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Exact solutions of the nilpotent Dirac equation

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Abstract. Only a few exact solutions are known for quantum mechanical equations, and some of these tend to be complicated, requiring special mathematical techniques and multiple particular cases. The relativistic nilpotent Dirac equation has the virtue of allowing highly streamlined solutions for all the existing solvable problems and also adding others not solvable analytically by any other method.

1. Introduction

There are relatively few exact solutions for quantum mechanical equations in any form. Typically, we have free particle, square well, Coulomb potential and harmonic oscillator. The nilpotent form has these solutions and the free particle has a specially interesting form. The nilpotent structure helps to clarify the structure but doesn't significantly reduce the complication. The Coulomb and two new solutions (one of which is a harmonic oscillator show the special power of the nilpotent method).

The nilpotent method is dependent solely on defining an operator and does not depend on an equation, though it produces a particular form of the Dirac equation when multiplied out. Here, we use a Clifford algebra that is a tensor product of real numbers, complex numbers, quaternions ($\boldsymbol{i}$,$\boldsymbol{j}$,$\boldsymbol{k}$) and multivariate vectors (for the momentum term $\mathbf{p}$ or ∇) rather than conventional γ matrices, and the conventional Dirac equation is then premultiplied by γ^5. The result is that energy, momentum and mass terms (or their equivalent operators) are separated by different quaternion coefficients in a kind of extension of the use of complex numbers for this purpose. In addition, the four terms in the Dirac spinor are distinguished only by the sign variations of their energy and momentum terms and there is only a single phase factor to which the operator applies. The operator becomes a row vector of the form

$$\left(\mp\boldsymbol{k}\frac{\partial}{\partial t}\mp i\boldsymbol{i}\nabla+\boldsymbol{j}m\right)$$

where the $\partial/\partial t$ and ∇ can be replaced by covariant derivatives including field terms. The operator is then applied to a phase factor which is uniquely determined to find the only column vector amplitude that will be nilpotent or square to zero. This is best illustrated using the free particle solution. Here the phase factor is $e^{-i(Et-\mathbf{p}.\mathbf{r})}$ and the operator is as above (without additional field terms). Then

$$\underset{row}{\left(\mp\boldsymbol{k}\frac{\partial}{\partial t}\mp i\boldsymbol{i}\nabla+\boldsymbol{j}m\right)}\underset{column}{(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)}\,e^{-i(Et-\mathbf{p}.\mathbf{r})}=0$$

becomes

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$

which is version of Einstein's energy-momentum-mass equation

$$E^2 - p^2 - m^2 = 0.$$

There exists a 5-dimensional Clifford algebra $R^{0,5}$ with metric signature (- - - - -) composed of a set of 5 generators $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{i}, \boldsymbol{j}, i)$: two sets of quaternion units $(\boldsymbol{i}, \boldsymbol{j})$ and one complex unit (i) at the foundation of the conventional Dirac equation. There exist six other versions of this 5-dimensional Clifford algebras that possess varying numbers of space-like terms that can also represent the Dirac equation. These other 5-dimensional Clifford algebras can be derived from the basic $R^{0,5}$ by repeatedly taking the tensor product of $R^{0,5}$ with complex i up to five times. Conventionally, the Dirac Equation was formulated as a 4×4 system of partial differential equations where $N_{operator}$ was a 4×4 matrix differential operator [1-23], $N_{phasefactor}$ was a 4×1 spinor and $0 = (0\,0\,0\,0)$ was a zero vector. The nilpotent Dirac equation reduces that 4×4 system of partial differential equations to a 1×1 partial differential equation. The single partial differential form of the Dirac equation leads to a scalar product in differential form because

$$(N_{operator} N_{phasefactor})^2 = 0$$

where

$$\text{vector } 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \text{scalar}\, 0 = 0.$$

2. Nilpotent solution method

1. Start with the boundary conditions for a free, $p^2 = E^2 - m^2$, reflected in the nilpotent Dirac operator as

$$N_{operator} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\nabla + \boldsymbol{j}m\right).$$

2. The correct boundary condition terms specific to the type of interactions the fermion is subject to into the appropriate quaternion bins of the nilpotent Dirac operator

$$N_{operator} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\nabla + \boldsymbol{j}m\right) \text{ for a free particle.}$$

3. Transfer the corresponding boundary conditions terms from the nilpotent Dirac operator

$$N_{operator} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\nabla + \boldsymbol{j}m\right) \text{ to the nilpotent phase factor } N_{phasefactor} = e^{-i(Et-\mathbf{p}.\mathbf{r})}.$$

