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On Singular Limits of Relative Entropies

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Abstract: In this paper we generalize a key result relating singular limits of certain relative entropies with index in the setting of conformal nets, which has played an important role recently in the mathematical theory of relative entropies in the context of Conformal Field Theory.

1. Introduction

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [12,26] for a partial list of references, and [18–20,20] for a partial list of recent mathematical work.

To explain the motivation of this paper, we shall refer the reader to the preliminary Sect. 2 for definitions and notations. Let $\mathcal{A}$ be a conformal net, and I_1, I_2 two intervals with disjoint closures. Let ω be the state corresponding the vacuum and let $\omega_1 \otimes \omega_2$ be the corresponding product state defined on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. The relative entropy $S_{\mathcal{A}}(\omega, \omega_1 \otimes \omega_2)$ contains rich information about the $\mathcal{A}$, especially when we consider the singular limit where the end points of I_1 and I_2 are close. For an example see Th. 4.2 of [20] where one can read of the central charge and global index of $\mathcal{A}$ from such limits under certain conditions. A key ingredient in the proof of this result is the following. Suppose $\mathcal{B} \subset \mathcal{A}$ is a subnet, then $S_{\mathcal{A}}(\omega, \omega_1 \otimes \omega_2) = S_{\mathcal{B}}(\omega, \omega_1 \otimes \omega_2) + S_{\mathcal{A}}(\omega, \omega \cdot E_1)$, where E_1 is a conditional expectation defined in Sect. 3. The result proved in Th. 4.4 of [20] (also stated as Th. 2.20 in [29]) is that when end points of I_1 and I_2 are close, $S_{\mathcal{A}}(\omega, \omega \cdot E_1)$ increase to limit $\ln[\mathcal{A} : \mathcal{B}]$. This is an interesting result connecting relative entropy with index, but under conditions that $\mathcal{B}$ is strongly additive and the index $[\mathcal{A} : \mathcal{B}] < \infty$. Though these conditions are satisfied for the cases considered in [20,29], it is a natural question to see if one can remove them. In fact in [2] the authors argue their computations agree with our Th. 4.4, but in their setting the index $[\mathcal{A} : \mathcal{B}] = \infty$ and Th. 4.4 is only proved under the assumption $[\mathcal{A} : \mathcal{B}] < \infty$. One of the goals of this paper is to remove

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the assumption $[\mathcal{A} : \mathcal{B}] < \infty$. In fact we are able to prove such a result in its full generality in Theorem 3.2.

It maybe surprising that one is able to prove such a general result, and we will explain briefly the ideas behind the proof, which is carried out in Sect. 3, and also indicate the preparations for the proof. The proof is divided into three parts. First we consider when $\mathcal{B} \subset \mathcal{A}$ is irreducible, and in this we consider separately the case when $[\mathcal{A} : \mathcal{B}] < \infty$ and $[\mathcal{A} : \mathcal{B}] = \infty$. For $[\mathcal{A} : \mathcal{B}] < \infty$ case, we improve upon the proof of Th. 4.4 of [20] by removing strong additivity condition on $\mathcal{B}$. This makes use of Möbius covariance and a more streamlined Theorem 2.38 in [29], which is also stated in Th. 3.1 for readers' convenience. For $[\mathcal{A} : \mathcal{B}] = \infty$ case, we use a result (cf. Lemma 2.3) first appeared [23] in the setting of type II_1 for subfactors of infinite index, and estimation on relative entropy in Prop. 2.4. The last case is when $\mathcal{B} \subset \mathcal{A}$ is not irreducible, and we reduce this case to essentially when $\mathcal{B}$ is trivial, and we prove a general result Th. 3.4 about any split Möbius net in Sect. 3.1. Even though Th. 3.4 is motivated as an intermediate step in the proof of Th. 3.2, it can be of independent interest.

2. Preliminaries

2.1. Spatial derivatives, relative entropy and index theory for general subfactors. Let ψ be a normal state on a von Neumann algebra M acting on a Hilbert space H and ϕ' be a normal faithful state on the von Neumann algebra M'. The Connes spatial derivative, usually denoted by $\frac{d\psi}{d\phi'}$, is a positive operator (cf. [10]) . We will use the simplified notation of [22] and write $\frac{d\psi}{d\phi'} = \Delta(\frac{\psi}{\phi'})$. If ψ is faithful , we have

$$\Delta(\frac{\psi}{\phi'})^{it} m \Delta(\frac{\psi}{\phi'})^{-it} = \sigma_t^{\psi}(m), \forall m \in M, \Delta(\frac{\psi}{\phi'})^{it} m \Delta(\frac{\psi}{\phi'})^{-it} = \sigma_{-t}^{\phi'}(m), \forall m \in M'$$

where σ_t^{ψ}, $\sigma_{-t}^{\phi'}$ are modular automorphisms.

$$[D\psi_1 : \psi_2]_t := \Delta(\frac{\psi_1}{\phi'})^{it} \Delta(\frac{\psi_2}{\phi'})^{-it}$$

is independent of the choice of ϕ' and is called **Connes cocycle**.

