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Solving the Dirac equation in central potential for muonic hydrogen atom with point-like nucleus

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E-mail: eshetudiriba444@gmail.com, gaaruma@gmail.com and Gashaw@aims.ac.za**Keywords:** Muon, muonic hydrogen, central potential, energy eigenvalue, probability density

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**Abstract**

Muon has properties very similar to an electron. For this reason, it is possible to replace one of the electrons in an atom by a muon to form a muonic atom. The main purpose of this study is to calculate the energy eigenvalues and to study the probability density of muonic hydrogen with point like nucleus. Numerical results have generated using Matlab software programming language. The reduced mass of muon has been used in order to correct the error incurred by the assumption that the nucleus of muonic hydrogen is point-like which in turn gives it an infinite mass. The energy eigenvalues for different states have been calculated using the rest and reduced masses of muon, and the result have been tabulated. According to these results, the relativistic quantum description is not responsible for the lamb shift. The Probability density shows that muon is much more likely to be found near the nucleus of hydrogen atom for the ground state when compared with the excited states.

1. Introduction

The muon is an elementary particle similar to the electron, with an electric charge of -1 in the unit of proton charge e and a spin of $1/2$, but with the rest mass of $m_\mu \cong 106\text{MeV}/c^2(207m_e)$. The muon, μ , and its associated neutrino ν_μ were first discovered in the decays of charged pions: $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ [1]. A muon is a lepton with properties very similar to an electron. For this reason, it is possible to replace one of the electrons in an atom by a muon to form a muonic atom. However, since the mass of a muon is 207 times larger than that of an electron, the radii of the muonic orbits are much smaller than those of electrons [2]. Consequently, the overlap between the muon orbits and the nucleus is much larger than in ordinary atoms and the energy levels can be significantly perturbed by the nuclear charge distribution [3].

Muonic atoms are observed for the most part by the means of x-ray radiation which they emit; this radiation decays with the half-life characteristics of muon. Since the muons approaches the nucleus very closely than electrons in an electronic atom, they can be used to study the details of nuclear charge distributions, the distribution of the nuclear magnetic moment within the nuclear volume and nuclear quadrupole deformation. The exotic atoms like muonic atoms can be used for the investigation of atomic nucleus [3, 4].

Since muon is heavier than the electron, and found in an atomic bound state, it is much closer to the proton than the electron, so proton size effects are greatly magnified. Despite the muon's limited $2.2\ \mu\text{s}$ lifetime, it was anticipated that the larger impact of the proton size on the energy levels would allow a 0.1% measurement of the proton charge radius. The effective potential that the muon experiences is significantly modified by the proton charge distribution. Therefore, a measurement of the 2P-2S Lamb shift could give a precise value for the proton charge radius [5].

The Muon thus cascades down to the lowest energy orbitals where it is in the direct vicinity of the nucleus at a distance $a_\mu \cong \frac{1}{207}a_e$, smaller than the corresponding Bohr radius of internal electrons. It therefore forms a hydrogen-like atom of charge Z around the nucleus, unconscious to the presence of other electrons at larger

distances from the nucleus. For heavy nuclei, the Bohr radius of a muonic atom is of the same order as the nuclear radius. The muon therefore penetrates the nucleus, having a 90% probability to be inside the nucleus in the ground state. Because of this, the study of muonic atom spectra gives useful information on the structure of nuclei, in particular on the charge (i.e. proton) distribution inside the nuclei [6].

Several studies were conducted on the energy correction of muonic hydrogen [7, 8]. But they did not calculate the energy eigenvalue and study the probability density of muonic hydrogen atom. Therefore in this paper we focus on modifying Dirac equation, calculation of the energy eigenvalues of muonic hydrogen with point-like proton for different states and we perform numerical calculations for muonic hydrogen with point-like proton to interpret the probability density of muonic hydrogen atom. The reduced mass of muon has been used in order to correct the error incurred by the assumption that the nucleus of a muonic hydrogen is point-like which is in turn gives it an infinite mass [9].

