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Localization, supersymmetric gauge theories and toric geometry

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Abstract

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Gauge theories is one of the most pervasive and important subject of modern theoretical physics, and there are still many things about them we do not understand. In particular dealing with strongly coupled theories where normal perturbative techniques do not apply is a fundamental open problem. In this thesis, we study a particular class of toy-models that have supersymmetry, which makes them much easier to deal with. We employ the mathematical technique of localization, which for supersymmetric theories lets us evaluate certain path integrals exactly and for any value of the coupling. This is used to study the 5d N=1 theories placed on toric Sasaki-Einstein manifolds and compute their partition functions, finding that they factorize into a product of contributions from each closed Reeb orbit of the manifold. This computation leads us to define two new hierarchies of special functions associated to these manifolds, and we study their properties. Finally, we use the 5d N=1 theories to construct new 4d N=2 theories on a large class of curved backgrounds. These theories have some interesting features, such as supporting both instantons and anti-instantons, and having a position-dependent complexified coupling constant.

Keywords: Supersymmetry, gauge theory, localization, toric geometry

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Dedicated to my parents

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Jian Qiu, Luigi Tizzano, Jacob Winding, Maxim Zabzine, *Gluing Nekrasov partition functions*, Commun.Math.Phys. 337 (2015) no.2, 785-816, [arXiv:1403.2945]

II Jian Qiu, Luigi Tizzano, Jacob Winding, Maxim Zabzine, *Modular properties of full abelian 5d SYM partition function*, JHEP 1603 (2016) 193, [arXiv:1511.06304]

III Jacob Winding, *Multiple elliptic gamma functions associated to cones*, submitted to Adv.Math., [arXiv:1609.02384]

IV Guido Festuccia , Jian Qiu , Jacob Winding, Maxim Zabzine, $\mathcal{N}=2$ *supersymmetric gauge theory on connected sums of* $S^2 \times S^2$, JHEP 1703 (2017) 026, [arXiv:1611.04868]

Papers not included in the thesis

V Luigi Tizzano, Jacob Winding, *Multiple sine, multiple elliptic gamma functions and rational cones*,submitted to J. Geometry Phys., [arXiv:1502.05996]

Contents

1. Introduction

Quantum field theory (QFT) is the framework that combines quantum mechanics and special relativity. It is the language of modern particle physics and a fundamental tool of any theoretical physicist, not only for particle physics but also in condensed matter and of course for string theory. QFT describes all the fundamental physics we know except gravity, and has been wildly successful at this. It has made the most precise predictions ever in the history of science, and been tested against a huge amount of experimental data. Despite these successes and several decades of intense study, there are still many things about QFT we do not understand. For example we do not know how to mathematically formulate the theory properly, except in very simple examples. The proper mathematical understanding of QFT is one of the Millennium problems [1], and is a question of great mathematical importance.

One way of formulating QFT is in terms of the so called path integral, which is an integral over the space of all field configurations. This is an infinite dimensional space and making mathematical sense of this integral is an open problem. But in theoretical physics, we care more about what works than about mathematical rigor, so even though we do not know exactly what the path integral is, we usually pretend that it is some ordinary integral and try to evaluate it, using techniques that work in finite dimensions. The favorite technique is that of the stationary phase approximation, which leads us to do a perturbative expansion and compute quantities order by order in the coupling constant. Over the years physicists have developed many powerful tools for dealing with this perturbation theory, including the famous Feynman diagrams, and for most applications these perturbative techniques work very well (something we know from comparisons with experiments). When the coupling constant is small, we can compute many orders of the perturbation series, and a very high numerical precision can be reached.

There are however cases where perturbation techniques are not applicable, mainly when the theory is strongly coupled, i.e. when the the coupling constant is of order 1. This happens for several physically important cases, like the description of the strong nuclear force, quantum chromodynamics (QCD), or the theory of high temperature superconductors. Understanding the behavior of strongly coupled quantum field theories is therefore one of the major problems of theoretical physics. The work of this thesis is part of a research program that tries to attack this problem using a mathematical result called the localization formula, or equivariant localization or just localization [2, 3, 4] . This is a mathematical result stating that for certain integrals, the stationary phase approximation is actually exact. That means that we can evaluate some special integrals exactly, without having to do any perturbative expansion. The result is called localization since it reduces the integral to a much more 'localized' object: the entire integral becomes a sum over the specific places where the phase is stationary.

This result is relevant to QFT since for some special field theories, namely those with supersymmetry, localization can be applied to the path integral. The localization formulas then reduce us from the infinite-dimensional integral to a well-defined finite-dimensional integral, that we in principal can compute explicitly, giving an exact result. This is quite impressive and can let us make sense of the path integral for these theories. Since it works for any value of the coupling constant, it potentially lets us understand new things about QFTs at strong coupling.

Of course, the supersymmetric theories that we can localize are not realistic, but they can still serve as useful toy models of the realistic non-supersymmetric theories that we ultimately care about. The research of this thesis studies some different examples of supersymmetric quantum field theories and computes their partition functions exactly using localization. This might also give us some further insight into the mathematical structures of supersymmetric QFTs, and the investigations leads us to introduce new special functions which are of mathematical interest.

Trying to understand strongly coupled QFTs is one motivation for the present work, but it can also be motivated from a string theory perspective. String theory is the best attempt at a theory of quantum gravity, and of a theory of everything, that we have found so far. It is also intimately connected to supersymmetric quantum field theories, including the theories that we study in this thesis.

To describe this connection, we give a lightning review of some of the basics of string theory. String theory is a generalization of quantum field theory that instead of describing point particles describe 1d extended objects, i.e. strings. When one describes this mathematically, a lot of internal consistency requirements appear, and they fix most of the theory. One uncover a very rigid mathematical structure, and for example the theory of general relativity follows from consistency, as does the fact that the theory needs to be supersymmetric. However everything is not uniquely fixed and there are five different consistent versions of (super-)string theory, that all live in 10 dimensions (another thing imposed by internal consistency). It was eventually understood that these are all connected by various dualities, and should be thought of as different limits of a mysterious, magical, mother theory called M-theory, introduced by Witten in the mid 1990's [5].

This M-theory is described in the low-energy limit by the unique 11d supergravity. It is known that it does not have strings as its fundamental objects. Instead its fundamental objects are higher dimensional extended objects, the 3d M2 brane and the 6d M5 brane. These M-branes are described by some world-volume theories, which are quantum field theories that describe their dynamics. The M2 brane is described by the ABJM theory [6] (after Aharony, Bergman, Jafferis and Maldacena), and the 6d theory is described by the mysterious $\mathcal{N} = (2, 0)$ 6d theories [7, 8], commonly referred to as just the (2,0) theories. These 6d theories are superconformal (meaning that it is both conformal and supersymmetric) and does not seem to have a Lagrangian description. The 5d supersymmetric theories that this thesis studies is very closely related to this $(2, 0)$ theory through dimensional reduction on a circle, and therefore studying it gives us a way to learn things about the 6d theory. So this is the part of motivation from a string theory perspective.

1.1 Organization of the thesis

This thesis is organized as follows. In chapter 2, we give a pedagogical introduction to the mathematical idea of localization, describing in some detail the localization formulas for the finite dimensional setting, and then demonstrating through an example how they are formally applied to an infinite-dimensional setting. Next in chapter 3 we introduce supersymmetric gauge theories, and in particular spell out some details of the theories in 5 and 4 dimensions that we eventually study. We then turn to the geometrical background that we need, discussing first toric geometry in chapter 4 and then Sasaki-Einstein geometry in the following chapter. In chapter 6 we describe how to put 5d $\mathcal{N} = 1$ theory on a curved Sasaki-Einstein background and how to use localization to compute its partition function. Next we describe the new special functions that this computation lead us to define, and the various factorization properties these functions satisfy. Finally in chapter 8 we discuss the procedure of dimensionally reducing gauge theories, and how this lets us construct new 4d $\mathcal{N} = 2$ theories on curved backgrounds by reducing 5d $\mathcal{N} = 1$ theories over a non-trivial circle fibration.

2. Localization

The main technique used in this thesis is a collection of mathematical results that goes under the name of localization formulas. These are formulas that lets us evaluate certain integrals exactly. They work by reducing the domain of integration from the original space, to something much smaller, thus “localizing” the integral. The most famous example of this is the Cauchy residue theorem from complex analysis, which reduces contour integrals to a sum of residues. That result is not quite the same as the ones we study here, since it depends on holomorphicity rather than the existence of a group action, but the idea is very similar.

The first localization formula using a group action was discovered by Duistermaat and Heckman [2], who gave a localization formula in the symplectic geometry setting. Berline and Vergne [4] and, independently, Atiyah and Bott [3] later proved the more general equivariant localization formula. Hence the main localization result sometimes goes by the name Berline-Vergne-Atiyah-Bott formula.

In recent years, this mathematical idea has seen a lot of use in theoretical physics, where it is used to study supersymmetric field theories, as we will explain in section 2.3.

2.1 Equivariant cohomology

We begin by introducing the basic set-up, that of equivariant cohomology, before turning to the localization formula and its proof. We assume that the reader is familiar with the notions of manifolds, differential forms and de Rham cohomology.

Consider a smooth n dimensional manifold X, and let G be a Lie group with an action on X, meaning a map

$$\psi : G \to \mathrm{Diff}(X), \tag{2.1}$$

which induces the evaluation map

$$\begin{aligned} \mathrm{ev}_\psi : X \times G &\to X, \\ (x,p) &\mapsto \psi_g(x). \end{aligned} \tag{2.2}$$

The map ψ should respect the group multiplication. A group action is called smooth if the evaluation map is smooth, free if it has no fixed points, and effective if only the identity element leaves the entire manifold fixed.

An example of a group action is the action of $U(1)$ on the two-sphere S^2, which rotates it around its axis. This action has two fixed points, the north and south pole, so it is not free, but it is effective.

The idea of equivariant cohomology is to construct some cohomology groups of X that takes into account the group action. The simplest case is if G acts freely (i.e. the group action has no fixed points). Then the quotient X/G is a smooth manifold and the de Rham cohomology $H^\bullet(X/G)$ is defined. We take this to be the G-equivariant cohomology $H_G^\bullet(X) = H^\bullet(X/G)$. If the action of G is not free, the situation is more subtle, and the above definition of the equivariant cohomology fails, since X/G is no longer a smooth manifold. In this case, there is a topological way of defining G-equivariant cohomology,

$$H_G^\bullet(X) = H^\bullet(X \times_G EG) = H^\bullet((X \times EG)/G), \tag{2.3}$$

where EG is the universal bundle of G [9]. This is a topological space related to G, defined by the two properties that it is contractible, and that G acts freely on EG. This definition gives a well-defined meaning to the equivariant cohomology of X, but it is not very explicit. To get a better understanding, and for performing computations, we will introduce an explicit model of this cohomology. There exists a number of different such models, including the Weil, Cartan and BRST (or Kalkman) models, see for example [10, 11, 12] for more detailed treatments and explanations of the relations between them. We will construct the Cartan model [13], since it is most closely related to the quantum field theory set-up.

2.1.1 The Cartan model

Let $\mathfrak{g}$ be the Lie algebra of G. We can write an element $\phi \in \mathfrak{g}$ on the form $\phi = \phi^a T_a$, where $\{T_a | a = 1, 2, \ldots, \dim(\mathfrak{g})\}$ is a basis of $\mathfrak{g}$, and the sum over a is implicit. The coordinates ϕ^a can be thought of as elements in the dual space $\mathfrak{g}^*$. For constructing the Cartan model we need the symmetric algebra of $\mathfrak{g}^*$, denoted $S(\mathfrak{g}^*)$. This is the algebra whose elements are polynomials in ϕ^a, so $S(\mathfrak{g}^*) \simeq \mathbb{R}[\phi^1, \ldots, \phi^{\dim(\mathfrak{g})}]$. We consider differential forms valued in $S(\mathfrak{g}^*)$, i.e. $\Omega^\bullet(X) \otimes S(\mathfrak{g}^*)$. Further, we care about such forms which are G-invariant, meaning that they are invariant under the induced group action on the form and the induced adjoint action on $S(\mathfrak{g}^*)$. We spell out what this means as follows. For every basis element T_a of the Lie algebra, the group action on X induces a vector field v_a on X, which points along the flow generated by the group elements e^{tT_a}, for t a real parameter. We can view v_a as representing the infinitesimal group action of T_a on X, and the induced group action on a differential form is the Lie derivative along v_a. The adjoint action on $S(\mathfrak{g}^*)$ by a basis element T_a is given by

$$f^b_{ac}\phi^c \frac{\partial}{\partial \phi^b}, \tag{2.4}$$

where f^b_{ac} is the structure constants of $\mathfrak{g}$. So the condition of a form $\alpha(\phi) \in \Omega^\bullet(X) \otimes S(\mathfrak{g}^*)$ being G-invariant is

$$L_{v_a}\alpha(\phi) + f^b_{ac}\phi^c \frac{\partial \alpha(\phi)}{\partial \phi^b} = 0. \tag{2.5}$$

We denote the space of G-invariant forms $(\Omega^\bullet(X) \otimes S(\mathfrak{g}^*))^G$.

For the forms in $\Omega^\bullet(X) \otimes S(\mathfrak{g}^*)$, we define the Cartan differential,

$$\begin{aligned} &d_G : \Omega^\bullet(X) \otimes S(\mathfrak{g}^*) \to \Omega^\bullet(X) \otimes S(\mathfrak{g}^*), \\ &d_G = d + \phi^a \iota_{v_a}, \end{aligned} \tag{2.6}$$

where d is the usual de Rham differential on the differential forms, and ι_{v_a} is the contraction of the differential form and the vector field generated by T_a. This differential squares to the sum of Lie derivatives along the vector fields v_a,

$$d_G^2 = \phi^a L_{v_a}. \tag{2.7}$$

The action of d_G^2 on G-invariant forms is zero, since for an invariant form $\alpha(\phi)$ we have, using (2.5),

$$d_G^2\alpha(\phi) = \phi^a L_{v_a}\alpha(\phi) = -\phi^a f_{ac}^b \phi^c \frac{\partial\alpha(\phi)}{\partial\phi^b} = 0 \tag{2.8}$$

by the antisymmetry of the structure constants. We can thus define the Cartan model of equivariant cohomology as the associated cohomology

$$H_G^\bullet(X) = H\big((\Omega^\bullet(X)\otimes S(\mathfrak{g}^*))^G, d_G\big). \tag{2.9}$$

An invariant form α is called equivariantly closed if $d_G\alpha = 0$, and equivariantly exact if it can be written as $\alpha = d_G\beta$.

The Cartan differential is not homogenous with respect to the degrees of differential forms, since it maps a p-form into a sum of a $p-1$ and $p+1$ form. If we write an element $\alpha \in \Omega^\bullet(X)\otimes S(\mathfrak{g}^*)$ as a sum of forms of various degrees,

$$\alpha(\phi) = \sum_{k=0}^{n} \alpha_k(\phi), \tag{2.10}$$

then the equation $d_G\alpha = 0$ becomes a set of equations relating the different degrees of the form:

$$d\alpha_{k-2}(\phi) + \phi^a \iota_a \alpha_k(\phi) = 0. \tag{2.11}$$

If X is even-dimensional, these conditions relate the top degree component to the 0-form part, and we will see this play a part in the localization formulas that we discuss below.

To make closer contact with the formalism used in physics, we now reformulate the Cartan model in the language of supergeometry [14]. Supergeometry is a generalization of normal differential geometry, where instead of having only the usual (even) coordinates one also introduces Grassmann (odd) coordinates. The usual, even, coordinates commute, $x^\mu x^\nu = x^\nu x^\mu$, and the Grassmann coordinates instead anti-commute, i.e. obey $\psi^\mu\psi^\nu = -\psi^\nu\psi^\mu$. In particular this implies $(\psi^\mu)^2 = 0$. In physics terms, quantities obeying anti-commutation relations are commonly called fermionic, while even quantities are called bosonic.

Integration of odd coordinates is defined algebraically, so that if we have n odd coordinates, their integration is given by

$$\begin{aligned} &\int \psi^1 d^n\psi = \int \psi^1\psi^2 d^n\psi = \ldots = \int \psi^1 \ldots \psi^{n-1} d^n\psi = 0, \\ &\int \psi^1\psi^2 \cdots \psi^n \, d^n\psi = 1, \end{aligned} \tag{2.12}$$

which mimics how we usually integrate forms in how only the "top" part gives something non-zero. This is called Berezin integration [15].

The primary example of a supermanifold is the odd tangent bundle ΠTX, which is a supermanifold incarnation of the ordinary tangent bundle. In ΠTX, the even coordinates x^μ are the usual coordinates on X, and we also have the odd coordinates ψ^μ, which correspond to the 1-forms dx^μ. Their multiplication should be thought of as the wedge product of forms. Differential forms on X correspond to the functions on ΠTX, $\Omega^\bullet(X) \simeq C^\infty(\Pi TX)$. There is a canonical integration measure on ΠTX given by $d^n x \; d^n\psi$. Dealing with ΠTX is a reformulation of the usual theory of differential forms on X.

In this supergeometry language, the Cartan differential is represented as a transformation on the coordinates (writing now δ for the Cartan differential)

$$\begin{aligned} &\delta x^\mu = \psi^\mu, \qquad \delta\psi^\mu = \phi^a v_a^\mu, \\ &\delta\phi^a = 0. \end{aligned} \tag{2.13}$$

Written in this way, physicists may recognize the form of a supersymmetry variation. This is precisely why we reformulated the model in terms of the supergeometry language. Note that if we took $\delta\psi^\mu = 0$, the transformation would instead be the ordinary de Rham differential.

In this language, an equivariantly closed form is a function $f \in C^\infty(\Pi TX)$ such that $\delta f(x, \psi) = 0$. In physics terminology we call δ the supersymmetry and say that f is supersymmetric.

2.2 Localization formulas

The localization formulas give us a way of computing integrals of equivariantly closed forms in terms of the fixed points of the group action. For simplicity, we first restrict our attention to the case of a single $U(1)$

action (i.e. a rotation) with a set of isolated fixed points. Let X be a manifold of dimension $2n$, equipped with a $U(1)$ action with isolated fixed points $\{x_p\}$, and let the vector field representing the group action be called v. The Lie algebra of $U(1)$ is one-dimensional, $\mathfrak{u}(1) \simeq \mathbb{R}$, so it has a single coordinate ϕ. For an equivariantly closed form $\alpha(\phi)$, its integral over X is exactly given by

$$\int_X \alpha(\phi) = \left(\frac{2\pi}{\phi}\right)^n \sum_p \frac{\alpha_0(\phi)|_{x_p}}{\sqrt{\det(v)|_{x_p}}}. \tag{2.14}$$

This is the Berline-Vergne-Atiyah-Bott formula [4, 3]. In the formula $\det(v)|_{x_p}$ means the determinant of the linearized action of v at the fixed point x_p, which is also known as the index of v at x_p.

We will now briefly review the proof of this formula, following an approach suggested by Witten [16]. We will work using the supergeometry language, and consider the odd tangent bundle ΠTX, with bosonic coordinates x^μ and fermionic coordinates $\psi^\mu \simeq dx^\mu$. Then the Cartan differential, $d_G = d + \phi\iota_v$, is represented by the "supersymmetry" δ acting as in equation (2.13), but with a single vector field v and a single coordinate ϕ. The integral over X is written in the supergeometry language as

$$Z = \int_X \alpha(\phi) = \int_{\Pi TX} d^{2n}x \; d^{2n}\psi \; \alpha(x, \psi, \phi). \tag{2.15}$$

Consider the following deformation of this integral

$$Z(t) = \int_{\Pi TX} d^{2n}x \; d^{2n}\psi \; \alpha(\phi) e^{-t\delta W(\phi)}, \tag{2.16}$$

where $W(x, \psi, \phi)$ is some function such that $\delta^2 W = 0$ and $t \in \mathbb{R}$. $Z(t)$ is actually independent of t, since

$$\begin{aligned} \frac{d}{dt} Z(t) &= -\int_{\Pi TX} d^{2n}x d^{2n}\psi \; \delta W(\phi) \; \alpha(\phi)) e^{-t\delta W(\phi)} \\ &= -\int_{\Pi TX} d^{2n}x d^{2n}\psi \; \delta(W\alpha e^{-t\delta W}) = 0, \end{aligned} \tag{2.17}$$

where we integrate by parts, use that $\delta\alpha = \delta^2 W = 0$ and finally apply Stokes theorem, which tells us that the integral of an exact form is zero in the absence of a boundary.

The integral we want to compute is given by $Z(0)$, but since $Z(t)$ is independent of t, we can compute it for any value of t. In particular, we can take the limit $t \to \infty$, in which case the integral is dominated by the points where $\delta W = 0$, if δW is semi-positive definite. Using this, we can compute the original integral exactly through the saddle point approximation around the points where $\delta W = 0$.

We have the freedom of choosing W, and an appropriate choice is given by

$$W = g_{\mu\nu}\psi^{\mu}(\delta\psi)^{\nu} = \phi g_{\mu\nu}\psi^{\mu}v^{\nu} \, , \tag{2.18}$$

where $g_{\mu\nu}$ is the metric on X. This is invariant under the group action as long as v preserves the metric, or in other words if v is Killing, so we assume this to be the case. One can always construct a G-invariant metric by averaging over the group action, so there is no loss of generality with this assumption. Next we compute δW,

$$\delta W = \phi\partial_{\rho}(g_{\mu\nu}v^{\mu})\psi^{\rho}\psi^{\nu} + \phi^2 g_{\mu\nu}v^{\mu}v^{\nu}. \tag{2.19}$$

This has a '2-form part' (quadratic in ψ's) and a '0-form' part, and the 0-form part is proportional to the square of the norm of the vector field v. With this choice of W, the only points on the manifolds which will not be exponentially suppressed in the large t limit are the points where v vanish, i.e. the fixed points of the group action.

Let us now consider the contribution of one isolated fixed point, x_p. Around this point we can pick local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ so that, with the fixed point being at the the origin, the metric takes the form

$$g = \sum_{i=1}^{n}(dx_i^2 + dy_i^2). \tag{2.20}$$

We can think of the local geometry as the product of n two-dimensional planes.

In these coordinates, any vector field generating a $U(1)$ action and that vanish at the origin will take the form of a sum of rotations of these planes, i.e.

$$v|_p = \sum_{i}^{n} \nu_i^p(x_i\partial_{y_i} - y_i\partial_{x_i}), \tag{2.21}$$

see figure 2.1. The numbers ν_i^p describe how fast each plane is rotated, and the the index or linearized determinant of v at this fixed point, is given by

$$\det(v)|_{x_p} = \prod_{i=1}^{n}(\nu_i^p)^2, \tag{2.22}$$

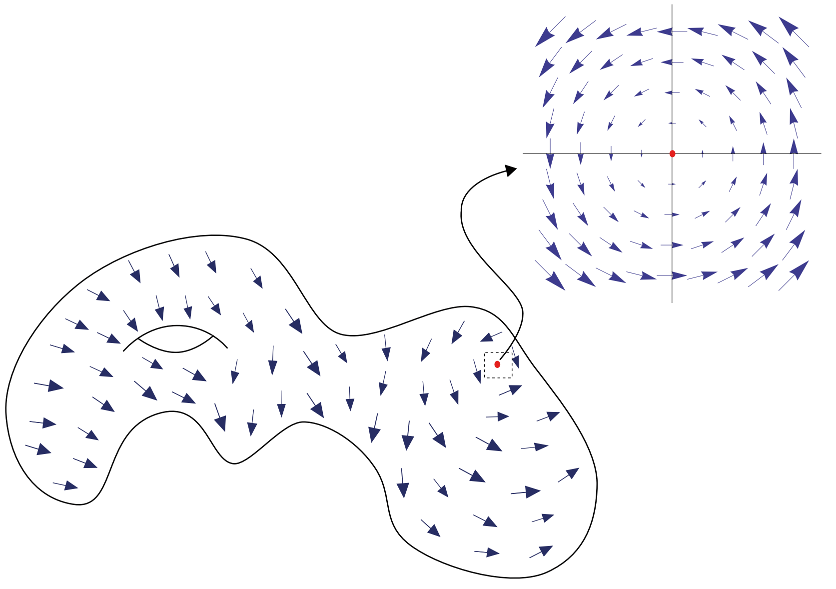

Figure 2.1. A sketch of a manifold with a vector field generating a $U(1)$ action, showing the local behavior of the vector field around a fixed point (marked by the red dot) in the coordinates of (2.20).

which we will find in the following computations. In these coordinates, the function W close to the fixed point takes the form

$$W(x, \psi, \phi) = \frac{\phi}{2} \sum_i \nu_i^p (x_i \psi_i^y - y_i \psi_i^x), \tag{2.23}$$

where we have also split the odd coordinates into two groups labelled by x and y. The localizing term δW takes the form

$$\delta W(x, \psi, \phi) = \phi \sum_i \nu_i^p \psi_i^x \psi_i^y + \frac{\phi^2}{2} \sum_i (\nu_i^p)^2 ((x_i)^2 + (y_i)^2). \tag{2.24}$$

To perform the saddle point approximation (which is exact), we rescale the coordinates

$$\tilde{x} = \sqrt{t} x, \qquad \tilde{\psi} = \sqrt{t} \psi. \tag{2.25}$$

The contribution from the fixed point we wish to compute is then

$$\lim_{t \to \infty} \int \alpha \left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}, \phi \right) e^{-t \delta W \left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}, \phi \right)} d^{2n} \tilde{x} d^{2n} \tilde{\psi}, \tag{2.26}$$

using that the measure on ΠTX is invariant under scalings, since the odd coordinates scale inversely compared to the bosonic ones.

