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Topical Review

A review of non-Lorentz invariant variable speed of light theories

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Abstract

This work re-derives and discusses non-Lorentz invariant variable speed of light (VSL) theories in the context of cosmological problems. Following a thorough introduction to the subject, an explicit solution demonstrating a possible dependence of the speed of light on the cosmological scale factor is presented and analyzed. The parameters of the initial ansatz, $c(t) = c_0 a^n$, are constrained by requiring the VSL formulation to be a solution to the flatness and horizon problems. The theoretical section is concluded with a derivation of the change of entropy in a VSL Universe. Even though such findings imply that the speed of light can vary only in non-flat spacetime, an adapted approach using the Generalized Second Law of Thermodynamics is shown to loosen this restriction. Further, in the experimental section, recent evidence for a temporally varying fine structure constant at $\approx 4\sigma$ significance is presented as a potential test for the VSL hypothesis. Overall, this work introduces and evaluates many aspects of non-Lorentz invariant VSL theories whilst encouraging future research and serving as a largely self-sufficient comprehensive overview paper.

Keywords: variable speed of light, fine structure constant, temporal variation, entropy

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Contents

It is usually assumed that the laws of nature have always been the same as they are now. There is no justification for this. The laws may be changing, and in particular quantities which are considered to be constants of nature may be varying with cosmological time.

Paul Dirac, 'On methods in theoretical physics', June 1968, Trieste

1. Introduction

The spatial and temporal consistency of the speed of light are considered a sacred fact in most cosmological evolution models. When discussing the conditions of the very early Universe, contemporary physics primarily relies on theoretical formulations and well-rounded hypotheses. Inflation—the widely accepted theory of superluminal expansion in the earliest stages of the Universe—solves the *horizon problem* without introducing variations on fundamental constants. Contrarily, as a possible contester of Inflation, variable speed of light (VSL) theories provide solutions to cosmological problems by introducing a temporally varying speed of light.

1.1. Varying dimensional constants

As with most varying constant theories, it is important to begin the discussion by addressing the fact that the speed of light is *not* a dimensionless parameter. As a result, a counter-argument for any VSL theory can be easily made by assuming that there exists a definition of units such that c remains constant all throughout space and time. This argument was proposed in J Duff's paper, which concludes that 'it is operationally meaningless and confusing to talk about time variation of arbitrary unit-dependent constants whose only role is to act as conversion factors' [1]. A similar analogy states that, if time were to be measured in some different manner, even Newton's equations would be rendered more complicated. However, within the context of our choice of units, the equations will essentially have the same meaning, just written differently. This point was raised by Poincaré who stated that the choice of units is a trivial necessity and units should be defined based on how convenient they are [2].

Another example given by Poincaré is the re-definition of c in Maxwell's equations, such that the speed of light remains constant in dielectric media. A substitution such as $c = \frac{c_0}{\sqrt{\epsilon}}$ will achieve this, however, it will significantly complicate the expression of Maxwell's laws. Such a substitution will require the addition of time derivatives, as well as gradients of ϵ, to Maxwell's equations [3]. Arguably, Poincareé will not be in favor of such a re-definition of c due to its inconvenience, however, this example conversely demonstrates the previous claim: given that c is a dimensional constant, units can always be defined in to render it non-varying.

1.2. VSL loophole: a variable fine structure constant

As the discussion above implies, a theory based on a varying dimensional parameter is tautological. However, in most VSL theories, the temporal inconsistency of c is postulated as a natural consequence of the variability of dimensionless parameters, such as the fine structure constant, α, or a ratio between $\frac{c(t)}{c(t')}$.

In a series of papers, which will be more thoroughly discussed in section 3, Murphy *et al* provide evidence for the possibility of a varying fine-structure constant [4]. In summary, their work focuses on quantifying $\frac{\Delta\alpha}{\alpha}$ in medium-redshift damped Lyman-α systems (DLAs). Initially, they reported statistical evidence for a smaller α at earlier times: $\frac{\Delta\alpha}{\alpha} = (-0.72 \pm 0.18) \times 10^{-5}$. Given the fine-structure constant's inverse proportionality to c, the Webb *et al* results appear promising for VSL theories. As remarked by Bekenstein, VSL theories can be considered interchangeable with 'variable α theories' (and in some instances, such as [5], 'variable gravitational coupling' theories).

Fundamentally, Webb's statistical results motivate the claim that the variability in α is caused by a temporally inconsistent c, $\hbar$, or e (or some combination of them). In the case of a varying *dimensionless* constant theory, a choice must be made as to which *dimensional* parameters are varying. This translates to fixing a system of parameters and units, thus establishing a consistent framework in which a theory can be developed [3]. In particular, VSL theories fix the values of the elementary charge, e, and the Planck constant, $\langle$. Therefore, in this framework, any variation observed in α is credited to a variable $c(t)$.

However, this choice is not entirely trivial; different varying 'constants' predict unique physical behaviors that can potentially be experimentally verified. While proposing a varying e theory, Beckenstein [6] implies that only in the context of electromagnetic field-to-matter coupling can the variability of e and c be treated as equivalent. This is demonstrated in [5], where all charges, including e, are set to vary. However, their variation is described by constant ratios that, theoretically, can be absorbed in a varying c parameter. Nevertheless, in all other senses, a VSL theory can be distinguished from a varying e theory since it tends to preserve the weak equivalence principle while breaking Lorentz invariance. In simple terms, the equivalence principle states that the gravitational and inertial mass of particles is the same, thus implying that the trajectory taken by any point-like particle in a non-accelerating frame under the influence of gravity is equivalent. This was tested even before the formulation of General Relativity through the measurement of the Eötvös parameter:

$$\frac{2|a_1 - a_2|}{a_1 + a_2}, \tag{1}$$

where a_1 and a_2 are the accelerations of the two test masses. Unlike VSL, as discussed by [3], a varying e theory predicts a non-zero Eötvös parameter, in the order of 10^{-13}. Current experimental constraints are unable to probe this value, however, [7] outlines the possibility for the STEP satellite to measure the Eötvös parameter to one part in 10^{18} which could eliminate variable e theories capable of explaining the Webb *et al* results. Additional experimental tests that distinguish a variable c from e are detailed in [8].

1.3. Brief catalogue of VSL theories

Presently, it is difficult to find a general consensus on the formulation of a unified VSL theory. In a broader sense, current efforts uniquely explore the implications of variable c. As classified in [3], VSL theories can be broadly catalogued into 'extreme', or 'hard breaking of Lorentz symmetry' [9, 10] (the main focus of this paper), 'bimetric' [11, 12], 'color dependent' [13], and 'Lorentz invariant' [14, 15].

As hinted above, the formulation of VSL theories is tied to the extent to which they violate the main principles of General Relativity: non-variability of c and Lorentz invariance (essentially implying that all observers moving with respect to each other within an inertial frame will agree upon the laws of physics; in other words, dynamical equations, such as Maxwell's field equations, of a system are preserved under the action of a transformation in the Lorentz group). For an example, bimetric theories, such as [11, 12], are Lorentz invariant. They are formulated on the idea that even though the speed of massless particles differs, all principles of Special Relativity are preserved in each sector. Furthermore, the fundamental building blocks of 'Lorentz invariant' and 'color dependent' VSL theories are self-explanatory and recommended further reading on this subject can be found in appendix C. In addition, appendix C.3 contains a catalogue of academic resources detailing recent research and discussion on various aspects of VSL theories.

2. Albrecht and Magueijo (AM) VSL model

2.1. Motivation

The Big Bang model demands a theory, such as Inflation, in order to explain (for instance) the horizon problem. Inflation—a period of superluminal expansion in the early Universe—provides an explanation for the observation that, what should have been causally disconnected regions, were in a causal contact at some point in the past. Alternatively, instead of changing the matter content of the Universe, AM [9] propose that a drastic change in the speed of light should lead to the same current space geometry and observations as Inflation. AM [9] provides no formal solution to $c(t)$, with a discussion revolving a step change in the speed of light at some time in the early Universe. It is essential to consider this initially, since it emphasizes the purpose behind a VSL theory in the simplest manner. Further on, in section 2.5, a more formal definition of $c(t)$ will be discussed, where $c(t)$ is modeled as a continuous decaying function.

2.2. Particle horizons, past light cones, and the Cosmic Microwave Background (CMB)

By definition, the comoving radius, r_h, is the distance covered by a particle had it been traveling at $\mathbf{c}$ since the beginning of the Universe. This manifests as a cosmological horizon, implying that any observer (at any given time) is able to see (and be affected by) only a finite region of space, with a radius r_h. This can be expressed using the conformal time, η, which is the time a photon will require to travel from the observer's current position to the furthest observable

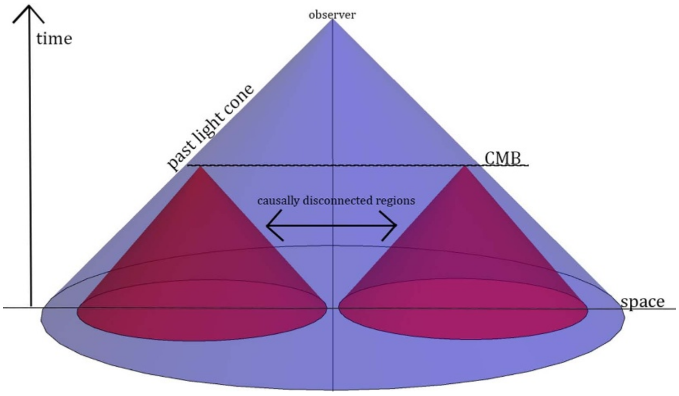

Figure 1. Past light cones in a Universe that started at the Big Bang.
Note: Diagram depicts the past light cone of an observer at current time and two causally disconnected events occurring at the moment of the CMB's release, stretching back to the Big Bang. As usual, c is represented by 45° lines.

distance (under the assumption that the Universe stops expanding). Mathematically, conformal time can be expressed as: $\eta = \int_0^t \mathrm{d}t'\ a(t')^{-1}$, where t is some time and $a(t)$ is the scale factor that will be further discussed in the context of the Friedmann equations. A trivial derivation of r gives: $c\int_0^\eta \mathrm{d}\tau = \int_0^{r_h} \mathrm{d}r \ \rightarrow \ r_h = c\eta$.

As previously noted, the principal motivation behind theories such as VSL and Inflation is the observed causal connection of spatial regions that, otherwise, should not have had time to interact. Essentially, each observer carries a *past light cone*, as depicted in figure 1, whose boundaries are determined by the speed of light—a set speed limit on the motion of known particles. Essentially, all events that can affect the observer (situated at the peak of the light cone) must originate within the bounds of the past light cone. This ability to interact, is, what was already referred to as, a causal connection. Light emitted outside the bounds of the past light cone can never be measured or seen by the observer (a crucial feature of the space-time diagram is that light travels at a $\frac{\pi}{4}$ angle). In close relation to the definition of a past light cone is the notion of a *particle horizon*, the largest possible proper distance that can be observed from a point in space-time. Essentially, the particle horizon marks the furthest away point, both spatially and temporally, from which an event that occurred in the past can affect an observer.

A more detailed discussion on the origin of the CMB can be found in appendix B.1. In essence, the CMB is a relic of radiation from the early Universe, roughly 380 000 years after the Big Bang. Among other things, it provides information on the density distribution and structure of the early Universe. It can be thought of as a 'frozen image' from the Universe at the moment when the radiation was emitted. The CMB temperature measurements have yielded a value of $2.725\,48 \pm 0.000\,57$ K [16]. This is a fairly isotropic value (same in all directions), with minor anisotropies discussed further. The apparent isotropy of the CMB between causally disconnected regions is the underlying premise behind the *horizon problem*.

2.3. The horizon problem: a powerful motivation for VSL theories

As previously discussed, the existence of particle horizons limits the ability of an event to affect a region of space if it occured outside the bounds of its light cone. It is expected, for two causally disconnected regions, as depicted by the two smaller cones in figure 1, to have no information about the properties (or existence) of each other (at the time when the CMB was released). The figure below demonstrates the essence of the horizon problem: an observer existing at current time can simultaneously view/measure regions of space that were causally disconnected at the moment when the CMB was released; furthermore, the CMB temperature isotropy observed among those regions led to the conclusion that in some way, spatial regions that should not have had time to interact since the Big Bang, were somehow in causal contact at a point in the past.

