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On the Unitary Representations of the Braid Group B_6

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Abstract: We consider a non-abelian leakage-free qudit system that consists of two qubits each composed of three anyons. For this system, we need to have a non-abelian four dimensional unitary representation of the braid group B_6 to obtain a totally leakage-free braiding. The obtained representation is denoted by ρ. We first prove that ρ is irreducible. Next, we find the points $y \in \mathbb{C}^*$ at which the representation ρ is equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\mu}_6(\pm i)$, an irreducible four dimensional representation of the braid group B_6. The representation $\hat{\mu}_6(\pm i)$ was constructed by E. Formanek to classify the irreducible representations of the braid group B_n of low degree. Finally, we prove that the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a hermitian positive definite matrix.

Keywords: braid group; unitarity; equivalence; irreducibility

1. Introduction

Due to Artin, the braid group B_n is represented in the group $Aut(F_n)$ of automorphisms of the free group F_n generated by $x_1, \ldots, x_n$. The matrix representation of B_n was published by W. Burau in 1936. This representation was known as a Burau representation. Since then, other matrix representations of B_n have been constructed. For more details, see [1].

Braid group unitary representations have been essential in topological quantum computations. To understand the d-dimensional systems in which anyons are exchanged, a lot of work has been made. The exchange of n anyons inside the qudit system, the d-dimensional analogues of qubits, has been governed by the braid group B_n which has $n-1$ generators $\tau_1, \ldots, \tau_{n-1}$. Here, τ_i exchanges the particle i with its neighbor, particle $i+1$.

When the topological charge of the qudits changes due to the braiding of the anyons from different qudits, a leakage of some of the information will occur in the computational Hilbert space, the fusion space of the anyons.

The leakage-free braiding of anyons has been under investigation for a while. To perform universal quantum computation without any leakage, the requirement would be to consider two-qubit gates. This would be very restrictive and this property can only be realized for two-qubit systems related to the Ising-like anyons model [2].

R. Ainsworth and J.K. Slingerland showed that a non-abelian, leakage-free qudit of dimension d involving n anyons is equivalent to a non-abelian d-dimensional representation of the braid group B_n. Here, n is the sum of the number of anyons n_1 inside the first qudit and the number of anyons n_2 inside the second qudit. As for the dimension d of the representation of B_n, it is the product of the dimensions d_1 and d_2 of the Hilbert spaces of the individual qudits.

Moreover, it was proved in [2] that the number of anyons per qubit is either 3 or 4. Thus, there are mainly 3 different types of two-qubit systems and a 4-dimensional representation of the corresponding braid group is constructed for each. Taking into account E. Formanek's result that there is no d-dimensional representation of B_n with $d < n-2$, it was verified in [2] that the only possible type of two-qubit system is having 2 qubits of which each is composed of 3 anyons.

This system is a non-abelian leakage-free qudit system of dimension 4 involving 6 anyons. It is equivalent to a non-abelian 4-dimensional representation of the braid group B_6. This representation is denoted by ρ. Since the number of anyons is 6, there are 5 elementary exchanges $\tau_1, \ldots, \tau_5$. The exchanges τ_1, τ_2, τ_4, and τ_5 satisfy the following relations:

$$\rho(\tau_i) = \rho_1(\tau_i) \otimes I_{d_2} \quad (1 \leq i \leq n_1 - 1)$$

and

$$\rho(\tau_i) = I_{d_1} \otimes \rho_2(\tau_i) \quad (n_1 + 1 \leq i \leq n - 1),$$

where ρ_1 and ρ_2 are the d_1 and d_2- dimensional representations of B_{n_1} and B_{n_2} on the Hilbert spaces of the first and second qudit respectively. I_{d_1} and I_{d_2} are the d_1 and d_2-dimensional identity matrices respectively. Here, $n_1 = n_2 = 3$ and $d_1 = d_2 = 2$.

The matrix $\rho(\tau_3)$ is constructed by imposing braid group relations. For more details, see [2].

In our work, we consider the unitary representation ρ and the irreducible representation $\hat{\mu}_6(\pm i)$ which is defined by E. Formanek in [3]. Both representations are 4-dimensional representations of the braid group B_6.