Using the local expression of δW, (2.24), we see that in these coordinates the exponential is quadratic and thus independent of t. The leading top-form contribution (i.e. the only top-form independent of t), will come from the exponential, since all the higher form components of α will come with factors of $1/\sqrt{t}$. This means that only the 0-form part of α, evaluated at the fixed point x_p, will enter at leading order in t, and we find that integral is given by

$$\int \alpha_0(\phi)|_{x_p} e^{-\phi \sum_{i=1}^n \nu_i^p \psi_i^x \psi_i^y - \frac{\phi^2}{2} \sum_{i=1}^n ((\nu_i^p)^2((x_i)^2+(y_i)^2))} d^{2n}x d^{2n}\psi \ . \tag{2.27}$$

This a Gaussian integral over a diagonalized quadratic form in both the odd and the even variables. Performing the integral gives

$$(2\pi)^n \frac{\alpha_0(\phi)|_{x_p} \prod_{i=1}^n \phi\nu_i^p}{\prod_{i=1}^n \phi^2(\nu_i^p)^2} = \left(\frac{2\pi}{\phi}\right)^n \frac{\alpha_0(\phi)|_{x_p}}{\prod_{i=1}^n \nu_i^p} = \left(\frac{2\pi}{\phi}\right)^n \frac{\alpha_0(\phi)|_{x_p}}{\sqrt{\det v|_{x_p}}}, \tag{2.28}$$

and after summing up contributions from all the fixed points, we find the Berline-Vergne-Atiyah-Bott formula (2.14).

One can be less explicit and avoid using specific local coordinates or particular choice of W, which will give us some important general insights. The term δW expanded around the fixed point will in general have terms of quadratic and higher orders, but in the the large t limit only the quadratic part matter and we write

$$t\delta W = \phi H_{\mu\nu}\tilde{x}^\mu \tilde{x}^\nu + \phi S_{\mu\nu}\tilde{\psi}^\mu \tilde{\psi}^\nu + \mathcal{O}(1/\sqrt{t}). \tag{2.29}$$

Here, $H_{\mu\nu}$ is symmetric and $S_{\mu\nu}$ is anti-symmetric. The "supersymmetry variations", (2.13), of $\tilde{x}, \tilde{\psi}$ are linearized in the large t limit (i.e. we expand the vector field around the fixed point, keeping only lowest order),

$$\delta\tilde{x}^\mu = \tilde{\psi}^\mu, \qquad \delta\tilde{\psi}^\mu = \phi\tilde{x}^\nu \partial_\nu v^\mu|_{x_p}, \qquad \delta\phi = 0. \tag{2.30}$$

Using this linearized supersymmetry, we can compute $\delta^2 W$, and find that the requirement that this is zero gives us the following equation relating S and H:

$$t\delta^2 W = 2\phi H_{\mu\nu}\tilde{\psi}^\mu \tilde{x}^\nu + 2\phi S_{\mu\nu}\tilde{x}^\rho \partial_\rho v^\mu|_{x_p} \tilde{\psi}^\nu = 0, \tag{2.31}$$

which implies (using the antisymmetry of S)

$$H_{\mu\nu} = S_{\mu\rho}\partial_\nu v^\rho|_{x_p}. \tag{2.32}$$

The same arguments as before applies to see that only the 0-form part of α contributes, and the integral around the fixed point x_p can be written

$$\int \alpha_0(\phi)|_{x_p} e^{-(\phi^2 H_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu + \phi S_{\mu\nu}\tilde{\psi}^\mu\tilde{\psi}^\nu)} d^{2n}\tilde{x}\, d^{2n}\tilde{\psi}\,. \tag{2.33}$$

This is a Gaussian integral over even and odd variables, and performing it we find the result

$$(2\pi)^n \frac{\alpha_0(\phi)|_{x_p}\mathrm{Pf}(\phi S)}{\sqrt{\det(\phi^2 H)}}. \tag{2.34}$$

Using (2.32) we can simplify this as

$$(2\pi)^n \frac{\alpha_0(\phi)|_{x_p}\mathrm{Pf}(S)}{\sqrt{\det(\phi S\cdot(\partial v)|_{x_P})}} = \left(\frac{2\pi}{\phi}\right)^n \frac{\alpha_0(\phi)|_{x_p}}{\sqrt{\det((\partial v)|_{x_P})}}, \tag{2.35}$$

which is the same as the previous result, but where the linearized action of v is written as a determinant of the matrix of first derivatives of v, $\partial_\mu v_\nu$. One can easily confirm that in the local coordinates (2.20) this gives the same result as before. The important lesson here is that the supersymmetry relates the bosonic and fermionic part of the Gaussian integral, so that a large part of the resulting determinants cancel and we are left with only the determinant of the linearized part of v. Since we here did not use the particular form of the localization term W, we see that no matter which term we pick, we will always find the same determinant, namely the determinant of δ^2.

2.2.1 Generalizations

The above is a valid proof for the case of a single $U(1)$ action with isolated fixed points on a compact manifold. There are many generalizations of this, and here we discuss some that are relevant for physics. For example, it is often the case (as we will see in the example of section 2.4) that we do not have isolated fixed points, but instead some continuous submanifold F that is fixed by the group action, called the localization locus. This usually happens when applying the localization formulas in the QFT setting. In this case instead of a sum over the discrete fixed points, we get an integral over the locus F. For an equivariantly closed form α on a manifold X with a group action, and for ϕ the coordinates on the Lie algebra, we have

$$\int_M \alpha(\phi) = \sum_F \int_F \frac{\alpha(\phi)}{e_N(F)(\phi)}, \tag{2.36}$$

where the sum runs over all connected components of the localization locus, and $e_N(F)$ is the equivariant Euler form of the normal bundle of F inside M. This should be thought of as the generalization of the index of the vector field, but where we only include the directions orthogonal to the locus, hence the normal bundle of F.

The localization formula can also be generalized to a 'more proper' supermanifold setting [17]. What we did above used supermanifolds of the form ΠTX, where the odd coordinates are exactly the differentials of the even coordinates. This is just a different way of describing differential forms on the ordinary manifold X. Instead we can start with a supermanifold $\mathcal{M}$, with n even coordinates x_0^μ and m odd coordinates x_1^i. We then construct the odd tangent bundle of this, $\Pi T\mathcal{M}$, where the tangent space part has n odd coordinates ψ_1^μ and m even coordinates ψ_0^i.

Let there be a $U(1)$ action on $\Pi T\mathcal{M}$, represented by a vector field v, which has odd and even parts and that we can write as

$$v = v_0^\mu \frac{\partial}{\partial x_0^\mu} + v_1^i \frac{\partial}{\partial \psi_1^i} \ , \tag{2.37}$$

where v_1^i are Grassmann odd parameters. The Cartan differential (or supersymmetry) then acts like

$$\begin{aligned} &\delta x_0^\mu = \psi_1^\mu, \quad \delta \psi_1^\mu = \phi v_0^\mu, \\ &\delta x_1^i = \psi_0^i, \quad \delta \psi_0^i = \phi v_1^i, \\ &\delta\phi = 0 \end{aligned} \tag{2.38}$$

where ϕ again is the coordinate on the S^1. Again (x_0, x_1) are the coordinates on our original manifold and we should think of (ψ_1, ψ_0) as their corresponding differentials. Here $\delta^2 = v$, which is an operator with both odd and even parts. In the localization formula the determinant of δ^2 becomes a superdeterminant. If we write v as the sum of its even and odd parts, $v = v_0 + v_1$, the linearized superdeterminant of v at a fixed point x_p is

$$\mathrm{sdet}(v)|_{x_p} = \frac{\det(v_0)|_{x_p}}{\det(v_1)|_{x_p}}. \tag{2.39}$$

The localization formula for this case with isolated fixed points then very closely mimics the usual Atiyah-Bott formula (2.14), and reads

$$\int_{\mathcal{M}} \alpha(\phi) = \left(\frac{2\pi}{\phi}\right)^{n-m} \sum_{p} \frac{\alpha_0(\phi)|_{x_p}}{\sqrt{\mathrm{sdet}(v)|_{x_p}}}. \tag{2.40}$$

This setting is the most relevant for supersymmetric field theories, where we need to formally generalize it to the infinite-dimensional setting. We discuss this in the next section.

2.3 Localization for quantum field theories

In quantum field theory, the quantities that we want to compute can be formulated in terms of a path integral, which is an integral over the space of field configurations. This space is infinite-dimensional and making proper sense of the path integral is a long-standing and very difficult mathematical problem, worth a million dollars to whomever solves it [1]. However in specific cases when our quantum field theories are of a particular form, we can formally use the localization formulas discussed above to define the path integral. For a modern introduction and overview of this field, see the recent collection of review articles [18].

In the QFT setting one quantity we might want to compute is the partition function of a field theory placed on a compact manifold M. Then the equivariant form α is written as $e^{iS[\phi]}$, where S is the action of the theory and ϕ denotes the collection of all fields. The space we are integrating over is now the space of field configurations $\mathcal{X}$, which is an infinite-dimensional supermanifold, where the even coordinates describe the bosonic degrees of freedom and the odd describe the fermionic degrees of freedom. The group action on this manifold is generally a combination of a gauge transformation and an isometry of the space-time manifold. The partition function is given by the path integral

$$Z = \int_{\mathcal{X}} \mathcal{D}\phi\, e^{iS[\phi]}. \tag{2.41}$$

Looking at the description of the equivariant differential in (2.13), we see that having an equivariant differential on this space means to have an operator δ that maps even coordinates (bosons) to odd coordinates (fermions) and vice versa. In physics terminology, this is called a supersymmetry, and theories whose action is invariant under this, $\delta S[\phi] = 0$, are called supersymmetric. We will explain supersymmetry a bit more in the next chapter.

The partition function is not the only possible observable we can compute; localization can also be used to compute other supersymmetric (or BPS) observables. Certain Wilson loops can be computed, and certain correlators between operators inserted at the fixed points of the torus action, but unfortunately localization is not a magic bullet that lets us compute anything we want even for the theories where we can apply it.

The existence of the supersymmetry δ means that the space of field configurations has the structure of some odd tangent bundle of a supermanifold $\mathcal{M}$, i.e. $\mathcal{X} = \Pi T\mathcal{M}$, with δ acting as the Cartan differential, mapping from the coordinates on $\mathcal{M}$ to their differentials. If further the various properties required for the localization formula are fulfilled, we can formally apply the formula (2.40) or some suitable generalization, and take this as a definition of what the path integral means. In such cases we can write down a localization term W and study the locus where $\delta W = 0$. This localization locus will often be finite-dimensional, so this lets us reduce the path integral to some well-defined finite dimensional integral. Typically one ends up with an integral over a space of matrices, which is referred to as a matrix model.

In the infinite-dimensional setting there are a number of subtleties that do not appear in finite dimensional cases. For example the operator δ^2 is now a differential operator with an infinite number of eigenmodes and eigenvalues, and its determinant needs to be regularized somehow. Another issue is that of zero modes, something that appear in many places across physics and mathematics. In general a zero mode is a solution to an eigenvalue problem with zero eigenvalue. That is to contrast with the "normal" modes, which are modes with non-zero eigenvalue under the same operator. In localization, the modes we care about are those of δ^2. Any field configuration can be written as some combination of all its modes, both normal and zero modes, and when performing the computation the zero modes of δ^2 have to be treated systematically, otherwise the determinant will typically be identically zero. Pestun did this for gauge theories on S^4 [19], by adding constant fields for each field with zero modes to the gauge fixing complex. In the example of section 2.4 , we will see another infinite-dimensional example where both regularizing a functional determinant and dealing with the zero modes are important.

2.3.1 Historical overview

The idea of applying the localization formulas to physics has been around for a long time and has a successful history in theoretical physics. Here, I will briefly review some of the important results of this line of investigation, going roughly in historical order. Of course, this is not a small subject and the review here cannot even attempt being complete. The idea is to give some historical context so that the reader can understand where the work of this thesis fits into the development of the field.

The idea of applying localization formulas to physical systems originated with Witten, who studied supersymmetric quantum mechanics and connected it with Morse theory [16], where he used a version of a localization formula. A bit later the idea was applied to the so called topological quantum field theories, which is an interesting class of field theories which do not have local degrees of freedom but only depend on the topology of the background manifold. These can be studied using localization techniques, that reduces the path integral to something much simpler, like a sum over indices of vector fields [20, 21].

An important example of these is the 4d $\mathcal{N} = 2$ theories, which can be turned into a topological field theory by a procedure known as topological twisting. We will very briefly outline this procedure in chapter 3. These theories were studied by Nekrasov [22, 23], who showed that once supersymmetric localization is applied, it reduces the path integral to an integral over the instanton moduli space. This integral can further by performed using localization techniques and the answer takes an interesting form as a sum over 2d partitions. This computation reproduced from first principles the Seiberg-Witten solution [24] of the $\mathcal{N} = 2$ 4d Yang-Mills theory, a very impressive feat. This partition function for instantons is known as the Nekrasov instanton function, and it makes appearances in many places when performing localization computations, as we will see in chapter 6 and in the articles I,II and IV.

Before 2007, localization calculations had been limited to the cases of topological field theories or supersymmetric quantum mechanics. This changed when Pestun [19] constructed non-topological $\mathcal{N} = 2$ supersymmetric gauge theory on the four-sphere, and used localization to compute a closed form expression for the partition function. He also computed the expectation values of certain supersymmetric circular Wilson loops, proving an older conjecture inspired by AdS/CFT that expectation values of circular supersymmetric Wilson loops in 4d $\mathcal{N} = 4$ theory are given by Gaussian matrix models [25]. Pestun started from $\mathcal{N} = 1$ gauge theory in 10d, and using dimensional reduction and a conformal mapping from $\mathbb{R}^4$ to S^4, he constructed the physical $\mathcal{N} = 2$ theory on the four-sphere. Then, applying the supersymmetric localization principle, he could reduce the infinite dimensional path integral to a finite dimensional integral, i.e. a matrix model. This provided the first explicit example of a non-topological supersymmetric theory placed on a curved background; and from this a new direction of research into supersymmetric field theories was spawned. Others repeated similar computations for a variety of theories in different dimensions, placed on different curved backgrounds. It is not possible or meaningful to try and list every such computation, and for a comprehensive introduction and review of the state of the art in the field we refer to the review volume [18]. The work presented in this thesis is a natural continuation of this research program, where we use the localization techniques to study 5d $\mathcal{N} = 1$ supersymmetric Yang-Mills theories placed on any toric Sasaki-Einstein manifold. In article IV we study 4d $\mathcal{N} = 2$ theories placed on a large class of 4d manifolds.

2.4 Localization of supersymmetric quantum mechanics

As an example of supersymmetric localization of an infinite-dimensional path integral, we will study the theory of supersymmetric quantum mechanics. This is simpler than the full QFT setting, but still demonstrates most of the subtleties of localization in an infinite dimensional setting. The calculation turns out to be closely related to the famous Atiyah-Singer index theorem [26].

The original idea to use equivariant localization for supesymmetric quantum mechanics was introduced by Witten in [16], where he used localization-type arguments to relate supersymmetric quantum mechanics with Morse theory, and suggested various possible generalizations. This was followed up by a number of people including Atiyah [27] and Jaffe et. al. [28]. Here we will follow the approach of Hietamäki, Morozov, Niemi and Palo [29], who used supersymmetric quantum mechanics to prove an instance of the Atiyah-Singer index theorem.

The model we will deal with is called $\mathcal{N} = 1/2$ supersymmetric quantum mechanics, and it describes a spinning particle moving on a curved manifold. Let (M, g) be a compact Riemannian manifold of dimension $2n$. Our model describes a spinning particle moving along some path in this manifold, described by $x : \mathbb{R} \to M$. We also have a $U(1)$ principal bundle P over M,[1] with some connection A, or in physics terms a background $U(1)$ gauge field, that we couple the particle to (see section 3.1 for a short explanation of these terms). Let $F = dA$ be the curvature of the connection (or the field strength). Then the action of this model takes the form

$$S[x,\psi] = \int_\gamma dt \left(\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\mu A_\mu - \frac{1}{2} \psi^\mu F_{\mu\nu} \psi^\nu \right. \\ \left. + \frac{1}{2} \psi^\mu \left(g_{\mu\nu} \partial_t + \dot{x}^\rho g_{\mu\sigma} \Gamma^\sigma_{\rho\nu} \right) \psi^\nu \right), \tag{2.42}$$

where ψ^μ are odd (Grassmann) coordinates dual to the even coordinates x^μ, and t is the time coordinate. Here all the background fields are evaluated at $x(t)$, i.e. when we write $g_{\mu\nu}$ it means $g_{\mu\nu}(x(t))$, and $A_\mu = A_\mu(x(t))$ and so on, which is important to keep in mind. The dot indicates time derivative as usual, and $\Gamma^\sigma_{\rho\nu}$ are the Christoffel symbols; so the third term can also be written as $\psi^\mu \nabla_t \psi_\mu$, where ∇ is the covariant derivative. This action is invariant under a supersymmetry δ that acts on the coordinates as

$$\delta x^\mu = \psi^\mu, \qquad \delta\psi^\mu = \dot{x}^\mu, \tag{2.43}$$

and we see that $\delta^2 = \partial_t$, the generator of time translation.

If we want to compute the partition function of this theory through the path integral, we need to regulate the allowed paths in some way, since otherwise the action diverges with the path length. We do this by taking time to be periodic, or in other words we take $x : S^1 \to M$. This is equivalent to requiring that the particle only travel around closed

[1] We could consider a general gauge group, but take $U(1)$ for simplicity.

loops on M. The space of all maps $S^1 \to M$, is called the loop space of M and is denoted LM. This is an infinite dimensional space. The action is now the integral over the circle.

The even coordinates x should now be thought of as loops on M, i.e. $x \in LM$, while the odd coordinates are anti-commuting loops on M, which should be thought of as the differential of x, $\psi \sim dx$. Thus the relevant space is the odd tangent bundle of the loop space, ΠTLM, which at least formally have a canonical integration measure. The path integral expression for the partition function is

$$Z_{\mathcal{N}=1/2} = \int_{\Pi TLM} \mathcal{D}x\ \mathcal{D}\psi\, e^{iS[x,\psi]}\ , \tag{2.44}$$

and to make sense of this path integral we will use the localization formulas.

First, let us see that the supersymmetry (2.43) is an equivariant differential on the space ΠTLM. For any functional $F[x]$ of closed paths in LM (for example the action), we define the functional derivative by the rule

$$\frac{\delta}{\delta x[t]} F[x[t']] = \delta(t-t')F'[x(t')], \tag{2.45}$$

and for the Grassmann valued paths in ΠTLM the functional differentiation is given by the anti-commutator

$$\left\{ \frac{\delta}{\delta \psi^\mu(t)}, \psi^\nu(t') \right\} = \delta^\nu_\mu \delta(t-t'). \tag{2.46}$$

Then the functional exterior derivative on ΠTLM is given by the operator

$$D = \int_{S^1} dt\ \psi^\mu(t) \frac{\delta}{\delta x^\mu(t)}, \tag{2.47}$$

which, thinking of ψ^μ's as "1-forms" takes us up one step in form degree, just like the usual exterior derivative. We can see that this is exactly captured by the transformation of coordinates $\delta x^\mu = \psi^\mu$. To define the equivariant differential, we also need the contraction operator, which is defined as

$$\iota_{\dot{x}} = \int_{S^1} dt\ \dot{x}^\mu \frac{\delta}{\delta \psi^\mu(t)}, \tag{2.48}$$

which "eats" a 1-form ψ^μ along the direction $\dot{x}^\mu$, and replaces it with $\dot{x}^\mu$. The equivariant exterior derivative is then

$$D_{\dot{x}} = D + \iota_{\dot{x}}, \tag{2.49}$$

and it is not hard to see that this is exactly encoded in the supersymmetry of (2.43). The square of $D_{\dot{x}}$ is the generator of time translations on ΠTLM,

$$D_{\dot{x}}^2 = \int_{S^1} dt \frac{d}{dt}, \tag{2.50}$$

so on a functional $F[x,\psi]$ it acts as

$$D_{\dot{x}}^2 W[x,\psi] = \int_{S^1} dt\ \frac{d}{dt} W[x,\psi]. \tag{2.51}$$

We want to consider the functionals that are invariant under this. From this discussion we see how this supersymmetric model on the loop space has the properties of the equivariant differential as described in section 2.1.

The action (2.42) is actually $D_{\dot{x}}$-exact, $S[x,\psi] = D_{\dot{x}}\Sigma[x,\psi]$, where $\Sigma[x,\psi]$ is a functional given by

$$\Sigma[x,\psi] = \int_{S^1} dt\ \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\nu(t) + A_\mu\right]\psi^\mu(t). \tag{2.52}$$

Σ is invariant under $D_{\dot{x}}^2$. Therefore, in the path integral (2.44) we can introduce a parameter T in front of the action, and the integral will be not depend on it[2]. Thus we can use the action itself as a localizing term. From looking at the first quadratic term in the action (2.42), $\frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, we can see that if we let $T \to \infty$, the path integral manifestly localizes onto $\dot{x}^\mu(t) = 0$. Those are the constant paths, or stationary particles, $x(t) = x(0) \in M \quad \forall t$, and the space of constant paths is exactly the original manifold M. Thus, after localizing we end up with a finite dimensional integral over the locus, i.e. over M.

We should next consider the fluctuations around the fixed points, i.e. around the constant loops. To do this we introduce the based loops on ΠTLM

$$x(t) = x_0 + \hat{x}(t), \qquad \psi(t) = \psi_0 + \hat{\psi}(t), \tag{2.53}$$

where (x_0, ψ_0) are the constant modes (or zero modes, as they are killed by ∂_t) and $\hat{x}, \hat{\psi}$ the non-constant fluctuations around them. The path integral measure decomposes into a finite-dimensional integral over the constant modes, and another infinite-dimensional integral over the fluctuations,

$$\mathcal{D}x\, \mathcal{D}\psi = d^{2n}x_0\ d^{2n}\psi_0\ \mathcal{D}\hat{x}\ \mathcal{D}\hat{\psi}\,. \tag{2.54}$$

[2]Note that we cannot however take $T = 0$, since it renders the integral undefined.

Mimicking the procedure in the proof in the finite dimensional case in section 2.2, we rescale the non-constant modes with $\sqrt{T}$,

$$(\hat{x}, \hat{\psi}) \to \left(\frac{\hat{x}}{\sqrt{T}}, \frac{\hat{\psi}}{\sqrt{T}} \right). \tag{2.55}$$

The measure again stays invariant under such a scaling, since the even and odd coordinates transform with opposite weight. Expanding the action to leading order in T gives

$$\begin{aligned} TS[x, \psi] = \int_{S^1} dt \, \Big(&\frac{1}{2} g_{\mu\nu} \dot{\hat{x}}^\mu \dot{\hat{x}}^\nu + \frac{1}{2} \hat{\psi}^i(t) \eta_{ij} \partial_t \hat{\psi}^j(t) - \frac{1}{2} \psi_0^\mu F_{\mu\nu}(x_0) \psi_0^\nu \\ &+ \frac{1}{2} R_{ij\mu\nu}(x_0) \psi_0^i \psi_0^j \dot{\hat{x}}^\mu \dot{\hat{x}}^\nu \Big) + \mathcal{O}(1/\sqrt{T}), \end{aligned} \tag{2.56}$$

where we have Taylor expanded the quantities around (x_0, ψ_0), and also used the inverse vielbein e^i_μ to write $\hat{\psi}^i = \hat{\psi}^\mu e^i_\mu(x_0)$. Here η_{ij} is the flat space metric: we choose local coordinates around x_0 and perform the Taylor expantion of the metric around it; this is also why $R_{ij\mu\nu}$, the Riemann tensor on M, appear. We interpret ψ^i_0 as the usual differential form dx^i at x_0, and write $\mathcal{R}_{\mu\nu}$ for the Riemann two-forms

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} R_{ij\mu\nu}(x_0) \psi_0^i \psi_0^j, \tag{2.57}$$

which is what appears in the expanded action.

Note that (by construction) in the $T \to \infty$ limit, the path integral over the fluctuating modes $\hat{x}, \hat{\psi}$ is Gaussian and on evaluation it will give us determinants. Performing the infinite dimensional Gaussian integral formally gives us the result

$$\int_{\Pi TM} d^n x_0 \, d^n \psi_0 \, e^{-\frac{1}{2} F_{\mu\nu}(x_0) \psi_0^\mu \psi_0^\nu} \frac{\mathrm{Pf}'(\partial_t)}{\sqrt{\det'(\delta^\nu_\mu \partial_t^2 + \mathcal{R}^\nu_\mu \partial_t)}}. \tag{2.58}$$

The integral is now over a finite-dimensional space, but the determinant factors here are over the differential operator ∂_t, and needs to be treated carefully. Looking at the determinant factors, the numerator comes from the fermionic integral over $\hat{\psi}$ and the denominator from integrating over $\hat{x}$. The $'$ denotes that we are excluding the zero modes, since these are dealt with separetly. We can see that there is partial cancellation between numerator and denominator, leaving us with the determinant factor

$$\frac{\mathrm{Pf}'(\partial_t)}{\sqrt{\det'(\delta^\nu_\mu \partial_t^2 + \mathcal{R}^\nu_\mu \partial_t)}} = \frac{1}{\sqrt{\det'(\delta^\nu_\mu \partial_t + \mathcal{R}^\nu_\mu)}}. \tag{2.59}$$

The question now is how to deal with the determinant of the differential operator ∂_t. The eigenfunctions of ∂_t with non-zero eigenvalue on the unit circle are $e^{2\pi ikt}$ for k a non-zero integer, which has the eigenvalue $2\pi ik$. $\mathcal{R}_{\mu\nu}$ is just an ordinary anti-symmetric matrix of 2-forms on M, so it can be skew-diagonalized into n 2×2 skew-diagonal blocks $\mathcal{R}^{(j)}$ with eigenvalues $\pm\lambda^j$, $j = 1, \ldots, n$. For each such block the determinant gets the contribution

$$\det'(\partial_t + \mathcal{R}^{(j)}) = \prod_{k\neq 0} (2\pi ik + \lambda_j)(2\pi ik - \lambda_j), \tag{2.60}$$

where the infinite product over k needs to be regularized. Let us consider half of the above, i.e. we consider only the eigenvalue $+\lambda_j$. The infinite product over k can be written

$$\begin{aligned} \prod_{k=1}^{\infty} (2\pi ik + \lambda_j)(-2\pi ik + \lambda_j) &= \prod_{k=1}^{\infty} (2\pi i)^2 (k + \frac{\lambda_j}{2\pi i})(k - \frac{\lambda_j}{2\pi i}) \\ &= \left[\prod_{k=1}^{\infty} (2\pi i)^2\right] \prod_{k=1}^{\infty} (k + \frac{\lambda_j}{2\pi i})(k - \frac{\lambda_j}{2\pi i}) \end{aligned} \tag{2.61}$$

Both these infinite divergent products can be assigned values using zeta-function regularization. Let us start with the second product. We can further rewrite it as

$$\prod_{k=1}^{\infty} (k^2 - \frac{\lambda_j^2}{(2\pi i)^2}) = \prod_{k=1}^{\infty} k^2 (1 - \frac{\lambda_j^2}{(2\pi ik)^2}) = \left[\prod_{k=1}^{\infty} k^2\right] \frac{\sinh \frac{\lambda_j}{2}}{\lambda_j/2}, \tag{2.62}$$

where we use the famous infinite product formula for sin due to Euler,

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right), \tag{2.63}$$

but where the factor of i turned it into the hyperbolic sine function.

