# *On the noncommutative geometry of semi-graded rings*

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**Title**

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**Título**

Sobre la geometría no conmutativa de anillos semi-graduados

**Abstract:** In this thesis, we establish several topological characterizations of the noncommutative spectrum of semi-graded rings by considering the notion of weak Zariski topology. With this aim, necessary or sufficient conditions to guarantee that families of these rings defined by endomorphisms and derivations are NI or NJ rings are formulated. We present results about the characterization of different types of elements of noncommutative rings such as idempotents, units, von Neumann regular, $\pi$-regular, and clean elements. We also investigate the notions of strongly harmonic and Gelfand rings over such families of semi-graded rings. Our results generalize treatments developed for commutative rings, skew polynomial rings, and several families of $\mathbb{N}$-graded rings, and contribute to the research on these topics that has been partially carried out in the literature.
On the other hand, we investigate the schematicness and the Serre-Artin-Zhang-Verevkin theorem for semi-graded rings. More exactly, for the Ore polynomials of higher order generated by homogeneous relations and skew Poincaré-Birkhoff-Witt extensions, we formulate necessary or sufficient conditions to guarantee the schematicness of these families of rings. We develop a noncommutative scheme theory for semi-graded rings that are not necessarily connected and $\mathbb{N}$-graded. With this theory, we prove the Serre-Artin-Zhang-Verevkin theorem for several families of non-$\mathbb{N}$-graded algebras that include different kinds of noncommutative rings appearing in ring theory and noncommutative algebraic geometry. Our treatment contributes to the research on this theorem developed in the literature.

**Resumen:** En esta tesis, establecemos diversas caracterizaciones topológicas del espectro no conmutativo de anillos semi-graduados al considerar la noción de topología débil de Zariski. Con este propósito, formulamos condiciones necesarias o suficientes para garantizar que familias de estos anillos definidos por endomorfismos y derivaciones sean anillos NI o anillos NJ. Presentamos resultados sobre la caracterización de diferentes tipos de elementos de anillos no conmmutativos tales como idempotentes, unidades, von Neumann regulares, $\pi$-regulares, y elementos limpios. También investigamos las nociones de anillo fuertemente armónico y de Gelfand sobre dichas familias de anillos semi-graduados. Nuestros resultados generalizan tratamientos desarrollados para anillos conmutativos, anillos de polinomios torcidos, y variadas familias de anillos $\mathbb{N}$-graduados, y contribuyen a la investigación sobre estos temas que ha sido llevada a cabo parcialmente en la literatura.
Por otra parte, investigamos la esquematicidad y el teorema de Serre-Artin-Zhang-Verevkin para anillos semi-graduados. Más exactamente, para los polinomios de Ore de orden superior generados por relaciones homogéneas y las extensiones torcidas de Poincaré-Birkhoff-Witt, formulamos condiciones necesarias o suficientes para garantizar la esquematicidad de estas familias de anillos. Desarrollamos una teoría de esquemas no conmutativa para anillos

semi-graduados que no son necesariamente conexos y $\mathbb{N}$-graduados. Con esta teoría, demostramos el teorema de Serre-Artin-Zhang-Verevkin para diversas familias de álgebras no $\mathbb{N}$-graduadas que incluyen diferentes clases de anillos no conmutativos que surgen en la teoría de anillos y la geometría algebraica no conmutativa. Nuestro tratamiento contribuye a la investigación sobre este teorema desarrollada en la literatura.

**Keywords:** *Semi-graded ring, skew polynomial ring, skew PBW extension, quadratic algebra, NI ring, NJ ring, idempotent element, unit element, Gelfand ring, harmonic ring, schematic algebra, noncommutative algebraic geometry, noncommutative scheme.*

**Palabras clave:** *Anillo semi-graduado, anillo de polinomios torcidos, extensión PBW torcida, álgebra cuadrática, anillo NI, anillo NJ, elemento idempotente, elemento unidad, anillo de Gelfand, anillo armónico, álgebra esquemática, geometría algebraica no conmutativa, esquema no conmutativo.*

# Dedicatory

*This thesis is dedicated to my parents, Esther and Alberto*

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# Contents

# Introduction

In the development of noncommutative settings, one of the most important differences between the theories of commutative and noncommutative spaces is that the former arise as rings of functions, and the second as rings of operators. Throughout history, different properties of rings and modules have been considered for these noncommutative objects, thinking not only in their noncommutative version compared to the commutative case, but also in the formulation of unknown properties. Several books on properties of noncommutative spaces can be found in the literature (e.g., Brown and Goodearl [27], Bueso et al. [28], Connes [35], Fajardo et al. [41], Goodearl and Warfield [49], McConnell and Robson [110], and Smith [156]).

On the other hand, the term *noncommutative algebraic geometry* is considered to be problematic in some sense for a general field of research. This is because the term "noncommutative algebraic geometry" suggests that there is some algebraic geometry happening, and that unlike usual algebraic geometry there is a level of noncommutativity involved. Following Belmans's ideas in his PhD thesis [19, p. xiii], "Different people have different reasons to dislike it." Let us see what he said:

1. Some results can be applied uniformly to commutative and noncommutative objects, suggesting the need for a rather clumsy name such as "not necessarily commutative algebraic geometry".

2. Other results are of a purely algebraic nature for noncommutative rings, but the tools required to obtain (or even state them) require algebraic geometry, but does this warrant the presence of the word geometry?

3. Or it could be that most of the geometry is abstracted away (maybe only at first sight), leaving very little for a usual algebraic geometer to recognise.

Related with these ideas, Ginzburg [44] identified two scales of noncommutative algebraic geometry:

1. noncommutative algebraic geometry *in the small*;

2. noncommutative algebraic geometry *in the large*.

In the first scale, Ginzburg means a *generalization* of commutative algebraic geometry, and in this way, he considers deformations and quantizations of commutative algebras. This is one of the topics in this thesis: the study of the left and right spectrum of a noncommutative ring with the notion of weak Zariski topology, as a generalization of the spectrum with Zariski topology in the commutative setting. This line of research allows us to recover classical results of the commutative setting.

With respect to the geometry *in the large*, Ginzburg means a *replacement* of commutative algebraic geometry that consists in the development of analogous notions that do not necessarily coincide with their corresponding in the commutative setting, and so we do not get back classical properties. The example in this thesis is the study of sheaves on the noncommutative site as an analogue of the sheaves on the projective scheme in the commutative case, and the corresponding Serre's theorem for schematic semi-graded rings.

On the structure of the thesis, this is not a classical monograph, but rather it is based on a collection of papers. Chapter 1 presents some ring-theoretical notions of semi-graded rings that are necessary in the next chapters. With the aim of showing their generality in areas such as ring theory, noncommutative algebraic geometry and noncommutative differential geometry, we include a non-exhaustive list of noncommutative algebras that are particular examples of these rings. Next, Chapter 2 contains several topological characterizations of the noncommutative spectrum of different families of semi-graded rings. Finally, in Chapter 3, we characterize the schematicness of some families of semi-graded rings, and formulate the Serre-Artin-Zhang-Verevkin theorem for schematic semi-graded rings.

**Notation and some terminology**

| Symbol | Meaning |
|---|---|
| $\mathbb{N}$ | The set of natural numbers including zero |
| $\mathbb{Z}$ | The set of integer numbers |
| $\mathbb{R}$ | The field of real numbers |
| $\mathbb{C}$ | The field of complex numbers |
| $R$ | Associative ring (not necessarily commutative) with identity |
| $K$ | Commutative ring with identity |
| $\Bbbk$ | Field |
| $R^*$ | The non-zero elements of $R$ |
| $Z(R)$ | The center of $R$ |
| $\mathrm{Idem}(R)$ | The set of idempotent elements of $R$ |
| $N(R)$ | The set of nilpotent elements of $R$ |
| $\mathrm{vnr}(R)$ | The set of von Neumann regular elements of $R$ |
| $\pi - r(R)$ | The set of $\pi$-regular elements of $R$ |
| $\mathrm{vnl}(R)$ | The set of von Neumann local elements of $R$ |
| $\mathrm{cln}(R)$ | The set of clean elements of $R$ |
| $U(R)$ | The set of units of $R$ |
| $N^*(R)$ | The upper radical of $R$, i.e., the sum of all nil ideals of $R$ |
| $J(R)$ | The Jacobson radical of $R$, i.e., the intersection of all maximal left ideals of $R$ |
| $N_*(R)$ | The prime radical of $R$, i.e., the intersection of all prime ideals of $R$ |
| $L(R)$ | The Levitzki radical of $R$, i.e., the sum of all locally nilpotent ideals of $R$ |
| $I \lhd_l R$ | $I$ is a left ideal of $R$ |

It is well-known that the following relations hold: $N_*(R) \subseteq L(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. Note that $N^*(R)$ is the unique maximal nil ideal of $R$ and

$$\begin{aligned} N^*(R) &= \{a \in R \mid RaR \text{ is a nil ideal of } R\} \\ &= \bigcap \{P \mid P \text{ is a strongly prime ideal of } R\}. \end{aligned}$$

For more details, see [69, 89, 108, 117].

Recall that for a ring $R$ and an element $r \in R$, $r$ is called a *left* (resp. *right*) *zero divisor of* $R$ if there exists $a \in R$ such that $ra = 0$ (resp. $ar = 0$). If $r$ is not a zero divisor, then it is called *regular*. $R$ is called a *domain* if it has no non-zero zero divisors. An element $r \in R$ is said to be a *nilpotent* element of $R$ if $r^n = 0$, for some $n \in \mathbb{N}$. The nilpotent elements of a ring are both left and right zero divisors. As is well-known, an element $r \in R$ is called an *idempotent* of $R$ if $r^2 = r$. If $r \neq 1$ is an idempotent element, then $r(1-r) = (1-r)r = 0$, whence $r$ is a left and right zero divisor. $R$ is said to be *reduced* if $N(R) = \{0\}$, and it is said to be *Abelian* if every idempotent of $R$ is central. $R$ is called *reversible* if $ab = 0$ implies $ba = 0$, where $a, b \in R$, and if $rst = 0$ implies $rts = 0$, where $r, s, t \in R$, then $R$ is said to be *symmetric*. $R$ is called *semicommutative* if for every pair of elements $a, b \in R$, we have that $ab = 0$ implies $aRb = 0$.

Throughout this thesis, the term ring means an associative (not necessarily commutative) ring with identity, and the term module means a left unital module (all localizations are considered by the left side).

## Statement of contributions

The chapters two and three in this thesis correspond to the following publications and preprints containing original results.

- **Chapter 2.** Suárez, H., **Chacón, A.**, Reyes, A. On NI and NJ skew PBW extensions. *Communications in Algebra* 50 (8) 3261–3275, 2022.
- **Chapter 2. Chacón, A.**, Higuera, S., Reyes, A. On types of elements, Gelfand and strongly harmonic rings of skew PBW extensions over weak compatible rings. Submitted for publication in *Arabian Journal of Mathematics* (*Springer*). Available online at `https://arxiv.org/abs/2212.09601`, 2022.
- **Chapter 3. Chacón, A.**, Reyes, A. On the schematicness of some Ore polynomials of higher order generated by homogenous quadratic relations. Accepted for publication in *Journal of Algebra and Its Applications*, 2023.
- **Chapter 3. Chacón, A.**, Reyes, A. Noncommutative scheme theory and the Serre-Artin-Zhang-Verevkin theorem for semi-graded rings. Submitted for publication. Available online at `https://arxiv.org/abs/2301.07815`, 2023.

# CHAPTER 1

# Semi-graded rings

In this chapter, we present the algebraic structures of interest in this thesis: the *semi-graded rings*. Also, we formulate some ring-theoretical notions that are necessary in the next chapters.

More exactly, Section 1.1 contains definitions and some key properties of semi-graded rings, finitely semi-graded rings and modules over these rings. Next, in Section 1.2 we present a list (not exhaustive) of noncommutative algebraic structures that are particular examples of semi-graded rings. Our aim in this section is to show explicitly the generality of these rings in the setting of ring theory, noncommutative algebraic geometry and noncommutative differential geometry. Section 1.3 recalls the notion of compatible ring with some of its properties. In Section 1.4, we consider the Armendariz property and some of its generalizations appearing in the literature. Finally, Section 1.5 contains some facts about the category of semi-graded rings.

## 1.1 Preliminaries and key properties

Lezama and Latorre [99] introduced the *semi-graded rings* as a generalization of $\mathbb{N}$-graded rings and several families of noncommutative rings of polynomial type that are not $\mathbb{N}$-graded (not in a trivial way). In that paper, they considered some notions of noncommutative algebraic geometry in the setting of semi-graded rings such as the Hilbert series, Hilbert polynomial and Gelfand-Kirillov dimension. As a matter of fact, in that paper, they extended the notion of noncommutative projective scheme to the context of semi-graded rings and generalized the well-known Serre-Artin-Zhang-Verevkin theorem (see also [96]). This result will be of interest for us in the Chapter 3.

Next, we recall briefly some definitions and results about semi-graded rings which are key in the following chapters.

**Definition 1.1 ([99, Definition 2.1]).** Let $R$ be a ring. $R$ is said to be *semi-graded* ($\mathsf{SG}$) if there exists a collection $\{R_n\}_{n\in\mathbb{Z}}$ of subgroups of the additive group $R^+$ such that the following conditions hold:

(i) $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

(ii) For every $m, n \in \mathbb{Z}$, $R_m R_n \subseteq \bigoplus_{k\leq m+n} R_k$.

(iii) $1 \in R_0$.

The collection $\{R_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* $R$, and we say that the elements of $R_n$ are *homogeneous of degree* $n$. We say that $R$ is *positively semi-graded* if $R_n = 0$ for every $n < 0$. If $R$ and $S$ are semi-graded rings and $f : R \to S$ is a ring homomorphism, then we say that $f$ is *homogeneous* if $f(R_n) \subseteq S_n$ for every $n \in \mathbb{Z}$.

Definitions 1.2 and 1.3 recall the notion of finitely semi-graded ring and finitely semi-graded algebra, respectively.

**Definition 1.2 ([99, Definition 2.4]).** A ring $R$ is called *finitely semi-graded* (FSG) if it satisfies the following conditions:

(i) $R$ is SG.

(ii) There exist finitely many elements $x_1, \dots, x_n \in R$ such that the subring generated by $R_0$ and $x_1, \dots, x_n$ coincides with $R$.

(iii) For every $n \geq 0$, $R_n$ is a free $R_0$-module of finite dimension.

**Definition 1.3 ([98, Definition 10]).** A $\Bbbk$-algebra $R$ is said to be *finitely semi-graded* (FSG) if the following conditions hold:

(i) $R$ is an FSG ring with semi-graduation given by $R = \bigoplus_{n\geq 0} R_n$ .

(ii) For every $m, n \geq 1$, $R_m R_n \subseteq R_1 \oplus \cdots \oplus R_{m+n}$.

(iii) $R$ is connected, i.e., $R_0 = \Bbbk$.

(iv) $R$ is generated in degree 1.

From Definition 1.3, it is straightforward to see that if $R$ is a FSG $\Bbbk$-algebra, then $R_+ := \bigoplus_{n\geq 1} R_n$ is a maximal ideal of $R$.

Notice that graded rings are SG, and finitely graded $\Bbbk$-algebras, PBW extensions [17], 3-dimensional skew polynomial rings [18], down-up algebras [21, 22], diffusion algebras [70] and skew PBW extensions [42] are examples of FSG rings [98, Proposition 1.17]. Definitions of these families of algebras and others are presented in Section 1.2.

Semi-graded rings and finitely semi-graded rings have been studied recently in the literature. For instance, Lezama et al. [96, 98] computed the set of point modules of finitely semi-graded rings. By considering the parametrization of the point modules for the quantum affine $n$-space, Lezama obtained the set of point modules for some important examples of non-$\mathbb{N}$-graded quantum algebras [94, Theorem 5.3].

Next, we present some results about modules in the setting of semi-graded rings.

**Definition 1.4 ([99, Definition 2.1]).** Let $R$ be an SG ring and let $M$ be an $R$-module. We say that $M$ is *semi-graded* (SG) if there exists a collection $\{M_n\}_{n\in\mathbb{Z}}$ of subgroups $M_n$ of the additive group $M^+$ such that the following conditions hold:

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$.

(ii) For every $m, n \in \mathbb{Z}$, $R_m M_n \subseteq \bigoplus_{k\leq m+n} M_k$.

The collection $\{M_n\}_{n\in\mathbb{Z}}$ is called *a semi-graduation of* $M$, and we say that the elements of $M_n$ are *homogeneous of degree* $n$.

$M$ is said to be *positively semi-graded* if $M_n = 0$, for every $n < 0$. Let $f : M \to N$ be a homomorphism of $R$-modules, where $M$ and $N$ are semi-graded $R$-modules. We say that $f$ is *homogeneous* if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$.

**Definition 1.5 ([99, Definition 2.3]).** Let $R$ be an SG ring, $M$ an SG $R$-module, and $N$ a submodule of $M$. We say that $N$ is a *semi-graded* (SG) *submodule of* $M$ if $N = \bigoplus_{n\in\mathbb{Z}} N_n$, where $N_n = M_n \cap N$. In this case, $N$ is an SG $R$-module.

**Proposition 1.6 ([99, Proposition 2.6]).** *If $R$ is an SG ring, $M$ is an SG $R$-module, and $N$ is a submodule of $M$, then the following conditions are equivalent:*

(1) *$N$ is a semi-graded submodule of $M$.*

(2) *For every $z \in N$, the homogeneous components of $z$ belong to $N$.*

(3) *$M/N$ is an SG $R$-module with semi-graduation given by*

$$(M/N)_n = (M_n + N)/N, \ n \in \mathbb{Z}.$$

**Remark 1.7.** Let $R$ be an SG ring and $M$ be an SG $R$-module. Then:

(i) If $N$ is an SG submodule of $M$, then the canonical map $M \to M/N$ is a homogeneous homomorphism.

(ii) If $\{M_i\}_{i\in I}$ is a family of SG submodules of $M$, then $\bigcap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are SG submodules of $M$.

Let $N$ be a subset of $M$. We define the SG *submodule generated by* $N$ as the intersection of all SG submodules of $M$ containing $N$, and we denote it as $\langle N\rangle^{\mathsf{SG}}$. If $N = \{n_1, \dots, n_l\}$, then we write $\langle N\rangle^{\mathsf{SG}} = \langle n_1, \dots, n_l\rangle^{\mathsf{SG}}$. We will say that $M$ is a *finitely generated* SG $R$-module if there exist finitely elements $m_1, \dots, m_t$ such that $M = \langle m_1, \dots, m_t\rangle^{\mathsf{SG}}$. If $M$

is simultaneously a module over different kinds of rings and there is risk of confusion, we write $\langle - \rangle_R^{\mathsf{SG}}$ to indicate the ring $R$ we are considering. If $N$ is an $\mathsf{SG}$ submodule of $M$, the notion of *finitely generated* $\mathsf{SG}$ submodule is defined in the natural way.

**Example 1.8.** Consider the well-known first Weyl algebra $A_1(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - 1\rangle$ over $\Bbbk$. Then:

(i) $A_1(\Bbbk)y$ is an $\mathsf{SG}$ submodule of $R$, whence $\langle y \rangle^{\mathsf{SG}} = A_1(\Bbbk)y$.

(ii) $1 \notin A_1(\Bbbk)x$, but due to the relation $yx = xy + 1$, it follows that $1 \in \langle x \rangle^{\mathsf{SG}}$.

**Definition 1.9.** If $R$ is a positively $\mathsf{SG}$ ring, for $t \in \mathbb{N}$ we define $R_{\geq t}$ as the intersection of all two-sided ideals that are $\mathsf{SG}$ submodules containing $\bigoplus_{k \geq t} R_k$.

Different properties of modules over families of semi-graded rings have been investigated by some people [103, 122, 137].

## 1.2 Some families of examples

Semi-graded rings extend several kinds of noncommutative rings of polynomial type such as Ore extensions [126, 127], families of differential operators generalizing Weyl algebras and universal enveloping algebras of finite dimensional Lie algebras [14, 17, 155], algebras appearing in mathematical physics [70, 139, 190], down-up algebras [21, 22, 85], ambiskew polynomial rings [78, 79], 3-dimensional skew polynomial rings [18, 133, 139, 149], PBW extensions [17], skew PBW extensions [42], and others. Ring-theoretical, algebraic and geometrical properties of semi-graded rings have been investigated in the literature (e.g., [11, 29, 56, 137, 138, 144, 145, 162, 164, 169] and references therein).

In this section, we present families of noncommutative rings that are particular examples of semi-graded rings with the aim of showing the generality of these objects, and the scope of the results presented in Chapters 2 and 3. For the completeness of the thesis, we include detailed references for every family of rings.

### 1.2.1 Skew polynomial rings and ambiskew polynomial rings

Skew polynomial rings (also known as Ore extensions) were introduced by Ore [126, 127] (Noether and Schmeidler [125] were interested in some kind of differential operator rings). Briefly, for $\sigma$ an endomorphisms of a ring $R$, a $\sigma$-derivation on $R$ is any additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$, for all $r, s \in R$ (strictly speaking, this is the definition of *left* $\sigma$*-derivation*, but we will not need the concept of *right* $\sigma$-derivation, which is any additive map $\delta : R \to R$ satisfying the rule $\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$). Notice that if $\sigma$ is the identity map on $R$, then $\sigma$-derivations are just ordinary derivations. The condition $\delta(1) = 0$ it follows from the skew product rule.

**Definition 1.10 ([126, 127], [49, p. 34]).** Let $R$ be a ring, $\sigma$ a ring endomorphism of $R$ and $\delta$ a $\sigma$-derivation on $R$. We will write $R[x; \sigma, \delta]$ provided

(i) $R[x;\sigma,\delta]$ containing $R$ as a subring;

(ii) $x$ is not an element of $R$;

(iii) $R[x;\sigma,\delta]$ is a free left $R$-module with basis $\{1, x, x^2, \dots\}$;

(iv) $xr = \sigma(r)x + \delta(r)$, for all $r \in R$.

Such a ring $R[x;\sigma,\delta]$ is called a *skew polynomial ring over* $R$, or an *Ore extension of* $R$. If $\sigma$ is the identity of $R$, then we write $R[x;\delta]$ and call it a *differential operator ring*. On the other hand, if $\delta$ is the zero map, then we write $R[x;\sigma]$ which is known as a *skew polynomial ring of endomorphism type*. Iterated skew polynomial rings are defined in the natural way. In the literature, we can find a lot of papers concerning ring-theoretical and module properties of Ore extensions. Some general details about these objects can be found in Brown and Goodearl [27], Goodearl and Warfield [49], and McConnell and Robson [110], and references therein.

Ore extensions are one of the most important techniques to define noncommutative algebras. Next, we illustrate this situation with Weyl algebras, some of its deformations, the $q$-Heisenberg algebra, and the quantum matrix algebra.

About the family of Weyl algebras $A_n(\Bbbk)$, in the literature it is common to find characterizations of these algebras as rings of differential operators. Surely, the most beautiful and excellent treatment about Weyl algebras is presented by Coutinho [37]. Briefly, the $n$th *Weyl algebra* $A_n(\Bbbk)$ over $\Bbbk$ is the $\Bbbk$-algebra generated by the $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ where

$$\begin{aligned} &x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, && 1 \le i < j \le n,\\ &y_jx_i = x_iy_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker's delta}, && 1 \le i, j \le n. \end{aligned}$$

From the relations defining the Weyl algebras, it follows that these cannot be expressed as skew polynomial rings of automorphism type (since the algebra is simple) but skew polynomial rings with non-trivial derivations.

Following Goodearl and Warfield [49, p. 36], for an element $q \in \Bbbk \setminus \{0\}$, $A_1^q(\Bbbk)$ denotes the $\Bbbk$-algebra presented by two generators $x$ and $y$ and the relation $xy - qyx = 1$, which is known as a *quantized Weyl algebra* over $\Bbbk$. Note that $A_1^q(\Bbbk) = A_1(\Bbbk) = \Bbbk[y][x; d/dy]$, when $q = 1$. If $q \neq 1$, then $A_1^q(\Bbbk) = \Bbbk[y][x;\sigma,\delta]$, where $\sigma$ is the $\Bbbk$-algebra automorphism given by $\sigma(f(y)) = f(qy)$, and $\delta$ is the $q$-*difference operator* (also known as *Eulerian derivative*)

$$\delta(f(y)) = \frac{f(qy) - f(y)}{qy - y} = \frac{\alpha(f) - f}{\alpha(y) - y},$$

as it can be seen in [49, Exercise 2N], so this algebra is not a skew polynomial ring of automorphism type.

A generalization of $A_1^q(\Bbbk)$ is given by the *additive analogue of the Weyl algebra* $A_q(q_1,\dots,q_n)$. For non-zero elements $q_1,\dots,q_n \in \Bbbk$, this algebra is generated by the indeterminates $x_1,\dots,x_n$ and $y_1,\dots,y_n$ satisfying the relations $x_jx_i = x_ix_j$, $y_jy_i = y_iy_j$, for every $1 \leq i,j \leq n$, $y_ix_j = x_jy_i$, for all $i \neq j$, and $y_ix_i = q_ix_iy_i + 1$, for $1 \leq i \leq n$. It is clear from these definitions that these algebras are not skew polynomial rings of automorphism type.

Another deformation of Weyl algebras was introduced by Giaquinto and Zhang [43] with the aim of studying the Jordan Hecke symmetry as a quantization of the usual second Weyl algebra. By definition, the *quantum Weyl algebra* $A_2(J_{a,b})$ is the $\Bbbk$-algebra generated by the indeterminates $x_1, x_2, \partial_1, \partial_2$, with relations (depending on parameters $a, b \in \Bbbk$)

$$\begin{aligned} x_1x_2 &= x_2x_1 + ax_1^2, & \partial_2\partial_1 &= \partial_1\partial_2 + b\partial_2^2 \\ \partial_1x_1 &= 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 &= -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2 \\ \partial_2x_1 &= x_1\partial_2, & \partial_2x_2 &= 1 - bx_1\partial_2 + x_2\partial_2. \end{aligned}$$

Note that if $a = b = 0$, then $A_2(J_{0,0})$ is precisely the second Weyl algebra $A_2(\Bbbk)$.

By definition, the *q-Heisenberg algebra* $\mathbf{H}_n(q)$ is the $\Bbbk$-algebra generated over $\Bbbk$ by the indeterminates $x_i, y_i, z_i$, for $1 \leq i \leq n$, subject to the relations

$$\begin{aligned} x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i, \quad z_jz_i = z_iz_j, & \qquad 1 \leq i < j \leq n, \\ x_iz_i - qz_ix_i = z_iy_i - qy_iz_i = x_iy_i - q^{-1}y_ix_i + z_i = 0, & \qquad 1 \leq i \leq n, \\ x_iy_j = y_jx_i, \quad x_iz_j = z_jx_i, \quad y_iz_j = z_jy_i, & \qquad i \neq j. \end{aligned}$$

It is easy to see that $\mathbf{H}_n(q)$ can be expressed as an iterated skew polynomial ring.

Given any $q \in \Bbbk \setminus \{0\}$, the corresponding *quantized coordinate ring* of the ring of matrices of size $2 \times 2$ with entries in $\Bbbk$, denoted by $M_2(\Bbbk)$, is the $\Bbbk$-algebra $O_q(M_2(\Bbbk))$ presented by four generators $x_{11}, x_{12}, x_{21}$, and $x_{22}$ and the six relations $x_{11}x_{12} = qx_{12}x_{11}$, $x_{12}x_{22} = qx_{22}x_{12}$, $x_{11}x_{21} = qx_{21}x_{11}$, $x_{21}x_{22} = qx_{22}x_{21}$, $x_{12}x_{21} = x_{21}x_{12}$, and $x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}$. This algebra, also known as the *coordinate ring of quantum* $2 \times 2$ matrices over $\Bbbk$, or the $2 \times 2$ *quantum matrix algebra* over $\Bbbk$, can be expressed as the skew polynomial ring $\Bbbk[x_{11}][x_{12}; \sigma_{12}][x_{21}; \sigma_{21}][x_{22}; \sigma_{22}, \delta_{22}]$ [49, Exercise 2V].

Jordan [77] introduced a certain class of iterated Ore extensions called *ambiskew polynomial rings*. These structures have been studied by Jordan et al. [78, 79] at various levels of generality that contain different examples of noncommutative algebras. Next, we recall briefly its definition.

Consider a commutative $\Bbbk$-algebra $B$, a $\Bbbk$-automorphism of $B$, and elements $c \in B$ and $p \in \Bbbk^*$. Let $S$ be the Ore extension $B[x; \sigma^{-1}]$ and extend $\sigma$ to $S$ by setting $\sigma(x) = px$. By [34, p. 41], there is a $\sigma$-derivation $\delta$ of $S$ such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B, \sigma, c, p)$ is the Ore extension $S[y; \sigma, \delta]$, whence the following relations hold:

$$yx - pxy = c, \quad \text{and, for all } b \in B, \quad xb = \sigma^{-1}(b)x \quad \text{and} \quad yb = \sigma(b)y. \tag{1.1}$$

Equivalently, $R$ can be presented as $R = B[y;\sigma][x;\sigma^{-1},\delta']$ with $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$, so that $xy - p^{-1}yx = -p^{-1}c$. If we consider the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, then we can see that the definition involves twists from both sides using $\sigma$; this is the reason for the name of the objects.

### 1.2.2 Universal enveloping algebras and PBW extensions

If $\mathfrak{g}$ is a finite dimensional Lie algebra over a commutative ring $K$ with basis $\{x_1, \dots, x_n\}$, then by the *Poincaré-Birkhoff-Witt theorem*, the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the algebra generated by $x_1, \dots, x_n$ subject to the relations $x_ir - rx_i = 0 \in K$, for every element $r \in K$, and $x_ix_j - x_jx_i = [x_i, x_j] \in \mathfrak{g}$, where $[x_i, x_j] \subseteq K + Kx_1 + \dots + Kx_n$, for all $1 \le i, j \le n$. As is well-known, in general these algebras are not skew polynomial rings even including non-zero trivial derivations. Some enveloping algebras can be expressed as skew polynomial rings; however, in these rings the derivations are non-trivial. Let us see an example.

Following [49, p. 40], the standard basis for the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is labelled $\{e, f, h\}$, where $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. In this way, the enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ is the $\Bbbk$-algebra presented by three generators $e, f, h$ and three relations $ef - fe = h$, $he - eh = 2e$, and $hf - fh = -2f$. If $R$ is the subalgebra of $U(\mathfrak{sl}_2(\Bbbk))$ generated by $e$ and $h$, then $R = \Bbbk[e][h;\delta_1] = \Bbbk[h][e;\sigma_1]$, where $\Bbbk[e]$ and $\Bbbk[h]$ are commutative polynomial rings, $\delta_1$ denotes the derivation $2e(d/de)$ on $\Bbbk[e]$, and $\sigma_1$ is the $\Bbbk$-algebra automorphism of $\Bbbk[h]$ with $\sigma_1(h) = h - 2$. Thus, $U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk[e][h;\delta_1][f;\sigma_2,\delta_2] = \Bbbk[h][e;\sigma_1][f;\sigma_2,\delta_2]$, where $\sigma_2(e) = e$, $\sigma_2(h) = h + 2$, $\delta_2(e) = -h$, and $\delta_2(h) = 0$ [49, Exercise 2S]. Other examples of universal enveloping algebras known as *parafermionic* and *parabosonic algebras* are presented in Section 1.2.8.

Notice that universal enveloping algebras above are PBW extensions over $K$ in the sense of Bell and Goodearl [17] (note that these authors presented another examples of enveloping rings related to enveloping universal algebras). In Remark 1.30 (iv), we will say some words about these extensions.

### 1.2.3 3-dimensional skew polynomial algebras and bi-quadratic algebras on 3 generators with PBW bases

Another kind of noncommutative rings which includes the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$ of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, the Dispin algebra $U(\mathfrak{osp}(1,2))$ and the Woronowicz's algebra $W_\nu(\mathfrak{sl}_2(\Bbbk))$ [183], is the family of *3-dimensional skew polynomial algebras*. These algebras were introduced by Bell and Smith [18] and are very important in noncommutative algebraic geometry and noncommutative differential geometry (e.g., [132, 133, 139, 141, 149] and references therein). Next, we recall its definition and classification.

**Definition 1.11 ([149, Definition C4.3]).** *A 3-dimensional skew polynomial algebra $A$* is a $\Bbbk$-algebra generated by the indeterminates $x, y, z$ restricted to relations $yz - \alpha zy = \lambda,\ zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that

(i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$;

(ii) standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\Bbbk$-basis of the algebra.

**Proposition 1.12 ([149, Theorem C.4.3.1]).** *If $A$ is a 3-dimensional skew polynomial algebra, then $A$ is one of the following algebras:*

(1) *if $|\{\alpha, \beta, \gamma\}| = 3$, then $A$ is defined by the relations $yz - \alpha zy = 0,\ zx - \beta xz = 0,\ xy - \gamma yx = 0$.*

(2) *if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then $A$ is one of the following algebras:*

(i) $yz - zy = z, \quad zx - \beta xz = y, \quad xy - yx = x;$

(ii) $yz - zy = z, \quad zx - \beta xz = b, \quad xy - yx = x;$

(iii) $yz - zy = 0, \quad zx - \beta xz = y, \quad xy - yx = 0;$

(iv) $yz - zy = 0, \quad zx - \beta xz = b, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - \beta xz = 0, \quad xy - yx = x;$

(vi) $yz - zy = z, \quad zx - \beta xz = 0, \quad xy - yx = 0,$

*where $a, b$ are any elements of $\Bbbk$. All non-zero values of $b$ give isomorphic algebras.*

(3) *If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $A$ is one of the following algebras:*

(i) $yz - \alpha zy = 0, \quad zx - \beta xz = y + b, \quad xy - \alpha yx = 0;$

(ii) $yz - \alpha zy = 0, \quad zx - \beta xz = b, \quad xy - \alpha yx = 0.$

*In this case, $b$ is an arbitrary element of $\Bbbk$. Again, any non-zero values of $b$ give isomorphic algebras.*

(4) *If $\alpha = \beta = \gamma \neq 1$, then $A$ is the algebra defined by the relations $yz - \alpha zy = a_1x + b_1,\ zx - \alpha xz = a_2y + b_2,\ xy - \alpha yx = a_3z + b_3$. If $a_i = 0$ $(i = 1, 2, 3)$, then all non-zero values of $b_i$ give isomorphic algebras.*

(5) *If $\alpha = \beta = \gamma = 1$, then $A$ is isomorphic to one of the following algebras:*

(i) $yz - zy = x, \quad zx - xz = y, \quad xy - yx = z;$

(ii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = z;$

(iii) $yz - zy = 0, \quad zx - xz = 0, \quad xy - yx = b;$

(iv) $yz - zy = -y, \quad zx - xz = x + y, \quad xy - yx = 0;$

(v) $yz - zy = az, \quad zx - xz = z, \quad xy - yx = 0;$

*Parameters $a, b \in \Bbbk$ are arbitrary, and all non-zero values of $b$ generate isomorphic algebras.*

### 1.2.4 Bi-quadratic algebras on 3 generators with PBW bases

Related with algebras generated by three indeterminates, recently Bavula [16] defined the *skew bi-quadratic algebras* with the aim of giving an explicit description of bi-quadratic algebras on 3 generators with PBW basis.

For a ring $R$ and a natural number $n \geq 2$, a family $M = (m_{ij})_{i>j}$ of elements $m_{ij} \in R$ $(1 \leq j < i \leq n)$ is called a *lower triangular half-matrix* with coefficients in $R$. The set of all such matrices is denoted by $L_n(R)$.

If $\sigma = (\sigma_1, \dots, \sigma_n)$ is an $n$-tuple of commuting endomorphisms of $R$, $\delta = (\delta_1, \dots, \delta_n)$ is an $n$-tuple of $\sigma$-endomorphisms of $R$ (that is, $\delta_i$ is a $\sigma_i$-derivation of $R$ for $i = 1, \dots, n$), $Q = (q_{ij}) \in L_n(Z(R))$, $\mathbb{A} := (a_{ij,k})$ where $a_{ij,k} \in R$, $1 \leq j < i \leq n$ and $k = 1, \dots, n$, and $\mathbb{B} := (b_{ij}) \in L_n(R)$, the *skew bi-quadratic algebra* (*SBQA*) $A = R[x_1, \dots, x_n; \sigma, \delta, Q, \mathbb{A}, \mathbb{B}]$ is a ring generated by the ring $R$ and elements $x_1, \dots, x_n$ subject to the defining relations

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \quad \text{for } i = 1, \dots, n, \text{ and every } r \in R, \tag{1.2}$$

$$x_i x_j - q_{ij} x_j x_i = \sum_{k=1}^{n} a_{ij,k} x_k + b_{ij}, \quad \text{for all } j < i. \tag{1.3}$$

In the particular case when $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for $i = 1, \dots, n$, the ring $A$ is called the *bi-quadratic algebra* (*BQA*) and is denoted by $A = R[x_1, \dots, x_n; Q, \mathbb{A}, \mathbb{B}]$. $A$ has *PBW basis* if $A = \bigoplus_{\alpha \in \mathbb{N}^n} R x^\alpha$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The following result classifies (up to isomorphism) the bi-quadratic algebras on three generators of Lie type, i.e., when $q_1 = q_2 = q_3 = 1$.

**Proposition 1.13 ([16, Theorem 1.4]).** *Let $A$ be an algebra of Lie type over an algebraically closed field $\Bbbk$ of characteristic zero. Then the algebra $A$ is isomorphic to one of the following (pairwise non-isomorphic) algebras:*

(1) $P_3 = \Bbbk[x_1, x_2, x_3]$*, a polynomial in three indeterminates.*

(2) $\mathcal{U}(\mathfrak{sl}_2(\Bbbk))$*, the universal enveloping algebra of the Lie algebra* $\mathfrak{sl}_2(\Bbbk)$.

(3) $\mathcal{U}(\mathfrak{H}_3))$*, the universal enveloping algebra of the Heisenberg Lie algebra* $\mathfrak{H}_3$.

(4) $\mathcal{U}(\mathcal{N})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = z, [x, z] = 0, [y, z] = 1$*, and the algebra* $\mathcal{U}(\mathcal{N})/\langle c-1\rangle$ *is a tensor product* $A_1 \otimes \Bbbk[x']$ *of its subalgebras, the Weyl algebra* $A_1(\Bbbk) = \Bbbk\{y, z\}/\langle [y, z] = 1\rangle$ *and the polynomial algebra* $[x']$ *where* $x' = x + \frac{1}{2}z^2$.

