - \mathcal{L}(i\tilde{\varphi}(s),\varphi(s))\right] + i\tilde{\varphi}(t)\varphi(t) \quad (124)$$

can be interpreted as the effective Liouville operator for a given model.

By taking into account spatial fluctuations due to the diffusion term (122), performing naive continuum limit and integrating by parts, we finally arrive at the total Liouville operator governing time evolution of directed percolation process

$$\begin{aligned}\mathcal{L}_{\mathrm{DP}} = -\int_0^t \mathrm{d}s \int \mathrm{d}\boldsymbol{x} \,[&i\tilde{\varphi}(s,\boldsymbol{x})\partial_s\varphi(s,\boldsymbol{x}) - iD_0\tilde{\varphi}(s,\boldsymbol{x})\partial^2\varphi(s,\boldsymbol{x}) \\ &- \mathcal{L}_{\mathrm{reac}}(i\tilde{\varphi}(s,\boldsymbol{x}),\varphi(s,\boldsymbol{x}))] + i\tilde{\varphi}(t)\varphi(t),\end{aligned} \quad (125)$$

where ∂^2 is the Laplace operator in d spatial dimensions (see Eq. (29)). Inserting the Liouville operator responsible for the reaction part (120), we get

$$\begin{aligned}\mathcal{L}_{\mathrm{DP}} = -\int_0^t \mathrm{d}s \int \mathrm{d}\boldsymbol{x} \Big[&i\tilde{\varphi}(s)\partial_s\varphi(s) - iD_0\tilde{\varphi}(s)\partial^2\varphi(s) - K_{10}(1-i\tilde{\varphi})\varphi \\ &- K_{12}(i\tilde{\varphi}-1)i\tilde{\varphi}\varphi - K_{21}(1-i\tilde{\varphi})i\tilde{\varphi}\varphi^2\Big] + i\tilde{\varphi}(t)\varphi(t),\end{aligned} \quad (126)$$

where, for brevity, we have suppressed spatial arguments in fields φ and $\tilde{\varphi}$. To get rid of the linear term, it is customary to redefine $\tilde{\varphi}$ field by letting $i\tilde{\varphi} - 1 \to \tilde{\varphi}$. The ensuing Liouville operator takes the form

$$\mathcal{L}_{\mathrm{DP}} = -\int_0^t \mathrm{d}s \int \mathrm{d}\boldsymbol{x} \Big[\tilde{\varphi}(s)\partial_s\varphi(s) - D_0\tilde{\varphi}(s)\partial^2\varphi(s) + (K_{10}-K_{12})\tilde{\varphi}\varphi$$

$$- K_{12}\tilde{\varphi}^2\varphi - K_{21}\tilde{\varphi}(\tilde{\varphi}+1)\varphi^2\Big]+\tilde{\varphi}(t)\varphi(t)+\varphi(0). \tag{127}$$

For RG applications, more relevant quantity than the Liouville operator is played by field-theoretic action since the latter enters as weight factor expressions for various physical quantities. It can be readily shown that transition from the Liouville operator (127) to corresponding field-theoretic action for directed percolation consists of two steps. First, the detailed derivation reveals [183] that the boundary terms (last two terms in the action (127)) drop out in actual calculations of physically relevant quantities such as average number, etc. This feature can be traced back to the formula (116), in particular, to two exponential factors. Therefore, we neglect boundary terms in what follows. Second, the quartic term can be discarded by the straightforward application of RG analysis, which reveals its IR irrelevance [41]. We have seen that integration for the response field $\tilde{\varphi}$ is implicitly carried over the imaginary axis ranging from $-i\infty$ to $+i\infty$. After some trivial rescaling and relabeling time variable in the usual sense, i.e., $s \to t$, we finally obtain famous De Dominicis–Janssen action functional of the DP process [7, 36, 41, 211]

$$\mathcal{S}_{\rm DP}(\tilde{\varphi},\varphi) = \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left\{\tilde{\varphi}(-\partial_t + D_0\partial^2 - D_0\tau_0)\varphi + \frac{D_0 u_0}{2}(\tilde{\varphi}^2\varphi - \tilde{\varphi}\varphi^2)\right\}, \tag{128}$$

where spatiotemporal arguments of fields φ and $\tilde{\varphi}$ are not written explicitly, u_0 plays the role of small coupling constant, and τ_0 represents the deviation from the criticality. For instance, in directed bond (site) percolation on lattice [48], the parameter $\tau = p - p_c$ corresponds to the deviation of the probability of open bonds p from its critical value p_c. In critical static theory, in the standard φ^4-model [6, 41], the temperature variable plays an analogous role. As has been already mentioned, in actual calculations, we employ a dimensional regularization, and therefore in all relevant expressions we explicitly write d as a general spatial dimension. Action (128), in fact, corresponds to the Reggeon field theory, which exhibits additional rapidity reversal symmetry $\varphi(t,\boldsymbol{x}) \leftrightarrow \tilde{\varphi}(-t,\boldsymbol{x})$. This has nontrivial consequences for scaling relations, in particular, it decreases the total number of independent critical indices from expected four to three [41, 211].

