SOME CONSIDERATIONS ABOUT NELSON'S DERIVATION OF SCHROEDINGER EQUATION

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§ 0. Schroedinger equation for a particle moving in a potential $V(x)$ in n-dimensions was rederived by Nelson by purely probabilistic considerations starting from Newtonian mechanics (see Nelson 1966, 1967). While there have been several attempts to generalize the theory to the relativistic domain and to the case of spinning particles (see DeLaPena-Auerbach 1969, 1971; Caubet 1975), the conceptual aspects of this approach have not encountered much consideration. I shall discuss the following points, in the following:

1) the contrast with the usual approach is entirely due to an improper use of the term "Markov process";

2) the measure on path space, defined by the stochastic process $x(t)$ is, by construction, beyond any experimental verification;

3) from a technical point of view, the theory provides the simplest method to derive Feynman-Kac formula for the Green's function.

Points (1) and (2) are perhaps of "public domain", but I have not seen them stated explicitly. Point (3) is essentially due to Albeverio and Höegh-Krohn (1973) and can be found in Ezawa, Klauder and Shepp (1974).

§ 1. The theory is given in terms of a stochastic process $x(t)$ which is characterized by the following requirements: i) $x(t)$ is a diffusion process with diffusion coefficient $\nu = \hbar/2m$, the drift coefficient $b(x(t),t)$ being irrotational; ii) the process admits time-reversal invariance; iii) for a suitable definition of mean acceleration this quantity must be proportional to the classical force $-\text{grad } V/m$. Following Nelson's arguments, we end up with a process which is characterized by two functions $R(x,t)$ and $S(x,t)$ satisfying a system of two coupled non-linear differential equations of 1^{st} order in time,

which is equivalent to Schroedinger equation if we put $\psi(x,t) = \exp\{R(x,t)+iS(x,t)\}$. The point is that the drift coefficient $b(x,t) = \hbar/m \operatorname{grad}(R + S)$ is not a preassigned vectorfield, but is a function of the initial state of the particle. The conclusion is that the whole process $x(t)$ appears as some kind of envelope of a family of diffusion processes, but it is not a Markov process in itself. In fact, the knowledge of $x(t_o)$ (as a random variable) is not sufficient to determine the process; to do that, it is necessary to know also the expectation of $\frac{x(t_o) - x(t_o-h)}{h}$ (conditional on $x(t_o)$) for arbitrarily small h; this is characteristic of a non-Markov process. The essential feature of Markov processes, the transition function, can be defined, but it is dependent on the initial state of the particle; this is qualitatively different from the diffusion processes which arise in applications (heat equation, classical statistical mechanics, etc.) and there is no point to insist that they are the same.

§ 2 . Let us suppose that a solution $\psi(x,t)$ of Schroedinger equation is known. Then we can write down the stochastic differential equation

$$dx(t) = b(x(t),t)\, dt + d\, w(t)$$

where $b(x,t) = \hbar/m \operatorname{grad}[\log|\psi| + \arg\psi]$ and $w(t)$ is the Wiener process with covariance $\hbar/m$. If we are able to solve this equation, the result is a stochastic process $x(t)$, that is a measure μ on path space. For example μ gives the probability that the trajectory of the particle belong to a certain cylinder set $C = \{x(t_1)\in \Gamma_1;\ x(t_2)\in \Gamma_2;\dots;\ x(t_n)\in \Gamma_n\}$. $\mu(C)$ is easily given in terms of the diffusion process $x(t)$, namely if $P(x,t;\Gamma,s)$ is the transition function, it holds:

$$\mu(C) = \int_{\Gamma_1}\cdots\int_{\Gamma_n} |\psi(x_1,t_1)|^2 dx_1\ P(x_1,t_1;dx_2,t_2)\dots P(x_{n-1},t_{n-1};dx_n,t_n)$$

The point that we want to stress is that this value of $\mu(C)$ is valid only if we do not try to check it! Actually a method to measure $\mu(C)$ consists in killing the particle at time $t_1, t_2,\dots,t_{n-1}$ if it is not in $\Gamma_1, \Gamma_2,\dots, \Gamma_{n-1}$ and counting the number of particles reaching Γ_n at time t_n. In this situation Quantum Mechanics gives the following expression for the probability of detecting the particle in Γ_n at time t_n:

$$P(x(t_n)\in\Gamma_n) =$$

$$\int_{\Gamma_n} dx_n \left| \int_{\Gamma_1}\dots\int_{\Gamma_{n-1}} \psi(x_1,t_1)G(x_1,t_1;x_2,t_2)\dots G(x_{n-1},t_{n-1};x_n,t_n)\, dx_1\dots dx_{n-1}\right|^2$$

Of course this value cannot coincide with $\mu(C)$ since this latter is an additive functional of sets Γ's, while P is not. There is no contradiction, however; according to Nelson's theory, we must determine μ starting from the wave funtion, which is not the original one $\psi(x,t)$ but has been "projected" in $\Gamma_1, \Gamma_2,\dots, \Gamma_{n-1}$ at times $t_1, t_2,\dots, t_{n-1}$. We have to calculate this new wave function $\tilde{\psi}(x,t)$ and then to solve the stochastic differential equation for the new stochastic process $\tilde{x}(t)$, thus obtaining the new measure $\tilde{\mu}$ holding in this case. The conclusion is that the measure μ cannot give any additional information which is not explicitly contained in the wave function; consequently the tremendous task of solving the stochastic differential equation for a given $\psi(x,t)$ is unnecessary.

§ 3. The third point we wish to discuss is the technical value of Nelson's approach. Let us consider the ground state $\varphi_0(x)$ of the Hamiltonian $H = p^2/2m + V(x)$, and let E_0 be the corresponding eigenvalue; $\varphi_0(x)$ can be taken to be real and nowhere vanishing. The Fokker-Planck equation for the process x(t) is given by

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}\left(\frac{\hbar}{m}\frac{\operatorname{grad}\varphi_0}{\varphi_0}\rho\right) + \frac{\hbar}{2m}\Delta\rho$$

Let $\rho = \varphi_0\psi$; it follows

$$-h\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + (V(x)-E_0)\psi$$

which is formally Schroedinger equation with imaginary time (Albeverio and Hoegh-Krohn, 1973). It follows that the fundamental solution of the Fokker-Plank equation is given by

$$P(x,t;y,s) = \frac{\varphi_0(y)}{\varphi_0(x)}\, G(x,-it;y,-is)$$

This means that we can obtain the Green's function G through the following steps: calculate the ground state; calculate the transition probability density for the process associated to the ground state;

analitically continue to imaginary times.

A general formula is known for the transition probability density (see Gihman-Skorohod, 1972); the result, in our case, is just Feynman-Kac formula

$$\langle x \mid e^{-t(H-E_o)} \mid y \rangle = \int_{\substack{w(o)=x \\ w(t)=y}} \left[\exp -\frac{1}{\hbar} \int_o^t V\,(w(\tau))\, d\tau \right] d\mu_w$$

This connection between $P(x,t;y,s)$ and the Green's function can be used to prove that $E(x(t)x(s))$ is just the analytic continuation of $\langle \varphi_o \mid T(x_{op}(t)x_{op}(s) \mid \varphi_o \rangle$ to imaginary times; in general, Feynman-Kac-Nelson formula can be obtained this way (Simon, 1974).

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