(5) $\mathcal{U}(\mathfrak{n}_2 \times \urcorner z) \cong \urcorner\{x, y, z\}/\langle [x, y] = y\rangle$*, and $z$ is a central element.*

(6) $\mathcal{U}(\mathcal{M})/\langle c-1\rangle \cong \Bbbk\{x, y, z\}/\langle [x, y] = y, [x, z] = 1, [y, z] = 0\rangle$ *and the algebra* $\mathcal{U}(\mathcal{M})/\langle c-1\rangle$ *is a skew polynomial algebra* $A_1(\Bbbk)[y; \sigma]$ *where* $A_1(\Bbbk) = \Bbbk\{x, z\}/\langle [x, z] = 1\rangle$ *is the Weyl algebra and $\sigma$ is an automorphism of* $A_1(\Bbbk)$ *given by the rule* $\sigma(x + 1)$ *and* $\sigma(z) = z$.

**Proposition 1.14 ([16, Theorem 2.1]).** *Up to isomorphism, there are only five bi-quadratic algebras on two generators:*

(1) *The polynomial algebra* $\Bbbk[x_1, x_2]$,

(2) *The Weyl algebra* $A_1(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_1x_2 - x_2x_1 = 1\rangle$,

(3) *The universal enveloping algebra of the Lie algebra* $\mathfrak{n}_2 = \langle x_1, x_2 \mid [x_2, x_1] = x_1\rangle$, $\mathcal{U}(\mathfrak{n}_2) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - x_1x_2 = x_1\rangle$,

(4) *The quantum plane* $\mathcal{O}_q(\Bbbk) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 = qx_1x_2\rangle$, *where* $q \in \Bbbk \setminus \{0, 1\}$, *and*

(5) *The quantum Weyl algebra* $A_1(q) = \Bbbk\{x_1, x_2\}/\langle x_2x_1 - qx_1x_2 = 1\rangle$, *where* $q \in \Bbbk \setminus \{0, 1\}$.

### 1.2.5 Diffusion algebras

Diffusion algebras were introduced formally by Isaev et al. [70] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice. However, its origin can be found in Krebs and Sandow [86].

**Definition 1.15.** ([70, p. 5817]) The *diffusion algebras type 1* are affine algebras $\mathcal{D}$ that are generated by $n$ indeterminates $D_1, \dots, D_n$ over $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_n}^{k_n}$ with $k_j \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and there exist elements $x_1, \dots, x_n \in \Bbbk$ such that for all $1 \le i < j \le n$, there exist $\lambda_{ij} \in \Bbbk^*$ such that

$$\lambda_{ij}D_iD_j - \lambda_{ji}D_jD_i = x_jD_i - x_iD_j. \tag{1.4}$$

Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t = 0$, for all $t = 1, \dots, n$, is a *multiparameter quantum affine* $n-$*space*.

Fajardo et al. [41] studied ring-theoretical properties of a graded version of these algebras.

**Definition 1.16.** ([41, Section 2.4]) The *diffusion algebras type 2* are affine algebras $\mathcal{D}$ generated by $2n$ variables $\{D_1, \dots, D_n, x_1, \dots, x_n\}$ over a field $\Bbbk$ that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1} B_{\alpha_2}^{k_2} \cdots B_{\alpha_n}^{k_n}$ with $B_{\alpha_i} \in \{D_1, \dots, D_n, x_1, \dots, x_n\}$, for all $i \le 2n$, $k_j \in \mathbb{N}$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, such that for all $1 \le i < j \le n$, there exist elements $\lambda_{ij} \in \Bbbk^*$ satisfying the relations

$$\lambda_{ij}D_iD_j - \lambda_{ji}D_jD_i = x_jD_i - x_iD_j. \tag{1.5}$$

Different physical applications of algebras type 1 and 2 have been studied in the literature. From the point of view of ring-theoretical, homological and computational properties, several thesis and papers have been published (e.g., [41, 52, 63, 93, 170]). For instance, notice that a diffusion algebra type 1 generated by $n$ indeterminates has Gelfand-Kirillov dimension $n$ since because of the PBW basis, the vector subspace consisting of elements of total degree at most $l$ is isomorphic to that of a commutative polynomial ring in $n$ indeterminates. Similarly, diffusion algebras type 2 have Gelfand-Kirillov dimension $2n$.

**Remark 1.17.** About the above definitions of diffusion algebras, we have the following facts:

(i) Isaev et al. [70] and Pyatov and Twarok [131] defined diffusion algebras type 1 by taking $\Bbbk = \mathbb{C}$. Nevertheless, for the results obtained in this thesis we can take any field not necessarily $\mathbb{C}$.

(ii) Following Krebs and Sandow [86], the relations (1.4) are consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta, \quad \text{for all} \quad \gamma, \delta = 0, 1, \dots, n-1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta} \in \Bbbk$, and $D_i$'s and $X_j$'s are operators of a particular vector space such that not necessarily $[D_i, X_j] = 0$ holds [86, p. 3168].

(iii) Hinchcliffe in his PhD thesis [63, Definition 2.1.1] considered the following notation for diffusion algebras. Let $R$ be the algebra generated by $n$ indeterminates $x_1, x_2, \dots, x_n$ over $\mathbb{C}$ subject to relations $a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j$, whenever $i < j$, for some parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$, for all $i < j$ and $b_{ij}, r_i \in \mathbb{C}$, for all $i < j$. He defined the *standard monomials* to be those of the form $x_n^{i_n} x_{n-1}^{i_{n-1}} \cdots x_2^{i_2} x_1^{i_1}$. $R$ is called a *diffusion algebra* if it admits a *PBW basis of these standard monomials.* In other words, $R$ is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for $R$. If all the elements $q_{ij} := \frac{b_{ij}}{a_{ij}}$'s are non-zero, then the diffusion algebras have a PBW basis in any order of the indeterminates [63, Remark 2.1.6].

Diffusion algebras of $n$ generators (also called *$n$-diffusion algebras*) are constructed in such a way that the subalgebras of three generators are also diffusion algebras. As we can see in Proposition 1.18, diffusion algebras type 1 of three generators can be classified into 4 families, $A, B, C$, and $D$, and these in turn are divided into classes as shown below (notice that this classification reflects the number of coefficients $x_s, s \in \{i, j, k\}$, being zero in comparison with the expression (1.4)).

**Proposition 1.18 ([131, p. 3270]).** *If $\mathcal{D}$ is a diffusion algebra type 1 generated by the indeterminates $D_i, D_j$ and $D_k$ with $i < j < k$, and $\Lambda \in \Bbbk$, then $\mathcal{D}$ belongs to some of the following classes of diffusion algebras:*

(1) The case of $A_I$:

$$\begin{aligned} gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j, \\ gD_iD_k - gD_kD_i &= x_kD_i - x_iD_k, \\ gD_jD_k - gD_kD_j &= x_kD_j - x_jD_k, \end{aligned}$$

*where $g \neq 0$.*

(2) The case of $A_{II}$:

$$g_{ij}D_iD_j = x_jD_i - x_iD_j,$$
$$g_{ik}D_iD_k = x_kD_i - x_iD_k,$$
$$g_{jk}D_jD_k = x_kD_j - x_jD_k,$$

*where* $g_{st} := g_s - g_t$ *with* $g_s \neq g_t$, *for all* $s < t$, *and* $s, t \in \{i, j, k\}$.

(3) The case of $B^{(1)}$:

$$g_jD_iD_j - (g_j - \Lambda)D_jD_i = -x_iD_j,$$
$$gD_iD_k - (g - \Lambda)D_kD_i = x_kD_i - x_iD_k,$$
$$g_jD_jD_k - (g_j - \Lambda)D_kD_j = x_kD_j,$$

*where* $g, g_j \neq 0$.

(4) The case of $B^{(2)}$:

$$g_{ij}D_iD_j = -x_iD_j,$$
$$g_{ik}D_iD_k - \lambda_{ki}D_kD_i = x_kD_i - x_iD_k,$$
$$g_{jk}D_jD_k = x_kD_j,$$

*where* $g_{ij}, g_{ik}, g_{jk} \neq 0$.

(5) The case of $B^{(3)}$.

$$gD_iD_j - (g - \Lambda)D_jD_i = x_jD_i - x_iD_j,$$
$$g_kD_iD_k = -x_iD_k,$$
$$(g_k - \Lambda)D_jD_k = -x_jD_k,$$

*where* $g \neq 0$ *and* $g_k \neq 0, \Lambda$.

(6) The case of $B^{(4)}$:

$$(g_i - \Lambda)D_iD_j = x_jD_i,$$
$$g_iD_iD_k = x_kD_i,$$
$$gD_jD_k - (g - \Lambda)D_kD_j = x_kD_j - x_jD_k,$$

*where* $g \neq 0$ *and* $g_i \neq 0, \Lambda$.

(7) The case of $C^{(1)}$:

$$g_jD_iD_j - (g_j - \Lambda)D_jD_i = -x_iD_j,$$
$$g_kD_iD_k - (g_k - \Lambda)D_kD_i = -x_iD_k,$$
$$g_{jk}D_jD_k - g_{kj}D_kD_j = 0,$$

*where* $g_j, g_k, g_{j,k} \neq 0$.

(8) The case of $C^{(2)}$*:*

$$\begin{aligned} g_{ij}D_iD_j - g_{ji}D_jD_i &= -x_iD_j, \\ g_{ik}D_iD_k - g_{ki}D_kD_i &= -x_iD_k, \\ D_jD_k &= 0, \end{aligned}$$

*where* $g_{ij}, g_{ik} \neq 0$.

(9) The case of $D$*: With* $q_{st} := \frac{g_{ts}}{g_{st}}$*, where* $s, t \in \{i, j, k\}$ *(recall that* $g_{st} \neq 0$*, for* $s < t$*), we have*

$$\begin{aligned} D_iD_j - q_{ji}D_jD_i &= 0, \\ D_iD_k - q_{ki}D_kD_i &= 0, \\ D_jD_k - q_{kj}D_kD_j &= 0. \end{aligned}$$

About the relationship between diffusion algebras and skew polynomial rings, if we consider the notation in Remark 1.17 (3), then a 3-diffusion algebra generated by the indeterminates $x_1, x_2, x_3$ is a skew polynomial ring over its 2-diffusion subalgebra generated by $x_2$ and $x_3$ [63, Lemma 2.2.1], where it is easy to see that a 2-diffusion algebra is a skew polynomial ring over the polynomial subalgebra generated by $x_2$. In general, an $n$-diffusion algebra (generated by the indeterminates $x_1, \dots, x_n$) is a skew polynomial ring over its $(n-1)$ diffusion subalgebra generated by $x_2, \dots, x_n$ [63, Remark 2.2.2].

Since a diffusion algebra on $n \geq 2$ generators is left Noetherian if and only if $q_{ij} \neq 0$, for all $i < j$ [63, Proposition 2.2.5], where $q_{ij}$ is given in Remark 1.17 (3), then every Noetherian 2-diffusion algebra is isomorphic to one of the following three types of algebra [63, Proposition 3.3.1]:

- *The quantum affine plane*, that is, the free algebra generated by the indeterminates $x_1$ and $x_2$ subject to the relation $x_1x_2 - qx_2x_1 = 0$, for some $q \in \mathbb{C} \setminus \{0\}$ (allowing the possibility $q = 1$).
- *The quantized Weyl algebra*, i.e., the free algebra generated by the indeterminates $x_1$ and $x_2$ subject to the relation $x_1x_2 - qx_2x_1 = 1$, for some $q \in \mathbb{C} \setminus \{0, 1\}$.
- *The universal enveloping algebra of the 2-d soluble Lie algebra*, that is, the free algebra generated by the indeterminates $x_1$ and $x_2$ subject to the relation $x_1x_2 - x_2x_1 = x_1$.

Related to Proposition 1.18, Hinchcliffe [63] proved the following result about classification of diffusion algebras assuming certain conditions on the coefficients of commutation of the indeterminates.

**Proposition 1.19 ([63, Proposition 3.1.4]).** *If* $q_{ij} \notin \{0, 1\}$*, for all* $i, j$*, then a diffusion algebra* $R$ *is isomorphic either to multiparameter quantum affine* $n$*-space or to the* $\mathbb{C}$*-algebra* *generated by the indeterminates* $x_1, x_2, x_3, \ldots, x_n$ *subject to relations*

$$\begin{aligned} x_1x_2 - q_{12}x_2x_1 &= 1, \quad \text{where} \quad q_{12} \neq 1, \\ x_1x_i - q_{1i}x_ix_1 &= 0, \quad \text{where} \quad q_{1i} \neq 1, \\ x_2x_i - q_{1i}^{-1}x_ix_2 &= 0, \\ x_ix_j - q_{ij}x_jx_i &= 0, \quad \text{for all } 3 \leq i < j. \end{aligned}$$

### 1.2.6 Generalized Weyl algebras and down-up algebras

Other algebraic structures that illustrate the results obtained in this thesis are the *generalized Weyl algebras* and *down-up algebras*. We briefly present the definitions and some relations between these algebras (see [77, 78, 79] for a detailed description).

Given an automorphism $\sigma$ and a central element $a$ of a ring $R$, Bavula [14] defined the *generalized Weyl algebra* $R(\sigma, a)$ as the ring extension of $R$ generated by the indeterminates $X^-$ and $X^+$ subject to the relations $X^-X^+ = a$, $X^+X^- = \sigma(a)$, and, for all $b \in R$, $X^+b = \sigma(b)X^+$, $X^-\sigma(b) = bX^-$. This family of algebras includes the classical Weyl algebras, primitive quotients of $U(\mathfrak{sl}_2)$, and ambiskew polynomial rings. Generalized Weyl algebras have been extensively studied in the literature by various authors (see [15, 16, 78], and references therein).

On the other hand, the *down-up algebras* $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, were defined by Benkart and Roby [21, 22] as generalizations of algebras generated by a pair of operators, precisely, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered set $P$. More exactly, consider a partially ordered set $(P, \prec)$ and let $\mathbb{C}P$ be the complex vector space with basis $P$. If for an element $p$ of $P$, the sets $\{x \in P \mid x \succ p\}$ and $\{x \in P \mid x \prec p\}$ are finite, then we can define the "down" operator $d$ and the "up" operator $u$ in $\text{End}_{\mathbb{C}}\, \mathbb{C}P$ as $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$, respectively (for partially ordered sets in general, one needs to complete $\mathbb{C}P$ to define $d$ and $u$). For any $\alpha, \beta, \gamma \in \mathbb{C}$, the *down-up algebra* is the $\mathbb{C}$-algebra generated by $d$ and $u$ subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2d + \gamma u$. A partially ordered set $P$ is called $(q, r)$-*differential* if there exist $q, r \in \mathbb{C}$ such that the down and up operators for $P$ satisfy both relations, and $\alpha = q(q+1), \beta = -q^3$, and $\gamma = r$. From [22], we know that for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha, \beta, \gamma) \simeq A(\alpha, \beta, \lambda\gamma)$. This means that when $\gamma \neq 0$, no problem if we assume $\gamma = 1$. For more details about the combinatorial origins of down-up algebras, see [21, Section 1].

Remarkable examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and some of its deformations introduced by Witten [181] and Woronowicz [183]. Related to the theoretical properties of these algebras, Kirkman et al. [85] proved that a down-up algebra $A(\alpha, \beta, \gamma)$ is Noetherian if and only if $\beta$ is non-zero. As a matter of fact, they showed that $A(\alpha, \beta, \gamma)$ is a generalized Weyl algebra and that $A(\alpha, \beta, \gamma)$ has a filtration for which the associated graded ring is an iterated Ore extension over $\mathbb{C}$.

Following [21, p. 32], if $\mathfrak{g}$ is a 3-dimensional Lie algebra over $\mathbb{C}$ with basis $x, y, [x, y]$ such that $[x, [x, y]] = \gamma x$ and $[[x, y], y] = \gamma y$, then in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ these relations are given by $x^2y - 2xyx + yx^2 = \gamma x$ and $xy^2 - 2yxy + y^2x = \gamma y$. Notice that $U(\mathfrak{g})$ is a homomorphic algebra of the down-up algebra $A(2, -1, \gamma)$ via the mapping $\phi : A(2, -1, \gamma) \to U(\mathfrak{g})$, $d \mapsto x, u \mapsto y$, and the mapping $\psi : \mathfrak{g} \to A(2, -1, \gamma)$, $x \mapsto d, y \mapsto u$, $[x, y] \mapsto du - ud$, extends by the universal property of $U(\mathfrak{g})$ to an algebra homomorphism $\psi : U(\mathfrak{g}) \to A(2, -1, \gamma)$ which is the inverse of $\psi$. Hence, $U(\mathfrak{g})$ is isomorphic to $A(2, -1, \gamma)$.

It is straightforward to see that $U(\mathfrak{sl}_2(\mathbb{C})) \cong A(2, -1, -2)$. Also, for the Heisenberg Lie algebra $\mathfrak{h}$ with basis $x, y, z$ where $[x, y] = z$ and $[z, x] = [z, y] = 0$, $U(\mathfrak{h}) \cong A(2, -1, 0)$.

Now, with the aim of providing an explanation of the existence of quantum groups, Witten [181, 182] introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\Bbbk))$. By definition, Witten's deformation is a unital associative algebra over a field $\Bbbk$ (which is algebraically closed of characteristic zero) that depends on a 7-tuple $\underline{\xi} = (\xi_1, \dots, \xi_7)$ of elements of $\Bbbk$. This algebra, denoted by $W(\underline{\xi})$, is generated by the indeterminates $x, y, z$ subject to the defining relations $xz - \xi_1 zx = \xi_2 x$, $zy - \xi_3 yz = \xi_4$, and $yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z$. From [21, Section 2], we know that a Witten's deformation algebra $W(\xi)$ with

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \quad \xi_1 = \xi_3, \quad \text{and} \quad \xi_2 = \xi_4, \tag{1.6}$$

is isomorphic to one down-up algebra. Notice that any down-up algebra $A(\alpha, \beta, \gamma)$ with not both $\alpha$ and $\beta$ equal to 0 is isomorphic to a Witten deformation algebra $W(\underline{\xi})$ whose parameters satisfy (1.6).

Since algebras $W(\underline{\xi})$ are filtered, Le Bruyn [90, 91] studied the algebras $W(\underline{\xi})$ whose associated graded algebras are Auslander regular. He determined a 3-parameter family of deformation algebras which are said to be *conformal* $\mathfrak{sl}_2$ *algebras* that are generated by the indeterminates $x, y, z$ over a field $\Bbbk$ subject to the relations given by $zx - axz = x$, $zy - ayz = y$, and $yx - cxy = bz^2 + z$. In the case $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with these three defining relations is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1+ac)$, $\beta = -ac^{-1}$ and $\gamma = -c^{-1}$. Notice that if $c = b = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}$, $\beta = 0$, and $\gamma = -a^{-1}$. As one can check, conformal $\mathfrak{sl}_2$ algebras are not Ore extensions.

Kulkarni [88] showed that under certain assumptions on the parameters, a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2(\Bbbk)$ algebra or to an iterated Ore extension. More exactly, following [88, Theorem 3.0.3] if $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $W(\underline{\xi})$ is isomorphic to one of the following algebras: (i) a conformal $\mathfrak{sl}_2$ algebra with generators $x, y, z$ and relations given above or (ii) an iterated Ore extension whose generators satisfy

- $xz - zx = x$, $zy - yz = \zeta y$, $yx - \eta xy = 0$, or
- $xw = \theta wx$, $wy = \kappa yw$, $yx = \lambda xy$, for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \Bbbk$.

Notice that iterated Ore extensions above are defined in the following way: (i) the Witten deformation algebra is isomorphic to $\Bbbk[z][y,\sigma_1][x,\sigma_2]$ where $\sigma_1$ is the automorphism of $\Bbbk[z]$ defined as $\sigma_1(z) = z-\zeta$, with $zy-yz=\zeta y$; $\sigma_2$ is the automorphism of $\Bbbk[z][y,\sigma_1]$ defined as $\sigma_2(y)=\eta^{-1}y$, $\sigma_2(z)=z+1$, which satisfies $xz-zx=x$ and $yx-\eta xy=0$. (ii) The Witten deformation algebra is isomorphic to $\Bbbk[w][y,\sigma_1][x,\sigma_2]$ where $\sigma_1$ is the automorphism of $\Bbbk[w]$ defined as $\sigma_1(w)=\kappa^{-1}w$ with $wy=\kappa yw$, and $\sigma_2$ is the automorphism of $\Bbbk[w][y,\sigma_1]$ defined as $\sigma_2(w)=\theta w$, $\sigma_2(y)=\lambda^{-1}y$ such that $wy=\kappa yw$ and $yx=\lambda xy$.

### 1.2.7 Quantum groups

The term *"quantum group"* was independently popularized by Drinfel'd [39] and Jimbo [75] around 1985. They used it to build solutions to the quantum Yang-Baxter equations. These "groups" represent certain special Hopf algebras which are *deformations* of the universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac–Moody algebra. Intuitively, a deformation is a family of algebras that depends "nicely" on a parameter $q$ such that we get back the initial structure for some special value of $q$. For example, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Choose a generic parameter $q$. Then, for each $q$, we have a Hopf algebra $U_q(\mathfrak{g})$, called the *quantum group* or the *quantized universal enveloping algebra*, whose structure tends to that of $U(\mathfrak{g})$ as $q$ approaches 1, it is same as the Hopf algebra $U(\mathfrak{g})$ [72].

We describe briefly this type of associative algebras introduced by Drinfel'd and Jimbo. Following [185], let $A=(a_{ij})$ be an integral symmetrizable $n\times n$ Cartan matrix, so that $a_{ii}=2$ and $a_{ij}\leq 0$, for $i\neq j$, and there exists a diagonal matrix $D$ with diagonal entries $d_i$ non-zero integers such that the product $DA$ is symmetric. Let $0\neq q\in\Bbbk$ so that $q^{4d_i}\neq 1$, for each $i$. Then the *quantum group* $U_q(A)$ is the $\Bbbk$-algebra generated by $4n$ elements, $E_i, K_i^{\pm 1}, F_i$, for $1\leq i,j\leq n$, subject to the following set of relations:

$$K = \{K_iK_j - K_jK_i, K_iK_i^{-1}-1, K_i^{-1}K_i-1, \tag{1.7}$$

$$E_jK_i^{\pm 1} - q^{\pm d_ia_{ij}}K_i^{\pm 1}E_j, K_i^{\pm 1}F_j - q^{\pm d_ia_{ij}}F_jq^{\pm d_ia_{ij}}\}, \tag{1.8}$$

$$T = \left\{E_iF_j - F_jE_i - \delta_{ij}\frac{K_i^2-K_i^{-2}}{q^{2d_i}-q^{-2d_i}}\right\}, \tag{1.9}$$

$$S^+ = \left\{\sum_{\mu=0}^{1-a_{ij}}(-1)^\mu\begin{bmatrix}1-a_{ij}\\ \mu\end{bmatrix}_t E_i^{1-a_{ij}-\mu}E_jE_i^\mu : i\neq j, t=q^{2d_i}\right\}, \tag{1.10}$$

$$S^- = \left\{\sum_{\mu=0}^{1-a_{ij}}(-1)^\mu\begin{bmatrix}1-a_{ij}\\ \mu\end{bmatrix}_t F_i^{1-a_{ij}-\mu}F_jF_i^\mu : i\neq j, t=q^{2d_i}\right\}, \tag{1.11}$$

where

$$\begin{bmatrix}m\\ n\end{bmatrix}_t = \begin{cases}\prod_{i=1}^{n}\frac{t^{m-i+1}-t^{i-m-1}}{t^i-t^{-i}}, & \text{for } m>n>0,\\ 1, & \text{for } n=0 \text{ or } n=m.\end{cases}$$

One of the basic properties of these algebras is that they have a triangular decomposition, i.e., $U_q(A) \cong U_q^+(A) \otimes U_q^0(A) \otimes U_q^-(A)$, where $U_q^+(A)$ (resp., $U_q^-(A)$) is the subalgebra of $U_q(A)$ generated by $E_i$ (resp., $F_i$), and $U_q^0(A)$ is the subalgebra of $U_q(A)$ generated by $K_i^{\pm 1}$ [72, Chapter 4].

### 1.2.8 Other families of quantum algebras

In this section, we recall the definitions of some examples of noncommutative rings known in the literature as *quantum algebras* or *quantized algebras*.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$ with basis $x_1, \dots, x_n$ and $U(\mathfrak{g})$ its enveloping algebra. The *homogenized enveloping algebra of* $\mathfrak{g}$ is $\mathcal{A}(\mathfrak{g}) := T(\mathfrak{g} \oplus \Bbbk z)/\langle R\rangle$, where $T(\mathfrak{g} \oplus \Bbbk z)$ denotes the tensor algebra, $z$ is a new indeterminate, and $R$ is spanned by the union of sets $\{z \otimes x - x \otimes z \mid x \in \mathfrak{g}\}$ and $\{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathfrak{g}\}$.

From [49, p. 41], for $q$ an element of $\Bbbk$ with $q \neq \pm 1$, the *quantized enveloping algebra* of $\mathfrak{sl}_2(\Bbbk)$ corresponding to the choice of $q$ is the $\Bbbk$-algebra $U_q(\mathfrak{sl}_2(\Bbbk))$ presented by the generators $E, F, K, K^{-1}$ and the relations $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, $KE = q^2EK$, and $KF = q^{-2}FK$. From [49, Exercise 2T], we know that $U_q(\mathfrak{sl}_2(\Bbbk))$ can be expressed as an iterated skew polynomial ring of the form $\Bbbk[E][K^{\pm 1}; \sigma_1][F; \sigma_2, \delta_2]$ [49, Exercise 2T], so that this algebra is not of automorphism type.

Following Yamane [184], if $q \in \mathbb{C}$ with $q^8 \neq 1$, the complex algebra $A$ generated by the indeterminates $e_{12}, e_{13}, e_{23}, f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2$ subject to the relations

$$
\begin{aligned}
&e_{13}e_{12} = q^{-2}e_{12}e_{13}, && f_{13}f_{12} = q^{-2}f_{12}f_{13}, && \\
&e_{23}e_{12} = q^{2}e_{12}e_{23} - qe_{13}, && f_{23}f_{12} = q^{2}f_{12}f_{23} - qf_{13}, && \\
&e_{23}e_{13} = q^{-2}e_{13}e_{23}, && f_{23}f_{13} = q^{-2}f_{13}f_{23}, && \\
&e_{12}f_{12} = f_{12}e_{12} + \frac{k_1^2 - l_1^2}{q^2 - q^{-2}}, && e_{12}k_1 = q^{-2}k_1e_{12}, && k_1f_{12} = q^{-2}f_{12}k_1, \\
&e_{12}f_{13} = f_{13}e_{12} + qf_{23}k_1^2, && e_{12}k_2 = qk_2e_{12}, && k_2f_{12} = qf_{12}k_2, \\
&e_{12}f_{23} = f_{23}e_{12}, && e_{13}k_1 = q^{-1}k_1e_{13}, && k_1f_{13} = q^{-1}f_{13}k_1, \\
&e_{13}f_{12} = f_{12}e_{13} - q^{-1}l_1^2e_{23}, && e_{13}k_2 = q^{-1}k_2e_{13}, && k_2f_{13} = q^{-1}f_{13}k_2, \\
&e_{13}f_{13} = f_{13}e_{13} - \frac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}}, && e_{23}k_1 = qk_1e_{23}, && k_1f_{23} = qf_{23}k_1, \\
&e_{13}f_{23} = f_{23}e_{13} + qk_2^2e_{12}, && e_{23}k_2 = q^{-2}k_2e_{23}, && k_2f_{23} = q^{-2}f_{23}k_2, \\
&e_{23}f_{12} = f_{12}e_{23}, && e_{12}l_1 = q^{2}l_1e_{12}, && l_1f_{12} = q^{2}f_{12}l_1, \\
&e_{23}f_{13} = f_{13}e_{23} - q^{-1}f_{12}l_2^2, && e_{12}l_2 = q^{-1}l_2e_{12}, && l_2f_{12} = q^{-1}f_{12}l_2, \\
&e_{23}f_{23} = f_{23}e_{23} + \frac{k_2^2 - l_2^2}{q^2 - q^{-2}}, && e_{13}l_1 = ql_1e_{13}, && l_1f_{13} = qf_{13}l_1, \\
&e_{13}l_2 = ql_2e_{13}, && l_2f_{13} = qf_{13}l_2, && e_{23}l_1 = q^{-1}l_1e_{23}, \\
&l_1f_{23} = q^{-1}f_{23}l_1, && e_{23}l_2 = q^{2}l_2e_{23}, && l_2f_{23} = q^{2}f_{23}l_2, \\
&l_1k_1 = k_1l_1, && l_2k_1 = k_1l_2, && k_2k_1 = k_1k_2, \\
&l_1k_2 = k_2l_1, && l_2k_2 = k_2l_2, && l_2l_1 = l_1l_2,
\end{aligned}
$$

is very important in the definition of the *quantized enveloping algebra of* $\mathfrak{sl}_3(\mathbb{C})$.

The *Lie-deformed Heisenberg* is the free $\mathbb{C}$-algebra defined by the commutation relations

$$\begin{aligned} q_j(1+i\lambda_{jk})p_k - p_k(1-i\lambda_{jk})q_j &= i\hbar\delta_{jk}, \\ [q_j, q_k] = [p_j, p_k] &= 0, \quad j,k = 1,2,3, \end{aligned}$$

where $q_j, p_j$ are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with $\lambda_k$ real parameters. If $\lambda_{jk} = 0$, then one recovers the usual Heisenberg algebra.

With the aim of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi [60] considered the $A_q^-$ algebra by using the free algebra $\mathbf{U}$. Following Berger [24, Example 2.7.7], this $\Bbbk$-algebra $\mathbf{U}$ is generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n$, and $\psi_1^*, \dots, \psi_n^*$, subject to the relations

$$\begin{aligned} \psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* &= 0, & 1 \le i < j \le n, \\ \omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* &= 0, & 1 \le i,j \le n, \\ \psi_i^*\psi_i - q^2\psi_i\psi_i^* &= -q^2\omega_i^2, & 1 \le i \le n. \end{aligned}$$

The *Non-Hermitian realization of a Lie deformed* defined by Jannussis et al. [71] is an important example of a non-canonical Heisenberg algebra considering the case of non-Hermitian (i.e., $\hbar = 1$) operators $A_j,\ B_k$, where the following relations are satisfied:

$$\begin{aligned} A_j(1+i\lambda_{jk})B_k - B_k(1-i\lambda_{jk})A_j &= i\delta_{jk}, \\ [A_j, B_k] &= 0 \ \ (j \neq k), \\ [A_j, A_k] = [B_j, B_k] &= 0, \end{aligned}$$

and,

$$\begin{aligned} A_j^+(1+i\lambda_{jk})B_k^+ - B_k^+(1-i\lambda_{jk})A_j^+ &= i\delta_{jk}, \\ [A_j^+, B_k^+] &= 0 \ (j \neq k), \\ [A_j^+, A_k^+] = [B_j^+, B_k^+] &= 0, \end{aligned} \tag{1.12}$$

with $A_j \neq A_j^+,\ B_k \neq B_k^+\ (j,k = 1,2,3)$. If the operators $A_j,\ B_k$ are in the form $A_j = f_j(N_j+1)a_j,\ B_k = a_k^+ f_k(N_k+1)$, where $a_j,\ a_j^+$ are leader operators of the usual Heisenberg-Weyl algebra, with $N_j$ the corresponding number operator ($N_j = a_j^+a_j,\ \langle N_j \mid n_j\rangle = \langle n_j|n_j\rangle$), and the structure functions $f_j(N_j+1)$ complex, then it is showed that $A_j$ and $B_k$ are given by

$$\begin{aligned} A_j &= \sqrt{\frac{i}{1+i\lambda_j}}\left(\frac{[(1-i\lambda_j)/(1+i\lambda_j)]^{N_j+1}-1}{(1-i\lambda_j)/(1+i\lambda_j)-1}\frac{1}{N_j+1}\right)^{\frac{1}{2}} a_j, \\ B_k &= \sqrt{\frac{i}{1+i\lambda_k}}a_k^+\left(\frac{[(1-i\lambda_k)/(1+i\lambda_k)]^{N_k+1}-1}{(1-i\lambda_k)/(1+i\lambda_k)-1}\frac{1}{N_k+1}\right)^{\frac{1}{2}}. \end{aligned}$$

Following Havliček et al. [59, p. 79], the $\mathbb{C}$-algebra $U'_q(\mathfrak{so}_3)$ is generated by the indeterminates $I_1, I_2$, and $I_3$, subject to the relations given by

$$I_2I_1 - qI_1I_2 = -q^{\frac{1}{2}}I_3, \quad I_3I_1 - q^{-1}I_1I_3 = q^{-\frac{1}{2}}I_2, \quad I_3I_2 - qI_2I_3 = -q^{\frac{1}{2}}I_1,$$

where $q$ is a non-zero element of $\mathbb{C}$. It is straightforward to show that $U'_q(\mathfrak{so}_3)$ cannot be expressed as an iterated Ore extension.

Zhedanov [190, Section 1] introduced the *Askey-Wilson algebra* $AW(3)$ as the algebra generated by three operators $K_0, K_1$, and $K_2$, that satisfy the commutation relations

$$[K_0, K_1]_\omega = K_2, [K_2, K_0]_\omega = BK_0 + C_1K_1 + D_1, \text{ and } [K_1, K_2]_\omega = BK_1 + C_0K_0 + D_0,$$

where $B, C_0, C_1, D_0$, and $D_1$ are the structure constants of the algebra, which Zhedanov assumes are real, and the $q$-commutator $[-,-]_\omega$ is given by $[\square, \triangle]_\omega := e^\omega\square\triangle - e^{-\omega}\triangle\square$, where $\omega \in \mathbb{R}$. Notice that in the limit $\omega \to 0$, the algebra AW(3) becomes an ordinary Lie algebra with three generators ($D_0$ and $D_1$ are included among the structure constants of the algebra in order to take into account algebras of Heisenberg-Weyl type). The relations defining the algebra can be written as

$$\begin{aligned}
e^\omega K_0K_1 - e^{-\omega}K_1K_0 &= K_2, \\
e^\omega K_2K_0 - e^{-\omega}K_0K_2 &= BK_0 + C_1K_1 + D_1, \\
e^\omega K_1K_2 - e^{-\omega}K_2K_1 &= BK_1 + C_0K_0 + D_0.
\end{aligned}$$

According to these relations that define the algebra, it is clear that AW(3) cannot be expressed as an iterated Ore extension.

With the purpose of introducing generalizations of the classical bosonic and fermionic algebras of quantum mechanics concerning several versions of the Bose-Einstein and Fermi-Dirac statistics, Green [50] and Greenberg and Messiah [51] introduced by means of generators and relations the *parafermionic* and *parabosonic algebras*. For the completeness of the thesis, briefly we recall the definition of each one of these structures following the treatment developed by Kanakoglou and Daskaloyannis [80]. Let $[\square, \triangle] := \square\triangle - \triangle\square$ and $\{\square, \triangle\} := \square\triangle + \triangle\square$.

Consider the $\Bbbk$-vector space $V_F$ freely generated by the elements $f_i^+, f_j^-$, with $i, j = 1, \ldots, n$. If $T(V_F)$ is the tensor algebra of $V_F$ and $I_F$ is the two-sided ideal $I_F$ generated by the elements $[[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon - \eta)^2\delta_{jk}f_i^\xi + \frac{1}{2}(\varepsilon - \xi)^2\delta_{ik}f_j^\eta$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j, k = 1, \ldots, n$, then the *parafermionic algebra* in $2n$ generators $P_F^{(n)}$ ($n$ parafermions) is the quotient algebra of $T(V_F)$ with the ideal $I_F$, that is,

$$P_F^{(n)} = \frac{T(V_F)}{\langle [[f_i^\xi, f_j^\eta], f_k^\varepsilon] - \frac{1}{2}(\varepsilon - \eta)^2\delta_{jk}f_i^\xi + \frac{1}{2}(\varepsilon - \xi)^2\delta_{ik}f_j^\eta \mid \xi, \eta, \varepsilon = \pm 1, i, j, k = 1, \ldots, n\rangle}.$$

It is well-known (e.g., [80, Section 18.2]) that a parafermionic algebra $P_F^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra $\mathfrak{so}(2n+1)$, i.e., $P_F^{(n)} \cong U(\mathfrak{so}(2n+1))$.

Similarly, if $V_B$ denotes the $\Bbbk$-vector space freely generated by the elements $b_i^+, b_j^-$, $i, j = 1, \dots, n$, $T(V_B)$ is the tensor algebra of $V_B$, and $I_B$ is the two-sided ideal of $T(V_B)$ generated by the elements $[\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon - \eta)\delta_{jk} b_i^\xi - (\varepsilon - \xi)\delta_{ik} b_j^\eta$, for all values of $\xi, \eta, \varepsilon = \pm 1$, and $i, j = 1, \dots, n$, then the *parabosonic algebra* $P_B^{(n)}$ in $2n$ generators ($n$ parabosons) is defined as the quotient algebra $P_B^{(n)}/I_B$, that is,

$$P_B^{(n)} = \frac{T(V_B)}{\langle [\{b_i^\xi, b_j^\eta\}, b_k^\varepsilon] - (\varepsilon - \eta)\delta_{jk} b_i^\xi - (\varepsilon - \xi)\delta_{ik} b_j^\eta \mid \xi, \eta, \varepsilon = \pm 1, i, j = 1, \dots, n \rangle}.$$

It is known that the parabosonic algebra $P_B^{(n)}$ in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$, that is, $P_B^{(n)} \cong U(B(0, n))$. For more details about *parafermionic* and *parabosonic algebras*, see [80, Proposition 18.2], and references therein.

### 1.2.9 Ore polynomials of higher order generated by homogeneous quadratic relations

For a ring $R$, as we saw in Section 1.2.1, the *Ore extensions* introduced by Ore [126, 127] consist of the uniquely representable elements $r_0 + r_1 x + \cdots + r_k x^k$, $k = k(r) = 0, 1, 2, \dots, r_i \in R$, with the commutation relation $xr = \sigma(r)x + \delta(r)$, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$. Different generalizations, called *skew Ore polynomials*, have been introduced and studied by Cohn [33, 34], Dumas [40], and Smits [157], considering the commutation relation $xr = \Psi_1(r)x + \Psi_2(r)x^2 + \cdots$, where the $\Psi$'s are endomorphisms of $R$. Nevertheless, there are cases of quadratic algebras such as Clifford algebras, Weyl-Heisenberg algebras, and Sklyanin algebras, in which this commutation relation is not sufficient to define the noncommutative structure of the algebras since a free non-zero term $\Psi_0$ is required (e.g., Ostrovskii and Samoilenko [128]). Precisely, *skew Ore polynomials of higher order* with commutation relation with this free term, that is, $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_n(r)x^n + \cdots$, were studied by Maksimov [105], where, for every $r, s \in R$, the free term $\Psi_0$ satisfies the relation

$$\Psi_0(rs) = \Psi_0(r)s + \Psi_1(r)\Psi_0(s) + \Psi_2(r)\Psi_0^2(s) + \cdots,$$

or the equivalent operator equation $\Psi_0 r = \Psi_0(r) + \Psi_1(r)\Psi_0 + \Psi_2(r)\Psi_0^2$, where $r$ is considered as the operator of left multiplication by $r$ on $R$. Notice that one may consider $\Psi_0$ as a singular differentiation operator with respect to $\Psi_1, \Psi_2, \dots$, but where $\Psi_1$ need not be an endomorphism of $R$.