We need to deal with the two infinite pre-factors, which we do through zeta regularization in the following way

$$\prod_{k=1}^{\infty} k^2 = \exp\left(\sum_{k=1}^{\infty} 2\log k\right) = \exp\left(2 \sum_{k=1}^{\infty} \left.\frac{\log k}{k^s}\right|_{s=0}\right) = \exp(-2\zeta'(0)) = 2\pi,$$

where ζ is the Riemann zeta function,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \tag{2.64}$$

for $s > 0$, which is analytically continued to $s = 0$. The derivative of this at zero is $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, which gives the final result in (2.64).

Similarly for the other infinite pre-factor, we find

$$\prod_{k=1}^{\infty}(2\pi i)^2 = (2\pi i)^{2(\sum_{k=1}^{\infty}\frac{1}{k^s})|_{s=0}} = (2\pi i)^{2\zeta(0)} = (2\pi i)^{-1}. \tag{2.65}$$

So in total, the regularized determinant is found to be

$$\det{}'(\delta_\mu^\nu \partial_t + \mathcal{R}_\mu^\nu) = (-i)^n \det\left[\frac{\sinh\frac{1}{2}\mathcal{R}}{\frac{1}{2}\mathcal{R}}\right] \tag{2.66}$$

where the determinant is of the Riemann curvature tensor. Disregarding the irrelevant overall constant, we have thus found the following expression for the partition function:

$$Z = \int dx_0\, d\psi_0\, e^{-\frac{i}{2}F_{\mu\nu}(x_0)\psi_0^\mu\psi_0^\nu}\sqrt{\det\frac{\frac{1}{2}\mathcal{R}}{\sinh\frac{1}{2}\mathcal{R}}}. \tag{2.67}$$

Interpreting ψ_0^μ as a basis of 1-forms on M, we can recognize that the factor involving F is the Chern character $\mathrm{ch}(F)$ of the $U(1)$ bundle, and the determinant factor is the $\hat{A}$-genus with respect to the curvature R, and the entire thing can be written as

$$Z = \int_M \mathrm{ch}(F) \wedge \hat{A}(R). \tag{2.68}$$

So the partition function is in the end given by the integral over these characteristic classes of P.

In fact, the partition function we are computing is the index of the Dirac operator on M coupled to the gauge field A, and the result that this can be written as this integral over characteristic classes is an instance of the Atiyah-Singer index theorem. The full explanation of this is outside the scope of this chapter; and the interested reader can find it in [29] and references therein.

3. Supersymmetric gauge theories

In this chapter we introduce the supersymmetric theories that this thesis study. We begin with a short general introduction to gauge theories, supersymmetry and their combination, supersymmetric gauge theories. Then we give a brief introduction to the 6d (2,0) theories, which, although we do not study them directly, provide some motivation and a unifying perspective of a lot of the supersymmetric field theories and their relations. These 6d theories are closely related to the maximally supersymmetric Yang-Mills theories in 5d, and we next introduce supersymmetric 5d gauge theories, the topic of articles I and II. Finally we briefly outline 4d $\mathcal{N} = 2$ theories, which is the subject of article IV.

3.1 Gauge theories

The canonical example of a gauge theory is given by Maxwells electrodynamics in four dimensions. In the classic formulation, this is the theory of the electric and magnetic fields, described by the Maxwell equations. In more modern language, the theory is formulated in terms of a gauge field A_μ, sometimes called the four-potential, since it is a four-vector and acts as a potential from which the electric and magnetic fields can be derived. The electric and magnetic fields are the components of the field strength tensor $F_{\mu\nu}$, which is given in terms of A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.1}$$

In the language of differential forms, $A = A_\mu dx^\mu$ is a 1-form on spacetime, and F is the exterior derivative of A, $F = dA$. The classical theory depends only on F and not on A itself, and F is invariant under the change of A by an exact form, $A \mapsto A + df$ for any function f. We call this shift a gauge transformation of A.

The Lagrangian that describes Maxwell electrodynamics is

$$\mathcal{L}_{\mathrm{EM}}[A, J] = -\frac{1}{4g^2} F \wedge \star F - A \wedge \star J. \tag{3.2}$$

where g is the coupling constant and $J = J_\mu dx^\mu$ is a four-current that should be understood as an abbreviation for potentially many terms describing other fields that carry electric charge. J is not a fundamental field itself. This Lagrangian plus the fact that $d^2 = 0$ gives us the Maxwell equations

$$\begin{aligned} d \star F &= g^2 \star J, \\ dF &= 0. \end{aligned} \tag{3.3}$$

In more mathematical terms, this gauge theory is constructed in terms of a principal $U(1)$ bundle over the space-time manifold M. An excellent introduction to this topic is the book [30]. The gauge field is the connection on the principal bundle.

A connection on a principal G-bundle is a gadget that define a notion of parallel transport on the bundle. It is represented by a 1-form on the total space P taking values in the Lie algebra $\mathfrak{g}$ of G and gives us a way of "connecting" the fibers over nearby points on the base manifold. It can also locally (but potentially not globally) be described by a one-form on the base, also taking values in the Lie algebra.

Let us first describe this in the example of a $U(1)$ bundle. Then $\mathfrak{g} \simeq \mathbb{R}$, so the connection is described locally by a real one form $A_\mu \in \Omega^1(M)$. Given this A_μ we define the associated covariant derivative as

$$D_\mu = \partial_\mu - iA_\mu. \tag{3.4}$$

Let $\gamma : [0, 1] \to M$ be a smooth path on M and s be a section of the principal bundle. We say that s is parallel if

$$D_{\dot{\gamma}(t)} s = \dot{\gamma}^\mu(t) D_\mu s = 0, \qquad t \in [0, 1], \tag{3.5}$$

where $\dot{\gamma}^\mu$ is the vector field along the direction of the path. This defines parallel transport and when elements of fibers over nearby points are to be considered 'the same', and thus how the different fibers are glued together. See figure 3.1 for a graphical representation of this.

The field strength $F = dA$ is called the curvature of the connection; and it can be thought of as measuring the "infinitesimal holonomy" of the connection, or in other words how much an element of a fiber is changed by being parallel-transported along an infinitesimally small loop.

So the Maxwell theory is a theory of connections on principal $U(1)$ bundles, possibly coupled to other fields living in associated vector bun-

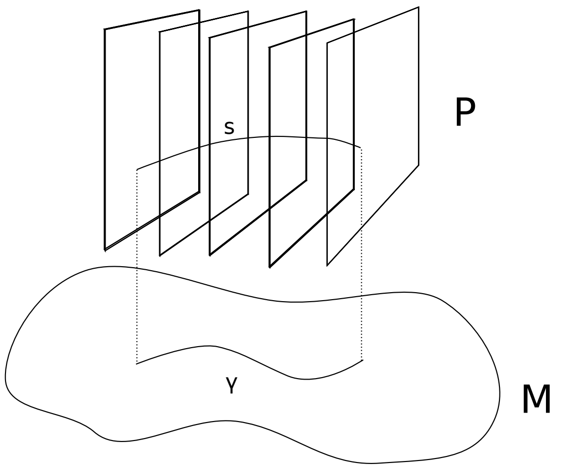

Figure 3.1. A sketch illustrating a principal bundle $P \to M$ and the notion of a section s over a path $\gamma(t)$ in M.

dles. This was generalized to other, nonabelian gauge groups by Yang and Mills [31], and this general type of theory is called Yang-Mills theory (abbreviated YM). Then one considers a G principal bundle over M, with a connection/gauge field A, locally described by a 1-form valued in the Lie algebra $\mathfrak{g}$. Let T^a be a basis of $\mathfrak{g}$, and

$$[T^a, T^b] = i f^{abc} T^c, \quad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{3.6}$$

where f^{abc} are the structure constants of $\mathfrak{g}$. We can then write the gauge field explicitly as $A = A^a_\mu T^a dx^\mu$. The field strength is given by

$$F = dA - i[A, A] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{kla} A^k_\mu A^l_\nu) T^a dx^\mu \wedge dx^\nu, \tag{3.7}$$

and the covariant exterior derivative now acts by $d_A = d - i[A, \cdot]$. We have $F = d_A A$, which is the appropriate way to define the curvature of a connection on a principal bundle for non-abelian gauge groups. F is again invariant under gauge transformations of A, which acts as $A \mapsto A + d_A\sigma$, where σ is a function taking values in $\mathfrak{g}$.

The appropriate generalization of the action is

$$\mathcal{L}_{\mathrm{YM}}[A, J] = -\mathrm{Tr}\left[\frac{1}{2g^2} F \wedge \star F + 2A \wedge \star J\right], \tag{3.8}$$

where the factor of 2 comes from the convention for the trace in (3.6). For the gauge group $SU(3)$ this describes the strong force and with the appropriate choice of matter fields it becomes quantum chromodynamics (QCD).

Yang-Mills theory can be defined in any dimension, but for the quantized version, it is only renormalizable for dimensions 4 and lower. This can be understood by looking at the dimension of the coupling constant g^2. In 4 dimensions g^2 is dimensionless, while in higher dimensions it acquires the dimension of length (or inverse mass). For example in 5d, $[g^2] = [r]$, implying that the theory is non-renormalizable by simple power counting. Nevertheless, the theories can be formulated and studied in higher dimensions, where we can treat them as effective theories.

3.2 Supersymmetry

Next we give a brief introduction to the idea of supersymmetry, before turning to the supersymmetric Yang-Mills theories. Supersymmetry was discovered independently by Golfand and Likhtman [32], Volkov and Akulov [33], and Gervais and Sakita [34], and it is a symmetry that maps bosons into fermions and vice versa. Because of the spin-statistics theorem, this means that it changes the spin of particles by a half. The spin of a particle classifies its representation of the Lorentz part of the Poincare algebra, so this means that supersymmetry interacts non-trivially with the space-time symmetry. This is very different from gauge symmetries and other so called internal symmetries, which do not 'talk to' space-time symmetries at all (their commutators with the generators of the Poincare algebra are trivial). There is a famous no-go theorem of Coleman and Mandula [35], which states that the only way of extending the Poincare algebra is through adding internal symmetries. Supersymmetry was invented as a way around this, and circumvents the result (which implicitly assumed that the additional generators would obey commutation relations) by adding anti-commuting (Grassmann) generators. Later it was shown by Haag, Lopuszanski and Sohnius [36] that this in fact is the most general possible non-trivial extension of the Poincare algebra, which in itself makes supersymmetry a compelling topic to study.

Supersymmetry extends the Poincare algebra by adding some number of anti-commuting generators Q^a, $a = 1, 2, \ldots, \mathcal{N}$, commonly called supercharges. Each Q^a is a spinor in the minimal spin representation that lets you construct a non-trivial extension of the Poincare algebra. So it has several components, and each component is called a supersymmetry. We will take 4d as our main example, in which case each Q^a is a Dirac spinor with 4 components, which we usually represent as a pair of 2-component Weyl spinors, writing $Q^a = (Q^a_\alpha, \bar{Q}^a_{\dot\alpha})$ where $\alpha, \dot\alpha = 1, 2$. So the $\mathcal{N} = 1$ theory in 4d has 4 supersymmetries.

The non-trivial parts of the algebra between the supercharges and the generators of the usual Poincare algebra is the commutator with the generators of the Lorentz group, which is fixed by requiring Q^a to be a spinor, and the anti-commutators between the supercharges themselves, which in our 4d example is given by

$$\begin{aligned} \{Q^a_\alpha, \bar{Q}^b_{\dot\beta}\} &= 2\delta^{ab}\sigma^\mu_{\alpha\dot\beta}P_\mu, \\ \{Q^a_\alpha, Q^b_\beta\} &= \{\bar{Q}^a_{\dot\alpha}, \bar{Q}^b_{\dot\beta}\} = 0, \end{aligned} \tag{3.9}$$

where $\sigma^\mu = (1, \vec{\tau})$ is a vector of 2×2 matrices, and $\vec{\tau}$ is the three Pauli matrices.[1] σ^μ is the 2-component representation of the gamma matrices, which gives the Clifford multiplication of spinors.We see that the supersymmetries square to a translation, showing that supersymmetry and spacetime symmetry is deeply connected. For a nice review of spinors and supersymmetry in various dimensions, we refer the reader to appendix B of the second volume of String theory by Polchinski [37], or the book by Wess and Bagger [38].

In ordinary field theory, a particle is defined as an irreducible representation of the Poincare algebra. This only really works for non-interacting theories; when the theory is interacting things are more complicated and it is not easy to describe the Hilbert space. An irreducible representation of the Poincare algebra is classified by two numbers: its spin and its mass.

When we extend the Poincare algebra to the super-Poincare, we again repeat a similar story and find its irreducible representations. These are called supermultiplets, and take the and take the form of a collection of different usual fields. Each supermultiplet necessarily include both bosons and fermions, and since the supersymmetry maps each boson to a fermion, a given supermultiplet will have equally many bosonic and

[1] For the experts we comment that this is for the simple case without central charges, which can be turned on in the second line of (3.9).

fermionic degrees of freedom. A supermultiplet is classified by its mass and its ‘component’ of maximal spin. The most commonly used supermultiplets are the vector multiplet (which includes a gauge field A_μ), the chiral multiplet and the hypermultiplet (which both describe matter fields and include scalars and spinors), and the gravity-multiplet, which includes the graviton and is used for supersymmetric theories of gravity. The exact field content of a given multiplet depends on the number of dimensions and the amount of supersymmetry one is considering, where more supersymmetries makes the supermultiplets bigger.

Each time a supersymmetry generator acts on a field with spin s, it maps it to another field with spin $s + \frac{1}{2}$. We also know that interacting fields with spin 2 have to behave like gravitons (shown by Weinberg [39]), which means that such theories are theories of gravity and thus outside the scope of normal quantum field theory. The supersymmetric theories that include the graviton are called supergravity theories [40]. Restricting ourselves to field theories with no graviton therefore places a limit on how many supersymmetries we can have, since if we have too many there is no way to construct a supermultiplet that only include fields with $s < 2$. In 4 dimensions, the maximal allowed supersymmetry is found to be 16 supersymmetries, corresponding to adding four Q^a's. This is called $\mathcal{N} = 4$, and the corresponding supersymmetric Yang-Mills theory is one of the most well studied theories of modern theoretical physics.

3.2.1 Supersymmetric Yang-Mills

The supersymmetric version of Yang Mills theory was first considered by Brink, Schwarz and Scherk [41] in 1977, and the topic has attracted a lot of attention ever since. A supersymmetric Yang-Mills (SYM) theory (SYM) is a supersymmetric field theory that contains a vector field A_μ, and whose action contains the usual YM term, $\mathrm{Tr}(F \wedge \star F)$. As mentioned, the vector field is part of the vector multiplet, which always will contain at least a fermion partner called the gaugino and possibly additional scalar fields. In addition to this one can add chiral or hypermultiplets that describe matter fields.

The more supersymmetries you have, the larger every supermultiplet becomes and the more complicated the actions look. However having more and more supersymmetry also constrains the theory more and more, so it is actually easier to study theories with more supersymmetry. Especially the maximally supersymmetric theories, like 4d $\mathcal{N} = 4$, are typically excellent models to study.

However here we instead give the least supersymmetric 4d SYM, the $\mathcal{N} = 1$ theory, as an example since it does not contain very many fields and has a simple action. The vector multiplet of this theory has the field content of a (real) gauge field A, and one complex 2-component (Weyl) spinor λ, the gaugino. A and λ are both valued in the Lie algebra $\mathfrak{g}$ of the gauge group; one often says that they take values in the adjoint representation. The supersymmetry variations on these fields is given by

$$\begin{aligned} \delta_\epsilon A_\mu &= i\epsilon^\dagger \bar{\sigma}_\mu \lambda - i\lambda^\dagger \bar{\sigma}_\mu \epsilon, \\ \delta_\epsilon \lambda &= F_{\mu\nu}\sigma^{\mu\nu}\epsilon, \end{aligned} \tag{3.10}$$

where F is the field strength of A and ϵ is a constant, complex Weyl spinor that acts as the parameter for the transformation. We see that δ maps us from bosons to fermions and vice versa, and the analogy with the Cartan model of equivariant cohomology of section 2.1.1 should be clear.

The corresponding Lagrangian that includes the Yang-Mills term $F \wedge \star F$ and some term for λ, such that the action is supersymmetric, i.e. $\delta_\epsilon S[A, \lambda] = 0$, is

$$\mathcal{L}_{\mathrm{SYM}}[A, \lambda] = -\frac{1}{2g^2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu} - i\lambda^\dagger \bar{\sigma}^\mu D_\mu \lambda\right], \tag{3.11}$$

where $D_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$ is the covariant derivative. In sections 3.4 and 3.5 we give two other examples of more complicated SYM theories, which have more supersymmetries, so their vector multiplet includes more fields. We next turn to one of the most interesting examples of supersymmetric theories, which however is not of SYM type.

3.3 The 6d (2,0) theory

The mysterious $\mathcal{N} = (2, 0)$ theories are one of the main motivations behind the investigations of this thesis. The notation $(2, 0)$ means that the theory has 2 supercharges $Q_{1,2}$, that are 6d Weyl spinors with the same chirality. A Weyl spinor in 6d has 4 complex components (or 8 real ones), so the theory has 16 supersymmetries. One can also consider (1,1) theories, and (1,0) theories, but the (2,0) theories are especially interesting. They are superconformal (meaning that they are both conformal and supersymmetric) 6d theories, and first appeared in Nahm's classification of superconformal theories [42], but did not originally attract very much attention. Then in 1996 Witten constructed 6d (2,0) theories with A, D and E gauge groups by putting type IIB string theory on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma_{ADE}$ [7, 8]. Witten also understood that the theory, with $SU(N)$ gauge group should describe the dynamics of $M2$-branes ending on a stack of N M5-branes; in a similar way that $SU(N)$ 4d SYM describes strings ending on a stack of $D3$-branes.

For the abelian theories, their field content is given by a 2-form B, 5 scalars Φ_I, $I = 1, \ldots, 5$, and a spinor Ψ. The 2-form is constrained to have a self-dual field strength,

$$H = dB = \star H, \tag{3.12}$$

a fact that makes the theory quite hard to study.

We do not know how to formulate the theory for non-abelian gauge groups, and it is not known if there exists a Lagrangian formulation of the non-abelian $(2, 0)$ theory. Since the theory is superconformal, it sits at fixed points of the RG-flow. Furthermore, it is believed that these fixed points are isolated, which further forbids any dimensionless parameters, meaning that there is no free parameters at all. Thus, there is no perturbative regime where one can study the theory, so most of the usual techniques do not apply. This makes studying the theory very hard.

By compactifying the (2,0) theories in various ways, one can construct a large number of interesting lower-dimensional supersymmetric QFTs, and also discover and understand dualities between them. Perhaps the most well known is the reduction over T^2, which gives $\mathcal{N} = 4$ SYM in 4d, which again is a superconformal theory. The invariance of the superconformal $(2, 0)$ theory under interchanging the two radii of T^2 is then a natural and geometric explanation of the S-duality of 4d $\mathcal{N} = 4$. The famous AGT correspondence [43] between 4d and 2d theories can also be understood from compactifying the 6d theories on (punctured) Riemann surfaces [44], and the newer 3d/3d duality [45, 46] can be understood in

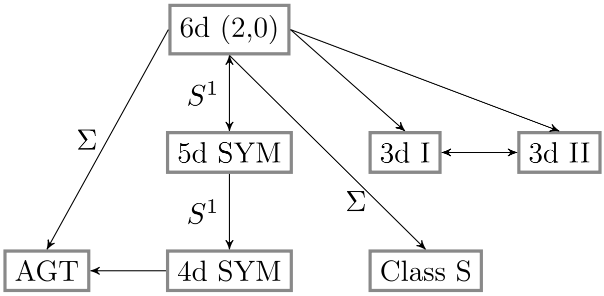

Figure 3.2. A sketch illustrating relationships between different theories.

a similar fashion by compactifying to three dimensions in two different ways. See figure 3.2 for a sketch of these various relations.

On dimensional reduction on a circle, one instead reaches the maximally supersymmetric 5d $\mathcal{N} = 2$ SYM theory. It has been conjectured [47] that actually no information is lost in this particular reduction, the idea being that the instanton contributions in 5d captures all of the information of the 6d theory. This is argued by comparing the spectrum of instantons in 5d MSYM with the Kaluza-Klein spectrum of compactified $(2, 0)$, and finding that with the identification

$$\frac{g_5^2}{8\pi^2} = R_6, \tag{3.13}$$

between the 5d YM coupling and the radius of the circle, they match. This matching is done for the case of two slightly separated M5 branes, corresponding to the gauge group $SU(2)$, but it is conjectured to hold in general.

One of the few ways we have of studying the 6d theory is through the $\mathrm{AdS}_7/\mathrm{CFT}_6$ duality [48]. Through this we know a few important things about it, such as that its free energy scale as N^3 [49]. This has also been reproduced by localization calculations with the theory placed on the sphere S^5 [50], which provides further evidence of the conjecture that the 5d SYM indeed captures the entirety of the $(2, 0)$ theory.

3.4 5d Super Yang-Mills

In 5d the maximally supersymmetric YM theories is the $\mathcal{N} = 2$ SYM, and as mentioned above this is believed to be equivalent to the 6d (2,0) theories. Therefore, the 5d SYM theories give us a tool with which we can study the 6d (2,0) theories. The field content of $\mathcal{N}=2$ theories can be described as a $\mathcal{N}=1$ vector multiplet plus a massless hypermultiplet in the adjoint representation, so here we describe the field content and supersymmetry of the 5d $\mathcal{N}=1$ theories. The introduction here is brief, considerably more details can be found in for example the review [51].

We should comment here that in 5d, the Yang-Mills coupling becomes dimensionfull, $[g_{5d}^2] = [m^{-1}]$, turning the theory non-renormalizable. Therefore naively, the 5d SYM theory is not a sensible theory to study. We can however make sense of it by thinking of them as low-energy effective theories, that have some UV completions. There is evidence that there exist strongly coupled superconformal $\mathcal{N}=1$ 5d theories [42, 52], that flow to the 5d SYM theories in the IR. So these are the UV completions of the $\mathcal{N}=1$ theories, while the $\mathcal{N}=2$ theories are believed to be completed by the 6d $(2,0)$ theories.

5d $\mathcal{N}=1$ theories have a vector multiplet containing the gauge field, and a hypermultiplet that can describe matter fields. Below, we describe their respective field contents, off shell supersymmetry and supersymmetric actions, for theories on flat space.

3.4.1 Vector multiplet

The vector multiplet in 5d $\mathcal{N}=1$ has the following field content: a gauge boson A_m, a symplectic Majorana spinor λ_I, a real scalar field σ, and auxilliary real scalars D_{IJ}, all in the adjoint representation of the gauge group. The indices I, J are in the fundamental of the $SU(2)_R$ symmetry that this theory enjoys, so $I, J = 1, 2$, and the symplectic Majorana condition is the reality condition one can impose on spinors in 5d, which reads

$$(\lambda_I^\alpha)^* = \epsilon^{IJ} C_{\alpha\beta} \lambda_J^\beta, \tag{3.14}$$

where $C_{\alpha\beta}$ is the charge conjugation matrix, and α, β are the spinor indices of $Spin(5) \simeq Spin(4)$. The auxiliary scalars $D_{IJ} = D_{JI}$ are introduced so that we can write the off-shell supersymmetry transformations that we need for localization.

The supersymmetry variations of the vector multiplet are given by

$$
\begin{aligned}
&\delta A_m = i\xi^I \Gamma_m \lambda_I, \\
&\delta\sigma = i\xi^I \lambda_I, \\
&\delta\lambda_I = -\frac{1}{2} F_{mn}\Gamma^{mn}\xi_I + (D_m\sigma)\Gamma^m \xi_I + D_I{}^J \xi_J, \\
&\delta D_{IJ} = -i(\xi_I \Gamma^m D_m \lambda_J) + [\sigma, \xi_I \lambda_J] + (I \leftrightarrow J).
\end{aligned}
\tag{3.15}
$$

where F is the field strength of A and D_m is the gauge covariant derivative. The constant symplectic Majorana spinor ξ_I parametrizes the supersymmetry. The supersymmetric Yang-Mills Lagrangian invariant under the above supersymmetry is

$$
\begin{aligned}
\mathcal{L}_{\text{vect}} = \frac{1}{g_{YM}^2} \mathrm{Tr}\Big[\frac{1}{2} F \wedge \star F - d_A\sigma \wedge \star d_A \sigma + \Big(-\frac{1}{2} D_{IJ} D^{IJ} \\
+ i\lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I]\Big) \mathrm{Vol}_5\Big] .
\end{aligned}
\tag{3.16}
$$

Here Vol_5 is the volume form. This action is written in 5d flat Euclidean space, since when we later put the theory on curved backgrounds, S^5 being the primary example, this is what we need. Note the 'wrong' sign of the scalar kinetic term $-d_A\sigma \wedge \star d_A\sigma$, which we can understand as follows. The above action can be reached starting from the flat 6d $\mathcal{N} = 1$ super-Yang-Mills theory in the Lorentzian signature, $\mathbb{R}^{5,1}$, and formally reducing along the time direction. This gives a 5d Euclidean theory but with the wrong sign of the kinetic term of σ, since it comes from the time component of a vector in 6d Minkowski space. When one computes the 5d partition function, σ has to be Wick rotated, i.e. we choose a contour where σ is imaginary instead of real. This makes the kinetic term have the correct sign and turns the action positive definite.