A more rigorous mathematical description of the horizon problem can be found in chapter 8.8 of [17]. An expanded version of this derivation, as well as a conceptual explanation, is presented in appendix A.1. Essentially, the particle horizon (comoving radius from the moment of the Big Bang until the emission of the CMB) can be approximated to $r_{\mathrm{CMB}} = 0.06 \times H_0^{-1}$. Based on this, any event occurring outside this boundary, at the moment of the Big Bang, is considered causally disconnected from a region in space at the time of the CMB. In figure 1 this is represented by the red light cones that do not intersect at any point in the past.

When measuring regions of the CMB, a current observer can see radiation from points in space that were causally disconnected at that time. This can be demonstrated by a derivation of Δr, the comoving distance between a current observer and a point on the CMB, as $\Delta r \approx 2 \times H_0^{-1}$. Noting that $\Delta r \gg r_{\mathrm{CMB}}$ (where Δr corresponds, in figure 1, to the observer's past light cone radius at the CMB line), it can be concluded that by measuring distant points on the CMB at current time, it is possible to see the properties of causally disconnected regions in the past.

However, the isotropy of the CMB temperature, or in other words, the thermal smoothness currently observed at all points, implies that disconnected regions of space (whose past light cones do not intersect) were in causal contact in the very early Universe. In a search for a theory that can explain the information exchange between distant patches of the Universe, different theories, such as Inflation and VSL, were formulated. Inflation proposes a solution of superluminal expansion of space, indicating that space was much more compact. However, instead of changing the geometry of space, VSL, more simply, suggests that instead of space expanding superluminously, light slowed down. In such a manner, a faster speed of light would imply that information would have been able to reach more distant patches of the Universe without the need for superluminal spatial expansion. The most basic application of this idea will be discussed in the following section, where $c(t)$ is assumed to have only two discrete values. Notably, this is not necessarily the case with VSL theories, and a continuous $c(t)$ can be found that provides solutions to the *horizon*, as well as *flatness problem*.

2.4. Simple VSL solution: an illustrative example in flat spacetime

Prior to a more rigorous derivation, it is important to discuss the aforementioned simplified, intuitive version of the formulation in [9, 10]. This is solely done for explanation purposes, with a more rigorous derivation being outlined in the following section 2.5. (Note: the notation used throughout this paper may differ from the original for the purpose of consistency and clarity)

In the simplified picture, [9] proposes that at some critical time, t_c, the speed of light instantaneously changed from c_+ to c_-, where $c_+ > c_-$. Vaguely, this would imply that prior to t_c the past light cone of any spacetime region would be stretched out, with light traveling at a smaller angle seen from the horizontal, *space*, axis. Consequently, regions that should have been causally disconnected, had intersecting past light cones in the early Universe. In turn, this enabled said regions to interact and exchange information. This could explain the isotropic temperature observed in the CMB among distant patches of space.

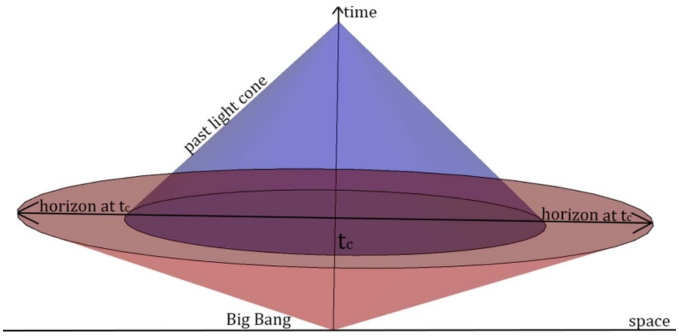

Figure 2. Conformal diagram of past light cone with VSL.
Note: Diagram depicts the shape of the past light cone and the horizon at t_c when $c_+ \to c_-$. As usual, c_- is represented by 45° lines.

The main feature of figure 2 is to demonstrate that an observer's particle horizon can discontinuously (in this case, where $c(t)$ is modeled by a step change) become larger at some critical time if the speed of light at $t < t_c$ was greater than what is currently measured. Another thing to note on the diagram is that, for the purpose of clarity, the entire past light cone is not drawn, just until the intersection, where the particle horizon increases. The red region represents the future light cone of a photon originating at the Big Bang, assuming it freely moved through space since $t = 0$. Due to the assumption that it traveled at speed c_+, the cone's radius is larger than the one associated with c_- (represented in blue), which is the speed of light measured at present time. In such a way, information in the pre-CMB Universe has the ability to travel faster and affect more distant regions of space, leading to a causal connection between those patches.

In order to quantify the ratio $\frac{c_+}{c_-}$, the comoving radius at t_c, r_c, can easily be expressed. Let η be the conformal time now and η_c the conformal time at $t = t_c$. Based on the formula derived in section 2.2, the comoving radius at t_c is $r_c = c_+\eta_c$. Similarly, let r_0 be the comoving radius from current time until the intersection at $t = t_c$. It can be easily seen that $r_0 = c_-(\eta - \eta_c)$. In order to satisfy the condition that $r_c \gg r_0$:

$$\frac{c_+}{c_-} \gg \frac{\eta}{\eta_c} - 1 \quad \approx \frac{c_+}{c_-} \gg \frac{\eta}{\eta_c} \quad \text{assuming } \eta \gg \eta_c. \tag{2}$$

Satisfying this condition enables distant regions of the Universe to be in causal contact at some point in the distant past. In order to get a very rough numerical order-of-magnitude estimate, a $\log_{10}$ can be applied on both sides. As done by [9], lets assume that at the critical time, η_c, takes the value of 10^{-32} s. This reflects a theoretical value determined for the end of the inflationary epoch. In reference to Inflation, this marks the final moment of superluminal spatial expansion and the beginning of a (much slower) universal expansion that is observed at current time. Hence, in the context of VSL theories, this value is appropriate for a rough estimate since it marks a theoretical turning point in the early Universe. As stated, unlike Inflation, VSL uses the idea that a larger speed of light resulted in the causal contact of distant regions, not superluminal spatial expansion. Instead of an inflationary epoch, during $t < 10^{-32}$ s, an increased c allowed for a vaster network of spatial patches to be causally connected in the early Universe. Essentially, it was not space that was more compact, but the traveling speed of information that was significantly greater.

To roughly estimate the ratio $\frac{c_+}{c_-}$, it is necessary to determine an approximate value for η. Since the only imposed condition is that $\eta \gg \eta_c$, as done by [9], η will be taken to be the conformal time at the matter-radiation equality, about 50 000 years after the Big Bang. This choice is convenient since the estimate of the redshift at the matter-radiation equivalence can be done through a trivial calculation (see appendix A.2). The equation for conformal time is $\eta = \int_0^t \mathrm{d}t'\, a(t')^{-1}$. Assuming a matter dominated Universe, $a(t) \propto t^{\frac{2}{3}}$. Therefore, at $t = t_{\mathrm{eq}}$:

$$\eta = \int_0^{t_{\mathrm{eq}}} \mathrm{d}t'\, a\left(t'\right)^{-1} \propto 3t_{\mathrm{eq}}^{\frac{1}{3}} \propto \sqrt{a\left(t_{\mathrm{eq}}\right)}. \tag{3}$$

(Note: WLOG, a more detailed, similar derivation can be found in appendix A.1, in the context of the horizon problem and comoving radii)

Further, applying the relation $a(t_{\mathrm{eq}}) = \frac{1}{1+z_{\mathrm{eq}}}$ gives an estimate of $\eta \approx \sqrt{\frac{1}{1+z_{\mathrm{eq}}}} \approx z_{\mathrm{eq}}^{-\frac{1}{2}}$. Plugging the values using a logarithmic scale gives a very rough numerical estimate of equation (2):

$$\log_{10} \frac{c_+}{c_-} \gg 32 - \frac{1}{2} \log_{10} z_{\mathrm{eq}} \approx 30. \tag{4}$$

Therefore, in this very simple illustrative example, which was derived for the purpose of presenting the chief idea behind VSL in a more intuitive manner, in order for equation (2) to hold, the speed of light needs to be roughly 30 orders of magnitude greater in the very early Universe. Similarly, the estimation performed in [9] agrees with the value calculated above. This result, however, should not be confused with the actual formulation of the VSL theories presented in [10], which models the speed of light as a continuous function dependent on the scale factor, $a(t)$. The following section presents a detailed discussion of this, and the constraints imposed on $c(t)$ derived as a consequence of the horizon and flatness problems.

2.5. AM VSL formulation for any spacetime curvature

2.5.1. Justification for a preferred cosmological reference frame. The VSL theory proposed in [9] and further discussed in [10] is based on the idea that the Universe has a preferred cosmological reference frame. Magueijo [3] points out that the cosmological reference frame is a suitable choice for a preferred frame, since it has witnessed all performed experiments and events. The experimental observation of dipole anisotropy in the CMB radiation [18] can be interpreted to imply that all observers are in motion with respect to the cosmic rest frame. In other words, the patterns detected in the CMB's thermal fluctuations can be thought of as largely due to an observer's Doppler shift relative to the cosmological frame (indeed, the anisotropy is also partially caused by primordial density fluctuations, but this is outside the scope of the current argument). Physically, this translates to the observation of slightly higher CMB temperatures in the direction of motion, as demonstrated by [19].

2.5.2. VSL and the Einstein field equation. The modified Albrecht-Maguejo (AM) model (developed in [10]) postulates that the Einstein field equations are valid in a variable c Universe, with minimal coupling in the cosmological frame. This approach requires the stress-mass tensor $T_{\mu\nu}$ to have a non-vanishing divergence. Physically, this implies that the canonical energy and momentum will not be conserved.

The modified action, initially proposed by [9], is:

$$\mathcal{S} = \int \mathrm{d}x^4 \left[\sqrt{-g}\left(\frac{\psi\,(R+2\Lambda)}{16\pi G} + \mathscr{L}_\mathrm{m}\right) + \mathscr{L}_\psi\right], \tag{5}$$

where $\mathscr{L}_\mathrm{m}$ is the matter field Lagrangian, Λ is the geometrical cosmological constant, $\mathscr{L}_\psi$ is the Lagrangian associated with the scalar field $\psi(x^\mu)$ that does not explicitly contain the metric. In order to have the Einstein field equations remain valid for the cosmological reference frame it is necessary to require $\psi(x^\mu) = c^4$ and $\delta_{g^{\mu\nu}}\mathscr{L}_\psi = 0$ [10]. Due to this condition, the Riemann tensor and the Ricci scalar can be derived in the usual way. Therefore, the Einstein field equations can be obtained following the original derivation. This is explicitly shown in appendix A.3. Essentially, after a trivial calculation, it can be seen that minimizing the action in equation (5) gives:

$$G_{\mu\nu} - g_{\mu\nu}\Lambda = \frac{8\pi G}{\psi}T_{\mu\nu}. \tag{6}$$

Where the stress-energy tensor is defined as:

$$T_{\mu\nu} = -\frac{\mathscr{L}_\mathrm{m}}{g}\frac{\delta g}{\delta g^{\mu\nu}} - 2\sqrt{-g}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g^{\mu\nu}} - \frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_\psi}{\delta g^{\mu\nu}}. \tag{7}$$

Those familiar with this derivation can identify that equation (7) has an additional term contributed by $\mathscr{L}_\psi$ (the usual derivation of the Einstein–Hilbert action defines $T_{\mu\nu} = -\frac{\mathscr{L}_\mathrm{m}}{g}\frac{\delta g}{\delta g^{\mu\nu}} - 2\sqrt{-g}\frac{\delta\mathscr{L}_\mathrm{m}}{\delta g^{\mu\nu}}$). A way to eliminate this term is to impose the condition for $\mathscr{L}_\psi$ not to contain the metric explicitly (nor be multiplied by it). This implies that $\delta\mathscr{L}_\psi = \frac{\delta\mathscr{L}_\psi}{\delta g^{\mu\nu}} = 0$, thus eliminating the additional term. In reference frames where the scalar field ψ is not constant, there will be additional $\partial_\mu\psi$ terms in the Ricci scalar and Riemann tensor [10].

Essentially, the modified AM formulation implies that the speed of light is modeled by a scalar field that is constant only in the cosmological reference frame. Since all observers are in relative motion to it, they measure a varying speed of light. The main purpose of the derivation above is to demonstrate that the modified AM model attempts to preserve the validity of Einstein's field equation under some conditions and thus minimize the damage done to currently accepted theories of spacetime geometry.