Later, Golovashkin and Maksimov [47] investigated the representation of algebras $Q(a, b, c)$ over a field $\Bbbk$ of characteristic zero defined by a *quadratic relation* in two generators $x, y$ given by

$$yx = ax^2 + bxy + cy^2, \tag{1.13}$$

as an algebra of Ore polynomials of higher degree with commutation relation (1.13) with $a, b, c$ belong to $\Bbbk$. As one can check, the algebra generated by the relation is represented in the form of an algebra of skew Ore polynomials of higher order if the elements $\{x^m y^n\}$ form a linear basis of the algebra. Hence, this algebra can be defined by a system of linear mappings $\Psi_0, \Psi_1, \Psi_2, \dots$ of the algebra of polynomials $\Bbbk[x]$ into itself such that for an arbitrary element $p(x) \in \Bbbk[x]$, $yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_k(p(x))y^k$, $k = k(p(x))$, $k = 0, 1, 2, \dots$ If this representation exists, then one can obtain the relations between the operators $\Psi_0, \Psi_1, \Psi_2, \dots$ They found conditions for such an algebra $Q(a, b, c)$ to be expressed as a skew polynomial with generator $y$ over the polynomial ring $\Bbbk[x]$ (cf. [46]), and proved that these conditions are equivalent to the existence of a PBW basis, i.e., basis of the form $\{x^m y^n\}$. Notice that this kind of algebras have been previously studied in the literature where its Poincaré series was calculated by Ufnarovskii [171].

Next, we recall briefly some of the results presented in [47] about PBW bases of these algebras which are useful in Chapter 3.

First of all, Golovashkin and Maksimov [47, Section 1] distinguished three types of algebras that can be occur from relation (1.13):

(i) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ form a PBW basis.

(ii) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ are linearly dependent (for instance, the algebra determined by the relation $yx = x^2 + xy + y^2$).

(iii) Algebras in which the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$, are linearly independent, but do not form a PBW basis (for instance, the algebra subjected to the relation $yx = x^2 - xy + y^2$).

Case (i) is of interest since in this situation the quadratic algebra is determined by the structural constants that arise when expanding the products $(x^k y^r)(x^l y^s)$ in terms of the basis $\{x^m y^n\}$. Nevertheless, it is more useful to use special linear mappings of the ring of polynomials $\Bbbk[x]$ rather than structural constraints. Let us see the details.

If the monomials $\{x^m y^n\}$ form a basis, then for every power $x^n \in \Bbbk[x]$, $yx^n$ has a unique expression given by

$$yx^n = \Psi_{0,n}(x) + \Psi_{1,n}(x)y + \cdots + \Psi_{m(n),n}(x)y^{m(n)}, \tag{1.14}$$

where $\Psi_{k,n}(x)$, for each $k$, are polynomials from $\Bbbk[x]$. Precisely, for $k = 0, 1, \dots,$ it can be defined a linear mapping $\Psi_k : \Bbbk[x] \to \Bbbk[x]$ given by $\Psi_k(x^n) = \Psi_{k,n}(x)$. If we define $x^0 = y^0 = 1_{\Bbbk}$, then $yx^0 = y \cdot 1 = 1 \cdot y + 0 \cdot y^2 + \cdots$ . By (1.14),

$$\Psi_0(1) = 0, \quad \Psi_1(1) = 1, \quad \Psi_k(1) = 0, \quad k = 2, 3, \dots \tag{1.15}$$

which shows that for every element $p(x) \in \Bbbk[x]$, there is a unique expresion given by

$$yp(x) = \Psi_0(p(x)) + \Psi_1(p(x))y + \cdots + \Psi_{m(p(x))}(p(x))y^{m(p(x))}. \tag{1.16}$$

Having in mind that $y^n(p(x)) = y(y^{n-1}p(x)) = \cdots = y(y(\cdots y(yp(x))))$, it follows that the values of the operators $\Psi_k$, $k = 0, 1, \ldots$, uniquely determine the algebra of skew polynomials generated by (1.13) in the case that $\{x^m y^n \mid m, n \in \mathbb{N}\}$.

**Remark 1.20.** A general algebra of skew polynomials with indeterminate $x$ over $R$ is also determined by certain linear operators $\Psi_k : R \to R$ such that for each $r \in R$, there is a unique representation $xr = \Psi_0(r) + \Psi_1(r)x + \cdots + \Psi_{m(r)}x^{m(r)}$. Of course, if $R$ is a ring with one generator, then the algebra has a PBW basis. If $m_0 = 1$, then we obtain the classical Ore extensions [127]. It it important to say that the associativity and uniqueness of the representation of the product $xr$ guarantee conditions on the set of operators $\{\Psi_k\}_{k\in\mathbb{N}}$ which are necessary and sufficient to determinate the algebra of skew polynomials (for more details, see [105, 106]).

Example 1.21 contains examples of skew polynomials with PBW basis over the ring $\Bbbk[x]$ generated by the quadratic homogeneous relation (1.13).

**Example 1.21 ([47, Section 1.3]).** The following two cases arise in the study of operators in functional analysis [128, 178]:

(i) $a = 0$ and $c \neq 0$. Here, expression (1.13) turns out to be $yx = bxy + cy^2$. If $x_1 := x$, $y_1 := cy$, then we obtain $y_1x_1 = bx_1y_1 + y_1^2$.

(ii) $a \neq 0$ and $c = 0$. Again, if $x_1 := x$, $y_1 := cy$, then (1.13) is equivalent to $y_1x_1 = bx_1y_1 + x_1^2$.

Denote by $T$ the operator of multiplying a polynomial $f(x)$, that is, $Tf(x) = xf(x)$. Let $D$ be the ordinary operator of differentiation, that is, $Dx^n = nx^{n-1}$, and $D_q$ be the operator of $q$-differentiation given by $D_qf(x) = \frac{f(x)-f(qx)}{x-qx}$, for every $q \in \Bbbk$. Of course, for $q = 1$, $D_1 = D$, while for $q = 0$, the operator $D_0 = \overline{D}$ is the operator of difference quotient given by $\overline{D}f(x) = \frac{f(x)-f(0)}{x}$. If one consider the operator of integration $J$, $Jx^n = \frac{1}{n+1}x^{n+1}$, the operator of $q$-integration $J_q$ defined by $J_qx^n = \frac{1-q}{1-q^{n+1}}x^{n+1}$, the Dirac operator $V_0$ given by $V_0f(x) = f(0)$, and the identity operator $I$, then the following relations hold at the basis $\{x^n \mid n \in \mathbb{N}\}$:

$$\overline{D}T = I, \quad T\overline{D} = I - V_0, \quad D_qJ_q = I, \quad J_qD_q = I - V_0, D_qX - qXD_q = I,$$

and

$$\overline{D}D_q = qD_q\overline{D} + \overline{D}^2, \text{ and } TJ_q = qJ_qT + J_q^2,$$

which are precisely the Cases (i) and (ii) considered in Example 1.21. If $q = 1$, we obtain the Weyl relation $DT - TD = I$, and the equivalent relations $\overline{D}D - D\overline{D} = \overline{D}^2$ and $TJ - JT = J^2$.

It is straightforward to see that the sets of operators $\{T^mD_q^n\}$ and $\{D_q^mT^n\}$, for $m, n \in \mathbb{N}$, are PBW bases of the algebra generated by the operators $T$ and $D_q$. Similarly, the sets $\{T^mJ_q^n\}$ and $\{J_qT^n\}$ ($\{\overline{D}^mD^n\}$ and $\{D^m\overline{D}^n\}$), where $m, n \in \mathbb{N}$, are PBW bases of the algebras generated by the operators $T$ and $J_q$ ($D$ and $\overline{D}$). It follows that for both Cases (i) and (ii) in Example 1.21, the sets $\{a^mb^n \mid m, n \in \mathbb{N}\}$ and $\{b^ma^n \mid m, n \in \mathbb{N}\}$ are bases of the algebra (1.13).

Notice that if $a = 0$ in (1.13), then the quadratic algebra is given by $yx = bxy + cy^2$. From [47, Expression (11)], we know that this is an algebra of skew polynomials determined by the operators $\Psi_0 = 0, \psi_1(x^n) = b^n x^n, \Psi_2(x^n) = \psi_1 c D_b(x^n), \Psi_3(x^n) = \Psi_1 c^2 D_b^2(x^n)$, that is, $\Psi_k = \Psi_1 c^{k-1} D_b^{k-1}$, for $k = 1, 2, \dots$

**Example 1.22 ([47, Section 1.5]).** When two of the three coefficients $a, b$, and $c$ are equal to zero, the resulting algebras are given by three types: (iii) $yx = ax^2$, (iv) $yx = cy^2$, and (v) $yx = bxy$. The set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a PBW basis for algebras (iii) and (iv), while the algebra (v) is an Ore extension.

Examples 1.21 and 1.22 guarantee that if $ab = 0$, then the quadratic algebra defined by (1.13) is an algebra of skew polynomials over $\Bbbk[x]$ and has a PBW basis $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [47, Proposition 1].

A useful tool in the study of the PBW bases for quadratic algebras defined by (1.13) is the matrix given by

$$M := \begin{pmatrix} b & a \\ -c & 1 \end{pmatrix},$$

which is called the *companion* matrix for (1.13) [47, Section 2.2]. If the lower-right elements of the matrices $M^l$ does not vanish for all $l \in \mathbb{N}$, then $Q(a, b, c)$ has a PBW basis of the form $\{x^m y^n \mid m, n \in \mathbb{N}\}$ [47, Proposition 4]. Equivalently, $Q(a, b, c)$ is the ring of skew polynomials over $\Bbbk[x]$ determined by the sequence of operators $\Psi_k$, $k = 0, 1, \dots$, satisfying the infinite relations

$$\begin{aligned} \Psi_0 X &= aX^2 + bX\Psi_0 + c\Psi_0^{(2)}, \\ \Psi_1 &= \qquad\quad bX\Psi_1 + c\Psi_1^{(2)}, \\ &\ \vdots \\ \Psi_k X &= \qquad\quad bX\Psi_k + c\Psi_k^{(2)}, \end{aligned}$$

where

$$\Psi_0^{(2)} = \Psi_0^2, \qquad \Psi_k^{(2)} = \Psi_0\Psi_k + \Psi_1\Psi_{k-1} + \cdots + \Psi_k\Psi_0,$$

see [47, Lemma 1]. General relations between $\Psi_j^{(k)}$ were formulated in [105, 106, 157].

If $b + ac \neq 0$ a necessary and sufficient condition for the monomials $\{x^m y^n \mid m, n \in \mathbb{N}\}$ form a basis of $Q(a, b, c)$ is precisely that the lower-right elements of the matrices $M^l$ does not vanish for all $l \in \mathbb{N}$. If $a = c = 1$ and $b = -1$, then the elements $\{x^m y^n \mid m, n \in \mathbb{N}\}$ are linearly independent but do not form a PBW basis of $Q(a, b, c)$ [47, Propositions 5 and 10].

### 1.2.10 Skew Poincaré-Birkhoff-Witt extensions

Skew Poincaré-Birkhoff-Witt extensions were defined by Gallego and Lezama [42] with the aim of generalizing Poincaré-Birkhoff-Witt extensions introduced by Bell and Goodearl [17] (Section 1.2.2) and Ore extensions of injective type defined by Ore [126, 127] (Section 1.2.1). Over the years, several authors have shown that skew PBW extensions also generalize families of noncommutative algebras such as those mentioned in the previous sections. Due to their generality, in Chapters 2 and 3 we prove several results about these objects that can be illustrated with algebras described in the previous sections. Precisely, the importance of skew PBW extensions is that they do not assume that the coefficients commute with the variables, and the coefficients do not necessarily belong to fields. As a matter of fact, skew PBW extensions contain well-known groups of algebras such as some types of $G$-algebras in the sense of Apel [9], Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, families of differential operator rings, and many other algebras of interest in noncommutative algebraic geometry and noncommutative differential geometry. Ring, theoretical and geometrical properties of skew PBW extensions have been presented in several papers [1, 11, 62, 98, 136, 163, 164].

**Definition 1.23 ([42, Definition 1]).** Let $R$ and $A$ be rings. We say that $A$ is a *skew PBW extension* over $R$ (also called a *$\sigma$-PBW extension of $R$*) if the following conditions hold:

(i) $R$ is a subring of $A$ sharing the same identity element.

(ii) There exist elements $x_1, \dots, x_n \in A \setminus R$ such that $A$ is a left free $R$-module with basis given by the set $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For $1 \leq i, j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n,$$

i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)} \in R$ with

$$x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \textstyle\sum_{k=1}^n r_k^{(i,j)} x_k.$$

From now on, we use freely the notation $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ to denote a skew PBW extension $A$ over a ring $R$ in the indeterminates $x_1, \dots, x_n$. $R$ will be called the *ring of coefficients* of the extension $A$.

**Remark 1.24 ([42, Remark 2]).** (i) Since $\text{Mon}(A)$ is a left $R$-basis of $A$, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 1.23 are unique.

(ii) If $r = 0$, it follows that $c_{i,0} = 0$. In fact, from $0 = x_i 0 = c_{i,0} + r_i$, with $r_i \in R$, we obtain $c_{i,0} = 0 = r_i$ for all $i$.

(iii) In Definition 1.23 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

(iv) Let $i < j$. From Definition 1.23 it follows that there exist elements $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, and hence $1 = c_{j,i} c_{i,j}$, that is, for each $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.

(v) Every element $f \in A \setminus \{0\}$ has a unique representation as $f = \sum_{i=0}^{t} c_i X_i$, with $c_i \in R \setminus \{0\}$ and $X_i \in \text{Mon}(A)$ for $0 \leq i \leq t$ with $X_0 = 1$. When necessary, we use the notation $f = \sum_{i=0}^{t} c_i Y_i$.

**Proposition 1.25 ([42, Proposition 3]).** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over $R$. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.*

We use the notation $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$ for the families of injective endomorphisms and $\sigma_i$-derivations, respectively, established in Proposition 1.25. For a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ over $R$, we say that the pair $(\Sigma, \Delta)$ is a *system of endomorphisms and $\Sigma$-derivations* of $R$ with respect to $A$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^{\alpha} := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^{\alpha} := \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes the classical composition of functions.

The next definition presents some particular examples of skew PBW extensions.

**Definition 1.26 ([42, Definition 4], [97, Definition 2.3 (ii)]).** Consider a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ over $R$.

(a) $A$ is called *quasi-commutative* if the conditions (iii) - (iv) in Definition (1.23) are replaced by the following:

  (iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.

  (iv') For every $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) $A$ is *bijective* if $\sigma_i$ is bijective, for every $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) If $\sigma_i$ is the identity map of $R$ for each $i = 1, \ldots, n$, then we say that $A$ is a skew PBW extension of *derivation type*. Similarly, if $\delta_i$ is zero, for every $i$, then $A$ is called a skew PBW extension of *endomorphism type*.

(d) $A$ is said to be *semi-commutative* if it is quasi-commutative and $x_i r = r x_i$, for each $i$ and every $r \in R$.

**Definition 1.27 ([42, Definition 6]).** Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over $R$.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) For $X = x^{\alpha} \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) If $f$ is an element as in Remark 1.24(v), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

**Proposition 1.28 ([42], Theorem 7).** *If $A$ is a polynomial ring with coefficients in $R$ with respect to the set of indeterminates $\{x_1, \ldots, x_n\}$, then $A$ is a skew PBW extension of $R$ if and only if the following conditions hold:*

(1) *For each* $x^\alpha \in \mathrm{Mon}(A)$ *and every* $0 \neq r \in R$, *there exist unique elements* $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ *and* $p_{\alpha,r} \in A$ *such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{1.17}$$

*where* $p_{\alpha,r} = 0$ *or* $\deg(p_{\alpha,r}) < |\alpha|$ *if* $p_{\alpha,r} \neq 0$. *Moreover, if* $r$ *is left invertible, so is* $r_\alpha$.

(2) *For each* $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, *there exist unique elements* $c_{\alpha,\beta} \in R \setminus \{0\}$ *and* $p_{\alpha,\beta} \in A$ *such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.18}$$

*where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$.

In $\mathrm{Mon}(A)$, we define the total order

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but there exists } i \text{ with } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases} \tag{1.19}$$

If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A \setminus \{0\}$ can be represented in a unique way as $f = c_1 x^{\alpha_1} + \cdots + c_t x^{\alpha_t}$, with $c_i \in R \setminus \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that $x^{\alpha_1}$ is the *leading monomial* of $f$ and we write $\mathrm{lm}(f) := x^{\alpha_1}$; $c_1$ is the *leading coefficient* of $f$, $\mathrm{lc}(f) := c_1$; and $c_1 x^{\alpha_1}$ is the *leading term* of $f$, $\mathrm{lt}(f) := c_1 x^{\alpha_1}$. It is clear that $\succeq$ is an *monomial order* in the sense of [42], i.e., the following conditions hold

(i) For every $x^\alpha, x^\beta, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$

$$x^\alpha \succeq x^\beta \implies \mathrm{lm}(x^\gamma x^\alpha x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\beta x^\lambda);$$

(ii) $x^\alpha \succeq 1$, for every $x^\alpha \in \mathrm{Mon}(A)$; and

(iii) $\succeq$ is degree compatible, i.e., $|\alpha| \geq |\beta| \implies x^\alpha \succeq x^\beta$.

**Example 1.29.** (i) Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a PBW extension over $R$. $A$ is a positively SG ring with graduation $A = \bigoplus_{n \in \mathbb{N}} A_n$, where

$$A_n := {}_R\langle x^\alpha \in \mathrm{Mon}(A) \mid \deg(x^\alpha) = n \rangle,$$

i.e., $A_n$ are the set of homogeneous polynomials of degree $n$.

**Remark 1.30.** (i) From Definition 1.23 (iv), it is clear that skew PBW extensions are more general than iterated skew polynomial rings. For example, universal enveloping algebras of finite dimensional Lie algebras and some 3-dimensional skew polynomial algebras in the sense of Bell and Smith [18] cannot be expressed as iterated skew polynomial rings but are skew PBW extensions. For quasi-commutative skew PBW extensions, these are isomorphic to iterated Ore extensions of endomorphism type [100, Theorem 2.3].

(ii) Skew PBW extensions of endomorphism type are more general than iterated Ore extensions of endomorphism type. Let us illustrate the situation with two and three indeterminates.

For the iterated Ore extension of endomorphism type $R[x;\sigma_x][y;\sigma_y]$, if $r \in R$ then we have the following relations: $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, and $yx = \sigma_y(x)y$. Now, if we have $\sigma(R)\langle x, y\rangle$ a skew PBW extension of endomorphism type over $R$, then for any $r \in R$, Definition 1.23 establishes that $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, and $yx = d_{1,2}xy + r_0 + r_1x + r_2y$, for some elements $d_{1,2}, r_0, r_1$ and $r_2$ belong to $R$. From these relations it is clear which one of them is more general.

If we have the iterated Ore extension $R[x;\sigma_x][y;\sigma_y][z;\sigma_z]$, then for any $r \in R$, $xr = \sigma_x(r)x$, $yr = \sigma_y(r)y$, $zr = \sigma_z(r)z$, $yx = \sigma_y(x)y$, $zx = \sigma_z(x)z$, $zy = \sigma_z(y)z$. For the skew PBW extension of endomorphism type $\sigma(R)\langle x, y, z\rangle$, $xr = \sigma_1(r)x$, $yr = \sigma_2(r)y$, $zr = \sigma_3(r)z$, $yx = d_{1,2}xy + r_0 + r_1x + r_2y + r_3z$, $zx = d_{1,3}xz + r_0' + r_1'x + r_2'y + r_3'z$, and $zy = d_{2,3}yz + r_0'' + r_1''x + r_2''y + r_3''z$, for some elements $d_{1,2}, d_{1,3}, d_{2,3}, r_0, r_0', r_0'', r_1, r_1', r_1'', r_2, r_2', r_2'', r_3, r_3', r_3''$ of $R$. As the number of indeterminates increases, the differences between both algebraic structures are more remarkable.

(iii) Ambiskew polynomial rings (Section 1.2.1) are skew PBW extensions over $B$, that is, $R(B, \sigma, c, p) \cong \sigma(B)\langle y, x\rangle$.

(iv) PBW extensions introduced by Bell and Goodearl [17] (Section 1.2.2) are particular examples of skew PBW extensions. More exactly, the first objects satisfy the relation $x_ir = rx_i + \delta_i(r)$, for every $i = 1, \ldots, n$, and each $r \in R$, and the elements $d_{ij}$ in Definition 1.23 (iv) are equal to the identity of $R$. As examples of PBW extensions, we mention the following: the enveloping algebra of a finite-dimensional Lie algebra; any differential operator ring $R[\theta_1, \ldots, \theta_1; \delta_1, \ldots, \delta_n]$ formed from commuting derivations $\delta_1, \ldots, \delta_n$; differential operators introduced by Rinehart; twisted or smash product differential operator rings, and others (for more details, see [17, p. 27]).

(v) 3-dimensional skew polynomial algebras and bi-quadratic algebras on three generators with PBW bases (Section 1.2.3) are skew PBW extensions.

(vi) The *Jordan Algebra* introduced by Jordan [76] is the free $\Bbbk$-algebra $\mathcal{J}$ defined by $\mathcal{J} := \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$. It is immediate to see that this algebra is not a skew polynomial ring of automorphism type but an easy computation shows that $\mathcal{J} \cong \sigma(\Bbbk[y])\langle x\rangle$.

(vii) The algebra $U_q'(\mathfrak{so}_3)$ is a skew PBW extension over $\Bbbk$, i.e., $U_q'(\mathfrak{so}_3) \cong \sigma(\Bbbk)\langle I_1, I_2, I_3\rangle$ [41, Example 1.3.3].

(viii) Using techniques such as those presented in [41, Theorem 1.3.1], it can be shown that AW(3) is a skew PBW extension of endomorphism type, that is, AW(3) $\cong \sigma(\mathbb{R})\langle K_0, K_1, K_2\rangle$.

**Proposition 1.31 ([100, Proposition 4.1]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. If $R$ is a domain, then $A$ is a domain.*

From Proposition 1.31 it follows that if $R$ is a domain and $f, g \in A \setminus \{0\}$, then $\deg(fg) = \deg(f) + \deg(g)$. With this observation it is clear the next result.

**Corollary 1.32 ([41, Corollary 3.2.2.]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. If $R$ is a domain, then $U(A) = U(R)$.*

**Proposition 1.33 ([41, Proposition 3.2.3.]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a ring $R$. If $R$ is a domain, then $J(A) = 0$.*

*Proof.* Let $f \in J(A)$. Then $1 - f \in U(A) = U(R)$, so $f \in R$, whence $J(A) \subseteq R$. Suppose that $J(A) \neq 0$ and let $0 \neq f \in J(A)$. Since $fx_n \in J(A)$, then $0 = \deg(fx_n) = \deg(f) + \deg(x_n) \geq 1$, which is a contradiction. Thus, $J(A) = 0$. □

From Definition 1.23, it follows that skew PBW extensions are not $\mathbb{N}$-graded rings in a non-trivial sense. With this in mind, Proposition 1.34 allows to define a subfamily of these extensions, the *graded skew PBW extensions* (Definition 1.35) that were introduced by Suárez [159]. Before presenting the definition, we recall that if $R = \bigoplus_{p\in\mathbb{N}} R_p$ and $S = \bigoplus_{p\in\mathbb{N}} S_p$ are $\mathbb{N}$-graded rings, then a map $\varphi : R \to S$ is called *graded* if $\varphi(R_p) \subseteq S_p$, for each $p \in \mathbb{N}$. For $m \in \mathbb{N}$, $R(m) := \bigoplus_{p\in\mathbb{N}} R(m)_p$, where $R(m)_p := R_{p+m}$.

**Proposition 1.34 ([159, Proposition 2.7(ii)]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If the following conditions hold:*

(1) *$\sigma_i$ is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded $\sigma_i$-derivation, for all $1 \leq i \leq n$, and*

(2) *$x_j x_i - d_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in Definition 1.23* (iv) *and $d_{i,j} \in R_0$,*

*then $A$ is an $\mathbb{N}$-graded algebra with graduation given by $A = \bigoplus_{p\geq 0} A_p$, where for $p \geq 0$, $A_p$ is the $\Bbbk$-space generated by the set*

$$\Big\{r_t x^{\alpha} \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^{\alpha} \in \mathrm{Mon}(A)\Big\}.$$

**Definition 1.35 ([159, Definition 2.6]).** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension over an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. If $A$ satisfies both conditions established in Proposition 1.34, then we say that $A$ is a *graded skew PBW extension over $R$*.

Some properties of graded skew PBW extensions can be found in [160, 161, 164, 168]. Next, we recall some examples of these objects.

**Example 1.36.** The Jordan plane, homogenized enveloping algebras (Section 1.2.8), and some classes of diffusion algebras (Section 1.2.5, [159, Examples 2.9]) are graded skew PBW extensions. If we assume the condition of PBW basis, then graded Clifford algebras defined by Le Bruyn [91] are also examples of graded skew PBW extensions. Let us see the details.

Following Cassidy and Vancliff [31], let $\Bbbk$ be an algebraically closed field such that $\text{char}(\Bbbk) \neq 2$ and let $M_1, \dots, M_n \in \mathbb{M}_n(\Bbbk)$ be symmetric matrices of order $n \times n$ with entries in $\Bbbk$. A *graded Clifford algebra* $\mathcal{A}$ is a $\Bbbk$-algebra on degree-one generators $x_1, \dots, x_n$ and on degree-two generators $y_1, \dots, y_n$ with defining relations given by

(i) $x_ix_j + x_jx_i = \sum_{k=1}^{n} (M_k)_{ij}y_k$ for all $i, j = 1, \dots, n$;

(ii) $y_k$ central for all $k = 1, \dots, n$.

Note that the commutative polynomial ring $R = \Bbbk[y_1, \dots, y_n]$ is an $\mathbb{N}$-graded algebra where $R_0 = \Bbbk, R_1 = \{0\}$, $y_1, \dots, y_n \in R_2$, and $R_i = \{0\}$, for $i \geq 3$. If we suppose that the set $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \in \mathbb{N}, i = 1, \dots, n\}$ is a left PBW $R$-basis for $\mathcal{A}$, then the graded Clifford algebra $\mathcal{A}$ is a graded skew PBW extension over the connected algebra $R$, that is, $\mathcal{A} \cong \sigma(R)\langle x_1, \dots, x_n\rangle$. Indeed, from the relations (i) and (ii) above, it is clear that $\sigma_i = \text{id}_R$, $\delta_i = 0$, $d_{i,j} = -1 \in R_0$, for $1 \leq i, j \leq n$, and $\sum_{k=1}^{n}(M_k)_{ij}y_k \in R_2$, where $d_{i,j}$ is given as in Definition 1.23 (iv). In this way, $\mathcal{A}$ is a bijective skew PBW extension that satisfies both conditions of Proposition 1.34.

Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$, and consider the sets $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$ as in Proposition 1.25. An ideal $I$ of $R$ is called $\Sigma$-*ideal* if $\sigma_i(I) = I$, for each $1 \leq i \leq n$. From Lezama et al. [97, Definition 2.1], $I$ is called $\Sigma$-*invariant* if $\sigma_i(I) \subseteq I$, and it is called $\Delta$-*invariant* if $\delta_i(I) \subseteq I$, for $1 \leq i \leq n$. If $I$ is both $\Sigma$ and $\Delta$-invariant, we say that $I$ is $(\Sigma, \Delta)$-*invariant*. Following Hashemi et al. [57], for $S \subseteq R$, we denote the set of all elements of $A$ with coefficients in $S$ by $S\langle x_1, \dots, x_n\rangle$. Definitions of $(\Sigma, \Delta)$-ideal $I$ and $I\langle x_1, \dots, x_n\rangle = I\langle x_1, \dots, x_n; \Sigma, \Delta\rangle$ introduced by Hashemi et al. [57] are the same those of $(\Sigma, \Delta)$-invariant ideal $I$ and $IA$, respectively, considered in [97]. Note that the terminology used by Nasr-Isfahani [118] is different, since for $\sigma$ an endomorphism of $R$, $\delta$ a $\sigma$-derivation of $R$, and $I$ an ideal of $R$, $I$ is called $\sigma$-*invariant* if $\sigma^{-1}(I) = I$, and it is called a $\delta$-*ideal* if $\delta(I) \subseteq I$; this same terminology is used in Nasr-Isfahani [117], [118].

**Remark 1.37.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a ring $R$ and $I$ an ideal of $R$.

(i) $I$ is $\Sigma$-invariant if and only if $\sigma^{\alpha}(I) \subseteq I$, for every $\alpha \in \mathbb{N}^n$.

(ii) $I$ is $\Delta$-invariant if and only if $\delta^{\alpha}(I) \subseteq I$, for every $\alpha \in \mathbb{N}^n$.

(iii) $I$ is $\Sigma$-ideal if and only if $\sigma_i^{-1}(I) = I$, for each $1 \leq i \leq n$. Also, if for each $1 \leq i \leq n$, $\sigma_i^{-1}(I) = I$ then $\sigma^{-\alpha}(I) = I$, for $\alpha \in \mathbb{N}^n$. Therefore, the definition of $\Sigma$-ideal given in this thesis coincides with the definition of $\Sigma$-invariant ideal given by Hashemi et al. [57, Definition 3.1] and the definition of $\alpha$-invariant ideal presented by Nasr-Isfahani [118, p. 5116].

Finally, we recall some results about quotient rings of skew PBW extensions (c.f. [97]) that are useful in Chapter 2, Section 2.1.

**Proposition 1.38 ([41, Proposition 5.1.2]).** *Let $R$ be a ring, $(\Sigma, \Delta)$ a system of endomorphisms and $\Sigma$-derivations of $R$, $I$ a proper two-sided ideal of $R$ and $\overline{R} := R/I$. If $I$ is $(\Sigma, \Delta)$-invariant then over $\overline{R}$ is induced a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphisms and $\overline{\Sigma}$-derivations defined by $\overline{\sigma_i}(\overline{r}) := \overline{\sigma_i(r)}$ and $\overline{\delta_i}(\overline{r}) := \overline{\delta_i(r)}$, $1 \leq i \leq n$.*

**Proposition 1.39 ([41, Proposition 5.1.6]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$ and $I$ a $(\Sigma, \Delta)$-invariant ideal of $R$. Then:*

(1) *$IA$ is an ideal of $A$ and $IA \cap R = I$. $IA$ is proper if and only if $I$ is proper. Moreover, if for every $1 \leq i \leq n$, $\sigma_i$ is bijective and $\sigma_i(I) = I$, then $IA = AI$.*

(2) *If $I$ is proper and $\sigma_i^{-1}(I) \subseteq I$, for every $1 \leq i \leq n$, then $A/IA$ is a skew PBW extension over $R/I$.*

## 1.3 Compatible rings

Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Krempa [87] defined $\sigma$ as a *rigid endomorphism* if $r\sigma(r) = 0$ implies $r = 0$, where $r \in R$. In this way, a ring $R$ is called *$\sigma$-rigid* if there exists a rigid endomorphism $\sigma$ of $R$. Ring-theoretical properties of $\sigma$-rigid rings can be found in [64, 87, 109], and references therein. In his PhD thesis and related papers, Annin [6, 7, 8] (c.f. Hashemi and Moussavi [58]), called a ring $R$ *$\sigma$-compatible* if for every $a, b \in R$, $ab = 0$ if and only if $a\sigma(b) = 0$; $R$ is called *$\delta$-compatible* if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$. Moreover, if $R$ is both $\sigma$-compatible and $\delta$-compatible, then $R$ is called *$(\sigma, \delta)$-compatible*. From [58, Lemma 2.2], we know that a ring $R$ is $(\sigma, \delta)$-compatible and reduced if and only if it is $\sigma$-rigid. Thus, compatible rings are more general than rigid rings.

Following Reyes [135, Definition 3.2], if $R$ is a ring and $\Sigma$ is a family of endomorphisms of $R$, then $\Sigma$ is called a *rigid endomorphisms family* if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, for every $r \in R$ and $\alpha \in \mathbb{N}^n$. A ring $R$ is called *$\Sigma$-rigid* if there exists a rigid endomorphisms family $\Sigma$ of $R$. Note that if $\Sigma$ is a rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. If we consider the family of injective endomorphisms $\Sigma$ and the family $\Delta$ of $\Sigma$-derivations of a skew PBW extension $A$ over a ring $R$ (Proposition 1.25), then $\Sigma$-rigid rings are reduced rings: if $R$ is a $\Sigma$-rigid ring and $r^2 = 0$, for $r \in R$, then $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$, and so $r = 0$, that is, $R$ is reduced. If $A$ is a skew PBW extension over a ring $R$, then $R$ is $\Sigma$-rigid if and only if $A$ is a reduced ring [135, Proposition 3.5].

Motivated by the notion of compatibility above, Hashemi et al. [56] and Reyes and

Suárez [143] introduced independently the $(\Sigma, \Delta)$-compatible rings as a natural generalization of $(\sigma, \delta)$-compatible rings. Briefly, for a ring $R$ with a finite family of endomorphisms $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and a finite family of $\Sigma$-derivations $\Delta := \{\delta_1, \ldots, \delta_n\}$, we say that $R$ is $\Sigma$-*compatible* if for each $a, b \in R$, $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, where $\alpha \in \mathbb{N}^n$. Similarly, we say that $R$ is $\Delta$-*compatible*, if for each $a, b \in R$, it follows that $ab = 0$ implies $a\delta^{\beta}(b) = 0$, where $\beta \in \mathbb{N}^n$. If $R$ is both $\Sigma$-compatible and $\Delta$-compatible, $R$ is called $(\Sigma, \Delta)$-*compatible*. By [143, Theorem 3.9] or [56, Lemma 3.5], $R$ is $(\Sigma, \Delta)$-compatible and reduced if and only if it is $\Sigma$-rigid. Once more again, compatible rings are more general than rigid rings.

Examples of skew PBW extensions over $(\Sigma, \Delta)$-compatible rings include quantizations of Weyl algebras (Section 1.2.1), PBW extensions defined by Bell and Goodearl [17] (Section 1.2.2), the family of 3-dimensional skew polynomial algebras [18, 133, 139] (Section 1.2.3), the class of diffusion algebras (Section 1.2.5), and other families of noncommutative algebras having PBW bases such as those presented in Section 1.2. Ring and module theoretic properties of these extensions over compatible rings have been investigated by some people (e.g. [73, 137, 138, 143, 147]).

Reyes and Suárez [145] defined the *weak compatible rings* with the aim of generalizing the $(\Sigma, \Delta)$-compatible rings above and the weak compatible rings introduced by Ouyang and Liu [129] in the setting of Ore extensions.

**Definition 1.40 ([145, Definition 4.1]).** Let $R$ be a ring with a finite family of endomorphisms $\Sigma$ and a finite family of $\Sigma$-derivations $\Delta$. We say that $R$ is *weak* $\Sigma$-*compatible* if for each $a, b \in R$, $a\sigma^{\alpha}(b) \in N(R)$ if and only if $ab \in N(R)$, where $\alpha \in \mathbb{N}^n$. Similarly, $R$ is called *weak* $\Delta$-*compatible* if for each $a, b \in R$, $ab \in N(R)$ implies $a\delta^{\beta}(b) \in N(R)$, where $\beta \in \mathbb{N}^n$. If $R$ is both weak $\Sigma$-compatible and weak $\Delta$-compatible, then $R$ is called *weak* $(\Sigma, \Delta)$-*compatible*.

The following two examples show that the weak $(\Sigma, \Delta)$-compatibility condition is a non-trivial generalization of compatible rings.

**Example 1.41 ([129, Example 2.5]).** Let $R$ be a reduced ring and $R_2$ the ring of upper triangular matrices. Consider the endomorphism $\sigma : R_2 \to R_2$ defined by $\sigma\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$, and let $\delta$ be the zero $\sigma$-derivation. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have that $R_2$ is not a $(\sigma, \delta)$-compatible ring. On the other hand, the set of nilpotent elements of $R_2$ consists of all matrices of the form $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$, for any $b \in R$. In this way, if $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & f \\ 0 & h \end{smallmatrix}\right) \in N(R_2)$, then $ae = ch = 0$. This implies that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} e & 0 \\ 0 & h \end{pmatrix} \in N(R_2).$$

By a similar argument, if $A\sigma(B) \in N(R_2)$, then $AB = 0$, for all $A, B \in R_2$. Therefore, we conclude that $R_2$ is a weak $(\sigma, \delta)$-compatible ring. Notice that the Ore extension $R_2[x; \sigma, \delta]$ is a skew PBW extension over $R_2$ which is weak $(\sigma, \delta)$-compatible.

**Example 1.42 ([163, Example 3.2]).** Let $\mathrm{S}_2(\mathbb{Z})$ be a subring of upper triangular matrices defined by $\mathrm{S}_2(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right) \mid a, b \in \mathbb{Z}\right\}$. Let $\sigma_1 = \mathrm{id}_{\mathrm{S}_2(\mathbb{Z})}$ be the identity endomorphism of $\mathrm{S}_2(\mathbb{Z})$, and consider $\sigma_2$ and $\sigma_3$ two endomorphisms defined by $\sigma_2\left(\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ 0 & a\end{smallmatrix}\right)$ and $\sigma_3\left(\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)$. Note that $\mathrm{S}_2(\mathbb{Z})$ is not $\sigma_3$-compatible, since for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$, we have $A\sigma_3(B) = 0$ but $AB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right) \neq 0$. Hence, $\mathrm{S}_2(\mathbb{Z})$ is not a $\Sigma$-compatible ring. In the same way, the set of nilpotent elements of $S_2(\mathbb{Z})$ consists of all matrices of the form $\left(\begin{smallmatrix} 0 & b \\ 0 & 0\end{smallmatrix}\right)$, for any $b \in \mathbb{Z}$. An argument similar to the previous example shows that $\mathrm{S}_2(\mathbb{Z})$ is a weak $\Sigma$-compatible ring, so we can consider a skew PBW extension $A = \sigma(\mathrm{S}_2(\mathbb{Z}))\langle x, y, z\rangle$ with three indeterminates $x, y$ and $z$ satisfying the conditions established in Definition 1.23.