3.4.2 Hypermultiplet

The hypermultiplet has the field content of 2 real scalars in a $SU(2)_R$ doublet q_I, a $SU(2)_R$ singlet spinor ψ and doublet of auxiliary scalars F_I. The hypermultiplet fields takes values in a representation R of the gauge group.

The supersymmetry algebra for the hyper coupled to a vector multiplet, takes the form

$$\begin{aligned}
\delta q_I &= -2i\xi_I\psi, \\
\delta\psi &= \Gamma^m\xi_I D_m q^I + i\xi_I\sigma q^I - \hat{\xi}_I F^I, \\
\delta F_I &= 2\hat{\xi}_I(i\Gamma^m D_m\psi + \sigma\psi + \lambda_K q^K).
\end{aligned} \tag{3.17}$$

Here $\hat{\xi}_I$ is a constant spinor satisfying

$$\xi_I\xi^I = \hat{\xi}_I\hat{\xi}^I, \quad \xi_I\hat{\xi}_J = 0, \quad \xi_I\Gamma^m\xi^I + \hat{\xi}_I\Gamma^m\hat{\xi}^I = 0. \tag{3.18}$$

The supersymmetric action for the hypermultiplet takes the form

$$\begin{aligned}
\mathcal{L}_{\text{hyp}} = \text{Tr}_R \big[&D_m q_I D^m q^I - \bar{q}_I\sigma^2 q^I - 2i\bar{\psi}\Gamma^m D_m\psi - 2\bar{\psi}\sigma\psi \\
&- i\bar{q}_I D^{IJ} q_J - 4\bar{\psi}\lambda_I q^I - \bar{F}_I F^I \big]
\end{aligned} \tag{3.19}$$

One can also give mass to the hypermultiplet. The easiest way of doing this is to introduce an auxiliary background vector multiplet $(\tilde{A}, \tilde{\sigma}, \tilde{\lambda}_I, \tilde{D}_{IJ})$, and view the mass terms of the hyper as coming from a non-zero vacuum expectation value of the scalar of this vector multiplet,

$$\langle\tilde{\sigma}\rangle = m, \quad \langle\tilde{A}\rangle = \langle\tilde{D}_{IJ}\rangle = \langle\tilde{\lambda}_I\rangle = 0, \tag{3.20}$$

where m is the mass matrix of the hypermultiplet. Introducing a mass term will change the supersymmetry variations (3.17), where the new mass terms will appear as a shift of σ.

If one introduces a massless hypermultiplet in the adjoint representation of the gauge group, the supersymmetry of the theory is enhanced to $\mathcal{N} = 2$. The soft supersymmetry breaking of this theory by turning on a non-zero mass is sometimes referred to as $\mathcal{N} = 2^*$.

In the articles I, II we study the $\mathcal{N} = 1$ theories described above, placing them on a curved compact manifold and computing their partition functions exactly using the methods of supersymmetric localization described in chapter 2. The procedure of placing a supersymmetric field theory on a curved manifold is described in section 3.6, but first we turn to 4d $\mathcal{N} = 2$ theories.

3.5 4d $\mathcal{N} = 2$ theories

Here we briefly describe the basics of 4d $\mathcal{N} = 2$ gauge theories, which is a large subject of great importance to various areas of physics and mathematics [19, 24, 22]. If one takes a 5d $\mathcal{N}=1$ theory described in the last section and places it on $\mathbb{R}^4 \times S^1$ and then shrinks the circle to zero, as we will describe in chapter 8, one will reach a 4d $\mathcal{N}=2$ theory. This is in fact the main tool we use in article IV, and because of this much of the story is very similar to the 5d $\mathcal{N}=1$ theories.

3.5.1 Vector multiplet

A 4d $\mathcal{N}=2$ vector multiplet contains the gauge field A_μ, a pair of (Dirac, i.e. 4-component) spinors λ_I, two real scalar fields σ, φ, and auxilliary scalars $D_{IJ} = D_{JI}$, all taking values in the Lie algebra of the gauge group. The index I is over the fundamental doublet of the $SU(2)$ R-symmetry that the theory enjoy. The supersymmetry transformations are

$$\begin{aligned}
&\delta A_\mu = i\xi_I \Gamma_\mu \lambda_I, \\
&\delta\lambda_I = -\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\xi_I + (\not{D}\sigma + \Gamma_5 \not{D}\varphi)\xi_I - i[\varphi,\sigma]\Gamma_5\xi_I - D_{IJ}\xi^J, \\
&\delta\sigma = i\xi_I\lambda^I, \qquad \delta\varphi = i\xi_I\Gamma_5\lambda^I, \\
&\delta D_{IJ} = -i\xi_I\not{D}\lambda_J - [\varphi, \xi_I\Gamma_5\lambda_J] + [\sigma, \xi_I\lambda_J] + (I \leftrightarrow J).
\end{aligned} \tag{3.21}$$

Again ξ_I are constant complex Dirac spinors. The two scalar fields are often combined into a single complex scalar $\Phi = \varphi + i\sigma$, and slash is shorthand for contraction with Γ_μ. The supersymmetric Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\text{vect}} = \frac{1}{g_{YM}^2}\text{Tr}\Bigg[&\frac{1}{2}F\wedge\star F + \frac{g_{YM}^2\theta}{16\pi^2}F\wedge F + (d_A\Phi)^\dagger\wedge\star(d_A\Phi) \\
+ &\left(-i\lambda\Gamma^\mu D_\mu\bar{\lambda} - i\sqrt{2}[\lambda,\lambda]\Phi^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\lambda}]\Phi^\dagger - \frac{1}{2}[\Phi^\dagger,\Phi]^2\right)\text{Vol}\Bigg]
\end{aligned} \tag{3.22}$$

where Vol_4 is the volume form. This action includes the famous θ-term, and it is natural to phrase things in terms of the complex coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}. \tag{3.23}$$

3.5.2 Hypermultiplet

The hypermultiplet, much like in 5d $\mathcal{N}=1$, contains a $SU(2)_R$-doublet of scalars q_I, a $SU(2)_R$ singlet ψ and a doublet of auxiliary scalars F_I, all in a representation R of the gauge group. They have the supersymmetry variations

$$\begin{aligned}
\delta q_I &= -2i\xi_I\psi, \\
\delta\psi &= \Gamma^\mu\xi_I D_\mu q^I + i\xi_I\sigma q^I - \hat{\xi}_I F^I, \\
\delta F_I &= 2\hat{\xi}_I(i\Gamma^\mu D_\mu\psi + \sigma\psi + \lambda_K q^K).
\end{aligned} \tag{3.24}$$

which is essentially the same as in 5d (since the reduction of both scalars and spinors is trivial from five to four dimensions, more on this in chapter 8). Again $\hat{\xi}$ is a constant spinor satisfying

$$\xi_I\xi^I = \hat{\xi}_I\hat{\xi}^I, \quad \xi_I\hat{\xi}_J = 0, \quad \xi_I\Gamma^m\xi^I + \hat{\xi}_I\Gamma^m\hat{\xi}^I = 0. \tag{3.25}$$

3.6 Rigid supersymmetry on curved backgrounds

Localization gives us a powerful tool with which to investigate quantum field theories, but it requires them to be placed on compact, curved manifolds (since on non-compact manifolds the partition function diverges with volume).[2] Therefore a natural and important question is to ask on which backgrounds we can place supersymmetric field theories. And how do we go about constructing such a theory? Specifically, given a compact Riemannian (i.e. Euclidean signature) manifold (M, g), when can we define a supersymmetric theory on it, and if we can, what is the theory? There are two known, different approaches to this problem. The first one is through a procedure called topological twisting, first considered by Witten [20, 54], and the second one is that of finding spaces that admit appropriately generalized "constant spinors". Topological twisting takes a theory, like the 4d $\mathcal{N} = 2$, and changes it in a particular way ("twists it") so that it becomes topological and can be placed on any background. This is an interesting and useful procedure, but for our purposes the second approach is more relevant. In the flat space, supersymmetry is parametrized by constant spinors, $\partial_\mu\xi = 0$, so the natural generalization to curved backgrounds is to consider covariantly constant spinors,

$$\nabla_\mu\xi = 0, \quad \text{where } \nabla_\mu = \partial_\mu - \frac{1}{4}\Gamma^{\mu\nu}\omega_{\mu\nu}. \tag{3.26}$$

[2]There are attempts at localizing theories on non-compact spaces [53], but there are various technical issues and how it works is not well understood at the moment.

Here $\omega_{\mu\nu}$ is the spin connection. For manifolds that admit covariantly constant spinors, the supersymmetry can be formulated by simply covariantizing everything, essentially promoting each derivative to a covariant derivative. The issue with this is that admitting covariantly constant spinors is a very strong condition on the background geometry: the only spaces with covariantly constant (or parallel, as they are also called) spinors are special holonomy manifolds. Those are the Calabi-Yau manifolds, hyperkähler manifolds, G_2-manifolds of dimension 7, and $Spin(7)$-manifolds in dimension 8. This is very restrictive, and does not admit any interesting examples in lower dimensions where we are interested in studying field theories. Therefore, people considered a deformation of the $\nabla_\mu \xi = 0$ condition, relaxing it into the so called Killing spinor equation which reads

$$\nabla_\mu \xi = K_\mu \xi, \tag{3.27}$$

where K_μ is a background field; usually taken to be a constant divided by the length parameter r that control the size of the compact manifold, so that it goes to zero in the flat space or large r limit. So people considered equations of the form

$$\nabla_\mu \xi = \frac{\alpha}{r} \Gamma_\mu \xi, \tag{3.28}$$

for some constant α. This is the equation one finds on the spheres, which was the first cases people considered; for correct choices of α the spheres admit solutions.

If the background admits such Killing spinors, and we modify our supersymmetry variations and action by adding specific terms with r^{-1} and r^{-2} dependence, we can define a supersymmetric theory on the curved background. The first explicit example of this was by Pestun on S^4 [19]. He used a conformal map between $\mathbb{R}^4$ and S^4 to find out how to modify his action and supersymmetry, and found that adding a small number of terms that went away in the large r limit allowed him to have a supersymmetric field theory on S^4. We will see examples of these terms when discussing 5d SYM theories on Sasaki-Einstein backgrounds in chapter 6.

For a while after Pestuns work, the people studying theories on other curved backgrounds in different dimensions did not have a systematic way of finding which terms to add to the action and variations. But for simple cases like spheres and other highly symmetrical spaces, you can make an ansatz of terms going like r^{-1} and r^{-2}, and fix their coefficients by directly checking supersymmetry algebra closure and invariance of the action.

Of course, we want to understand things better than this, so thankfully it was not long until the question was addressed in a systematic way by Festuccia and Seiberg [55], who used off-shell supergravity to give a general prescription. Their idea was to find a supergravity solution for the particular manifold in question, that is, fix the metric and then solve the supergravity equations of motions to find the corresponding values of the other fields in the supergravity multiplet. Finally take a rigid limit, where the gravitational degrees of freedom freeze and decouple, and we are left with the desired rigid supersymmetric field theory on the correct background. The extra terms in the supersymmetry transformations and in the action are then naturally understood as the background fields from the supergravity solution. In supergravity, the parameters of the supersymmetry variations is the gravitino, which goes to a fixed background value in the rigid limit. Since the background fields have to be supersymmetric, we have to set the supersymmetry variation of the gravitino to zero, $\delta\Psi = 0$ and solve for the gravitino. This equation is exactly the (generalized) Killing spinor equation, and the gravitino solutions are the Killing spinors that will parametrize the rigid supersymmetry.

So the Festuccia-Seiberg procedure gives us a systematic and conceptually clear way of finding supersymmetric backgrounds and constructing the rigid supersymmetric field theories on them. Of course actually solving the Killing spinor equation, or finding out when solutions exist, is not generally an easy task.

People have used this procedure to study many different cases, for example 4d $\mathcal{N} = 1$ theories with $U(1)_R$ symmetry, 3d $\mathcal{N} = 2$ theories with $U(1)_R$, 2d $\mathcal{N} = (2, 2)$ theories and so on [56, 57, 58] For some of these cases, there is a full classification of possible supersymmetric backgrounds, like for the 4d $\mathcal{N} = 1$ with $U(1)_R$ symmetry [59, 60]. In this case, given any complex Riemannian manifold one can always solve the Killing spinor equation. Requiring more than one supercharge gives us more constraints on the geometry. For other settings, the problem of classifying all supersymmetric backgrounds are still open. The relevant example for us is the case of $\mathcal{N} = 1$ in 5d, where it is known that we can always construct such theories on Sasaki-Einstein manifolds (see chapter 5), but the full classification of all possible backgrounds is not known (although there is informed conjectures, see [61, 62]).

4. Toric Geometry

In this and the next chapter we leave physics for a while, and introduce some concepts from geometry that we need, starting with the rich and interesting subject of toric geometry. Broadly this is the study of spaces with torus actions. Toric geometry can be viewed from a number of different perspectives; symplectic geometry, algebraic geometry, combinatorics, gauged linear sigma models and so on. Here we will introduce the subject from the symplectic geometry viewpoint. For more details and proofs, we refer the reader to the book by Fulton [63].

4.1 Basic concepts

A symplectic manifold is a pair (M, ω), where M is a manifold and ω is a closed 2-form on M that is non degenerate at every point of M. A vector field X on M is called symplectic if $L_X\omega = 0$, and Hamiltonian if

$$\iota_X \omega = dH, \tag{4.1}$$

for some function H, which we call the Hamiltonian function for X. The typical example of a symplectic manifold is $\mathbb{C}^r$ equipped with the canonical symplectic form

$$\omega = \frac{i}{2}\sum_{i=1}^{r} d\bar{z}_i \wedge dz_i = \sum_{i=1}^{r} dx_i \wedge dy_i, \tag{4.2}$$

where $z_i = x_i + iy_i$.

Another fundamental example is the cotangent bundle of some manifold, T^*N. Letting q^i be local coordinates on N and p^i be the corresponding coordinates on the covectors, then the canonical symplectic form is given locally by

$$\omega = \sum_i dq^i \wedge dp^i. \tag{4.3}$$

A fundamental result of symplectic geometry is the Darboux theorem, which states that any two symplectic manifolds of the same dimensions are locally symplectomorphic to each other. This means that there always exists a choice of local coordinates such that the symplectic form takes the form of (4.3). In particular this means that the symplectic structure carries no local data.

Next we turn to group actions on symplectic manifolds. Let M have a G-action for G a compact Lie group, let $\mathfrak{g}$ be the Lie algebra of G, and take an element $X \in \mathfrak{g}$. Denote by $X^{\#}$ the vector field on M generated by $\{\exp(tX)|t \in \mathbb{R}\} \subset G$, i.e. the vector field that generates the flow in the direction of X.

If there exists a map

$$\mu : M \to \mathfrak{g}^* \ , \tag{4.4}$$

such that

$$d\langle \mu(p), X\rangle = \iota_{X^{\#}}\omega, \tag{4.5}$$

and the map μ is equivariant with respect to the given group action ψ, then the group action is called Hamiltonian. Here, equivariant means that

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu, \qquad \forall g \in G, \tag{4.6}$$

i.e. that the group action and the moment map "commute" in the appropriate sense. This means that the group action and the symplectic structure are compatible. The map μ is called the moment map. One should think of $\langle \mu(p), X\rangle$ as the component of μ along the direction of X, and the condition (4.5) says that this should be the Hamiltonian function for the vector field $X^{\#}$.

For toric geometry, the relevant Lie group is a torus, $\mathbb{T}^n = S^1 \times \cdots \times S^1 = (S^1)^n$. Its Lie algebra and its dual are both identified with $\mathbb{R}^n$. Hence, if a manifold M has a Hamiltonian $\mathbb{T}^n$-action, the moment map is a map $\mu : M \to \mathbb{R}^n$. It further satisfies that for each basis vector e_i of $\mathbb{R}^n$, the function $\mu_i = \langle \mu, e_i\rangle$ is a Hamiltonian function for the corresponding vector field v_i on M. The moment map is invariant under the torus action.

For an effective $\mathbb{T}^n$ action, the dimension of M needs to be at least $2n$. We can now give the precise meaning of toric in the symplectic geometry setting, namely: a connected symplectic manifold is toric if it is equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold.

A technical warning is in place here. In this thesis, the word toric is not always used strictly in the sense defined here, but sometimes we use it in the imprecise sense of "a manifold with a torus action". The precise meaning of toric also is slightly different for Sasaki-Einstein manifolds, as we describe in the next chapter. Hopefully with this warning in mind, the meaning should be clear from the context.

Let us make all the above definitons a bit more concrete by working out a canonical example in some detail.

Example ($\mathbb{C}^r$) : Consider again the canonical example of a symplectic manifold, $\mathbb{C}^r$. This has a natural $\mathbb{T}^r$ action, acting on the coordinates as a phase rotation, $z_j \mapsto e^{i\theta_j} z_j$. This action is symplectic, meaning that it preserves the symplectic form, $g^*\omega = \omega$ for all $g \in \mathbb{T}^r$. It is also effective, and hamiltonian, which we see explicitly below. The angles $(\theta_1, \ldots, \theta_r)$ gives an element in the Lie algebra $\mathfrak{g} \simeq \mathbb{R}^r$ of the torus. Let $\{e_i\}_{i=1}^r$ be the usual basis vectors on $\mathbb{R}^r$; then the action of e^i on $\mathbb{C}^r$ is represented by the vector field $v_i = y_i \partial_{x_i} - x_i \partial_{y_i}$. We then define the following moment map

$$\begin{aligned} \mu : \mathbb{C}^r &\to (\mathbb{R}^r)^* \simeq \mathbb{R}^r, \\ \mu(z_1, \ldots, z_r) &= \frac{1}{2}(|z_1|^2, \ldots, |z_r|^2). \end{aligned} \tag{4.7}$$

This is clearly invariant under the torus action.

For the basis vector e_i is is easy to see that the corresponding component μ_i of the moment map is precisely its Hamiltonian vector field, since

$$d\mu^i = d(\frac{1}{2}|z_i|^2) = \frac{1}{2}(\bar{z}_i dz_i + z_i d\bar{z}_i) = x_i dx_i + y_i dy_i \tag{4.8}$$

which is the same as

$$\iota_{X^i}\omega = \iota_{y_i\partial_{x_i} - x_i\partial_{y_i}}(\sum_{i=1}^r dx_i \wedge dy_i) = x_i dx_i + y_i dy_i. \tag{4.9}$$

This shows that the torus action is Hamiltonian, so $\mathbb{C}^r$ is a toric manifold.

Moment maps and Hamiltonian actions are very important in symplectic geometry, and they help us understand the geometry of toric manifolds, primarily through the following theorem of Atiyah [64]. Let

Figure 4.1. The moment polytopes for S^2 and $\mathbb{P}^2$.

(M, ω) be a connected symplectic manifold, and let $\mathbb{T}^n$ be a torus with a hamiltonian action on M, with moment map μ. Then the image of μ is the convex hull of the images of the fixed points of the action. This image will thus be a polytope, called the moment polytope.

We have seen already the example of $\mathbb{C}^r$, which is not compact. Its image under the moment map is the cone $\mathbb{R}^r_{\geq 0} = \{x_i > 0 | i = 1, \dots r\}$. A typical compact example to consider is the the sphere $S^2 = \mathbb{P}^1$ with the standard $\mathbb{T}^1 = U(1)$ action acting as a rotation. This group action has two fixed point, the north and south pole, and so the moment polytope will be an interval.

A second example is the complex projective space $\mathbb{P}^2$, which can be described by coordinates $(z_1, z_2, z_3) \in \mathbb{C}^3 \backslash \{0\}$ under the identification

$$(\lambda z_1, \lambda z_2, \lambda z_3) \sim (z_1, z_2, z_3), \qquad \lambda \in \mathbb{C}^*. \tag{4.10}$$

This has the torus action $\mathbb{T}^3/\mathbb{T}^1 = \mathbb{T}^2$, where each factor of the $\mathbb{T}^3$ can be thought of as rotating one of the different complex coordinates and you then quotient by the identification above. We can represent the remaining $\mathbb{T}^2$ action as the phase rotation of say z_1 and z_2, in which case the moment map takes the form

$$\mu = \left(\frac{|z_1|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2}, \frac{|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2} \right). \tag{4.11}$$

This group action has three fixed points, when two of the complex coordinates are zero. On these three fixed points the above moment map takes the values $(0, 0), (1, 0)$ and $(0, 1)$, and the moment polytope will be a triangle, see figure 4.1. All higher dimensional projective spaces are also good examples, and the moment polytope of $\mathbb{P}^r$ is the r-simplex.

These polytopes are examples of so called Delzant polytopes, which is a polytope $\Delta \in \mathbb{R}^n$ fulfilling the following conditions:

- Simplicity: there are n edges meeting at every vertex.

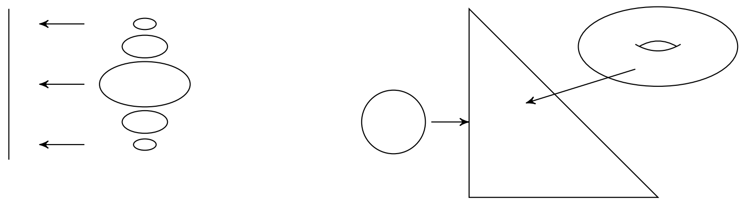

Figure 4.2. A sketch of how one can think of toric manifolds as degenerating torus fibrations over the moment polytope.

- Rationality: the edges meeting at a vertex are generated by rational vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$.
- Smoothness: at each vertex, the vectors describing the edges form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

A famous result in toric geometry is that for any compact symplectic toric manifold its moment polytope Δ is Delzant. Something stronger is in fact true: there is a one-to-one correspondence between compact toric manifolds and their Delzant polytopes. All the relevant toric data is contained in the rational polytope.

A useful way of thinking about a toric manifold is as a $\mathbb{T}^n$ fibration over its moment polytope, where as you go to a boundary, one of the circles degenerate. For example for S^2, its polytope is a line interval, over which a single S^1 is fibered, which shrinks to zero size as you go to the ends of the interval, see figure 4.2. For $\mathbb{P}^2$, over the interior of the triangle one has a $\mathbb{T}^2$ fibration, and as one goes to an edge of the triangle, one of the circles degenerate, and when you go to a vertex, the entire torus degenerates. The local geometry along an edge can be thought of as $\mathbb{R}^2 \times S^2$, while the local geometry at the vertex is $\mathbb{R}^4$.

4.2 Symplectic reduction

Another useful view on toric geometry is given by the procedure of symplectic reduction or symplectic quotient. This is a general procedure where a symplectic manifold M with a Hamiltonian group action of a group G can be reduced to a lower dimensional space, using the group action and the moment map,

$$M \mathbin{/\!/} G = \mu^{-1}(t)/G, \tag{4.12}$$

where $t \in \mathfrak{g}$ is called the Kähler parameter. The reduction preserves the symplectic structure, so it gives us a way of constructing new symplectic manifolds. If the manifold further is Kähler (see the next chapter for the definition) it also preserves the Kähler structure, so it also goes by the name Kähler reduction.

We specialize this to the case of a torus action on some complex space $\mathbb{C}^{k+n}$. Let $z_1, \ldots, z_{k+n}$ be the complex coordinates on $\mathbb{C}^{k+n}$, and consider the torus $\mathbb{T}^k$ acting on them with the following group action:

$$z_j \mapsto e^{iQ_a^j \theta_a} z_j, \quad a = 1, \ldots, k. \tag{4.13}$$

The numbers Q_a^j are integers, called the charges of the torus action. The group action (4.13) has the following moment map:

$$\mu_a(z_1, \ldots, z_k) = \sum_{j=1}^{k+n} Q_a^j |z_j|^2, \quad a = 1, \ldots, k\ . \tag{4.14}$$

Finally, introduce a set of k real numbers t_a, the Kähler parameters, and then consider the following space:

$$M = \mathbb{C}^{k+n} \mathbin{/\!/} \mathbb{T}^k = \left(\bigcap_{a=1}^{k} \mu_a^{-1}(t_a) \right) /U(1)^k. \tag{4.15}$$

If the torus action acts freely on $\bigcap_a \mu_a^{-1}(t_a)$, M will be a smooth symplectic toric manifold of real dimension $2n$. If there are fixed points, one can instead get an orbifold or a cone. The notation $\mathbb{C}^{k+n} \mathbin{/\!/} \mathbb{T}^k$ is often used for the symplectic reduction. The Kähler parameters in the above are useful in that they might let us avoid the fixed points of the torus action.