2.6. VSL solution to the flatness problem

2.6.1. Generic $c(t)$ case: solving the Friedmann equations with a variable c. Current observations claim that, on a large scale, the Universe is relatively flat. For this to be the case, there is a required critical density of matter, ρ_c. Even the slightest deviation from $\rho_\mathrm{c} = \frac{3H^2}{8\pi G}$ (found by plugging $K=0$, denoting flat space, and $H = \frac{\dot{a}}{a}$ in the Friedmann equations) will result in curved space geometry at current time.

In this section, it will be demonstrated that the AM (VSL) theory proposed in [9], explains a fine-tuning mechanism that is capable of driving the early Universe towards the critical density ρ_c. The Friedmann equations (see appendix B.2), when incorporating a generic $c(t)$ become:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2(t)}{a^2}, \tag{8}$$

$$\ddot{a} = -\frac{4\pi Ga}{3}\left(\frac{3p}{c^2(t)} - \rho\right), \tag{9}$$

where a is the *dimensionless* scale parameter, ρ and p are the density and pressure of matter, respectively, and K is the curvature parameter (see appendix B.2 for a table detailing the relation between the value of K and the Robertson–Walker metric geometry). Differentiating equation (8) and plugging it back into equation (9) gives:

$$\dot{\rho}+3\frac{\dot{a}}{a}\left(\rho+\frac{p}{c^2}\right)=\frac{\dot{c}}{c}\frac{3Kc^2}{4\pi Ga^2}. \tag{10}$$

It is important to mention that any term potentially involving the cosmological constant, Λ, has been taken to equal 0. These terms are closely related to the vacuum energy in Einstein's equation. Incorporating this would require the density to be set equal to $\rho=\rho_{\rm m}+\rho_{\Lambda}$ (where $\rho_{\rm m}$ is the density of mass and ρ_{Λ} is the density in the cosmological constant). AM [9] acknowledges that a non-zero value for ρ_{Λ} (where $\rho_{\Lambda}=\frac{\Lambda c^2}{8\pi G}$) can be incorporated in the equation above without an impact on the underlying principles of the theory. A conceptual explanation of the cosmological constant can be found in appendix B.3. For the purpose of simplicity, however, in the derivations presented further Λ is equated to 0. This renders it mathematically easier to analyze a variable c theory in the context of the Friedmann equations.

In an analysis of equation (10), [9] notes that it provides a fine-tuning mechanism for the early Universe to approach ρ_c, given that it started with a value of ρ that is a small perturbation from ρ_c. As previously demonstrated, [9] required the speed of light to be much larger in the past, therefore $\frac{\dot{c}}{c}<0$. Initially, if the Universe started with $\rho<\rho_c$, then $K<0$ and the right-hand side of equation (10) will cause an increase in the mass density of the Universe (it will be positive, leading to a more positive $\dot{\rho}$). Oppositely, if $\rho>\rho_c$, $K>0$, then mass will be 'destroyed' (the mass density will decrease), driving the Universe to the critical density where $K=0$. A more formal discussion and derivation related to this mechanism can be found in [20]. (Note: the derivation in [20] was performed using the density parameters of matter and the cosmological term, meaning a non-zero Λ Universe was assumed. This is not the case in the discussion above, however, the derivation still applies. Further discussion on the Λ term will be presented in section 2.6.2.)

This was demonstrated by [9] in a different manner as well. Defining a quantity $\Omega=\frac{\rho}{\rho_c}$, it is evident that $\Omega<1$ if $K<0$, $\Omega=1$ if $K=0$, and $\Omega>1$ if $K>0$.

In this formulation, the Universe is modeled by a perfect fluid. The equation of state for a perfect fluid is $p=(\gamma-1)\rho c^2$, where γ is a constant, related to the characteristic thermal speed of molecules (for non-relativistic matter $\gamma\approx 1$, for vacuum energy $\gamma=0$, see appendix B.3, and for relativistic matter $\gamma\approx\frac{4}{3}$). Substituting this into equation (10) (dividing by ρ and using the definition for $\rho_c=\frac{3\dot{a}^2}{8\pi Ga^2}$) gives:

$$\frac{\dot{\rho}}{\rho}+3\frac{\dot{a}}{a}\gamma=2\frac{\Omega-1}{\Omega}\frac{\dot{c}}{c}. \tag{11}$$

This can be re-arranged to give an expression for $\frac{\dot{\rho}}{\rho}$. The same can be done to express $\frac{\dot{\rho}_c}{\rho_c}$, by setting $K=0$ in equation (10):

$$\frac{\dot{\rho}_c}{\rho_c}+3\frac{\dot{a}}{a}\left(1+\Omega\left(3\gamma-2\right)\right)=0. \tag{12}$$

In order to examine the change in Ω, we can express:

$$\dot{\Omega}=\frac{\rho_c\dot{\rho}-\rho\dot{\rho}_c}{\rho_c^2}=\Omega\left(\frac{\dot{\rho}}{\rho}-\frac{\dot{\rho}_c}{\rho_c}\right). \tag{13}$$

As done in [9], equations (11) and (12) can be used to find an expression of $\dot{\Omega}$ in terms of other parameters:

$$\dot{\Omega} = \frac{\dot{a}}{a}\Omega\left(\Omega - 1\right)\left(\gamma - 2\right) + 2\frac{\dot{c}}{c}\left(\Omega - 1\right). \tag{14}$$

As noted in [9] the condition of $\Omega = 1$ (or $\rho = \rho_c$) is not stable for any (standard Big Bang) matter field that satisfies $0 \leqslant \gamma < \frac{2}{3}$ (this comes from the strong energy condition, see appendix C.4.1). However, the additional term $2\frac{\dot{c}}{c}(\Omega - 1)$, can drive a perturbed Ω to 1 in the early Universe. This process of fine-tuning can be more easily seen if a variable $\epsilon = \Omega - 1$ is defined, showing that it will be driven to 0. If $|\frac{\dot{c}}{c}| \gg \frac{\dot{a}}{a}$, neglecting the first term gives $\ln(\epsilon) = 2\ln(c) + (\text{const.})$ or $\epsilon \propto c^2$. AM [9] acknowledges that given this proportionality relation, assuming that the Universe started off with a matter density perturbed from the critical value, for instance, $\Omega > 1$ (implying that $\epsilon > 0$), the decrease in the speed of light would result in $\epsilon \ll 1$.

It is necessary to expand upon the implications of the derivation above. [9] acknowledge that VSL prevents a Universe with 'natural initial conditions' (such as $\epsilon \approx \pm 1$, a small perturbation from $\epsilon = 0$) to either re-collapse soon after the Big Bang ($\epsilon \approx 1$), or become an empty, Milne Universe, devoid of matter and radiation ($\epsilon \approx -1$). This is a consequence of the driving mechanism derived above, where the second term in equation (14) will push any ϵ value (that is a small perturbation from 0) to $\epsilon = 0$. As mentioned in [9], this is a feature unique to VSL, and Inflation is unable to 'save' Universes with an initial condition of $\epsilon = -1$.

2.6.2. Defining an expression for $c(t)$: constraining the ansatz in relation to the flatness problem. The following method was explored in [10], where a time-evolved $c(t)$ is more rigorously defined as:

$$c\left(t\right) = c_0 a^n, \tag{15}$$

where c_0 and n are constants and a is the dimensionless scale factor.

In the previous section, it was demonstrated that a VSL theory can provide a fine-tuning mechanism to drive a perturbed Universe to the critical density. In this section, an explicit definition for $c(t)$ will be found in order to demonstrate that, in the context of the flatness and horizon problems, the ansatz in equation (15) is consistent.

Notably, unlike [9, 10], a variable G will not be considered in this paper. This is justifiable since the explicit solutions for $c(t)$, as well as the entire discussion regarding the flatness and horizon problems in cosmology, are not impacted by a variable G. This can easily be confirmed by assuming a temporal G dependence and re-deriving the solution starting from the Friedman equations. The final constraints for the parameter n in the equation $c(t)$ will be equivalent to a constant G approach.

Notably, the LHS of equation (11) can be treated as an expanded quotient rule and rewritten as:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^{3\gamma}\right)}{\rho a^{3\gamma}}. \tag{16}$$

This mathematical trick used in [10] allows the rewriting of equation (10) (also plugging in $p = (\gamma - 1)\rho$ and noticing that $c\dot{\rfloor} = c_0^2 n a^{n+n-1}\dot{a}$) as:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^{3\gamma}\right)}{\rho a^{3\gamma}} = \frac{3Knc_0^2 a^{n+n-1}\dot{a}}{4\pi G\rho a^2}.$$

This can be equivalently re-written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^{3\gamma}\right) = \frac{3nKc_0^2 a^{2n+3(\gamma-1)}\dot{a}}{4\pi G}. \tag{17}$$

Integrating the solution (with ξ as an integration constant) gives:

$$\rho a^{3\gamma} = \frac{3Kc_0^2}{4\pi G}\cdot\frac{na^{2(n-1)+3\gamma}}{2(n-1)+3\gamma} + \xi \quad \rightarrow \quad \rho = \frac{3Kc_0^2}{4\pi G}\cdot\frac{na^{2(n-1)}}{2(n-1)+3\gamma} + \xi a^{-3\gamma}. \tag{18}$$

Prior to discussing the consequence of this, plugging it back into equation (8) gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\left(\frac{3Kc_0^2}{4\pi G}\cdot\frac{na^{2(n-1)}}{2(n-1)+3\gamma} + \xi a^{-3\gamma}\right)G}{3} - \frac{Kc_0^2 a^{2n}}{a^2}, \tag{19}$$

$$\therefore \left(\frac{\dot{a}}{a}\right)^2 = Kc_0^2\frac{2-3\gamma}{2(n-1)+3\gamma}a^{2n-2} + \xi' a^{-3\gamma}. \tag{20}$$

The flatness condition is to have $K = 0$ at large a. This translates to a flat Universe cosmological time after the Big Bang, such as the one we observe now. The coefficients in front of each term are not necessarily relevant in this discussion, since the main argument in [10] is to derive a value for n so the term containing K becomes negligible at large a. This can be done in the following manner:

$$a^{-3\gamma} \gg a^{2(n-1)} \quad \rightarrow \quad -3\gamma \gg 2(n-1) \tag{21}$$

$$\therefore n \ll \frac{1}{2}(2-3\gamma). \tag{22}$$

This result matches the solution derived in [10], $n \leqslant \frac{1}{2}(2-3\gamma)$. This result emphasizes that it is necessary for the curvature term (the one multiplied by K) to never overtake the expansion term, since this will result in a largely non-flat space geometry. It can be argued that the solution in [10] allows for $n = \frac{1}{2}(2-3\gamma)$. However, in this case, the term multiplying K will blow up, due to the denominator becoming 0. Additionally, for values $n \lesssim \frac{1}{2}(2-3\gamma)$, a similar behavior is expected. Therefore it is more appropriate to state n as $n \ll \frac{1}{2}(2-3\gamma)$.

Previously it was indicated that in order to analyze, in isolation, the form of $c(t)$ needed to satisfy the current observations of a flat Universe, $\Lambda = 0$ was assumed. A trivial calculation can show that the substitution of $\rho = \rho_m + \rho_\Lambda$ and $p = p_\mathrm{m} - p_\Lambda$ (with the equation of state $p_\Lambda = -\rho_\Lambda c(t)^2$, since this corresponds to vacuum energy, and is derived through the relation $T^{\mu\nu} = -\rho_\Lambda g^{\mu\nu}$, discussed in appendix B.3) in equation (10) will result in an additional $\dot{\rho}_\Lambda$ term on the left-hand side. Plugging in the definition for ρ_Λ and differentiating it with respect to t leads to the term $\dot{\rho}_\Lambda = \frac{nc_0^2 a^{2n-1}\dot{a}\Lambda}{4\pi G}$. Following the integration steps outlined above generates the equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(3(3\gamma+2n-2)(4G\pi(2n+3\gamma)+c_0^2\Lambda n a^{2n}\right)^{-1} \times \left(\xi'' a^{-3\gamma} + 24c_0^2 GK\pi n(2n+3\gamma)a^{2n-2}\right) - Kc_0^2 a^{2n-2} + \frac{c_0^2 a^2 \Lambda}{3}, \tag{23}$$

which reduces to equation (20) when $\Lambda = 0$ (the last term comes from the Friedman equation, equation (8), modified to include a non-zero Λ, where $\rho = \rho_m + \rho_\Lambda$). Notably, this result includes the constant G, which [10] sets to vary in order to provide a solution for n in the context of a non-zero cosmological constant. However, this is well outside the scope of this paper, which is primarily concentrated on a varying c. Further reading can be found in [10], where a varying $G(t)$ constraints the value of n to $n \leqslant -\frac{3\gamma}{2}$. This result is of importance for a non-0 Λ Universe. Further comments on this result can be found in the discussion section (section 2.8.1).