**Proposition 1.43 ([145, Proposition 4.2]).** *If $R$ is a weak $(\Sigma, \Delta)$-compatible ring, then the following assertions hold:*

(1) *If $ab \in N(R)$, then $a\sigma^{\alpha}(b), \sigma^{\beta}(a)b \in N(R)$, for all elements $\alpha, \beta \in \mathbb{N}^n$.*

(2) *If $\sigma^{\alpha}(a)b \in N(R)$, for some element $\alpha \in \mathbb{N}^n$, then $ab \in N(R)$.*

(3) *If $a\sigma^{\beta}(b) \in N(R)$, for some element $\beta \in \mathbb{N}^n$, then $ab \in N(R)$.*

(4) *If $ab \in N(R)$, then $\sigma^{\alpha}(a)\delta^{\beta}(b), \delta^{\beta}(a)\sigma^{\alpha}(b) \in N(R)$, for every $\alpha, \beta \in \mathbb{N}^n$.*

Proposition 1.44 shows that if $R$ is reduced, then the notions of compatible ring and weak compatible ring coincide (c.f. [143, Theorem 3.9]).

**Proposition 1.44 ([145, Theorem 4.5]).** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension, then the following statements are equivalent:*

(1) *$R$ is reduced and weak $(\Sigma, \Delta)$-compatible.*

(2) *$R$ is $\Sigma$-rigid.*

(3) *$A$ is reduced.*

## 1.4 Armendariz rings

A commutative ring $R$ is called *Armendariz* (the term was introduced by Rege and Chhawchharia [134]) if for polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m$ of $R[x]$ which satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, for every $i, j$. As we can appreciate, the interest of this notion lies in its natural and its useful role in understanding the relation between the annihilators of the ring $R$ and the annihilators of the polynomial ring $R[x]$. For instance, in [10, Lemma 1] Armendariz showed that a reduced ring always satisfies this condition (recall that reduced rings are Abelian, and also semiprime, i.e., its prime radical is trivial). For the Ore extensions, the notion of Armendariz has been also studied. Commutative and non-commutative treatments have been investigated in several papers (e.g., [10, 134, 5, 84, 68, 65, 92, 109], and references therein).

With the purpose of generalizing the notion of $\sigma$-rigid ring and studying some ring-theoretical properties, several notions of *skew* Armendariz rings have been established in the literature (c.f. [65, 113, 119]). Let us see the details.

Following [123, Definitions 3.4 and 3.5], let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. We say that $R$ is a $(\Sigma, \Delta)$-*Armendariz ring* if for polynomials $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ in $A$, the equality $fg = 0$ implies $a_iX_ib_jY_j = 0$, for every $i, j$. We say that $R$ is a $(\Sigma, \Delta)$-*weak Armendariz ring* if for linear polynomials $f = a_0 + a_1x_1 + \cdots + a_nx_n$ and $g = b_0 + b_1x_1 + \cdots + b_nx_n$ in $A$, the equality $fg = 0$ implies $a_ix_ib_jx_j = 0$, for every $i, j$.

Note that every $\Sigma$-rigid ring is a $(\Sigma, \Delta)$-skew Armendariz ring [123, Proposition 3.6].

Now, following [140, Definitions 3.1 and 3.2], let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. $R$ is called a $\Sigma$-*skew Armendariz* ring if for elements $f = \sum_{i=0}^{m} a_iX_i$ and $g = \sum_{j=0}^{t} b_jY_j$ in $A$, the equality $fg = 0$ implies $a_i\sigma^{\alpha_i}(b_j) = 0$, for all $0 \leq i \leq m$ and $0 \leq j \leq t$, where $\alpha_i = \exp(X_i)$. $R$ is called a *weak* $\Sigma$-*skew Armendariz* ring, if for elements $f = \sum_{i=0}^{n} a_ix_i$ and $g = \sum_{j=0}^{n} b_jx_j$ in $A$ $(x_0 := 1)$, the equality $fg = 0$ implies $a_i\sigma_i(b_j) = 0$, for all $0 \leq i, j \leq n$ $(\sigma_0 := \mathrm{id}_R)$.

Note that every $\Sigma$-skew Armendariz ring is a weak $\Sigma$-skew Armendariz ring. If $A$ is a skew PBW extension over a $\Sigma$-rigid ring $R$, then $R$ is $\Sigma$-skew Armendariz [140, Proposition 3.4]. The converse of this assertion is false as the following remark shows.

**Remark 1.45.** We have the following facts:

- Consider the commutative ring $R = \left\{\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z},\ t \in \mathbb{Q}\right\}$ and the automorphism $\sigma$ of $R$ given by $\sigma\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}$. In [65, Example 1], it was shown that $R$ is $\sigma$-skew Armendariz but not a $\sigma$-rigid ring. Since $\Sigma$-rigid and $\Sigma$-skew Armendariz are generalizations of $\sigma$-rigid and $\sigma$-skew Armendariz, respectively, this example shows that there exist an example of a $\Sigma$-skew Armendariz ring which is not $\Sigma$-rigid.
- Let $\mathbb{Z}_2[x]$ be the commutative polynomial ring over $\mathbb{Z}_2$, and $\sigma$ the endomorphism of $\mathbb{Z}_2[x]$ defined by $\sigma(f(x)) = f(0)$. Then $\mathbb{Z}_2[x]$ is $\sigma$-skew Armendariz but not $\sigma$-rigid [65, Example 5].

From the facts above, we can establish the following relations

$$\Sigma\text{−rigid} \subsetneqq (\Sigma, \Delta)\text{−Armendariz} \subsetneqq (\Sigma, \Delta)\text{−weak Armendariz},$$

$$\Sigma\text{−rigid} \subsetneqq \Sigma\text{−skew Armendariz} \subsetneqq \text{weak } \Sigma\text{−skew Armendariz},$$

$$(\Sigma, \Delta)\text{−Armendariz} \subsetneqq \Sigma\text{−skew Armendariz},$$

$$(\Sigma, \Delta)\text{−weak Armendariz} \subsetneqq \text{weak } \Sigma\text{−skew Armendariz}.$$

Reyes and Suárez [142] introduced a generalization of weak $\Sigma$-skew Armendariz, the *weak skew-Armendariz* rings.

**Definition 1.46 ([142, Definition 4.1]).** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. We say that $R$ is a *skew-Armendariz* ring if for polynomials $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ in $A$, $fg = 0$ implies $a_0b_k = 0$, for each $0 \leq k \leq t$.

Note that every Armendariz ring is skew-Armendariz, where $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$ $(1 \leq i \leq n)$, and every $\Sigma$-skew Armendariz ring is also a skew-Armendariz ring. If $R$ is $\Sigma$-rigid, the elements $d_{i,j}$ are invertible (Definition 1.23 (iv)), and they are at the center of $R$, then $R$ is skew-Armendariz, by [135, Proposition 3.6].

**Definition 1.47 ([142, Definition 4.2]).** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. We say that $R$ is a *weak skew-Armendariz* ring if for linear polynomials $f = a_0 + a_1x_1 + \cdots + a_nx_n$, and $g = b_0 + b_1x_1 + \cdots + b_nx_n$ in $A$, $fg = 0$ implies $a_0b_k = 0$, for every $0 \leq k \leq n$.

We can see that every skew-Armendariz ring is weak skew-Armendariz. However, a weak Armendariz ring is not necessarily Armendariz. As an illustration of this fact in the case of Ore extensions, see [92, Example 3.2]. Of course, every weak $\Sigma$-skew Armendariz ring is a weak skew-Armendariz ring. In this way, we have the relations

$$\Sigma\text{−rigid} \subsetneqq (\Sigma, \Delta)\text{−Armendariz} \subsetneqq \Sigma\text{−skew Armendariz} \subsetneqq \text{skew} - \text{Armendariz},$$

$$\Sigma\text{−rigid} \subsetneqq (\Sigma, \Delta)\text{−weak Armendariz} \subsetneqq \text{weak } \Sigma\text{−skew Armendariz},$$

and of course,

$$\text{weak } \Sigma\text{−skew Armendariz} \subsetneqq \text{weak skew} - \text{Armendariz}.$$

In this way, the results presented in [142] for skew-Armendariz and weak skew-Armendariz rings generalize all results established in the previous papers [123, 135, 140], for skew PBW extensions, and in particular, for Ore extensions of injective type.

From [142, Theorem 4.4], we know that if $A$ is a skew PBW extension over a ring $R$, then the following statements are equivalent: (i) $R$ is reduced and skew-Armendariz, (ii) $R$ is $\Sigma$-rigid, and (iii) $A$ is reduced.

On the other hand, Ouyang et al. [130] introduced the notion of skew $\pi$-Armendariz ring as follows: If $R$ is a ring with an endomorphism $\sigma$ and a $\sigma$-derivation $\delta$, then $R$ is called *skew $\pi$-Armendariz ring* if for polynomials $f(x) = \sum_{i=0}^{l} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ of $R[x;\sigma,\delta]$, $f(x)g(x) \in N(R[x;\sigma,\delta])$ implies that $a_ib_j \in N(R)$, for each $0 \leq i \leq l$ and $0 \leq j \leq m$. Skew $\pi$-Armendariz rings are more general than skew Armendariz rings when the ring of coefficients is $(\sigma,\delta)$-compatible [130, Theorem 2.6], and also extend $\sigma$-Armendariz rings defined by Hong et al. [66] considering $\delta$ as the zero derivation.

Ouyang and Liu [129] showed that if $R$ is a weak $(\sigma,\delta)$-compatible and NI ring (a ring $R$ is called *NI* if $N(R) = N^*(R)$), then $R$ is skew $\pi$-Armendariz ring [129, Corollary 2.15].

Reyes [136] formulated the analogue of skew $\pi$-Armendariz ring in the setting of skew PBW extensions. For a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ over a ring $R$, we say that $R$ is *skew* $\Pi$-*Armendariz ring* if for elements $f = \sum_{i=0}^{l} a_i X_i$ and $g = \sum_{j=0}^{m} b_j Y_j$ belong to $A$, $fg \in N(A)$ implies $a_i b_j \in N(R)$, for each $0 \leq i \leq l$ and $0 \leq j \leq m$. If $R$ is reversible and $(\Sigma, \Delta)$-compatible, then $R$ is skew $\Pi$-Armendariz ring [136, Theorem 3.10]. This result was generalized to skew PBW extensions over weak $(\Sigma, \Delta)$-compatible and NI rings as the following proposition shows.

**Proposition 1.48 ([145, Theorems 4.7 and 4.9]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$.*

(1) *If $f = \sum_{i=0}^{m} a_i X_i$ and $g = \sum_{j=0}^{t} b_j Y_j$ are elements of $A$, then $fg \in N(A)$ if and only if $a_i b_j \in N(R)$, for all $i, j$.*

(2) *For every idempotent element $e \in R$ and a fixed $i$, we have $\delta_i(e) \in N(R)$ and $\sigma_i(e) = e + u$, where $u \in N(R)$.*

**Proposition 1.49 ([162, Theorem 3.3]).** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible ring. $R$ is NI if and only if $A$ is NI. In this case, it is clear that $N(A) = N(R)A$.*

In Chapter 2, we present new results about relationships between NI skew PBW extensions and different types of elements of these rings.

## 1.5 Category of semi-graded rings

We define the *category* SGR *of semi-graded rings* whose objects are the semi-graded rings and morphisms are the homogeneous ring homomorphisms. For a semi-graded ring $R$, $\mathsf{SGR} - R$ will denote the *category of semi-graded modules over* $R$ whose objects are precisely the semi-graded modules over $R$, and the morphisms are the homogeneous $R-$homomorphisms. It is straightforward to see that $\mathsf{SGR} - R$ is preadditive, and that the zero object of the category is the trivial module. Notice that the finitely generated SG submodules are the finitely generated objects of the category $\mathsf{SGR} - R$.

**Proposition 1.50.** *Let $f : M \to N$ be a morphism in $\mathsf{SGR} - R$. Then $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are* SG *submodules of $M$ and $N$, respectively.*

*Proof.* Let $m = m_1 + \cdots + m_k \in \mathrm{Ker}(f)$, where $m_i \in M_{n_i}$ and $n_i \neq n_j$, for all $i \neq j$. Then $0 = f(m) = f(m_1) + \cdots + f(m_k)$. Since $f$ is homogeneous, it follows that $f(m_i) \in N_{n_i}$, and therefore $f(m_i) = 0$. We conclude that $\mathrm{Ker}(f)$ is an SG submodule of $M$.

In a similar way, one can see that $\mathrm{Im}(f)$ is an SG submodule of $N$. □

From Proposition 1.50, it follows that for a morphism $f : M \to N$ in $\mathsf{SGR} - R$, $N/\mathrm{Im}(f)$ is an SG submodule of $N$. This guarantees that the category $\mathsf{SGR} - R$ has kernels and cokernels. If $f$ is a monomorphism in $\mathsf{SGR} - R$, then $f$ is the kernel of the canonical homomorphism $j : N \to N/\mathrm{Im}(f)$. If $f$ is a epimorphism of $\mathsf{SGR} - R$, then $f$ is the cokernel of the inclusion $i : \mathrm{Ker}(f) \to M$. In this way, the category $\mathsf{SGR} - R$ is normal and conormal.

Now, if $\{M_i\}_{i\in I}$ is a family of objects of $\mathsf{SGR} - R$, then their direct sum $\bigoplus\limits_{i\in I} M_i$ is a semi-graded ring with semi-graduation given by

$$\left(\bigoplus_{i\in I} M_i\right)_p := \bigoplus_{i\in I}(M_i)_p,\ p \in \mathbb{Z}.$$

As one can see, this object with the natural inclusions is precisely the *coproduct of the familiy of objects* $\{M_i\}_{i\in I}$ in $\mathsf{SGR} - R$. Thus, the category $\mathsf{SGR} - R$ is cocomplete and Abelian. The subobjects of $\mathsf{SGR} - R$ are (up to isomorphism) the submodules with the inclusion, and the intersection and the sum are the usual. With this, it is clear that $\mathsf{SGR} - R$ is a locally small $\mathtt{Ab5}$ category.

As it occurs in the graded setting, we consider the object $Q := \bigoplus\limits_{n\in\mathbb{Z}} \prod\limits_{i\in I} (M_i)_n$ as the possible product of the objects in the category $\mathsf{SGR} - R$. It is easy to check that $Q$ is a subgroup of $\prod\limits_{i\in I} M_i$. To verify that $Q$ is a submodule, we use the following characterization:

$$Q = \left\{(m_i) \in \prod_{i\in I} M_i \mid \text{ there exist } a, b \in \mathbb{Z} \text{ with } m_i \in \bigoplus_{k=a}^{b}(M_i)_k,\ \text{for all } i \in I\right\}.$$

If $(m_i) \in \prod\limits_{i\in I}(M_i)_m$ and $r \in R_n$, then one might expect to define $b$ as the sum $m + n$. Nevertheless, since it is not possible to find a precise value for $a$, then it is not easy to ensure a lower bound for the expression above. Thus, apparently $Q$ is not a submodule.

In order to remedy this situation, consider the largest submodule of $\prod\limits_{i\in I} M_i$ contained in $Q$ which is compatible with the semi-graduation. Let

$$C := \left\{N \mid N \text{ is a submodule of } \prod_{i\in I} M_i \text{ and } N = \bigoplus_{n\in\mathbb{Z}} N_n,\ \text{where } N_n = \prod_{i\in I}(M_i)_n \cap N\right\}. \tag{1.20}$$

Notice that $C$ is non-empty since $\bigoplus\limits_{i\in I} M_i \in C$, and if $N \in C$ then $N \subseteq Q$. Let $P := \sum\limits_{N\in C} N$. It is clear that $P$ is a submodule of $\prod\limits_{i\in I} M_i$ contained in $Q$.

**Proposition 1.51.** *$P$ is an* $\mathsf{SG}$ *$R$-module with semi-graduation given by* $P_n = \prod\limits_{i\in I}(M_i)_n \cap P$.

*Proof.* Since $P$ and $\prod\limits_{i\in I}(M_i)_n$ are subgroups of $\prod\limits_{i\in I} M_i$, then $P_n$ is a subgroup of $P$. It is clear that $\sum\limits_{n} P_n \subseteq P$, and that the sum is direct. Let $x \in P$. There exist $N_1, \dots, N_k \in C$ and $x_j \in N_j$, $1 \le j \le k$, such that $x = x_1 + \cdots + x_k$. Since $N_j \in C$, then $N_j = \bigoplus (N_j)_n$. Thus $x_j = \sum\limits_{m\in J} y_{j,m}$ where $J$ is a finite index set and $y_{j,m} \in (\prod(M_i)_m) \cap N_j$. Then $\sum\limits_{j=1}^{k} y_{j,m} \in P_m$

and $x = \sum_{j=1}^{k} \sum_{m\in J} y_{j,m} = \sum_{m\in J} \sum_{j=1}^{k} y_{j,m} \in \bigoplus P_n$. From this, we have that $P = \bigoplus P_n$.

Let $r \in R_m$ and $x = (a_i) \in P_n$, with $a_i \in M_i$. Since $P$ is a submodule, $rx \in P = \bigoplus P_n$. Therefore, $rx = x_1 + \cdots + x_k = (ra_i)$, for some $x_j \in P_{l_j}$. Note that in the $i$-th component when we compare degrees in $M_i$, if $l_j > m + n$ then $x_j = 0$, whence $rx \in \bigoplus_{k\leq n+m} P_k$. We conclude that $P$ is an $\mathsf{SG}$ $R$-module. □

From the reasoning above, it follows that $P$ is the greatest element of $C$. Let $\pi_i : \prod_{i\in I} M_i \to M_i$ the $i$-th projection. Then $\pi_i|_P$ is a homogeneous $R$-morphism. If no confusion arises, we only write $\pi_i$ instead $\pi_i|_P$.

**Proposition 1.52.** *Let $M$ be an $\mathsf{SG}$ $R$-module, $\{f_i : M \to M_i\}$ a family of homogeneous homomorphisms and $\prod_{i\in I} f_i : M \to \prod_{i\in I} M_i$. Then $\operatorname{Im}(\prod_{i\in I} f_i) \subseteq P$.*

*Proof.* Let $N = \operatorname{Im}(\prod_{i\in I} f_i)$ and $N_n = \prod_{i\in I}(M_i)_n \cap N$. It is clear that $\bigoplus_{n\in\mathbb{Z}} N_n \subseteq N$. Let $m \in M_k$. Since every morphism $f_i$ is homogeneous, then $f_i(m) \in (M_i)_k$, whence $\big(\prod_{i\in I} f_i\big)(m) = (f_i(m)) \in N_k$. From this, $N \subseteq \bigoplus_{n\in\mathbb{Z}} N_n$, and so $N \in C$. Thus, $N \subseteq P$. □

From the discussion above, it follows the next theorem.

**Theorem 1.53.** *Let $R$ be an $\mathsf{SG}$ ring and $\{M_i\}_{i\in I}$ be a family of $\mathsf{SG}$ $R$-modules. Then $(P, \pi_i)_{i\in I}$ is the product of $\{M_i\}_{i\in I}$ in $\mathsf{SGR} - R$.*

We write $P := \prod_{i\in I}^{\mathsf{sgr}} M_i$. Then the category $\mathsf{SGR} - R$ is complete.

As we saw above, given a family of $\mathsf{SG}$ $R$-modules $\{M_i\}_{i\in I}$, we have the relations

$$\bigoplus_{i\in I} M_i \subseteq \prod_{i\in I}^{\mathsf{sgr}} M_i \subseteq \bigoplus_{n\in\mathbb{Z}} \prod_{i\in I} (M_i)_n \subseteq \prod_{i\in I} M_i.$$

If the family $\{M_i\}_{i\in I}$ is finite, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$, which guarantees that we have a desirable description of $\prod_{i\in I}^{\mathsf{sgr}} M_i$. In general, it is easy to describe the elements of $\bigoplus_{n\in\mathbb{Z}} \prod_{i\in I}(M_i)_n$, but not for the elements of $\prod_{i\in I}^{\mathsf{sgr}} M_i$. For this reason, we want to find sufficient conditions to guarantee that $\prod_{i\in I}^{\mathsf{sgr}} M_i = \bigoplus_{n\in\mathbb{Z}} \prod_{i\in I}(M_i)_n$.

From the definition of $\prod_{i\in I}^{\mathsf{sgr}} M_i$, we see that $\prod_{i\in I}^{\mathsf{sgr}} M_i = \bigoplus_{n\in\mathbb{Z}} \prod_{i\in I}(M_i)_n$ is equivalent to $\bigoplus_{n\in\mathbb{Z}} \prod_{i\in I}(M_i)_n \in C$, ($C$ was defined in expression (1.20)) if and only if $\bigoplus_{n\in\mathbb{Z}} \prod_{i\in I}(M_i)_n$ is a submodule of $\prod_{i\in I} M_i$, or equivalently, for all $(n,k) \in \mathbb{Z}^2$ and each elements $r \in R_n$ and $m \in \prod (M_i)_k$, there exists $a \in \mathbb{Z}$ such that $rm \in \bigoplus_{k=a}^{n+k} (M_i)_k$. In general, $a$ could depend of $r$ and $m$, but we focus when $a$ only depends of $n$ and $k$. This motivates the following definition.

**Definition 1.54.** Let $R$ be an $\mathsf{SG}$ ring, $M$ an $\mathsf{SG}$ $R$-module and $\beta : \mathbb{Z}^2 \to \mathbb{Z}$ a map. We say that $M$ is $\beta$-*bounded* ($\beta - \mathsf{SG}$) if for every element $(n,m) \in \mathbb{Z}^2$, we have the inclusion

$$R_m M_n \subseteq \bigoplus_{k=\beta(m,n)}^{m+n} M_k.$$

If we consider the category of $\beta-\mathsf{SG}$ modules over $R$ (with morphisms the homogeneous $R$-homomorphisms), denoted by $\beta - \mathsf{SGR} - R$, then this is a full subcategory of $\mathsf{SGR} - R$, which is closed fot submodules, kernels, cokernels, products and coproducts. As a matter of fact, notice that if $\{M_i\}_{i\in I}$ is a family of objects of $\beta-\mathsf{SGR}-R$, then $\prod_{i\in I}^{\mathsf{sgr}} M_i = \bigoplus_{n\in\mathbb{Z}} \prod_{i\in I} (M_i)_n$.

**Remark 1.55.** (i) If $\beta = 0$, then the $\beta$-SG modules are the positively semi-graded modules.

(ii) Let $\beta_1, \beta_2 : \mathbb{Z}^2 \to \mathbb{Z}$ be functions. If $\beta_1 \leq \beta_2$ then $\beta_2 - \mathsf{SGR} - R$ is a full subcategory of $\beta_1 - \mathsf{SGR} - R$.

As a particular case, if $k \geq 0$ then we define $\beta_k(n,m) = n + m - k$, whence the modules belonging to $\beta_k - \mathsf{SG}$ are those whose their expansion have a length less or equal than $k$. Notice that if $R$ is an $\mathbb{N}$-graded ring, then $\beta_0 - \mathsf{SGR} - R$ is precisely the category of $\mathbb{N}$-graded modules over $R$.

# CHAPTER 2

# Elements and topology of some families of semi-graded rings

In this chapter, we establish several topological characterizations of the noncommutative spectrum of different families of semi-graded rings.

With this aim, necessary or sufficient conditions to guarantee that some of these families defined by endomorphisms and derivations are NI or NJ rings are presented in Section 2.1. Theorems 2.5, 2.10, 2.14, 2.18, Propositions 2.9, 2.24, 2.25, and Corollary 2.15 are the original results that contribute to the study of ideals and radicals of skew PBW extensions which has been partially carried out (e.g., [57, 97, 104, 122, 124, 146]), and establish ring-theoretical properties for noncommutative rings not considered in the literature such as those presented in Chapter 1. We generalize some results appearing in Jiang et al., [74] and Nasr-Isfahani [117, 118] for skew polynomial rings and $\mathbb{N}$-graded rings.

On the other hand, Section 2.2 presents results concerning the characterization of different types of elements of noncommutative rings over compatible rings such as idempotents, units, von Neumann regular, $\pi$-regular, and clean elements. The original results in this section are Theorems 2.28, 2.31, 2.35, 2.37, 2.39, 2.40, and 2.41. We extend several results presented by Hashemi et al. [55] for skew polynomial rings and Hamidizadeh et al. [52, 147] for skew Poincaré-Birkhoff-Witt extensions (c.f. [62, 73]).

Next, in Section 2.3, we investigate the notions of strongly harmonic and Gelfand ring over families of semi-graded rings. Propositions 2.42 and 2.43, and Theorems 2.45 and 2.48 are the original results in this section.

By using some results of the previous sections, Section 2.4 presents different results concerning the noncommutative spectrum of some families of semi-graded rings. There, the original results are Propositions 2.51, 2.55, 2.56, 2.59, 2.60 and 2.61.

Finally, Section 2.5 presents some ideas for a future work.

## 2.1 NI and NJ rings

First, we recall some ring-theoretical notions which are necessary for the rest of chapter.

A ring $R$ is called *nil-semisimple* if it has no nonzero nil ideals. Recall that nil-semisimple rings are semiprime [69, p. 187]. A proper right ideal (resp. left ideal, ideal) $P$ of $R$ is *right prime* (resp. *left prime*, *prime*) if for every $a, b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$. A proper ideal $P$ of $R$ is called *completely prime* if $ab \in P$ implies that $a \in P$ or $b \in P$ (equivalently, $R/P$ is a domain); and $P$ is said to be *strongly prime* if $R/P$ is nil-semisimple. Note that maximal ideals and completely prime ideals are strongly prime, and any strongly prime ideal contains a minimal strongly prime ideal.

Several kinds of rings are defined in terms of their set of nilpotent elements. For example, a ring $R$ is called *NI* if its set $N(R)$ of nilpotent elements coincides with its upper radical $N^*(R)$. If $N(R) = J(R)$, then $R$ is called *NJ*. $R$ is said to be *2-primal* if $N(R)$ is equal to the prime radical $N_*(R)$ of $R$. $R$ is called *weakly 2-primal* if $N(R)$ coincides with its Levitzki radical $L(R)$. Some examples of NJ rings are nil rings, division rings, Boolean rings, commutative Jacobson rings, commutative affine algebras over a field $\Bbbk$, semi-Abelian $\pi$-regular rings, locally finite Abelian rings [74, Example 2.5]. Every reduced regular ring is NJ [74, Proposition 2.11]. Note that $\mathbb{Z}[[x]]$ is a domain and hence NI with $N(\mathbb{Z}[[x]]) = \{0\}$, but $J(\mathbb{Z}[[x]]) = x\mathbb{Z}[[x]] \neq \{0\}$, and so $\mathbb{Z}[[x]]$ is not an NJ ring [74, Example 2.2]. This example shows that NI are not included in NJ rings.

NI and NJ rings have recently been investigated by several authors. For instance, Hwang et al. [69] studied the structure of NI rings related to strongly prime ideals and showed that minimal strongly prime ideals can be lifted in NI rings [69, Theorem 2.3]. They proved that for an NI ring $R$, $R$ is *weakly pm* (every strongly prime ideal of $R$ is contained in a unique maximal ideal of $R$) if and only if the topological space of maximal ideals of $R$ is a retract of the topological space of strongly prime ideals of $R$, or equivalently, if the topological space of strongly prime ideals of $R$ is normal [69, Theorem 3.7]. Also, they proved that $R$ is weakly pm if and only if $R$ is *pm* (every prime ideal of $R$ is contained in a unique maximal ideal of $R$) when $R$ is a symmetric ring [69, Theorem 3.8].

Concerning skew polynomial rings, Bergen and Grzeszczuk [23] studied the Jacobson radical of skew polynomial rings of derivation type $R[x; \delta]$ when the base ring $R$ has no nonzero nil ideals. They proved that if $R$ is an algebra with no nonzero nil ideals satisfying the ascending chain condition (acc) condition on right annihilators of powers, then $J(R[x; \delta]) = 0$ [23, Theorem 2]. In the case that $R$ is a semiprime algebra where every non-zero ideal contains a normalizing element, then $J(R[x; \delta]) = 0$ [23, Theorem 3]. Related to this topic, Nasr-Isfahani [117] gave necessary and sufficient conditions for a skew polynomial ring of derivation type $R[x; \delta]$ to be semiprimitive when $R$ has no nonzero nil ideals [117, Corollary 2.2]. He also proved that $J(R[x; \delta]) = N(R[x; \delta]) = N(R)[x; \delta]$ if and only if $N(R)$ is a $\delta$-ideal of $R$ (i.e., $N(R)$ is an ideal of $R$ and $\delta(N(R)) \subseteq N(R)$) and $N(R[x; \delta]) = N(R)[x; \delta]$ [117, Proposition 2.7]. Now, according to [117, Proposition 2.8], if $R[x; \delta]$ is NI then $J(R[x; \delta]) = N(R[x; \delta]) = N(R)[x; \delta] = N^*(R)[x; \delta] = N^*(R[x; \delta])$.

Later, Nasr-Isfahani [118] computed the Jacobson radical of an NI $\mathbb{Z}$-graded ring $R = \bigoplus_{i\in\mathbb{Z}} R_i$. He showed that $J(R) = N(R)$ if and only if $R$ is NI ring and $J(R) \cap R_0$ is nil [118, Theorem 2.4]. He also proved that $R[x;\sigma]$ is NJ if and only if $R[x;\sigma]$ is NI and $J(R[x;\sigma]) \cap R \subseteq N(R)$ [118, Corollary 2.5 (1)]. For a skew polynomial ring $R[x;\sigma,\delta]$, he showed that $R[x;\sigma,\delta]$ is NI and $N(R)$ is $\sigma$-rigid (i.e, $r\sigma(r) \in N(R)$ implies $r \in N(R)$, where $r \in R$) if and only if $N(R)$ is a $\sigma$-invariant ideal of $R$ ($N(R)$ is an ideal and $\sigma^{-1}(N(R)) = N(R)$) and $N(R[x;\sigma,\delta]) = N(R)[x;\sigma,\delta]$, and equivalently, $N(R)$ is a $\sigma$-rigid ideal of $R$ and $N^*(R[x;\sigma,\delta]) = N^*(R)[x;\sigma,\delta]$ [118, Theorem 3.1].

Han et al., [53] showed that if $R$ is an NI ring, $a, b \in R$, and $ab \in Z(R)$, then $ab - ba \in N^*(R)$, and there exists $l \geq 1$ such that $(ab)^n = (ba)^n$, for every $n \geq l$ [53, Theorem 1.3 (3)]. They also proved that for the ideal $I$ of $R$ generated by the subset $\{ab - ba \mid a, b \in R, \text{ such that } ab \in Z(R)\}$, if $R$ is NI then $I$ is nil and $R/I$ is an Abelian NI ring [53, Theorem 1.3 (5)].

Jiang et al., [74] studied the relationship between NJ rings and some families of rings. They investigated extensions as Dorroh, Nagata, and Jordan. For a ring $R$ and an automorphism $\sigma$ of $R$, they proved that if $R$ is weakly 2-primal $\sigma$-compatible, then $R[x;\sigma]$ is NJ [74, Theorem 3.10 (1)], and if $R$ is a weakly 2-primal $\delta$-compatible ring, then $R[x;\delta]$ is NJ [74, Theorem 3.12 (1)]. Moreover, they considered some topological conditions for NJ rings and showed relations between algebraic and topological notions [74, Section 4].

For a ring $R$, 2-primal implies weakly 2-primal. $R$ is reduced if and only if $R$ is nil-semisimple and NI, or equivalently, $R$ is semiprime and 2-primal [69, p. 187]. As we can check, $R$ is 2-primal if and only if $N_*(R) = N^*(R) = N(R)$. Shin [152, Proposition 1.11] proved that the set of nilpotent elements of a ring $R$ coincides with its prime radical if and only if every minimal prime ideal of $R$ is completely prime. Thus, $R$ is 2-primal if and only if every minimal prime ideal of $R$ is completely prime, or equivalently, $R/N_*(R)$ is reduced.

The following relations are well-known: reduced $\Rightarrow$ semicommutative $\Rightarrow$ NI, but the converses do not hold (see [32, 83] for more details). Shin [152, Lemma 1.2 and Theorem 1.5] established that semicommutative rings are 2-primal, and hence semicommutative rings are NI. A ring $R$ is called *right* (respectively, *left*) *duo* if every right (respectively, left) ideal is an ideal. A ring $R$ is called *right* (respectively, *left*) *quasi-duo* if every maximal right (respectively, left) ideal is an ideal.

Note that domains are reduced rings, reduced rings are symmetric, symmetric rings are reversible, and reversible rings are semicommutative, but the converses are not true in general [108]. Shin [152, Lemma 1.2] showed that right (left) duo rings are semicommutative. Therefore, NI rings contain several families of rings such as domains, reduced rings, symmetric rings, semi-commutative rings, reversible rings, one-sided duo rings, 2-primal rings and NJ rings (see [107] for a detailed description). Köthe's conjecture establishes that the upper nilradical contains every nil left ideal holds for NI rings [69, p. 192]. Equivalent definitions for NI rings are presented in Proposition 2.1.

**Proposition 2.1 ([69, Lemma 2.1]).** *For a ring $R$, the following conditions are equivalent.*

(1) *$R$ is NI.*

(2) *$N(R)$ is an ideal.*

(3) *Every subring (possibly without identity) of $R$ is NI.*

(4) *Every minimal strongly prime ideal of $R$ is completely prime.*

(5) *$R/N^*(R)$ is a reduced ring.*

(6) *$R/N^*(R)$ is a symmetric ring.*

### 2.1.1 NI rings

This section contains the original results of the thesis about the NI property for the family of skew PBW extensions considered in Section 1.2.10. We start with Proposition 2.3 which follows directly from [104, Theorem 3.9] and [146, Proposition 4.4]. Recall that a ring $R$ is *locally finite* if every finite subset in $R$ generates a finite semigroup multiplicatively.

The following result shows the relation between NI rings and invariant ideals.

**Proposition 2.2.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. If $A$ is NI, then $N(R)$ is a $(\Sigma, \Delta)$-invariant ideal.*

*Proof.* It is clear that $N(R)$ is a $\Sigma$-invariant ideal. On the other hand, consider an element $a \in N(R)$. Since $a$ and $\sigma_i(a)$ are elements of $N(A)$ and $N(A)$ is an ideal of $A$, then $\delta_i(a) = x_i a - \sigma_i(a)x_i \in N(A)$, that is, $\delta_i(a) \in N(R)$. Therefore, $N(R)$ is a $(\Sigma, \Delta)$-invariant ideal. □

**Proposition 2.3.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. If $R$ satisfies one of the following conditions,*

(1) *$R$ is 2-primal and $(\Sigma, \Delta)$-compatible, or*

(2) *$R$ is locally finite, $(\Sigma, \Delta)$-compatible and $\Sigma$-skew Armendariz,*

*then $A$ is an NI ring.*

**Proposition 2.4.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is an NI skew PBW extension over $R$, then $R$ is an NI ring, $N(R) \subseteq J(R)$, $N(A) \subseteq J(A)$, and therefore $N(R) \subseteq J(A)$.*

*Proof.* From Definition 1.23 (i), we know that $R$ is a subring of $A$. Since $A$ is NI, Proposition 2.1 implies that $R$ is NI. Now, since the Jacobson radical of a ring contains the nil ideals, by Proposition 2.1 we have that $N(A)$ and $N(R)$ are nil ideals of $A$ and $R$, respectively. Hence, $N(R) \subseteq J(R)$ and $N(A) \subseteq J(A)$. Since $N(R) \subseteq N(A)$, then $N(R) \subseteq J(A)$. □

From Propositions 2.3 and 2.4, we deduce that if $R$ is locally finite, $(\Sigma, \Delta)$-compatible and $\Sigma$-skew Armendariz, then $R$ is NI. This result has been proved by Reyes and Suárez [146, Theorem 4.3].

Since weak $(\Sigma, \Delta)$-compatible rings are more general than $(\Sigma, \Delta)$-compatible rings, and NI rings are more general than 2-primal rings, the following theorem generalizes [57, Theorem 4.11], and some other results of [57, 104] formulated for skew PBW extensions over 2-primal $(\Sigma, \Delta)$-compatible rings. From now on, we need to assume that the elements $d_{i,j}$ in Definition 1.23 (iv) are central in $R$.

**Theorem 2.5.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible ring $R$, then $R$ is NI if and only if $A$ is NI.*

*Proof.* Suppose that $R$ is an NI ring. Let us first see that $N(R)$ is a $(\Sigma, \Delta)$-invariant. By Proposition 2.1, $N(R)$ is an ideal of $R$. For a fixed $i$, if $\sigma_i(r) \in \sigma_i(N(R))$, where $r \in N(R)$, then $\sigma_i(r^k) = \sigma_i(r)^k = 0$, for some positive integer $k$. Thus $\sigma_i(r) \in N(R)$, i.e., $\sigma_i(N(R)) \subseteq N(R)$. Now, for $r \in N(R)$, $\delta_i(r) \in N(R)$, since $R$ is weak $(\Sigma, \Delta)$-compatible, whence $\delta_i(N(R)) \subseteq N(R)$. Since $N(R)$ is a $(\Sigma, \Delta)$-invariant ideal or $R$, by [97, Proposition 2.6 (i)] we have $N(R)\langle x_1, \dots, x_n\rangle$ is an ideal of $A$.

Let us show that $N(R)\langle x_1, \dots, x_n\rangle = N(A)$. From [145, Theorem 4.6], $f = \sum_{i=0}^{t} a_i X_i \in N(R)\langle x_1, \dots, x_n\rangle$ if and only if $a_i \in N(R)$, for $0 \leq i \leq t$, if and only if $f = \sum_{i=0}^{t} a_i X_i \in N(A)$. Therefore, $N(A)$ is an ideal of $A$, and by Proposition 2.1, $A$ is an NI ring.

Conversely, if $A$ is an NI ring, then by Proposition 2.4 we have that $R$ is an NI ring. □

If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a $(\Sigma, \Delta)$-compatible ring $R$, then $N(R)$ is $\Sigma$-rigid. Indeed, for $r \in R$ and $\alpha \in \mathbb{N}^n$, if $r\sigma^{\alpha}(r) \in N(R)$, then $r^2 \in N(R)$ [144, Lemma 2], whence $r \in N(R)$.

For the next result, recall that a ring $R$ is said to be *Dedekind finite* if $ab = 1$ implies $ba = 1$, where $a, b \in R$.

**Proposition 2.6.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is an NI skew PBW extension over $R$, then the following assertions hold:*

(1) *$N(R)$ and $N(A)$ are completely semiprime.*

(2) *Left (resp. right) invertible elements in $A$ are units.*

*Proof.* (1) Since $A$ and $R$ are NI rings, then $N(R)$ is an ideal of $R$ and $N(A)$ is an ideal of $A$. If $r^2 \in N(R)$ then $(r^2)^k = r^{2k} = 0$, for some positive integer $k$, and so $r \in N(R)$. Analogously, if $f^2 \in N(A)$ then $f \in N(A)$. Therefore, $N(R)$ and $N(A)$ are completely semiprime.