Suppose M acts on a Hilbert space H and ω is a vector state given by $\Omega \in H$. The relative entropy (cf. 5.1 of [22] or [1]) in this case is $S(\omega, \phi) = -\langle \ln \Delta(\phi/\omega')\Omega, \Omega\rangle$ where ω' is the vector state on M' defined by vector Ω and $\Delta(\phi/\omega') := \frac{d\phi}{d\omega'}$ is Connes spatial derivative. When Ω is not in the support of ϕ we set $S(\omega, \phi) = \infty$.

A list of properties of relative entropies that will be used later can be found in [22] (cf. Th. 5.3, Th. 5.15 and Cor. 5.12 [22]):

Theorem 2.1. *(1) Let M be a von Neumann algebra and M_1 a von Neumann subalgebra of M. Assume that there exists a faithful normal conditional expectation E of M onto M_1. If ψ and ω are states of M_1 and M, respectively, then $S(\omega, \psi \cdot E) = S(\omega \restriction M_1, \psi) + S(\omega, \omega \cdot E)$;*

(2) Let be M_i an increasing net of von Neumann subalgebras of M with the property $(\bigcup_i M_i)'' = M$. Then $S(\omega_1 \restriction M_i, \omega_2 \restriction M_i)$ converges to $S(\omega_1, \omega_2)$ where ω_1, ω_2 are two normal states on M;

(3) Let ω and ω_1 be two normal states on a von Neumann algebra M. If $\omega_1 \geq \mu\omega$, then $S(\omega, \omega_1) \leq \ln \mu^{-1}$;

(4) Let ω and ϕ be two normal states on a von Neumann algebra M, and denote by ω_1 and ϕ_1 the restrictions of ω and ϕ to a von Neumann subalgebra $M_1 \subset M$ respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$;

(5) Let ϕ be a normal faithful state on $M_1 \otimes M_2$. Denote by ϕ_i the restriction of ϕ to $M_i, i = 1, 2$. Let ψ_i be normal faithful states on $M_i, i = 1, 2$. Then

$$S(\phi, \psi_1 \otimes \psi_2) = S(\phi_1, \psi_1) + S(\phi_2, \psi_2) + S(\phi, \phi_1 \otimes \phi_2)$$

Let $E : M \to N$ be a normal faithful conditional expectation onto a subalgebra N. $E^{-1} : N' \to M'$ is in general an operator valued weight which verifies the following equation: for any pair of normal faithful weights ψ on N and ϕ' on M' we have

$$\Delta(\frac{\psi E}{\phi'}) = \Delta(\frac{\psi}{\phi' E^{-1}})$$

Kosaki (cf. [14]) defined index of E, denoted by $\text{Ind}E$ to be $E^{-1}(1)$. When 1 is in the domain of E^{-1}, we say that E has finite index. When both N, M are factors and E has finite index, we have the (cf. [23]) Pimsner-Popa inequality

$$E(m) \geq \lambda m, \forall m \in M_+,$$

where $\lambda = (\text{Ind}E)^{-1}$ is the best constant such that the inequality holds. The action of the modular group $\sigma_t^{\psi E}$ on $N' \cap M$ is independent of the choice of ψ. When E is the minimal conditional expectation such action is trivial on $N' \cap M$. Also the compositions of minimal conditional expectations are minimal (cf. [15]).

Lemma 2.2. *Let $\lambda > 0$. Then $E(m) \geq \lambda m, \forall m \in M_+$, where M_+ denote the set of positive elements of M, if and only if $||E(m)|| \geq \lambda ||m||, \forall m \in M_+$.*

Proof. The proof is given in [4] and we include it here for reader's convenience. One direction is trivial. Let us assume that $||E(m)|| \geq \lambda ||m||, \forall m \in M_+$. For any $y \in M$, let $b_n = (E(y^*y) + \frac{1}{n})^{\frac{-1}{2}}, n \geq 1$. Then we have

$$b_n y^* y b_n \leq ||b_n y^* y b_n|| \leq ||y^* y|| ||b_n^2|| \leq \lambda^{-1} ||E(y^* y)|| ||(E(y^* y) + \frac{1}{n})^{-1}|| \leq \lambda^{-1},$$

and it follows that

$$y^* y \leq b_n^{-1} \lambda^{-1} b_n^{-1} = (E(y^* y) + \frac{1}{n})\lambda^{-1}, \forall n \geq 1.$$

Now let $n \to \infty$ we have finished the proof. □

The following result was first proved in the case of type II_1 setting by [23]. Unfortunately the proof there does not work in the case of type III_1 we are interested in. We give a proof following [4].