Let us note that in order to derive a action functional for general reaction problems it is also possible to proceed in a different fashion. A common approach is to employ the coherent-state representation known from quantum many-particle systems [181]. A detailed exposition appertaining to reaction–diffusion systems can be found in Ref. [49].

On the other hand, in Ref. [212], an additional derivation was put forward based on the formalism of Green functions relying on functional methods [76]. Such approach yields a renormalizable field-theoretic action in a transparent and effective way. In Ref. [50], further supplementary details are discussed.

For the sake of completeness, we note that for certain models, it is possible to construct a field-theoretic action by direct phenomenological considerations, which heavily depends on a particular problem and can be justified only a posteriori. Directed

percolation belongs to such systems and the form of action (128) can be obtained by several guiding principles:

1. The only slow variable of the theory is the density of sick individuals $\varphi(t, \boldsymbol{x})$.
2. Time evolution can be characterized as that of a Markov process.
3. The individual is infected, and the corresponding rate is a function of the local density of sick individuals φ. After a certain time amount, the sick individuals spontaneously recover.
4. The state with no sick individuals ($\varphi = 0$) is considered absorbing and could not be left. Effectively, the spread of percolation terminates.
5. The disease proliferates solely through a diffusion process.
6. The effect of fast degrees of freedom can be captured by a proper choice of stochastic variable that reflects the absorbing condition.

Employing these principles one is naturally led to the following the stochastic differential equation:

$$\partial_t \varphi = D_0 \partial^2 \varphi - D_0 \tau_0 \varphi - \frac{D_0 u_0}{2} \varphi^2 + \sqrt{\varphi} f, \tag{129}$$

where $f = f(t, \boldsymbol{x})$ is the noise term. As such, stochastic differential equations with multiplicative noise are ambiguous and should be augmented with proper interpretation. The standard choice is the Itô prescription [70, 87]. The random noise field $f(t, \boldsymbol{x})$ is assumed to be a Gaussian random field with zero mean $\langle \zeta \rangle = 0$ and a pair correlation function obeying

$$\langle f(t, \boldsymbol{x}) f(t, \boldsymbol{x}') \rangle = u_0 D_0 \delta(t - t') \delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{130}$$

where the parenthesis $\langle \cdots \rangle$ stands for an averaging procedure over all possible realizations of the given stochastic process (Gaussian noise distribution). The multiplicative property of the noise variable in governing equation (129) is a direct consequence of the presence of a unique absorbing state in which all fluctuations have to cease once this state is reached. In contrast to turbulence, here, the source for randomness is entirely due to the intrinsic nature of underlying processes and does not correspond to some additional source of energy input.

We also readily see that the field-theoretic model (128) is indeed tantamount to the Langevin equation (129), since the action (128) can be readily related to the action (16) through the following substitution (see Eq. (1))

$$K = -D_0 \partial^2 + D_0 \tau_0, \quad U = -\frac{g \varphi^2}{2}, \quad B = \sqrt{\varphi}. \tag{131}$$

As has been pointed out at the beginning of Sect. 5, similar to critical statics, non-equilibrium systems such as (128) might exhibit scale-free behavior in critical region as well. Then, power-law dependencies are expected in scaling functions with exponents that depend only on gross features of the system, e.g., space dimension and symmetry properties [7, 48]. There are certain differences between dynamic (as DP and BAP) and

static models. In contrast to the equilibrium system, there is an important extension of scaling analysis. As has been already encountered previously in Sect. 3.2.1 dynamic models are described by two independent correlation lengths, in addition to the length scale related spatial correlations there is an additional scale $\xi_\parallel$ describing temporal correlations. Typically, in statistical physics, the models are not isotropic in the sense of models in relativistically invariant theories. On the contrary, they exhibit so-called strong anisotropic scaling with time variable playing a distinguished role [36, 47].