2. Results and discussion

2.1. Minimal coupling to the electromagnetic field

Dirac equation in an electromagnetic field with scalar potential A_0 and vector potential \vec{A} is

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\Psi = 0 \quad (2.1)$$

By using the separation of variables the wave function becomes [10]

$$\Psi(\vec{x}, t) = \begin{pmatrix} \chi \\ \phi \end{pmatrix} e^{-iEt} \quad (2.2)$$

Where χ and ϕ are Pauli spinors. Now we assume that the contribution comes from the vector potential \vec{A} is zero ($\vec{A} = 0$). So that the Dirac equation becomes the coupled differential equations of Pauli spinors [9]

$$(E - m - eA_0)\chi = \frac{1}{r} \frac{\vec{\sigma} \cdot \vec{r}}{r} \left[-ir \frac{\partial}{\partial r} + i\vec{\sigma} \cdot \vec{L} \right] \phi \quad (2.3)$$

$$(E + m - eA_0)\phi = \frac{1}{r} \frac{\vec{\sigma} \cdot \vec{r}}{r} \left[-ir \frac{\partial}{\partial r} + i\vec{\sigma} \cdot \vec{L} \right] \chi \quad (2.4)$$

Using the following ansatz [11]

$$\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} f(r)\mathcal{Y}_{jlm}(\theta, \phi) \\ ig(r)\mathcal{Y}_{jl'm}(\theta, \phi) \end{pmatrix} \quad (2.5)$$

Where \mathcal{Y}_{jlm} a two is row spherical spinor and the quantum number l and l' represents the upper and lower components of the Dirac spinors. From equation (2.5) we have $\chi = f\mathcal{Y}_{jlm}$ and $\phi = ig\mathcal{Y}_{jl'm}$ such that equations (2.3) and (2.4) become

$$(E - m - V(r))f\mathcal{Y}_{jlm} = \frac{1}{r} \frac{\vec{\sigma} \cdot \vec{r}}{r} \left[-ir \frac{\partial}{\partial r} + i\vec{\sigma} \cdot \vec{L} \right] ig\mathcal{Y}_{jl'm} \quad (2.6)$$

$$(E + m - V(r))ig\mathcal{Y}_{jl'm} = \frac{1}{r} \frac{\vec{\sigma} \cdot \vec{r}}{r} \left[-ir \frac{\partial}{\partial r} + i\vec{\sigma} \cdot \vec{L} \right] f\mathcal{Y}_{jlm} \quad (2.7)$$

We neglect the angular wave functions and replace the partial derivative by ordinary derivative since all terms only depend on radial wave function. The radial wave equations become

$$\left(\frac{d}{dr} + \frac{1 - \kappa}{r} \right) g - (m - E + V(r))f = 0 \quad (2.8)$$

$$\left(\frac{d}{dr} + \frac{1 + \kappa}{r} \right) f - (m + E - V(r))g = 0 \quad (2.9)$$

Where $V(r) = eA_0$ which is the central potential for point like proton, κ is the generalized quantum number, m and E are the rest mass and total energy of moun respectively.

Table 1. The energy eigenvalues of muonic hydrogen with point-like proton for different states.

Principal quantum numbers	Orbital quantum numbers	Total angular quantum numbers	Shell	Energy eigenvalues for rest mass of muon in (MeV)	Energy eigenvalues for reduced mass of muon in (MeV)
$n = 1$	$l = 0$	$j = 1/2$	$1s_{1/2}$	105.655560758	94.961941514
$n = 2$	$l = 0$	$j = 1/2$	$2s_{1/2}$	105.657671813	94.963838904
	$l = 1$	$j = 1/2$	$2p_{1/2}$	105.657671813	94.963838904
		$j = 3/2$	$2p_{3/2}$	105.657671831	94.963838912
$n = 3$	$l = 0$	$j = 1/3$	$3s_{1/2}$	105.658063750	94.964190275
	$l = 1$	$j = 1/2$	$3p_{1/2}$	105.658063750	94.964190275
		$j = 3/2$	$3p_{3/2}$	105.658063753	94.964190277

2.2. Exact solution to the coupled radial dirac equation

The coulomb potential is given by [12]

$$V(r) = \frac{-Z\alpha}{r} \quad (2.10)$$

This form of potential leads us to find the well-known quantized energy eigenvalue of the form

$$E_{nj} = \frac{m}{\sqrt{1 + \frac{(Z\alpha)^2}{\left(n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}\right)^2}}} \quad (2.11)$$

Where $\alpha = \frac{1}{137}$, the fine structure constant and $n = 1, 2, \dots$, the Principal quantum number.