Lemma 2.3. *(1) Suppose that $||E(P)|| \geq \lambda > 0$ for all nonzero projections P in M. Then $\text{Ind}E \leq \frac{1}{\lambda}$;*

(2) If $\text{Ind}E = \infty$ then for any $\epsilon > 0$, one can find a non-zero projection $P \in M$ such that $||E(P)|| \leq \epsilon$.

Proof. Ad (1): By Lemma 2.2 It is sufficient to check that $||E(m)|| \geq \lambda ||m||, \forall m \in M_+$ or equivalently $||E(m)|| \geq \lambda, \forall m \in M_+$ with $||m|| = 1$. For any $1 > \epsilon > 0$, let P be the spectral projection of m corresponding to interval $[1 - \epsilon, 1]$. Since $||m|| = 1$, P is nonzero, and $m \geq (1 - \epsilon)P$. We have $E(m) \geq (1 - \epsilon)E(P)$, and $||E(m)|| \geq (1 - \epsilon)||E(P)|| \geq (1 - \epsilon)\lambda$ for all $1 > \epsilon > 0$. Ad(2) : It follows immediately from (1)
□

2.2. A result on relative entropy.

Proposition 2.4. *Suppose that* $a \in M_+$, $||a|| \leq 1$, *and* $\omega(a) \geq \epsilon_0, \phi(a) \leq \frac{\epsilon_0^2}{4m}$ *where* $1 > \epsilon_0 > 0$ *, and m is a positive integer. Then* $S(\omega, \phi) \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$.

Proof. By assumption the spectrum $\sigma(a)$ is contained in $[0, 1]$. When restricting ω, ϕ to the von Neumann algebra generated by a, we can think of ω, ϕ as defined by two probability measures μ_1 and μ_2 on $[0, 1]$ such that $\omega(a) = \int_0^1 x du_1(x), \phi(a) = \int_0^1 x du_1(x)$. Let p be the spectral projection of a corresponding to interval $[\epsilon_0/2, 1]$. Then

$$\epsilon_0/2 + \omega(p) \geq \int_0^{\epsilon_0/2} x du_1(x) + \int_{\epsilon_0/2}^1 x du_1(x) = \omega(a) \geq \epsilon_0,$$

hence $\omega(p) \geq \epsilon_0/2$. From

$$\frac{\epsilon_0^2}{4m} \geq \phi(a) = \int_0^1 x du_2(x) \geq \int_{\epsilon_0/2}^1 x du_2(x) \geq \epsilon_0/2 \int_{\epsilon_0/2}^1 du_2(x) = \epsilon_0/2\phi(p),$$

we have $\phi(p) \leq \frac{\epsilon_0}{2m}$. On the other hand, when restricted to the two dimensional algebra generated by p, we have

$$S(\omega, \phi) = \omega(p) \ln \frac{\omega(p)}{\phi(p)} + \omega(1-p) \ln \frac{\omega(1-p)}{\phi(1-p)}.$$

It follows that

$$\omega(p) \ln \frac{\omega(p)}{\phi(p)} + \omega(1-p) \ln \frac{\omega(1-p)}{\phi(1-p)} \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$$

where $-e^{-1}$ is the minimal value of the function $x \ln x$ on $(0, 1]$. □

2.3. Nets and subnets.

2.3.1. Nets This section is contained in [7,13]. We refer to [7,13] for more details and proofs.

We shall denote by Möb the Möbius group, which is isomorphic to $SL(2, \mathbb{R})/\mathbb{Z}_2$ and acts naturally and faithfully on the circle S^1.

By an interval of S^1 we mean, as usual, a non-empty, non-dense, open, connected subset of S^1 and we denote by $\mathcal{I}$ the set of all intervals. If I is an interval on the circle on a complex plane with two end points a, b, $r_I := |b - a|$ is called the length of I. If $I \in \mathcal{I}$, then also $I' \in \mathcal{I}$ where I' is the interior of the complement of I. Two intervals are *disjoint* if their closure are disjoint. A finite set of intervals are disjoint if any two different intervals from the set are disjoint.

A *net* $\mathcal{A}$ *of von Neumann algebras on* S^1 is a map

$$I \in \mathcal{I} \mapsto \mathcal{A}(I)$$

from the set of intervals to the set of von Neumann algebras on a (fixed) Hilbert space $\mathcal{H}$ which verifies the *isotony property*:

$$I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$$

where $I_1, I_2 \in \mathcal{I}$.