In the vicinity of the critical point, we expect for directed percolation that correlation lengths diverge satisfying some power-law dependence $\xi_\perp \sim |\tau_0|^{-\nu_\perp}, \quad \xi_\parallel \sim |\tau_0|^{-\nu_\parallel}$, where τ_0 was defined in Eq. (128). Exponents $\nu_\perp$ and $\nu_\parallel$ usually assume different values. It is convenient to define the dynamic exponent z as a fraction $z = \nu_\parallel/\nu_\perp$. Then in the critical region, the correlation lengths in the time and space direction are simply related as $\xi_\parallel \sim \xi_\perp^z$. From this, we can also infer that time and space scales are related by similar relation $t \sim |\boldsymbol{x}|^z$. In physics, two specific values are commonly encountered: $z = 1$ corresponding to Lorentz-covariant theory and $z = 2$ to diffusive spreading, respectively. Sub-diffusive propagation is characterized by values of the exponent $z > 2$, and super-diffusive propagation by values while $1 < z < 2$, respectively. As with any other static critical exponent, z is also a universal parameter characterizing a given universality class.

One additional exponent is needed for complete knowledge of the scaling behavior of the DP process at criticality. Usually, it is the exponent β, that quantitatively characterizes the scaling of the order parameter ρ, i.e., the density of particles in the active phase of a macroscopic system, in the critical region. The asymptotic form of this relation reads $\rho \sim (p_c - p)^\beta \propto (-\tau_0)^\beta$. One can directly apply scaling analysis [48] with the resulting critical exponents $\beta = 1, \quad \nu_\perp = \frac{1}{2}, \quad \nu_\parallel = 1$. These values correspond to the mean-field theory, which is valid above the upper critical dimension $d_c = 4$ [41].

The natural task is to calculate these exponents in physically relevant space dimensions. In this context, various methods have been used in the past. Analytical results for the critical exponents of the DP process are currently available up to the two-loop approximation [36, 41, 213]. From a more general perspective, perturbative calculations are feasible only for a few first terms in a loop expansion because the complexity of the calculation increases dramatically with each loop order. Practically all perturbative results for non-equilibrium models are confined to the two-loop order. Therefore, one of the practical aims we pursue is to develop numerical and algorithmic techniques to obtain reliable higher loop corrections. Recently, existing results have been improved up to the three-loop order [214]. The third-order perturbation theory presents a nontrivial task. It can be argued that its difficulty is an order of magnitude larger than the difficulty of the previous order. First, the number of Feynman graphs is much greater and next, there are also technical issues related to a proper extraction of the divergent parts and numerical evaluation of the remaining parts of Feynman diagrams. For illustration purposes, let us consider a renormalized two-point response function

$$W_{\psi\tilde{\psi}} = \langle \psi(t, \boldsymbol{x})\tilde{\psi}(0, \boldsymbol{0})\rangle. \tag{132}$$

This quantity satisfies the RG equation, which takes the form of first-order partial differential equation

$$\left(\mu\partial_\mu + \beta_g\partial_g - \tau\gamma_\tau\partial_\tau - D\gamma_D\partial_D - \gamma_\psi - \gamma_{\tilde{\psi}}\right) W_{\psi\tilde{\psi}} = 0, \tag{133}$$

where β_g is the beta function corresponding to the charge $g = u^2$, and γ_Q, $Q \in \{\tau, D, \psi, \tilde{\psi}\}$ are anomalous dimensions of parameters and fields.

RG functions can be determined by perturbation theory and usually, such a procedure is accompanied by technical difficulties. In the critical region, function (133) takes the asymptotic form

$$W_{\psi\tilde{\psi}}(t, \boldsymbol{x}, \tau) = t^{-2\Delta_\psi/\Delta_\omega} f\left(\frac{\boldsymbol{x}}{t^{1/\Delta_\omega}}, \frac{\tau}{t^{-\Delta_\tau/\Delta_\omega}}\right). \tag{134}$$

We stress that the critical exponents $\nu_\perp$, $\nu_\parallel$, and z defined previously can be expressed through the critical dimensions Δ_ω, Δ_τ and their ratios as follows $\nu_\perp = 1/\Delta_\tau$, $\nu_\parallel = \Delta_\omega/\Delta_\tau$ and $z = \Delta_\omega$.