First, we prove that the unitary representation $\rho : B_6 \to GL_4(C)$ is irreducible.

As the representation ρ is proved to be irreducible, it follows that it is equivalent to the tensor product of a one-dimensional representation $\chi(y)$ and the irreducible 4-dimensional representation $\hat{\mu}_6(\pm i)$, where $y \in \mathbb{C}^*$. For more details, see [3].

We then determine the points $y \in \mathbb{C}^*$ at which the two representations ρ and $\chi(y) \otimes \hat{\mu}_6(\pm i)$ are equivalent.

Finally, we show that the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a hermitian positive definite matrix.

2. Preliminaries

Definition 1 (See [4]). *The braid group on n strings, B_n, is the abstract group with presentation* $B_n = \{\sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, \ldots, n-2\}$.

The Hecke algebra representation of B_6 was constructed by V.F.R. Jones in [5]. E. Formanek obtained a low-degree representation of B_6 by conjugating the representation constructed by V.F.R. Jones by a certain permutation matrix. For more details, see [3].

Definition 2 (See [3]). *The representation $\mu_6 : B_6 \to GL_5(\mathbb{Z}[t^{\pm 1}])$ is given by:*

$$\mu_6(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & -t \\ 0 & -t & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix}, \qquad \mu_6(\sigma_2) = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mu_6(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & -t & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mu_6(\sigma_4) = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix},$$

and

$$\mu_6(\sigma_5) = \begin{pmatrix} 1 & 0 & -t & 0 & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

The restriction of μ_6 to B_5 is denoted by $\mu_5 : B_5 \to GL_5(\mathbb{C})$. For $z \in \mathbb{C}^*$, the complex specializations $\mu_6(z)$ and $\mu_5(z)$ are obtained from μ_6 and μ_5 respectively by letting $t = z$.

Definition 3 (See [3]). *A representation $\rho : B_n \to GL_r(\mathbb{C})$ is of Burau type if $r \geq 2$ and it is equivalent to an irreducible representation $\chi(y) \otimes \beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ or $\chi(y) \otimes \hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$ where $\chi(y)$ is a one-dimensional representation, $\beta_n(z)$ is the reduced Burau representation, and $\hat{\beta}_n(z)$ is the composition factor of the reduced Burau representation.*

Definition 4 (See [3]). *Let $y \in \mathbb{C}^*$. The representation $\chi(y) : B_n \to \mathbb{C}^*$ is the one-dimensional representation defined by $\chi(y)(\sigma_i) = y$.*

Theorem 1 (See [3]).
1. *For $z \in \mathbb{C}^*$, $\mu_5(z)$ and $\mu_6(z)$ are irreducible unless z is a root of $(t^2 + t + 1)(t^2 + 1)$;*
2. *If z is a root of $t^2 + t + 1$, then the composition factors of $\mu_6(z)$ are $\chi(-z)$ and $\hat{\beta}_6(z)$ and the composition factors of $\mu_5(z)$ are $\chi(-z)$ and $\beta_5(z)$;*
3. *If z is a root of $t^2 + 1$, then the composition factors of $\mu_6(z)$ are $\chi(1)$ and an irreducible representation $\hat{\mu_6}(z) : B_6 \to GL_4(\mathbb{C})$, where $\hat{\mu_6}(z)(\sigma_1)$ has eigen values $1, 1, -z, -z$. The composition factors of $\mu_5(z)$ are $\chi(1)$ and an irreducible representation $\hat{\mu_5}(z)$, which is the restriction of $\hat{\mu_6}(z)$ to B_5.*

Theorem 2 (See [3]). *Let $\rho : B_6 \to GL_r(\mathbb{C})$ be an irreducible representation, where $2 \leq r \leq 5$. Then one of the following is true.*

1. *The representation ρ is of a Burau type;*
2. *For some $y \in \mathbb{C}^*$, ρ is equivalent to $\chi(y) \otimes \hat{\mu_6}(\pm i) : B_6 \to GL_4(\mathbb{C})$. Distinct pairs $(y, \pm i)$ give inequivalent representations;*
3. *For some $y, z \in \mathbb{C}^*$, ρ is equivalent to $\chi(y) \otimes \mu_6(z) : B_6 \to GL_5(\mathbb{C})$, where z is not a root of $(t^2 + t + 1)(t^2 + 1)$.*