A *Möbius covariant net* $\mathcal{A}$ of von Neumann algebras on S^1 is a net of von Neumann algebras on S^1 such that the following properties $1 - 4$ hold:

1. Möbius covariance: *There is a strongly continuous unitary representation U of* Möb *on $\mathcal{H}$ such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)\,, \qquad g \in \text{Möb},\ I \in \mathcal{I}\,.$$

We will write $\alpha_g(a) := U(g)^* a U(g), \forall a \in \mathcal{A}(I)$.
2. Positivity of the energy: *The generator of the rotation one-parameter subgroup $\theta \mapsto U(\text{rot}(\theta))$ (conformal Hamiltonian) is positive*, namely U is a positive energy representation.
3. Existence and uniqueness of the vacuum: *There exists a unit U-invariant vector Ω (vacuum vector), unique up to a phase, and Ω is cyclic for the von Neumann algebra $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.*

A *Möbius covariant net* $\mathcal{A}$ is nontrivial if $\mathcal{A}(I) \neq \mathbb{C}, \forall I \in \mathcal{I}$. For simplicity we will also call a *Möbius covariant net* $\mathcal{A}$ simply as a *Möbius net*.

Now we recall some definitions from [13] . Recall that $\mathcal{I}$ denotes the set of intervals of S^1. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \bigvee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \bigvee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I.

A *Conformal net* $\mathcal{A}$ of von Neumann algebras on S^1 is a net of von Neumann algebras on S^1 such that the above properties 2−4 hold, and 1 is replaces by conformal covariance:

Conformal covariance. There exists a projective unitary representation U of $\text{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \text{Diff}(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \text{Diff}(I'), \end{aligned}$$

where $\text{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $\text{Diff}(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$. By [21] a conformal net is split.

Let $\mathcal{A}$ be a Möbius net and I an interval. By removing a point p from I, we get two intervals I_1, I_2 contained in I such that $I_1 \cup I_2 \cup \{p\} = I$. Let ϕ be a linear functional defined on the algebra $M := \{a_1a_2, \forall a_1 \in \mathcal{A}(I_1), a_2 \in \mathcal{A}(I_2)\}$ by $\phi(a_1a_2) = \phi(a_1)\phi(a_2), \forall a_1 \in \mathcal{A}(I_1), a_2 \in \mathcal{A}(I_2)$. Note that M is dense in $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. The following Proposition is essentially contained in [24], and we include a proof in our setting:

Proposition 2.5. *Assume that $\mathcal{A}$ is nontrivial. Then ϕ can not be extended to a normal state on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$.*

Proof. We will argue by contradiction, assuming that ϕ can be extended to a normal state on $\mathcal{A}(I_1)\vee\mathcal{A}(I_2)$. By Möbius covariance we can assume that $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and denote by g_t the one parameter family subgroup of Möb such that $g_t.x = e^{-t}x$. Note that when $t \geq 0$, $g_t.I_1 \subset I_1$, $g_t.I_2 \subset I_2$, and $g_t.(-1, 0) = (-e^{-t}, 0)$, $g_t.(0, , 1) = (0, e^{-t})$. Choose any $a_1 \in \mathcal{A}((-1, 0)), a_2 \in \mathcal{A}((0, 1))$. Since closed balls of $\text{B}(\mathcal{H})$ is compact in weak topology, we can find a subnet $\alpha_{g_{t_i}}(a_1a_2)$ that converges weakly to b, where $t_i \to \infty$. b commutes with $\mathcal{A}((-e^{-t_i}, e^{-t_i}))'$ for $t_i \to \infty$, by [11] we conclude that $b \in \mathbb{C}$. Note that $\omega(\alpha_{g_{t_i}}(a_1a_2))) = \omega(a_1a_2)$ since the vacuum is Möb invariant, $b = \omega(a_1a_2)$. Since ϕ is normal on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ by our assumption, we have $\lim_{i\to\infty} \phi(\alpha_{g_{t_i}}(a_1a_2))) = \phi(b) = \omega(a_1)\omega(a_2) = \omega(a_1a_2)$. We have

$$\langle a_2\Omega, a_1^*\Omega\rangle = \langle \omega(a_2)\Omega, a_1^*\Omega\rangle, \forall a_1 \in \mathcal{A}((-1,0))$$

Since Ω is cyclic for $\mathcal{A}((-1,0))$ (cf. [11]) we conclude that $a_2 = \omega(a_2), \forall a_2 \in \mathcal{A}((0,1))$ which is absurd. □

2.4. Subnets. Let $\mathcal{A}$ be a Möbius net. By a *Möbius subnet* (cf. [17]) we shall mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotonic

$$\mathcal{B}(I_1) \subset \mathcal{B}(I_2), I_1 \subset I_2,$$

and Möbius covariant with respect to the representation U, namely

$$U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$$

for all $g \in \text{Möb}$ and $I \in \mathcal{I}$. When no confusion arises we will call a *Möbius subnet* simply a *subnet*. Note that by Lemma 13 of [17] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{A}(I) \to \mathcal{B}(I)$ such that E_I preserves the vector state given by the vacuum of $\mathcal{A}$, and if E_J restricts to E_I if $I \subset J$. Let P be the projection onto the closed subspace spanned by $\mathcal{B}(I)\Omega$. The $E_I(a)P = PaP, \forall a \in \mathcal{A}(I)$.