In the case of the DP process on a lattice [48], the response function is the primary building block for a definition of the number of active sites $N(t, \tau)$ propagate that from the original seed

$$N(t, \tau) = \int \mathrm{d}^d x \; W_{\psi\psi'}(t, \boldsymbol{x}, \tau) \sim t^\theta F\left(\tau t^{\Delta_\tau/\Delta_\omega}\right). \tag{135}$$

This quantity is relevant for seed simulations [48]. The latter relation in Eq. (135) stands for corresponding asymptotic behavior, controlled by the critical exponent θ and valid at criticality. The critical exponent θ can be expressed using critical dimensions of field ψ and frequency ω by the following formula:

$$\theta \equiv \frac{d - 2\Delta_\psi}{\Delta_\omega}. \tag{136}$$

Another dynamic quantity is the mean square radius $R^2(t)$ of propagating particles from the origin at time $t = 0$, which can be introduced by the following formula:

$$R^2(t, \tau) = \frac{\int \mathrm{d}^d x \; \boldsymbol{x}^2 W_{\psi\psi'}(t, \boldsymbol{x}, \tau)}{\int \mathrm{d}^d x \; W_{\psi\psi'}(t, \boldsymbol{x}, \tau)} \sim t^{2/z}, \tag{137}$$

where $z = \Delta_\omega$ and the latter expression holds in the vicinity of the critical point, which corresponds to the limit $\tau \to 0$.

Let us note that the complete three-loop calculation has been recently accomplished [214], where the actual technical details can be found. The explicit expressions for critical indices up to ε^3 precision read

$$z = 2 - \frac{\varepsilon}{12} - 0.02920905(2)\varepsilon^2 + 0.029207(4)\varepsilon^3, \tag{138}$$

$$\nu_{\parallel} = 1 + \frac{\varepsilon}{12} + 0.02238280(2)\varepsilon^2 - 0.008169(3)\varepsilon^3, \tag{139}$$

$$\nu_{\perp} = \frac{1}{2} + \frac{\varepsilon}{16} + 0.021097832(11)\varepsilon^2 - 0.009594(2)\varepsilon^3, \tag{140}$$

$$\theta = \frac{\varepsilon}{12} + 0.037509726(13)\varepsilon^2 - 0.032978(3)\varepsilon^3, \tag{141}$$

where the number in brackets corresponds to a numerical error stemming from applied integration procedures.

Here, we have been mainly concerned with the paradigmatic model of directed percolation. However, there are many ways in which related classes of problems can be further generalized and investigated by means of RG methods [36, 41, 213, 215–217].

5.2 Irreversible reaction processes

Next, we briefly discuss a different class of reaction–diffusion models that also exhibit intriguing scaling behavior. In contrast to percolation models, their common feature is the presence of strong spatiotemporal correlations solely due to the inherent chemical kinetics, i.e., in these models scaling behavior is generated even without presence of a phase transition. The models can be again analyzed using the Doi–Peliti approach with subsequent field-theoretic treatment. For brevity, we concentrate on the main prototypical classes. A more exhausting exposition can be found in the existing literature [36, 49].

5.2.1 Single-species reactions

Probably, the simplest nontrivial example of reaction kinetics [36, 179, 188] is provided by a single-species pair annihilation process $A + A \to \varnothing$, where A is a reacting classical particle and $\varnothing$ is an inert particle that does not react anymore nor it affects remaining particles A. The A particles are subject to chaotic diffusive motion and may react after the mutual collision with rate K_0.

The usual approach to such kind of problems is based on the use of the kinetic rate equation. It leads to a self-consistent description analogous to the mean-field approximation in the theory of critical phenomena. Its basic assumption is that the particle density is spatially homogeneous (fluctuations in the concentration field are neglected). This homogeneity can be thought of as a consequence of either infinite mobility of the reactants or of a very small probability that a chemical reaction occurs when reacting entities meet each other [178, 179]. On the other hand, if the particle mobility becomes sufficiently small, or equivalently, if the microscopic reaction probability becomes large enough there is a possible transition to a new regime where it is more probable that the given particle reacts with local neighbors than with distant particles. This behavior is known as the diffusion-controlled regime [179]. For the pair annihilation process, an assumption of the density homogeneity leads to the following equation for the mean number density $\partial_t n(t) = -K_0 n^2(t)$, from which we infer a long-time asymptotic decay as $n(t) \sim t^{-1}$. We observe that the decay exponent does

not depend on the space dimension. This is a usual situation observed in the mean-field theory. However, it turns out [36, 218] that in lower space dimensions $d \leq 2$ the assumption of spatially uniform density, or equivalently of negligible density fluctuations of reacting particles, is not appropriate. If we assume that A particles are subject to diffusive spreading with diffusion constant D, the typical time interval for two particles at a mutual distance r to collide scales as $t \sim r^2/D$. Due to the reentrant property [219] of random walk in low space dimensions we expect the volume $V(t)$ covered by diffusing particles to scale as $V(t) \sim r^d$. From this, we immediately get that the particle density $n(t)$ should scale according to $n(t) \sim r(t)^{-d} \sim (Dt)^{-d/2}$ in $d < 2$ spatial dimensions. This naively corresponds to a slower decay than the mean-field prediction $n(t) \sim (K_0 t)^{-1}$. At the critical dimension, logarithmic corrections are expected $n(t) \sim \ln(Dt)/Dt$.