4. Calculate the nilpotent amplitude (eigenvalue) $N_{amplitude} = (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ from the action of the nilpotent operator onto the phase factor in the quaternion state vector:

$$N_{operator} N_{phasefactor} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\nabla + \boldsymbol{j}m\right) e^{-i(Et-\mathbf{p}.\mathbf{r})}.$$

This methodology automatically satisfies the nilpotent condition:

$$(N_{operator} N_{phasefactor})^2 = A^2 = (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)^2 = 0$$

and therefore provides the full analytical solution to any fermion acting within the field of another point source in the form of the nilpotent amplitude

$$N_{amplitude} = (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m).$$

5. Inserting the nilpotent amplitude $N_{amplitude} = (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ for an interacting fermion into a mathematical software package will provide the transition from the full analytical solution to numerical predictions that can be used to assess the accuracy of the nilpotent methodology.

3. Eigenvalue Spectrum of a Dirac Particle in a One-Dimensional Square-Well Potential

We calculate the spectrum of eigenvalues for a Dirac particle in a square-well potential of depth $V_0 \leq 0$ and width a. To accomplish this we need to decompose the real axis z into three intervals depending on the presence or absence of the potential $-V_0$. The one dimensional nilpotent Dirac operator and nilpotent phase factor for intervals 1 and 3 reads as follows

$$N_{operator} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial z} + \boldsymbol{j}m\right)$$

$$N_{phasefactor} = e^{-i(Et - \mathbf{p}.\mathbf{r})}$$

The potential $-V_0$ is present in interval 2 and the one dimensional nilpotent Dirac operator and nilpotent phase factor

$$N_{operator} = \left(\pm i\boldsymbol{k}\left(\frac{\partial}{\partial t} - V_0\right) \pm \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial z} + \boldsymbol{j}m\right)$$

$$N_{phasefactor} = e^{-i(Et - \mathbf{p}.\mathbf{r})}$$

$N_{operator}N_{phasefactor} = \left(\pm i\boldsymbol{k}\frac{\partial}{\partial t} \pm \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial z} + \boldsymbol{j}m\right)e^{-i(Et-\mathbf{p}.\mathbf{r})}$ for Region 1, $z \leq -a/2$, and Region 3, $z \geq a/2$ and boundary conditions: $p_1^2 = E^2 - m^2$.

$$\psi_1(z) = F(\pm i\boldsymbol{k}E \pm \boldsymbol{i}p_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{z})} + F'(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\boldsymbol{p}_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{z})}$$
$$\psi_3(z) = H(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\boldsymbol{p}_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{z})} + H'(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\boldsymbol{p}_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{z})}$$

$N_{operator}N_{phasefactor} = \left(\pm i\boldsymbol{k}\left(\frac{\partial}{\partial t} - V_0\right) \pm \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial z} + \boldsymbol{j}m\right)e^{-i(Et-\boldsymbol{p}.\boldsymbol{r})}$ for Region 2, $-a/2 \leq z \leq a/2$, and boundary conditions: $p_2^2 = (E - V_0)^2 - m^2$.

$$\psi_2(z) = G\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{z})} + G'\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{z})}$$

Since the Noether current and orientation of the spin are not changed by the boundary conditions the quaternion state vectors equate to each other on each side of the boundaries:

$$\psi_1(-a/2) = \psi_2(-a/2)$$

$$F(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_1 + \boldsymbol{j}m)e^{-(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{a}/2)} + F'(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_1 + \boldsymbol{j}m)e^{-(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{a}/2)} =$$
$$G\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{-(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{a}/2)} + G'\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{-(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$\psi_2(a/2) = \psi_3(a/2)$$
$$G\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{a}/2)} + G'\left(i\boldsymbol{k}(E - V_0) + \boldsymbol{i}\mathbf{k}p_2 + \boldsymbol{j}m\right)e^{(\boldsymbol{i}\boldsymbol{p}_2\boldsymbol{a}/2)} =$$
$$H(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{a}/2)} + H'(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}p_1 + \boldsymbol{j}m)e^{(\boldsymbol{i}\boldsymbol{p}_1\boldsymbol{a}/2)}$$