Definition 2.6. Let $\mathcal{A}$ be a Möbius net and $\mathcal{B} \subset \mathcal{A}$ a subnet. We say $\mathcal{B} \subset \mathcal{A}$ is of finite index if $\mathcal{B}(I) \subset \mathcal{A}(I)$is of finite index for some (and hence all) interval I. The index will be denoted by $[\mathcal{A} : \mathcal{B}]$.

If $\mathcal{A}$ is a conformal net, then $\text{Vir}_{\mathcal{A}}$ is defined given by $\text{Vir}_{\mathcal{A}}(I) = U(\text{Diff}(I))''$. $\text{Vir}_{\mathcal{A}} \subset \mathcal{A}$ is also a subnet and $\text{Vir}_{\mathcal{A}}$ is isomorphic to certain Virasoro net studied in [6]. We will refer to $\text{Vir}_{\mathcal{A}}$ as the *Virasoro subnet* of $\mathcal{A}$. For a subnet $\mathcal{B} \subset \mathcal{A}$, the relative commutants are defined as $\mathcal{C}(I) := \mathcal{B}(I)' \cap \mathcal{A}, \forall I$. $\mathcal{B} \subset \mathcal{A}$ is called **irreducible** if $\mathcal{C}(I) := \mathcal{B}(I)' \cap \mathcal{A}(I) = \mathrm{C}$ for some interval I. By (1) of the next lemma this condition implies $\mathcal{C}(I) := \mathcal{B}(I)' \cap \mathcal{A}(I) = \mathrm{C}, \forall I$. The next Lemma follows immediately from definitions.

Lemma 2.7. *(1)* $U(g)\mathcal{C}(I)U(g^*) = \mathcal{C}(g.I), \forall g \in \text{Möb}$;
(2) Assume that $\mathcal{B}$ is strongly additive, then we have the isotony property*:*

$$I_1 \subset I_2 \Rightarrow \mathcal{C}(I_1) \subset \mathcal{C}(I_2)$$

where $I_1, I_2 \in \mathcal{I}$. Denote by $\mathcal{H}_{\mathcal{C}}$ the closure of $\vee_{I\in\mathcal{I}}\mathcal{C}(I)\Omega$. Then the map $I \to \mathcal{C}(I)$ on $\mathcal{H}_{\mathcal{C}}$ gives an irreducible Möbius net on $\mathcal{H}_{\mathcal{C}}$.

Without strong additivity of $\mathcal{B}$, it is not clear if the the relative commutants $\mathcal{C}(I)$ verify the isotony condition as in Lemma 2.7. This problem was discussed in [16] and has an affirmative answer under certain conditions. However we can always define a $\mathcal{D}(I) := \mathcal{B}' \cap \mathcal{A}(I), \forall I \in \mathcal{I}$ where $\mathcal{B} := \vee_{I\in\mathcal{I}}\mathcal{B}(I)$. Note that $\mathcal{D}(I) \subset \mathcal{C}(I), \forall I \in \mathcal{I}$, and in fact is the largest subnet of $\mathcal{A}$ which verifies this condition (cf. [16]). By slightly abusing terminology we shall refer to $\mathcal{D}$ as the *coset* of $\mathcal{B} \subset \mathcal{A}$. Such a theory has been studied in [27] in concrete models.

Definition 2.8. We say a subnet $\mathcal{B} \subset \mathcal{A}$ verifies condition A if either $\mathcal{B} \subset \mathcal{A}$ is irreducible or if the net $\mathcal{D}$ defined above is nontrivial.

From definition it is clear that if $\mathcal{B}$ is strongly additive, then $\mathcal{B} \subset \mathcal{A}$ verifies condition A. As another example, assume that a compact group G acts properly on $\mathcal{A}$ such that the fixed point net $\mathcal{A}^G$ is a subnet (cf. [28]). By Proposition 2.1 of [5] $\mathcal{A}^G \subset \mathcal{A}$ is irreducible, and hence verfies condition A. If $\mathcal{B}$ is $\mathrm{Vir}_{\mathcal{A}}$, then $\mathcal{B} \subset \mathcal{A}$ is irreducible and verifies condition A. Note that if the central charge of $\mathcal{A}$ is greater than 1 $\mathrm{Vir}_{\mathcal{A}}$ is not strongly additive (cf. [3]). The following was communicated to us by Sebastiano Carpi:

Lemma 2.9. *Let $\mathcal{A}$ be a conformal net, and let $\mathcal{B} \subset \mathcal{A}$ be a subnet. Then $\mathcal{B} \subset \mathcal{A}$ verifies condition A.*

Proof. It is sufficient to check that if the net $\mathcal{D}$ defined above is trivial, then $\mathcal{B} \subset \mathcal{A}$ is irreducible. If the net $\mathcal{D}$ is trivial, by Prop. 2.4 of [9] (also see a brief discussion around equation (51) of [8]) $\mathrm{Vir}_{\mathcal{A}}$ is contained in $\mathcal{B}$, and by Prop. 3.7 of [6] the proof is complete. □