2.7. Horizon problem

At large $a(t)$, $K \to 0$, resulting in the relatively flat Universe observed today. Then, equation (20) evidently becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \xi a^{-3\gamma} \quad \to \quad \dot{a} = \sqrt{\xi a^{2-3\gamma}} \quad \to \quad a(t) \propto t^{\frac{2}{3\gamma}}. \tag{24}$$

This condition makes sense since, for example, in a matter-dominated era, $\gamma = 1$ and $a(t) \propto t^{\frac{2}{3}}$. It is significant that this approximate solution does not incorporate the variability of c. This implies that $c(t)$ primarily impacts the early evolution of the Universe.

To provide a more intuitive understanding of the solution to the Horizon problem presented in [10], it is important to return to the discussion of comoving coordinates and proper distance. By definition, proper distance is the physical separation between two points at constant cosmological time. Unlike comoving distance, the cosmological expansion is factored into the calculation for proper distance.

Let's label the proper distance to the particle horizon, at some time t, as $d_h(t)$. Fundamentally, the horizon problem reduces to the fact that causally disconnected regions of space (at the moment of the CMB radiation emission) can be observed to have isotropic CMB temperature (see a detailed discussion in appendix A.1).

An approach to solving the horizon problem is to postulate that the scale factor grows faster (or as fast) as the proper distance to the horizon. Having the scale factor grow equally as fast will imply that all points in space were causally connected since the Big Bang (and still are, which, albeit is not what is observed). An $\alpha(t)$ that grows much faster than $d_h(t)$ implies that causally connected regions in the past are not anymore; the predicted proper distance (at some later time) between them will be less than their expansion away from each other. In other words, regions of space that are not in causal contact at some later time, were once causally connected. Essentially, this resembles the principle behind Inflation, stating that superluminal expansion ($a(t) \gg d_h(t)$) occurred in the early Universe.

Any proper distance, d_h, at time t, can be found by a $c(t)\cdot t$ proportionality. Using equation (15) (eliminating the constants to express proportionality) and equation (24) a condition for n can be derived, as shown in [10]:

$$d_h(t) \propto c(t)\, t \propto a^n t \propto t^{1+\frac{2n}{3\gamma}} \quad \rightarrow \quad d_h(t) \propto t^{1+\frac{2n}{3\gamma}}. \tag{25}$$

If it is required for $a(t) \gg d_h(t)$, a value for n can be found through the following inequality:

$$t^{\frac{2}{3\gamma}} \gg t^{1+\frac{2n}{3\gamma}} \quad \rightarrow \quad \frac{2}{3\gamma} \gg 1 + \frac{2n}{3\gamma}, \tag{26}$$

$$\therefore n \ll \frac{1}{2}(2-3\gamma). \tag{27}$$

Evidently, the constraint on n is equivalent to the one derived in relation to the flatness problem. This demonstrates that an explicit temporally-dependent equation for c can be found that solves both cosmological problems. Much more work is needed to determine the extent of theoretical implications VSL can have. Given that a constant speed of light is a factor in many physical theories, a solution to the flatness and horizon problem is not an indication of validity.

2.8. Discussion and counterarguments: theory

2.8.1. Evaluating the implications of n. As demonstrated above, an ansatz for a varying c of the form $c = c_0 a^n$ with $n \ll \frac{1}{2}(2-3\gamma)$ provides a solution for the horizon, as well as the flatness problem. As previously mentioned, the value of γ is closely related to the contents of space.

Initially, the early Universe was dominated by relativistic matter (radiation). In this case, where $\gamma = \frac{4}{3}$, $n \ll -1$. Similarly, dust (ordinary matter) filled Universe will have $\gamma = 1$, and therefore, $n \ll -\frac{1}{2}$. This resonates with the VSL postulate that the decrease of c (in relation to the expansion of space) should be greater in the early Universe, which was radiation-dominated. Conversely, if the Universe was (non-relativistic) matter-dominated, then the speed of light should have a less steep decrease in order for the ansatz to be a solution satisfying the conditions for the horizon and flatness problem.

The more interesting result happens when $\gamma = 0$ is considered, which is characteristic of a Universe with non-zero vacuum energy (dark energy). If the solution from section 2.6.2 is considered, then there arises a possibility for $n > 0$. In other words, setting $\gamma = 0$ implies $n \ll 1$. This allows for an increase in $c(t)$ as the Universe expands. Undeniably, this suggests that the upper limit on n should be lower. This issue is addressed in [10], by introducing a variable $G(t)$ and solving the Friedmann equations in a similar manner as demonstrated above. In short, the re-derived constraint is $n \leqslant -\frac{3\gamma}{2}$, ensuring that $n \ll 0$ in all cases (γ is defined to be always positive, or 0).

Interestingly, such a variation in G does not affect the derivation in section 2.6.2. Equation (17) will simply be of the form $\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^{3\gamma} G) = \frac{3nKc_0^2 a^{2n+3(\gamma-1)}\dot{a}}{4\pi}$, which will provide the same final result. The main difference occurs when the non-zero Λ solution is considered.

Evidently, the temporal variation in c proposed by [9, 10] implies that any curvature will become negligible (with the accelerating expansion of the Universe) as long as $c(t)$ drops rapidly enough. However, considering only standalone variations in c does not provide the necessary constraint on n given a non-zero Λ Universe. This invites further investigation. As is demonstrated in [10], it is possible to derive a formulation of the AM VSL theory, in the limit of a constant c in order to have a scalar-tensor theory describing a variation in $G(t)$. However, only solutions that have an equation of state resembling black body radiation are allowed to persist. Essentially, this implies that a variable c theory cannot be converted into a variable $G(t)$ theory without implications. Moreover, the set limit on n contains the value 0, even in a $\Lambda \neq 0$ Universe. This would result in the possibility for $c = c_0$, which describes a Universe with a constant light speed. Therefore, as remarked by [10], the constraint on n in the $c(t)$ equation can also be seen as a generalization of Inflation (more particularly, the conditions that allow for an inflationary resolution to the horizon/flatness problems).

2.8.2. Counterargument and a proposed resolution: the entropy problem. As proposed in [21], in flat, $K = 0$, and closed, $K > 0$, Universes, the Second Law of Thermodynamics demands $\dot{c}(t) = 0$, whereas in an open Universe, with $K < 0$, the speed of light can only decrease. To demonstrate this [21] assumes a Universe where the particle number is conserved, regardless of expansion (and any consequences of the fine-tuning mechanism described in section 2.6.1). In this case, it is implied that $N = n_\rho a^3$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$, where N is the total number of particles, n_ρ is the comoving number density of particles (to distinguish it from n, that denotes the exponential value in equation (15)), and a is the scalar factor, as before. It can be easily seen that applying the chain rule and substituting the Hubble Parameter, $H \equiv \frac{\dot{a}}{a}$, gives:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = a^3 \left(3Hn_\rho + \dot{n}_\rho\right) = 0. \tag{28}$$

Since $a^3 \neq 0$, it must be required for the particle number density to obey $3Hn_\rho + \dot{n}_\rho = 0$. As done in [21], applying Gibbs' equation for an adiabatic system (no heat transfer) and $\frac{\dot{n}_\rho}{n_\rho} = -3H$ gives:

$$n_\rho T\dot{S} = \dot{\rho} - \frac{\dot{n}_\rho}{n_\rho}\left(\rho + \frac{p}{c(t)^2}\right) = \dot{\rho} + 3H\left(\rho + \frac{p}{c(t)^2}\right) = \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c(t)^2}\right), \tag{29}$$

where S is the entropy of the system and T is the temperature in Kelvin.

Evidently, the right-hand side of the equality above (equation (29)) is equivalent to equation (10). A simple substitution gives an equation for the change in entropy as a consequence of varying c:

$$\dot{S} = \frac{\dot{c}}{c}\frac{3Kc^2}{4\pi Ga^2 n_\rho T}. \tag{30}$$

Initially, without referencing the equation above, it is crucial to consider the expected behavior of entropy given a variable c. For example, a larger c would increase an observer's past light cone, resulting in more information from the early Universe reaching a point in space-time. In turn, the inverse relation between entropy and information implies that this should lead to a decrease in S. Conversely, since c is considered to decrease over time (in mathematical terms $\dot{c} < 0$), the behavior of the temporal change in entropy will resemble an opposite relation, $\dot{S} > 0$. This is consistent with the Second Law of Thermodynamics, which states that the entropy of the Universe must always increase.

A complication to VSL arises when equation (30) is considered. Evidently, in VSL theories the prefactor, $\frac{\dot{c}}{c}$, is defined to be negative. If it is required for $\dot{S} > 0$, the only possible equality for K that satisfies this is $K < 0$, corresponding to an open Universe with $\rho < \rho_c$. This result puts into question the validity of the flat Universe solutions discussed in sections 2.6.2 and 2.7.

However, other entropy contributions can result in lesser restrictions than those imposed by equation (30). As elaborated by [21], in cosmologies that model the Universe as a perfect fluid, the total entropy can be modeled by the equation $\dot{S}_\mathrm{f} + \dot{S}_\mathrm{h} \geqslant 0$. Here $\dot{S}_\mathrm{f}$ represents the entropy of a fluid enclosed by a horizon (analogous to $\dot{S}$ in equation (30)), whereas $\dot{S}_\mathrm{h}$ is the entropy of the horizon itself. The equality states that the total entropy cannot decrease with time.

In summary, [21] derives $\dot{S}_\mathrm{f}$ and $\dot{S}_\mathrm{h}$, in the limit of $t \to \infty$, as:

$$\dot{S}_\mathrm{f} \propto \begin{cases} t^{2\frac{n+1}{1-n}} \ln^3 t\,, & -1 < n < -\frac{1}{2}, \\ t^{\frac{1}{2}\frac{2n+3}{1-n}} \ln^3 t\,, & -\frac{1}{2} < n < 0, \end{cases} \quad ,$$

$$\dot{S}_\mathrm{h} \propto t^{\frac{5}{1-n}} \ln^2 t \qquad -1 < n < 0.$$

In both ranges of n (which includes both ordinary and relativistic matter), $\dot{S}_\mathrm{h}$ dominates over $\dot{S}_\mathrm{f}$ and the equality $\dot{S}_\mathrm{f} + \dot{S}_\mathrm{h} \geqslant 0$ is satisfied. Essentially, the change in entropy of a perfect fluid, through which the Universe is modeled, at large t, will be dominated by the entropy of the particle horizon that surrounds it, thus satisfying the Generalized Second Law of Thermodynamics equation, $\dot{S}_\mathrm{f} + \dot{S}_\mathrm{h} \geqslant 0$. Chimento *et al* [21] states this as an alternative approach to deriving entropy in VSL that loosens the constraints implied by equation (30).

Although the behavior of entropy under VSL may not explicitly prohibit the existence of a flat Universe, the discussion above demonstrates the care needed in order to make concrete claims from a varying c theory.

3. Experimental evidence for a varying fine structure constant

3.1. Methods and background

3.1.1. Quasars and redshift. A series of experiments and data analysis conducted by Webb *et al* [4, 22–25] demonstrated the possibility for a variation in the fine structure constant. Primarily, they focused on quasar spectroscopy as a method to examine physical laws and constants at medium to high redshift. Essentially, quasars are one of the most luminous objects in the Universe. The most distant ones observed were formed less than a billion years after the Big Bang. (Wang *et al*'s paper [26] in 2021 discovered quasar J0313-1806 at redshift $z_\mathrm{em} = 7.624$, dating it back to approximately 670 million years after the Big Bang, which is the most distant/oldest quasar currently measured) Additionally, quasars are considered to be powered by accretion onto supermassive black holes [27]. *Accretion* refers to a process in which matter, under the influence of a Supermassive Black Hole's immense gravitational pull, begins to spiral inward towards the center of the Black Hole. Throughout this process, a surrounding accretion disk is formed. In simple terms, as the matter particles lose gravitational potential energy (they spiral towards the center of the Black Hole) their energy is converted into heat, resulting in the emission of large amounts of radiation, thus making the object appear bright.

In the context of cosmology, redshift refers to a measure of the stretching of the wavelength of light. The mathematical definition is, $z = \frac{\lambda_\mathrm{o} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}}$, where λ_o is the wavelength observed from Earth, whereas λ_e is the wavelength emitted from the source. Current consensus among cosmologists explains redshift as a consequence of the Universe's spatial expansion that leads to the stretching (redshifting) of light traveling cosmological distances. However, within the context of VSL theories, this effect can be partly credited to $\Delta c(t) < 0$, or, in other words, the deceleration of the speed of light over large timescales.