We convert this quantized energy eigenvalue of point like proton into Matlab source code to generate numerical results. By using Matlab software, we can tabulate the energy eigenvalue for different states of muonic hydrogen as shown in table 1. In the calculations, we used the rest mass of muon $m = 105.6583755$ MeV and the reduced mass of muon $= 94.96447137$ MeV, the fine structure constant $\alpha = 1/137$ and the atomic number $Z = 1$.

Lamb shift is a difference in energy between two energy eigenvalue of $2S_{1/2}$ and $2P_{1/2}$ states of the hydrogen atom. From the table, the lamb shift is zero. That means relativistic quantum description does not responsible for lamb shift. Therefore, it is quantum electrodynamics that can provide Lamb shift.

The explicit form of the radial wave functions in equations (2.8) and (2.9) is [7]

$$f = \frac{1}{2} Ne^{-\frac{\rho}{2}} \rho^{s-1} [M(s - \gamma, 2s + 1, \rho) - L_- M(s - \gamma + 1, 2s + 1, \rho) - \Lambda_+ \rho^{-2s} M(-s - \gamma, -2s + 1, \rho) - \Lambda_- L_+ \rho^{-2s} M(-s - \gamma + 1, -2s + 1, \rho)] \quad (2.12)$$

$$g = \frac{1}{2} N \zeta e^{-\frac{\rho}{2}} \rho^{s-1} [M(s - \gamma, 2s + 1, \rho) + L_- M(s - \gamma + 1, 2s + 1, \rho) - \Lambda_+ \rho^{-2s} M(-s - \gamma, -2s + 1, \rho) + \Lambda_- L_+ \rho^{-2s} M(-s - \gamma + 1, -2s + 1, \rho)] \quad (2.13)$$

Where

$$\Lambda_{\pm} = \frac{\Gamma(2s + 1)}{\Gamma(-2s + 1)} \frac{\Gamma\left(\frac{1}{2} - s - \beta_{\pm}\right)}{\Gamma\left(\frac{1}{2} + s - \beta_{\pm}\right)}; L_{\pm} = \frac{s \pm \gamma}{\kappa - \gamma/E'}; E' = \frac{E}{m}$$

And

$$\gamma = \frac{Z\alpha E'}{\sqrt{1 - E'^2}}; \zeta = \sqrt{\frac{m - E}{m + E}}; s = \sqrt{\kappa^2 - (Z\alpha)^2}$$

2.3. The probability density

The wave function in block matrix is given by

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} f(r) \mathcal{Y}_{jlm}(\theta, \phi) \\ ig(r) \mathcal{Y}_{jl'm}(\theta, \phi) \end{pmatrix}$$

The probability density can be

$$\rho(\vec{r}) = |\psi|^2 = \psi^\dagger \psi \quad (2.14)$$

The conjugate transpose becomes

$$\psi^\dagger = (\psi^T)^* = (\chi^\dagger \phi^\dagger)$$

Then the probability density can be rewritten as

$$\rho(\vec{r}) = \psi^\dagger \psi = \chi^\dagger \chi + \phi^\dagger \phi \quad (2.15)$$

But from block matrix given earlier we have

$$\chi^\dagger = f^* \mathcal{Y}_{jlm}^\dagger, \quad \phi^\dagger = -ig^* \mathcal{Y}_{jl'm}^\dagger$$

Equation (2.15) becomes

$$\rho(\vec{r}) = f^* f \mathcal{Y}_{jlm}^\dagger \mathcal{Y}_{jlm} + g^* g \mathcal{Y}_{jl'm}^\dagger \mathcal{Y}_{jl'm}$$

Since f and g are real valued functions of r we obtain

$$\rho(\vec{r}) = f^2 \mathcal{Y}_{jlm}^\dagger \mathcal{Y}_{jlm} + g^2 \mathcal{Y}_{jl'm}^\dagger \mathcal{Y}_{jl'm}$$