(2) If $f \in A$ is left (resp. right) invertible, then $gf = 1$ (resp. $fh = 1$), for some $g, h \in A$. Since $A$ is NI, $A$ is Dedekind finite and so $fg = 1$ (resp. $hf = 1$).

□

**Remark 2.7.** Reyes and Rodríguez [138, Theorem 3.14] presented a relation between skew PBW extensions of endomorphism type and the notion of Dedekind finite by considering a *skew notion of McCoy ring*.

**Proposition 2.8.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type over $R$ and $I \subseteq R$. Then $I\langle x_1, \dots, x_n\rangle$ is an ideal of $A$ if and only if $I$ is a $\Delta$-invariant (and therefore $(\Sigma, \Delta)$-invariant) ideal of $R$.*

*Proof.* If $I\langle x_1, \dots, x_n\rangle$ is an ideal of $A$, then $I$ is an ideal of $R$. Let $\delta_i(r) \in \delta_i(I)$ such that $r \in I$. Then $x_i r = r x_i + \delta_i(r) \in I\langle x_1, \dots, x_n\rangle$, for each $1 \le i \le n$. As $-r x_i \in I\langle x_1, \dots, x_n\rangle$ then $\delta_i(r) \in I\langle x_1, \dots, x_n\rangle$; in particular, $\delta_i(r) \in I$. This means that $I$ is a $\Delta$-invariant ideal. Since $A$ is of derivation type, $I$ is a $\Sigma$-invariant ideal.

The converse follows from [97, Proposition 2.6 (i)]. □

**Proposition 2.9.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of derivation type over $R$, then the following assertions hold:*

(1) *$N(R)$ is $\Sigma$-rigid.*

(2) *If $A$ is NI, then $N(R)$ and $N^*(R)$ are $\Sigma$-rigid ideals.*

(3) *For every completely prime $P$ of $A$, $P \cap R$ is a completely prime ideal of $R$.*

(4) *$A$ is NI if and only if $N(R)$ is a $\Delta$-invariant ideal of $R$ and $N(A) = N(R)\langle x_1, \dots, x_n\rangle$.*

*Proof.* (1) If $r \in R$ satisfies $r^2 = r\sigma^{\alpha}(r) \in N(R)$, then $r \in N(R)$, i.e., $N(R)$ is $\Sigma$-rigid.

(2) If $A$ is NI, then $R$ is NI and therefore $N(R)$ is an ideal of $R$. By (1), $N(R)$ is $\Sigma$-rigid, so $N(R)$ is a $\Sigma$-rigid ideal. Since $N(R) = N^*(R)$, then $N^*(R)$ is a $\Sigma$-rigid ideal.

(3) From [124, Theorem 1], for every completely prime ideal $P$ of $A$, $P \cap R$ is a completely prime ideal of $R$.

(4) If $A$ is NI, then by Proposition 2.4 $R$ is NI, and so $N(R)$ is an ideal of $R$. Let $r \in N(R) \subseteq N(A)$. Since $A$ is NI, $N(A)$ is an ideal of $A$ and $x_i r = r x_i + \delta_i(r) \in N(A)$, for each $1 \le i \le n$. As $-r x_i \in N(A)$, then $\delta_i(r) \in N(A) \cap R = N(R)$, which means that $N(R)$ is a $\Delta$-invariant ideal. From [57, Proposition 4.1], $N(A) \subseteq N(R)\langle x_1, \dots, x_n\rangle$. For the other inclusion, let $f = a_0 + a_1 X_1 + \cdots + a_t X_t \in N(R)\langle x_1, \dots, x_n\rangle$, with $a_i \in N(R) \subseteq N(A)$, for $0 \le i \le t$. Since $N(A)$ is an ideal of $A$, $a_0, a_1 X_1, \dots, a_t X_t \in N(A)$, and so $f = a_0 + a_1 X_1 + \cdots + a_t X_t \in N(A)$.

Conversely, if $N(R)$ is a $\Delta$-invariant ideal, Proposition 2.8 guarantees that $N(A) = N(R)\langle x_1, \dots, x_n\rangle$ is an ideal of $A$. Proposition 2.1 implies that $A$ is an NI ring.

□

The following result is one of the most important of the thesis. This extends Nasr-Isfahani [118, Theorem 3.1].

**Theorem 2.10.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a ring $R$, then the following statements are equivalent:*

(1) *$A$ is NI and $N(R)$ is $\Sigma$-rigid.*

(2) *$N(R)$ is a $\Sigma$-ideal of $R$ and $N(A) = N(R)\langle x_1, \dots, x_n\rangle$.*

(3) *$N(R)$ is a $\Sigma$-rigid ideal of $R$ and $N^*(A) = N^*(R)\langle x_1, \dots, x_n\rangle$.*

*Proof.* (1) $\Rightarrow$ (2) Suppose that $A$ is NI and $N(R)$ is $\Sigma$-rigid. By Proposition 2.4, $R$ is NI and so $N(R)$ is an ideal of $R$. From Proposition 1.25, we know that every $\sigma_i$ is injective, whence $r \in N(R)$ if and only if $r^k = 0$, for some positive integer $k$, if and only if $\sigma_i(r^k) = (\sigma_i(r))^k = 0$ if and only if $\sigma_i(r) \in N(R)$, and equivalently, $r \in \sigma_i^{-1}(N(R))$, which shows that $N(R)$ is a $\Sigma$-ideal of $R$.

With the aim of showing that $N(A) = N(R)\langle x_1, \dots, x_n\rangle$, before consider the following facts.

Note that $N(R)$ is $\Delta$-invariant; indeed, if $r \in N(R) \subseteq N(A)$, then $x_i r = \sigma_i(r)x_i + \delta_i(r) \in N(A)$. Since $N(R)$ is a $\Sigma$-ideal, then $\sigma_i(r) \in N(R) \subseteq N(A)$, and so $\sigma_i(r)x_i \in N(A)$, which implies that $\delta_i(r) \in N(A)$, that is, $\delta_i(r) \in N(R)$. By [97, Proposition 2.2 (i)], the system of endomorphisms and $\Sigma$-derivations $(\Sigma, \Delta)$ induces over $R/N(R)$ a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphisms and $\Sigma$-derivations defined by $\overline{\sigma}_i(\overline{r}) := \overline{\sigma_i(r)}$ and $\overline{\delta}_i(\overline{r}) := \overline{\delta_i(r)}$, $1 \leq i \leq n$. Since $N(R)$ is proper, [97, Proposition 2.6 (ii)] implies that $A/N(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over $R/N(R)$. Note that $R/N(R)$ is a $\overline{\Sigma}$-rigid ring, since if $\overline{r\sigma}^{\alpha}(\overline{r}) = 0$, then $\overline{r\sigma^{\alpha}(r)} = 0$, and so $r\sigma_i(r) \in N(R)$. Having in mind that $N(R)$ is $\Sigma$-rigid, $r \in N(R)$ and so $\overline{r} = 0$. By [142, Theorem 4.4], $A/N(R)\langle x_1, \dots, x_n\rangle$ is a reduced ring (c.f. [41, Theorem 6.1.9]).

Let us see that $N(A) \subseteq N(R)\langle x_1, \dots, x_n\rangle$. If $f \in N(A)$, then $f^k = 0$, for some positive integer $k$. Thus $\overline{f^k} = \overline{f}^k = 0$ in $A/N(R)\langle x_1, \dots, x_n\rangle$, and so $\overline{f} \in N\left(A/N(R)\langle x_1, \dots, x_n\rangle\right) = 0$. Hence $f \in N(R)\langle x_1, \dots, x_n\rangle$, that is, $N(A) \subseteq N(R)\langle x_1, \dots, x_n\rangle$.

For the another inclusion, if $f = r_0 + r_1X_1 + \cdots + r_tX_t \in N(R)\langle x_1, \dots, x_n\rangle$, where $r_0, r_1, \dots, r_t \in N(R) \subseteq N(A)$, since $N(A)$ is an ideal of $A$, then $r_0, r_1X_1, \dots, r_tX_t \in N(A)$, and therefore $f = r_0 + r_1X_1 + \cdots + r_tX_t \in N(A)$.

(2) $\Rightarrow$ (1) Let $r \in N(R) \subseteq N(A)$. Then $rx_i \in N(R)\langle x_1, \dots, x_n\rangle$, for $1 \leq i \leq n$, whence $x_ir = \sigma_i(r)x_i + \delta_i(r) \in N(R)\langle x_1, \dots, x_n\rangle = N(A)$. Thus $\delta_i(r) \in N(R)$, i.e., $N(R)$ is $\Delta$-invariant. Since $N(R)$ is proper, [97, Proposition 2.6 (i)] implies that $N(R)\langle x_1, \dots, x_n\rangle$ is an ideal of $A$ and $AN(R) \subseteq N(R)\langle x_1, \dots, x_n\rangle$. Using that $N(R)\langle x_1, \dots, x_n\rangle = N(A)$, we obtain that $N(A)$ is an ideal of $A$, and hence $A$ is NI.

To show that $N(R)$ is $\Sigma$-rigid it is enough to see that for $r \in R$ and $1 \leq i \leq n$, $r\sigma_i(r) \in N(R)$ implies that $r \in N(R)$. If for $r \in R$, $r\sigma_i(r) \in N(R)$, $1 \leq i \leq n$, then $r\sigma_i(r)x_i \in N(A) = N(R)\langle x_1, \dots, x_n\rangle$, $1 \leq i \leq n$, and so $\sigma_i(r)x_ir = \sigma_i(r)\sigma_i(r)x_i + \sigma_i(r)\delta_i(r) \in N(R)\langle x_1, \dots, x_n\rangle$. Thus $\sigma_i(r^2) \in N(R) = \sigma_i^{-1}(N(R))$, since $N(R)$ is $\Sigma$-ideal. Then $r^2 \in N(R)$ and so $r \in N(R)$.

(1) $\Rightarrow$ (3) If $A$ is NI, then $N^*(A) = N(A)$, and by Proposition 2.4, $R$ is NI, i.e., $N^*(R) = N(R)$. From implication (1) $\Rightarrow$ (2) we have that $N(A) = N(R)\langle x_1, \ldots, x_n\rangle$, and so $N^*(A) = N(A) = N(R)\langle x_1, \ldots, x_n\rangle = N^*(R)\langle x_1, \ldots, x_n\rangle$.

(3) $\Rightarrow$ (2) Suppose that $N(R)$ is a $\Sigma$-rigid ideal of $R$. By the same argument as in the proof of (1) $\Rightarrow$ (2), we have that $N(R)$ is a $\Sigma$-ideal of $R$. If $r \in N(R)$, then $x_i r = \sigma_i(r)x_i + \delta_i(r) \in AN(R) \subseteq N(R)\langle x_1, \ldots, x_n\rangle$, and so $\delta_i(r) \in N(R)$, for $1 \leq i \leq n$, which shows that $N(R)$ is $\Delta$-invariant. Using the same argument as in the proof of (1) $\Rightarrow$ (2), we see that $N(A) \subseteq N(R)\langle x_1, \ldots, x_n\rangle$. Since $N(R)$ is an ideal then $R$ is NI, and so $N^*(R) = N(R)$. By assumption $N^*(R)\langle x_1, \ldots, x_n\rangle = N^*(A)$, whence $N(R)\langle x_1, \ldots, x_n\rangle = N^*(R)\langle x_1, \ldots, x_n\rangle = N^*(A) \subseteq N(A)$. □

A particular case of Theorem 2.10 is formulated in the following corollary.

**Corollary 2.11 ([118, Theorem 3.1]).** *Let $R$ be a ring, $\sigma$ an endomorphism of $R$, and $\delta$ a $\sigma$-derivation of $R$. Then the following statements are equivalent:*

(1) *$R[x; \sigma, \delta]$ is NI and $N(R)$ is $\sigma$-rigid.*

(2) *$N(R)$ is a $\sigma$-ideal of $R$ and $N(R[x; \sigma, \delta]) = N(R)[x; \sigma, \delta]$.*

(3) *$N(R)$ is a $\sigma$-rigid ideal of $R$ and $N^*(R[x; \sigma, \delta]) = N^*(R)[x; \sigma, \delta]$.*

### 2.1.2 NJ rings

In this section, we present the original results of the thesis about the NJ property for skew PBW extensions. We start with Proposition 2.12 that follows directly from the definitions of NI and NJ rings.

**Proposition 2.12.** *If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is an NJ skew PBW extension over a ring $R$, then $R$ is NI and $J(A) \cap R = N(R)$.*

From [41, Propositions 3.2.1 and 3.2.3], it follows the next result.

**Proposition 2.13.** *Skew PBW extensions over domains $R$ are NJ rings, and hence, NI rings.*

**Theorem 2.14.** *If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a graded skew PBW extension over $R = \bigoplus_{n\in\mathbb{N}} R_n$, then $A$ is NJ if and only if $A$ is NI and $J(A) \cap R_0$ is a nil ideal.*

*Proof.* Suppose that $A$ is NI and $J(A) \cap R_0$ is nil. By [159, Remark 2.10 (i)], $R_0 = A_0$, whence $J(A) \cap R_0 = J(A) \cap A_0$ is nil. From [118, Theorem 2.4], $J(A) = N(A)$, i.e., $A$ is an NJ ring.

Conversely, if $A$ is an NJ ring, then $A$ is NI. Since $R_0$ is a subring, $J(A) \cap R_0$ is an ideal, and as $A$ is an NJ ring, $J(A) = N(A)$. In this way, $J(A) \cap R_0 \subseteq J(A) = N(A)$, and hence $J(A) \cap R_0$ is a nil ideal. □

**Corollary 2.15.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a graded skew PBW extension over a connected algebra $R = \bigoplus_{n\in\mathbb{N}} R_n$, then $A$ is NJ if and only if $A$ is NI.*

*Proof.* Since $R$ is connected, [159, Remark 2.10 (i)] implies that $A_0 = R_0 = \Bbbk$, whence $J(A) \cap A_0 = \{0\}$ is nil. The result follows from Theorem 2.14. □

**Corollary 2.16.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension over $R$. Then $A$ is NJ if and only if $A$ is NI and $J(A) \cap R$ is a nil ideal of $R$.*

*Proof.* From [168, Proposition 2.6], we know that quasi-commutative skew PBW extensions where the ring $R$ has the trivial graduation are graded skew PBW extensions, so [159, Remark 2.10 (i)] guarantees that $A_0 = R_0 = R$. Thus, the assertion follows from Theorem 2.14. □

**Corollary 2.17.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is an NJ quasi-commutative skew PBW extension over $R$ and $J(R) \subseteq J(A)$, then $R$ is NJ.*

*Proof.* By Proposition 2.12, we have that $J(A) \cap R = N(R)$. As $J(R) \subseteq J(A)$, $J(R) \subseteq N(R)$. Since $A$ is NJ, then it is NI, so by Proposition 2.4, $N(R) \subseteq J(R)$. This fact shows that $R$ is NJ. □

The next theorem presents similar results to [74, Theorem 3.10 (1)].

**Theorem 2.18.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension over $R$. If $R$ is a weakly 2-primal weak $\Sigma$-compatible ring, then $A$ is NJ.*

*Proof.* Since $A$ is quasi-commutative, $\delta_i = 0$ for $1 \le i \le n$, and so $A$ is weak $\Delta$-compatible. If $R$ is weakly 2-primal then it is NI, and by Theorem 2.5, $A$ is NI. Let us show that $J(A) \cap R$ is nil. If $r \in J(A) \cap R$, then $rx_1 \in J(A)$. From [168, Proposition 2.6], $A$ is a graded skew PBW extension with the trivial graduation of $R$, and so $rx_1$ is a homogeneous component of $J(A)$ with degree 1. By [118, Lemma 2.3 (2)], $rx_1 \in N(A)$. In this way, [145, Theorem 4.6] implies that $r \in N(R)$. The assertion follows from Corollary 2.16. □

**Remark 2.19.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$.

(i) If $R$ is 2-primal and $(\Sigma, \Delta)$-compatible, then by Proposition 2.3 (1) we have that $A$ is NI, that is, $N^*(A) = N(A)$. Now, from [104, Theorem 4.11], $J(A) = N_*(A)$, whence $N(A) = J(A)$, and thus $A$ is NJ.

(ii) If $R$ is locally finite and weak $\Sigma$-skew Armendariz, then $R$ is NJ. More exactly, since $R$ is weak $\Sigma$-skew Armendariz, [142, Proposition 4.9] implies that $R$ is Abelian. Now, by assumption $R$ is locally finite, so [67, Proposition 2.5] guarantees that $N(R) = J(R)$, that is, $R$ is an NJ ring.

(iii) If $A$ is NJ and $J(R)$ is nil (or $N^*(R) = J(R)$ or $R/N^*(R)$ is semiprimitive), then $R$ is NJ. Notice that if $A$ is NJ, Proposition 2.12 shows that $R$ is NI, and if $J(R)$ is nil (or $N^*(R) = J(R)$ or $R/N^*(R)$ is semiprimitive), then by [74, Proposition 2.3], $R$ is NJ.

**Proposition 2.20.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type over $R$. Then the following assertions hold:*

(1) *If $A$ is NI and $R$ is $\Delta$-compatible and right duo, then $A$ is NJ.*

(2) *$A$ is NI and $I \subseteq N(R)$ if and only if $A$ is NJ, where $I \subseteq R$ is the set of all coefficients of all terms of all elements of $J(A)$.*

*Proof.* (1) If $A$ is NI, then $N(A) = N^*(A) \subseteq J(A)$. For the other inclusion, if $f = a_0 + a_1X_1 + \cdots + a_tX_t \in J(A)$, then $fx_n = a_0x_n + a_1X_1x_n + \cdots + a_tX_tx_n \in J(A)$ and therefore $1 + fx_n = 1 + a_0x_n + a_1X_1x_n + \cdots + a_tX_tx_n$ is a unit of $A$. Since $R$ is $(\Sigma, \Delta)$-compatible and right duo, [52, Theorem 4.7] guarantees that $a_0, a_1, \dots, a_t \in N(R)$. So, $f = a_0 + a_1X_1 + \cdots + a_tX_t \in N(R)\langle x_1, \dots, x_n\rangle$. As $A$ is NI, Proposition 2.9 (4) implies that $N(R)\langle x_1, \dots, x_n\rangle = N(A)$, that is, $f \in N(A)$.

(2) Suppose that $A$ is NI and $I \cap R \subseteq N(R)$. By Proposition 2.9 (4), $N(A) = N(R)\langle x_1, \dots, x_n\rangle$. If $f = a_0 + a_1X_1 + \cdots + a_tX_t \in J(A)$, then by assumption $a_k \in N(R)$, for $0 \leq k \leq t$, whence $a_kX_k \in N(R)\langle x_1, \dots, x_n\rangle$, for every $k$. Since $A$ is NI, then $N(A)$ is an ideal of $A$, and hence $f = a_0 + a_1X_1 + \cdots + a_tX_t \in N(A)$.

For the converse, suppose that $A$ is NJ. Then $A$ is NI, and so Proposition 2.9 (4) implies that $N(A) = N(R)\langle x_1, \dots, x_n\rangle$. Now, if $r \in I$, then for some $f = a_0 + a_1X_1 + \cdots + a_tX_t \in J(A)$ there exists $1 \leq k \leq t$ such that $r = a_k$. Since $J(A) = N(A) = N(R)\langle x_1, \dots, x_n\rangle$, it follows that $r \in N(R)$.

□

The following theorem is another important result of the thesis.

**Theorem 2.21.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type over $R$. Then $A$ is NI if and only if $A$ is NJ.*

*Proof.* By Proposition 2.20 (2), it is enough to prove that every coefficient of each term of every polynomial of $J(A)$ is nilpotent. Let $f = r_0 + r_1X_1 + \cdots + r_tX_t \in J(A)$. Since $A$ is NI, then $N(A)$ is an ideal of $A$, and by Proposition 2.9 (4), $N(R)$ is a $\Delta$-invariant ideal of $R$ and $N(A) = N(R)\langle x_1, \dots, x_n\rangle$. Since $A$ is of derivation type, $N(R)$ is a $\Sigma$-ideal. By [97, Proposition 2.6 (ii)], $A/N(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over $R/N(R)$. By considering the notation of the proof of [97, Proposition 2.6 (ii)], and identifying $\overline{x_i}$ with $x_i$, $1 \leq i \leq n$, we use $A/N(A) = A/N(R)\langle x_1, \dots, x_n\rangle \cong \overline{\sigma}(R/N(R))\langle x_1, \dots, x_n\rangle$ to denote such an extension. Now, by [97, Proposition 2.2 (i)], the systems of endomorphisms and $\Sigma$-derivations $(\Sigma, \Delta)$ induce over $R/N(R)$ a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphisms and $\Sigma$-derivations defined by $\overline{\sigma}_i(\overline{r}) := \overline{\sigma_i(r)}$ and $\overline{\delta}_i(\overline{r}) := \overline{\delta_i(r)}$, $1 \leq i \leq n$. If $\overline{r}\overline{\sigma}_i(\overline{r}) = \overline{r\sigma_i(r)} = 0$, then $r\sigma_i(r) \in N(R)$ and so $r \in N(R)$, since by Proposition 2.9 (1), $N(R)$ is $\Sigma$-rigid.

Therefore, $\overline{r} = 0$ and so $R/N(R)$ is $\overline{\Sigma}$-rigid. By [143, Theorem 3.9], $R/N(R)\langle x_1, \dots, x_n\rangle \cong A/N(A)$ is $(\Sigma, \Delta)$-compatible (c.f. [41, Proposition 6.2.4]). Now, as $f \in J(A)$, then $\overline{f} := f + N(A) \in J(A)/N(A) = J(A/N(A))$ because Proposition 2.4 establishes that $N(A) \subseteq J(A)$. So, $\overline{f}x_n = \overline{r_0}x_n + \overline{r_1}X_1x_n + \cdots + \overline{r_t}X_tx_n \in J(\overline{\sigma}(R/N(R)\langle x_1, \dots, x_n\rangle)$, with $\overline{r_k} := r_k + N(R)$, $1 \leq k \leq n$. Having in mind that $\overline{f}x_n \in J(\overline{\sigma}(R/N(R)\langle x_1, \dots, x_n\rangle)$, then $\overline{1} + \overline{f}x_n = \overline{1} + \overline{r_0}x_n + \overline{r_1}X_1x_n + \cdots + \overline{r_t}X_tx_n$ is a unit of $\overline{\sigma}(R/N(R)\langle x_1, \dots, x_n\rangle$. Thus, by [104, Remark 4.9 (ii)], $\overline{r_k} \in N(R/N(R))$, $0 \leq k \leq n$. Since $R/N(R)$ is reduced, we have that $\overline{r_k} = 0$, for each $0 \leq k \leq n$. In this way, $r_k \in N(R)$, for each $0 \leq k \leq n$. The result follows from Proposition 2.20 (2). □

**Remark 2.22.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type over $R$. By Propositions 2.9 and 2.20, and Theorem 2.21, if $A$ is NI then

$$N(A) = N^*(A) = N(R)\langle x_1, \dots, x_n\rangle = N^*(R)\langle x_1, \dots, x_n\rangle = J(A).$$

We immediately have the following corollary.

**Corollary 2.23 ([117, Proposition 2.8]).** *Let $R$ be a ring and $\delta$ a derivation of $R$. If $R[x;\delta]$ is NI, then $J(R[x;\delta]) = N(R[x;\delta]) = N(R)[x;\delta] = N^*(R)[x;\delta] = N^*(R[x;\delta])$.*

The next result, Proposition 2.24, extends [118, Corollary 3.2].

**Proposition 2.24.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of derivation type over $R$, then the following statements are equivalent:*

(1) *$A$ is NJ.*

(2) *$A$ is NI.*

(3) *$R$ is NI and $N(A) = N(R)\langle x_1, \dots, x_n\rangle$.*

(4) *$R$ is NI and $N^*(A) = N^*(R)\langle x_1, \dots, x_n\rangle$.*

*Proof.* (1) ⇔ (2) It follows from Theorem 2.21.

(2) ⇒ (3) If $A$ is NI, then by Proposition 2.4 we have that $R$ is NI. From Proposition 2.9 (4), $N(A) = N(R)\langle x_1, \dots, x_n\rangle$.

(3) ⇒ (2) If $R$ is NI, then $N(R)$ is an ideal of $R$, and by Proposition 2.9 (4), $N(R)$ is $\Delta$-invariant and therefore $A$ is NI.

(3) ⇒ (4) If $R$ is NI, then $N(R)$ is a $\Sigma$-ideal. Theorem 2.10 (2) ⇒ (3) asserts that $N^*(A) = N^*(R)\langle x_1, \dots, x_n\rangle$.

(4) ⇒ (3) Since $R$ is NI, then $N(R)$ is an ideal of $R$. Now, by Proposition 2.9 (1), $N(R)$ is $\Sigma$-rigid. Theorem 2.10 (3) ⇒ (2) implies that $N(A) = N(R)\langle x_1, \dots, x_n\rangle$. □

The next proposition establishes similar results to [118, Corollary 3.3].

**Proposition 2.25.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a quasi-commutative skew PBW extension over $R$, then the following statements are equivalent:*

(1) *$A$ is NJ and $N(A) = N(R)\langle x_1, \ldots, x_n\rangle$.*

(2) *$N(R)$ is a $\Sigma$-ideal of $R$ and $N(A) = N(R)\langle x_1, \ldots, x_n\rangle$.*

(3) *$A$ is NI and $N(R)$ is $\Sigma$-rigid.*

(4) *$N(R)$ is $\Sigma$-rigid ideal of $R$ and $N^*(A) = N^*(R)\langle x_1, \ldots, x_n\rangle$.*

*Proof.* (1) $\Rightarrow$ (2) If $A$ is NJ, then we know that $A$ is NI, and therefore $R$ is NI, so $N(R)$ is an ideal of $R$. In the proof of Theorem 2.10 (1) $\Rightarrow$ (2), it was shown that $N(R)$ is a $\Sigma$-ideal.

(2) $\Rightarrow$ (3) It follows from Theorem 2.10 (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) If $r \in J(A) \cap R$, then $rx_1 \in J(A)$. From [168, Proposition 2.6], $A$ is a graded skew PBW extension with the trivial graduation of $R$. Thus $rx_1$ is a homogeneous component of $J(A)$ with degree one. By [118, Lemma 2.3 (2)], we have that $rx_1 \in N(A)$. Since $A$ is NI and $N(R)$ is $\Sigma$-rigid, then Theorem 2.10, implication (1) $\Rightarrow$ (2), guarantees that $N(A) = N(R)\langle x_1, \ldots, x_n\rangle$. Since $rx_1 \in N(A)$, it follows that $r \in N(R)$, and thus $J(A) \cap R$ is a nil ideal of $R$. Corollary 2.16 implies that $A$ is NJ.

(3) $\Leftrightarrow$ (4) This is precisely the content of Theorem 2.10 (2) $\Leftrightarrow$ (3). □

## 2.2 Idempotents, units, von Neumann regular, and clean elements

In 1936, von Neumann [179] introduced the von Neumann regular rings as an algebraic tool for studying certain lattices and some properties of operator algebras. Briefly, an element $a \in R$ is called *von Neumann regular* if there exists an element $r \in R$ with $a = ara$. A ring $R$ is called *von Neumann regular* if every of its elements is von Neumann regular. These rings are also known as *absolutely flat rings* due to their characterization in terms of modules. von Neumann regular rings are of great importance in areas such as topology and functional analysis. More precisely, the prime spectrum of a commutative von Neumann ring establishes relationships with different types of compactifications and homomorphisms of the prime spectrum, and the prime spectrum of its ring of idempotent elements (see [48, 89, 150, 151] for more details). These facts show the close relationship of the von Neumann rings with Boolean rings ($R$ is called *Boolean* whenever $\text{Idem}(R) = R$).

Following Lam [89], the element $a \in R$ is said to be a *$\pi$-regular* element of $R$ if $a^m r a^m = a^m$, for some $r \in R$ and $m \geq 1$. The element $a$ is called *strongly $\pi$-regular* if there exist elements $n \in \mathbb{N}$ and $r \in R$ such that $a^n = a^{n+1}r$; if $n = 1$, then we say that $a$ is *strongly regular*. In the natural way, the *strongly regular*, *$\pi-$regular* and *strongly* $\pi-$regular rings are defined. We consider the set of von Neumann regular elements $\text{vnr}(R)$ and the set of $\pi$-regular elements $\pi - r(R)$. It is clear that $\text{Idem}(R) \subseteq \text{vnr}(R) \subseteq \pi - r(R)$. It is straightforward to see that the implications Boolean $\Rightarrow$ von Neumann regular $\Rightarrow$ $\pi$-regular hold. A beautiful treatment about von Neumann regular rings can be found in Goodearl [48] (c.f. Lam [89]).

On the other hand, Contessa [36] introduced the notion of von Neumann local. An element $a \in R$ is called a *von Neumann local element* if either $a \in \text{vnr}(R)$ or $1-a \in \text{vnr}(R)$. Following Nicholson [120], an element $a \in R$ is a *clean element* if $a$ is the sum of a unit and an idempotent of $R$. Let $\text{vnl}(R)$ be the set of von Neumann local elements and $\text{cln}(R)$ the set of clean elements. If $\text{cln}(R) = R$, then $R$ is called a *clean ring* [120]. Examples of clean rings are the exchange rings and semiperfect rings. Several characterizations of clean elements have been established by different authors [81, 121]. Finally, if $\text{vnl}(R) = R$, then $R$ is said to be a *von Neumann local* ring [55].

Concerning skew polynomial rings, Hashemi et al., [55] investigated characterizations of different elements over skew polynomial rings by using the notion of *compatible ring* introduced by Annin [6, 7, 8] (Section 1.3). With these ring-theoretical notions, Hashemi et al., [55] characterized the unit elements, idempotent elements, von Neumann regular elements, $\pi$-regular elements, and also the von Neumann local elements of the skew polynomial ring $R[x;\sigma,\delta]$ when the base ring $R$ is a right duo $(\sigma,\delta)$-compatible. Hamidizadeh et al. [52] characterized the above types of elements of skew PBW extensions over compatible rings in the sense of Hashemi et al. [56] and Reyes and Suárez [143] (Section 1.3), and generalized the results obtained by Hashemi et al. [55].

Since Reyes and Suárez [145] introduced the *weak* $(\Sigma,\Delta)$*-compatible rings* as a natural generalization of compatible rings above and the *weak* $(\sigma,\delta)$*-compatible rings* defined by Ouyang and Liu [129] for Ore extensions, an immediate and natural task is to study the types of elements described above of these extensions by considering this weak notion of compatibility, and hence to investigate if it is possible to extend all results established in [52, 55] to a more general setting. This is the purpose of this section.

Briefly, we recall that Ouyang and Liu [129] characterized the nilpotent elements of skew polynomial rings over a weak $(\sigma,\delta)$-compatible and NI ring. Reyes and Suárez [145] extended this result for skew PBW extensions as the following proposition shows. We assume that the elements $d_{i,j}$ from Definition 1.23 (iv) are central and invertible in $R$.

**Proposition 2.26 ([145, Theorem 4.6]).** *If $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a skew PBW extension over a weak $(\Sigma,\Delta)$-compatible and NI ring, then $f = \sum_{i=0}^{m} a_iX_i \in N(A)$ if and only if $a_i \in N(R)$, for all $0 \leq i \leq m$.*

Next, we formulate analogue results to the obtained for the case of skew polynomial rings [55], and the skew PBW extensions over right duo rings [52].

Proposition 2.27 generalizes [52, Theorem 4.5].

**Proposition 2.27.** *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension over a weak $(\Sigma,\Delta)$-compatible and NI ring $R$, and consider $f = \sum_{i=0}^{l} a_iX_i$ and $g = \sum_{j=0}^{m} b_jY_j$ nonzero elements of $A$ such that $fg = c \in R$. If $b_0$ is a unit of $R$, then $a_1, a_2, \dots, a_l$ are nilpotent elements of $R$.*

*Proof.* Assume that $b_0$ is a unit element of $R$. Let us show that $a_1, a_2, \dots, a_l$ are all nilpotent. Since $R$ is NI ring and weak $(\Sigma,\Delta)$-compatible, we have $N(R)$ is a $(\Sigma,\Delta)$-invariant ideal of

$R$. Hence, $\overline{R} = R/N(R)$ is a reduced ring and also weak $(\overline{\Sigma}, \overline{\Delta})$-compatible. By Proposition 1.44, $\overline{R}$ is a $\overline{\Sigma}$-rigid ring. Since $fg = c \in R$, we have $\overline{f}\overline{g} = \overline{c}$ in $\sigma(\overline{R})\langle x_1, \dots, x_n\rangle$, and hence $\overline{a_0}\overline{b_0} = \overline{c}$ and $\overline{a_i}\overline{b_j} = \overline{0}$, for each $i + j \geq 1$, by [52, Proposition 4.2]. Therefore, we get $\overline{a_i} = 0$ for each $i \geq 1$, since $b_0$ is a unit, whence $a_i$ is nilpotent for every $i \geq 1$. □

We establish the following characterization of the units of a skew PBW extension. Theorem 2.28 generalizes [52, Theorem 4.7].

**Theorem 2.28.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$, then an element $f = \sum_{i=0}^{m} a_iX_i \in A$ is a unit of $A$ if and only if $a_0$ is a unit of $R$ and $a_i$ is nilpotent, for every $1 \leq i \leq m$.*

*Proof.* Suppose that $R$ is a weak $(\Sigma, \Delta)$-compatible and NI ring. This implies that $N(R)$ is a $(\Sigma, \Delta)$-invariant ideal of $R$, whence $\overline{R} = R/N(R)$ is reduced and weak $(\overline{\Sigma}, \overline{\Delta})$-compatible. Proposition 1.44 implies that $\overline{R}$ is $\overline{\Sigma}$-rigid, and $\overline{A} = A/N(A)$ is a skew PBW extension over $\overline{R}$ by Proposition 1.39.

Consider $f = \sum_{i=0}^{l} a_iX_i$ a unit element of $A$. There exists $g = \sum_{j=0}^{m} b_jY_j \in A$ such that $fg = gf = 1$, which implies that $\overline{f}\overline{g} = \overline{g}\overline{f} = \overline{1}$ in $\overline{A}$, and so $\overline{a_0}\overline{b_0} = \overline{b_0}\overline{a_0} = \overline{1}$ and $\overline{a_i}\overline{b_j} = 0$, for each $i + j \geq 1$ by [52, Proposition 4.2]. Then $\overline{a_0}$ and $\overline{b_0}$ are units of $\overline{R}$ and $a_1, \dots, a_l \in N(R)$. Since $N(R) \subseteq J(R)$ and $\overline{a_0}$ is a unit element of $\overline{R}$, we have that $a_0 \in U(R)$.

Conversely, let $a_0$ be a unit element and $a_1, \dots, a_l$ be nilpotent elements of $R$. Then $\sum_{i=1}^{l} a_iX_i \in N(A)$ by Proposition 2.26. Also, we get $N(A) \subseteq J(A)$ since $A$ is NI, and so $\sum_{i=1}^{l} a_iX_i \in J(A)$. Therefore, we have $f = \sum_{i=0}^{l} a_iX_i$ is a unit element of $A$. □

As a consequence of the characterization of the units and nilpotent elements of a skew PBW extension over weak $(\Sigma, \Delta)$-compatible rings, we obtain Corollary 2.29 and Proposition 2.30.

**Corollary 2.29.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$, then $U(A) = U(R) + N(R)A$.*

**Proposition 2.30.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$, then $A$ is NJ.*

*Proof.* Since $A$ is a NI ring, then $N(A) \subseteq J(A)$ by Proposition 1.49. Additionally, if $f$ is an element of $J(A)$ with $f = \sum_{i=0}^{m} a_iX_i$, we obtain that $1 + fx_n = 1 + \sum_{i=0}^{m} a_iX_ix_n$ is a unit element of $A$. Theorem 2.28 shows that the coefficients $a_0, a_1, \dots, a_m \in N(R)$, and hence $f \in N(A)$ by Proposition 2.26. Thus, we conclude $N(A) = J(A)$. □

About idempotent elements of skew PBW extensions over weak $(\Sigma, \Delta)$-compatible NI rings, the next result generalizes [52, Theorem 4.9].

**Theorem 2.31.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $f = \sum_{i=0}^{l} a_iX_i$ is an idempotent element of $A$, then*

*$a_i \in N(R)$, for each $1 \leq i \leq l$, and there exists an idempotent element $e \in R$ such that $\overline{a_0} = \overline{e} \in R/N(R)$.*

*Proof.* Let $f = \sum_{i=0}^{l} a_i X_i$ be an idempotent element of $A$ and consider the element $g = 1 - f = (1 - a_0) - \sum_{i=1}^{l} a_i X_i$. Then $fg = 0 \in N(R)$. Hence, by Proposition 1.48 (1), we have $a_i a_i = a_i^2 \in N(R)$ for $1 \leq i \leq l$ and $a_0(1 - a_0) \in N(R)$. The former means that $a_i \in N(R)$ for $i \geq 1$, and the last assertion implies that $a_0 - a_0^2 \in N(R)$. By [89, Theorem 21.28], there exists an idempotent $e \in R$ such that $a_0 - e \in N(R)$, that is, $\overline{a_0} = \overline{e} \in R/N(R)$. □

**Remark 2.32.** Note that in any ring $R$ if $e$ is and idempotent, then $1 - 2e$ is invertible since $(1 - 2e)^2 = 1$.

**Lemma 2.33.** *Let $R$ be any ring, $f, e \in \mathrm{Idem}(R)$ and $s \in N(R)$. If $f = e + s$ and $es = se$, then $s = 0$.*

*Proof.* If $s \neq 0$ then there exists $k \geq 2$ such that $s^k = 0$ and $s^{k-1} \neq 0$. Since $f$ is idempotent, $0 = f(1 - f) = (e + s)(1 - e - s) = s - 2es - s^2$. Thus $s^2 = (1 - 2e)s$, and multiplying by $s^{k-2}$ we have $0 = s^k = (1 - 2e)s^{k-1}$. Since $1 - 2e$ is invertible, it follows that $0 = s^{k-1}$, which is a contradiction. Hence $n = 0$. □

**Proposition 2.34.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$. If $e \in \mathrm{Idem}(R)$, then $e \in Z(A)$.*

*Proof.* Fix $1 \leq i \leq n$. By using Proposition 1.48 (2), there exists $u \in N(R)$ such that $\sigma_i(e) = e + u$. On the other hand, since $R$ is Abelian, we have $eu = ue$ and $\sigma_i(e) \in \mathrm{Idem}(R)$, which implies that $u = 0$ and $\sigma_i(e) = e$ by Lemma 2.33. Hence, $\delta_i(e) = \delta_i(e^2) = \sigma_i(e)\delta_i(e) + \delta_i(e)e = 2e\delta_i(e)$, i.e., $(1 - 2e)\delta_i(e) = 0$, whence $\delta_i(e) = 0$. Finally, since $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for all $1 \leq i \leq n$, then $e$ commutes with the $x_i$'s, and therefore $e \in Z(A)$. □

Next, we formulate a result that describes the idempotent elements of skew PBW extensions over Abelian NI rings. Theorem 2.35 generalizes [52, Theorem 4.10].