It is important to note, however, that the redshift (z) discussed further in the context of Webb *et al* is that of absorption rather than emission. For the purpose of consistency the subscript in z_{abs} will be omitted.

Absorption lines on quasar spectra occur when light emitted from a distant quasar passes through absorption clouds prior to reaching measuring devices on Earth. As a result, the absorption redshift, z, will be less than the emission one, z_{em}. Intuitively, since the absorption clouds are closer to Earth than the quasar, the moment at which absorption occurs is temporally less distant in the past than the instant of emission. As previously discussed, redshift can be interpreted as a measure of the amount of time light has been subjected to the expansion of space. A greater redshift indicates that the emission/absorption occurred further in the past.

3.2. *Implications of a redshift-dependent fine structure constant*

As previously mentioned, the fine structure constant α is proportional to c^{-1}. If, as VSL theories claim, the speed of light has decreased since the early Universe, it is expected to observe a temporally varying α such that $\Delta\alpha(t) > 0$ (for $\Delta\alpha(t) = \alpha_{\mathrm{now}} - \alpha_{\mathrm{past}}$, which is opposite from the definition in Webb *et al*). Notably, an observation of objects at high redshift (distant) is, in essence, synonymous with looking at a younger Universe, or probing the past. Hereby, a redshift-dependent fine structure constant implies a temporal (or spatial) variation in α. From VSL it is expected to observe a smaller value for α at higher redshifts (implying a smaller fine structure constant in the past).

This contributes to many potential consequences, one of which is the accepted redshift for the emission of the CMB radiation (a conceptual section on the CMB can be found in appendix B.1). Since the ionization energy of hydrogen-like atoms is dependent on α, a VSL theory implies that this value could shift. Essentially, the CMB emission corresponds to the Universe cooling down to temperatures that allow for the formation of atomic nuclei, such as Hydrogen. If the ionization energy of Hydrogen differed in the past, the CMB could have been emitted at a different time than currently theorized. This can have implications for cosmological models of the Universe's evolution.

Additionally, as a consequence of modified energy levels, all atomic spectra will be perturbed when measured at redshift. In other words, the energy levels of atoms depend on α (for hydrogen-like atoms $E_n \propto \alpha^2$) and any change in the fundamental constant will result in their perturbation. In relation to absorption spectra, this implies that at redshift (or equivalently, in the past), the position of the absorption lines for each atomic species will differ from currently measured positions. This is closely related to the study performed by Webb *et al* which is further discussed in this section, where the absorption spectra of atoms were measured in the search for a varying α.

3.3. *Many-multiplet method employed by Webb et al*

An expanded theoretical background of this method can be found in [4]. Essentially, the idea of a multiplet in particle physics is a state space for the internal degrees of freedom of a particle (such as spin, color, isospin, hypercharge etc). Mathematically this can be represented as a vector space associated with a group of continuous symmetries (ex. SU(2) for quantum isospin, SU(3) for color change in QCD etc). In the context of spectroscopy, however, a multiplet can appear in the absorption spectra if an electron transition occurs between closely spaced energy levels. These energy levels are due to a particle's fine or hyperfine structure (a brief overview of these concepts can be found in appendix B.4).

As derived in [4], the energy equation for a transition from the ground state, within a particular multiplet of energy levels, at a given redshift, z, can be written as:

$$E_z = E_c + Q_1 Z_n^2 \left[\left(\frac{\alpha_z}{\alpha_0} \right)^2 - 1 \right] + K_1 (\mathbf{LS}) Z_n^2 \left(\frac{\alpha_z}{\alpha_0} \right)^2 + K_2 (\mathbf{LS})^2 Z_n^4 \left(\frac{\alpha_z}{\alpha_0} \right)^4, \quad (31)$$

where α_0 is the current value of the fine structure constant, whereas α_z is the value at a specific redshift, z. In the equation above, $\mathbf{L}$ and $\mathbf{S}$ are the electron's total orbital angular momentum and total spin respectively. Q_1 and $K_{1,2}$ are relativistic corrections previously computed. E_c is the energy of the configuration center and Z_n is the nuclear charge. By assuming a possible non-zero variation of α with respect to the redshift, Webb *et al* [4] simplify this equation as:

$$k_z = k_0 + q_1 x + q_2 y \qquad \begin{cases} x \equiv \left(\frac{\alpha_z}{\alpha}\right)^2 - 1 \\ y \equiv \left(\frac{\alpha_z}{\alpha}\right)^4 - 1 \end{cases}, \quad (32)$$

where k_z is the wavenumber in the rest-frame of the absorption cloud at redshift z, whereas k_0 is the wavenumber measured at Earth.

In the paper(s) detailing their statistical results [4, 22, 23], Webb *et al* use the many-multiplet method and data from Quasi-Stellar Objects (QSO) absorption spectra in order to numerically estimate a value for $\frac{\Delta\alpha}{\alpha}$, defined as $\frac{\alpha_z - \alpha_0}{\alpha_0}$.

3.4. Initial results for a redshift-dependent fine structure constant by Webb et al

In a series of experiments involving statistical data analysis dating early 2000s, Webb *et al* provided possible evidence for a smaller fine structure constant in the early Universe [4, 22, 23]. Notably, this adheres to the prediction postulated by VSL theories that, given a $c(t)$ that decreases over time, α is expected to increase. They reported an average value of $\frac{\Delta\alpha}{\alpha} = (-0.72 \pm 0.18) \times 10^{-5}$ [4] found by employing the aforementioned observational many-multiplet technique. Essentially, this enhances the sensitivity of previous measurements by studying relativistic transitions to different ground states. This was done by using absorption lines in quasar spectra (QSO) at medium redshift. As claimed by Webb *et al* the many-multiplet method is more sensitive than the alkali-doublet method previously used in [28]. They conclude, in summary, that their current results are advantageous to prior computations since the many-multiplet method allows for an order of magnitude gain in sensitivity compared to the alkali doublet (AD) method [4] (a brief comparison of the MM method and AD method can be found in appendix B.5).

Using spectograph measurements of DLAs with redshifts of $0.5 < z_{\text{abs}} < 1.8$ [22] and $0.9 < z_{\text{abs}} < 3.5$ [4], their final results are summarized in table I from [23]:

The table above demonstrates their results and the claim that the MM method proves to be more sensitive than the AD method.

3.4.1. Initial results: discussion and wider scientific commentary.

Due to the definition of $\frac{\Delta\alpha}{\alpha}$, if α_z (measured at some redshift) is less than α_0 (measured in the laboratory), $\frac{\Delta\alpha}{\alpha}$ will be negative. Based on table 1, this is the case for all calculated values in the range $0.5 < z < 3.0$. As elaborated before, measuring distant quasars is equivalent to looking at a younger Universe. Therefore, the results indicate that the fine structure constant was possibly smaller in the past, in other words, $\alpha_z < \alpha_0$.

Table 1. Webb *et al* [23], summary of results. Values of $\frac{\Delta\alpha}{\alpha}$ are weighted in units of 10^{-5}. N_{abs} is the number of absorption systems in each sample. As before, MM and AD indicate the *many multiplet* and *alkali doublet* method. Reprinted table with permission from [23], Copyright 2001 by the American Physical Society.

Sample	Method	N_{abs}	Redshift	$\frac{\Delta\alpha}{\alpha}$
FeII/MgII	MM	28	$0.5 < z < 1.8$	-0.70 ± 0.23
NiII/CrII/ZnII	MM	21	$1.8 < z < 3.5$	-0.76 ± 0.28
SiIv	AD	21	$2.0 < z < 3.0$	-0.5 ± 1.3

It is important to note, that if the above reasoning is followed, it is expected for $\frac{\Delta\alpha}{\alpha}$ to be more negative at higher redshift. The results presented in table 1 make it difficult to draw a trend conclusion since the error range is significant compared to the mean value. However, even though the methods differ, the results obtained are consistent given the much larger standard deviation on the SiIv sample. Furthermore, the data in table 1 indicates a $\frac{\Delta\alpha}{\alpha} \neq 0$ at the 4.1σ significance. This implies that there is a low chance the variability in α is due to a random or identified systematic errors. As [4] concludes, at the 4σ confidence level, there is a possibility for a lower fine structure constant in the past. Evidently, this result favors a variable c theory. The temporal variation in α, however, is later re-evaluated through the results presented in [25].

Prior to discussing newer data collected and analyzed by the same group of scientists that, albeit, complicates the discussion on VSL, it is noteworthy to mention the wider scientific commentary regarding the initial Webb *et al*'s results. The overall acceptance was positive, followed by the formulation of different theories intended to explain the findings or their consequences. In addition to variable c and e theories, these results motivated the formulation of spatially variable h theories, such as the one proposed by [29, 30]. Conversely, some theories, like the one discussed in [31], refrained from crediting the variability of α to a changing constant, rather argued for the existence of an ultralight particle that couples to neutrons through the α dependence of their masses. Experimental sensitivity currently is unable to rule out such a theory.

Even though the 'Webb results' sparked a discussion on variable constant theories, arguments have been placed forward, such as the one presented in [1], that the dimensionality of fundamental constants renders any discussion of their variability meaningless. Additionally, [1] indirectly states that an interpretation of the variability in α as a result of VSL (or a varying e, h theory) is ultimately illogical. However, VSL theorist, Magueijo, remarks on this claim, calling it an 'equally meaningless statement' [3]. Magueijo [3] explains that if a variation in α is observed, it must be caused by the temporal (or in some theories—spatial) dependence of some 'constant' (or a combination thereof), all of which have dimensions. As previously mentioned, Beckenstein [6] and Magueijo [3, 8] dispute the claims that a variable c theory can be translated to a variable e theory due to the different physical consequences they predict. Therefore, if a variation in α is caused by a temporal dependence of some fundamental constant, there exists a set of experiments to determine which 'constant' is varying.

3.5. Recent results for a redshift-dependent fine structure constant by Webb et al

In a series of more recently published papers by Webb *et al* [24, 25], they conclude that when probing a different region of the Universe, the evolution of α exhibits an inverse relation from their previous measurements. They address possible inconsistencies in their data since, for the

Table 2. Webb *et al* [24], comparison to previous results. Values of $\langle \frac{\Delta\alpha}{\alpha} \rangle$ are weighted in units of 10^{-5}.

Redshift	HIRES data: $\langle \frac{\Delta\alpha}{\alpha} \rangle$	VLT data: $\langle \frac{\Delta\alpha}{\alpha} \rangle$
$z < 1.8$	-0.54 ± 0.12	-0.06 ± 0.16
$z > 1.8$	-0.74 ± 0.17	$+0.61 \pm 0.20$

purpose of these calculations, was taken from a different observatory, Very Large Telescope (VLT). Similarly to their previous work, they use the manu-multiplet method. The following table 2 summarizes and compares the previous results (averaged over all of High Resolution Echelle Spectrometer (HIRES) data, discussed in [4, 24, 25]) to the recent VLT measurements:

Evidently, the results shown above complicate the discussion on VSL. The inversely proportional relation of α and c implies that in order to have a larger speed of light in the past, it is required for α to be smaller at larger redshifts (in other words, $\Delta\alpha$, defined as $\alpha_z - \alpha_0$, should become more negative at greater redshift values), assuming c is the only quantity that varies. Under this assumption, the 'Webb results' [4, 22, 23], published in the early 2000s, were cited by VSL theorists, such as Magueijo [3], as possible evidence for a VSL. However, the more recent findings from the VLT data suggest that for $z > 1.8$, the value for $\frac{\Delta\alpha}{\alpha}$ could be positive with greater confidence. Evidently, these results are not in favor of a VSL theory, but rather evoke a possible counter-argument that more than one 'constant' is varying, thus resulting in the $\frac{\Delta\alpha}{\alpha}$ trend observed. Additionally, depending on the choice of constants, the physical implications of such models will vastly differ. Conversely, Webb *et al* propose a different explanation for their results, that does not involve the variability of fundamental constants.

3.5.1. Webb et al final conclusions. A summary figure of their results, as well as a more detailed description of the dipole fit, is presented in [25]. After combining the VLT and HIRES datasets, Webb *et al* conclude that the apparent sign changes in the $\frac{\Delta\alpha}{\alpha}$ calculations indicate that a weighted mean model is not a good description of the VLT data [25]. They proceed to define an angular dipole fit, where it is acknowledged that any possible redshift dependence is encoded in the angular amplitude, which quantifies the magnitude of change. However, no specific function is provided to describe this relationship.