To provide a faithful confirmation of these results, it is necessary to employ more sophisticated methods. To obtain a proper continuum model a similar procedure to that for directed percolation can be followed. The major difference is caused by the underlying reaction processes. For pair annihilation, the corresponding reaction term (compare to (119) and (120)) takes the form $R_{n\to m} = K_0(n+1)(n+2)\delta_{n,m+2}$. The ensuing field-theoretic action reads

$$\begin{aligned}\mathcal{S}(\psi^\dagger,\psi) = &-\int_0^\infty \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\,\{\psi^\dagger\partial_t\psi - D_0\psi^\dagger\partial^2\psi + \lambda_0 D_0[2\psi^\dagger + (\psi^\dagger)^2]\psi^2\} \\ &+ n_0\int \mathrm{d}\boldsymbol{x}\,\psi^\dagger(0,\boldsymbol{x}),\end{aligned} \tag{142}$$

where λ_0 corresponds to the rescaled rate K_0 and n_0 denotes the initial number density. By brief inspection, we see that the "noise" term $\propto (\psi^\dagger)^2\psi^2$ in this action corresponds to the imaginary noise [205] in the equivalent Langevin equation. This reflects dynamic anti-correlations between reacting A particles, because mutual annihilation leaves behind vacant zones that effectively leads to slowing down of the subsequent reaction as the remaining particles have to first cross through appeared vacancies. In this regard, let us note that the naive construction of the field-theoretic model for the annihilation process based purely on Langevin equation with real noise leads to imprecise estimation of the upper critical dimension [220].

The model (142) was renormalized in Ref. [221] and later on comprehensively analyzed by Lee [218] using the field-theoretic RG method, which corroborated that the decaying exponent for the particle density equals $-d/2$ in low space dimensions. This analysis entails nontrivial resummation over the RG relevant parameter n_0. Technical complications can be to some extent circumvented using functional Legendre transform of the generating functional of renormalized Green functions [212]. The topic of multiscaling was studied for this particular reaction scheme [222], and actual experimental realization was found in effectively one-dimensional nanotube systems [223].

There are several possibilities for making a model for pair annihilation process more realistic. The majority of reactions necessarily occur in a liquid or gaseous environment. Hence, it is natural to investigate how scaling behavior in reaction systems is affected by additional effects of the ambient environment. In this regard, various

models have been introduced [36, 49]. It can be envisioned that the environment is subject to evolve according to its own dynamical rules and exerting additional drift on reacting particles. Most of the RG analyses of the effect of random drift on the pair annihilation process have been carried out for the case of a quenched random drift field. Potential random drift with long-range [224, 225] and short-range correlations [226] have been studied as well as turbulent flow (i.e., quenched solenoidal random field) with potential disorder [227, 228]. For a more realistic description of a turbulent flow, time-dependent velocity field would be more appropriate. In Ref. [228], dynamic disorder with a given Gaussian distribution has been considered, whereas the most ambitious approach based on a velocity field generated by the stochastic Navier–Stokes equation was considered in Ref. [212]. As we have seen in Sect. 4.2, the situation for the Navier–Stokes equation near the critical dimension $d_c = 2$ of the pure reaction model is even more intriguing from RG point of view. Calculations in Ref. [212] were performed in the one-loop approximation and later on improved in Ref. [229] to the two-loop precision. The RG method predicts instability of pair annihilation process with respect to ever-present fluctuations of the ambient environment, i.e., the previously found asymptotic behavior induced by density fluctuations only [218] in the diffusion-controlled regime is unstable to any velocity fluctuations including thermal fluctuations. The asymptotic behavior of the system is governed by four stable fixed points [212, 229], at which the anomalous behavior is influenced by both drift and density fluctuations, respectively. Further, it may be readily seen from the inspection of the Feynman graphs, in the one-loop approximation, there is no influence of the velocity fluctuations on the renormalization of the interaction vertices. However, the influence of higher order terms of the perturbation series can have a significant effect on the critical properties. A two-loop calculation was performed in Ref. [229], in which, also a renormalized integro-differential equation for the number density was put forward which directly takes into account the effect of density and velocity fluctuations at next-to-leading order. The solution of this equation was analyzed within perturbation theory and later on also by numerical means [230].