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Equating the coefficients E, p, m of the corresponding algebraic operators $i\boldsymbol{k}$, $\boldsymbol{i}\mathbf{k}$ and $\boldsymbol{j}$ eliminates them from the calculations and creates a system of linear scalar wave functions partitioned into three different variable domains for E, p, m

$$\begin{aligned}
\psi_1(-a/2) &= \psi_2(-a/2)\\
FEe^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'Ee^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} &= G(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}\\
Fp_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'p_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} &= Gp_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}\\
Fme^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'me^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} &= Gme^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}\\
\psi_2(a/2) &= \psi_3(a/2)\\
G(E-V_0)e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'(E-V_0)e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} &= HEe^{(i\boldsymbol{p}_1\boldsymbol{a}/2)} + H'Ee^{(i\boldsymbol{p}_1\boldsymbol{a}/2)}\\
Gp_2e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'p_2e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} &= Hp_1e^{(i\boldsymbol{p}_1\boldsymbol{a}/2)} + H'p_1e^{(i\boldsymbol{p}_1\boldsymbol{a}/2)}\\
Gme^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'me^{(i\boldsymbol{p}_2\boldsymbol{a}/2)} &= Hme^{(i\boldsymbol{p}_1\boldsymbol{a}/2)} + H'me^{(i\boldsymbol{p}_1\boldsymbol{a}/2)}
\end{aligned}$$

Moving G' to the left hand side and dividing by the common G' term gives the system this form for the energy

$$\psi_1(-a/2) = \psi_2(-a/2)$$
$$FEe^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'Ee^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} = G(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'(E-V_0)e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$FEe^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'Ee^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} - G(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} = G'(E-V_0)e^{(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$\frac{FEe^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} + \frac{F'Ee^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} - \frac{G(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} = \frac{G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'(E-V_0)e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}$$
$$[F/G'][E/(E-V_0)]e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} + [F'/G'][E/(E-V_0)]e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} - [G/G'] = 1$$

For the momentum:

$$Fp_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'p_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} = Gp_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$Fp_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'p_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} - Gp_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} = G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$\frac{Fp_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} + \frac{F'p_1e^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} - \frac{Gp_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} = \frac{G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'p_2e^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}$$
$$[F/G'][p_1/p_2]e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} - [F'/G'][p_1/p_2]e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} - [G/G'] = 1$$

And for the mass:

$$Fme^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'me^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} = Gme^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} + G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$Fme^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} + F'me^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)} - Gme^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)} = G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}$$
$$\frac{Fme^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} + \frac{F'me^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} - \frac{Gme^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} = \frac{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}$$
$$\frac{Fme^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} + \frac{F'me^{-(i\boldsymbol{p}_1\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} - \frac{Gme^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}}{G'me^{-(i\boldsymbol{p}_2\boldsymbol{a}/2)}} = 1$$
$$[F/G']e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} + [F'/G']e^{(i(\boldsymbol{p}_2-\boldsymbol{p}_1)\boldsymbol{a}/2)} - [G/G'] = 1$$

For the next transition

$$\psi_2(a/2) = \psi_3(a/2)$$

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the energy terms give

$$G(E-V_0)e^{(\boldsymbol{ip_2a/2})}+G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}=HEe^{(\boldsymbol{ip_1a/2})}+H'Ee^{(\boldsymbol{ip_1a/2})}$$

$$G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}=HEe^{(\boldsymbol{ip_1a/2})}+H'Ee^{(\boldsymbol{ip_1a/2})}-G(E-V_0)e^{(\boldsymbol{ip_2a/2})}$$

$$\frac{G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}}{G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}}=\frac{HEe^{(\boldsymbol{ip_1a/2})}}{G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}}+\frac{H'Ee^{(\boldsymbol{ip_1a/2})}}{G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}}-\frac{G(E-V_0)e^{(\boldsymbol{ip_2a/2})}}{G'(E-V_0)e^{(\boldsymbol{ip_2a/2})}}$$

$$1=[H/G']E/(E-V_0)e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']E/(E-V_0)e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']$$

while the momentum terms produce

$$Gp_2e^{(\boldsymbol{ip_2a/2})}+G'p_2e^{(\boldsymbol{ip_2a/2})}=Hp_1e^{(\boldsymbol{ip_1a/2})}+H'p_1e^{(\boldsymbol{ip_1a/2})}$$

$$G'p_2e^{(\boldsymbol{ip_2a2})}=Hp_1e^{(\boldsymbol{ip_1a/2})}+H'p_1e^{(\boldsymbol{ip_1a/2})}-Gp_2e^{(\boldsymbol{ip_2a/2})}$$