Conclusively, Webb *et al* claim that their analysis provides evidence for a spatially varying α in both VLT and HIRES datasets [25]. Based on their results, the variation of α across the sky increases in amplitude at greater distances from Earth (note: this claim does not reference the sign of $\frac{\Delta\alpha}{\alpha}$, which fluctuates in their results, implying, at best, an inconclusive result in relation to VSL theories).

After combining the two datasets, [25] claims that the fine structure constant appears larger in the southern direction, referenced to Earth, and smaller in the northern, by a value on the order of 10^{-5}. This conclusion, however, is not in favor of a VSL theory, such as the ones discussed above, that solely relies on the temporal variation in c. This is true under the assumption that α, based on the results of [24, 25], does not exhibit any temporal dependence.

3.6. Overview of experimental constraints, systematic errors, and counter-claims

3.6.1. Possible systematic errors in Webb et al data. An extended catalogue of possible systematic errors is discussed in section 3 of [28], where it is concluded that only two are non-negligible: isotropic abundance variation and atmospheric dispersion. For reference, other possible sources of error considered are: wavelength mis-calibration, systematic line blending with unknown species, differential isotropic saturation, and hyperfine structure effects.

The isotopic abundance variation is a phenomenon that could mimic a variation in α at different redshifts. Webb *et al* assume throughout their data analysis that the isotopic abundances in all species are equivalent to those observed on Earth. As noted in [28], a paper published by Timmes and Clayton [32], indicates that the abundance of ^{30}Si and ^{29}Si decreases with a lower metallicity. Webb *et al* note that in their data, when considering Silicon doublets at high redshift, the metallicity is considered to be $\lesssim -1$ [28]. Therefore, if the isotopic abundance of Si is significantly different at high redshifts than the one assumed, the data analysis can lead to a false-positive, thus resulting in an incorrect conclusion of a non-zero $\frac{\Delta\alpha}{\alpha}$.

Atmospheric dispersion is a significant cause for concern since the initial data, that was used to generate the results discussed in section 3.4. The experiment was carried out prior to August 1996 on HIRES, when the spectrometer was not yet equipped with an image rotator (device intended to compensate for Earth's rotation and keep stellar objects in the same position relative to the measuring device). This is discussed in detail in [4, 28], however, briefly, if the spectrograph is not positioned perpendicular to the horizon, then light of different wavelengths will reach the detector at different angles, causing potential inaccuracies in the absorption spectrum.

3.6.2. Questioning the validity of Webb et al's results. In a recent paper published by A Songaila and L L Cowie, a conclusion was reached that spectroscopic measurements of quasar absorption lines, such as the ones used in Webb *et al*'s research, are currently incapable of unambiguously detecting a variation of the fine structure constant using the many-multiplet method [33]. As Songaila and Cowie note, laboratory measurements detailed in [34] have demonstrated no indication of a temporally varying α, whereas analytical methods, such as the one discussed in Gould *et al* [35], have established a range of $(-0.11 \leqslant \frac{\Delta\alpha}{\alpha} \leqslant 0.24) \times 10^{-7}$, which is much lower than the values obtained by Webb *et al*.

In their study, Songalia and Cowie analyze data from three quasars, at medium-redshifts in the range of $2.30 < z_{\mathrm{em}} < 3.03$. They use the same telescope (Keck I/HIRES) as Webb *et al* did for their initial dataset. In their detailed calibration analysis, they notice a drift in the HIRES wavelength calibration, which can significantly impact results for elements, such as Fe II and Mg II, that have widely spaced absorption lines. Essentially, given that the analysis involves a comparison of absorption lines and their relative positions, even a minor error due to wavelength calibration drift can cause inaccuracies in the computations, with the expected relative error being higher for widely spaced absorption lines.

Additionally, it was determined that there is a linear deviation that proved significant between sky line and Thorium–argon (ThAr) calibration. In all of their Keck I/HIRES data collection, Webb *et al* used a ThAr lamp emission spectra before and after exposure, in order to calibrate the wavelength [4, 36]. Thorium and argon emit light at well known wavelengths that cover a wide range of the spectrum, therefore it is considered a reliable calibration device. Similarly, sky line calibration uses emission lines from Earth's atmosphere that spread over a wide range of the emission spectrum. As demonstrated in section 3 of [33], the discrepancy between ThAr and sky line calibration is linear, which introduces a significant systematic error. When accounting for the gradient found ($\frac{\mathrm{d}(\Delta v)}{\mathrm{d}\lambda} = (-2.91 \pm 0.66) \times 10^{-5}\,\mathrm{km\,s^{-1}\,\AA^{-1}}$, where v is the velocity shift—a quantity related to the Doppler effect which results in redshift due to the relative motion of the source and observer if they are moving away from each other), [33] conclude that for elements with widely spaced absorption lines (such as Mg II and Fe II, used by Webb *et al* [4, 36]), the measurement of $\frac{\Delta\alpha}{\alpha}$ using HIRES data and ThAr calibration will exhibit a negative bias. Considering this, the 'Webb results' presented in [36] should have a significance of $\approx 3.5\sigma$, rather than their prior statement of 4.7σ [33].

3.7. Discussion and counterarguments: experimental evidence

Undeniably, the discussion on the variability of α is a rather complicated one. Initial data provided by Webb *et al* [4, 22, 23, 36], as discussed in section 3.4.1, implied the existence of a temporal dependence of α. These results were cited by VSL theorists (such as [3]), as a justification and a potential validation. Moreover, the report that α was possibly lower in the past, encouraged research on varying constant theories, especially VSL.

Songaila and Cowie [33]'s discussion on the current lack of ability to draw formative conclusions using QSO spectra from HIRES and the many-multiplet method puts into question the validity of the vast 'Webb results' collection. If unaddressed systematic errors impact the results significantly, then the temporal/spatial variability of α should be stated at a lower σ confidence than the one initially argued by Webb *et al*.

The more recent results by Webb *et al* [24, 25], indicate a potential spatial variability in α. In their calculations, $\frac{\Delta\alpha}{\alpha}$ is shown to be positive in some cases, which disfavors VSL theories outlined above; a larger α in the past can not be explained by a theory that relies only on the variability of c, where $\Delta c(t) < 0$.

Throughout the analysis of all papers detailing the Webb results it came to my attention that in the visual representation of their data, which can be seen in [25], most *triangles*, representing quasars common to samples taken from both telescopes, have a variation in the range of 0. Moreover, most *squares*, which represent data taken from the VLT telescope, are blue, which indicates that the variability of $\frac{\Delta\alpha}{\alpha}$ on those datapoints is >0. Conversely, even though there is a large amount of *circles* that show $\frac{\Delta\alpha}{\alpha} > 0$, most statistically significant ones are red, suggesting $\frac{\Delta\alpha}{\alpha} < 0$. Therefore, it is natural to question whether the difference in $\frac{\Delta\alpha}{\alpha}$ values can be caused due to a calibration issue between the telescopes. This observation of a trend in the results that may result in a major systematic error is not addressed by [24, 25]. In summary, mostly all significant data points (that lie in the range $2\sigma - 3\sigma$) for the VLT telescope demonstrate a negative $\frac{\Delta\alpha}{\alpha}$, whereas for the HIRES/Keck I a positive $\frac{\Delta\alpha}{\alpha}$, which may be caused by a calibration error. Therefore, it is crucial to be careful when discussing the validity of these results. Accordingly, a more through study of the datasets is warranted and further research on this subject is advised. In relation to VSL, it is difficult to determine whether such a theory is favored by the 'Webb results' given the new VLT dataset. Conclusively, in order to determine whether a variable α is a real effect, more sensitive experimental equipment is required that can produce data at a greater confidence level with a lower systematic error.

4. Summarizing discussion: a personal take

Throughout this paper, I took an objective look at many aspects of a VSL theory. Needless to say, the implications of a varying c theory are far-reaching, making this an extensive undertaking. The primary appeal of the AM formulation is its simplicity, and the ability to solve cosmological problems straightforwardly. It could be even stated that the behavior (and form) of $c(t)$ is consequential of the solution to the horizon/flatness problem since the constraint on the parameter n was derived thusly. Moreover, even for a generic $c(t)$, the expression of the Friedmann equations hinted at the possibility for mass creation and annihilation in the early

Universe, leading to a flat spacetime with matter density matching the critical value. This is a bold, yet intriguing claim made by [9].

A major question that arises is: what is the nature of the mechanism which could drive a rapid decrease in c in the early Universe? AM [9] leaves this unanswered, however, the more formal definition of $c(t)$ presented by [10] implies that temporal change in the speed of light is not abrupt, but modeled by a continuous function that depends on the scale factor. Personally, unlike Bimetric VSL theories [11, 12], I found this approach rather comprehensive and less mathematically challenging. The parameter n in the proposed ansatz for $c(t)$ is constrained through solutions for cosmological problems that allow for the possibility of $n = 0$. Essentially, if n is taken to be 0, no variation in the speed of light will be present. Consequently, instead of looking at this VSL theory as a competitor to Inflation, it can be interpreted as an extension of it. This point is raised in [10], where it is stated that the VSL formulation essentially generalizes the conditions for the inflationary solution to the flatness/horizon problems.

Moreover, extensive care is taken to reduce the possible damage done to underlying theories. The proposed action that leads to the Einstein field equation (through the conventional derivation, which was done as an exercise) in the preferred cosmological reference frame where the scalar field, ψ, is constant, is an indicator of this. Arguably, this approach could be motivated by Maxwell's equations in dielectric media, making it conceptually easier to grasp.

Another appealing aspect of this theory is that the constraint for n derived through the horizon and flatness problems is consistent. Furthermore, when considering a $\Lambda \neq 0$ Universe (and a varying G), the revised constraint falls below the upper limit of the $\Lambda = 0$ derivation. Mathematically, the variability of G is relevant only when considering the $\Lambda \neq 0$ Universe, since equation (20) is invariant under a change of G. This result requires further investigation; a noteworthy derivation is performed by [10] where it is shown that in the limit of a constant c, a scalar-tensor theory of a variable G will provide similar solutions to cosmological problems (with strict limitations, however).

Prior to researching the experimental aspects of VSL, I was unaware of the magnitude the discussion surrounding Webb *et al*'s results has. The initial results, published in the early 2000s, sparked a discussion on temporally varying constants (c, e, and even a spatially varying h). Even though their 4.7σ confidence has been put into question due to systematic errors, their initial results seemed promising and consistent. However, recent data has influenced Webb *et al* to revise their hypothesis and claim that α is spatially varying, rather than temporally. A graphical overview of their data, presented in [25], evoked some personal skepticism of the result. As previously mentioned, most data points at the 3σ level seem to exhibit a dependency on the telescope used for collection (Keck telescope data seems to trend towards $\frac{\Delta\alpha}{\alpha} < 0$, whereas VLT data favors $\frac{\Delta\alpha}{\alpha} > 0$). This could be the result of some systematic error that is not accounted for. Evidently, this invites further investigation.

As a concluding remark, I must acknowledge that any VSL theory requires extensive research toward the development of a theoretical framework that is consistent with all current observations. Since the consistency of c is assumed in many well-accepted theories and postulates, a VSL theory should be able to make the same predictions alongside solutions to cosmological problems. Any experimental result that favors VSL will need to produce results at a high significance, possibly larger than the 5σ threshold, in order to prove that the effect is real. Personally, even though a varying speed of light theory seems largely unconvincing, it is a fascinating proposal that demonstrates the extensive effect modifying one of the foundational principles can have on physics.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Derivations

A.1. Horizon problem: derivation of non-overlapping particle horizons

The main purpose of this derivation is to demonstrate that by looking at widely spaced out regions in the CMB, an observer at current time is measuring regions of space that were outside each-other's past light cone at the moment of the CMB emission. For a more detailed derivation see chapter 8.8 in [17].