The annihilation reaction $A + A \to \varnothing$ can be also considered in a more general setup [210, 231–233]. In particular, it is possible to include random sinks and sources for A particles to sustain a stationary state. In the majority of papers, this is performed using an additive shot noise in the Langevin equation of the given stochastic problem. Since the investigation here starts from the master equation approach, it is not quite justified for our purposes. There is no unique way to introduce random sources into the master equation corresponding to the random noise term of the mean-field (Langevin) description. Probably, the simplest permissible choice, described in detail in the literature [180], which is tantamount to adding reactions $A \to X$ and $Y \to A$ [232, 233], where Y and X are particle baths of the source and the sink, respectively. On the basis of the analysis of canonical scaling dimensions, effective actions can be constructed [232], which are then the starting point for an RG analysis of the critical behavior of the considered systems. In all cases, the effect of random sources and sinks to the large scale, the long-time behavior of the Green functions is significant and changes the universality class of the model. Instead of the universality class of the pair annihilation reaction, the asymptotic behavior of the model with random sources

and sinks belongs to the universality class of the Gribov process in the critical case and to a modified Gribov process in the critical limit of the noncritical model.

5.2.2 Persistence in coupled reaction schemes

Some specific types of the reaction–diffusion models are instances of the widely studied persistence problem [234]. Persistence can be introduced simply as the probability that the fluctuating non-equilibrium field does not change sign up to a given time t [234, 235]. Remarkably, certain problems of this type are related to the zero-temperature dynamics of Ising and Potts model and can be, in fact, recast in terms of irreversible reaction processes [235–237]. In these particular models, the aim is to study probability $P(t)$ that a given site has not been run over a domain wall. Surprisingly $P(t)$ displays power-law behavior $P(t) \sim t^{-\theta}$ in the limit of large time, where θ is a universal scaling exponent. For reaction–diffusion problems, persistence is thus tantamount to the probability that no reaction takes place at a particular site up to time t. On the other hand, the generalization of the latter looks at the persistence of a pattern present in the initial configuration [235, 238, 239]. It covers the behavioral characteristics of many different models [239, 240]. Among others, it describes the survival of a test particle in a random environment, which can be represented for example by diffusing traps [241, 242], by different reaction–diffusion systems [243–245].

We consider a specific system with two reacting particle species A and B, with corresponding diffusion constants D_{nA} and D_{nB}, undergoing following reaction scheme

$$A + B \to A, \quad A + A \to \begin{cases} \emptyset, & \text{with probability } p, \\ A, & \text{with probability } 1 - p. \end{cases} \tag{143}$$

Similarly to the previous studies [235, 245], we are interested in a case, for which both particle species are mobile. Rate equations for reaction process (143) take the following form: $\partial_t a = D_A \partial^2 a - \lambda a^2$, $\partial_t b = D_B \partial^2 b - \lambda' ab$, where $a = a(t)$, $b = b(t)$ denote particle densities, D_A, D_B denote diffusion constants, and λ, λ' are the reaction rates. For brevity, it is customary to introduce two parameters $\delta = D_B/D_A$ and $Q = 1/(2-p)$, respectively.

Due to rather general considerations concerning the diffusion process only, the upper critical dimension for such reaction scheme is $d_c = 2$ [218]. Above the upper critical dimension $d > d_c$, the resulting A particle density decreases with time as $a \sim 1/(\lambda t)$ [218], whereas the B particle density decays with different exponent as $b \sim t^{-\theta}$. At or below the upper critical dimension $d \le d_c$, the fluctuations become relevant, and it is no longer feasible to determine θ from the rate equations. Instead, one has to resort to field-theoretic action [245]

$$\mathcal{S}(\phi) = \int_0^\infty \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}\boldsymbol{x} \Big[\psi_A^\dagger(\partial_t - \partial^2)\psi_A + \psi_B^\dagger(\partial_t - \delta\partial^2)\psi_B + \lambda_0 \psi_A^\dagger \psi_A^2 + \lambda_0 \psi_A^{\dagger 2} \psi_A^2 + \lambda_0' Q \psi_B^\dagger \psi_A \psi_B + \lambda_0' \psi_A^\dagger \psi_B^\dagger \psi_A \psi_B \Big] - \int_{-\infty}^\infty \mathrm{d}\boldsymbol{x}\, (a_0 \psi_A^\dagger + b_0 \psi_B^\dagger),$$

where $\phi = \{\psi_A^\dagger, \psi_A, \psi_B^\dagger, \psi_B\}$ is a total set of fields. By analyzing this action using the RG method improved quantitative predictions can be obtained. Calculations reveal [245, 246] that $\theta = \theta_S + \gamma_b^*$, where $\theta_S = d((1+\delta)/2)^{d/2}/(2-p)$ denotes the contribution given by Smoluchowski approximation [246], and $\gamma_b^* \propto \varepsilon$ is the anomalous dimension of the density field $b(t, \boldsymbol{x})$ [245].