$$\frac{G'p_2e^{(\boldsymbol{ip_2a/2})}}{G'p_2e^{(\boldsymbol{ip_2a/2})}}=\frac{Hp_1e^{(\boldsymbol{ip_1a/2})}}{G'p_2e^{(\boldsymbol{ip_2a/2})}}+\frac{H'p_1e^{(\boldsymbol{ip_1a/2})}}{G'p_2e^{(\boldsymbol{ip_2a/2})}}-\frac{Gp_2e^{(\boldsymbol{ip_2a/2})}}{G'p_2e^{(\boldsymbol{ip_2a/2})}}$$

$$1=[H/G'][p_1/p_2]e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G'][p_1/p_2]e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']$$

and the mass terms

$$Gme^{(\boldsymbol{ip_2a/2})}+G'me^{(\boldsymbol{ip_2a/2})}=Hme^{(\boldsymbol{ip_1a/2})}+H'me^{(\boldsymbol{ip_1a/2})}$$

$$G'me^{(\boldsymbol{ip_2a/2})}=Hme^{(\boldsymbol{ip_1a/2})}+H'me^{(\boldsymbol{ip_1a/2})}-Gme^{(\boldsymbol{ip_2a/2})}$$

$$\frac{G'me^{(\boldsymbol{ip_2a/2})}}{G'me^{(\boldsymbol{ip_2a/2})}}=\frac{Hme^{(\boldsymbol{ip_1a/2})}}{G'me^{(\boldsymbol{ip_2a/2})}}+\frac{H'me^{(\boldsymbol{ip_1a/2})}}{G'me^{(\boldsymbol{ip_2a/2})}}-\frac{Gme^{(\boldsymbol{ip_2a/2})}}{G'me^{(\boldsymbol{ip_2a/2})}}$$

$$1=[H/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']$$

With the following substitutions

$$E/(E-V_0)=Q$$

$$p_1/p_2=p$$

these 6 equations

$$[F/G'][E/(E-V_0)]e^{(\boldsymbol{i(p_2-p_1)a/2})}+[F'/G'][E/(E-V_0)]e^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[F/G'][p_1/p_2/p_2]e^{(\boldsymbol{i(p_2-p_1)a/2})}-[F'/G'][p_1/p_2/p_2]e^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[F/G']e^{(\boldsymbol{i(p_2-p_1)a/2})}+[F'/G']e^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[H/G']E/(E-V_0)e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']E/(E-V_0)e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

$$[H/G'][p_1/p_2]e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G'][p_1/p_2]e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

$$[H/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

become

$$[F/G']Qe^{(\boldsymbol{i(p_2-p_1)a/2})}+[F'/G']Qe^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[F/G']pe^{(\boldsymbol{i(p_2-p_1)a/2})}-[F'/G']pe^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[F/G']e^{(\boldsymbol{i(p_2-p_1)a/2})}+[F'/G']e^{(\boldsymbol{i(p_2-p_1)a/2})}-[G/G']=1$$

$$[H/G']Qe^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']Qe^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

$$[H/G']pe^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']pe^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

$$[H/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}+[H'/G']e^{(\boldsymbol{i(p_1-p_2)a/2})}-[G/G']=1$$

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These constitute two 3×3 system of equations, which can be solved for the normalisation coefficients as a matrix equation $A\mathbf{x} = b$ using a computer algebra software package. Unlike in the conventional formalism the nilpotent method offers a direct solution method for solving for the *normalisation coefficients only.*

$$A_1\mathbf{x} = b$$

$$\begin{bmatrix} Q & Q & -1 \\ p & p & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} [F/G']e^{(\boldsymbol{i}(\boldsymbol{p}_2 - \boldsymbol{p}_1)\boldsymbol{a}/2)} \\ [F'/G']e^{(\boldsymbol{i}(\boldsymbol{p}_2 - \boldsymbol{p}_1)\boldsymbol{a}/2)} \\ [G/G'] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A_2\mathbf{x} = b$$

$$\begin{bmatrix} Q & Q & -1 \\ p & p & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} [H/G']e^{(\boldsymbol{i}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\boldsymbol{a}/2)} \\ [H'/G']e^{(\boldsymbol{i}(\boldsymbol{p}_1 - \boldsymbol{p}_2)\boldsymbol{a}/2)} \\ [G/G'] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Here we can take $G' = 1$ to normalize and make the matrix solvable. For the special case of the infinite square well potential, we can simply equate Q to 0. The square well potential is a well-known textbook case. The significance of the nilpotent version, however, is that it resolves very simply into a procedure for finding normalization coefficients. The nilpotent method has reduced problem solving in quantum mechanics to a uniform procedure of a linear algorithmic nature for calculating normalisation coefficients. In a similar problem also completed by the authors, Eigenvalues of the Dirac Equation in a One-Dimensional Square Potential Well with Scalar Coupling, the nilpotent methodology has provided a full analytical solution.