**Theorem 2.35.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$, and $f = \sum_{i=0}^{n} a_i X_i$ an element of $A$. If $f^2 = f$, then $f = a_0 \in \mathrm{Idem}(R)$.*

*Proof.* Let $f = \sum_{i=0}^{l} a_i X_i \in A$ such that $f^2 = f$. By using Proposition 2.31, we have $a_i \in N(R)$, for each $1 \leq i \leq l$, and there exist an idempotent element $e \in R$ and a nilpotent element $b \in R$ such that $a_0 = e + b$. In this way, if we consider $h = b + \sum_{i=1}^{l} a_i X_i \in A$, then $f = e + h$. Finally, we have $h \in N(A)$ by Proposition 2.26 and $he = eh$ by using Proposition 2.34. Therefore, we conclude $h = 0$ by Lemma 2.33, and so $f = a_0$. □

The next corollary extends [52, Corollaries 4.11 and 4.12].

**Corollary 2.36.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$, then* $\text{Idem}(A) = \text{Idem}(R)$*, and so $A$ is an Abelian ring.*

In [52, Theorem 4.14], the von Neumann regular elements of skew PBW extensions over right duo rings were characterized. Next, we formulate a generalization of this theorem.

**Theorem 2.37.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$, then* $\text{vnr}(A)$ *consists of the elements of the form $\sum_{i=0}^{m} a_iX_i$ where $a_0 = ue$, $a_i \in eN(R)$ for every $i \geq 1$, some $u \in U(R)$ and $e \in \text{Idem}(R)$.*

*Proof.* By Corollary 2.36, $A$ is an Abelian ring. Additionally, $f \in \text{vnr(A)}$ if and only if $f = ue$, for some $u \in U(A)$ and $e \in \text{Idem}(A)$ [55, Proposition 4.2]. Hence, the result follows from Corollaries 2.29 and 2.36. □

As a consequence of [13, Theorem 1 and Lemma 2], we obtain the following description of the $\pi$-regular elements over Abelian rings.

**Proposition 2.38.** *Let $R$ be an Abelian ring. Then $r$ is a $\pi$-regular element of $R$ if and only if there exist $e \in \text{Idem}(R)$ and $u \in U(R)$ such that $er = eu$ and $(1 - e)r \in N(R)$.*

By using the previous proposition, we can describe the $\pi$-regular elements in a skew PBW extension over weak $(\Sigma, \Delta)$-compatible and Abelian NI ring. Theorem 2.39 generalizes [52, Theorem 4.15].

**Theorem 2.39.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$, then*

$$\pi - r(A) = \left\{ \sum_{i=0}^{l} a_iX_i \in A \mid a_0 \in \pi - r(R),\ a_i \in N(R),\ \textit{for } i \geq 1 \right\}.$$

*Proof.* Let $f = \sum_{i=0}^{l} a_iX_i$ be an element of $\pi - r(A)$. Since $A$ is Abelian, there exist elements $e \in \text{Idem}(A) = \text{Idem}(R)$ and $u \in U(A)$ such that $ef = eu$ and $(1 - e)f \in N(A)$, by Proposition 2.38. Then $ea_0 = eu'$ and $ea_i \in N(R)$, for some $u' \in U(R)$ and for all $1 \leq i \leq l$ by Theorem 2.28. Additionally, we have $(1 - e)a_i \in N(R)$, for all $0 \leq i \leq l$, by Proposition 2.26. Thus, Proposition 2.38 shows that $a_0 \in \pi - r(R)$ and $a_i \in N(R)$, for $1 \leq i \leq l$.

On the other hand, suppose that $a_0 \in \pi - r(R)$ and $a_i \in N(R)$, for all $i \geq 1$. By using Proposition 2.38, there exist $e \in \text{Idem}(R)$ and $u \in U(R)$ such that $ea_0 = eu$ and $(1 - e)a_0 \in N(R)$. This implies that $ef = ea_0 + \sum\limits_{i=1}^{l} ea_iX_i = e\left(u + \sum\limits_{i=1}^{n} a_iX_i\right) = eu'$ and $(1 - e)f = \sum\limits_{i=0}^{l}(1 - e)a_iX_i$ where $u' = u + \sum_{i=1}^{l} a_iX_i$. Since $N(R)$ is an ideal of $R$, $(1 - e)a_i \in N(R)$, and so $(1 - e)f \in N(A)$ by Proposition 2.26, and $u' \in U(A)$ by Theorem 2.28. Therefore, Proposition 2.38 guarantees that $f \in \pi - r(A)$. □

Theorem 2.40 extends [52, Theorem 4.16].

**Theorem 2.40.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$, then* $\text{vnl}(A)$ *consists of the elements of the form $\sum_{i=0}^{l} a_i X_i$, where either $a_0 = ue$ or $a_0 = 1 - ue$, $a_i \in eN(R)$, for every $i \geq 1$, some element $u \in U(R)$ and $e \in \text{Idem}(R)$.*

*Proof.* It follows from Theorem 2.37 and [55, Theorem 6.1 (2)]. □

In [52, Theorem 4.17], the clean elements of skew PBW extensions over right duo rings were characterized. We present a generalization of this result.

**Theorem 2.41.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and Abelian NI ring $R$. Then*

$$\text{cln}(A) = \left\{ \sum_{i=1}^{l} a_i X_i \in A \mid a_0 \in \text{cln}(R),\ a_i \in N(R) \right\}.$$

*Proof.* The result follows from Proposition 1.49 and Corollaries 2.29 and 2.36. □

## 2.3 Gelfand and strongly harmonic rings

Mulvey [116] introduced the *strongly harmonic rings* with the aim of generalizing the Gelfand duality from $C^*$-algebras to rings (not necessarily commutative). Borceux and Van den Bossche [25] modified the definition of strongly harmonic rings and defined the *Gelfand rings*. A ring $R$ is called *Gelfand* (resp. *strongly harmonic*) if for each pair of distinct maximal right ideals (resp. maximal ideals) $M_1$, $M_2$ of $R$, there are right ideals (resp. ideals) $I_1$, $I_2$ of $R$ such that $I_1 \not\subseteq M_1$, $I_2 \not\subseteq M_2$ and $I_1 I_2 = 0$. Equivalently, $R$ is a Gelfand ring (resp. strongly harmonic) if for each pair of distinct maximal right ideals (resp. maximal ideals) $M_1$, $M_2$ of $R$, there are elements $r \notin M_1$, $s \notin M_2$ of $R$ such that $rRs = 0$. Gelfand rings and strongly harmonic rings have been investigated by different authors such as Borceux et al. [25, 26], Carral [30], Demarco and Orsatti [38], Mulvey [114, 115, 116], Sun [166, 167], Zhang et al. [186]. Additionally, Gelfand rings are tied to the Zariski topology over a ring, which allows to characterize different properties of the prime spectrum and the maximal spectrum of a ring [2].

Continuing with the study of Gelfand rings and their relationship with topological spaces, Mulvey [114] obtained a generalization of Swan's theorem concerning vector bundles over a compact topological space. He established an equivalence between the category of modules over a Gelfand ring and the category of modules over the corresponding compact ringed space. The algebraic $K$-theory of commutative Gelfand rings has been studied by Carral [30] by showing relationships between the stable rank over Gelfand rings and the covering dimension of the maximal ideal space. These results are analogous to those corresponding for Noetherian rings with respect to the Krull dimension.

Our purpose in this section is to study Gelfand and Harmonic rings in the setting of skew PBW extensions over weak $(\Sigma, \Delta)$-compatible rings.

**Proposition 2.42.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. Then $A/J(A)$ is a Gelfand ring (resp. strongly harmonic) if and only if for each pair of distinct maximal right ideals (resp. maximal ideals) $M_1$, $M_2$ of $A$, there exist elements of $R$, $r \notin M_1$ and $s \notin M_2$ such that $rRs \subseteq N(R)$.*

*Proof.* Let $M_1$, $M_2$ be a pair of distinct maximal right ideals of $A$. By using that $A/J(A)$ is a Gelfand ring, there exist elements $f, g$ of $A$ such that $f = \sum_{i=0}^{m} a_iX_i \notin M_1$, $g = \sum_{j=0}^{l} b_jX_j \notin M_2$ and $fAg \subseteq J(A) = N(A)$. Since $f \notin M_1$, $a_t \notin M_1$ for some coefficient of $f$ with $1 \leq t \leq m$. In this way, we also have $b_s \notin M_2$ for some coefficient of $g$ with $1 \leq s \leq l$. Since $fRg \subseteq fAg \subseteq N(A)$, then $fcg \in N(A)$ and $a_icb_j \in N(R)$, for all $i, j$, and for every $c \in R$, by Proposition 2.26. Thus, if we consider $r = a_t$ and $s = b_s$, the result follows.

For the other implication, let $M_1$, $M_2$ be a pair of distinct maximal right ideals of $A$. Since $rRs \subseteq N(R)$, for some elements $r \notin M_1$ and $s \notin M_2$, then $rAs \subseteq N(A)$, by Proposition 2.26. This implies that $A/J(A)$ is a Gelfand ring.

The proof of the strongly harmonic case is analogous. □

**Proposition 2.43.** *If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$, then $A$ is not a local ring.*

*Proof.* Suppose that $A$ is a local ring. By using Theorem 2.28, we have that $x_n$ is not an unit of $A$. In this way, since $J(A) = N(A)$ we have that $x_n \in N(A)$, which contradicts Proposition 2.26. Hence $A$ is not local. □

**Example 2.44.**

(i) In the Example 1.41, we have that the Ore extension $R_2[x; \sigma, \delta]$ is not a local ring by Proposition 2.43.

(ii) Consider the skew PBW extension $A = \sigma(\mathrm{S}_2(\mathbb{Z}))\langle x, y, z\rangle$ over the ring $\mathrm{S}_2(\mathbb{Z})$ in Example 1.42. Proposition 2.43 shows that $A$ is not a local ring.

**Theorem 2.45.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $N(R)$ is a prime ideal of $R$, then $A/J(A)$ is not a Gelfand ring.*

*Proof.* Suppose that $A/J(A)$ is a Gelfand ring. By using Proposition 2.43, there exist at least two maximal right ideals $M_1, M_2$ of $A$. Additionally, since $A/J(A)$ is a Gelfand ring, there exist $r, s \in R$ such that $r \notin M_1$, $s \notin M_2$ and $rRs \subseteq N(R)$ by Proposition 2.42. Since $N(R)$ is a prime ideal of $R$, we have $r \in N(R)$ or $s \in N(R)$. Finally, since $N(R) \subseteq N(A) = J(A) \subseteq M_1, M_2$, then $r \in M_1$ or $s \in M_2$ which is a contradiction. Hence, we conclude $A/J(A)$ is not a Gelfand ring. □

**Corollary 2.46.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $N(R)$ is a prime ideal of $R$, then $A/N_*(A)$ is not a Gelfand ring.*

**Example 2.47.** If we consider the Theorem 2.45 and the Corollary 2.46, we conclude $A/J(A)$ and $A/N_*(A)$ are not Gelfand rings where $A$ is the Ore extension $R_2[x;\sigma,\delta]$ over the ring of upper triangular matrices $R_2$ in Example 1.41, or the skew PBW extension $A = \sigma(\mathrm{S}_2(\mathbb{Z}))\langle x, y, z\rangle$ over the ring $\mathrm{S}_2(\mathbb{Z})$ in Example 1.42.

**Theorem 2.48.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $N(R)$ is a prime ideal then $A/J(A)$ is strongly harmonic if and only if $A$ has a unique maximal ideal.*

*Proof.* If $A$ has a unique maximal ideal, then $A/J(A)$ has a unique maximal ideal and therefore $A/J(A)$ is strongly harmonic. If $A$ has at least two maximal ideals, the proof of the Theorem 2.45 guarantees that $A/J(A)$ is not a strongly harmonic ring. □

## 2.4 Weak Zariski and Zariski topologies

Following Zhang et al. [188], let $\mathrm{Id}_r(R)$ (resp. $\mathrm{Id}_l(R)$, $\mathrm{Id}_2(R)$) be the set of all right ideals (resp. left ideals, ideals) of $R$, and we write $\mathrm{Spec}_r(R)$ (resp. $\mathrm{Spec}_l(R)$, $\mathrm{Spec}(R), \mathrm{Max}_r(R)$, $\mathrm{Max}_l(R)$, $\mathrm{Max}(R)$, $\mathrm{Cspec}(R)$, and $\mathrm{Minspec}(R)$) for the set of all right prime ideals (resp. all left prime ideals, all prime ideals, all maximal right ideals, all maximal left ideals, all maximal ideals, all completely prime ideals, all minimal prime ideals) of $R$. Clearly, $\mathrm{Max}(R), \mathrm{Cspec}(R), \mathrm{Minspec}(R) \subseteq \mathrm{Spec}(R) = \mathrm{Spec}_r(R) \cap \mathrm{Spec}_l(R)$. Notice that, in general, the *prime right spectrum* $\mathrm{Spec}_r(R)$ in this sense is different from the *right spectrum* $\mathrm{Spec}_r(R)$ in Rosenberg [149, Section 1.5] since the right radical $\mathrm{rad}_r(R)$ (i.e., the intersection of all right ideals in the right spectrum $\mathrm{Spec}_r(R)$ of any ring $R$ coincides with its Levitzki radical $L(R)$ by [149, Theorem 4.10.2], and in general, Levitzki radical $L(R)$ and prime radical $N_*(R)$ are distinct.

Let $U_r(I) = \{P \in \mathrm{Spec}_r(R) \mid P \not\supseteq I\}$, where $I \in \mathrm{Id}_r(R)$. Then $U_r(0)$ is just the empty set and $U_r(R)$ is $\mathrm{Spec}_r(R)$. If $\zeta(R)$ denotes the collection of all subsets

$$U = \bigcup_{\alpha\in\Gamma} \left[\bigcap_{i=1}^{s_\alpha} U_r(L_i^\alpha)\right]$$

of $\mathrm{Spec}_r(R)$, where $L_i^\alpha \in \mathrm{Id}_r(R)$ and $s_\alpha (\alpha \in \Gamma)$ are all positive integers, then $\zeta(R)$ contains the empty set and $\mathrm{Spec}_r(R)$, and $\zeta(R)$ is closed under arbitrary unions and finite intersections. Therefore, $\mathrm{Spec}_r(R)$ is called a space with a *weak Zariski topology* [186, 187]. Analogously, we endow $\mathrm{Spec}_l(R)$ with a weak Zariski topology.

The Zariski topology on Spec(R) has a sub-base $\{U(I) \mid I \in \mathrm{Id}_2(R)\}$, where $U(I) = U_r(I) \cap \mathrm{Spec}(R)$. It is easy to see that this sub-base is a base. Similarly, $\mathrm{Spec}_r(R)$ has a sub-base $\{U_r(I) \mid I \in \mathrm{Id}_r(R)\}$, but that sub-base does not form a base for $\mathrm{Spec}_r(R)$. In the case where this sub-base is a base, i.e., if for any $L_1, L_2 \in \mathrm{Id}_r(R)$, there is at least one $H \in \mathrm{Id}_r(R)$ with $U_r(L_1) \cap U_r(L_2) = U_r(H)$, then we say that $\mathrm{Spec}_r(R)$ is a space with a Zariski topology and $R$ is called a *right top ring* (for more details, see Zhang et al., [186, 187] and references therein). If a ring $R$ is not right quasi-duo, then $\mathrm{Spec}_r(R)$ is a space with a weak Zariski topology but not with Zariski topology [187, Example 2.3]. Similarly we have the left version.

For $U_r(I)$ above, let $V_r(I) = \text{Spec}_r(R) \setminus U_r(I)$, $U(I) = U_r(I) \cap \text{Spec}(R)$, and $V(I) = V_r(I) \cap \text{Spec}(R)$, where $I$ is a right ideal of $R$. $U_l(I)$ and $V_l(I)$ are similarly defined. Zhang et al., [187] called an idempotent $e \in R$ *clopen* if $e \notin P$ implies $1-e \in P$, for any prime ideal $P$ of $R$, i.e., $U(ReR) = V(R(1-e)R)$. A ring $R$ is called *clopen* if every its idempotent is clopen. As it is clear, commutative rings are clopen. In [187, Theorem 3.6], it was shown that right top rings, left top rings, 2-primal rings, and Abelian rings are all clopen.

McDonald [111, Theorem IV. 3] proved that for a commutative ring $R$ there is a natural bijection between the set of all idempotents in $R$ and the clopen sets in Spec($R$). Zhang et al., [187, Proposition 3.7] showed that for an Abelian ring $R$, the Boolean lattice of all idempotents in $R$ and the lattice of all clopen sets in $\text{Spec}_r(R)$ are isomorphic. Recall that a topological space is said to be *connected* if it is not the union of two disjoint nonempty closed sets. The following results were proved in [187, Theorems 3.5, 3.9 and 3.11]:

**Proposition 2.49 ([187, Theorem 1.1]).** *Let $R$ be any ring. Then the following results hold:*

(A) *For any clopen subset $U$ in* $\text{Spec}_r(R)$ *(resp.* $\text{Minspec}(R) \cap \text{Max}_r(R)$, $\text{Spec}(R)$, $\text{Minspec}(R) \cup \text{Max}(R)$*), there exists an idempotent element $e$ in $R$ such that* $U = U_r(eR) \cap \text{Spec}(R) = V_r((1-e)R) \cap \text{Spec}(R)$ *(resp.* $\text{Minspec}(R) \cap \text{Max}_r(R)$, $\text{Spec}(R)$, $\text{Minspec}(R) \cup \text{Max}(R)$*).*

(B) *The following statements are equivalent:*

  (1) $\text{Spec}(R)$ *is connected.*

  (2) $\text{Spec}_r(R)$ *is connected.*

  (3) $\text{Spec}_l(R)$ *is connected.*

  (4) *The only clopen idempotents in $R$ are 0 and 1.*

  (5) $\text{Minspec}(R) \cup \text{Max}(R)$ *is connected.*

  (6) $\text{Minspec}(R) \cup \text{Max}_r(R)$ *is connected.*

**Proposition 2.50.** *If $R$ is a domain, then* $\text{Spec}(R)$ *is connected.*

*Proof.* Since idempotent elements different than the identity of $R$ are zero divisor, then the only idempotent elements are 0 and 1. □

**Proposition 2.51.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a domain $R$. Then* $\text{Spec}(R)$ *and* $\text{Spec}(A)$ *are connected.*

*Proof.* By Proposition 1.31 $A$ is a domain, so the result follows from Proposition 2.50. □

**Proposition 2.52.** *If $R$ is a skew Armendariz ring, then $R$ is a clopen ring.*

*Proof.* Since $R$ is a weak skew Armendariz ring, by [142, Proposition 4.9] $R$ is Abelian. From [187, Theorem 3.6], it follows that $R$ is clopen. □

**Proposition 2.53.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over $R$. If $R$ is a skew Armendariz ring, then $A$ is a clopen ring.*

*Proof.* By [142, Proposition 4.10], $A$ is Abelian, and from [187, Theorem 3.6], we deduce that $A$ is clopen. □

**Proposition 2.54.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a skew Armendariz ring $R$, then* $\text{Idem}(R) = \text{Idem}(A)$.

*Proof.* It follows from [142, Proposition 4.8]. □

Propositions 2.49, 2.52, 2.53 and 2.54 imply the next result.

**Proposition 2.55.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a skew Armendariz ring $R$, then the following statements are equivalent:*

(1) $\text{Spec}(R)$ *is connected.*

(2) $\text{Spec}(A)$ *is connected.*

(3) *The only idempotents in $R$ are 0 and 1.*

**Proposition 2.56.** *Let $A$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible Abelian and NI ring $R$. Then the following statements are equivalent:*

(1) $\text{Spec}(R)$ *is connected.*

(2) $\text{Spec}(A)$ *is connected.*

(3) *The only idempotents in $R$ are 0 and 1.*

*Proof.* By Corollary 2.36, $\text{Idem}(A) = \text{Idem}(R)$ and $A$ is Abelian. Thus, $R$ and $A$ are clopen rings. The result follows from Proposition 2.49. □

A topological space is called *normal* if given any disjoint closed sets $E$ and $F$, there are open sets $U \supseteq E$ and $V \supseteq F$ that are also disjoint. Related to the normality of the spectrum, Zhang et al., [188] considered several conditions. Some of these are the following:

(C1) $R/N_*(R)$ is a Gelfand ring;

(C2) $R/N_*(R)$ is a strongly harmonic ring;

(C6) $\text{Max}(R)$ is a retract of $\text{Spec}(R)$;

(C7) $\text{Spec}(R)$ is a normal space;

(C8) $R$ is a strongly harmonic ring;

(C9) Max($R$) is a retract of Minspec($R$) $\cup$ Max($R$);

(C10) Minspec(R) $\cup$ Max($R$) is a normal space;

(C11) Max($R$) is a Hausdorff space;

(R2) $\text{Max}_r(R)$ is a retract of $\text{Spec}_r(R)$;

(R3) $\text{Spec}_r(R)$ is a normal space;

(R4) $\text{Max}_r(R)$ is a retract of Minspec($R$) $\cup$ Max($R$);

(R5) Minspec($R$) $\cup$ $\text{Max}_r(R)$ is a normal space;

(R6) $R$ is a right quasi-duo ring and Spec($R$) is a normal space;

(R7) $R$ is a right quasi-duo ring and Cspec($R$) is a normal space.

Demarco and Orsatti [38] proved that for a commutative ring, (C6) - (C7) are equivalent. Simmons [153, 154] added the further equivalent condition (C8).

Later, Sun [165, 167, 166] proved the following facts:

- (C6) - (C7) are equivalent for the class of 2-primal rings. (C6) - (C8) are equivalent for the class of symmetric rings;
- For the class of neo-commutative rings (Kaplansky [82] defined a neo-commutative ring $R$ to be one where the product of any two finitely generated ideals of $R$ is finitely generated), (R2) - (R3) are equivalent and (C6) - (C7) are equivalent.

Zhang et al., [188] proved the following results, where (L2)-(L7) are the left dual conditions of (R2)-(R7):

- (C1), (R2) - (R6), and (L2) - (L6) are equivalent [188, Theorem 3.6].
- (C2) $\Leftrightarrow$ (C7) $\Leftrightarrow$ (C10) $\Leftrightarrow$ the conjuction of (C9) and (C11) [188, Theorem 3.6].
- For the class of 2-primal rings, (C1), (R2) - (R7), and (L2) - (L7) are equivalent and (C2), (C6) - (C7), and (C9) - (C10) are equivalent [188, Theorem 4.2].

Due to its importance, we highlight the following equivalencies of those shown by Zhang et al., [188].

**Proposition 2.57 ([188, Theorem 3.4]).** *Let $R$ be any ring. Then the following statements are equivalent:*

(1) *$R/N_*(R)$ is a Gelfand ring.*

(2) *$\text{Spec}_r(R)$ is a normal space.*

(3) $\mathrm{Spec}_l(R)$ *is a normal space.*

**Proposition 2.58 ([188, Theorem 3.5]).** *Let $R$ be any ring. Then the following statements are equivalent:*

(1) $R/N_*(R)$ *is a strongly harmonic ring;*

(2) $\mathrm{Spec}(R)$ *is a normal space.*

**Proposition 2.59.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a ring $R$. If $R$ is a domain then $\mathrm{Spec}_r(A)$ is not a normal space.*

*Proof.* Since $R$ is a domain, then $J(A) = N(A) = N_*(A) = 0$ by Proposition 1.33. Hence, $A = A/N_*(A)$.

Suppose that $A$ is a Gelfand ring. If there exist two different maximal right ideals $M_1, M_2$, then there are right ideals $I_1 \not\subseteq M_1$ and $I_2 \not\subseteq M_2$ such that $I_1 I_2 = 0$. Since $A$ is a domain, then $I_1 = 0 \subseteq M_1$ or $I_2 = 0 \subseteq M_2$, which is a contradiction. Then $A$ has a unique maximal right ideal $M$, i.e. $A$ is a local ring. Since $x_n$ is not a unit, then $x_n \in M = J(A) = 0$, a contradiction. We conclude that $A$ is not a Gelfand ring, and the result follows from Proposition 2.57. □

**Proposition 2.60.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $N(R)$ is a prime ideal of $R$, then $\mathrm{Spec}_r(A)$ is not a normal space.*

*Proof.* The assertion follows from Corollary 2.46 and Proposition 2.57. □

**Proposition 2.61.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a weak $(\Sigma, \Delta)$-compatible and NI ring $R$. If $N(R)$ is a prime ideal of $R$, then $\mathrm{Spec}(A)$ is a normal space if and only if $A$ has a unique maximal ideal.*

*Proof.* If $A$ has a unique maximal, then $A/N_*(A)$ is a strongly harmonic ring, so the result follows from Proposition 2.58.

Conversely, if $A$ has more than one maximal ideal, then by Theorem 2.48 $A/J(A)$ is not strongly harmonic, and neither is $A/N_*(A)$. □

## 2.5 Future work

In this chapter, we have studied NI and NJ rings, kinds of elements, Gelfand and strongly harmonic rings, and the noncommutative spectrum for several families of semi-graded rings defined by endomorphisms and derivations. However, for the another examples of semi-graded rings, the questions about these properties are open. In this sense, a natural task is to investigate what happens in these families of noncommutative rings.

On the other hand, since some people have studied the notion of compatibility for modules over Ore extensions and skew PBW extensions (e.g., [4, 8, 103, 122, 137]), we think as future work to investigate a classification of several types of elements in extended modules over skew PBW extensions. Following this idea and the notions of *strongly harmonic and Gelfand modules* introduced by Medina-Bárcenas et al. [112], it will be interesting to study these modules over semi-graded rings.

# CHAPTER 3

## Schematicness of semi-graded rings

In this chapter, we investigate the schematicness and the Serre-Artin-Zhang-Verevkin theorem for semi-graded rings.

With this aim, Section 3.1 presents a little of history of schematic algebras defined by Van Oystaeyen and Willaert [174] in the setting of connected $\mathbb{N}$-graded rings. We recall some key definitions, important results and remarkable examples of these algebras.

In Section 3.2, we consider the Ore polynomials of higher order generated by homogeneous relations (Section 1.2.9). We formulate Theorems 3.14, 3.15, 3.16 and 3.21 that contain necessary or sufficient conditions to guarantee the schematicness of this family of polynomials.

Section 3.3 presents a noncommutative scheme theory for semi-graded rings following the ideas introduced by Van Oystaeyen and Willaert [174], but now in the setting of non-$\mathbb{N}$-graded rings which are not necessarily connected. With this theory, we prove the Serre-Artin-Zhang-Verevkin theorem (Theorem 3.62) for several families of non-$\mathbb{N}$-graded algebras which include different kinds of noncommutative rings appearing in ring theory and noncommutative algebraic geometry. For the skew PBW extensions, Propositions 3.65 and 3.66 establish sufficient conditions to guarantee their schematicness. Examples 3.67 and 3.68 show that the theory presented by Lezama [96, 99] about Serre-Artin-Zhang-Verevkin theorem and the one developed in this chapter are independent.

The results in this chapter contribute to the study of algebraic geometry for noncommutative objects (e.g., [61, 95, 101, 132, 133] and references therein).

Finally, Section 3.4 presents some ideas for a future work.

## 3.1 Serre's theorem and graded schematic algebras

We recall briefly some notions of algebraic geometry which are key in the proof of *Serre's theorem.*

Following Hartshorne [54], if $C = \Bbbk \oplus C_1 \oplus C_2 \oplus \cdots$ is a positively graded commutative Noetherian ring generated in degree one, consider $Y = \text{Proj } C$ and $Y(f) = \{\mathfrak{p} \in Y \mid f \notin \mathfrak{p}\}$, the Zariski open set corresponding to a homogeneous element $f \in C$. It is well-known that there is a finite subset $\{f_i \mid f_i \in C_1\}$ such that $Y = \bigcup_i Y(f_i)$. Equivalently, for every choice of $d_i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $(C_+)^n \subseteq \sum_i C f_i^{d_i}$. In this way, for any finitely generated graded $C$-module $M$, we have

$$\begin{aligned}\Gamma_*(M) &:= \bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \widetilde{M(n)}) \\ &= Q_{\kappa_+}(M) = \varprojlim_i Q_{f_i}(M),\end{aligned}$$

where $\widetilde{M(n)}$ denotes the sheaf of modules associated to the shifted module $M(n)$ and $Q_{f_i}(M)$ is the localization of $M$ at $\{1, f_i, f_i^2, \dots\}$. Of course,

$$Q_f(M) = \varprojlim_i Q_{ff_i}(M),$$

where the inverse systems are defined as $q \leq h$ if and only if $Y(g) \subseteq Y(h)$. This is precisely the key fact to prove *Serre's theorem*: the category of coherent $\mathcal{O}_Y$-modules is equivalent with a certain quotient category.

In the noncommutative setting, for a *noncommutative positively graded Noetherian* $\Bbbk$-algebra $R = \Bbbk \oplus R_1 \oplus R_2 \oplus \cdots$ with $R = \Bbbk[R_1]$ (notice that $R$ is connected, that is, $R_0 = \Bbbk$), Van Oystaeyen and Willaert [174] presented his interpretation of Serre's theorem for algebras with enough Ore sets called *schematic algebras*. With the aim of presenting the key ideas developed by them, we start by recalling some notions of torsion theory that we will use freely throughout the chapter. For more details, we refer to Goldman [45], Stenstrom [158] or Van Oystaeyen [172].

**Definition 3.1 ([174, Section 2]).** Let $\mathscr{L}$ be a set of left ideals of an arbitrary ring $R$. $\mathscr{L}$ is said to be *a filter* if it satisfies the following conditions:

$T_1$: If $I \in \mathscr{L}$ and $I \subseteq J$, then $J \in \mathscr{L}$.

$T_2$: If $I, J \in \mathscr{L}$, then $I \cap J \in \mathscr{L}$.

$T_3$: If $I \in \mathscr{L}$ and $a \in R$, then $(I : a) := \{r \in R \mid ra \in I\} \in \mathscr{L}$.

The functor $\kappa : R - \text{Mod} \to R - \text{Mod}$ defined by

$$\kappa(M) = \{m \in M \mid \text{ there exists } I \in \mathscr{L} \text{ with } Im = 0\},$$

is a *left exact preradical*, that is, a left exact subfunctor of the identity functor on the category $R-\text{Mod}$. A module $M$ satisfying $\kappa(M) = M$ is called a *$\kappa$-torsion module*, and if $\kappa(M) = 0$, then $M$ is said to be a *$\kappa$-torsion-free module*. It is straightforward to see that the family of torsion modules are closed under quotient objects and coproducts, while the torsion-free modules are closed under subobjects and products.

The filter $\mathscr{L}$ is called *idempotent* (also called a *Gabriel topology*) when it satisfies the following condition:

$T_4$: If $I \lhd_l R$ and there exists $J \in \mathscr{L}$ such that for all $a \in J$ the relation $(I : a) \in \mathscr{L}$ holds, then $I \in \mathscr{L}$.

Condition $T_4$ implies that $\mathscr{L}$ is closed under products and that the functor $\kappa$ is a radical, that is, $\kappa(M/\kappa(M)) = 0$, for all $M \in R-\text{Mod}$.

**Proposition 3.2.** *If $R$ is a left Noetherian ring and $J_1 \supseteq J_2 \supseteq \cdots$ is a descending chain of two-sided ideals of $R$, then the set*

$$\mathscr{A} = \{I \lhd_l R \mid \text{ there exist elements } n, m \in \mathbb{N} \text{ with } (J_m)^n \subseteq I\},$$

*is an idempotent filter.*

*Proof.* $T_1$: If $(J_m)^n \subseteq I$ and $I \subseteq I'$, then it is clear that $(J_m)^n \subseteq I'$.

$T_2$: If $I, I' \in \mathscr{A}$, then there exist elements $n_1, n_2, m_1, m_2 \in \mathbb{N}$ such that $(J_{m_1})^{n_1} \subseteq I$ and $(J_{m_2})^{n_2} \subseteq I'$. If we consider $n := \max\{n_1, n_2\}$ and $m := \max\{m_1, m_2\}$, then it follows that $(J_m)^n \subseteq I \cap I'$.

$T_3$: If $I \in \mathscr{A}$, then there exist elements $n, m \in \mathbb{N}$ such that $(J_m)^n \subseteq I$. Fix $a \in R$ and let $r \in (J_m)^n$. Since $(J_m)^n$ is an ideal of $R$, then $ra \in (J_m)^n \subseteq I$, and so $r \in (I : a)$, that is, $(J_m)^n \subseteq (I : a)$.

$T_4$: Let $I \lhd_l R$ and $J \in \mathscr{A}$ such that for all $a \in J$, $(I : a) \in \mathscr{A}$. Since $J \in \mathscr{A}$, there exist elements $n, m \in \mathbb{N}$ with $(J_m)^n \subseteq J$. By assumption $R$ is left Noetherian, so $(J_m)^n$ is finitely generated by some elements $a_1, \dots, a_l$. Notice that $a_i \in J$, for $1 \leq i \leq l$, whence $(I : a_i) \in \mathscr{A}$. In this way, there exist elements $k_i, j_i \in \mathbb{N}$ such that $(J_{j_i})^{k_i} \subseteq (I : a_i)$. If $k := \max\{k_i\}_{1\leq i\leq l}$ and $j := \max\{j_i\}_{1\leq i\leq l}$, then $(J_j)^k \subseteq (I : a_i)$, for every $1 \leq i \leq l$.

Let $r \in (J_j)^k$ and $s \in (J_m)^n$. There exist elements $r_1, \dots, r_l \in R$ such that $s = \sum_{i=1}^{l} r_i a_i$, and so $rs = \sum_{i=1}^{l} r r_i a_i$. Since $(J_j)^k$ is an ideal of $R$, we have that $r r_i \in (J_j)^k$, for all $i$. Thus $r r_i a_i \in I$, whence $rs \in I$. It follows that $(J_{j+m})^{k+n} \subseteq (J_j)^k (J_m)^n \subseteq I$.

□

Consider a ring $R$, $\mathscr{L}$ an idempotent filter of left ideals of $R$ and its associated radical $\kappa$. For an $R$-module $M$, we recall the quotient module $Q_\kappa(M)$ of $M$ (Definition 3.4). With this aim, we introduce the Definition 3.3.

**Definition 3.3.** Let $M \in R - \text{Mod}$. Consider the family $\Omega_M$ of pairs $(I, f)$ with $I \in \mathscr{L}$ and $f : I \to M$ an $R$-homomorphism. We define the relation $\sim$ on $\Omega_M$ as $(I_1, f_1) \sim (I_2, f_2)$ if and only if there exists an element $J \in \mathscr{L}$ such that $J \subseteq I_1 \cap I_2$ and $f|_J = g|_J$.

It is straightforward to see that $\sim$ is an equivalence relation. The equivalence class of the element $(I, f)$ is denoted as $[I, f]$, and the set of equivalence classes will be written as $M_{\mathscr{L}}$. For two elements $[I, f], [J, g] \in M_{\mathscr{L}}$, we define their sum as $[I, f]+[J, g] = [I\cap J, f+g]$. It is easy to see that this sum is well defined and that $(M_{\mathscr{L}}, +)$ is an Abelian group.

It is also easy to see that if $I, J \in \mathscr{L}$ and $f \in \text{Hom}(I, R)$, then $f^{-1}(J) \in \mathscr{L}$. In this way, when we take elements $[I, f] \in R_{\mathscr{L}}$ and $[J, g] \in M_{\mathscr{L}}$, we can define the product of these elements as $[I, f] \cdot [J, g] = [f^{-1}(J), g \circ f]$. Notice that this product is well defined, and so $R_{\mathscr{L}}$ is actually a ring with identity $[R, \text{id}_R]$. Thus, $M_{\mathscr{L}}$ is a left $R_{\mathscr{L}}$-module.

Let $m \in M$. We define the application $\beta(m) : R \to M$ given by $\beta(m)(r) = rm$. It is well-known that $\beta : M \to \text{Hom}(R, M)$ is an isomorphism of $R$-modules. If we consider $\varphi_M : M \to M_{\mathscr{L}}$ defined by $\varphi_M(m) = [R, \beta(m)]$, then it follows that $\varphi_R$ is a ring homomorphism, and so we can consider $R_{\mathscr{L}}$ and $M_{\mathscr{L}}$ as $R$-modules with the action given by $r[I, f] := [R, \beta(r)][I, f]$. Note that $\varphi_M$ is actually a homomorphism of $R$-modules. Since $\text{Ker}(\varphi_M) = \kappa(M)$, the fundamental isomorphism theorem guarantees that $\varphi_M(M) \cong M/\kappa(M)$. In this way, if $\kappa(M) = 0$ then we can embed $M$ into $M_{\mathscr{L}}$.

For an element $\xi \in M_{\mathscr{L}}$ given by $\xi = [I, f]$, and an element $a \in I$, notice that $a\xi = [R, \beta(f(a))] = \varphi_M(f(a))$, which shows that $I\xi \subseteq \varphi_M(M)$, that is, $\text{Coker}(\varphi_M)$ is a $\kappa$-torsion module.

Considering the notation and terminology above, we present the definition of the quotient module of an object $M$ in $R - \text{Mod}$.

**Definition 3.4.** The *quotient module of* $M$ with respect to $\kappa$ is defined as $Q_\kappa(M) = (M/\kappa(M))_{\mathscr{L}}$. Since $\mathscr{L}$ is idempotent, it follows that $\kappa\left(M/\kappa(M)\right) = 0$. Hence, we can embed $M/\kappa(M)$ into $Q_\kappa(M)$.

Equivalently, the *quotient module* of $M$ with respect to $\kappa$ is given by

$$Q_\kappa(M) = \varinjlim_{I\in\mathscr{L}} \text{Hom}_R(I, M/\kappa(M)).$$

$Q_\kappa(M)$ turns out to be a module over the ring $Q_\kappa(R)$.