Concerning the reaction part of the reaction–diffusion system (143), the main interest lies in a calculation of the decay exponent θ for B particle density influenced by the effect of the anomalous diffusion motion of particles. Furthermore, recently, it was realized [245] that the additional anomalous dimension ϕ is needed due to a nontrivial renormalization of the particle correlation function for B particles

$$C^s_{BB}(t, \boldsymbol{x}) \equiv \frac{\langle \psi_B(t, \boldsymbol{x}) \psi_B(t, \boldsymbol{0}) \rangle - \langle \psi_B(t) \rangle^2}{\langle \psi_B(t) \rangle^2} \sim t^\phi f(x/\sqrt{t}). \tag{144}$$

The underlying technical reason reflects the necessity of additional multiplicative renormalization of b^2 operator. which existence was pointed out in Ref. [245]. On the other hand, no extra anomalous dimension is needed for other types of scaled correlation functions, namely C^s_{AA} [218] and C^s_{AB} [245]. They are only the functions of $x/\sqrt{t}$ [245]. The existence of $\phi = \gamma_b^* - \gamma_{b^2}^*/2$ is thus a consequence of the noncorrespondence of the renormalization constants $Z_{BB} \neq (Z_b)^2$. The renormalization constant Z_{BB} is connected with the renormalization of the unscaled correlation function $C_{BB}(t, \boldsymbol{x}) = \langle \psi_B(t, \boldsymbol{x}) \psi_B(t, \boldsymbol{0}) \rangle - \langle \psi_B(t, \boldsymbol{x}) \rangle^2$ in the numerator of Eq. (144), whereas the B particle density renormalization is ensured by constant Z_b. The results of the RG calculations have been verified by numerical means [247]. Later on, this model was also generalized for the case of Lévy flights [248, 249]. Effect of convective velocity fluctuations was studied in Refs. [250, 251]. In particular, it was shown that the presence of ever-present ambient thermal fluctuations have the same influence of asymptotic behavior as was observed for pair annihilation reaction–diffusion model [212], and was discussed in the previous Sect. 5.2.1, i.e., fixed point governed by purely by the reaction processes is shown to be unstable with respect to ambient thermal fluctuations.

6 Conclusion

In this paper, we reviewed a number of selected problems that arise when the field-theoretic approach, especially renormalization theory, renormalization group and more advanced tools are applied to the problems of non-equilibrium statistical physics, stochastic differential equations and turbulence. In this conclusion, we briefly discuss a few issues not touched upon in the text and outline possible directions of future research.

It is well known that critical behavior of equilibrium systems is highly sensitive to various external perturbations such as, for example, environment motion. The same is even more true for non-equilibrium dynamical systems where laminar or turbulent flow can change scaling behavior making it trivial (the mean-field one) or, on the contrary, give rise to new regimes of it [252–255]. In practice, real systems can hardly

be isolated from the influence of surrounding medium, like turbulent motion in the atmosphere, in the ocean or, especially, in forest fires. That is why it is important to account for its effects when considering models of kinetic roughening and nonlinear diffusion. This problem was explored in the series of works [256–263] for various dynamical models of equilibrium and non-equilibrium phase transitions and in Refs. [264–270] for various models of self-organized critical systems, erosion of landscapes and surface roughening. In these cases, the use of multiple ε expansions is necessary.

In a standard approach, a multiplicatively renormalizable field theoretic model involves finite number of local interactions and, thus, finite number of corresponding coupling constants [19]. However, there are models of random rough surfaces [271–274], landscape erosion [270, 275–278], nonlinear diffusion [279, 280], self-organized criticality [281–284] and neural activity [285, 286] whose renormalization inevitably gives rise to infinite number of the counterterms and thus, infinite number of coupling constants.

In quantum-field theory, such models traditionally faced skeptical attitude as being non-renormalizable and therefore for having no predictive power. However, now the situation is changing, although a generally accepted interpretation is still not achieved, in particular, because of a wide variety of such models in comparison with usual renormalizable ones. For a recent discussion, see, e.g., Ref. [287] and references therein.

In the present cases, the RG analysis of properly extended infinite-dimensional Pavlik's model for fluctuating interface [272] and Pastor-Satorras–Rothman model of eroded landscape [277] reveals two-dimensional surfaces of fixed points with non-universal scaling exponents dependent on the choice of a point on the surfaces. These results can be refined using functional renormalization group, and some additional considerations [278]. When the interaction with moving medium (e.g., turbulent one) is included, predictions of the RG analysis become much more definite and universal exact values for the critical exponents can be derived [274, 279, 280].