4. Coulomb potential

More physically significant examples come from the interactions of point particles, some of which are not so far investigated. The simplest case of one point-particle in the field of another is the pure Coulomb interaction. A Coulomb term is a necessary consequence of any such interaction, because when we use the Dirac prescription to express the operator in terms of polar coordinates to specify the spherical symmetry of a point source, we immediately find that we need

$$\nabla \to \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \pm i\frac{j + 1/2}{r}.$$

and

$$(\boldsymbol{i}\boldsymbol{k}E - \boldsymbol{i}\boldsymbol{i}\nabla + \boldsymbol{j}m) \to \left(\boldsymbol{i}\boldsymbol{k}E - \boldsymbol{i}\boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j + 1/2}{r}\right) + \boldsymbol{j}m\right).$$

The nilpotent operator now becomes

$$\left(\pm \boldsymbol{i}\boldsymbol{k}\left(E - \frac{A}{r}\right) \mp \boldsymbol{i}\boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j + 1/2}{r}\right) + \boldsymbol{j}m\right).$$

The calculation now takes only six lines. It depends on finding a phase factor which makes the amplitude nilpotent. So we start with the standard expression:

$$\phi = e^{-ar} r^{\gamma} \sum_{\nu=0} a_\nu r^\nu$$

Applying the operator to ϕ, and squaring to 0 for nilpotency, we obtain:

$$4\left(E - \frac{A}{r}\right)^2 = -2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} + i\frac{j + \frac{1}{2}}{r}\right)^2 - 2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} - i\frac{j + \frac{1}{2}}{r}\right)^2 + 4m^2.$$

In turn, we then equate constant terms; and, following standard procedures, terms in $1/r^2$, with $\nu = 0$; and, assuming a power series terminating at n', coefficients of $1/r$ for $\nu = n'$, to give

$$a = \sqrt{m^2 - E^2}$$

$$\left(\frac{A}{r}\right)^2 = -\left(\frac{\gamma+1}{r}\right)^2 + \left(\frac{j+\frac{1}{2}}{r}\right)^2$$

$$2EA = -2\sqrt{m^2 - E^2}(\gamma + 1 + n'),$$

Here, the terms in $(j + \frac{1}{2})$, with two positive and two negative, cancel out when we sum four multiplications. Putting the three equations equations together yields

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{A^2}{(\gamma+1+n')^2}}} = \frac{1}{\sqrt{1 + \frac{A^2}{(\sqrt{(j+\frac{1}{2})^2 - A^2} + n')^2}}}$$

which, when $A = Ze^2$, is instantly recognisable as the hyperfine structure formula for a one-electron nuclear atom.

5. Coulomb plus linear potential

Since the Coulomb term is required for any point-particle, then all possible cases for point-particles can be considered using various cases of Coulomb plus other potentials. The constant potential is dealt with simply by shifting the value of E, so the first case of interest is when the additional potential is a linear function of distance, say Br. The complete operator is then:

$$\left(\pm \boldsymbol{k}\left(E + \frac{A}{r} + Br\right) \mp \boldsymbol{i}\left(\frac{\partial}{\partial t} + \frac{1}{r} \pm i\frac{j+\frac{1}{2}}{r}\right) + \boldsymbol{ij}m\right).$$

By comparing with the pure Coulomb potential, we could suggest a phase factor is of the form:

$$\phi = \exp\left(-ar - br^2\right) r^\gamma \sum_{\nu=0} a_\nu r^\nu .$$

When we apply the operator to this, and square to zero to maintain the nilpotent condition, we obtain:

$$E^2 + 2AB + \frac{A^2}{r^2} + B^2r^2 + \frac{2AE}{r} + 2BEr = m^2$$

$$-\left(a^2 + \frac{(\gamma+\nu+..+1)^2}{r^2} - \frac{(j+\frac{1}{2})^2}{r^2} + 4b^2r^2 + 4abr - 4b(\gamma+\nu+..+1) - \frac{2a}{r}(\gamma+\nu+..+1)\right)$$