Example (S^2): As an example, let us construct S^2 using symplectic reduction. We start from $(z_1, z_2) \in \mathbb{C}^2$, with the group action of $U(1)$,

$$(z_1, z_2) \mapsto e^{i\theta}(z_1, z_2). \tag{4.16}$$

Now we have a single $U(1)$, so there is a single charge vector $Q = [1, 1]$. The group action has the moment map

$$\mu = |z_1|^2 + |z_2|^2. \tag{4.17}$$

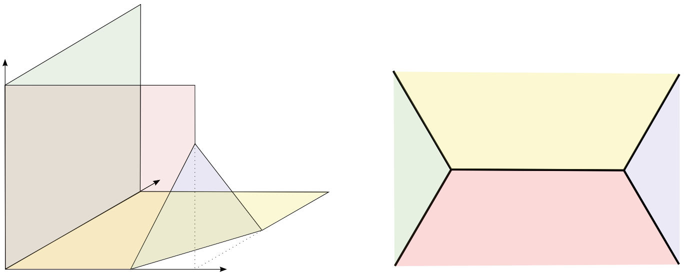

Figure 4.3. In the left figure, the interior region of the four bounding planes is the moment map image of the resolved conifold. The actual manifold should be understood as a $\mathbb{T}^3$ fibration over this, where one circle shrinks as one approaches a face. The right picture is the projection of this, giving what is commonly referred to as the toric diagram of the CY 3-fold.

We have a single Kähler parameter to t, and performing the symplectic quotient we get

$$\mathbb{C}^2//U(1) = \{(z_1, z_2)|\ |z_1|^2 + |z_2|^2 = t\}/\{(z_1, z_2) \sim e^{i\theta}(z_1, z_2)\} = S^2, \tag{4.18}$$

which is describing S^2 as the base of the Hopf fibration of an S^3 with radius controlled by t. We see that the presence of the Kähler parameter is crucial, since the $U(1)$ action has a fixed point at the origin.

Example (resolved conifold, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$): As a second example we construct the resolved conifold, which is the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. This space is a toric, non-compact Calabi-Yau manifold (see next chapter for a definition); and it is a very important example in the context of mirror symmetry [65]. We start from $\mathbb{C}^4$, with a single $U(1)$ action with charges $[1, 1, -1, -1]$ and some Kähler parameter t. The group action has the moment map

$$\mu = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2, \tag{4.19}$$

and for our reduction we consider

$$\mu^{-1}(t) = \{(z_1, z_2, z_3, z_4)|\ |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = t\}. \tag{4.20}$$

Take as coordinates $x = |z_2|^2$, $y = |z_3|^2$, $z = |z_4|$, then $|z_1|^2 = t - x + y + z$. We see from their definition that x, y, z and $t - x + y + z$ all have to be positive quantities, which tells us over which region x, y, z

are allowed to run over. This is the non-compact region of $\mathbb{R}^3$ bounded by the planes $x = 0, y = 0, z = 0$ and $t - x + y + z = 0$, see figure 4.3. To describe $\mu^{-1}(t)$, we have a $\mathbb{T}^4$ fibration over this space; where as we go to one of the bounding planes, one of the circles degenerate (shrink to zero size). However by construction, the combination of circles given by the charges $[1, 1, -1, -1]$ does not degenerate anywhere, so it gives a free action on $\mu^{-1}(t)$, and we can further quotient by this $U(1)$. This gives the resolved conifold, which we can view as the remaining $\mathbb{T}^3$ fibration, where some combination of circles degenerate on each bounding plane. The precise combination of circles is given by the normal to the bounding plane, something that we explain further in the next chapter when discussing toric Sasaki-Einstein manifolds. The resolved conifold is topologically the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ [66], and it is a toric Calabi-Yau manifold.

The charges Q_a^i encode all the data of a toric manifold, and are thus often referred to as the toric data. Given these charges, there is a simple condition for telling if a toric manifold is Calabi-Yau. A toric manifold is CY if and only if its charges satisfy $\sum_{i=1}^{k} Q_a^i = 0$ for all a.

5. Sasaki-Einstein geometry

In this chapter we introduce a hierarchy of geometrical structures in even and odd dimensions. This includes the Sasaki-Einstein manifolds, which are the odd-dimensional sibling to the more famous Calabi-Yau manifolds in even dimensions. In this thesis we are interested in Sasaki-Einstein manifolds since they allow Killing spinor solutions, which as discussed in section 3.6, allows one to construct supersymmetric field theories on them. They therefore provide a nice source of curved backgrounds where we can study supersymmetric field theories. To make it possible to perform explicit calculations, we further restrict ourselves to toric Sasaki-Einstein manifolds, so that techniques described in the last chapter can be applied. These have a nice description in terms of rational convex cones, as we describe in section 5.3.2.

There is a correspondence between geometrical structures in even and odd dimensions, as the following table illustrate:

Structure	Even	Odd	Structure
ω	Symplectic	Contact	κ
g, ω, J	Kähler	Sasaki	g, κ, J
$R_{\mu\nu} = 0, \omega, J$	Calabi-Yau	Sasaki-Einstein	$R_{\mu\nu} = \lambda g_{\mu\nu}, \kappa, J$
$R_{\mu\nu} = 0, \omega, I, J, K$	Hyperkähler	Tri-Sasaki	$R_{\mu\nu} = \lambda g_{\mu\nu}, \kappa, I, J, K$

As you go down the table, you are adding more and more structure. ω is a symplectic form, κ is a contact one-form, I, J, K are complex structures, g is a Riemannian metric and $R_{\mu\nu}$ is its Ricci tensor. In the following sections, we will explain these structures and what compatibility conditions they have to satisfy for the different cases shown in the table.

5.1 Contact structure

A contact structure is a structure in odd dimensions that we can think of as the odd dimensional version of a symplectic structure. For a more comprehensive introduction to the topic see for example [67].

Let M be a $2n+1$ dimensional manifold, equipped with a one-form κ that satisfies the non-degeneracy condition that $\kappa \wedge (d\kappa)^n$ is nowhere vanishing, i.e. it is a volume form on M. Then κ is called the contact one-form, and its kernel defines a hyperplane distribution in the tangent bundle TM. From the non-degeneracy condition, we see that $d\kappa$ is a non-degenerate closed 2-form on the kernel hyperplane of κ. This is one way of seeing that the contact structure is closely related to the symplectic structure.

We also define a vector field R, which is uniquely determined by the conditions

$$\iota_R \kappa = 1, \qquad \iota_R d\kappa = 0, \tag{5.1}$$

and is called the Reeb vector field of the contact structure.

Example ($\mathbb{R}^{2n+1}$) The canonical example of a contact structure is the following one on $\mathbb{R}^{2n+1}$. Let $(x^1, \dots, x^n, y^1, \dots, y^n, z)$ be our coordinates, then the canonical contact 1-form and Reeb are given by

$$\kappa = dz - \sum_{i=1}^{n} y^i dx^i, \quad R = \partial_z. \tag{5.2}$$

We see that

$$d\kappa = \sum_{i=1}^{n} dx^i \wedge dy^i, \tag{5.3}$$

which is exactly the canonical symplectic form on $\mathbb{R}^{2n}$.

Contact structure, precisely like symplectic structure, does not give any interesting local data. Every contact structure looks locally the same, i.e. we can always pick local coordinates such that the contact structure can be written in the form of (5.2). This result is called the Darboux theorem for contact structures.

If M also is equipped with a metric g, we usually want to consider the cases when the metric and the contact structure is compatible, in the sense that the following holds:

- the metric is preserved along the flow of the Reeb, $L_R g = 0$, and

- there exists an almost complex structure J on the kernel hyperplane of κ such that $g(X,Y) = d\kappa(X, JY)$ for all $X, Y \in \ker(J)$.

The triplet (M, g, κ) is then called a K-contact manifold.

From the definition of a contact structure we see that it defines a volume form, $\kappa \wedge (d\kappa)^n$. If we also have a metric this gives us another canonical volume form, that we call Vol. For a K-contact manifold, one can show that these two are related as

$$\text{Vol} = \frac{(-1)^n}{2^n n!} \kappa \wedge (d\kappa)^n. \tag{5.4}$$

In particular for a 5d contact manifold we have $\text{Vol} = \frac{1}{8}\kappa \wedge d\kappa \wedge d\kappa$.

The contact structure gives us a decomposition of the differential forms into what we call horizontal and vertical forms, $\Omega^\bullet(M) = \Omega^\bullet_V \oplus \Omega^\bullet_H$. These are defined by the two projectors

$$P_V = \kappa \wedge \iota_R, \quad P_H = 1 - P_V, \tag{5.5}$$

which are projectors since R is normalized. Intuitively a differential form is vertical if it has a 'leg' along κ, and horizontal if it does not. Further, a differential form α is called basic if it is horizontal and invariant under the Reeb vector flow, i.e.

$$\iota_R \alpha = 0, \qquad \mathcal{L}_R \alpha = 0. \tag{5.6}$$

We denote by $\Omega^p_B(M)$ the set of basic p-forms on M, which is a subset of the horizontal forms. It is easy to see that if α is basic then so is $d\alpha$. Therefore we can define the restriction of the exterior derivative to basic forms, $d_B = d|_{\Omega^\bullet_B}$, and use it to define the basic cohomology $H^\bullet_B(M)$.

5.2 Sasaki geometry

Next, we introduce a specialization of the contact manifolds, called Sasaki, which is the odd-dimensional sibling of Kähler structure. An even-dimensional manifold X is Kähler if it has three structures, namely a Riemannian metric g, a symplectic form ω and a complex structure J, which satisfy the compatibility condition

$$g(U, V) = \omega(U, JV)\ , \tag{5.7}$$

for any vector fields U, V.

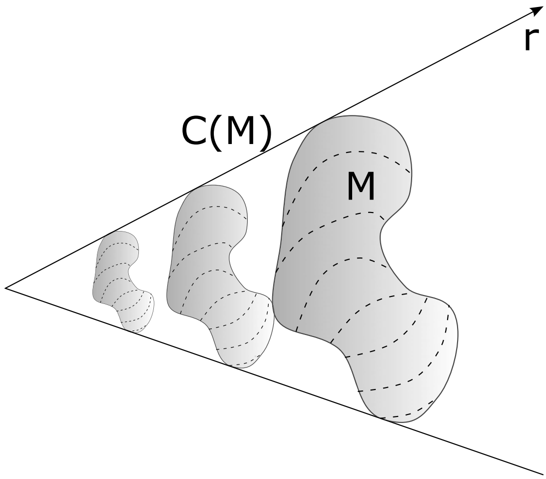

Figure 5.1. The metric cone over M. The coordinate r is along the cone direction and at each value of r there is a copy of M, increasing in size with r.

To define a Sasaki manifold, we first introduce the notion of the metric cone over a manifold (M, g). This is the manifold $C(M) = M \times \mathbb{R}_{\geq 0}$, equipped with the cone metric

$$ds^2_{C(M)} = dr^2 + r^2 ds^2_M. \tag{5.8}$$

M is called the base of the cone. See figure 5.1 for an illustration. The basic example is that the metric cone over the n-sphere is flat $n+1$ space, $C(S^n) = \mathbb{R}^{n+1}$.

A manifold is called Sasaki if its metric cone is Kähler. On the cone $C(M)$ we have the homothetic vector field $r\partial_r$, pointing in the cone direction, and we also have a complex structure J. The Reeb vector on $C(M)$ is defined to be

$$\textsc{r} = J(r\partial_r), \tag{5.9}$$

which is nowhere vanishing (away from the origin,$r = 0$); and by definition is orthogonal to the cone direction. Similarly we can define a 1-form κ on $C(M)$ through

$$\kappa = d(\log r). \tag{5.10}$$

Given the cone $C(M)$, we can view M as being embedded into the set with $r = 1$, and from the cone metric and definition of the Reeb, we see that the Reeb on the cone restricted to $r = 1$ gives a normalized vector on M. This is the Reeb vector of the contact structure on M. Similarly the restriction of κ on $C(M)$ to $r = 1$ gives the contact one-form on M. The complex structure on $C(M)$ also descends into a complex structure on the hyperplanes of the kernel of κ on M. The contact structure on a Sasaki manifold is K-contact.

One can use the complex structure to further refine the complex of basic forms that we introduced in the last section. The complex structure gives a holomorphic grading of the horizontal forms,

$$\Omega^k_H(M) = \bigoplus_{p+q=k} \Omega^{p,q}_H(M), \tag{5.11}$$

which carries over to the basic forms. We can then split the basic exterior derivative into its two components $d_B = \partial_B + \bar{\partial}_B$, which are the basic Dolbeault operators, acting as $\partial_B : \Omega^{p,q}_B \to \Omega^{p+1,q}_B$. This gives the basic Dolbeault complex $(\Omega^{p,\bullet}_B, \bar{\partial}_B)$, which has the associated basic Dolbeault cohomology. This is also called the Kohn-Rossi cohomology and is denoted $H^{p,\bullet}_B(M)$. We will use this when computing one-loop partition functions for theories placed on Sasaki-Einstein manifolds.

5.3 Sasaki-Einstein geometry

Now we come to the case of Sasaki-Einstein manifolds, which are the odd-dimensional version of Calabi-Yau manifolds. A metric is Einstein if the Ricci tensor is proportional to the metric,

$$R_{mn} = \lambda g_{mn}, \tag{5.12}$$

which has this name since it implies that the metric solves the vaccum Einstein equations (with a cosmological constant λ). A manifold is Sasaki-Einstein (SE) if it is both Sasaki and Einstein. If M has dimension $2n-1$ and is Sasaki-Einstein, then it turns out that $\lambda = 2(n-1)$. A direct calculation proves that the corresponding cone metric is Ricci-flat, implying that the cone is Calabi-Yau (CY). So an equivalent definition of an SE manifold is to say that its metric cone is CY.

The canonical examples of SE manifolds are the odd dimensional spheres, S^3, S^5 and so on, equipped with their round metrics. The metric cones over these are $\mathbb{C}^n$, equipped with the flat metric.

When classifying possible SE structures (or more generally, contact structures), it turns out that the behavior of the Reeb vector field play a crucial role. In particular, it is useful to look at the behavior of the orbits of the Reeb. Suppose that all orbits of R are closed, i.e. are circles. This means that R generates an isometric $U(1)$ action on M, and since the Reeb is nowhere zero this action has to be locally free, meaning that the isotropy group of any point must be finite. This kind of SE manifolds are called quasi-regular. If further all points have trivial isotropy, then the $U(1)$ action is free and the SE manifold is called regular. If M is regular the quotient $M/U(1)$ is a good manifold, while in quasi-regular case it is an orbifold. If the orbits of R do not all close, the manifold is said to be irregular. The generic orbit of R is then a real line, and no well-defined quotient exists.

5.3.1 Killing spinors

As mentioned in the beginning of this chapter, the reason why SE manifolds are relevant for this thesis is because they admit Killing spinors. So let us specify what a Killing spinor is and state the result about SE manifolds.

Let (M, g) be a Riemannian spin manifold, call the spin bundle E, and let ψ be a spinor on M, i.e. a smooth section of the spin bundle. Then ψ is called a Killing spinor if for some constant α it satisfies

$$\nabla_V \psi = \alpha V \cdot \psi, \tag{5.13}$$

for every vector field V. ∇ denotes the covariant spinor derivative, which include the spin connection, and the dot-product between V and ψ denotes Clifford multiplication. In notation more familiar to physicists we would write

$$\nabla_m \psi = \alpha \Gamma_m \psi, \tag{5.14}$$

where Γ_m are the gamma-matrices in the dimension of M. If the constant α is real, ψ is called a real Killing spinor, and if it is imaginary ψ is called imaginary.

For SE manifolds M there is the following important result. A simply connected SE manifold admits at least 2 linearly independent real Killing spinors, with $\alpha = \frac{1}{2}, -\frac{1}{2}$ if the dimension of M is of the form $4p - 3$ (dimensions $1, 5, 9, \ldots$), and $\alpha = \frac{1}{2}, \frac{1}{2}$ if the dimension is of the form $4p - 1$ (dimensions $3, 7, 11, \ldots$). So any simply connected SE manifold is spin and admits at least 2 Killing spinors.

In article I, we use 5d SE manifolds. For these, there is the following result, which follow from Smale's classification [68]. Any compact simply connected spin 5-manifold with no torsion in $H_2(M, \mathbb{Z})$ is diffeomorphic to $\#n(S^3 \times S^2)$, where $\#n$ denotes n connected sums. The connected sum of two manifolds is constructed by removing a small ball inside each of them, and then gluing together the two boundary spheres. Although this involves the choice of balls, the result is unique up to homeomorphism.

5.3.2 Toric Sasaki-Einstein manifolds

The main geometrical setting of articles I,II and IV is that of toric SE manifolds. The reason for requiring a torus action is that it allows us to use the localization formula to compute things explicitly, which is much harder in the general case.

A SE manifold M is said to be toric if there is an effective, holomorphic and Hamiltonian torus action on the corresponding CY cone $C(M)$, with the Reeb vector field in the Lie algebra of the torus, meaning that it can be written as a linear combination of the $U(1)$ actions. Here one identifies the Lie algebra and the corresponding vector fields on $C(M)$ that generates its group action. The word holomorphic in the above definition means that the complex structure is invariant under the torus action. So studying toric SE manifolds is the same as studying toric CY manifolds, which is a well studied topic. In particular it is well known that toric CY manifolds have to be non-compact.

The definition implies that there exists a torus invariant moment map μ,

$$\mu : C(M) \to \mathfrak{t}_n^* \simeq \mathbb{R}^n, \tag{5.15}$$

and as reviewed in chapter 4, the image of this,

$$C_\mu = \mu(C(M)), \tag{5.16}$$

will be a convex cone in $\mathbb{R}^n$, that we call the moment map cone. The base of this cone is a Delzant polytope, which is the polytope associated to the SE manifold M.

All the relevant data of the geometry is contained in the moment polytope, and we now consider some different ways this cone can be presented. For example, we can specify its inward pointing normals $\{v_1, \ldots, v_m\}$ and define it as

$$C_\mu = \{y \in \mathbb{R}^n | y \cdot v_a \geq 0, \; a = 1, \ldots, m\} \subset \mathbb{R}^n. \tag{5.17}$$

We may assume that the set of normals is minimal, and that they all are taken to be in $\mathbb{Z}^n$. This cone is also good, meaning that at every codimension k face, the k associated normals $v_{i_1}, \ldots, v_{i_k}$ can be completed in to an $SL_n(\mathbb{Z})$ matrix (for an explanation, see appendix B of I). The inward normals are related to the charges Q_a^j used for symplectic reduction in section 4.2 by the relations

$$\sum_{j=1}^{m} v_j Q_a^j = 0, \quad \forall a. \tag{5.18}$$

Since the metric cone over M is CY, we know that $\sum_j Q_a^j = 0 \; \forall a$. From (5.18) this can be translated into a condition for the inward normals, namely that there exists a vector $\xi \in \mathbb{Z}^n$ such that

$$\xi \cdot v_i = 1, \quad \forall i. \tag{5.19}$$

This is known as the 1-Gorenstein condition, and it implies that the normals are all coplanar. We can choose coordinates such that $\xi = [1, 0, 0, \ldots, 0]$, thus setting all the first components of our normals to 1.

The SE manifold is mapped to the hyperplane $\mu(M) = \mu(\{r = 1\} \times M)$, which is given by the intersection of $\{y \cdot \mathrm{R} = 1/2\}$ and C_μ. This intersection has the form of a compact $n-1$-dimensional Delzant polytope, that we call the base of the cone. Just like for compact toric manifolds, a useful picture for $C(M)$ is as the $\mathbb{T}^n$ fibration over C_μ, where again a circle degenerates as you hit a face of the cone. The inward normal of the face describe which combination of circles that degenerate as you approach it.

Similarly, the SE manifold M can be pictured as a $\mathbb{T}^n$ fibration over the base of the cone. Note here that this is slightly different from the compact symplectic toric case: the base is of dimension $n-1$ and there is a $\mathbb{T}^n$ fibration over it. As one goes to the faces, again circles degenerate, but now at the vertices there will always be one circle that is still non-degenerate. So where earlier the local geometry close to vertex was $\mathbb{C}^n$, it is now $\mathbb{C}^n \times S^1$ instead. In the case of an irregular toric SE manifold,

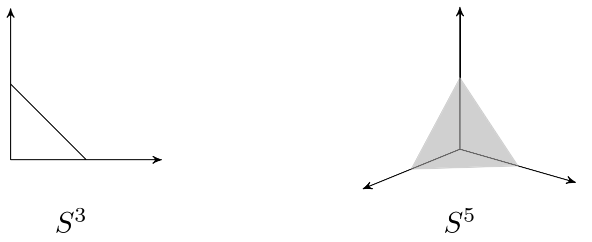

Figure 5.2. The moment map cones and polytopes of S^3 and S^5.

the closed orbits of the Reeb will exactly correspond to the vertices of this polytope.

Let us consider the example of S^3 viewed as a toric SE manifold. The cone over S^3 is $\mathbb{R}^4$, and its moment map cone is $\mathbb{R}^2_{\geq 0}$. The Reeb in explicit coordinates of the moment map cone is given by $(1,1)$, so the base polytope is the interval between $(0,1/2)$ and $(1/2,0)$, see figure 5.2. One can then think of S^3 as a $\mathbb{T}^2$ fibration over this interval, where one of the two circles degenerate as you go to the two ends of the interval. This also means that we can take a linear combination of the two circles and find a nowhere degenerate circle. This is the usual Hopf fibration $S^1 \to S^3 \to S^2$.

For S^5 the story is very similar. The moment map cone is instead $\mathbb{R}^3_{\geq 0}$. The base of this, representing S^5, is the triangle between the vertices $(1/2,0,0), (0,1/2,0)$ and $(0,0,1/2)$. Again this is the Hopf fibration $S^1 \to S^5 \to \mathbb{P}^2$.

Note that for 3d toric SE manifolds, the moment map cones are 2d, and there is not that much choice involved in choosing such a cone, at least not compared to the case of higher dimensions, since all 2d cones have exactly two sides. For 5d toric SE manifolds, the moment map cone is 3d, which allows for cones with three or more faces. So we have much more freedom in higher dimensions. These statements are equivalent to the fact that the only toric SE spaces in 3d are S^3 and the lense spaces $L(p,q)$, which are quotients of S^3 by $\mathbb{Z}_p$. In 5 and higher dimensions, we have an infinite number of infinite families of different manifolds, corresponding to rational convex polytopes. In 5 dimensions, the most well known example (other than S^5 and its lens spaces) is that of the

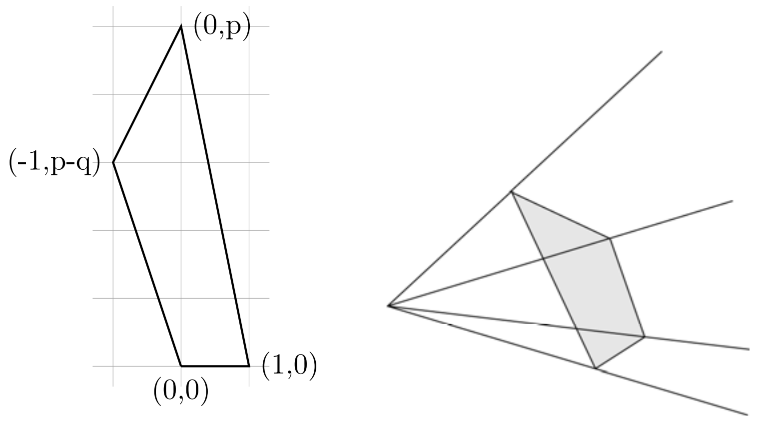

Figure 5.3. The polygon base of $Y^{p,q}$ and its moment cone.

$Y^{p,q}$ spaces, which correspond to 4-sided polytopes. We describe those in some detail in the next section.

5.3.3 The example of $Y^{p,q}$

The $Y^{p,q}$ spaces, constructed by Martelli and Sparks [69], are toric 5d SE spaces where an explicit metric is known. The parameters $p > q > 0$ are two coprime integers, and the spaces all have the topology of $S^2 \times S^3$. They have a free $U(1)$ action and can be viewed as a non-trivial S^1 fibration over $S^2 \times S^2$. The cone over $Y^{p,q}$ is a toric CY manifold, which can be constructed by symplectic reduction of $\mathbb{C}^4$ with a $U(1)$ acting with charges $Q = [-p, p+q, -p, p-q]$. The case of $p = 1,\ q = 0$ is slightly outside the $Y^{p,q}$ family, and gives the conifold that we discussed in the last chapter. In figure 5.3 we sketch the moment cone and the base polytope of $Y^{p,q}$.

The explicit metric on $Y^{p,q}$ is given by

$$\begin{aligned} ds^2 =& \frac{1-y}{6}(d\theta + \sin^2\theta d\phi^2) + \frac{dy^2}{w(y)q(y)} + \frac{q(y)}{9}[d\psi - \cos\theta d\phi]^2 \\ &+ w(y)\left[d\alpha + \frac{a-2y+y^2}{6(a-y^2)}[d\psi - \cos\theta d\phi]\right]^2 \end{aligned} \tag{5.20}$$

where

$$w(y) = \frac{2(a-y^2)}{1-y},$$
$$q(y) = \frac{a-3y^2+2y^3}{a-y^2}.$$

The coordinates run over the following ranges

$$0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \quad y_1 \le y \le y_2, \quad 0 \le \psi \le 2\pi, \quad 0 < \alpha < 2\pi l,$$

and the constant a is chosen in the range $0 < a < 1$. The constant l is determined in terms of a and p, q, as described below. Then the equation $q(y) = 0$ has one negative and two positive roots, and y_1 is taken to be the negative root and y_2 to be the smallest positive root. The coordinate α runs over the S^1 fiber, and (θ, ϕ, y, ψ) describe the $S^2 \times S^2$ base. Our choices of a and y_1, y_2 makes sure that $w(y) > 0$ everywhere, so that the circle fiber never degenerates.

To get the toric $Y^{p,q}$ manifold, we need to pick a such that $y_2 - y_1 = \frac{3q}{2p}$; which Martelli and Sparks show that you can always do for any coprime $p > q$. The roots and the integers p, q and l are related through the relations

$$\frac{y_1 - 1}{3ly_1} = p + q, \quad \frac{1 - y_2}{3ly_2} = p - q, \tag{5.21}$$

which can be used to fix l after setting p, q and a.

The Reeb vector field in these coordinates is given by

$$R = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha}, \tag{5.22}$$

which has constant unit norm.