Where \mathcal{Y}_{jlm} is spin angular function in two component form given by [13]

$$\mathcal{Y}_{jlm} = \mathcal{Y}_{lm}^{j=l \pm 1/2} = N \begin{pmatrix} \pm L_\pm Y_l^{m-\frac{1}{2}} \\ L_\mp Y_l^{m+\frac{1}{2}} \end{pmatrix}, \quad \mathcal{Y}_{jl'm}^{j=l' \pm 1/2} = N' \begin{pmatrix} \pm L'_\pm Y_{l'}^{m-\frac{1}{2}} \\ L'_\mp Y_{l'}^{m+\frac{1}{2}} \end{pmatrix}$$

Where

$$N = \frac{1}{\sqrt{2l+1}}, \quad N' = \frac{1}{\sqrt{2l'+1}}$$

And

$$L_\pm = \sqrt{\frac{l \pm m + 1/2}{2l+1}}, \quad L'_\pm = \sqrt{\frac{l' \pm m + 1/2}{2l'+1}}$$

The complex conjugate transpose of the above spin angular functions become

$$\mathcal{Y}_{jlm}^\dagger = N^* (\pm L_\pm^* Y_l^{m-\frac{1}{2}*} L_\mp^* Y_l^{m+\frac{1}{2}*}), \quad \mathcal{Y}_{jl'm}^\dagger = N^* (\pm L'_\pm^* Y_{l'}^{m-\frac{1}{2}*} L'_\mp^* Y_{l'}^{m+\frac{1}{2}*})$$

Now we have

$$\mathcal{Y}_{jlm}^\dagger \mathcal{Y}_{jlm} = N^2 (L_\pm^2 Y_l^{m-\frac{1}{2}*} Y_l^{m-\frac{1}{2}} + L_\mp^2 Y_l^{m+\frac{1}{2}*} Y_l^{m+\frac{1}{2}})$$

And

$$\mathcal{Y}_{jl'm}^\dagger \mathcal{Y}_{jl'm} = N'^2 (L'_\pm^2 Y_{l'}^{m-\frac{1}{2}*} Y_{l'}^{m-\frac{1}{2}} + L'_\mp^2 Y_{l'}^{m+\frac{1}{2}*} Y_{l'}^{m+\frac{1}{2}})$$

The probability density now written as

$$\rho(\vec{r}) = N^2 f^2 (L_\pm^2 Y_l^{m-\frac{1}{2}*} Y_l^{m-\frac{1}{2}} + L_\mp^2 Y_l^{m+\frac{1}{2}*} Y_l^{m+\frac{1}{2}}) + N'^2 g^2 (L'_\pm^2 Y_{l'}^{m-\frac{1}{2}*} Y_{l'}^{m-\frac{1}{2}} + L'_\mp^2 Y_{l'}^{m+\frac{1}{2}*} Y_{l'}^{m+\frac{1}{2}}) \quad (2.16)$$

We interested in the probability density that depends on the radial coordinate. Therefore, integrating overall direction, we have

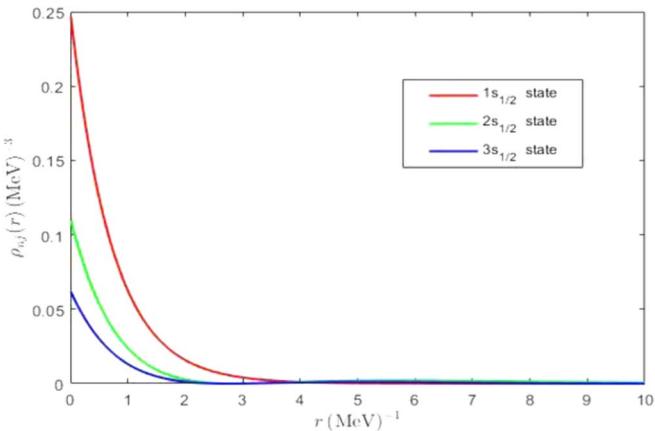


Figure 1. Probability density as a function of radial distance for $1s_{1/2}$, $2s_{1/2}$ and $3s_{1/2}$ states of Muonic hydrogen atom.