3. Irreducibility of $\rho : B_6 \to GL_4(\mathbb{C})$

The construction of a two-qubit system with a minimum amount of leakage has been of great interest. The only two-qubit system that can be realized without leakage is the system of two 3-anyon qubits. This system is equivalent to a 4-dimensional representation of the braid group B_6. This representation which was constructed in [2] is denoted by ρ.

In this section, we prove that $\rho : B_6 \to GL_4(\mathbb{C})$ is irreducible. We denote τ_i, the exchange of the i^{th} and $(i+1)^{th}$ anyon, by σ_i where $1 \leq i \leq 5$.

Definition 5 (See [2]). *The representation $\rho : B_6 \to GL_4(\mathbb{C})$is defined as follows:*

$$\rho(\sigma_1) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix}, \qquad \rho(\sigma_2) = \begin{pmatrix} \frac{1}{a-a^3} & 0 & c & 0 \\ 0 & \frac{1}{a-a^3} & 0 & c \\ -c & 0 & \frac{1}{\bar{a}-\bar{a}^3} & 0 \\ 0 & -c & 0 & \frac{1}{\bar{a}-\bar{a}^3} \end{pmatrix},$$

$$\rho(\sigma_3) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \bar{x} & 0 & 0 \\ 0 & 0 & \bar{x} & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \qquad \rho(\sigma_4) = \begin{pmatrix} \frac{1}{f-f^3} & e & 0 & 0 \\ -e & \frac{1}{\bar{f}-\bar{f}^3} & 0 & 0 \\ 0 & 0 & \frac{1}{f-f^3} & e \\ 0 & 0 & -e & \frac{1}{\bar{f}-\bar{f}^3} \end{pmatrix},$$

and

$$\rho(\sigma_5) = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & \bar{f} & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & \bar{f} \end{pmatrix}.$$

In [2], it was shown that the multiplicities among the eigen values of the generators of the braid group result in the formation of topological charges during the fusion of the anyons in the system. As the system considered in our work is leakage-free, the eigen values of the generators σ_i of the representation ρ should be the same. Thus, $f = a$ or $f = \bar{a}$ and $e = c = \sqrt{1 - \frac{1}{2-a^2-\bar{a}^2}}$.

By simple computations, the relation $\rho(\sigma_2)\rho(\sigma_3)\rho(\sigma_2) = \rho(\sigma_3)\rho(\sigma_2)\rho(\sigma_3)$ yields the equation $a^2 = -\bar{a}^2$. But, $a = e^{i\theta}$. Therefore, $e^{2i\theta} = -e^{-2i\theta}$. This implies that $e^{4i\theta} = -1$. Consequently, $e^{8i\theta} = 1$. That is, a must be a primitive eighth root of unity. Furthermore, $c = \sqrt{1 - \frac{1}{2-a^2-\bar{a}^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$.

Note that since a is a primitive eighth root of unity, $a^8 = 1$ and $a^2 \neq 1$. Then, $a^3 \neq a$. Consequently, $\bar{a}^3 \neq \bar{a}$, $f \neq f^3$, and $\bar{f} \neq \bar{f}^3$. This emphasizes that the defined matrices $\rho(\sigma_i)$, $1 \leq i \leq 5$, are well-defined.

Now we study the irreducibility of ρ. For simplicity, we denote $\rho(\sigma_i)$ by σ_i for $1 \leq i \leq 5$.

Lemma 1. *The representation $\rho : B_6 \to GL_4(\mathbb{C})$ has no non trivial proper invariant subspaces of dimension* 1.

Proof. Let S be a proper invariant subspace of dimension 1. We consider all the possible cases.

Case 1: $S = \langle e_i \rangle$, $i = 1, 2, 3, 4$.