Another interesting phenomena occurring in such stochastic problems are dimensional transmutation and non-conventional scaling behavior [288, 289], statistical restoration of broken symmetries [290] and dynamically induced nonlinear interaction with the environment [267]. An overview of these interesting effects can be found in Ref. [291].

Let us conclude this review with a brief discussion of the large-d limit in models of turbulence and turbulent advection. The role of expansion parameter in the ordinary perturbation there is played by the Reynolds number $Re \gg 1$. The standard RG resummation substitutes it with the exponent $y = O(1)$ from the random force correlation function (71) and (72).

The results of the RG analysis are reliable and internally consistent for asymptotically small y, while the possibility of their extrapolation to the physical value $y \to 4$ and thus their relevance for the real fluid turbulence is not so obvious. New physical effects are encountered as y increases, and they can be easily lost or misrepresented if the expansion in y is too naively applied. One of them is related to the so-called sweeping effects, leading to strong dependence of the velocity correlation functions on the outer turbulent scale [97–100]. These effects lead to IR divergences in the diagrams of perturbation theory which, formally speaking, manifest themselves as poles

at some finite values of y. Such poles and their physical interpretation were discussed, e.g., in Refs. [37, 38, 168, 279]. Some other problems in extrapolation to real finite values of y are mentioned in Ref. [170].

Thus, it is desirable to have an alternative expansion. A promising candidate for the expansion parameter is $1/d$, an attractive idea that was repeatedly introduced; see, e.g., Refs. [110, 170, 292–296]. One can hope that for $d \to \infty$ intermittency and anomalous scaling disappear or acquire a simple "calculable" form and the finite-dimensional turbulence can be studied within the regular expansion around this "solvable" limit [292]. In contrast to the well-known Berlin–Kac spherical model for the large n limit of the $O(n)$-symmetric models of critical phenomena, no closed model for the limit $d \to \infty$ of the stochastic Navier–Stokes equation was obtained yet. However, drastic simplifications in the non-stationary renormalized perturbation theory [292] and the RG-resummed series in y [170, 295] were found. This allowed to extend the expansions in y up to the orders y^3 [170] and y^4 [295].

Probably, the most encouraging result is the calculation of anomalous exponents in the Kraichnan model in the order $O(1/d)$ [108]; see Eq. (27) in Sect. 3. Discouraging is the fact that the $O(1/d^2)$ correction is not yet derived. The reason is probably that the $O(1/d)$ term is contained entirely in the $O(\xi)$ term of the expansion in ξ, while $O(1/d^2)$ is most likely an infinite series starting with ξ^2.

The following lament is unavoidable here. In 1990s, the Kraichnan model leads to remarkable progress in the theory of turbulence: the anomalous exponents were calculated on the base of a dynamical model within controlled approximations and regular expansions. However, it remains underrated and/or overlooked by the specialists in the field theory as an example of highly nontrivial field-theoretic model with rich internal context. Indeed, it can be attacked by a variety of approaches: non-perturbative results for second-order correlation functions, accurate numerical simulations, RG+OPE scenario, physically motivated closures etc. The expansion around $\xi = 0$ is available up to ξ^4, while systematic expansions in $1/d$ and $\sqrt{(2-\xi)}$ are still lacking: only the leading-order results [108, 116] are available.

Various modifications and generalizations, like advection of vector fields [110, 296] or inclusion of small-scale anisotropy [82], provide examples of mixing of relevant composite fields (dangerous operators), relations between exact and perturbative results, simplifications for large d and so on.

In the Introduction to this review, we tried to underline fruitful interactions between quantum-field theory and statistical physics on the example of the theory of critical behavior. We believe that exhaustive investigation of problems mentioned in this concluding section provide a good challenge for theoretical research in the future.

Acknowledgements The authors have greatly benefited from mutual collaborations and many valuable discussions with L.Ts. Adzhemyan, N.M. Gulitskiy, M.V. Kompaniets, N.M. Lebedev, M.Yu. Nalimov and M.M. Tumakova.

Author contributions All the authors contributed to the conception, drafting, critical revision and literature search for this review article.

Funding Open access funding provided by The Ministry of Education, Science, Research and Sport of the Slovak Republic in cooperation with Centre for Scientific and Technical Information of the Slovak Republic. The work was supported by VEGA grant No. 1/0297/25 of the Ministry of Education, Science,

Research and Sport of the Slovak Republic. The work of P.I.K. was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS", project 22-1-3-33-1.

Data availability This review article contains no original research data. For reused research data, readers should refer to the publications cited throughout the text.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose. The authors declare no conflict of interest.

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