Following [158], recall that an $R$-module $E$ is $\kappa$-*injective* if for every $R$-module $M$ and each submodule $N$ such that $\kappa(M/N) = M/N$, every $R$-homomorphism $N \to E$ can be extended to an $R$-homomorphism $M \to E$. We say that $E$ is $\kappa$-*closed* (also known as *faithfully* $\kappa$-*injective*) if the extension of the homomorphism is unique. It is straightforward to see that $E$ is $\kappa$-closed if and only if $E$ is $\kappa$-injective and $\kappa$-torsion-free. By using these notions, we can characterize $Q_\kappa(M)$ in the following way: $Q_\kappa(M)$ is the unique $\kappa$-closed module containing $N = M/\kappa(M)$ such that $Q_\kappa(M)/N$ is $\kappa$-torsion.

**Example 3.5 ([174, p. 111]).** (i) Consider $S$ a left Ore set in an arbitrary ring $R$.

The set

$$\mathscr{L}(S) = \{I \lhd_l R \mid I \cap S \neq \emptyset\}$$

is an idempotent filter. If $\kappa_S$ denotes its corresponding radical and $Q_S(M)$ is the module of quotients of $M$, then it is straightforward to see that $Q_S(M)$ is isomorphic to $S^{-1}M$, i.e., the classical Ore localization of $M$ at $S$.

(ii) If $R = \bigoplus_{k\geq 0} R_k$ is a positively graded Noetherian ring and $R_+$ denotes the two-sided ideal $\bigoplus_{k\geq 1} R_k$, by Proposition 3.2, the set

$$\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \text{there exists } n \in \mathbb{N} \text{ with } (R_+)^n \subseteq I\}$$

is an idempotent filter. The corresponding radical is denoted by $\kappa_+$.

From the treatment above, having in mind that the filter $\mathscr{L}(\kappa_+)$ is idempotent, Van Oystaeyen and Willaert [174] formed the *quotient category* $(R, \kappa_+)$-gr, that is, the full subcategory of $Q_{\kappa_+}(R)$-gr consisting of modules of the form $Q_{\kappa_+}(M)$ for some graded $R$-module $M$. Notice that $(R, \kappa_+)$-gr is equivalent to the full subcategory of $R$-gr consisting of the $\kappa_+$-closed modules and define Proj $R$ as the Noetherian objects in $(R, \kappa_+)$-gr. Since they wanted to describe the objects of Proj $R$ by means of objects of usual module categories in the same way as for commutative algebras, they need modules determined by Ore localizations. This is the content of the following definition.

**Definition 3.6 ([174, Definition 1]).** The noncommutative positively graded Noetherian $\Bbbk$-algebra $R = \Bbbk \oplus R_1 \oplus R_2 \oplus \cdots$ with $R = \Bbbk[R_1]$ is *schematic* if there is a finite set $I$ of homogeneous left Ore sets of $R$ such that for every $S \in I$, $S \cap R_+ \neq \emptyset$, and such that one of the following equivalent properties is satisfied:

(i) for each $(r_S)_{S\in I} \in \prod_{S\in I} S$, there exists $m \in \mathbb{N}$ such that $(R_+)^m \subseteq \sum_{S\in I} Rr_S$,

(ii) $\bigcap_{S\in I} \mathscr{L}(S) = \mathscr{L}(\kappa_+)$,

(iii) $\bigcap_{S\in I} \kappa_S(M) = \kappa_+(M)$, for all $M \in R-\text{Mod}$,

(iv) $\bigwedge_{S\in I} \kappa_S = \kappa_+$ where $\bigwedge$ denotes the infimum of torsion theories.

In [174, 176], Van Oystaeyen and Willaert constructed the noncommutative site, a category with coverings on which sheaves can be defined, and formulated the Serre's theorem. Examples 3.7 and 3.9 contain remarkable examples of schematic algebras.

**Example 3.7.** Recall that if $R$ is a *positively* filtered $\Bbbk$-algebra by the family $(F_nR)_{n\geq 0}$ (i.e., $F_0R = \Bbbk$), $\sigma : R \to G(R)$ is the principal symbol map, and $\widehat{R}$ is the Rees-ring of $R$, it is well-known that $G(R)$ and $\widehat{R}$ are positively graded and there is a canonical central element $X$ in $\widehat{R}$ of degree 1 such that $\widehat{R}/\langle X\rangle \cong G(R)$. If $\widehat{R}$ is Noetherian, this is equivalent to $G(R)$ being Noetherian or the filtration of $R$ being Zariskian. Notice that if $S$ is a multiplicatively set in $R$ such that $\sigma(S)$ is a multiplicative set in $G(R)$, then $\widehat{R} = \{sX^{\deg \sigma(S)} \mid s \in S\}$ is a multiplicative set (consisting of homogeneous elements) in $\widehat{R}$. For more details about graded rings, Zariskian filtrations and Rees rings, see Li and Van Oystaeyen's book [102].

For $R$ positively filtered by $(F_nR)_{n\geq 0}$, if $G(R)$ is schematic then $\widehat{R}$ is schematic [177, Theorem 1]. In this way, since for an almost commutative ring $R$ there exists a filtration on $R$ such that $G(R)$ is commutative, it follows that its Rees-ring is schematic. For example, the algebra $R$ generated by three elements $x, y$ and $z$ of degree 1 with relations $xy - yx = z^2$, $xz - zx = 0$, and $yz - zy = 0$, is schematic since it is the Rees-ring of the first Weyl algebra $A_1(\Bbbk)$ with respect to the Bernstein-filtration (this algebra is known as the *homogenized Weyl algebra*).

Van Oystaeyen and Willaert [177, p. 199] said that "it is probably not true that the class of schematic algebras is closed under iterated Ore extensions since Ore sets in a ring $R$ need not be Ore in an Ore extension $R[x;\sigma,\delta]$". Nevertheless, the following proposition shows that under suitable conditions, these extensions are schematic.

**Proposition 3.8 ([177, Theorem 3]).** *Given a positively graded ring $R$ which is generated by $R_1$ and which is schematic by means of Ore sets $S_i$, given $\sigma$ a graded automorphism of $R$ and $\delta$ a $\sigma$-derivation of degree 1, then for all $s_i \in \prod S_i$ and for all $m \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that*

$$(R[x;\sigma,\delta]_+)^p \subseteq M := \sum_i R[x;\sigma,\delta]s_i + R[x;\sigma,\delta]x^m,$$

*where $R[x;\sigma,\delta]$ denotes the Ore extension considered with graduation $(Rx[x;\sigma,\delta])_n = \bigoplus_{k=0}^{n} R_k x^{n-k}$.*

Proposition 3.8 is one of the results that Van Oystaeyen and Willaert [177] used to show that the algebras in Example 3.9 are schematic.

**Example 3.9 ([177, Examples 2-5]).** (i) The *coordinated ring of quantum* $2\times 2$-*matrices* $\mathscr{O}_q(M_2(\mathbb{C}))$ with $q \in \mathbb{C}$ is generated by elements $a, b, c$ and $d$ subject to the relations

$$\begin{aligned} ba &= q^{-2}ab, & ca &= q^{-2}ac, & bc &= cb,\\ db &= q^{-2}bd, & dc &= q^{-2}cd, & ad - da &= (q^2 - q^{-2})bc. \end{aligned}$$

(ii) *Quantum Weyl algebras* $A_n^{\overline{q},\Lambda}$ defined by Alev and Dumas [3] are given by an $n\times n$ matrix $\Lambda = (\lambda_{ij})$ with $\lambda_{ij} \in \Bbbk^*$ and a row vector $\overline{q} = (q_1, \dots, q_n)$, where $q_i \neq 0$ for every $i$, the algebra is generated by elements $x_1, \dots, x_n, y_1, \dots, y_n$ subject to relations $(i < j)$ given by

$$\begin{aligned} x_ix_j &= \mu_{ij}x_jx_i, & x_iy_j &= \lambda_{ji}y_jx_i, & y_jy_i &= \lambda_{ji}y_iy_j,\\ x_jy_i &= \mu_{ij}y_ix_j & x_jy_j &= 1 + q_jy_jx_j + \sum_{i<j}(q_i - 1)y_ix_i, & & \end{aligned}$$

where $\mu_{ij} = \lambda_{ij}q_i$.

(iii) *Three dimensional Sklyanin algebras* $A_{\Bbbk}$ *over a field* $\Bbbk$ according to Artin et al. [12] are graded $\Bbbk$-algebras generated by three homogeneous elements $x, y$ and $z$ of degree 1 satisfying the relations

$$axy + byx + cz^2 = 0, \quad ayz + bzy + cx^2 = 0, \quad \text{and} \quad azx + bxz + cy^2 = 0,$$

where $a, b, c \in \Bbbk$.

(iv) Color Lie super algebras defined by Rittenberg and Wyler [148].

**Remark 3.10.** Of course, there are examples of non-schematic algebras. If we take the graded algebra $\Bbbk\{x, y\}/\langle yx - xy - x^2\rangle$ and suppose that $\text{char}(\Bbbk) = 0$, then its subalgebra generated by $y$ and $xy$ is not left schematic [177, p. 203].

**Proposition 3.11 ([177, Lemma 2]).** *If $R$ is a graded $\Bbbk$-algebra such that its center $Z(R)$ is Noetherian and such that $R$ is a finitely generated $Z(R)$-module, then $R$ is schematic.*

## 3.2 Ore polynomials of higher order generated by homogenous quadratic relations

In this section, we investigate the schematicness of these algebras having in mind that the existence of a PBW basis facilitates the calculation of the center of the algebras in some cases, whence Proposition 3.11 is very useful to guarantee their schematicness. Notice that by definition, these quadratic algebras are connected and graded with $\deg x = \deg y = 1$. Following Golovashkin and Maksimov [47] we separate the cases $ac = 0$ and $ac \neq 0$. In the first case, by symmetry, we only consider $c = 0$.

### 3.2.1 Case $c = 0$

Consider the algebras $Q(a, b, 0)$. From Example 1.21 we know that the set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ forms a PBW basis of these algebras. Since the behavior of the algebras changes depending on whether these coefficients are zero or not, we consider three separate cases: (i) $b = 0$; (ii) $b \neq 0 = a$; and (iii) $a, b \neq 0$. Of course, the results for the algebras $Q(0, b, c)$ are analogous.

#### 3.2.1.1 Case $b = 0$

As one can check, the algebra $Q(a, 0, 0)$ (possibly with $a = 0$) is non-Noetherian [189, Proposition 1.14(b)], and therefore it is non-schematic. Nevertheless, we can check if these algebras satisfy the left schematic property. By using the commutative rule $yx = ax^2$, it is straightforward to see that $y^n x^k = a^n x^{n+k}$, for all $n \geq 0$ and $k \geq 1$.

**Lemma 3.12.** *Let $A = Q(a, 0, 0)$. If $S$ is a homogeneous non-trivial left Ore-set of $A$, then $S \cap Ay \neq \emptyset$.*

*Proof.* Let $p \in S \cap A_+$. It follows that $p = \sum_{i=0}^{n} a_i x^i y^{n-i}$, with $n \geq 1$. Notice that $(ax-y)x = 0$. If $a_0 = 0$ then $(ax-y)p = 0$. By the left cancellability property of $S$, there exists $s \in S$ given by $s = \sum_{i=0}^{m} b_i x^i y^{m-i}$ such that $s(ax-y) = 0$, whence $asx = sy$. More exactly,

$$\begin{aligned} asx &= a\left(\sum_{i=0}^{m} b_i x^i y^{m-i}\right) x \\ &= a\sum_{i=0}^{m} b_i a^{m-i} x^{m+1} \\ sy &= \sum_{i=0}^{m} b_i x^i y^{m-i+1}, \end{aligned}$$

which shows that $b_i = 0$, for $0 \leq i \leq m$, that is, $s = 0$, which is a contradiction. Hence, $a_0 \neq 0$.

Now, since $S$ is left Ore-set, there exist $s \in S$ and $r \in A$ such that $sx = rp$. Note that $s$ is homogeneous, and without loss of generality, we can take $r$ as an homogeneous element. Let $s = \sum_{i=0}^{t} b_i x^i y^{m-i}$ and $r = \sum_{j=0}^{k} c_j x^j y^{k-j}$. Then

$$\begin{aligned} \sum_{i=0}^{m} a^{m-i} b_i x^{m+1} &= sx \\ &= rp \\ &= \left(\sum_{j=0}^{k} c_j x^j y^{k-j}\right)\left(\sum_{i=0}^{n} a_i x^i y^{n-i}\right) \\ &= \sum_{j=0}^{k}\sum_{i=1}^{n} a^{k-j} a_i c_j x^{k+i} y^{n-i} + \sum_{j=0}^{k} a_0 c_j x^j y^{k+n-j}. \end{aligned}$$

It follows that $a_0 c_j = 0$, for $0 \leq j \leq k$. Since $a_0 \neq 0$, $c_j = 0$, for $0 \leq j \leq k$, which implies that $r = 0$ and so $\sum_{i=0}^{m} a^{m-i} b_i = 0$.

Since $s \in S$, $s^2 \in S$, where

$$\begin{aligned} s^2 &= \left(\sum_{j=0}^{m} b_j x^j y^{m-j}\right)\left(\sum_{i=0}^{m} b_i x^i y^{m-i}\right) \\ &= \sum_{j=0}^{m}\sum_{i=1}^{m} a^{m-j} b_i b_j x^{m+i} y^{m-i} + \sum_{j=0}^{m} b_0 b_j x^j y^{2m-j}, \end{aligned}$$

which shows that the coefficient of $x^{2m}$ is $\left(\sum_{j=0}^{m} a^{m-j} b_j\right) b_m = 0$. Therefore $s^2 \in S \cap Ay$. □

**Remark 3.13.** Using the right Ore condition it is easy to see that $Q(0,0,0)$ has no homogeneous non-trivial Ore-sets.

The next result guarantees that the algebras $Q(a,0,0)$ are not left schematic.

**Theorem 3.14.** *$Q(a,0,0)$ does not satisfy the left schematic property.*

*Proof.* Let $A = Q(a,0,0)$ and $S_1,\dots,S_m$ a finite family of homogeneous non-trivial left Ore-sets of $A$. By Lemma 3.12, there exists $s_i \in S_i \cap Ay$, for $1 \le i \le m$. Hence, $\sum_{i=1}^{m} As_i \subseteq Ay$. For any $n \in \mathbb{N}$, $x^n \in (A_+)^n \setminus Ay$, we have $(A_+)^n \not\subseteq \sum_{i=1}^{m} As_i$. □

#### 3.2.1.2 Case $b \neq 0 = a$

The algebra $Q(0,b,0)$ is known in the literature as the *quantum plane* or the *Manin's plane*.

**Theorem 3.15.** *The algebra $Q(0,b,0)$ is schematic.*

*Proof.* The assertion follows from a more general result presented in Proposition 3.65. □

#### 3.2.1.3 Case $a, b \neq 0$

We fix some notation. For $b \in \Bbbk$ and $k \geq 1$, we write

$$[k]_b := \sum_{i=0}^{k-1} b^i \quad \text{and} \quad [k]_b! := \prod_{i=1}^{k} [i]_b.$$

Notice that if $b \neq 1$, then $[k]_b = \frac{b^k-1}{b-1}$ and $[k]_1 = k$. From [20, Proposition 1], we know that in $Q(a,b,0)$ the following commutation rules hold:

$$yx^k = b^k x^k y + a[k]_b x^{k+1},$$

$$y^k x = \sum_{r=0}^{k} \frac{[k]_b!}{[k-r]_b!} b^{k-r} a^r x^{r+1} y^{k-r}.$$

**Theorem 3.16.** *We have the following cases for $Z(Q(a,b,0))$, the center of $Q(a,b,0)$:*

(1) *If $b$ is not a root of unity, then $Z(Q(a,b,0)) = \Bbbk$.*

(2) *If $b = 1$ and $\operatorname{char}(\Bbbk) = 0$, then $Z(Q(a,b,0)) = \Bbbk$.*

(3) *If $b = 1$ and $\operatorname{char}(\Bbbk) = t$, then $Z(Q(a,b,0)) = \Bbbk[x^t, y^t]$.*

(4) *If $b$ is a primitive $t$-th root of unity, then $Z(Q(a,b,0)) = \Bbbk[x^t, y^t]$.*

*In cases (3) and (4), $Q(a,b,0)$ is schematic by Proposition 3.11.*

*Proof.* Since $Q(a,b,0)$ is graded and generated by the homogeneous elements $x$ and $y$, it is enough to find the homogeneous centralizers of these elements. With this aim, let

$p = \sum_{i=0}^{n} a_i x^i y^{n-i} \in Z(y)$. Then

$$\begin{aligned}\sum_{i=0}^{n} a_i x^i y^{n-i+1} &= py \\ &= yp \\ &= \sum_{i=1}^{n} \left(a_i b^i x^i y^{n-i+1} + a_i a[i]_b x^{i+1} y^{n-i}\right) + a_0 y^{n+1}.\end{aligned}$$

Comparing the corresponding coefficients, we can see that

$$\begin{aligned}a_1 &= a_1 b, \\ a_2 &= a_2 b^2 + a_1 a[1]_b, \\ a_3 &= a_3 b^3 + a_2 a[2]_b, \\ &\vdots \\ a_i &= a_i b^i + a_{i-1} a[i-1]_b, \\ &\vdots \\ a_{n-1} &= a_{n-1} b^{n-1} + a_{n-2} a[n-2]_b, \\ a_n &= a_n b^n + a_{n-1} a[n-1]_b, \\ 0 &= a_n a[n]_b.\end{aligned}$$

(1) From expressions above, it follows that $a_i = 0$ for $i = 1, \dots, n$, whence $Z(y) = \Bbbk[y]$.

(2) In this case, the equations are expressed as

$$\begin{aligned}0 &= a_1 a \\ 0 &= 2a_2 a \\ &\vdots \\ 0 &= (i-1)a_{i-1} a \\ &\vdots \\ 0 &= (n-2)a_{n-2} a \\ 0 &= (n-1)a_{n-1} a \\ 0 &= n a_n a\end{aligned}$$

Since $\mathrm{char}(\Bbbk) = 0$, then $a_i = 0$ for $i = 1, \dots, n$. Hence, $Z(y) = \Bbbk[y]$.

(3) We obtain the same equations as in the previous case. Since $\mathrm{char}(\Bbbk) = t$, then the coefficients $a_{kt}$ are not necessarily zero, so $Z(y) \subseteq \Bbbk[x^t, y]$.

(4) From expressions above we obtain that $0 = a_1 = a_2 = \cdots = a_{t-1}$. On the $t$-th line we have $a_t = a_t + a_{t-1} a[t-1]_b = a_t$, whence $a_t$ is not necessarily zero. On the next line, $a_{t+1} = a_{t+1} b + a_t a[t]_b$, and since $b^t = 1$, then $[t]_b = 0$ which shows that $a_{t+1} = 0$.

Continuing in this way, we can see that the only coefficients that are not necessarily zero are the coefficients $a_{kt}$. Thus $Z(y) \subseteq \Bbbk[x^t, y]$.

Now, we look for when $p$ commutes with $x$. For cases (1) and (2) we have $p = a_0 y^n$, and by the commuting rule for $y^n x$ it is clear that $p$ does not commutes with $x$. In this way, $Z(Q(a,b,0)) = \Bbbk$.

For cases (3) and (4), let $p = \sum_{i=0}^{k} a_i x^{it} y^{n-it}$, with $n \geq kt$. It must be fulfilled that

$$\begin{aligned}\sum_{i=0}^{k} a_i x^{it+1} y^{n-it} &= xp \\ &= px \\ &= \sum_{i=0}^{k} a_i x^{it} y^{n-it} x \\ &= \sum_{i=0}^{k} \sum_{r=0}^{n-it} a_i \frac{[n-it]_b!}{[n-it-r]_b!} b^{n-it-r} a^r x^{it+r+1} y^{n-it-r}.\end{aligned}$$

Let $j$ be the smallest number such that $a_j \neq 0$. On the right side of the expression, the term $x^{jt+2} y^{n-jt-2}$ only appears once (when $j = i, r = 1$) with coefficient $a_j [n - jt]_b b^{n-jt-1}$, and on the left side this elements does not appear. Hence $0 = a_j [n - jt]_b b^{n-jt-1}$. Since $a_j, b \neq 0$, then $[n - jt]_b = 0$.

For case (3), we have $0 = [n - jt]_b = (n - jt)1$, and so $n - jt \in t\mathbb{Z}$, which shows that $n \in t\mathbb{Z}$.

For case (4), $[n - jt]_b = 0$ implies that $b^{n-jt} = 1$, whence $n - jt \in t\mathbb{Z}$, and therefore $n \in t\mathbb{Z}$.

As we saw, in both cases $p \in \Bbbk[x^t, y^t]$, and so $Z(Q(a,b,0)) \subseteq \Bbbk[x^t, y^t]$. From the commuting rule of $y^k x$ it is clear that in these cases $y^t \in Z(x)$.

We conclude that $Z(Q(a,b,0)) = \Bbbk[x^t, y^t]$. □

The following tables summarize the results obtained above.

| Case $ab = 0$ | | |
|---|---|---|
| Quadratic algebra | Center | Schematic |
| $yx = 0$ | $\Bbbk$ | No |
| $yx = ax^2$ | $\Bbbk$ | No |
| $yx = bxy$ | $\Bbbk$ or $\Bbbk[x^t, y^t]$ | Yes |

| Algebra $yx = ax^2 + bxy$ | | |
|---|---|---|
| Case | Center | Schematic |
| $b$ is not a root of unity | $\Bbbk$ | ? |
| $b = 1$, $\operatorname{char}(\Bbbk) = 0$ | $\Bbbk$ | ? |
| $b = 1$, $\operatorname{char}(\Bbbk) = t$ | $\Bbbk[x^t, y^t]$ | Yes |
| $b^t = 1$ | $\Bbbk[x^t, y^t]$ | Yes |

**Remark 3.17.** As we can see in the last table, for the cases in which the center is trivial the schematicity of the algebras is a pending task.

### 3.2.2 One example with $ac \neq 0$

As mentioned in the preliminaries, Golovashkin and Maksimov proved that if $b = -1$ and $ac = 1$, then the set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ do not form a PBW basis of $Q(a, b, c)$. In this section we study what happen in the case $b = -1$ and $ac \neq 1$.

**Proposition 3.18.** *The set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a PBW basis of $Q(a, -1, c)$.*

*Proof.* For this algebra, consider the companion matrix given by

$$M = \begin{pmatrix} -1 & a \\ -c & 1 \end{pmatrix}.$$

We have that $M^2 = (1 - ac)I$, and so it is easy to see that for all $n \in \mathbb{N}$, the lower-right term of $M^n$ is a power of $1 - ac$. Since $ac \neq 1$, then the lower-right term does not vanish. From the results presented in Section 3.2 we conclude that the set $\{x^m y^n \mid m, n \in \mathbb{N}\}$ is a basis of $Q(a, -1, c)$. □

**Lemma 3.19.** *In $Q(a, -1, c)$ we have the following commutation rules:*

(1) *If $k$ is even, then $yx^k = x^k y$ and $y^k x = xy^k$.*

(2) *If $k$ is odd, then $yx^k = ax^{k+1} - x^k y + cx^{k-1}y^2$ and $y^k x = ax^2 y^{k-1} - xy^k + cy^{k+1}$.*

*Proof.* We prove this fact by induction for $yx^k$; the proof for $y^k x$ is similar. The cases $k = 0, 1$ are straightforward. Before the inductive step, we need to check that $y^2 x = xy^2$. Let us see:

$$\begin{aligned}
y^2 x &= y(ax^2 - xy + cy^2) \\
&= ayx^2 - yxy + cy^3 \\
&= a(ax^2 - xy + cy^2)x - (ax^2 - xy + cy^2)y + cy^3 \\
&= a^2x^3 - axyx + acy^2x - ax^2y + xy^2 - cy^3 + cy^3 \\
&= a^2x^3 - ax(ax^2 - xy + cy^2) + acy^2x - ax^2y + xy^2 \\
&= a^2x^3 - a^2x^3 + ax^2y - acxy^2 + acy^2x - ax^2y + xy^2 \\
&= xy^2 - acxy^2 + acy^2x.
\end{aligned}$$

We have $(1 - ac)y^2x = (1 - ac)xy^2$, and since $ac \neq 1$, then $y^2x = xy^2$.

For the inductive step, suppose that $k + 1$ is even and the result holds for $k$ odd. We have:

$$\begin{aligned} yx^{k+1} &= yx^k x \\ &= (ax^{k+1} - x^k y + cx^{k-1}y^2)x \\ &= ax^{k+2} - x^k yx + cx^{k-1}y^2 x \\ &= ax^{k+2} - x^k(ax^2 - xy + cy^2) + cx^k y^2 \\ &= x^{k+1}y. \end{aligned}$$

Now, suppose that $k + 1$ is odd and the result holds for $k$ even. Then

$$\begin{aligned} yx^{k+1} &= yx^k x \\ &= x^k yx \\ &= x^k(ax^2 - xy + cy^2) \\ &= ax^{k+2} - x^{k+1}y + cx^k y^2, \end{aligned}$$

which concludes the proof. □

**Proposition 3.20.** $Z(Q(a, -1, c)) = \Bbbk[x^2, y^2]$.

*Proof.* Since $Q(a, -1, c)$ is graded and generated by the homogeneous elements $x$ and $y$, it is enough to find the homogeneous centralizers of these elements.

Let $p = \sum_{i=0}^{n} a_i x^i y^{n-i} \in Z(Q(a, -1, c))$, $I = \{x \in \mathbb{N} \mid x \leq n \text{ and } x \text{ is odd}\}$ and $J = \{x \in \mathbb{N} \mid x \leq n \text{ and } x \text{ is even}\}$. Suppose that $n$ is odd.

$$\begin{aligned} px &= \left( \sum_{i \in I} a_i x^i y^{n-i} + \sum_{i \in J} a_i x^i y^{n-i} \right) x \\ &= \sum_{i \in I} a_i x^{i+1} y^{n-i} + \sum_{i \in J} a_i x^i (ax^2 y^{n-i-1} - xy^{n-i} + cy^{n-i+1}) \\ &= \sum_{i \in I} a_i x^{i+1} y^{n-i} + \sum_{i \in J} a_i a x^{i+2} y^{n-i-1} - a_i x^{i+1} y^{n-i} + a_i c x^i y^{n-i+1} \\ xp &= \sum_{i \in I} a_i x^{i+1} y^{n-i} + \sum_{i \in J} a_i x^{i+1} y^{n-i} \end{aligned}$$

Since $px = xp$, then $\sum_{i \in J} a_i a x^{i+2} y^{n-i-1} - 2a_i x^{i+1} y^{n-i} + a_i c x^i y^{n-i+1} = 0$. Taking the coefficients of even powers of $x$, we have

$$\begin{aligned} a_0c &= 0 \\ a_0a + a_2c &= 0 \\ a_2a + a_4c &= 0 \\ &\vdots \\ a_{n-3}a + a_{n-1}c &= 0 \\ a_{n-1}a &= 0 \end{aligned}$$

From this, we have that $a_i = 0$, for all $i \in J$. In a similar way, from the equality $py = yp$, we have that $a_i = 0$ for all $i \in I$. Therefore $p = 0$,which is a contradiction since $\deg p = n$. Then $n$ cannot be odd.

With $n$ even, both equalities $xp = px$ and $yp = py$ imply

$$\sum_{i \in I} a_i a x^{i+2} y^{n-i-1} - 2a_i x^{i+1} y^{n-i} + a_i c x^i y^{n-i+1} = 0.$$

Then $a_i = 0$, for all $i \in I$. Hence $p = \sum_{i \in J} a_i x^i y^{n-i}$. Note that all powers of $x$ and $y$ are even, thus $p \in \Bbbk[x^2, y^2]$. It follows that $Z(A) \subseteq \Bbbk[x^2, y^2]$.

From the rule of commutation for even powers, it is clear that $x^2, y^2 \in Z(A)$. We conclude that $Z(A) = \Bbbk[x^2, y^2]$. □

Propositions 3.11 and 3.20 imply the following result:

**Theorem 3.21.** *$Q(a, -1, c)$ is schematic.*

**Remark 3.22.** It is possible to obtain more examples of algebras with basis PBW bases. By [47, Proposition 4], it is straightforward to verify that if the field $\Bbbk$ has an ordered subfield $\mathbb{F}$, then the algebra $Q(a, b, c)$ has a PBW basis in the following cases:

(i) $a, b, c \in \mathbb{F}$ such that $a, b \geq 0$ and $c < 0$.

(ii) $b = 0$ and $a, c \in \mathbb{F}$ with $c = -a$.

Characterize the schematicness of these algebras could be a topic of future interest.

## 3.3 Noncommutative scheme theory

In this section, we present a noncommutative scheme theory for semi-graded rings following the ideas introduced by Van Oystaeyen and Willaert [174] but now in the setting of rings that are non-$\mathbb{N}$-graded and connected.

### 3.3.1 Localization of semi-graded rings

In this section, we want to formalize several constructions concerning semi-graded rings which are necessary to formulate Serre's theorem.

With the aim of defining *good* Ore sets (Definition 3.23), for $R$ an SG ring and an element $n \in \mathbb{Z}$, we consider the following sets:

$$\begin{aligned} R'_n &= \{r \in R_n \mid \text{ for all } m \in \mathbb{Z}, \text{ and for all } h \in R_m, rh \in R_{n+m}\}, \\ R''_n &= \{r \in R'_n \mid \text{ for all } m \in \mathbb{Z}, \text{ and for all } h \in R_m, hr \in R_{n+m}\}, \\ R' &= \bigcup_{n \in \mathbb{Z}} R'_n, \\ R'' &= \bigcup_{n \in \mathbb{Z}} R''_n. \end{aligned}$$

**Definition 3.23.** Let $R$ be an SG ring and consider a left Ore set $S$ of $R$. We say that $S$ is *good* if the following conditions hold:

(i) $S \subseteq R''$, and,

(ii) if $s \in S$ and $r \in R'$, then there exist elements $u \in R'$ and $v \in S$ such that $us = vr$.

From Definition 3.23 it follows that for any elements $s_1, \ldots, s_k \in S$, there exist $r_1, \ldots, r_k \in R'$ such that $r_i s_i = r_j s_j \in S$, for every $i, j$.

**Definition 3.24.** Let $R$ be an SG ring and $M$ an SG $R$-module. We say that $M$ is *localizable semi-graded* (LSG) if for every element $(n, m) \in \mathbb{Z}^2$, the inclusion $R'_n M_m \subseteq M_{n+m}$ holds.

**Proposition 3.25.** *Let $R$ be an* SG *ring, $S$ a good left Ore set, and $M$ an* LSG *$R$-module. Then $S^{-1}M$ is an* LSG *$R$-module with semi-graduation given by*

$$(S^{-1}M)_n = \left\{ \frac{f}{s} \mid f \in \bigcup_{k \in \mathbb{Z}} M_k, \ \deg(f) - \deg(s) = n \right\}.$$

*Proof.* First of all, let us show that $(S^{-1}M)_n$ is a subgroup of $S^{-1}M$. It is clear that $0 = \frac{0}{1} \in (S^{-1}M)_n$ and that $(S^{-1}M)_n$ have additive inverses. Consider elements $\frac{p}{s}, \frac{q}{t} \in (S^{-1}M)_n$. Then $\deg(p) - \deg(s) = \deg(q) - \deg(t) = n$. There exist elements $u \in R'$ and $v \in S$ such that $us = vt \in S$. Note that $\deg(u) + \deg(s) = \deg(v) + \deg(t)$. Since $u, v \in R'$, it follows that $up$ and $vq$ are homogeneous elements satisfying $\deg(up) = \deg(u) + \deg(p) = \deg(v) + \deg(q) = \deg(vq)$, whence $\frac{p}{s} + \frac{q}{t} = \frac{up+vq}{vt}$ is a homogeneous elements of degree $\deg(v) + \deg(q) - (\deg(v) + \deg(t)) = n$.

Since it is clear that $S^{-1}M$ is the sum of the subgroups $(S^{-1}M)_n$, let us show that the sum is direct. Consider the sum

$$\sum_{i=1}^{k} \frac{m_i}{s_i} = 0$$

of homogeneous elements of $S^{-1}M$ with different degrees, that is, $\deg(m_i) - \deg(s_i) \neq \deg(m_j) - \deg(s_j)$, for $i \neq j$. There exist elements $r_1, \ldots, r_k \in R'$ such that $r_i s_i = r_j s_j$ for all $i, j$, which implies that

$$0 = \sum_{i=1}^{k} \frac{m_i}{s_i} = \frac{\sum_{i=1}^{k} r_i m_i}{r_1 s_1}.$$

Hence, there exists an element $s \in S$ such that $0 = s\sum_{i=1}^{k} r_i m_i = \sum_{i=1}^{k} s r_i m_i$. Since $s, r_i \in R'$ and $m_i$ is homogeneous for $0 \leq i \leq k$, then every one of the terms above is homogeneous. Using that $r_i s_i = r_j s_j$, we have $\deg(r_i)+\deg(s_i) = \deg(r_j)+\deg(s_j)$, whence $\deg(s) + \deg(r_i) + \deg(m_i) \neq \deg(s) + \deg(r_j) + \deg(m_j)$, which shows that $sr_i m_i = 0$. Thus, $0 = \frac{r_i m_i}{r_i s_i} = \frac{m_i}{s_i}$.

Now, let us see that $R_a(S^{-1}M)_b \subseteq \bigoplus_{k \leq a+b}(S^{-1}M)_k$. Let $r \in R_a$ and $\frac{m}{s} \in (S^{-1}M)_b$. There exist elements $r' \in R$ and $s' \in S$ such that $r's = s'r$. Since $s, s' \in R''$ and $r$ is homogeneous, we can take the element $r'$ being homogeneous. Then $\deg(r') = \deg(s') + \deg(r) - \deg(s)$, and using that $r\frac{m}{s} = \frac{r'm}{s'}$ and $r'm \in \bigoplus_{k \leq \deg(r')+\deg(m)} M_k$, it follows that $\frac{r'm}{s'} \in \bigoplus_{k \leq \deg(r')+\deg(m)-\deg(s')}(S^{-1}M)_k$. Since $\deg(r') + \deg(m) - \deg(s') = \deg(r) + \deg(m) - \deg(s) = a + b$, then $r\frac{m}{s} \in \bigoplus_{k \leq a+b}(S^{-1}M)_k$. This fact proves that $S^{-1}M$ is an SG $R$-module.

Notice that if we above consider the element $r \in R'_a$, then we can take $r' \in R'$, whence $r'm \in M_{\deg(r')+\deg(m)}$, and so $r\frac{m}{s} \in (S^{-1}M)_{a+b}$. This shows that $S^{-1}M$ is an LSG $R$-module. □

The next result shows that the localization of an SG ring by considering a good Ore set is again an SG ring.

**Proposition 3.26.** *Let $R$ be an* SG *ring and $S$ a good left Ore set. Then $S^{-1}R$ is an* SG *ring with semigraduation given by*

$$(S^{-1}R)_n = \left\{ \frac{f}{s} \mid f \in \bigcup_{k \in \mathbb{Z}} R_k,\ \deg(f) - \deg(s) = n \right\}.$$

*Proof.* It is clear that $R$ is an LSG $R$-module, so $S^{-1}R$ is an SG $R$-module with the semigraduation above, so $S^{-1}R = \bigoplus_{k \in \mathbb{Z}}(S^{-1}R)_k$. It is easy to see that $1 = \frac{1}{1} \in (S^{-1}R)_0$. We only have to show that $(S^{-1}R)_n(S^{-1}R)_m \subseteq \bigoplus_{k \leq n+m}(S^{-1}R)_k$.

Let $\frac{r_1}{s_1} \in (S^{-1}R)_n$ and $\frac{r_2}{s_2} \in (S^{-1}R)_m$. There exist elements $u \in R$ and $v \in S$ such that $vr_1 = us_2$, which implies that $\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{ur_2}{vs_1}$. Again, since $s_2, v \in R''$ and $r_1$ is homogeneous, we can take $u$ as an homogeneous element. Hence, $ur_2 \in \bigoplus_{k \leq \deg(u)+\deg(r_2)} R_k$, and so $\frac{ur_2}{vs_1} \in \bigoplus_{k \leq \deg(u)+\deg(r_2)-\deg(v)-\deg(s_1)}(S_{-1}R)_k$, i.e., $\frac{ur_2}{vs_1} \in \bigoplus_{k \leq n+m}(S^{-1}R)_k$. □

**Proposition 3.27.** *Let $R$ be an* SG *ring, $S$ a good left Ore set and $M$ an* LSG *$R$-module. Then $S^{-1}M$ is an* SG *$S^{-1}R$-module.*

*Proof.* We know that $S^{-1}M$ is an $S^{-1}R$-module and an $\mathsf{SG}$ $R$-module, which implies the direct sum $S^{-1}M = \bigoplus_{k\in\mathbb{Z}}(S^{-1}M)_k$. In this way, we just have to prove that $(S^{-1}R)_n(S^{-1}M)_m \subseteq \bigoplus_{k\leq n+m}(S^{-1}M)_k$. Consider elements $\frac{r}{s_1} \in (S^{-1}R)_n$ and $\frac{a}{s_2} \in (S^{-1}M)_m$. There exist elements $u \in R$ and $v \in S$ such that $vr = us_2$, which implies that $\frac{r}{s_1}\frac{a}{s_2} = \frac{ua}{vs_1}$. Again, since $s_2, v \in R''$ and $r$ is homogeneous, we can take $u$ as an homogeneous element. Hence, $ua \in \bigoplus_{k\leq \deg(u)+\deg(a)} R_k$, and so $\frac{ua}{vs_1} \in \bigoplus_{k\leq n+m}(S^{-1}R)_k$. □

We define $\mathsf{LSG} - R$ as the full subcategory of $\mathsf{SGR} - R$ whose objects are the $\mathsf{LSG}$ modules. This subcategory is closed for subobjects, quotients and coproducts, so it is complete Abelian.

### 3.3.2 Schematicness of semi-graded rings

Following Van Oystaeyen and Willaert's ideas developed in [174], in this section we define the notion of *schematicness* in the setting of semi-graded rings. For a positively $\mathsf{SG}$ ring $R$, we define $R_+ = \bigoplus_{k\geq 1} R_k$ and we say that a left Ore set $S$ is *non-trivial* if $S \cap R_+ \neq \emptyset$.

We start with the following observation:

**Remark 3.28.** If $R$ is a positively $\mathsf{SG}$ left Noetherian ring, then Proposition 3.2 shows that

$$\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \text{there exist } n, m \in \mathbb{N} \text{ with } (R_{\geq m})^n \subseteq I\}$$

is an idempotent filter. The corresponding left exact radical is denoted by $\kappa_+$ and $Q_{\kappa_+}(M)$ is the module of quotients of $M$.