For simplicity, we take $i = 1$. Since S is invariant, it follows that $\sigma_2(e_1) = \begin{pmatrix} \frac{1}{a-a^3} \\ 0 \\ -c \\ 0 \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

Case 2: $S = \langle e_i + ue_{i+1} \rangle$, $i = 1, 3$,$u \in \mathbb{C}^*$.

For simplicity, we take $i = 1$. Since S is invariant, it follows that $\sigma_2(e_1 + ue_2) = \begin{pmatrix} \frac{1}{a-a^3} \\ \frac{u}{a-a^3} \\ -c \\ -cu \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

Thus, there are no non trivial proper invariant subspaces of dimension 1. □

Lemma 2. *The representation $\rho : B_6 \to GL_4(\mathbb{C})$ has no non trivial proper invariant subspace of dimension* 2.

Proof. Let S be a proper invariant subspaces of dimension 2. We consider all the possible cases.

Case 1: $S = \langle e_i, e_{i+1} \rangle$, $i = 1, 3$.

For simplicity, we take $i = 1$. Since S is invariant, it follows that $\sigma_2(e_1) = \begin{pmatrix} \frac{1}{a-a^3} \\ 0 \\ -c \\ 0 \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

Case 2: $S = \left\langle e_i, e_j \right\rangle$, $i = 1, 2$, $j = 3, 4$.

For simplicity, we take $i = 1$. Since S is invariant, it follows that $\sigma_4(e_1) = \begin{pmatrix} \frac{1}{f-f^3} \\ -e \\ 0 \\ 0 \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

Case 3: $S = \langle e_i, e_1 + ue_2 \rangle$, $i = 3, 4$,$u \in \mathbb{C}^*$.

For simplicity, we take $i = 3$. Since S is invariant, it follows that $\sigma_4(e_3) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{f-f^3} \\ -e \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

Case 4: $S = \langle e_i, e_3 + ve_4 \rangle$, $i = 1, 2$,$u \in \mathbb{C}^*$.

For simplicity, we take $i = 1$. Since S is invariant, it follows that $\sigma_4(e_1) = \begin{pmatrix} \frac{1}{f-f^3} \\ -e \\ 0 \\ 0 \end{pmatrix} \in S$.

This implies that $e = 0$. But, $e = c$. Thus, $c = 0$, a contradiction.

Case 5: $S = \langle e_1 + ue_2, e_3 + ve_4 \rangle$, $u, v \in \mathbb{C}^*$.

Since S is invariant, it follows that $\sigma_3(e_1 + ue_2) = \begin{pmatrix} x \\ \bar{x}u \\ 0 \\ 0 \end{pmatrix} \in S$. This implies that $a = \bar{a}$. That is, $a^2 - \bar{a}^2 = 0$.

But, from the equation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$, we have $a^2 + \bar{a}^2 = 0$. Thus, $2a^2 = 0$ which gives a contradiction.

Thus, there are no non trivial proper invariant subspaces of dimension 2. □

Now, we state the theorem of irreducibility.

Theorem 3. *The representation* $\rho : B_6 \to GL_4(\mathbb{C})$ *is irreducible.*

Proof. By Lemma 1 and Lemma 2, there are no proper invariant subspaces of dimensions 1 and 2. Clearly, the representation ρ is unitary, that is $\sigma_i\sigma_i^* = I_4$ for $1 \leq i \leq 5$.

We note that if the representation is unitary, then the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. As there is no proper invariant subspace of dimension 1, there is no proper invariant subspace of dimension 3.

As a result, all the possible proper subspaces are not invariant. Consequently, ρ is irreducible. □

4. The Representations ρ and $\chi(y) \otimes \hat{\mu_6}(\pm i)$ Are Equivalent

By Theorem 3, the representation ρ is irreducible. The eigen values of $\rho(\sigma_i)$ for $1 \leq i \leq 5$ are different from those of $\hat{\beta}_4(z)$, the composition factor of the reduced Burau representation. Therefore, the representation ρ is not equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\beta}_4(z)$. That is, ρ is not of a Burau type.

Moreover, ρ is a 4-dimensional representation. Consequently, Theorem 2 implies that the representation ρ is equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$.