3. Singular Limits

Let $\mathcal{A}$ be a conformal net and I_1, I_2 are two disjoint intervals. We set

$$\omega_1 \otimes \omega(AB) = \langle \Omega \otimes \Omega, A \otimes B\, \Omega \otimes \Omega \rangle, \quad \forall A \in \mathcal{A}(I_1),\ B \in \mathcal{A}(I_2)\,.$$

Since $\mathcal{A}$ is split (cf. [21]), $\omega_1 \otimes \omega_2$ extends to a normal state on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. The mutual information we will compute is $S(\omega, \omega_1 \otimes \omega_2)$. When we wish to emphasize the underlying net, we will also write the mutual information as $S_{\mathcal{A}}(\omega, \omega_1 \otimes \omega_2)$. When $\mathcal{B} \subset \mathcal{A}$ is a subnet, we write $S_{\mathcal{B}}(\omega, \omega_1 \otimes \omega_2)$ the mutual information for the net $\mathcal{B}$ obtained by restricting $\omega, \omega_1 \otimes \omega_2$ from $\mathcal{A}$ to $\mathcal{B}$. Note that $\mathcal{B}$ is split. Let F_1 be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ to $\mathcal{B}(I_1) \vee \mathcal{A}(I_2)$ such that $F_1(xy) = E_I(x)y, \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2)$, and let E_1 be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ to $\mathcal{A}(I_1) \vee \mathcal{B}(I_2)$ such that $E_1(xy) = xE_I(y), \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2)$. First we observe that $\omega \cdot E_1 = \omega \cdot F_1$. In fact $\forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2)$,

$$\omega \cdot F_1(xy) = \langle E_I(x)y\Omega, \Omega\rangle = \langle PE_I(x)Py\Omega, \Omega\rangle = \langle E_I(x)E_I(y)\Omega, \Omega\rangle$$

where we have used $PE_I(x)P = E_I(x)P$, $PE_I(y)P = E_I(y)P$, and $P\Omega = \Omega$. Similarly $\omega \cdot E_1(xy) = \omega(E_I(x)E_I(y))$. We also note that $E_{g.I}(\alpha_g(x)) = \alpha_g(E_I(x)))$ for all $x \in \mathcal{A}(I)$. Here interval I is any interval that contains $I_1 \cup I_2$.

Note that $\omega_1 \otimes \omega_2 \cdot E_I \otimes E_I = \omega_1 \otimes \omega_2$ on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$, by Th. 2.1 we have

$$S_{\mathcal{A}}(\omega, \omega_1 \otimes \omega_2) = S_{\mathcal{B}}(\omega, \omega_1 \otimes \omega_2) + S_{\mathcal{A}}(\omega, \omega \cdot F_1) = S_{\mathcal{B}}(\omega, \omega_1 \otimes \omega_2) + S_{\mathcal{A}}(\omega, \omega \cdot E_1) \tag{1}$$

It is usually an interesting problem to study the limiting properties of relative entropies when intervals get close together. One can find such studies in §3 and §4 of [20]. The following Theorem (cf. Th. 2.38 of [29]) plays an important role:

Theorem 3.1. *Assume that M_n is an increasing sequence of factors act on a fixed Hilbert space, $N_n \subset M_n$ are subfactors and ω is a vector state associated with a vector Ω. Suppose that $E_n : M_n \to N_n, n \geq 1$ is a sequence of conditional expectations such that when restricting to M_n, $E_{n+1} = E_n, n \geq 1$, and $\mathrm{Ind}E_n = \lambda^{-1}$ is a positive real number independent of n. If strong operator closure of $\cup_n N_n$ contains M_1, then*

$$\lim_{n\to\infty} S(\omega, \omega E_n) = -\ln \lambda$$

The following is a generalization of Th. 4.4 of [20] (We note that there is a missing log in Th. 4.4 of [20]):

Theorem 3.2. *Assume that $\mathcal{A}$ is a conformal net and $\mathcal{B} \subset \mathcal{A}$ is a subnet. Let I_1 and I_2 be two intervals obtained from an interval I by removing an interior point, and let $J_n \subset I_2, n \geq 1$ be an increasing sequence of intervals such that*

$$\bigcup_n J_n = I_2, \quad \bar{J}_n \cap \bar{I}_1 = \emptyset \,.$$

Let E_n be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ to $\mathcal{A}(I_1) \vee \mathcal{B}(J_n)$ such that $E_n(xy) = xE_I(y), \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$. Then

$$\lim_{n\to\infty} S(\omega, \omega \cdot E_n) = \ln[\mathcal{A} : \mathcal{B}] \,.$$

Proof. Let F_n be the conditional expectation from $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$ to $\mathcal{B}(I_1) \vee \mathcal{A}(J_n)$ such that $F_n(xy) = E_I(x)y, \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$.