$$\begin{aligned}\rho(r) &= \int d\Omega \rho(r, \theta, \phi), \quad d\Omega = \sin \theta d\theta d\phi \\ &= \int d\Omega [f^2 N^2 (L_{\pm}^2 Y_l^{m-\frac{1}{2}*} Y_l^{m-\frac{1}{2}} + L_{\mp}^2 Y_l^{m+\frac{1}{2}*} Y_l^{m+\frac{1}{2}}) \\ &\quad + g^2 N'^2 (L_{\pm}^{\prime 2} Y_{l'}^{m-\frac{1}{2}*} Y_{l'}^{m-\frac{1}{2}} + L_{\mp}^{\prime 2} Y_{l'}^{m+\frac{1}{2}*} Y_{l'}^{m+\frac{1}{2}})]\end{aligned}$$

Since the term that carry the angular term is only spherical harmonics Y_l^m , the integration explicitly written as

$$\begin{aligned}\rho(r) &= f^2 N^2 \left[L_{\pm}^2 \int d\Omega Y_l^{m-\frac{1}{2}*} Y_l^{m-\frac{1}{2}} + L_{\mp}^2 \int d\Omega Y_l^{m+\frac{1}{2}*} Y_l^{m+\frac{1}{2}} \right] \\ &\quad + g^2 N'^2 \left[L_{\pm}^{\prime 2} \int d\Omega Y_{l'}^{m-\frac{1}{2}*} Y_{l'}^{m-\frac{1}{2}} + L_{\mp}^{\prime 2} \int d\Omega Y_{l'}^{m+\frac{1}{2}*} Y_{l'}^{m+\frac{1}{2}} \right]\end{aligned}\quad (2.17)$$

The orthonormalization condition is given by [14]

$$\int d\Omega Y_l^{m*} Y_{l'}^{m'} = \delta_{l,l'} \delta_{m,m'} \quad (2.18)$$

By using the above orthonormalization relation equation (2.17) becomes

$$\begin{aligned}\rho(r) &= f^2 N^2 [L_{\pm}^2 + L_{\mp}^2] + g^2 N'^2 [L_{\pm}^{\prime 2} + L_{\mp}^{\prime 2}] \\ &= \frac{1}{2l+1} f^2 + \frac{1}{2l'+1} g^2\end{aligned}\quad (2.19)$$

Where

$$L_{\pm}^2 + L_{\mp}^2 = 1, \quad N^2 = \frac{1}{2l+1}, \quad N'^2 = \frac{1}{2l'+1}$$

In terms of the total angular momentum probability density becomes

$$\rho_{nj} = \begin{cases} \frac{1}{2j} \left(f^2 + \frac{j}{j+1} g^2 \right), & \text{for } j = l + \frac{1}{2} \\ \frac{1}{2j+2} \left(f^2 + \frac{j+1}{j} g^2 \right), & \text{for } j = l - \frac{1}{2} \end{cases} \quad (2.20)$$

2.3.1. The numerical results of probability density for muonic hydrogen atom

The probability density in equation (2.20) can be converted into Matlab source code to produce numerical result. We have used MATLAB software to generate the graph of probability density for different states of muonic hydrogen atom. The probability density is obtained up to some normalization constant. The mass used in the calculation is in mega electron volt (MeV), which is the reduced mass of muon = 94.96447137 MeV.

Figure 1 illustrates the probability density for $1s_{1/2}$ (red line), $2s_{1/2}$ (green line) and $3s_{1/2}$ (blue line) energy levels of muonic hydrogen atom with point like proton. The graph shows that as the radial distance increase the probability density appear to decrease. When we compare with excited state, the ground state muon is much more likely to be found near the nucleus than the excited states.