Setting $c = 1$, for the purpose of simplicity, the comoving radius (where $a(t)$ is the scale factor of the Universe) between two points in time (t_1 and t_2) can be found as:

$$r_{t_1 \to t_2} = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{a(t)} = 3t_0^{\frac{2}{3}} \left(t_2^{\frac{1}{3}} - t_1^{\frac{1}{3}} \right). \tag{33}$$

In equation (33), it was assumed that the Universe is matter dominated which entails $a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$, where $a_0 = a(t_0) = 1$ (t_0 is the time elapsed since the Big Bang). We can also express this in terms of t: $t = t_0 a(t)^{\frac{3}{2}}$.The Hubble parameter, defined as $H(t) = \frac{\dot{a}(t)}{a(t)}$ in this case can be written as $\frac{2}{3t}$. The present Hubble parameter (for which the value has been measured), H_0, can be found by plugging in t_0:

$$H_0 = \frac{2}{3t_0} \quad \to \quad t_0 = \frac{2}{3H_0}. \tag{34}$$

Let $a_1 = a(t_1)$ and $a_2 = a(t_2)$. Then, using the derivation for t above, $t_1 = t_0 a_1^{\frac{3}{2}}$ and $t_2 = t_0 a_2^{\frac{3}{2}}$. Substituting for t_0, t_1, t_2 in equation (33), gives:

$$r_{t_1 \to t_2} = \frac{2}{H_0} \left(\sqrt{a_2} - \sqrt{a_1} \right). \tag{35}$$

Essentially, if the comoving radius is computed by setting $t_1 = 0$, then it is equivalent to the radius of the particle horizon. For a region of space, at the time of the CMB, the radius of the particle horizon is:

$$r_{\mathrm{CMB}} = \frac{2}{H_0} \left(\sqrt{a_{\mathrm{CMB}}} - \cancelto{0}{\sqrt{a(t=0)}} \right) \quad \to \quad r_{\mathrm{CMB}} = \frac{2}{H_0} \sqrt{a_{\mathrm{CMB}}}. \tag{36}$$

Plugging the approximate numerical value for $a_{\text{CMB}} = 0.0009$ (it can be found using the formula $a = \frac{1}{1+z}$, with the redshift, $z_{\text{CMB}} \approx 1089$) gives:

$$r_{\text{CMB}} \approx 0.06 \times H_0^{-1}. \tag{37}$$

The value above is the radius of the particle horizon at some point during the emission of the CMB. In order to demonstrate that current observers can see causally disconnected patches at the time of the CMB, it is necessairy to compute the comoving radius, Δr, from the CMB to present time. This value will demonstrate the radius of the spatial region (at the time of the CMB) a current observer can measure. Following the prior derivation, Δr is:

$$\Delta r = \frac{2}{H_0}\left(a\left(t = t_0\right) - \sqrt{a_{\text{CMB}}}\right) = \frac{2}{H_0}\left(1 - \sqrt{a_{\text{CMB}}}\right), \tag{38}$$

$$\therefore \Delta r \approx 2 \times H_0^{-1}. \tag{39}$$

In figure 1, Δr corresponds to the radius of the blue cone at the dashed line labeling the moment of the CMB, whereas r_{CMB} refers to the comoving radius of the two red cones (separately). Evidently, since $\Delta r \gg r_{\text{CMB}}$, if a current observer measures distant regions in the CMB, it is equivalent to looking at patches of space that were causally disconnected at the moment of CMB emission.

A.2. Redshift at the matter-radiation equality

The scaling of matter and radiation density in the Universe can be written in relation to the scale factor $a(t)$. As the Universe expands in three spatial dimensions ρ_{matter} drops proportionally to $a(t)^{-3}$. For energy, there is volumetric expansion as well as energy reduction that is $\propto a(t)^{-1}$. Letting the current matter and energy densities (and thus setting the initial conditions, knowing that $a(t_0) = 1$, where t_0 is the current time) be ρ_m^o and ρ_r^o, respectively, gives the equations:

$$\rho_{\text{radiation}}\left(t\right) = \frac{\rho_r^o}{a\left(t\right)^{-4}} \qquad \rho_{\text{matter}}\left(t\right) = \frac{\rho_m^o}{a\left(t\right)^{-3}}. \tag{40}$$

Since the relation between redshift, z, and the expansion factor is $a(t) = \frac{1}{z+1}$ the equations, at the matter-radiation equality point, can be written as:

$$\rho_{\text{radiation}} = \rho_r^o\left(1+z\right)^4 \qquad \rho_{\text{matter}} = \rho_m^o\left(1+z\right)^3, \tag{41}$$

$$\rho_{\text{radiation}} = \rho_{\text{matter}} \quad \rightarrow \quad \rho_r^o\left(1+z_{\text{eq}}\right)^4 = \rho_m^o\left(1+z_{\text{eq}}\right)^3, \tag{42}$$

$$\therefore z_{\text{eq}} = \frac{\rho_m^o}{\rho_r^o} - 1 \approx 3400, \tag{43}$$

which is the numerical result when using the WMAP reported valued for Ω_m and Ω_r. An explanation of the ΛCDM model is outside the scope of this paper, for a more conceptual explanation of this see [37].

A.3. From action To Einstein field equations

The following derivation is done with reference to [17] (without setting $c = 1$). The action, as defined in [10], is:

$$\mathcal{S} = \int \mathrm{d}x^4 \left[\sqrt{-g}\left(\frac{\psi\left(R+2\Lambda\right)}{16\pi G} + \mathscr{L}_{\text{m}}\right) + \mathscr{L}_{\psi}\right].$$

Let:

$$\int \mathrm{d}x^4\ \sqrt{-g}\mathscr{L}_{\mathrm{m}} = \delta_m,$$
$$\int \mathrm{d}x^4\ \mathscr{L}_\psi\ = \delta_\psi,$$
$$\Omega = \frac{\psi}{16\pi G}.$$

Using the principle of least action $\delta\mathcal{S} = 0$ and the notation: $\frac{\delta X}{\delta g^{\mu\nu}} = \delta X$:

$$\delta\mathcal{S} = \int \mathrm{d}x^4\ \left[\Omega \frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\,(R+2\Lambda) + \delta\mathcal{S}_m + \delta\mathcal{S}_\psi\right] = 0, \tag{44}$$

$$\text{Expanding}: \frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\,(R+2\Lambda)$$
$$\to \delta\left(\sqrt{-g}R\right) + 2\Lambda\delta\sqrt{-g} \overset{\text{prod. rule}}{=} \left[R\delta\sqrt{-g} + \sqrt{-g}\delta R\right] + 2\Lambda\delta\sqrt{-g}.$$
$$\text{Using the chain rule we get}: \delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g.$$
$$\text{Substituting}: -\frac{R}{2\sqrt{-g}}\delta g + \sqrt{-g}\delta R - \frac{\Lambda}{\sqrt{-g}}\delta g.$$

Substituting into equation (44)

$$\Omega\int \mathrm{d}x^4\ \left[-\frac{R}{2\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}R - \frac{\Lambda}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g\right]$$
$$= -\int \mathrm{d}x^4\ \frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathscr{L}_{\mathrm{m}} - \int \mathrm{d}x^4\ \frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_\psi.$$

Now solving: $\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathscr{L}_{\mathrm{m}}$

$$\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathscr{L}_{\mathrm{m}} \overset{\text{prod. rule}}{=} \mathscr{L}_{\mathrm{m}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g} + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{\mathrm{m}}$$
$$\overset{\text{chain rule}}{=} -\frac{\mathscr{L}_{\mathrm{m}}}{2\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{\mathrm{m}}.$$

Substituting the result and equating the integrands gives:

$$\Omega\left[-\frac{R}{2\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}R - \frac{\Lambda}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g\right]$$
$$= \frac{\mathscr{L}_{\mathrm{m}}}{2\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}g - \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{\mathrm{m}} - \frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_\psi \times \frac{2}{\sqrt{-g}\Omega} \to \frac{R}{g}\frac{\delta}{\delta g^{\mu\nu}}g + 2\frac{\delta}{\delta g^{\mu\nu}}R + \frac{2\Lambda}{g}\frac{\delta}{\delta g^{\mu\nu}}g$$
$$= \frac{1}{\Omega}\left(-\frac{\mathscr{L}_{\mathrm{m}}}{g}\frac{\delta}{\delta g^{\mu\nu}}g - 2\sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{\mathrm{m}} - \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_\psi\right).$$

Defining the **energy-momentum tensor** as:

$$T_{\mu\nu} = -\frac{\mathscr{L}_{\mathrm{m}}}{g}\frac{\delta}{\delta g^{\mu\nu}}g - 2\sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{\mathrm{m}} - \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_\psi.$$

Dividing everything by 2 and combining the equations gives:

$$\frac{R}{2g}\frac{\delta g}{\delta g^{\mu\nu}}+\frac{\delta R}{\delta g^{\mu\nu}}+\frac{\Lambda}{g}\frac{\delta g}{\delta g^{\mu\nu}}=\frac{1}{2\Omega}T_{\mu\nu}. \tag{45}$$

Solving this will require a term-by-term derivation. Initially the term $\frac{\delta g}{\delta g^{\mu\nu}}$ will be solved. The following linear algebra identities will be used: let A be an $n\times n$ matrix:

$$\begin{aligned}
&\ln(\det A)=tr(A),\\
&\delta\ln(\det A)=\delta tr(A),\\
&\delta\det(A)=\det(A)\,tr(\delta\ln(A))=\det(A)\,tr\left(A^{-1}\delta A\right).\\
&\text{Let}:\begin{cases}\det(A)=g\\ A=g_{\mu\nu}\\ A^{-1}=g^{\mu\nu}\end{cases},\\
&\rightarrow\delta g=g\;tr\left(g^{\mu\nu}\delta g_{\mu\nu}\right)=gg^{\mu\nu}\delta g_{\mu\nu}.
\end{aligned}$$

The identity $\delta g_{\mu\nu}g^{\mu\nu}=0$ gives:

$$\begin{aligned}
\delta g_{\mu\nu}g^{\mu\nu}&=g_{\mu\nu}\delta g^{\mu\nu}+g^{\mu\nu}\delta g_{\mu\nu}=0,\\
&\therefore g^{\mu\nu}\delta g_{\mu\nu}=-g_{\mu\nu}\delta g^{\mu\nu}.
\end{aligned}$$

Applying this gives:

$$\begin{aligned}
&\delta g=-gg_{\mu\nu}\delta g^{\mu\nu},\\
&\therefore\frac{\delta g}{\delta g^{\mu\nu}}=-gg_{\mu\nu}.
\end{aligned}$$

Plugging back into equation (45):

$$\begin{aligned}
-\frac{R}{2g}gg_{\mu\nu}+\frac{\delta R}{\delta g^{\mu\nu}}-\frac{\Lambda}{g}gg_{\mu\nu}&=\frac{1}{2\Omega}T_{\mu\nu},\\
\therefore-\frac{R}{2}g_{\mu\nu}+\frac{\delta R}{\delta g^{\mu\nu}}-\Lambda g_{\mu\nu}&=\frac{1}{2\Omega}T_{\mu\nu}.
\end{aligned}$$

The final step is solving $\frac{\delta R}{\delta g^{\mu\nu}}$:

$$\delta R=R_{\mu\nu}\delta g^{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}. \tag{46}$$

The following derivation will make use of the following property:

$$\nabla_\alpha M^\lambda_{\mu\nu}=\partial_\alpha M^\lambda_{\mu\nu}+\Gamma^\beta_{\alpha\beta}M^\lambda_{\mu\nu}-\Gamma^\lambda_{\mu\beta}M^\beta_{\nu\alpha}-\Gamma^\beta_{\nu\alpha}M^\lambda_{\mu\beta}.$$

By definition $R_{\mu\nu}$ is:

$$\begin{aligned}
R_{\mu\nu}&=\partial_\alpha\Gamma^\alpha_{\mu\nu}-\partial_\nu\Gamma^\alpha_{\mu\alpha}+\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha},\\
\overset{\text{product rule}}{\rightarrow}\delta R_{\mu\nu}&=\partial_\alpha\delta\Gamma^\alpha_{\mu\nu}-\partial_\nu\delta\Gamma^\alpha_{\mu\alpha}+\Gamma^\alpha_{\mu\nu}\delta\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\mu\beta}\delta\Gamma^\beta_{\nu\alpha}+\Gamma^\beta_{\alpha\beta}\delta\Gamma^\alpha_{\mu\nu}-\Gamma^\beta_{\nu\alpha}\delta\Gamma^\alpha_{\mu\beta}.
\end{aligned}$$

Applyign the property:

$$\nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} + \Gamma^\beta_{\alpha\beta}\delta\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\beta}\delta\Gamma^\beta_{\nu\alpha} - \Gamma^\beta_{\mu\alpha}\delta\Gamma^\alpha_{\mu\beta},$$
$$\nabla_\nu \delta\Gamma^\alpha_{\mu\alpha} = \partial_\nu \delta\Gamma^\alpha_{\mu\alpha} - \Gamma^\alpha_{\mu\nu}\delta\Gamma^\beta_{\alpha\beta}.$$