By Eq. (1) $S(\omega, \omega \cdot E_n) = S(\omega, \omega \cdot F_n)$ and we find it more convenient to work with $S(\omega, \omega \cdot F_n)$. We also note that $E_{g.I}(\alpha_g(x)) = \alpha_g(E_I(x)))$ for all $x \in \mathcal{A}(I)$. Let $\mu = \lim_{n\to\infty} S(\omega, \omega \cdot F_n)$. By Mobius covariance it is clear that μ is independent of the choice of I_1, I_2. We can assume that $I_1 = (-\infty, 0), I_2 = (0, 1)$ and we can choose $J_n = (1/n^2, 1)$. Applying dilation which is multiplication by positive integer n to I_1, J_n we can now assume $I_1 = (-\infty, 0), J_n = (1/n, n)$. We will compute μ using $I_1 = (-\infty, 0), J_n = (1/n, n)$.

Case I: $\mathcal{B}(I_1)' \cap \mathcal{A}(I_1) = \mathbb{C}$, and $[\mathcal{A} : \mathcal{B}] < \infty$. We will apply Th. 3.1 in this case for $M_n = \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$, $N_n = \mathcal{B}(I_1) \vee \mathcal{A}(J_n)$. Note that $\vee_n N_n = \mathcal{B}(I_1) \vee_n \mathcal{A}(J_n) = \mathcal{B}(I_1) \vee \mathcal{A}((0, \infty))$. Since $\mathcal{A}((0, \infty) = \mathcal{A}(I_1)'$ and $\mathcal{B}(I_1)' \cap \mathcal{A}(I_1) = \mathbb{C}$, we conclude that $M_1 \subset \vee_n N_n$ and Th. 3.1 implies our Theorem in this case.

Case II: $\mathcal{B}(I_1)' \cap \mathcal{A}(I_1) = \mathbb{C}$, and $[\mathcal{A} : \mathcal{B}] = \infty$. Let $m, 1 > \epsilon_0 > 0$ be as in Prop. 2.4. Since $\mathrm{Ind} E_{I_1} = \infty$, by Lemma 2.3 we can choose a projection $e_1 \in \mathcal{A}(I_1)$ such that $||F_1(e_1)|| < \frac{\epsilon_0^2}{4m}$. On the other hand since $\mathcal{A}(I_1)$ is a type III factor we can find unitary $u \in \mathcal{A}(I_1)$ such that $\omega(u^* e_1 u) > 2\epsilon_0$. Since $M_1 \subset \vee_n N_n$, we can find u_n in the unit ball of N_n such that $u_n \to u, u_n^* \to u^*$ in strong operator topology for some sequence of n which goes to infinity. Here we note that $\mathcal{H}$ is separable due to the split property of $\mathcal{A}$ (cf. [21]). Let $P_n = u_n e_1 u_n^*$. Then $P_n \in M_n$ converges to ue_1u^* strongly. It follows that when n is large enough we have

$$\omega(P_n) > \epsilon_0, \omega F_n(P_n) = \omega(u_n F_1(e_1) u_n^*) \leq ||F_1(e_1)|| < \frac{\epsilon_0^2}{4m}.$$

By Prop. 2.4 we have $S(\omega, \omega \cdot F_n) \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$. Therefore $\mu \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$ for any $m \geq 1$. Let m go to ∞ we have shown $\mu = \infty$ in this case.

Case III: Assume that $\mathcal{B} \subset \mathcal{A}$ is not irreducible. Then the index $[\mathcal{A}, \mathcal{B}] = \infty$ by Lemma 14 of [17]. By Lemma 2.9 the coset $\mathcal{D} = \mathcal{B}' \cap \mathcal{A}$ is nontrivial. We note that for any interval J, $\mathcal{B}(J) \vee \mathcal{D}(J)$ is isomorphic to $\mathcal{B}(J) \otimes \mathcal{D}(J)$ (cf. [16]), and ω on $\mathcal{B}(J) \vee \mathcal{D}(J)$ is a tensor product of its restriction on $\mathcal{B}(J)$ and $\mathcal{D}(J)$. Restricting $\omega, \omega \cdot F_n$ to $\mathcal{B} \vee \mathcal{D}$, by (5) of Th. 2.1 we have $S_{\mathcal{B}\vee\mathcal{D}}(\omega, \omega \cdot F_n) = S_{\mathcal{D}}(\omega, \omega_{I_1} \otimes \omega_{J_n})$. Since $\mathcal{A}$ is split, the subnet $\mathcal{D}$ is also split. Our Theorem follows from Theorem 3.4. □

The case considered in [2] corresponds to the case when $\mathcal{A}$ is a strongly additive net, and $\mathcal{B}$ is the fixed point net of $\mathcal{A}$ under the action of a compact group. Theorem 3.2 holds for such $\mathcal{B} \subset \mathcal{A}$.