The positive and negative $i(j + 1/2)$ terms here again cancel over the four solutions. If the power series terminates, as before, we can equate the respective coefficients of r^2, r and $1/r$ to give:

$$B^2 = -4b^2$$

$$2BE = -4ab$$

$$2AE = 2a(\gamma + \nu + 1)$$

from which we obtain:

$$b = \pm\frac{iB}{2}$$

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$$a = \mp iE$$
$$\gamma + \nu + 1 = \mp iA.$$

For the ground state (with $\nu = 0$) we have a phase factor:

$$\phi = \exp\left(\pm iEr \mp iBr^2/2\right) r^{\mp iA-1}.$$

The Coulomb plus linear potential has the particular interest of being the probably form of the strong interactions. Here, an imaginary exponential form of ϕ would suggest asymptotic freedom, and a free fermion phase factor. It is convenient also to restructure he complex r^γ term as a component phase, $\chi(r) = \exp(\pm iA\ln(r))$, so

$$\phi = \frac{\exp(kr + \chi(r))}{r},$$

where

$$k = \pm iE \mp iBr/2.$$

and $\chi(r)$ varies less rapidly with r than the rest of ϕ. At high energies, and small r, the first term in k is dominant, tending towards a free fermion solution, or asymptotic freedom. At low energies and large r, the second term will dominate. The confining potential Br suggests infrared slavery. Thus, this solution appears to have the characteristics of the strong interaction.

6. Coulomb plus any other potential

Remarkably, there is only one other solution for a point-particle in the field of another, for any added potential other than a linear one gives the same result, and it doesn't matter how many additional terms there are. So let us suppose we set up a nilpotent Dirac operator with a Coulomb potential proportional to $1/r$ and another potential of the form Cr^n, where n is an integer greater than 1 or less than –1.

$$\left(\boldsymbol{k}\left(E - \frac{A}{r} - Cr^n\right) + \boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j+\frac{1}{2}}{r}\right) + \boldsymbol{ij}m\right).$$

As in the previous cases, we need the phase factor which will make the amplitude nilpotent. From the standard Coulomb solution we can guess that the phase factor is of the form:

$$\phi = \exp\left(-ar - br^{n+1}\right) r^\gamma \sum_{\nu=0} a_\nu r^\nu$$

Again, we apply the operator to this factor and square to zero, assuming a termination in the series:

$$4\left(E - \frac{A}{r} - Cr^n\right)^2 = -2\left(-a + (n+1)br^n + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} + i\frac{j+\frac{1}{2}}{r}\right)^2$$
$$-2\left(-a + (n+1)br^n + \frac{\gamma}{r} + \frac{\nu}{r} + \frac{1}{r} + i\frac{j+\frac{1}{2}}{r}\right)^2 + 4m^2.$$

The constant terms equate to

$$a = \sqrt{m^2 - E^2}$$

Those in r^{2n}, with $\nu = 0$, lead to:

$$C^2 = -(n+1)^2 b^2$$
$$b = \pm \frac{iC}{n+1}.$$

And, from coefficients of r, where $\nu = 0$:

$$AC = -(n+1)b(1+\gamma),$$

$$(1+\gamma) = \pm iA.$$

If we now equate coefficients of $1/r^2$ and coefficients of $1/r$, for a power series terminating in $\nu = n'$, we find that

$$A^2 = -(1+\gamma+n')^2 + \left(j+\frac{1}{2}\right)^2$$

and

$$-EA = a(1+\gamma+n').$$

The last three equations combine to produce:

$$\left(\frac{m^2-E^2}{E^2}\right)(1+\gamma+n')^2 = -(1+\gamma+n')^2 + \left(j+\frac{1}{2}\right)^2$$

$$E = -\frac{m}{j+\frac{1}{2}}\left(\pm iA + n'\right).$$

Integral values of n' give us the evenly spaced energy levels of a harmonic oscillator. For a fermionic particle, we can assume that the term A required for spherical symmetry is connected with the random direction of fermion spin, and provide it with the half-unit value $\left(\pm\frac{1}{2}i\right)$, or $\left(\pm\frac{1}{2}i\hbar c\right)$, so completing the expression for the fermionic simple harmonic oscillator:

$$E = -\frac{m}{j+\frac{1}{2}}\left(\frac{1}{2}+n'\right).$$

The fact that this solution remains valid for any combination of nonlinear potentials shows that it is possible to find a complete analytical solution of the point particle in the field of any other.

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