This explicit writing of the geometry can be connected to the description of $Y^{p,q}$ in terms of its toric data in the following way. The toric action on a manifold with topology of $S^2 \times S^3$ will have (at least) 4 closed orbits, sitting over one of the two poles of S^2 and one of the two poles of S^3. By the logic described above, each closed orbit gives a vertex of the moment map polytope, so it will have 4 vertices. From the metric, we can see that the poles of S^2 are where $\theta = 0$ or $\theta = \pi$, while the poles of S^3 is given by $y = y_1$ or $y = y_2$. The edges of the polytope corresponds to the sets $\{\theta = 0\}, \{\theta = \pi\}, \{y = y_1\}$ and $\{y = y_2\}$. We can then find the vectors that generate the $U(1)$ rotations that degenerate at the different edges. Or in other words, combinations of $\partial_\phi, \partial_\psi$ and ∂_α whose norm vanish at one of these edges. Doing this, we find

$$v_1 = \partial_\phi + \partial_\psi, \quad v_2 = \partial_\psi + \frac{p-q}{2l}\partial_\alpha, \quad v_3 = -\partial_\phi + \partial_\psi, \quad v_4 = \partial_\psi - \frac{p+q}{2l}\partial_\alpha, \tag{5.23}$$

which degenerate at $\theta = 0, y = y_1, \theta = \pi$ and $y = y_2$ respectively. Here, we rescale ∂_α by $1/l$ so that it has a period of 2π.

The vectors $v_1, \ldots, v_4$ are precisely the inward normals of the moment map cone. They are however written in a slightly bad basis, since the orbits of ∂_ϕ and ∂_ψ do not close everywhere (but of course ∂_α, pointing along the fiber, does), and to make closer contact with the usual description we should write things in terms of regular $U(1)$'s. A basis with this property is instead

$$e_1 = \partial_\phi + \partial_\psi, \quad e_2 = -\partial_\phi + \frac{p-q}{2l}\partial_\alpha, \quad e_3 = -\frac{1}{l}\partial_\alpha, \tag{5.24}$$

in which our four vectors take the form

$$\begin{aligned}
v_1 &= e_1 && [1,0,0], \\
v_2 &= e_1 + e_2 && [1,1,0], \\
v_3 &= e_1 + 2e_2 + (p-q)e_3 && [1,2,p-q], \\
v_4 &= e_1 + e_2 + pe_3 && [1,2,p].
\end{aligned} \tag{5.25}$$

In particular we see that the normals written in this basis satisfies the 1-Gorenstein condition, with Gorenstein vector $\xi = [1,0,0]$, which shows that the manifold is SE (or that its cone is CY).

From these normals we can also find the charges Q. Since our vectors live in 3d and we have 4 of them, we only have a single charge vector $Q = [a,b,c,d]$. We can find this by solving $\sum_{j=1}^4 Q^j v_j = 0$ for Q: this gives 3 equations, but up to overall multiplication it fixes the charges for us, and we indeed find $Q = [-p, p+q, -p, p-q]$, as stated above.

5.4 Tri-Sasaki and hyperkähler manifolds

Let us here very briefly mention the last entry in the table of structures given in the introduction of this chapter. They are interesting and special manifolds, but not directly related to the work presented in this thesis.

A hyperkähler manifold is a CY manifold of dimension $4k$, which in addition to the complex structure J has a whole 2-sphere of complex structures with respect to which the metric is Kähler. In particular it admits three distict complex structures I, J, K that satisfy the quaternion algebra, i.e. the relations

$$I^2 = J^2 = K^2 = IJK = -1. \tag{5.26}$$

A tri-Sasaki manifold is a $4k-1$ dimensional manifold whose metric cone is hyperkähler. An interesting example is the seven-sphere.

6. 5d SYM on Sasaki-Einstein manifolds

Having introduced the necessary background, we give an introduction and summary of articles I and II. We first give an overview of how to put 5d $\mathcal{N} = 1$ SYM theories on toric SE manifolds, and how to write the fields in a cohomological complex. Then we sketch the localization computation of the partition function. The same topics are covered in I, II and for example [51, 70, 71].

6.1 Super-Yang-Mills theory on SE manifolds

We consider a compact simply connected 5d SE manifold M, and remember from above that this is spin and admits (at least) two linearly independent Killing spinors. It will be convenient for us to group these two together into a doublet ξ_I, with indices $I, J, \ldots$, taking values in $1, 2$. This is a fundamental $SU(2)$ doublet, where the $SU(2)$ is the R-symmetry of our theory. The $SU(2)$-indices are raised using the antisymmetric symbol ϵ^{IJ}, taken with $\epsilon^{12} = +1$, and lowered with ϵ_{IJ}, $\epsilon_{12} = -1$. We can write the two Killing spinor equations as

$$\nabla_m \xi_I = t_{IJ} \Gamma_m \xi^J, \tag{6.1}$$

where t_{IJ} is an $SU(2)_R$ triplet of scalars, which takes the constant value

$$t_{IJ} = \frac{i}{2r} (\tau_3)_{IJ}, \tag{6.2}$$

where r is a dimensionfull parameter controlling the size of the manifold and τ_3 is the third Pauli matrix. We can choose any linear combination of the Pauli matrices here. The Killing spinor ξ_I satisfies the symplectic Majorana condition, and act as the parameter for our supersymmetry variations. The Reeb vector of the SE manifold is related to these Killing spinors through

$$R^m = -\xi_I \Gamma^m \xi^I, \tag{6.3}$$

which is why the Reeb appears in the square of the supersymmetry. The dual of this we call $\kappa = g(R)$, which is the contact one-form. Now, let us write down the supersymmetry variations for the vector multiplet on this curved background. They are almost the same as for flat space as reviewed in section 3.4, but with added terms involving the new background field t_{IJ}:

$$\begin{aligned}
&\delta A_m = i\xi_I \lambda_m \lambda^I, \\
&\delta\sigma = i\xi_I \lambda^I, \\
&\delta\lambda_I = -\frac{1}{2}(\Gamma^{mn}\xi_I F_{mn} + (\Gamma^m \xi_I) D_m \sigma - \xi^J D_{JI} + 2t_I{}^J \xi_J \sigma, \\
&\delta D_{IJ} = -i\xi_I \Gamma^m D_m \lambda_J + [\sigma, \xi_I \lambda_J] + it_I{}^K \xi_K \lambda_J + (I \leftrightarrow J) \ .
\end{aligned} \tag{6.4}$$

Notice here that since t_{IJ} has a factor of $1/r$, in the flat space limit of $r \to \infty$ the terms involving $t_I{}^J$ will go away and the supersymmetry will go to the flat space version (3.15).

The supersymmetric action for the vector multiplet on this curved background is

$$\begin{aligned}
S_{\text{vec}} = \frac{1}{(g_{YM})^2} \int\limits_M \text{Vol}_M \, \text{Tr}\Big[\frac{1}{2}F_{mn}F^{mn} - D_m\sigma D^m \sigma - \frac{1}{2}D_{IJ}D^{IJ} \\
+ 2\sigma t^{IJ} D_{IJ} - 10 t^{IJ} t_{IJ} \sigma^2 + i\lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] - i\, t^{IJ} \lambda_I \lambda_J\Big] \ ,
\end{aligned} \tag{6.5}$$

which is the flat space action plus extra terms involving t_{IJ}. We can treat the hypermultiplet in a very similar way, the interested reader can find the details in paper I and [71, 51]. In this chapter we will for simplicity go through the calculations for the case of a theory with a vector multiplet and no hypers.

6.2 Cohomological complex

We can use the Killing spinor to make a change of field variables, so that we can write all the fields in terms of differential forms instead of spinors. This makes the structure and geometry of the supersymmetry easier to understand, and it will also help with the localization computation that we will sketch in section 6.3.

To do this, we construct the following forms

$$\begin{aligned}\Psi_m &= \xi_I \Gamma_m \lambda^I, \\ \chi_{mn} &= \xi_I \Gamma_{mn} \lambda^I - (\kappa \wedge \Psi)_{mn}.\end{aligned} \tag{6.6}$$

Ψ is an arbitrary 1-form, while χ is a horizontal 2-form w.r.t. the Reeb, $\iota_R \chi = 0$. It further satisfies

$$\iota_R \star \chi = \chi, \tag{6.7}$$

which can be proved using Fierz identities and properties of the Killing spinor. We call this condition horizontal self-duality. The map (6.6) between forms and spinors is invertible, so there is no loss of information in formulating everything in terms of forms instead of spinors. The inverse map is given by

$$\lambda_I = \Gamma^m \xi_I \Psi_m - \frac{1}{2} \xi^J (\xi_J \Gamma^{mn} \xi_I) \chi_{mn}. \tag{6.8}$$

So this is a good change of variables, and we have not performed a topological twist, which is the most famous example where spinors gets mapped to forms.

In these cohomological variables the supersymmetry variations take the form

$$\begin{aligned}\delta A &= i\Psi, \\ \delta \Psi &= -i\iota_R F + d_A \sigma, \\ \delta \sigma &= -i\iota_R \Psi, \\ \delta \chi &= H, \\ \delta H &= -iL_R^A \chi - [\sigma, \chi].\end{aligned} \tag{6.9}$$

Here, $L_R^A = \iota_R d_A + d_A \iota_R$ is the covariant Lie derivative along the Reeb vector field, and it is clear that $\delta^2 = -iL_R^A + iG_\sigma$, where G_σ denotes a gauge transformation with parameter σ. H is an auxiliary bosonic 2-form, defined from the relation $H = \delta\chi$, that enjoy the same properties as χ. Essentially H contains the degrees of freedom of the auxiliary scalars D_{IJ}.

In terms of these variables we can write the action in the following form:

$$S_{\text{vec}} = \frac{1}{g_{YM}^2} \left[CS_{3,2}(A + \sigma\kappa) + i\mathrm{Tr} \int_M \kappa \wedge d\kappa \wedge \Psi \wedge \Psi \right] + \delta W_{\text{vec}}, \tag{6.10}$$

where

$$
\begin{aligned}
CS_{3,2}(A) &= \mathrm{Tr}\int_M \kappa\wedge F\wedge F, \\
W_{\text{vec}} &= \frac{1}{g_{YM}^2}\mathrm{Tr}\int_M \left[\Psi\wedge\star(-\iota_R F - d_A\sigma) + \chi\wedge\star(2F-\frac{1}{2}H)\right. \\
&\quad \left. +\kappa\wedge d\kappa\wedge(\sigma\chi)\right].
\end{aligned}
\tag{6.11}
$$

This way of writing it makes it easy to evaluate the action on the localization locus, since we know that δ-exact quantities cannot contribute. The non-exact part is the supersymmetrization of $CS_{3,2}$. $CS_{3,2}$ is the 5d lift of 3d Chern-Simons theory using the contact structure, which is more clearly seen if one integrates by parts. One can also think of it as the lift of the topological θ-term $F\wedge F$ in 4d.

6.3 Localization

As explained in chapter 2, to perform the localization computation, we have to add an appropriate localization term, and find the localization locus where it vanishes. Then we have to compute the 1-loop determinant of δ^2 for the fluctuations around this locus.

For the localization term, we choose something that looks almost like W_{vec} in (6.11), but we drop the last term and take

$$
W = \mathrm{Tr}\int_M \left[\Psi\wedge\star(-\iota_R F + d_A\sigma) + \chi\wedge\star(2F-\frac{1}{2}H)\right], \tag{6.12}
$$

and then add $-t\delta W$ to the action as a localizing term. This has the bosonic part

$$
\delta W|_{\text{bos}} = \mathrm{Tr}\int_M \left[\iota_R F\wedge\star\iota_R F - d_A\sigma\wedge\star d_A\sigma - \frac{1}{2}H\wedge\star H + F_H^+\wedge\star H\right], \tag{6.13}
$$

where F_H^+ indicates the horizontally self-dual part of F, which appears since it is the only component of $F\wedge\star H$ that is non-zero.

Looking at both $\delta W|_{\text{bos}}$ and at the action, we can see that we should pick an integration contour where σ is imaginary, so that the $(d_A\sigma)^2$-terms become positive. Formally we take $\sigma\to i\sigma$. Further, H is auxiliary, it has no kinetic term and only appear quadratically, so it can be integrated out and we find

$$\delta W|_{\text{bos}} = \text{Tr} \int_M \left[\iota_R F \wedge \star \iota_R F + d_A \sigma \wedge \star d_A \sigma + F_H^+ \wedge \star F_H^+\right], \qquad (6.14)$$

which is a sum of three positive squares. Thus the localization locus is given by solutions to

$$F_H^+ = 0, \quad \iota_R F = 0, \quad d_A \sigma = 0. \qquad (6.15)$$

The last equation says that σ is covariantly constant, and the first two equations are known as the contact instanton equation [70]. They can equivalently be be written as

$$\star F = -\kappa \wedge F, \qquad (6.16)$$

i.e. that F is horizontally anti-self dual. This can be thought of as a 5d lift of the 4d anti-self-dual instanton equation, $\star F = -F$, using the contact structure. The contact instanton equation is closely related to this more famous 4d equation, but not as much is known about it.

One can also consider the self-dual contact instanton equation, $\star F = \kappa \wedge F$, but it is in fact not as natural, since the anti-self-dual equation implies the Yang-Mills equation, $d_A \star F = 0$, while the self-dual equation does not. This is because $d\kappa$ is horizontally self-dual, so that $d\kappa \wedge F_H^- = 0$, which implies that if F satisfies the anti-self dual equations we have

$$d_A \star F = -d_A(\kappa \wedge F) = -d\kappa \wedge F = 0. \qquad (6.17)$$

This is different from in 4d, where both instantons and anti-instantons automatically satisfies the vacuum Yang-Mills equation.

The system of equations (6.15) is not elliptic, which means that studying it is a slightly unconventional business. There is however a lift of this into another set of equations, called the Haydys-Witten equation which is a good elliptic system. These equations were first proposed by Witten [72] in an attempt to understand Khovanov knot homology from a field theory point of view. They were also independently constructed by Haydys [73]. These equations can be understood from a 5d field theory perspective as coming from the localization locus of a twisted $\mathcal{N} = 2$ theory [74].

To continue our localization computation, we should classify all the solutions to (6.15), which however proves to be a difficult problem to fully solve. On S^4, Pestun [19] was able to prove that for his locus, the only smooth solution was $F = 0$ and σ covariantly constant, and then he argued that on the north and south pole, there was also point-like instantons and anti-instantons. For our case, we can show that the only smooth solutions are $F = 0$ and σ covariantly constant, but we have no proper argument for why the only allowed singular solutions are those of localized contact instantons sitting at the closed Reeb orbits. There are however some energy arguments about why this should be the case, and it is in line with general conjectures on instantons on toric backgrounds [75], so we conjecture that it is the case. The localized contact instantons will contribute to the non-perturbative part of the partition function; giving one factor of the appropriate Nekrasov partition function on $\mathbb{R}^4 \times S^1$ per closed orbit.

On the locus of smooth solutions, which up to gauge fixing is given by $A = 0$ and $\sigma = ia$ constant, we evaluate the action and find the classical action

$$\begin{aligned} S_{\text{vec}}(a) = \frac{1}{g_{YM}^2} CS_{3,2}(\kappa(ia)) = \frac{1}{g_{YM}^2} \text{Tr} \int_M \kappa \wedge d\kappa \wedge d\kappa (ia)^2 \\ = -\frac{8\text{Vol}(M)}{g_{YM}^2} \text{Tr}[a^2], \end{aligned} \tag{6.18}$$

remembering that the volume form on M is taken to be $\frac{1}{8}\kappa \wedge d\kappa \wedge d\kappa$. This will act as the Gaussian damping factor for the matrix model that we get from the localization procedure, where the integral is over all matrices in $\mathfrak{g}$.

6.3.1 1-loop determinant

Having found the localization locus, the next step is to compute the linearized determinant of $\delta^2 = -iL_R^A + G_{ia}$ around it. As explained in section 2.3, when applying this procedure to a gauge theory one should deal properly with the issues of gauge fixing and zero modes. However here we will not be perfectly rigorous, and take a number of shortcuts.

Looking at the cohomological complex (6.9), we observe that with the further change of variables $\Phi = \iota_R A + \sigma$, which is the true gauge parameter, the complex takes the simple form

$$\begin{aligned} &\delta X = X', \qquad \delta X' = (-iL_R + iG_\Phi)X, \\ &\delta\Phi = 0, \end{aligned} \tag{6.19}$$

where we use the collective notation for our fields, $X = \{A, \chi\}$ and $X' = \{\Psi, H\}$. We should now think of X as the coordinates on the infinite dimensional supermanifold of fields, and X' as the corresponding 1-forms. The gauge connection A is an even coordinate and χ is an odd coordinate. The Atiyah-Bott localization formula should be understood for the supermanifold setting, as explained in section 2.2.1, which means that the determinant becomes a superdeterminant.

We will not treat the issues of gauge fixing and zero modes in full detail. There is a rigorous procedure for this, as performed by Pestun on S^4 [19], which involves the normal BRST procedure, in which one introduces ghosts fields $c, \bar{c}, b$, as well as introducing further fields $a, c_0, \bar{c}_0, b_0$ that deal with zero modes. Here, we will only say that we use the gauge freedom to fix Φ to the value $\Phi = a$, which on the locus with $A = 0$ is constant across the manifold. This incurs a Fadeev-Popov determinant from the ghost fields.

For the vector multiplet after adding the ghosts, the supermanifold parametrized by the coordinates X is

$$\mathcal{V} = \mathcal{A}(M, \mathfrak{g}) \oplus \Pi\Omega^{2;+}_H(M, \mathfrak{g}) \oplus \Pi\Omega^0(M, \mathfrak{g}) \oplus \Pi\Omega^0(M, \mathfrak{g}), \tag{6.20}$$

where the first factor is the space of gauge connections, the second factor is from the fermionic 2-form χ, which is horizontally self-dual, and the last two factors are from the ghost fields used for gauge fixing. We write Π in front to show when something has the odd grading, i.e. when the field is fermionic. The entire space of fields that we integrate over is the odd tangent bundle, $\Pi T\mathcal{V}$, where the fields of X' are the coordinates on the tangent bundle part.

Since we are computing the determinant around the locus of $A = 0$ and $\sigma = \text{constant}$, we should linearize our description of $\Pi T\mathcal{V}$ around it. The only non-affine part of $\mathcal{V}$ is the space of connections, which after linearizing around $A = 0$ is locally described by the space of 1-forms $\Omega^1(M, \mathfrak{g})$. So the linearization of $\mathcal{V}$ around the locus is

$$v = \Omega^1(M, \mathfrak{g}) \oplus \Pi\Omega^{2;+}_H(M, \mathfrak{g}) \oplus \Pi\Omega^0(M, \mathfrak{g}) \oplus \Pi\Omega^0(M, \mathfrak{g}), \tag{6.21}$$

and the determinant we should compute is

$$\frac{1}{\sqrt{\mathrm{sdet}_v(-iL_R+iG_a)}}=\sqrt{\frac{\det_{\Omega_H^{2,+}}(-iL_R+iG_a)(\det_{\Omega^0}(-iL_R+iG_a))^2}{\det_{\Omega^1}(-iL_R+iG_a)}}. \tag{6.22}$$

Using the contact structure that our SE manifold has, we can decompose these spaces into their horizontal and vertical parts, and then further use the complex structure to decompose the horizontal parts,

$$\Omega^1=\Omega^1_V\oplus\Omega^1_H=\Omega^1_V\oplus\Omega_H^{(1,0)}\oplus\Omega_H^{(0,1)}, \tag{6.23}$$

$$\Omega_H^{2,+}=\Omega^2_{d\kappa}\oplus\Omega_H^{(2,0)}\oplus\Omega_H^{(0,2)}, \tag{6.24}$$

where $\Omega^2_{d\kappa}$ denotes the part of the horizontal 2-form along $d\kappa$. From the properties of the Reeb, it is easy to see that the operator whose determinant we are computing, $L(a)=-iL_R+iG_a$, respects this decomposition, meaning that we can write the determinant as

$$\frac{1}{\mathrm{sdet}_v L(a)}=\frac{\det_{\Omega_H^{(2,0)}}(L(a))\det_{\Omega_H^{(0,2)}}(L(a))\big(\det_{\Omega^{(0,0)}}(L(a))\big)^3}{\det_{\Omega_H^{(1,0)}}(L(a))\det_{\Omega_H^{(0,1)}}(L(a))\det_{\Omega^{(0,0)}}(L(a))}$$

$$=\left(\frac{\det_{\Omega^{(0,0)}}(L(a))\det_{\Omega_H^{(2,0)}}(L(a))}{\det_{\Omega_H^{(1,0)}}(L(a))}\right)\left(\frac{\det_{\Omega^{(0,0)}}(L(a))\det_{\Omega_H^{(0,2)}}(L(a))}{\det_{\Omega_H^{(0,1)}}(L(a))}\right),$$

where we have used that since the spaces $\Omega^1_V,\Omega^2_{d\kappa}$ can be described as a function multiplying a fixed differential form, they are both isomorphic to the space of 0-forms $\Omega^{(0,0)}$. We thus find the complex of holomorphic and anti-holomorphic forms, which are isomorphic, so up to a phase we only need to compute the superdeterminant of the complex $\Omega_H^{(0,\bullet)}$. That is

$$\frac{1}{\sqrt{\mathrm{sdet}_v(L(a))}}=\left|\frac{\det_{\Omega^{(0,0)}}(L(a))\det_{\Omega_H^{(2,0)}}(L(a))}{\det_{\Omega_H^{(1,0)}}(L(a))}\right|. \tag{6.25}$$

These spaces of differential forms are infinite dimensional, but the presence of supersymmetry guarantees that there will be massive cancellations in the above super determinant. The issue is how to understand and classify these cancellations, and for this purpose we remember from section 5.1 that the complex $\Omega^{(0,\bullet)}$ has a differential, namely the basic Dolbeault operator $\bar\partial_H$. We will next argue that only the modes in the associated Dolbeault cohomology, $H^{(0,\bullet)}_{\bar\partial_H}$, called the Kohn-Rossi cohomology, will contribute: the contributions of all other modes will cancel.

This is seen as follows. For any element in $\Omega^{0,p}$ that is not closed, the differential of this gives a mode in $\Omega^{0,p+1}$. In the toric case that we consider, it is also clear that $\bar{\partial}_H$ and $L(a)$ commute: meaning that the mode α and the mode $\bar{\partial}_H\alpha$ have the same eigenvalue under $L(a)$. Therefore, all non-closed modes will cancel when we perform the super-determinant. Similarly any exact mode can be written as $\bar{\partial}_H\beta$, and its contribution is cancelled by the mode β. So the only modes that are not cancelled is exactly the modes in the Kohn-Rossi cohomology, and what we need to compute is

$$\frac{\det_{H^{0,0}_{\bar{\partial}}}(L(a))\det_{H^{0,2}_{\bar{\partial}}}(L(a))}{\det_{H^{(0,1)}_{\bar{\partial}}}(L(a))}. \tag{6.26}$$

This can be computed using various methods, including using an index theorem for transversally elliptic operators or some heat kernel methods etc., but to our knowledge the simplest way was introduced by Schmude [76]. He pointed out that for the toric SE manifolds we are considering, there is a very explicit description of the modes in the various cohomologies in terms of the toric data. To see how this works, first let α be some representative in $H^{(0,\bullet)}_{\bar{\partial}}$. We can then assume that it has a charge vector $\vec{q}$ under the $U(1)^3$ action on M. When we then write the Reeb as a linear combination of the three $U(1)$ actions, $R = \sum_{a=1}^{3} R^a e_a$, we have

$$L_R\alpha = i(\vec{R}\cdot\vec{q})\alpha, \tag{6.27}$$

where $\vec{R}$ means the three components (R^1, R^2, R^3).

Let t be the coordinate along the cone direction of $C(M)$. Then the Dolbeault differential on the cone is related to $\bar{\partial}_H$ as

$$\bar{\partial}^6 = \frac{1}{2}(t^{-1}dt - i\kappa)(L_{t\partial_t} + iL_R) - \frac{i}{2}d\kappa\iota_{t\partial_t} + \bar{\partial}_H. \tag{6.28}$$

We can now extend the representative α to a form on $C(M)$ by taking $\tilde{\alpha} = t^{\vec{R}\cdot\vec{q}}\alpha$. Plugging this into the expression for $\bar{\partial}^6$ one can see that it is killed, and hence $\bar{\partial}^6$-closed. Similarly an exact form is mapped to a $\bar{\partial}^6$-exact closed by the same map, so it gives us a good map between the cohomologies,

$$H^{(0,\bullet)}_{\bar{\partial}_H}(M) \to H^{(0,\bullet)}_{\bar{\partial}^6}. \tag{6.29}$$

Note also that this map is the identity on $t = 1$, which is where we think of M as being embedded into $C(M)$. This shows that the restriction map that restricts a form to the surface $t = 1$, is onto. Since $C(M)$ is CY, we know that $H^{(0,1)}_{\bar\partial^6}(C(M)) = 0$, so from that the restriction is onto it follows that $H^{(0,1)}_{\bar\partial}(M) = 0$ as well. Next, we consider the functions $H^{0,0}_{\bar\partial}$. The elements in $H^{(0,0)}_{\bar\partial^6}(C(M))$ are the holomorphic functions on $C(M)$, so the calculation becomes the counting of holomorphic functions on $C(M)$ weighted by their $U(1)^3$ charges. This is well known problem considered in the context of AdS/CFT on $AdS \times M$, where the holomorphic functions correspond to supersymmetric operators in the chiral ring [77, 78]. The monomials that generate the holomorphic functions on $C(M)$ are in one-to-one correspondence with the integer lattice points within the moment map cone $C_\mu(M) = \mu(C(M))$. And we can read of the charges under $U(1)^3$ from the coordinates of the lattice point that represent the monomial. Explicitly, a lattice point in $C_\mu(M)$ with coordinates (n_1, n_2, n_3) (in an appropriate basis of the U(1) actions) represents the monomial $z_1^{n_1} z_2^{n_2} z_3^{n_3}$. Each coordinate z_i has the eigenvalue R_i under the Reeb; which is just the component of the Reeb vector written in this basis of $U(1)$'s. Then the eigenvalue of the monomial is given by $n_1 R_1 + n_2 R_2 + n_3 R_3 = \vec{n} \cdot R$.