Figure 2 depicts the probability density for $2p_{1/2}$ and $3p_{1/2}$ states are zero at the origin, and start to increase with the radial distance, attaining maximum at $r = 2.9 \text{ MeV}^{-1}$ and $r = 2.5 \text{ MeV}^{-1}$, subsequently decreasing

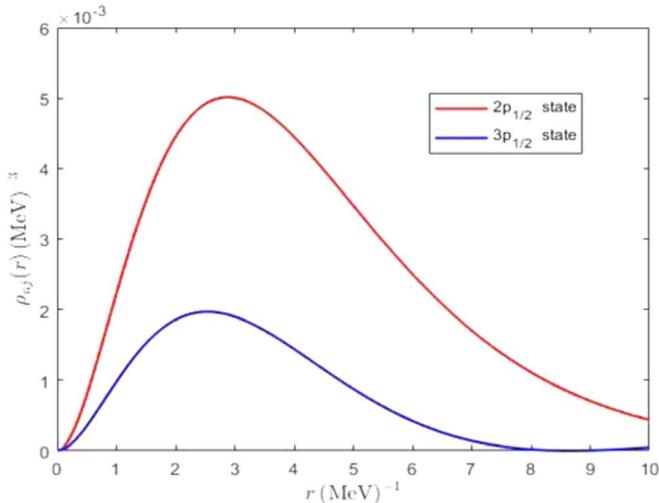


Figure 2. Probability density as a function of radial distance for $2p_{1/2}$ and $3p_{1/2}$ states of Muonic hydrogen atom.

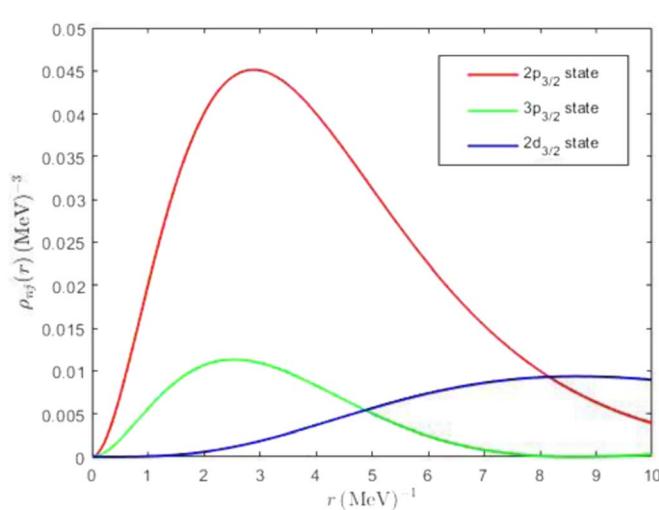


Figure 3. Probability density as a function of radial distance for $2p_{3/2}$, $3p_{3/2}$ and $3d_{3/2}$ states of Muonic hydrogen.

as the radial distance increases. It is evident from the results that the probability of finding a muon in $2p_{1/2}$ and $3p_{1/2}$ excited states near the nucleus is almost zero. But they are highly likely to be found above 2 MeV^{-1} and below 3 MeV^{-1} away from the nucleus.

In figure 3, the result shows that: (1) these states equally have the vanishing probability to be found near the origin; (2) as the radial distance increases up to 2 MeV^{-1} , the probability density of $2p_{3/2}$ shows a rapid increment slightly followed by $3p_{3/2}$ while $3d_{3/2}$ having no significant change; (3) above 2 MeV^{-1} , $2p_{3/2}$ and $3p_{3/2}$ attain maximum and begin to decrease while $3d_{3/2}$ shows slight increase attaining maximum at 8.13 MeV^{-1} where it dominates both $2p_{3/2}$ and $3p_{3/2}$ states.

3. Conclusion

We have presented the numerical results by using Matlab programming language to generate the energy eigenvalues for point like nucleus. But it has proven that it is possible to find the energy eigenvalue and probability density for finite size nucleus. Our calculations have considered the reduced mass of muon in order to correct the error incurred by the assumption that the nucleus of a muonic hydrogen is point-like which in turn gives it an infinite mass. From our calculation, the difference in energy eigenvalues between $2s_{1/2}$ and $2p_{1/2}$ cannot provide us the lamb shift. And also, we conclude that muon has greater chance to be found near the

nucleus of hydrogen atom for ground state and less chance to be found near the nucleus when the muon is in excited states.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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