Remark 3.3. For simplicty we have restricted to local nets instead of graded nets in this paper. Theorem 3.2 can be easily generalized to the Z_2 graded setting, for example by simply passing to the Z_2 fixed point subnet which is local.

3.1. Proof of Theorem 3.4.

Theorem 3.4. *Assume that $\mathcal{A}$ is a Möbius net with split properrty . Let I_1 and I_2 be two intervals obtained from an interval I by removing an interior point, and let $J_n \subset I_2, n \geq 1$ be an increasing sequence of intervals such that*

$$\bigcup_n J_n = I_2, \quad \bar{J}_n \cap \bar{I}_1 = \emptyset \,.$$

Let ϕ_n be the tensor states such that $\phi_n(xy) = \omega(x)\omega(y), \forall x \in \mathcal{A}(I_1), y \in \mathcal{A}(J_n)$.
Then

$$\lim_{n\to\infty} S_{\mathcal{A}(I_1)\vee\mathcal{A}(J_n)}(\omega, \phi_n) = \infty.$$

Proof. We will argue by contradiction. Note that by monotonicity $S_{\mathcal{A}(I_1)\vee\mathcal{A}(J_n)}(\omega, \phi_n)$ increase with n. For simplicity we will drop the subscript $\mathcal{A}(I_1)\vee\mathcal{A}(J_n)$ in the following. Assume that

$$\lim_{n\to\infty} S(\omega, \phi_n) = \mu < \infty.$$

Under this assumption we will show that ϕ_n can be extended to a normal state on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$, which contradicts Prop. 2.5. Denote by $M := \cup_n \mathcal{A}(I_1) \vee \mathcal{A}(J_n)$. Since $\mathcal{A}(I_1)\vee\mathcal{A}(J_n)$ is an increasing sequence of von Neumann algebras, M is a strongly dense $*$ subalgebra of $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. Let ϕ be the linear functional on M which restricts to ϕ_n on $\mathcal{A}(I_1)\vee\mathcal{A}(J_n)$. Note that ϕ is faithful on M since it is faithful on each $\mathcal{A}(I_1)\vee\mathcal{A}(J_n)$.

We apply GNS construction to (M, ϕ). Let H_ϕ be the corresponding Hilbert space. We will write $\tilde{m}$ for the operator on H_ϕ which represents the action of M, and $\hat{m}$ the vector corresponding to m in H_ϕ. We note that $\hat{M}$ is dense in H_ϕ and similarly $M\Omega$ is dense in $\mathcal{H}$. Denote by W the strong operator closure of $\tilde{M}$ on H_ϕ. Let $F(\tilde{m}) = m, \forall m \in M$. F is an isomorphism on each $\mathcal{A}(I_1) \vee \mathcal{A}(J_n)$. We need to show that F extends to a homorphism from W to $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. First we show we can extend F to W. It is sufficient to do this on the unit ball $(W)_1$ of W. Since the unit ball of $(\tilde{M})_1$ is dense in $(W)_1$ by §2 of [25], it is sufficient to show that if a sequence $\tilde{a}_n \in (\tilde{M})_1$ converges strongly to 0, then a_n converges strongly to 0 for any $||a_n| \leq 1, \forall n$. Suppose that a_n does not converge strongly to 0, then one can find a vector $b\Omega \in \mathcal{H}$ for some $b \in (M)_1$, a positive number ϵ_0 such that we have a subsequence a_{n_k} such that $\langle a_{n_k} b\Omega, a_{n_k} b\Omega\rangle = \omega(b^* a^*_{n_k} a_{n_k} b) \geq \epsilon_0$. On the other hand since $\tilde{a}_n$ converges strongly to 0, for any positive integer m we can find n_k large enough such that $\langle a_{n_k}\hat{b}, a_{n_k}\hat{b}\rangle = \phi(b^* a^*_{n_k} a_{n_k} b) \leq \frac{\epsilon_0^2}{4m}$. By Prop. 2.4 we have $S(\omega, \phi_{n_k}) \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$. It follows that $\mu \geq S(\omega, \phi_{n_k}) \geq \frac{\epsilon_0 \ln m}{2} - e^{-1} + \ln(1 - \frac{\epsilon_0}{2m})$ for any positive integer m, a contradiction. Hence F extends to a normal homorphism from W to $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. Since M is dense in $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$, it follows that $F(W) = A(I_1) \vee \mathcal{A}(I_2)$. Let W_1 be the kernel of F. This is a two sided weakly closed $*$ ideal of W, so there is a central projection $Q \in W$ such that $W_1 = QW$.

When restricting to $(1-Q)W$, F is an isomorphism of $(1-Q)W$ and $A(I_1) \vee \mathcal{A}(I_2)$ and hence ϕ extends to a normal state on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$. □

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Data Availability All data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest The author declares that there are no conflict of interest regarding the publication of this manuscript.

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