Re-writing the Ricci tensor based on the equations above and plugging into equation (46) gives:

$$\delta R_{\mu\nu} = \nabla_a \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha}.$$

The variation of the Ricci scalar is therefore:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_a \delta\Gamma^\alpha_{\mu\nu} g^{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha} g^{\mu\nu}.$$

Re-labeling the indices (for the second term only):

$$\nu \to \alpha,$$
$$\alpha \to \nu,$$
$$\to \delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha \left(\delta\Gamma^\alpha_{\mu\nu} g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} g^{\mu\alpha}\right).$$

The second term $\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu} g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} g^{\mu\alpha})$ is a divergence term. In the integral of the action, $S = \int \mathrm{d}x^4 (\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu} g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} g^{\mu\alpha}))$, the divergance does not contribute, in other words, $S = \int \mathrm{d}x^4 (\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu} g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu} g^{\mu\alpha})) = 0$:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu},$$
$$or: \frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}.$$

Substituting into equation (45):

$$-\frac{R}{2} g_{\mu\nu} + \frac{\delta R}{\delta g^{\mu\nu}} - \Lambda g_{\mu\nu} = \frac{1}{2\Omega} T_{\mu\nu} \quad \to \quad -\frac{R}{2} g_{\mu\nu} + R_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{1}{2\Omega} T_{\mu\nu},$$
$$\therefore R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{1}{2\Omega} T_{\mu\nu}.$$

By the definition of the Einstein tensor, $G_{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$

Substituting this in, alongside Ω which was defined as $\frac{\psi}{16\pi G}$, where $\psi(x^\mu) = c^4$:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{\psi}.$$

Appendix B. Brief explanation of concepts and derivations

B.1. CMB radiation

The CMB Radiation was accidentally discovered by Penzias and Wilson [38] in 1965. In their paper, they detail the discovery of a higher than expected temperature on their antenna. Forthwith, Dicke proposed an explanation [39] that the radiation detected is a remnant from the extremely hot early Universe. A more rigorous mathematical discussion on the CMB, the observed anisotropies, and dark matter can be found in [40].

More conceptually, the CMB is relic radiation from the Universe when the temperatures were approximately 3000 degrees Kelvin. Soon after the Big Bang, until the Universe was about 380 000 years old, the high temperatures allowed for a rapidly-moving bath of particles, where collisions were frequent. At that epoch, ordinary mater (vaguely: electrons, protons, and neutrons) was coupled to photons meaning that a ray of light did not have the ability to travel long distances due to the hot and dense particle-radiation bath of the early Universe. After a period of expansion and cooling, the Universe reached the necessary temperatures to enable the formation of atomic nuclei and hydrogen. At this stage, ordinary matter decoupled from the photons (which implies that the amount of free electrons that absorbed and emitted radiation was lower), light could travel further and the Universe became transparent. These light rays, emitted prior to the formation of the first atoms, are what we currently detect as the CMB Radiation.

B.2. Friedmann equations

For a more detailed derivation see chapters 8.2–8.3 in [17]. Conceptually, the Friedmann equations are derived from the Robertson–Walker metric (after defining a dimensionless scale factor $a(t)$). Since this metric is defined for any function $a(t)$, it can be plugged into Einstein's equation in order to derive the Friedmann equations. One of the main characteristics of this derivation is that the Universe is modeled as a perfect fluid (which is isotropic).

The two Friedmann equations are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{a^2} \tag{47}$$

$$\ddot{a} = -\frac{4\pi Ga}{3}\left(\frac{3p}{c^2} - \rho\right). \tag{48}$$

These equations relate the density and pressure of the Universe to the expansion. The value for K determines the geometry of the Robertson–Walker. Setting $K = 0$ gives the critical density, ρ_c, that would result is a flat Universe. The following table summarizes the K values and their implications:

$\rho > \rho_c \longleftrightarrow K < 0 \longleftrightarrow$ open geometry
$\rho = \rho_c \longleftrightarrow K = 0 \longleftrightarrow$ flat geometry
$\rho < \rho_c \longleftrightarrow K > 0 \longleftrightarrow$ closed geometry

B.3. Cosmological constant

Commonly, when performing physical experiments, only the change in energy from one state to another is measurable. As noted by [17], equations of motion for a particle in potential $V(x)$

would be equivalent to a particle moving in potential $V(x)+V_0$ (for some real constant V_0). However, in gravitational physics, the precise value of the energy is significant, not just the difference between two states. This indicated the possibility of (then) yet unmeasured constant energy of the vacuum itself.

In the formulation of General Relativity, Einstein noted the possibility for *vacuum energy*, which is interchangeable with Λ. This energy density, which exhibits Lorentz invariance, is characteristic to empty space. A generalized metric including the vacuum energy density, ρ_Λ, can be written as: $-\rho_\Lambda g^{\mu\nu}$. Since it is common to model the Universe as a perfect fluid, a comparison with the energy-momentum tensor of a perfect fluid gives the equation of state $p_\Lambda = -\rho_\Lambda c^2$.

This is appropriate since Lorentz invariance implies that the energy-momentum tensor of the vacuum should be proportional to the generalized metric $g^{\mu\nu}$ (this implies that the vacuum does not have a preferred direction). For reference, the perfect fluid energy-momentum tensor is $T^{\mu\nu} = (\rho + \frac{p}{c^2})U^\mu U^\nu + pg^{\mu\nu}$. Equating the term that does not include the 4-velocities (since it is a vacuum, the first term can be neglected) gives the equation of state shown above.

The vacuum energy can be interchanged to the concept of dark energy, with the equation of state $p_\Lambda = -\rho_\Lambda c^2$.

(A more detailed explanation of the cosmological constant, Λ, can be found in chapter 4.5 of [17].)

B.4. Fine and hyperfine structure

A comprehensive theoretical overview of the Fine and Hyperfine splitting in atoms (Hydrogen in particular) can be found in chapter 7.3–7.5 of Griffiths [41]. Essentially, the Fine Structure of atoms results in a splitting of spectral lines due to relativistic corrections to the Schrödinger. This can be measured as the Zeeman effect, resulting from the interactions between the electron's spin and the magnetic field generated by the electron's orbit around the nucleus.

Hyperfine Structure (whose correction is of lesser order than the Fine Structure contributions, as noted by Griffiths in chapter 7.3 [41]), on the other hand, results from an interaction between the magnetic field generated by the electron's orbit and the nuclear spin. Both Fine and Hyperfine structure result in the splitting of energy levels. Energy levels in a multiplet are typically associated with electrons that have the same n (principal quantum number) and l (orbital angular momentum), but differ in s (spin) or j (total angular momentum).

B.5. AD and many multiplet method: comparison

The theoretical background of the AD method is presented in more detail in Murphy *et al* [28]. The primary purpose of this method is to analyze doublet separations in quasar (QSO) absorbtion spectra. Murphy *et al* used the following equation as a constraint on $\frac{\Delta\alpha}{\alpha}$, previously derived by Varshalovich *et al* (qtd. [28]):

$$\frac{\Delta\alpha}{\alpha} = \frac{c_r}{2}\left[\frac{\left(\Delta\lambda\cdot\lambda^{-1}\right)_z}{\left(\Delta\lambda\cdot\lambda^{-1}\right)_0} - 1\right] \tag{49}$$

where $(\frac{\Delta\lambda}{\lambda})_z$ and $(\frac{\Delta\lambda}{\lambda})_0$ are the relative doublet separations in the absorption cloud at some redshift, z, and as measured in the laboratory respectively. The constant $c_r \approx 1$ accounts for higher order relativistic corrections.

Similarly to the MM method discussed in section 3.3, the AD method employed by Murphy *et al* relies on equation (32). In their initial paper [28], they increase the precision of the AD method by a factor of 3.3 from prior calculations. Essentially, this was achieved by eliminating systematic errors, using improved wavelength measurements in the laboratory, and better QSO absorbtion data quality. However, as they note in their discussion, the MM method, which is based on the same equation, differs by using multiple transitions from many multiplets of different ions. In this manner, the q_1 (which can be positive or negative given the nature of the ground and excited states of the ion) and q_2 coefficients are found to be more sensitive to the properties of the ions giving an order of magnitude greater sensitivity than the AD method. In summary, MM allows for more sensitivity over AD, permits the usage of all transitions appearing in the absorption spectra (resulting in enhanced precision due to a larger amount of usable data points), and minimizes systematic errors (discussed in section 3.6.1) through the comparison of transitions with opposite sign q_1 coefficients.

B.6. Celestial sphere

The celestial sphere is an imaginary, concentric to Earth (or some other observing point), sphere with an arbitrarily large radius. The primary purpose is to ease the mapping of astronomical objects. Each object is projected onto the inner surface of the celestial sphere. It is important to note that in such a manner the linear distance from the observer to the object is not considered, only the angular separation.

Appendix C. Brief catalogue of related readings

C.1. Varying constant theories

C.1.1. VSL: related to the AM formulation. → Albrecht A and Magueijo J 1999 Time varying speed of light as a solution to cosmological puzzles [9]

→ Barrow J D 1999 Cosmologies with varying light speed [10]

C.1.2. VSL: bimetric formulation. → Clayton M A and Moffat J W 1999 Dynamical mechanism for varying light velocity as a solution to cosmological problems [11]

→ Kursunoglu B N 2002 *Quantum Gravity, Generalized Theory of Gravitation, and Superstring Theory-Based Unification* [12]

C.1.3. VSL: color dependent speed of light, theory and experiment. *Theory:*

→ Amelino-Camelia G 2002 Relativity in spacetimes with short-distance structure governed by an observer-independent (Planckian) length scale [13]

→ Manida S N 1999 Fock–Lorentz transformations and time-varying speed of light [42]

Experimental:

→ Ellis J *et al* 2000 A search in gamma-ray burst data for nonconstancy of the velocity of light [43]

C.1.4. VSL: lorentz invariant. → Moffat J W 1993 Superluminary Universe: a possible solution to the initial value problem in cosmology [14]

→ Moffat J W 2005 Variable speed of light cosmology and bimetric gravity: an alternative to standard inflation [15]

→ Magueijo J 2000 Covariant and locally Lorentz-invariant varying speed of light theories [5]

C.1.5. Dual theory: variable c (AM formulation) and e. → Barrow J D and Magueijo J 1998 Varying-α theories and solutions to the cosmological problems [44]

C.1.6. VSL and string theory. → Kiritsis E 1999 Supergravity, D-brane probes and thermal super Yang-Mills: a comparison [45]

→ Alexander S H S 2000 On the varying speed of light in a brane-induced FRW universe [46]

C.2. VSL: experimental

C.2.1. 'Webb results': chronological. → Webb J K *et al* 1998 A search for time variation of the fine structure constant [22]

→ Murphy M T *et al* 2001 Possible evidence for a variable fine-structure constant from QSO absorption lines: motivations, analysis and results [4]

→ Murphy M T *et al* 2001 Further constraints on variation of the fine-structure constant from alkali-doublet QSO absorption lines [28]

→ Webb J K *et al* 2001 Further evidence for cosmological evolution of the fine structure constant [23]

→ Murphy M T *et al* 2003 Further evidence for a variable fine-structure constant from Keck/HIRES QSO absorption spectra [36]

→ Webb J K *et al* 2011 Indications of a spatial variation of the fine structure constant [24]

→ King J A *et al* 2012 Spatial variation in the fine-structure constant—new results from VLT/UVES [25]

C.3. Recent VSL research

→ Eaves R E 2021 Constraints on variation in the speed of light based on gravitational constant constraints [47]

→ Lee S 2023 A viable varying speed of light model in the RW metric [48]

→ Cuzinatto R *et al* 2023 Dynamical analysis of the covarying coupling constants in scalar–tensor gravity [49]

→ Bassani P M 2023 Varying constants and the Brans–Dicke theory: a new landscape in cosmological energy conservation [50]

→ Gupta R 2020 Cosmology with relativistically varying physical constants [51]

→ Lee S 2023 The cosmological evolution condition of the Planck constant in the varying speed of light models through adiabatic expansion [52]

C.4. Supplementary recommended readings

C.4.1. Energy conditions overview. → Curiel E 2017 A primer on energy conditions [53]

C.4.2. Extensive overview of VSL theories. → Magueijo J 2003 New varying speed of light theories [3]

C.4.3. Reading on cosmology and general relativity. → Carroll S 2004 *Spacetime and Geometry: An Introduction to General Relativity* [17]

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