We can also use this explicit map to realize that the restriction map is injective, since two different monomials with difference charges cannot cancel each other when restricted to $t = 1$.

This gives us the determinants over the 0-forms, but we also need to consider $H^{(0,2)}_{\bar\partial^6}$. This can actually be mapped to $H^{0,0}$, in the following way. Let $\bar\rho \in \Omega^{(0,2)}_H$, satisfying $\partial_H \rho = 0$. Since the rank of $\Omega^{0,2}_H$ is one, once we've picked a representative $\bar\rho$, any form in it can be written as $\bar f \bar\rho$ for some function ρ, and to find $H^{0,2}_{\bar\partial_H}$ we only need to remove the exact ones. If the form $\bar f \bar\rho$ is co-exact with respect to $\bar\partial_H$, that is $\bar\partial^\dagger_H(\bar f \bar\rho) = 0$, then $\bar f \bar\rho$ is orthogonal to all exact $(0, 2)$ forms, which means that this condition is a way of removing all exact forms. But we can see that

$$\bar\partial^\dagger_H(\bar f \bar\rho) = -g^{pq}(\partial_p \bar f)\bar\rho_{qr} dx^r, \tag{6.30}$$

which is zero if and and only if $\partial_H \bar f = 0$. Hence $\bar f$ is the complex conjugate of some holomorphic function $f \in H^{(0,0)}_{\bar\partial_H}$, and we have constructed the isomorphism between the holomorphic functions and the (0,2) forms given by $f \mapsto \bar f \bar\rho$.

The eigenvalue under L_R of the form $\bar{f}\bar{\rho}$ is given by minus the eigenvalue of f (because of the complex conjugation) plus the charge of $\bar{\rho}$. The form $\bar{\rho}$ is taken to be the reduction of the unique $(0,3)$-form on the CY cone over M; and its eigenvalue is given by the Gorenstein vector ξ that satisfies $\xi \cdot v_i = 1\ \forall i$, where v_i are the inwards normals of the moment map cone $C = C_\mu(M)$. This means that the eigenvalues of the (0,2) form modes are given by $-(\vec{m}+\xi)\cdot R$ for $\vec{m} \in C\cap\mathbb{Z}^3$. Because of the property $\xi \cdot v_i = 1$, shifting by ξ precisely excludes all the lattice points on the faces of C, meaning that one can also write this without the shift as $-\vec{m}\cdot R$ for $\vec{m} \in C^\circ \cap \mathbb{Z}^3$, where C° is the interior of the cone. This description is valid also for more general cones where the 1-Gorenstein condition is not fulfilled and ξ does not exist.

The determinant we are computing is thus given as the product over all the lattice points inside the moment map cone of our manifold and we find the following infinite product:

$$\prod_{\vec{n}\in C\cap\mathbb{Z}^3} (a+\vec{n}\cdot R)(\xi\cdot R - a + \vec{n}\cdot R) \equiv S_3^C(a|R), \tag{6.31}$$

where we define this to be the generalized triple sine function S_3^C associated to the cone C. This function is a generalization of the usual triple sine [79], which is the 1-loop determinant one finds on the squashed S^5, where $C = \mathbb{R}^3_{\geq 0}$. The infinite product as written is obviously divergent, and needs to be understood in a proper, zeta-regularized way. We will give an overview of zeta regularization and the definition and properties of this function in chapter 7, and the details can be found in articles III and V.

Here the determinant has only been taken over the 'spatial' part, L_R, and we are sloppily treating the Lie algebra element a as just a number. To be more specific, we can decompose the Lie algebra into its root spaces

$$\mathfrak{g} = \bigoplus_\beta \mathfrak{g}_\beta, \tag{6.32}$$

where β runs over all the roots of $\mathfrak{g}$. Then the eigenvalue of some $a \in \mathfrak{g}$ on $\mathfrak{g}_\beta$ is given by $i\langle a,\beta\rangle$, and we can write the determinant over the Lie algebra as

$$\det{}_{\mathrm{adj}} S_3^C(a|\vec{R}) = \prod_\beta S_3^C(i\langle a,\beta\rangle|\vec{R}). \tag{6.33}$$

In the above, we have also not been careful with the zero modes. The scalars or 0-forms have zero modes (the constant functions), and those needs to be excluded. Keeping track of these more carefully (see [71]) gives that the determinant for the vector multiplet is given by

$$\frac{1}{\sqrt{\det_{\mathcal{V}}(L(a))}} = \frac{\det'_{\text{adj}} S_3^C(a|\vec{R})}{\det'_{\text{adj}}(-G_a)} \tag{6.34}$$

where the $'$ means that we exclude zero modes of the determinant over the adjoint, and the denominator cancels the contributions from the zero modes of the constant functions. We then note that $\det'_{\text{adj}}(-G_a)$ is the usual Vandermonde determinant that appears when changing from a matrix integral over $\mathfrak{g}$ to its Cartan subalgebra $\mathfrak{t}$, so by performing this change we can precisely cancel this factor. The perturbative answer for a single vector multiplet on a toric SE manifold M is therefore

$$Z_{\text{pert}} = \int_{\mathfrak{t}} da\, e^{-\frac{8\text{Vol}(M)}{g_{YM}^2}\text{Tr}[a^2]} \det'_{\text{adj}}\, S_3^C(ia|\mathrm{R}). \tag{6.35}$$

One can repeat the above story almost word for word for the computation of the 1-loop determinant of a hypermultiplet. The main difference is in how the map between spinors and forms go: for the hyper we find bosonic 2-forms and functions, and a fermionic 1-form, which leads to that its contribution is that of $(S_3^C)^{-1}$. The mass of the hyper appear as a shift of its eigenvalue. The details of this can be found in I or in [71, 51], and the perturbative answer when including a hypermultiplet in representation R is given by

$$Z_{\text{pert}} = \int_{\mathfrak{t}} da\, e^{-\frac{8\text{Vol}(M)}{g_{YM}^2}\text{Tr}[a^2]} \frac{\det'_{\text{adj}}\, S_3^C(ia|\mathrm{R})}{\det_R\, S_3^C(i(a+m)+\xi\cdot \mathrm{R}/2|\mathrm{R})}. \tag{6.36}$$

6.4 Factorization

In addition to deriving the perturbative answer on any toric SE manifold, in paper I we also prove a factorization property of the perturbative partition function. Specifically we show that the generalized triple sine function can be written as a product of a number of copies of another special function called the multiple q-factorial, that is introduced in chapter 7. The factorization takes the form

$$S_3^C(z|\vec{R}) = e^{-\frac{\pi i}{6}B_{3,3}^C(z|\vec{R})} \prod_f \left(e^{2\pi i\beta_f z}|e^{2\pi i\beta_f \epsilon_f^1}, e^{2\pi i\beta_f\epsilon_f^2}\right)_\infty, \tag{6.37}$$

where the product runs over the closed orbits of the Reeb, or equivalently the vertices of the moment map polytope of M. The multiple q-factorial is the 1-loop determinant for a vector multiplet on the background $\mathbb{C}^2 \times_{\epsilon^1,\epsilon^2} S^1_\beta$. Here β is the radius of the S^1, which is non-trivially fibered over $\mathbb{C}^2$ as described by the following identification

$$[\theta + 2\pi, z_1, z_2] \simeq [\theta, e^{2\pi i\beta\epsilon^1} z_1, e^{2\pi i\beta\epsilon^2} z_2]. \tag{6.38}$$

This is a 5d version of the Ω-background used by Nekrasov in the context of instanton counting [22, 23]. In the factorization formula (6.37), the parameters β_f, ϵ^i_f are dictated by the local geometry at the closed orbit, and they can be read off from the toric data, i.e. from the cone C. That is, the factorization tells us that the 1-loop partition function splits into a product of flat space partition functions, one from each neighborhood of the different closed Reeb orbits, with parameters describing how the local geometry of M looks at the closed orbit. The exact map between the parameters and the toric data is described in article I.

This structure of the partition function has previously been observed in many different cases, including for 2d theories [80], 3d theories on S^3 and $S^2 \times S^1$ [81, 82] and of course by Pestun for S^4 [19]. Geometrically one can understand this as a decomposition of the manifold into a number of flat space patches of the form $\mathbb{R}^{2n}$ or $\mathbb{R}^{2n} \times S^1$, that the partition function respects. The building block from one patch is called a holomorphic block, and they are interesting mathematical objects. For example, holomorphic blocks can be understood from the perspective of the AGT correspondence, where they are related to particular CFT correlators, see [83] for a review.

6.5 Instanton contributions

As mentioned above, we conjecture that the only instanton solutions that will contribute to the partition function will be those localized around the closed Reeb orbits. This is in line with what was observed on S^4 by Pestun [19], who found point-like instantons living at the two fixed points of his torus action; and it is also what Nekrasov has conjectured in general [75]. The partition function contribution from a point-like instanton sitting at the origin of $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_\beta$ is given by the 5d Nekrasov instanton partition function [23, 84]. So we add one such contribution for each closed orbit with the appropriate parameters and conjecture that the full answer on any toric SE manifols is

$$Z^{\text{full}} = \int_{\mathfrak{t}} da e^{-\frac{8\text{Vol}(M)}{g_{YM}^2}\text{Tr}[a^2]} \frac{\det'_{\text{adj}} S_3^C(ia|\vec{R})}{\det_R S_3^C(i(a+m)+\vec{\xi}\cdot\vec{R}/2|\vec{R})} \times \prod_f Z_{\text{inst}}^{\mathbb{C}^2\times S^1}(a,m|\beta_f,\epsilon_f^1,\epsilon_f^2). \tag{6.39}$$

The product again runs over closed Reeb orbits. It is more natural to write all of this in the factorized form,

$$Z^{\text{full}} = \int_{\mathfrak{t}} da \prod_f \left[e^{-\frac{1}{g_{YM}^2}P(a,m|\beta_f,\epsilon_f^i)} Z_{\text{Nek}}^{\mathbb{C}^2\times S^1}(a,m|\beta_f,\epsilon_f^i)\right], \tag{6.40}$$

where $Z_{\text{Nek}}^{\mathbb{C}^2\times S^1}$ is the full Nekrasov partition function on this background, including the perturbative and instanton part. The function $P(a,m|\beta_f,\epsilon_f^i)$ is the effective potential of the local flat space theory on this background.

So the entire answer is written in a factorized form, where each factor is a flat space answer. One can think of this as a gluing of patches of $\mathbb{C}^2\times S^1$, where the parameter a is fixing the boundary conditions at infinity for each patch. In article II we further investigated the full answer for abelian theories, and we discuss this next.

6.5.1 Abelian instantons

For gauge group $U(N)$, the Nekrasov instanton partition function takes the form of a sum over a N-tuplet of 2d partitions (Young diagrams). This representation is not particularly helpful when studying what happens when you glue a number of them together, as we conjectured above. However for the case of a $U(1)$, $\mathcal{N}=2^*$ theory (i.e. a theory with a vector multiplet and a hyper in the adjoint with mass m), Carlsson, Nekrasov and Okounkov were able to perform the sum explicitly over partitions [85], and arrive at a closed-form expression for the instanton partition function, writing it as an infinite product,

$$
\begin{aligned}
&Z_{\text{inst}}^{\mathbb{C}^2\times S^1}(m,\beta,\epsilon_1,\epsilon_2) = \\
&\prod_{j,k,l=1}^{\infty} \frac{(1-e^{2\pi i\beta((j+1)\epsilon_1+k\epsilon_2+k+1)})(1-e^{2\pi i\beta(j\epsilon_1+(k+1)\epsilon_2+k+1)})}{(1-e^{2\pi i\beta(m+j\epsilon_1+k\epsilon_2+k+1)})(1-e^{2\pi i\beta(-m+(j+1)\epsilon_1+(k+1)\epsilon_2+k+1)})} \\
&= \frac{(q_1Q|q_1,q_2,Q)_\infty(q_2Q|q_1,q_2,Q)_\infty}{(MQ|q_1,q_2,Q)_\infty(M^{-1}Qq_1q_2|q_1,q_2,Q)_\infty},
\end{aligned}
\tag{6.41}
$$

where in the last line we use the exponentiated variables, $q_i = e^{2\pi i\beta\epsilon_i}$, $Q = e^{2\pi i\beta}$ and $M = e^{2\pi i\beta m}$. This expression is helpful when investigating what happens under gluing, and in the paper II we show that using this formula we can combine the perturbative answer and the conjectured factorized instanton contributions in a natural way. This was previously studied by Lockhart and Vafa [86], who considered abelian $\mathcal{N}=2^*$ theory on S^5. For a $U(1)$ theory, there is no integration at all, and on S^5, one finds that the full answer have the simple form

$$
Z^{\text{full}}_{U(1)}(m,\mathrm{R}) = \frac{G_2'(0|\mathrm{R})}{G_2(im|\mathrm{R})},
\tag{6.42}
$$

where G_2 is the double elliptic gamma function. This is defined as

$$
\begin{aligned}
G_2(z|\underline{\omega}) &= \prod_{n\in\mathbb{Z}^3_{\geq 0}} (1-e^{2\pi i(z+n\cdot\underline{\omega})})(1-e^{2\pi i(\omega_1+\omega_2+\omega_3-z+n\cdot\underline{\omega})}) \\
&= (e^{2\pi iz}|e^{2\pi i\omega_1},e^{2\pi i\omega_2},e^{2\pi i\omega_2})_\infty(e^{2\pi i(\omega_1+\omega_2+\omega_3-z)}|e^{2\pi i\omega_1},e^{2\pi i\omega_1},e^{2\pi i\omega_1})_\infty,
\end{aligned}
\tag{6.43}
$$

and this special function is closely related to the triple sine function, see chapter 7. This has a single zero at $z=0$, so in equation (6.42), the $'$ indicates that we remove this zero mode.

The G_2 function enjoys a modular property which relates its values at points related by $SL_3(\mathbb{Z})$ transformations, taking the form

$$
G_2(z|\underline{\omega}) = e^{\frac{2\pi i}{4!}B_{4,4}(z|\underline{\omega},-1)}\prod_{i=1}^{3} G_2(z_i|\underline{\omega}_i),
\tag{6.44}
$$

where $z_i = z/\omega_i$ and $\underline{\omega}_i = (\frac{\omega_j}{\omega_i},\frac{\omega_k}{\omega_i},-\frac{1}{\omega_i})$ for $j\neq k\neq i$. From a physics perspective this exactly the statement of factorization: the flat space full partition function for a $U(1)$ theory is also given by a G_2 function.

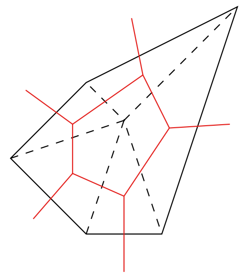

Figure 6.1. A sketch illustrating the triangulation of the polytope and the dual diagram in red. For each triangle (or outgoing leg) we associate a G_2 function, for each shared face (or internal line) we associate a G_1 and for each internal vertex (each loop) we put a G_0.

So there is something interesting going on here, where the S^5 partition function is related to the flat space one with just a rescaling of the variables.

The story can also be generalized for any toric SE manifold: one defines the generalized double elliptic gamma function G_2^C associated to a cone C as

$$G_2^C(z|\underline{\omega}) = \prod_{n \in C \cap \mathbb{Z}^3_{\geq 0}} (1 - e^{2\pi i (z + n \cdot \underline{\omega})})(1 - e^{2\pi i (\omega_1 + \omega_2 + \omega_3 - z + n \cdot \underline{\omega})}), \quad (6.45)$$

and then the full answer is the same as (6.42) but with G_2^C instead of G_2. The generalized double elliptic gamma function also enjoy a modular or factorization property, factorizing into one copy of the usual G_2 function for each closed Reeb orbit,

$$G_2^C(z|\underline{\omega}) = e^{\frac{2\pi i}{4!} B^C_{4,4}(z|\underline{\omega}, -1)} \prod_f G_2(\beta_f z | \beta_f \epsilon^1_f, \beta_f \epsilon^2, -\beta_f), \quad (6.46)$$

where $B^C_{4,4}$ is a generalized Bernoulli polynomial that encode the geometry of the cone. This again corresponds to one flat space contribution from each closed orbit, and gives evidence for our conjecture about only localized instantons contributing, at least for the case of an abelian theory.

The other interesting thing observed in article II is that of some gluing rules that the full abelian partition function satisfy. On the level of the cone, we can subdivide the cone into a number of simplicial cones, or equivalently, we triangulate the polytope base of the cone. Then for each triangle we associate a (normal) G_2 function with parameters determined by the geometry of the triangle. For each shared face between two triangles we associate a G_1 function, and for each internal point where multiple triangles meet, we associate a G_0 function. The product of all of these then gives us the generalized G_2^C function that builds up our partition function.

On the level of the special functions, this is just an observation of how to compensate for the various over and under countings of modes when subdividing the cone. But it might be interesting from a physic perspective. The G_1 function can be thought of as the index of some theory on $S^3 \times S^1$, and G_0 looks like the index of a on $S^1 \times S^1$. And the decomposition correspond geometrically to gluing together the 5d manifold out of $\mathbb{R}^4 \times S^1$ patches, and where their boundaries meet, we find what looks like the degrees of freedom of a 4d theory on $S^3 \times S^1$ (which is the topology of the boundary), and where there is overlap of the form $S^1 \times S^1 = T^2$ we find the degrees of freedom of a 2d theory on this space.

This is a different gluing story than that of the factorization described in 6.4, and seems closely related to localization results in the presence of defects [87, 88]. This is an interesting observation hinting at a deeper structure relating the partition function, defects and the cutting and gluing of manifolds, and is something that should be investigated further. It would for example be interesting to see if a similar structure can be found for partition functions in other dimension, and for non-abelian theories.

7. Special functions

In this chapter, we provide some background and brief overview of the results of articles III and V. They are about two new hierarchies of special functions, the generalized multiple sine and multiple elliptic gamma functions that was briefly introduced in the last chapter. The "usual" elliptic gamma function is an elliptic generalization of the Euler gamma function, which was introduced by Ruijsenaar [89], and studied by Felder and Varchenko [90]. This is a meromorphic function Γ of 3 variables (z, τ, σ) which is symmetric in τ and σ, and characterized by a difference equation involving the Jacobi theta function,

$$\Gamma(z+\sigma|\tau,\sigma) = \theta_0(z|\tau)\Gamma(z|\tau,\sigma). \tag{7.1}$$

Felder and Varchenko proved several identities that this function satisfies, most interestingly a modular three-term relation, relating its values at points related by $SL_3(\mathbb{Z})$ transformations, acting as a fractional linear transformation on the periods (τ, σ). These identities can be interpreted as a generalization of the behavior of modular forms.

The elliptic gamma function fits naturally into a hierarchy of multiple elliptic gamma functions as first defined by Nishizawa [91], and these are closely related to the hierarchy of multiple sine functions [79]. Both these hierarchies were studied by Narukawa [92] who gave integral representations of them and proved that they in general satisfy an interesting factorization property, which for the multiple elliptic gamma function can be thought of as a kind of generalized modular property.

From the computations in articles I and II, we found it natural to define the two new special functions that we named the generalized triple sine and the generalized elliptic gamma function. These mimic the original definitions closely, but the functions are now associated to a non-compact toric manifold, or equivalently a rational cone of appropriate dimension. The original functions are recovered when the cones are taken to be the positive quadrant of $\mathbb{R}^n$, $\mathbb{R}^n_{\geq 0}$. In article III we properly define these new hierarchies and write down their integral representations. We then prove that these generalized functions also satisfy interesting factorization or modularity properties, labelled by the choice of cone.

7.1 Q-factorials, multiple sine functions and elliptic gamma functions

Here we introduce a number of special functions, starting with the q-Pochhammer symbol, also called a q-shifted factorial, which is defined as

$$(x|q)_n = \prod_{k=0}^{n-1}(1 - xq^k), \quad (x|q)_0 = 1. \tag{7.2}$$

It can be extended to an infinite product,

$$(x|q)_\infty = \prod_{k=0}^{\infty}(1 - xq^k), \tag{7.3}$$

which is an analytic function in x and q as long as $|q| < 1$, and that can be extended to $|q| > 1$ by defining it to be

$$(x|q)_\infty = \prod_{k=0}^{\infty}(1 - xq^{-(k+1)})^{-1}, \tag{7.4}$$

in this case. This function is important in many areas of mathematics. It is one of the major building blocks when construction of q-analogs, in the combinatorics of counting partitions, and it shows up when studying the geometric Langlands. This function also comes up in a variety of physics contexts, and was introduced to physicists as the quantum dilogarithm by Faddeev and Kashaev [93], who used it when studying 2d quantum CFTs and solvable lattice models in 2 and 3 dimensions. They proved that it satisfies an interesting pentagon identity. For us, this function is relevant because the 1-loop determinant for a supersymmetric gauge theory in 3d on the background $\mathbb{R}^2 \times_q S^1$ can be written in terms of q-shifted factorials, $(x|q)_\infty^{\pm 1}$.

One can generalize this and define the multiple q-shifted factorial, which depends on more variables $q_1, \ldots, q_k$. For $|q_1|, \ldots, |q_k| < 1$ this is defined as

$$(x|q_1,\ldots,q_k)_\infty = \prod_{i_1,\ldots,i_k=0}^{\infty}(1-xq_1^{i_1}\cdots q_k^{i_k}). \tag{7.5}$$

Just as for the single variable case, the definition can be extended for when say $|q_1|,|q_2|,\ldots,|q_j|>1$, and the rest $|q_{j+1}|,\ldots,|q_k|<1$, as

$$(x|q_1,\ldots,q_k)_\infty = \prod_{i_1,\ldots,i_k=0}^{\infty}(1-xq_1^{-(i_1+1)}\cdots q_j^{-(i_j+1)}q_{j+1}^{i_{j+1}}\cdots q_k^{i_k})^{(-1)^j},$$

but the function is not defined when any $|q_j|=1$. As we have seen in chapter 6, the multiple q-factorial with two q's shows up as the building block of the 1-loop determinant for a 5d theory on $\mathbb{R}^4\times S^1$.

Let us next discuss the idea of zeta function regularization. When we write infinite products that are divergent, they have to be understood in a proper way, which is through zeta function regularization, a technique that lets us assign to them a finite value, using some zeta function. We now explain this construction in the setting that we need to define the multiple sine functions. For $\omega_1,\ldots,\omega_r,z\in\mathbb{C}$ such that they are all in the same half-plane of $\mathbb{C}$, for example that their imaginary parts are all greater than zero, the multiple Hurwitz zeta function is defined as [94]

$$\zeta_r(s,z|\underline{\omega}) = \sum_{n_1,\ldots,n_r=0}^{\infty}(z+n\cdot\underline{\omega})^{-s}, \tag{7.6}$$

for $\mathbb{R}(s)>r$ and where $n\cdot\underline{\omega}=n_1\omega_1+\cdots+n_r\omega_r$. This can be analytically continued for all $s\in\mathbb{C}$ as a meromorphic function and is holomorphic at $s=0$. Then the multiple gamma function Γ_r is defined as

$$\Gamma_r(z|\underline{\omega}) = \exp\left[\frac{\partial}{\partial s}\zeta_r(s,z|\underline{\omega})|_{s=0}\right]. \tag{7.7}$$

This is the Barnes multiple gamma function up to a multiplicative constant. We think of Γ_r as the regularized version of a divergent infinite product and write

$$\prod_{n_1,\ldots,n_r=0}^{\infty}(z+n\cdot\underline{\omega}) = \Gamma_r(z|\underline{\omega})^{-1}. \tag{7.8}$$

Then we define the multiple sine functions S_r as

$$S_r(z|\underline{\omega}) = \Gamma_r(z|\underline{\omega})^{-1}\Gamma_r(|\underline{\omega}|-z|\underline{\omega})^{(-1)^{r-1}}, \tag{7.9}$$

where $|\underline{\omega}| = \omega_1 + \omega_2 + \ldots + \omega_r$. Physicists will often write the infinite product instead, defining S_r as

$$S_r(z|\underline{\omega}) = \prod_{n_1,\ldots,n_r=0}^{\infty} (z + n \cdot \underline{\omega})(|\underline{\omega}| - z + n \cdot \underline{\omega})^{(-1)^{r-1}}, \tag{7.10}$$

leaving the zeta regularization implicit.

The multiple sine functions were first introduced by Kurokawa, and form a hierarchy generalizing the usual sine function, which is the first member: $S_1(z|\omega) = 2\sin\frac{z}{\omega}$. They show up in physics when computing 1-loop determinants of theories placed on odd-dimensional spheres, for S^3 one finds the double sine S_2 [95], for S^5 one finds the triple sine [96, 71], and for S^7 the quadruple sine appear [97, 98]. In the next subsection we define a generalization of the multiple sine functions that show up when the 1-loop determinant is computed for general toric manifolds.

The multiple sine functions have Weirstrass product representations [99]. To keep formulas manageable, we consider the case when all $\omega_i = 1$ and write $S_r(z|1, 1, \ldots, 1) = S_r(z)$. Then the Weirstrass representation of S_1 is

$$S_1(z) = 2\pi z \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), \tag{7.11}$$

which is the famous infinite product representation of the sine function due to Euler. The double sine have the representation

$$S_2(x) = 2\pi x e^{-x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{n+1} \left(1 - \frac{x}{n}\right)^{-n+1} e^{-2x}, \tag{7.12}$$

and so on for the higher multiple sines as well.