Note that, by Theorem 1, the representation $\hat{\mu}_6(z)$ is irreducible for $z = \pm i$ since the roots of the polynomial $t^2 + 1$ are clearly $\pm i$.

In this section, we determine the points $y \in \mathbb{C}^*$ at which the representations ρ and $\chi(y) \otimes \hat{\mu}_6(\pm i)$ are equivalent.

Since the two representations are equivalent, the determinants of the matrices $(\chi(y) \otimes \hat{\mu}_6(\pm i))(\sigma_i)$ and $\rho(\sigma_i)$ are equal for $1 \leq i \leq 5$.

By simple computations, the determinant $Det[(\chi(y) \otimes \hat{\mu}_6(\pm i))(\sigma_i)] = -y^4$ and $Det[\rho(\sigma_i)] = 1$ for all $i = 1, 2, 3, 4, 5$. Thus, $-y^4 = 1$. This implies that $y = \pm\sqrt{(\pm i)}$.

As a result, the two considered representations are equivalent at the following points: $y_1 = \sqrt{i}$, $y_2 = \sqrt{-i}$, $y_3 = -\sqrt{i}$, and $y_4 = -\sqrt{-i}$, where i is the complex number such that $i^2 = -1$.

5. Unitarity of $\chi(y) \otimes \hat{\mu}_6(\pm i) : B_6 \to GL_4(\mathbb{C})$

As the representation ρ is proved to be unitary and equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$, the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a matrix M.

In this section, we find the matrix M and we prove that M is a hermitian and positive definite.

Definition 6 (See [3]). *The representation $\hat{\mu}_6(\pm i) : B_6 \to GL_4(\mathbb{C})$ is defined as follows:*

$$\sigma_1 \mapsto \begin{pmatrix} \pm i & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & \pm i & 0 & 0 \\ 0 & \pm i & 0 & 0 \\ 0 & 0 & 1 & \pm i \\ 0 & 0 & 0 & \pm i \end{pmatrix},$$

$$\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pm i & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \sigma_4 \mapsto (1/2) \begin{pmatrix} 1 \pm i & 0 & -1 \pm i & 0 \\ 0 & 1 \pm i & 0 & -1 \pm i \\ -1 \pm i & 0 & 1 \pm i & 0 \\ 0 & -1 \pm i & 0 & 1 \pm i \end{pmatrix},$$

and

$$\sigma_5 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix}.$$

Now, we state the following theorem:

Theorem 4. *The images of the generators of B_6 under $\hat{\mu}_6(\pm i)$ are unitary relative to a hermitian positive definite matrix M.*

Proof. Let,

$$M = \begin{pmatrix} 2 & 1-i & 0 & 0 \\ 1+i & 2 & 0 & 0 \\ 0 & 0 & 2 & 1-i \\ 0 & 0 & 1+i & 2 \end{pmatrix}.$$

Here, i is the complex number such that $i^2 = -1$ and M is an invertible matrix whose determinant equals 4.

For simplicity, we denote $(\hat{\mu}_6(\pm i))(\sigma_i)$ by σ_i for $1 \leq i \leq 5$.

By direct computations, $\sigma_1 M \sigma_1^* = \sigma_2 M \sigma_2^* = \sigma_3 M \sigma_3^* = \sigma_4 M \sigma_4^* = \sigma_5 M \sigma_5^* = M$, where σ_i^* is the complex conjugate transpose of σ_i for $1 \leq i \leq 5$. Therefore, the representation $\hat{\mu}_6(\pm i)$ is a unitary relative to the matrix M.

Let M^* be the complex conjugate transpose of M. Clearly, $M^* = M$. This implies that the invertible matrix M is hermitian.

By computations, the eigen values of the matrix M are $2 + \sqrt{2}$ and $2 - \sqrt{2}$. Clearly, both values are positive. Consequently, M is a positive definite matrix.

As a result, the representation $\hat{\mu}_6(\pm i)$ is a unitary relative to an invertible hermitian positive definite matrix M. □

Note that the unitarity of the representation $\hat{\mu}_6(\pm i)$ relative to the matrix M clearly implies that the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is also a unitary relative to the same matrix M.

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