The Weirstrass representation can be understood as an alternative (but equivalent) way of regularizing of the infinite product, as we now explain for the example of S_2. Starting from its divergent infinite product, that we can put in the form

$$\begin{aligned} \mathcal{P} &= \prod_{n_1,n_2=0}^{\infty} \frac{(z + n_1 + n_2)}{(2 - z + n_1 + n_2)} \\ &= z \prod_{t=1}^{\infty} (z + t)^{t+1} (z - t)^{1-t} \end{aligned} \tag{7.13}$$

The idea is now to remove the divergent parts of this infinite product, so we have to understand what they are. We do this by taking the logarithm of $\mathcal{P}$:

$$\log \mathcal{P} = \log z + \sum_{t=1}^{\infty} \left[(t+1)\log(z+t) + (1-t)\log(-z+t)\right], \tag{7.14}$$

and expanding this for large t to find the divergences, we find

$$\log z + \sum_{t=1}^{\infty} \left[2\log t + 2z + \mathcal{O}\big(\frac{z}{t}\big)\right], \tag{7.15}$$

where the two terms inside the sum tells us how the product is diverging. To arrive at a finite product, we need to compensate each factor of the product with the factor that precisely cancel these divergences. The $2z$ part is compensated by multiplying each factor inside the product by e^{-2z}, and the $2\log t$ term tells us to multiply each factor by t^{-2}. So a regularized version of $\mathcal{P}$ is

$$\begin{aligned} & z\prod_{t=1}^{\infty} t^{-2}(z+t)^{t+1}(-z+t)^{1-t}e^{-2z} \\ &= z\prod_{t=1}^{\infty}\left(1+\frac{z}{t}\right)^{t+1}\left(1-\frac{z}{t}\right)^{1-t}e^{-2z}, \end{aligned} \tag{7.16}$$

This is now a convergent product, but in order to not change the original value, we should compensate for the extra factors that we have inserted. This means that we have to divide the above expression by the regularized values of $\prod_t t^{-2}$ and $\prod_t e^{-2z}$. The product over t^{-2} we have already computed in equation (2.64), and it is $(2\pi)^{-1}$, and the product over e^{-2z} is e^{z} by the same computation as in (2.65). So in total the correct regularized version of $\mathcal{P}$ is

$$2\pi z e^{-z}\prod_{t=1}^{\infty}\left(1+\frac{z}{t}\right)^{t+1}\left(1-\frac{z}{t}\right)^{1-t}e^{-2z}, \tag{7.17}$$

and we see that this regularization procedure correctly recovers the above Weirstrass representation of S_2 (7.12).

Next, we give some important properties of the multiple sine functions. They satisfy a generalized periodicity relation, where they are periodic with periods ω_i 'up to' a lower-degree multiple sine:

$$S_r(z+\omega_i|\underline{\omega}) = S_{r-1}(z|\underline{\omega}^{-}(i))^{-1}S_r(z|\underline{\omega}). \tag{7.18}$$

Since there is no S_0, this is just normal periodicity for the usual sine function S_1. For us, the most important property the multiple sine functions satisfy is the following factorization property:

$$S_r(z|\underline{\omega}) = e^{(-1)^r \frac{\pi i}{r!} B_{r,r}(z|\underline{\omega})} \prod_{k=1}^{r} (e^{2\pi i \frac{z}{\omega_k}} | e^{2\pi i \frac{\omega_1}{\omega_k}}, \ldots, \widecheck{e^{2\pi i \frac{\omega_k}{\omega_k}}}, \ldots)_\infty, \quad (7.19)$$

which applies as long as $\mathrm{Im}(\frac{\omega_i}{\omega_j}) \neq 0$ for all $i \neq j$. Here, $B_{r,r}$ is a polynomial in z of degree r, called a multiple Bernoulli polynomial. These are defined from a generating series as

$$(-1)^r \frac{t^r e^{zt}}{\prod_{j=1}^{r}(1 - e^{t\omega_j})} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(z|\underline{\omega}) \ , \quad (7.20)$$

which closely mimics the generating series for the Bernoulli numbers. The multiple sine functions are also closely related to the multiple elliptic gamma functions, which we turn to next.

The multiple elliptic gamma functions [91] are generalizations of the usual gamma function, which depends on more variables in an elliptic way. The first example is that of the original elliptic gamma function, that was studied by Felder and Varchenko [90]. This is a meromorphic function of three variables z, τ_1, τ_2 defined by the following infinite product (for $\mathrm{Im}\,(\tau_i) > 0, \ \forall i$)

$$\Gamma(z, \tau_1, \tau_2) = \prod_{j,k=0}^{\infty} \frac{(1 + e^{2\pi i(\tau_1 + \tau_2 - z + j\tau_1 + k\tau_2)})}{(1 - e^{2\pi i(z + j\tau_1 + k\tau_2)})}. \quad (7.21)$$

We can here recognize the infinite q-Pochhammer function defined above, so we can also define the elliptic gamma function as

$$\Gamma(z, \tau_1, \tau_2) = \frac{(e^{2\pi i(\tau_1 + \tau_2 - z)} | e^{2\pi i\tau_1}, e^{2\pi i\tau_2})_\infty}{(e^{2\pi i z} | e^{2\pi i\tau_1}, e^{2\pi i\tau_2})_\infty}, \quad (7.22)$$

which is defined for any region of τ_1, τ_2 away from $\mathrm{Im}(\tau_i) = 0$.

The elliptic gamma function shows up in various contexts, for example in statistical mechanics when studying the Ising model [100], and in mathematics when studying solutions to elliptic qKZB difference equations [101], which is the motivation of the study in [90]. Felder and Varchenko derive a number of properties that the elliptic gamma functions enjoy, most importantly a “modular” three-term relation, which connects the values of $\Gamma(z, \tau_1, \tau_2)$ at points related by $SL_3(\mathbb{Z})$ transformations acting on (τ_1, τ_2) as a fractional linear transformation. Explicitly, this relation reads

$$\Gamma(z, \tau_1, \tau_2) = e^{\frac{2\pi i}{3!} B_{3,3}(z|\tau_1,\tau_2,-1)} \Gamma(\frac{z}{\tau_1}|\frac{\tau_2}{\tau_1}, -\frac{1}{\tau_1}) \Gamma(\frac{z}{\tau_2}|\frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2}), \quad (7.23)$$

where $B_{3,3}$ again is a multiple Bernoulli polynomial, now cubic is z. This relation have an interpretation in terms of a generalization of Jacobi modular forms.

There is a natural generalization of the elliptic gamma function to a function depending on more variables, called the multiple elliptic gamma functions, which form a hierarchy that includes the Jacobi theta function and the elliptic gamma functions. These were defined by Nishizawa [91], and they are closely related to the multiple sine functions discussed above.

Let $x = e^{2\pi i z}$ and $q_i = e^{2\pi i \tau_i}, i = 0, \ldots, r$. Then the multiple elliptic gamma functions are defined as

$$G_r(z|\underline{\tau}) = (x|\underline{q})_\infty^{(-1)^r} (q_0 q_1 \cdots q_r x^{-1}|\underline{q})_\infty \ . \quad (7.24)$$

In this hierarchy, the Jacobi theta function is given by $\theta_0(z|\tau) = G_0(z|\tau)$, and the elliptic gamma function is given by $\Gamma(z, \tau_0, \tau_1) = G_1(z|\tau_0, \tau_1)$.

The multiple elliptic gamma functions enjoy the modularity property

$$G_r(z|\underline{\tau}) = e^{\frac{2\pi i}{(r+2)!} B_{r+2,r+2}(z|\underline{\tau},-1)} \prod_{k=0}^{r} G_r\left(\frac{z}{\tau_k}\middle| \frac{\tau_1}{\tau_k}, \cdots, \frac{\breve{\tau_k}}{\tau_k}, \ldots, \frac{\tau_r}{\tau_k}, -\frac{1}{\tau_k}\right). \quad (7.25)$$

This includes the modular property for the Jacobi theta function and the elliptic gamma functions.

Another important property the multiple elliptic gamma functions satisfy is a recurrence relation, or a type of generalized periodicity in the τ_i-parameters, which reads

$$G_r(z + \tau_i|\underline{\tau}) = G_{r-1}(z|\underline{\tau}^-(i)) G_r(z|\underline{\tau}), \quad (7.26)$$

where $\underline{\tau}^-(i) = (\tau_0, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_r)$.

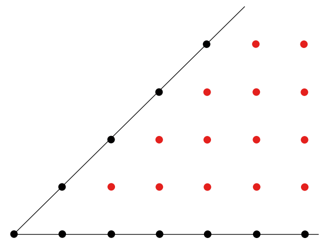

Figure 7.1. A 2d cone C, where the red dots indicate the lattice points in the interior, i.e. $C^\circ \cap \mathbb{Z}^2$, and the black dots are the lattice points on the faces. For this particular cone we can restrict to only interior lattice points by shifting with the vector $(2,1)$.

7.2 Generalized multiple sine functions

We generalize the above story in a particular way, guided by the localization calculation on 5d toric SE manifolds reviewed in chapter 6. For this purpose, we think of the ordinary multiple sine and elliptic gamma functions as being associated to odd-dimensional spheres, and our goal is to generalize this to other toric manifolds.

Recall the concept of the moment map polytope from chapter 4 and the moment map cone of section 5.3.2. For the spheres S^{2r-1}, the moment map cones are given by $\mathbb{R}^r_{\geq 0}$, and the infinite products that define both the multiple sine and multiple elliptic gamma functions runs over all lattice points within this cone. So the generalization that we consider is to replace this with an arbitrary cone C of dimension r, which we require to be rational and good. These conditions ensures that the cone corresponds to a smooth toric manifold.

For such a cone C, we define the associated generalized multiple sine function, written in the Weirstrass representation as

$$S_r^C(z|\underline{\omega}) = \prod_{n \in C \cap \mathbb{Z}^r} (z + n \cdot \underline{\omega}) \prod_{n \in (C)^\circ \cap \mathbb{Z}^r} (-z + n \cdot \underline{\omega})^{(-1)^{r-1}}, \qquad (7.27)$$

where C° denotes the interior of the cone, i.e. it doesn't include points on its boundary, see figure 7.1. This is the correct generalization of the shift by $\omega_1 + \ldots + \omega_r$ that occur for the usual multiple sine functions. When there exists a vector ξ such that $\xi \cdot v_i = 1$ for all inward normals v_i of C, i.e. when the cone is 1-Gorenstein, the shift corresponding to restricting to only interior points is given by $\xi \cdot \underline{\omega}$.

The infinite product in this definition is again to be understood as a zeta-regularized product, and the precise procedure for this is explained in III. There, we also prove the most important property of these functions enjoy, which is a factorization property that closely mimics the one for the ordinary multiple sines:

$$S_r^C(z|\underline{\omega}) = e^{(-1)^r \frac{\pi i}{r!} B_{r,r}^C(z|\underline{\omega})} \prod_{\rho \in \Delta_1^C} (x_\rho, \underline{q}_\rho)_\infty. \tag{7.28}$$

Here, Δ_1^C is the 1d edges of C, which also corresponds to the vertices of the moment polytope of our toric manifold. To each such vertex, there is a natural way to associate an $SL_r(\mathbb{Z})$ element, and this acts as a linear fractional transformation on $(z|\underline{\omega})$ to give us the correct local variables $(x_\rho|\underline{q}_\rho)$. $B_{r,r}^C$ is a generalized Bernoulli polynomial, which depends on the geometrical data of the cone/manifold. The details of all of this is described in article III.

7.3 Generalized multiple gamma functions

We can repeat a very similar story for the generalized multiple elliptic gamma functions. Again, for a cone C of dimension r, we can define the generalized multiple elliptic gamma function as

$$G_{r-1}^C(z|\underline{\tau}) = \prod_{n \in C \cap \mathbb{Z}^r} (1 - e^{2\pi i(z + n\cdot\underline{\tau})})^{(-1)^{r-1}} \prod_{n \in C^\circ \cap \mathbb{Z}^r} (1 - e^{2\pi i(-z + n\cdot\underline{\tau})}). \tag{7.29}$$

This function again satisfies a number of properties mimicking the usual multiple elliptic gamma functions, and perhaps most importantly we again find a factorization property. For each generator $\rho \in \Delta_1^C$ of the cone there is a natural way of associating a $SL_r(\mathbb{Z})$ element K_ρ, essentially by completing the $r-1$ normal vectors of ρ into a $SL_r(\mathbb{Z})$ matrix. Then the factorization of G_{r-1}^C can be written as

$$G_{r-1}^C(z|\underline{\tau}) = e^{\frac{2\pi i}{(r+1)!} B_{r+1,r+1}^{\hat{C}}(z|\underline{\tau},-1)} \prod_{\rho \in \Delta_1^C} K_\rho^* G_{r-1}(z|\underline{\tau}), \tag{7.30}$$

where K_ρ^* acts as a fractional linear transformation on the arguments of G_{r-1}. For details of this see III.

8. Dimensional reduction

The idea of constructing new theories by starting from a higher dimensional theory and reducing goes back to Kaluza [102] and Klein [103]. They were seeking a unification of electromagnetism and gravity, and came up with the ingenious idea of postulating an extra fifth dimension and consider only gravity (i.e. general relativity) on this five dimensional space. By taking the extra dimension to be a small circle, the effective theory in 4d then looks like gravity coupled to electromagnetism. This procedure is called to dimensionally reduce on a circle, and it is a very appealing idea. Of course we can also consider more extra dimensions and consider them to have other geometries, in which case the resulting effective theories depends in various ways on the geometry of the extra dimensions.

Since string theory lives in 10 dimensions but we only observe 4, dimensional reduction is a core part of the subject. This topic is called string compactifications, and it is a very rich subject with a lot of interesting physics and mathematics. In the present thesis we do not consider the dimensional reduction of supergravity or string theory, but only of supersymmetric field theories. This is an interesting topic in itself and sheds light on how different theories are connected. Especially the various dimensional reductions of the mysterious 6d (2,0) theory has lead to a number of interesting results, such as constructions of new theories [104] and the discovery of various field theory dualities [43, 44]. It has also explained some previously known dualities, like S-duality of 4d $\mathcal{N} = 4$ theory that can be naturally understood from the compactification of 6d (2,0) on a torus.

The topic of article IV is closely related to the reduction of the (2,0) theory on a non-trivial elliptic fibration, as studied in for example [105]. As we will see, since the fibration is non-trivial the resulting 4d theory will have a space-dependent complexified coupling.

8.1 Dimensional reduction on $\mathbb{R}^n \times S^1$

Let us first sketch how dimensional reduction on a trivially fibered circle on flat space work, and what it does to the different fields of a SYM theory. The starting manifold is $\mathbb{R}^n \times S^1$, and we want to see what happens when we take the circle to be really small. The resulting theory will effectively look like a n-dimensional theory on $\mathbb{R}^n$, and we are interested in what the field content and action of that theory is given that we know the theory in $n+1$ dimensions.

Let (x^μ, α), $\mu = 1, \ldots, n$ be the coordinates on $\mathbb{R}^n \times S^1$, and let r be the radius of the circle. Consider a massless scalar ϕ on $\mathbb{R}^n \times S^1$. Its field configuration can be expanded in Fourier modes along the circle direction,

$$\phi(x, \alpha) = \sum_{k=-\infty}^{\infty} \phi_k(x) e^{ik\alpha/r}. \tag{8.1}$$

This shows that the momentum is quantized along the compact circle direction and given by k/r. The wave-equation that the massless scalar satisfies, $(\partial_\mu \partial^\mu + \partial_\alpha \partial^\alpha)\phi = 0$ becomes

$$(\partial_\mu \partial^\mu - \frac{k^2}{r^2})\phi_k(x) = 0, \tag{8.2}$$

which shows that the modes ϕ_k become an infinite tower of n-dimensional scalar fields, with masses $m_k^2 = k^2/r^2$. As we take r small, and consider the effective theory at energies much smaller than r^{-1}, only the massless modes with $k = 0$ remain. This is the mode which is independent of the circle direction. Essentially the same computation can be made for fields of any spin, so when we dimensionally reduce we keep only the modes that are independent of the circle direction.

The types of fields that we want to reduce are scalars, vector fields and spinors. We have seen that a scalar goes into a lower-dimensional scalar, and that we only keep the mode independent of the circle direction. The gauge field A is decomposed as $A_m = (A_\mu, A_\alpha)$, and on reduction A_μ becomes a 1-form in n dimensions, and the last component becomes a scalar in the lower-dimensional theory, $A_\alpha = \varphi$.

For spinors, what happens depends on if the dimension n, is even or odd. This is because the number of components of a spinor in dimensions $n = 2k+1$ and $n = 2k$ are both 2^k. We mainly care about the reduction from five to four dimensions, and the spinor in 5d has 4 components and can be trivially pushed down to a 4d 4-component spinor, so that we have $\lambda^{5d} = \lambda^{4d}$. If we instead are reducing from for example 4d to 3d, one Dirac spinor in 4d becomes a pair of 2-component spinors in 3d, $\lambda^{4d} \to (\lambda_1^{3d}, \lambda_2^{3d})$. Of course this also depends on the reality conditions on the spinors and is in general a representation theory question of decomposing spinor representations under the lower-dimensional Lorentz subgroup.

To work out how the terms in the action reduces is straightforward: one writes every piece in terms of its lower-dimensional components, and then drops every term where something is differentiated along the circle direction. For example the field strength 2-form F for an abelian theory is

$$\begin{aligned} F = (\partial_\mu A_\nu - \partial_\mu A_\nu) dx^\mu \wedge dx^\nu + (\partial_\alpha A_\mu - \partial_\mu A_\alpha) d\alpha \wedge dx^\mu \\ = F_{4d} - \partial_\mu \varphi d\alpha \wedge dx^\mu \end{aligned} \tag{8.3}$$

so from the kinetic term of the gauge field we get

$$F \wedge \star F = (F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \varphi)(\partial^\mu \varphi)) \mathrm{Vol}. \tag{8.4}$$

The other terms in SYM actions reduce in a similar fashion. Next we discuss the case that is relevant for article IV, which is when we start with a compact manifold that has the structure of a non-trivial circle fibration over a 4d base.

8.2 Dimensional reduction of theories on compact manifolds

As we have seen in the above, dimensional reduction is straightforward if the space is a trivial circle fibration over flat space. If we instead consider a non-trivial S^1 fibration, it might still be possible to reduce. For the case of non-compact manifolds, this was studied by Scherk and Schwarz [106], where they find that the non-trivial fibration can introduce some mass terms. We are instead interested in compact geometries, so we have to investigate whether there are any obstructions to reducing, and when a consistent reduction can be performed. In article IV this is investigated in some detail and we find a number of mathematical conditions for when such a reduction can be performed.

In general, the various fields that we want to reduce are sections of vector bundles over our manifolds, so we study when vector bundles and sections of them can be pushed down. Let $E \to M$ be a vector bundle over the manifold M, and let M have the structure of a S^1 fibration over a base manifold B, $S^1 \to M \to B$. Then E can be pushed down to a bundle over B if there exists a trivialization of E over M where the transition functions are invariant along the circle direction. This condition on the bundle can be rephrased as a condition that its holonomy along the fibration circle is trivial. Further, any section s of E that satisfies $D_\alpha s = 0$, where α is the coordinate along the circle, can then also be pushed down and viewed as a section of the bundle over B.

The above is the formal requirements on fibrations and the bundles, but in particular, we have to consider when the spin bundle can be pushed down. We now specify to the situation of article IV, when we care about reduction from 5d to 4d. Then the spin-bundle can only be pushed down when the 4d base B is a spin-manifold, otherwise there is an obstruction and the spin bundle of M may instead be pushed down to a spin$^{\mathrm{c}}$ bundle on B (which always exist). An example of where this happens is the Hopf fibration $S^1 \to S^5 \to \mathbb{P}^2$, since $\mathbb{P}^2$ is not spin. But one can still in principle do the reduction and find supersymmetric $\mathcal{N} = 2$ theories on such non-spin base manifolds, but one has to deal with some subtleties related to having only a spin$^{\mathrm{c}}$ bundle. In the paper IV we do not consider these cases, but leave it for future investigations.

Let us now briefly explain how dealing with a non-trivial connection changes the reduction. For concreteness, we start with a 5d manifold and write its metric in coordinates adapted to the fibration, so that it takes the form

$$ds_5^2 = ds_4^2 + e^{2\phi}(d\alpha + b)^2, \tag{8.5}$$

where α again is the coordinate along the fiber, b is the connection one-form for the fibration and e^ϕ is the radius of the fiber. b and ϕ are both constant along the fiber, but they can vary along the base. Now let $\beta = e^{-2\phi} g_M(\partial_\alpha) = d\alpha + b$; which up to normalization is the dual one-form of the vector along the fiber.

As explained above we consider fields constant along the fiber. In particular the 5d gauge field $\hat{A}$ then takes the form $\hat{A} = A + \varphi\beta$, where A is the gauge field on B and φ is its component along the fiber, which is an adjoint scalar constant along the fiber. Plugging this expression into the curvature we obtain

$$\hat{F} = F + (d_A\varphi) \wedge \beta + \varphi d\beta, \tag{8.6}$$

and, using the inverse metric we can write the 5d YM term in 4d quantities, which will tell us what will appear in the 4d action

$$\hat{F} \wedge \star_5 \hat{F} = e^{\phi}(d\alpha + \beta) \wedge [(F + \varphi d\beta) \wedge \star_4 (F + \varphi d\beta) + 2e^{-2\phi}(d_A\varphi) \wedge \star_4 (d_A\varphi)].$$

We see from this that both the connection β and the position-dependent radius e^{ϕ} enter in non-trivial ways; the term $\varphi^2(d\beta)^2$ is something like a mass term for φ, and e^{ϕ} will give us a position dependent coupling constant.

Reducing the fields as described above is consistent with supersymmetry as long as the Killing spinor is constant along the fiber. Then we can reduce the supersymmetry variations and the action and get a 4d supersymmetric theory, which is what we do in article IV. We also classify the 5d SE manifolds with a free $U(1)$ action where the base is spin, and find that the corresponding 4d manifolds all have the topology of $\#_n(S^2 \times S^2)$, n-fold connected sums of $S^2 \times S^2$. The resulting theories have $\mathcal{N} = 2$ supersymmetry, and a position dependent coupling constant as shown above. We also find their partition functions by expanding the 5d 1-loop determinant along the fiber and only keeping the constant modes, and we discuss the instanton contributions. An interesting feature of the theories we construct is that they support instantons living on half the torus fixed points, and anti-instantons on the other half. This is the same as what Pestun found on S^4 [19] and shows that the theories we construct are not related to the equivariantly twisted theories studied in [107].

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Svensk sammanfattning

Inom modern fysik är kvantfältteori ett viktigt verktyg som används inom många olika ämnesområden. Den utgör grunden för partikelfysiken, används för att beskriva material inom fasta tillståndets fysik, och är också intimt förknippat med strängteori. En viss typ av fältteorier, så kallade gaugeteorier, är speciellt viktiga och beskriver elektromagnetism, den svaga samt den starka kraften. Därför är det ett viktigt problem att förstå deras beteende.

Så länge gaugeteorierna är svagt växelverkande, vilket är fallet för elektromagnetism och den svaga kraften, så har fysiker över åren utvecklat diverse kraftfulla metoder för att beräkna olika storheter och förstå teoriernas dynamik. Men när en teori är starkt växelverkande, vilket är fallet med den starka kraften som beskrivs av kvantkromodynamic (QCD) och för teorin som beskriver högtemperaturs supraledare, så fungerar inte dessa metoder längre och nya angreppssätt behövs. Ett sätt att försöka förstå starkt växelverkande teorier är att studera teorier med mer symmetrier. Dessa är inte längre realistiska, men kan ändå agera som användbara förenklade modeller som kan låta oss förstå mer om de involverade mekanismerna.

En speciellt användbar klass av sådana mer symmetriska teorier är de med en speciell typ av symmetri, s.k. supersymmetri. Denna symmetri begränsar teorierna på olika sätt och gör dem enklare att studera. I denna avhandling studerar vi supersymmetriska gaugeteorier och använder oss av en specifik matematisk teknik, lokalisering, som kan användas för dessa teorier. Denna teknik låter oss beräkna olika storheter exakt, även när teorierna är starkt växelverkande. Vi använder denna teknik för att studera teorier i fem och fyra dimensioner, och våra beräkningar leder oss också till att formulera en ny klass av speciella funktioner.

Supersymmetriska gaugeteorier är också intimt förknippade med strängteori, och en alternativ motivation för forskningen i denna avhandling kommer därifrån. Det finns fem olika strängteorier som alla lever i 10 rumtidsdimensioner, och de är alla besläktade med en unik teori i 11 dimensioner, som kallas M-teori. Denna teori har inte längre strängar som sina fundamentala objekt, utan istället högre-dimensionellla membran, de så kallade M2 och M5 branen. Dessa beskrivs i sin tur av olika supersymmetriska fältteorier, och i synnerhet de 6d teorier som beskriver M5 branen är väldigt speciella och intressanta. Dessutom är de svåra att studera direkt, då våra vanliga verktyg inte fungerar. Dessa 6d teorier är nära besläktade med de 5d och 4d teorier vi undersöker i denna avhandling, och hoppet är att om vi kan förstå dessa bättre kan vi också lära oss någonting nytt om de mystiska 6d teorierna.

I artikel I och II studerar vi supersymmetriska 5d gauge-teorier placerade på toriska Sasaki-Einstein mångfalder, med hjälp av lokaliseringstekniken, och beräknar deras tillståndssummor. Detta leder oss till att definiera en ny hierarki av speciella funktioner som vi studerar i artikel III och V och som uppvisar intressanta faktoriseringsegenskaper. I artikel IV använder vi de 5d teorier som vi studerat för att konstruera nya supersymmetriska 4d teorier, och vi använder våra resultat från 5d för att beräkna deras tillståndssummor. Vi finner att dessa teorier har en positionsberoende kopplingskonstant och att vårt svar innehåller bidrag både från så kallade instantons och anti-instantons, något som tidigare observerats för teorier på fyrsfären S^4.

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