**Definition 3.29.** Let $R$ be a positively $\mathsf{SG}$ left Noetherian ring. $R$ is called (*left*) *schematic* if there is a finite set $I$ of non-trivial good left Ore sets of $R$ such that for each $(x_S)_{S\in I} \in \prod_{S\in I} S$, there exist $t, m \in \mathbb{N}$ such that $(R_{\geq t})^m \subseteq \sum_{S\in I} Rx_s$.

The following result illustrates some characterizations of being schematic (c.f. Definition 3.6).

**Proposition 3.30.** *Let $R$ be a positively $\mathsf{SG}$ left Noetherian algebra and $S_1, \ldots, S_n$ a finite set of non-trivial good left Ore sets of $R$. The following conditions are equivalent:*

(1) *For each $(x_1, \ldots, x_n) \in \prod_{i=1}^n S_i$, there exist elements $t, m \in \mathbb{N}$ such that $(R_{\geq t})^m \subseteq \sum_{S\in I} Rx_s$.*

(2) *Let $I \lhd_l R$. If $I$ has no trivial intersection with every $S_i$, then $I$ contains a power of $R_{\geq t}$ for some $t \in \mathbb{N}$.*

(3) *$\bigcap_{i=1}^n \mathscr{L}(S_i) = \mathscr{L}(\kappa_+)$.*

*Proof.* The equivalence (1) ⇔ (2) and the implication (3) ⇒ (1) are straightforward.

(1) ⇒ (3) Let $I \in \bigcap_{i=1}^n \mathscr{L}(S_i)$. There exist elements $x_1, \ldots, x_n$ such that $x_i \in I \cap S_i$, for every $i$. Thus $\sum_{i=1}^n Rx_i \subseteq I$, and there exist $t, m$ with $(R_{\geq t})^m \subseteq I$, which shows that $I \in \mathscr{L}(\kappa_+)$.

Now, let $I \in \mathscr{L}(\kappa_+)$. There exist $t, m$ such that $(R_{\geq t})^m \subseteq I$. By using that $S_i \cap R_+ \neq \emptyset$, there exist elements $s_i \in S_i$ such that $\deg(s_i) \geq 1$, for all $i$. Hence $s_i^t \in R_{\geq t}$, $s_i^{tm} \in (R_{\geq t})^m \subseteq I$, and therefore $I \cap S_i \neq \emptyset$. This shows that $I \in \bigcap_{i=1}^n \mathscr{L}(S_i)$. □

If $R$ is schematic by considering the *good* left Ore sets $S_i$, then $\bigcap_{i=1}^n \kappa_{S_i}(M) = \kappa_+(M)$, for every $R$-module $M$. If $M$ is an $\mathsf{LSG}-$module, then for each $i = 1, \dots, n$ we have that $\kappa_{S_i}(M)$ is an $\mathsf{SG}$ submodule, and so $\kappa_+(M)$ is also an $\mathsf{SG}$ submodule. These facts imply that $M/\kappa_+(M)$ is an $\mathsf{SG}$ $R$-module, and so a submodule of $Q_{\kappa_+}(M)$. The idea is to show that $Q_{\kappa_+}(M)$ is semi-graded. For the remainder of the section we will take $\mathscr{L} := \mathscr{L}(\kappa_+)$.

Let us start by taking an $\mathsf{LSG}$ $R$-module $M$ such that $\kappa_+(M) = 0$. It is clear that $Q_{\kappa_+}(M) = M_{\mathscr{L}}$ and $\varphi_M(M) \cong M$. Thus, $\varphi_M(M)$ is a submodule of $M_{\mathscr{L}}$ which is an $\mathsf{SG}$ $R$-module where $\varphi_M(m)$ is homogeneous of degree $k$ if and only if $m$ is homogeneous of degree $k$. If we want $M_{\mathscr{L}}$ to be an $\mathsf{LSG}$ $R$-module, it must be satisfied that if $\xi$ is homogeneous of degree $k$, then for every $s \in R'$, the element $s\xi \in (\varphi_M(M))_{\deg(s)+k}$. Now, since there exists $I \in \mathscr{L}$ with $I\xi \subseteq \varphi_M(M)$, the following definition makes sense.

**Definition 3.31.** Let $\xi \in M_{\mathscr{L}}$. We say that the element $\xi$ is *homogeneous of degree* $k$ if there exists $I \in \mathscr{L}$ such that $I\xi \subseteq \varphi_M(M)$, and for every element $s \in I \cap R'$, $s\xi \in (\varphi_M(M))_{\deg(s)+k}$.

Notice that if the condition above is satisfied for $I$, then it also holds for every $J \subseteq I$. The following lemma shows that this condition is true for ideals containing $I$.

**Remark 3.32.** Since the good Ore sets $S_i$ are non-trivial, there exist elements $s_i' \in S_i \cap R_+$ for $i = 1, \dots, n$, whence $\alpha_i = \deg(s_i') > 0$. If we define $m := \operatorname{lcm}\{\alpha_i\}_{1 \leq i \leq n}$ and $s_i'' := (s_i')^{m/\alpha_i}$, then we obtain $s_i'' \in S_i \cap R_+$, and all of them have the same degree. Now, if we consider an element $I \in \mathscr{L}$, there exist $t, n \in \mathbb{Z}$ such that $(R_{\geq t})^n \subseteq I$. Thus, $s_i = (s_i'')^{tn} \in I \cap S_i$, which implies that $\sum_{i=1}^n Rs_i \subseteq I$. In this way, for each $I \in \mathscr{L}$ there exist elements $s_i \in S_i$, all with the same positive degree, satisfying the relation $\sum_{i=1}^n Rs_i \subseteq I$.

**Lemma 3.33.** *Let $I, J \in \mathscr{L}$ be ideals such that $I \subseteq J$ and $I\xi, J\xi \subseteq \varphi_M(M)$. If for every $s \in I \cap R'$ the element $s\xi \in (\varphi_M(M))_{\deg(s)+k}$, then the same property holds for each $s \in J \cap R'$.*

*Proof.* Let $s \in J \cap R'$. Then $s\xi \in \varphi_M(M)$ and so there exist homogeneous elements $\xi_j$, $j = l_1, \dots, l_r$ of $\varphi_M(M)$ with $\xi_j \in (\varphi_M(M))_j$ and such that $s\xi = \sum \xi_j$. As we said before, if the property holds for $I$ then it is true for any ideal contained in $I$, so we can take $I = (R_{\geq t})^m$, for some $t, m \in \mathbb{N}$. From above, there exist $s_i \in S_i$ for $i = 1, \dots, n$ such that $\sum Rs_i \subseteq I$ and $\deg(s_i) = \beta$ for each $1 \leq i \leq n$. In particular, every element $s_i \in I$ whence $s_i s \in I$ (recall that $I$ is a two-sided ideal). By assumption, $s_i s\xi \in (\varphi_M(M))_{\deg(s)+\beta+k}$ for each $i$.

On the other hand, if we consider the expression $s_i s\xi = \sum s_i\xi_j$ in terms of homogeneous elements of $\varphi_M(M)$, then for each $j \neq k + \deg(s)$ the equality $s_i\xi_j = 0$ holds. Since this is true for every $i$, it follows that $\xi_j \in \bigcap_{i=1}^n \kappa_{S_i}(\varphi_M(M)) = \kappa_+(\varphi_M(M)) = 0$ (recall that $\kappa_+(M) = 0$). Therefore, $s\xi = \xi_{\deg(s)+k}$. □

From Lemma 3.33, it is sufficient to guarantee the property by considering any ideal $I$ such that $I\xi \subseteq \varphi_M(M)$. Our purpose is to give a more simple method to verify that the element $\xi$ is homogeneous. Let $\xi = [I, f]$. Since $I\xi \subseteq \varphi_M(M)$, the element $\xi$ is homogeneous of degree $k$ if and only if for each $s \in I \cap R'$ the element $s\xi = [R, \beta(f(s))] = \varphi_M(f(s)) \in (\varphi_M(M))_{\deg(s)+k}$, or equivalently, for all $s \in I \cap R'$, the element $f(s) \in M_{\deg(s)+k}$.

For a morphism $f : I \to M$, we will say that $f$ is *homogeneous of degree* $k$ if for each $s \in I \cap R'$, the element $f(s)$ is homogeneous of degree $\deg(s) + k$. Hence, $[I, f]$ is homogeneous of degree $k$ (in $M_{\mathscr{L}}$) if and only if $f$ is homogeneous of degree $k$. Let $(M_{\mathscr{L}})_k$ be the family of homogeneous elements of degree $k$. It is clear that $(M_{\mathscr{L}})_k$ is a subgroup and $\varphi_M(M_k) \subseteq (M_{\mathscr{L}})_k$.

**Remark 3.34.** We will say that the morphism $f : I \to M$ is *strongly homogeneous of degree* $k$ if for every homogeneous element $s \in I$, the element $f(s)$ is homogeneous of degree $\deg(s) + k$. It is clear that in the setting of graded rings, the notions of homogeneous morphism and strongly homogeneous morphism coincide.

On the other hand, $[I, f]$ it will be called *strongly homogeneous of degree* $k$ if some of its representative elements is strongly homogeneous of degree $k$. Let $(\overline{M_{\mathscr{L}}})_k$ be the family of strongly homogeneous elements of degree $k$. It is straightforward to see that $\overline{R_{\mathscr{L}}} = \bigoplus(\overline{R_{\mathscr{L}}})_k$ is a graded ring and $\overline{M_{\mathscr{L}}} = \bigoplus(\overline{M_{\mathscr{L}}})_k$ is an $\overline{R_{\mathscr{L}}}$-graded module. Note also that if $s \in R''$ then $\varphi_R(s) \in \overline{R_{\mathscr{L}}}$; in particular $\overline{R_{\mathscr{L}}}$, is an extension of the graded ring $\varphi_R(\bigoplus R''_k)$. As it is clear, $\overline{R_{\mathscr{L}}}$ is an $R$-submodule of $R_{\mathscr{L}}$ if and only if $R$ is graded. This last remark shows that in the setting of non-graded rings is not appropriate to consider strongly homogeneous morphisms.

**Proposition 3.35.** *The sum* $\sum(M_{\mathscr{L}})_k$ *is direct.*

*Proof.* Let $[I_i, f_i] \in (M_{\mathscr{L}})_{k_i}$ for $i = 1, \dots, m$ with $k_i \neq k_j$ if $i \neq j$. Notice that if $\sum[I_i, f_i] = 0$, then there exists $J \subseteq \bigcap I_i$, $J \in \mathscr{L}$, such that $(\sum f_i)|_J = \sum f_i|_J = 0$. We can take $J = \sum Rs_j$ for some $s_j \in S_j$. Let $s \in J \cap R'$ with $\deg(s) = l$. Then $0 = (\sum f_i)(s) = \sum f_i(s)$, and since $f_i(s)$ is homogeneous of degree $l + k_i$ and all elements $k_i$ are different, then we have a sum of homogeneous elements of different degrees equal to zero, whence $f_i(s) = 0$, for each $i$. In particular, $f_i(s_j) = 0$, for all $i, j$. Therefore, $f_i(x) = 0$ for all $x \in J$, and so $[J, f_i|J] = [I_i, f_i] = 0$. □

Let $[I, f] \in M_{\mathscr{L}}$ with $I = \sum_{i=1}^{n} Rs_i$, for some elements $s_i \in S_i \cap R''_k$. Since there are finitely $s_i$'s, we can consider that the homogeneous decompositions of the elements $f(s_i)$ have the same length, say $f(s_i) = \sum_{t=\alpha}^{\beta}(f(s_i))_{t+k}$, where $(f(s_i))_j$ is the $j$-th homogeneous component of $f(s_i)$. By taking $f_t(s_i) = (f(s_i))_{t+k}$, we have $f(s_i) = \sum_{t=\alpha}^{\beta} f_t(s_i)$. For elements $t = \alpha, \dots, \beta$, we define the maps $f_t : I \to M$ in the natural way as $f_t(\sum a_i s_i) = \sum a_i f_t(s_i)$. However, we have to prove that these maps are well defined. This is the content of the following proposition.

**Proposition 3.36.** $f_t$ *is well defined for every element* $t = \alpha, \dots, \beta$.

*Proof.* We divide the proof in three parts.

- Suppose that $0 = \sum a_i s_i$, with $a_i \in R_{k_1}$ for every $i$ (recall that $s_i \in R''_k$). Fix $i$. Since $s_j \in R''$ for each $1 \leq j \leq n$, there exist elements $u_j \in R'$ and $v_j \in S_i$ such that $u_j s_i = v_j s_j$. In particular, $\deg(u_j) = \deg(v_j)$, and since $u_j, v_j \in R'$, $u_j f(s_i) = v_j f(s_j)$, and $M$ is LSG, if we compare the homogeneous components of the same degree, then we obtain that $u_j f_t(s_i) = v_j f_t(s_j)$, for each $\alpha \leq t \leq \beta$.

  Now, by using that $v_1 \in S_i$ and $a_1 \in R$, there exist elements $b_1 \in R$ and $c_1 \in S_i$ such that $b_1 v_1 = c_1 a_1$. Repeating this argument with the elements $v_2$ and $c_1 a_2$, we find that $b_2 \in R$ and $c_2 \in S_i$ satisfy the equality $b_2 v_2 = c_2 c_1 a_2$. Continuing in this way, for every $1 \leq j \leq n$ we will find elements $b_j \in R$ and $c_j \in S_i$ such that $b_j v_j = \prod_{i=1}^{j} c_i a_j$ (notice that the elements $b_i$'s can be taken homogeneous). If we define $c := \prod_{i=1}^{n} c_i \in S_i$ and $d_j = \prod_{i=j+1}^{n} c_i b_j$, then we have $d_j v_j = c a_j$, for every $1 \leq j \leq n$. Hence $0 = c \sum_{j=1}^{n} a_j s_j = \sum_{j=1}^{n} d_j v_j s_j = \sum_{j=1}^{n} d_j u_j s_i = r s_i$, where $r = \sum_{j=1}^{n} d_j u_j$. Note that the elements $d_j u_j$ are homogeneous of the same degree, which implies that $r$ is also homogeneous. Since $0 = r s_i$, by the first condition of the noncommutative localization, there exists an element $s \in S_i$ such that $sr = 0$.

  Now, considering the equalities

$$sc \sum_{j=1}^{n} a_j f_t(s_j) = s \sum_{j=1}^{n} d_j v_j f_t(s_j) = s \sum_{j=1}^{n} d_j u_j f_t(s_i) = sr f_t(s_i) = 0,$$

  it follows that $\sum_{j=1}^{n} a_j f_t(s_j) \in \kappa_{S_i}(M)$. Since this holds for every element $i$, we have that $\sum_{j=1}^{n} a_j f_t(s_j) \in \bigcap \kappa_{S_i}(M) = \kappa_+(M) = 0$, whence $\sum_{j=1}^{n} a_j f_t(s_j) = 0$.

- Suppose that $0 = \sum_{i=1}^{n} a_i s_i$ (the elements $a_i$'s are not necessarily homogeneous). Since there are only finitely elements $a_i$'s, we can consider the sum $a_i = \sum_{j=l_1}^{l_2} b_{i,j}$, with $b_{i,j} \in R_j$. In this way, $0 = \sum_{i=1}^{n} \sum_{j=l_1}^{l_2} b_{i,j} s_i = \sum_{j=l_1}^{l_2} \sum_{i=1}^{n} b_{i,j} s_i$. Now, using that $\sum_{i=1}^{n} b_{i,j} s_i \in R_{j+k}$ is the homogeneous component of degree $j+k$, it follows that $0 = \sum_{i=1}^{n} b_{i,j} s_i$. By the first part above, we can assert that $\sum_{i=1}^{n} b_{i,j} f_t(s_i) = 0$, whence $0 = \sum_{j=l_1}^{l_2} \sum_{i=1}^{n} b_{i,j} f_t(s_i) = \sum_{i=1}^{n} \sum_{j=l_1}^{l_2} b_{i,j} f_t(s_i) = \sum_{i=1}^{n} a_i f_t(s_i)$, for $\alpha \leq t \leq \beta$.

- Let $r$ be an element of $\sum R s_i$. Suppose that we have two expressions for $r$ given by $r = \sum a_i s_i = \sum b_i s_i$. Then $0 = \sum (a_i - b_i) s_i$. By the second part above, $\sum (a_i - b_i) f_t(s_i) = 0$, and so $\sum a_i f_t(s_i) = \sum b_i f_t(s_i)$, for $\alpha \leq t \leq \beta$. This means that the expression for $f_t(r)$ does not depend of the decomposition of $r$.

□

From the proof of Proposition 3.36 it follows that the maps $f_t$'s are $R-$homomorphisms. The next proposition establishes that these are homogeneous of degree $t$.

**Proposition 3.37.** *The map $f_t$ is homogeneous of degree $t$.*

*Proof.* Consider $s \in I \cap R'$ with $\deg(s) = l$. Let $(f_t(s))_m$ be the homogeneous component of degree $m$ in the expression of $f_t(s)$. For a fixed $i$, there exist elements $v_i \in S_i$ and $u_i \in R'$ such that $u_i s_i = v_i s$, which implies that $v_i f_t(s) = u_i f_t(s_i)$. Since $f_t(s_i) \in M_{t+k}$, $u_i, v_i \in R''$

and $M$ is $\mathsf{LSG}$, when we compare the homogeneous components of these elements, we have that if $m \neq t+l$ then $v_i(f_t(s))_m = 0$, whence $(f_t(s))_m \in \kappa_{S_i}(\varphi_M(M))$. Since this fact holds for every $i$, it follows that $(f_t(s))_m \in \bigcap \kappa_{S_i}(\varphi_M(M)) = \kappa_+(\varphi_M(M)) = 0$. Therefore, $f_t(s) = (f_t(s))_{t+l}$, which asserts that $f_t$ is homogeneous of degree $t$. □

Propositions 3.35, 3.36 and 3.37 imply the following important result.

**Proposition 3.38.** *If $M$ is an $\mathsf{LSG}$ $R$-module with $\kappa_+(M) = 0$, then $Q_{\kappa_+}(M) = M_{\mathscr{L}}$ is an $\mathsf{LSG}$ $R$-module with semigraduation given by*

$$M_{\mathscr{L}} = \bigoplus_k (M_{\mathscr{L}})_k.$$

**Theorem 3.39.** *If $M$ is an $\mathsf{LSG}$ $R-$module, then $Q_{\kappa_+}(M)$ is an $\mathsf{LSG}$ $R-$module.*

*Proof.* It follows from Proposition 3.38 and the fact that $\kappa_+(M/\kappa_+(M)) = 0$ and $Q_{\kappa_+}(M) = Q_{\kappa_+}(M/\kappa_+(M))$. □

### 3.3.3 Serre-Artin-Zhang-Verevkin theorem

In this section, we prove the Serre-Artin-Zhang-Verevkin theorem for semi-graded rings (Theorem 3.62) using a different approach than the one presented by Lezama [96, 99].

Briefly, this theorem was partially formulated by Lezama and Latorre [99, Theorem 6.12] where it was assumed that the semi-graded left Noetherian ring is a domain. Nevertheless, as is well-known, the Serre–Artin–Zhang–Verevkin theorem for finitely graded algebras does not include this restriction, so that this assumption was eliminated by Lezama [96, Theorem 1.24] (see also [41, Section 18.4, Theorem 18.5.13]). More exactly, he proved the theorem for an $\mathsf{SG}$ ring $R = \bigoplus_{n\geq 0} R_n$ satisfying the following conditions:

(C1) $R$ is left Noetherian;

(C2) $R_0$ is left Noetherian;

(C3) for every $n$, $R_n$ is a finitely generated left $R_0$-module;

(C4) $R_0 \subset Z(R)$.

Notice that condition (C4) implies that $R_0$ is a commutative Noetherian ring.

Universal enveloping algebras of finite-dimensional Lie algebras, some quantum algebras with three generators, and some examples of 3-dimensional skew polynomial algebras [18, 133, 139] illustrate the Serre–Artin–Zhang–Verevkin theorem [96, Example 1.26] and [41, Example 18.5.15].

We start with the following preliminary result.

**Lemma 3.40.** *Let $R$ be a positively* SG *left Noetherian ring and $S$ a non-trivial left Ore set of $R$. Then $\mathscr{L}(\kappa_+) \subseteq \mathscr{L}(S)$.*

*Proof.* Let $I \in \mathscr{L}(\kappa_+)$. There exist elements $t, n \in \mathbb{N}$ such that $R^n_{\geq t} \subseteq I$. Since $S$ is non-trivial, there exists $s \in S$ with $\deg(s) \geq 1$, whence $s^{tn} \in R^n_{\geq t}$. This fact shows that $S \cap I \neq \emptyset$. □

Lemma 3.40 says that if $M$ is an $R$-module and $S$ is a non-trivial left Ore set of $R$, then $\kappa_+(M) \subseteq \kappa_S(M)$.

**Lemma 3.41.** *Let $R$ be a positively* SG *left Noetherian ring and $S$ a non-trivial good left Ore set. If $M$ is an* LSG *$R$-module, then $S^{-1}(M) \cong S^{-1}(Q_{\kappa_+}(M))$.*

*Proof.* Let

$$f\colon S^{-1}M \longrightarrow S^{-1}(M/\kappa_+(M))$$
$$\frac{m}{s} \longmapsto \frac{\overline{m}}{s}.$$

It is clear that $f$ is surjective. Let $\frac{m}{s} \in \operatorname{Ker}(f)$. Then $\frac{\overline{m}}{s} = 0$, and so there exists $s' \in S$ such that $s'\overline{m} = 0$, i.e., $s'm \in \kappa_+(M) \subseteq \kappa_S(M)$. Hence, there exists $s'' \in S$ with $s''s'm = 0$, and since $s''s' \in S$, it follows that $\frac{m}{s} = 0$. Therefore, $S^{-1}(M) \cong S^{-1}(M/\kappa_+(M))$.

Now, let

$$g\colon S^{-1}(M/\kappa_+(M)) \longrightarrow S^{-1}(Q_{\kappa_+}(M))$$
$$\frac{\overline{a}}{s} \longmapsto \frac{h(\overline{a})}{s}.$$

Where $h$ is the isomorphisms between $M/\kappa_+(M)$ and $\varphi_M(M)$. Since $h$ is injective, so $g$ also is. Let $\frac{\xi}{s} \in S^{-1}(Q_+(M))$. Then there exist elements $t, n \in \mathbb{N}$ such that $(R^n_{\geq t})\xi \subseteq \varphi_M(M)$. Since $S$ is non-trivial, repeating the argument above in the proof of Lemma 3.40 we can assert that there exists $s' \in S$ such that $s'' = (s')^{tn} \in R^n_{\geq t}$. In this way, $s''\xi \in \varphi_M(M)$, and so there exist $m \in M$ such that $s''\xi = \varphi_M(m)$, whence $g\left(\frac{\overline{m}}{s''s}\right) = \frac{h(\overline{m})}{s''s} = \frac{\varphi_M(m)}{s''s} = \frac{s''\xi}{s''s} = \frac{\xi}{s}$. We conclude that $S^{-1}(Q_+(M)) \cong S^{-1}(M/\kappa_+(M)) \cong S^{-1}(M)$. □

For the rest of this section, $R$ denotes a schematic ring (recall that by Definition 3.29 $R$ is left Noetherian). Consider the full subcategory $(R, \kappa_+) -$ LSG of LSG $- R$ whose objects are the $\kappa_+$-closed modules. If $M$ is an $R$-module $\kappa_+$-closed and $N$ is a submodule of $M$, then $N$ is $\kappa_+$-closed if and only if $M/N$ is $\kappa_+$-torsion-free [158, Proposition 4.2, Chapter IX]. Hence, it is clear that the intersection of $\kappa_+$-closed modules is $\kappa_+$-closed. This fact allows us to consider the submodule $\kappa_+$-closed generated by a subset of $M$. If we define

$$N^c = \{x \in M \mid (N : x) \in \mathscr{L}(\kappa_+)\},$$

then it is clear that $N^c$ is the submodule $\kappa_+$-closed generated by $N$, and in fact, $N^c = M$ if and only if $M/N$ is $\kappa_+$-torsion.

Notice that in the category $(R, \kappa_+) - \mathsf{LSG}$ the subobjects are the submodules $\mathsf{LSG}$-$\kappa_+$-closed, that are closed under arbitrary intersections. The submodule $\mathsf{LSG}$-$\kappa_+$-closed generated by $X \subseteq M$ will be denoted as $\langle X\rangle^{\mathsf{SG}-\kappa}$. We will say that $M$ is $\mathsf{LSG}$-$\kappa_+$-*finitely generated* if there exists a finite set $X \subseteq M$ with $\langle X\rangle^{\mathsf{SG}-\kappa} = M$. Let $\mathrm{Proj}(R)$ be the full subcategory of $(R, \kappa_+) - \mathsf{LSG}$ consisting of $\mathsf{LSG}$-$\kappa_+$-finitely generated modules.

**Proposition 3.42.** *If $N$ is an $\mathsf{SG}$ submodule of $M$, then $N^c$ also is.*

*Proof.* Let $m = m_1 + \cdots + m_k \in N^c$ with $m_i \in M_{l_i}$. There exists $I \in \mathscr{L}(\kappa_+)$ such that $I \subseteq (N : m)$. Since $R$ is schematic by the good left Ore set $S_i$, $i = 1, \dots, n$, say, then there exist elements $s_i \in S_i$ with $\sum Rs_i \subseteq I$, whence $s_i m \in N$ for all $i$. Since $N$ is $\mathsf{SG}$ and $s_i \in R''$, then $s_i m_j \in N$, for each $i, j$. Thus, $\sum_{i=1}^{n} Rs_i \subseteq (N : m_j)$, which shows that $m_j \in N^c$. □

From these facts we have the equality $\langle X\rangle^{\mathsf{SG}-\kappa} = \left(\langle X\rangle^{\mathsf{SG}}\right)^c$ for each $X \subseteq M$. In this way, $M$ is $\mathsf{LSG}$-$\kappa_+$-finitely generated if and only if there exist a finite set $X \subseteq M$ such that $\left(\langle X\rangle^{\mathsf{SG}}\right)^c = M$, or equivalently, $M/M_1$ is $\kappa_+$-torsion with $M_1 = \langle X\rangle^{\mathsf{SG}}$.

Next, we want to define the notion of *noncommutative site.*

**Definition 3.43.** Let $\mathscr{O}$ be the set of non-trivial good left Ore sets of $R$ and $\mathscr{W}$ the free monoid on $\mathscr{O}$. We define the category $\underline{\mathscr{W}}$ as follows: the objects of $\underline{\mathscr{W}}$ are the elements of $\mathscr{W}$, while for two words $W$ and $W'$ we define the morphisms of $\underline{\mathscr{W}}$, denoted by $\mathrm{Hom}(W', W)$, as a singleton $\{W' \to W\}$ if there exists an increasing injection from the letters of $W$ to the letters of $W'$, i.e., $W = S_1, \dots S_n$ and $W' = V_0 S_1 V_1 S_2 V_2 \dots S_n V_n$ for some letters $S_i$ and some (possibly empty) words $V_i$. In other case, $\mathrm{Hom}(W', W)$ is defined to be empty.

It is easy to see that $\underline{\mathscr{W}}$ is a thin category. We denote the empty word as 1, which is the final object of the category.

If $W = S_1 \dots S_n \in \mathscr{W} \setminus \{1\}$ and $M$ is an $\mathsf{LSG}$ $R$-module, we define

$$Q_W(M) = S_n^{-1}R \otimes_R \cdots \otimes_R S_1^{-1}R \otimes_R M.$$

Lemma 3.41 asserts that if $W \neq 1$, then $Q_W(M) \cong Q_W(Q_{\kappa_+}(M))$.

If $W = S_1 \dots S_n \in \mathscr{W} \setminus \{1\}$, we say that $w \in W$ if $w = s_1 \dots s_n$ with $s_i \in S_i$. We associate a set of left ideals to $W$, namely

$$\mathscr{L}(W) = \{I \lhd_l R \mid \text{ there exists } w \in W \text{ such that } w \in I\}.$$

We define $\mathscr{L}(1) = \mathscr{L}(\kappa_+)$.

**Lemma 3.44.** *Let $W \in \mathscr{W} \setminus \{1\}$ and $w, w' \in W$. Then there exists $w'' \in W$ such that $w'' = aw$ and $w'' = bw'$, for some elements $a, b \in R$.*

*Proof.* We prove the assertion by induction on the length of elements of $W$. If $W = S_1$, then by the Ore's condition there exist elements $a \in R$ and $b \in S_1$ such that $aw = bw' \in S_1$.

Suppose that the assertion holds for every element of length $k$. Let $W = S_1 \dots S_{k+1}$, $\tilde{W} = S_2 \dots S_{k+1}$, $w = s_1 \dots s_{k+1}$, $w' = s'_1 \dots s'_{k+1} \in W$, $x = s_2 \dots s_{k+1}$, and $x' = s'_2 \dots s'_{k+1}$. By the inductive step, there exist elements $a, b \in R$ such that $ax = bx' \in \tilde{W}$. Since $S_1$ is a left Ore set, then there exist $s''_1 \in S_1$ and $a_1 \in R$ such that $a_1 s_1 = s''_1 a$. Hence, $a_1 w = a_1 s_1 x = s''_1 ax = s''_1 bx' \in W$. Again, by the Ore's condition, there exist $s^*_1 \in S_1$ and $b_1 \in R$ such that $b_1 s'_1 = s^*_1 s''_1 b$, whence $b_1 w' = b_1 s'_1 x' = s^*_1 s''_1 bx' = s^*_1 a_1 w \in W$. □

**Remark 3.45.** Lemma 3.44 can be extended to a finite collection of words, i.e., if $w_1, \dots, w_i \in W$, then there exist $a_1, \dots, a_n \in R$ such that $a_1 w_1 = a_2 w_2 = \dots = a_n w_n \in W$.

**Lemma 3.46.** *Let $W \in \mathscr{W} \setminus \{1\}$, $w \in W$ and $a \in R$. There exist elements $w' \in W$ and $b \in R$ with $w'a = bw$.*

*Proof.* We prove by induction on the length of words of $W$. If $W = S_1$, then the assertion is precisely the Ore's condition.

Suppose that the lemma holds for each element of length $k$. Let $W = S_1 \dots S_{k+1}$, $\tilde{W} = S_2 \dots S_{k+1}$, $w = s_1 \dots s_{k+1}$, and $x = s_2 \dots s_{k+1}$. By the inductive step, there exist elements $x' \in \tilde{W}$ and $b \in R$ such that $x'a = bx$. Since $S_1$ is an Ore set, there exist $s'_1 \in S_1$ and $b' \in R$ such that $s'_1 b = b' s_1$, whence $s'_1 x' a = s'_1 bx = b' s_1 x = b' w$. □

Lemmas 3.44 and 3.46 allow us to conclude that $\mathscr{L}(W)$ is a filter. In the case $W \neq 1$, we will call $\kappa_W$ the *pre-radical* associated to $\mathscr{L}(W)$. It is straightforward to see that for every $\mathsf{LSG}$ $R$-module $M$, the following equality holds

$$\kappa_W(M) = \{m \in M \mid \text{ there exists } w \in W \text{ such that } wm = 0\} = \operatorname{Ker}(M \to Q_W(M)).$$

Following [174, p. 113], a *global cover* is a finite subset $\{W_i \mid i \in I\}$ of $\mathscr{W}$ such that $\bigcap_{i \in I} \mathscr{L}(W_i) = \mathscr{L}(\kappa_+)$. For $W \in \mathscr{W}$, $\operatorname{Cov}(W)$ is defined as the set of all sets of the morphisms of $\underline{\mathscr{W}}$ of the form $\{W_i W \to W \mid i \in I\}$, where $\{W_i \mid i \in I\}$ is a global cover. It is clear that $\{1\}$ is a global cover that will be called trivial. Notice that the schematic condition guarantees the existence of at least one non-trivial global cover. This collection of coverings is not a Grothendieck topology of $\underline{\mathscr{W}}$, but satisfies similar conditions (Proposition 3.48) that allow us to talk about sheaves on $\underline{\mathscr{W}}$. For this reason, Van Oystaeyen and Willaert called the category $\underline{\mathscr{W}}$ with this coverings the *noncommutative site* (c.f. [176]).

The proof of the following lemma is analogous to the setting of graded rings [174, Lemma 1]. We include it for the completeness of the thesis.

**Lemma 3.47.** *If $\{W_i \mid i \in I\}$ is a global cover, then for all $V \in \mathscr{W}$,*

$$\bigcap_{i \in I} \mathscr{L}(W_i V) = \mathscr{L}(V).$$

*Proof.* If $I \in \mathscr{L}(V)$, there exists $v \in V$ such that $v \in I$. Let $w_i \in W_i$, we have that $w_i v \in W_i V$ and $w_i v \in I$ thus $I \in \mathscr{L}(W_i V)$. From this, $\mathscr{L}(V) \subseteq \bigcap_{i \in I} \mathscr{L}(W_i V)$.

Let $I \in \bigcap_{i\in I} \mathscr{L}(W_iV)$, for each $i$ there exist $v_i \in V$ and $w_i \in W_i$ such that $w_iv_i \in I$. by Remark 3.45 there exist $a_1, \ldots, a_n \in R$ and $v \in V$ such that $v = a_iv_i$, for each $i$. By Lemma 3.46, we obtain that there exist $w_i' \in W_i$ and $b_i \in R$ with $w_i'a_i = b_iw_i$. Since $w_i' \in \sum Rw_i'$ and $\{W_i \mid i \in I\}$ is a global cover, there exist elements $n, t \in \mathbb{N}$ such that $R^n_{\geq t} \subseteq \sum Rw_i'$. Multiplying by $v$ we obtain $\left(R^n_{\geq t}\right) v \subseteq I$. If $S$ is the first letter of $V$, there exists $s \in S \cap R^n_{\geq t}$. Finally, $sv \in I$ and $sv \in V$, and so $I \in \mathscr{L}(V)$. □

**Proposition 3.48.** *The category $\underline{\mathscr{W}}$ together with the sets* $\mathrm{Cov}(W)$ *for any element* $W \in \underline{\mathscr{W}}$ *satisfy the following properties:*

$G_1$ : $\{W \to W\} \in \mathrm{Cov}(W)$,

$G_2$ : $\{W_i \to W \mid i \in I\} \in \mathrm{Cov}(W)$ *and* $\forall i \in I : \{W_{ij} \to W_i \mid j \in I_i\} \in \mathrm{Cov}(W_i)$ *then* $\{W_{ij} \to W_i \to U \mid i \in I, j \in I_i\} \in \mathrm{Cov}(W)$,

$G_3$ : *if* $\{W_iW \to W \mid i \in I\} \in \mathrm{Cov}(W)$, $W' \to W \in \underline{\mathscr{W}}$, *and if we define* $W_iW \times_W W' = U_iW'$ *then* $\{W_iW \times_W W' \to W' \mid i \in I\} \in \mathrm{Cov}(W')$.

*Proof.* $G_1$ holds since $\{1\}$ is a global cover, $G_2$ is a direct consequence of Lemma 3.47 and $G_3$ is clear. □

Definitions 3.49 and 3.50 introduces the notion of presheaf and sheaf, respectively, in the setting of the category $\underline{\mathscr{W}}$.

**Definition 3.49.** A *presheaf* $\mathscr{F}$ on $\underline{\mathscr{W}}$ is a contravariant functor from $\underline{\mathscr{W}}$ to the category $\mathsf{LSG} - R$ such that for all $W \in \underline{\mathscr{W}}\backslash\{1\}$, the sections $\mathscr{F}(W)$ of $\mathscr{F}$ on $W$ is an $\mathsf{SG}$ $S^{-1}R$-module, where $S$ denotes the last letter of $W$ and $\mathscr{F}(1)$ is an $\mathsf{SG}$ $Q_{\kappa_+}(R)$-module.

Since $\mathscr{F}(1)$ denotes the global sections, we will denote it as $\Gamma_*(\mathscr{F})$. We abbreviate $\mathscr{F}(V \to W)$ as $\rho_V^W : \mathscr{F}(W) \to \mathscr{F}(V)$; if $W = 1$, then we will write $\rho_V$ instead of $\rho_V^1$.

**Definition 3.50.** A presheaf $\mathscr{F}$ on $\underline{\mathscr{W}}$ is a *sheaf* if it satisfies the following two properties:

(i) *Separatedness*: for all element $W \in \mathscr{W}$ and each global cover $\{W_i \mid i \in I\}$, if $m \in \mathscr{F}(W)$ satisfies that for every $i \in I$, $\rho^W_{W_iW}(m) = 0$ in $\mathscr{F}(W_iW)$, then $m = 0$.

(ii) *Gluing*: for all $W \in \mathscr{W}$ and each global cover $\{W_i \mid i \in I\}$, given $(m_i) \in \prod_i \mathscr{F}(W_iW)$ satisfying
$$\rho^{W_iW}_{W_iW_jW}(m_i) = \rho^{W_jW}_{W_iW_jW}(m_j), \quad \text{for all } (i,j) \in I \times I,$$
there exists an element $m \in \mathscr{F}(W)$ such that
$$\rho^W_{W_iW}(m) = m_i, \text{ for all } i \in I.$$

**Remark 3.51.** A presheaf $\mathscr{F}$ is a sheaf if and only if for every word $W$ and each global cover $\{W_i \mid i \in I\}$, $\mathscr{F}(W)$ (with the arrows given by $\mathscr{F}$) is the limit of the diagram

$$\begin{array}{ccc} \mathscr{F}(W_iW) & \longrightarrow & \mathscr{F}(W_iW_jW) \\ & \times & \\ \mathscr{F}(W_jW) & \longrightarrow & \mathscr{F}(W_jW_iW) \end{array} \tag{3.1}$$

*Proof.* Suppose that $\mathscr{F}$ is a sheaf. Let $M$ be an $\mathsf{SG}$ $R$-module with morphisms $f_i : M \to \mathscr{F}(W_iW)$ which are compatibles with the morphisms $\rho^{W_iW}_{W_iW_jW}$ and $\rho^{W_iW}_{W_JW_iW}$. Consider an element $m \in M$. By using this compatibility, we have that the element $(f_i(m))$ of $\prod_i \mathscr{F}(W_iW)$ satisfies the equality

$$\rho^{W_iW}_{W_iW_jW} f_i(m) = \rho^{W_jW}_{W_iW_jW} f_j(m), \quad \text{for all } (i,j) \in I \times I.$$

In this way, there is a unique element $m' \in \mathscr{F}(W)$ such that $\rho^{W}_{W_iW}(m') = f_i(m)$, for each $i \in I$. If we define the map $\beta : M \to \mathscr{F}(W)$ as $\beta(m) = m'$, then it is clear that $\beta$ is a homogeneous $R$-homomorphism and it is the only one that satisfies the equality $\rho^{W}_{W_iW} \circ \beta = f_i$, for all $i \in I$. Hence, $\mathscr{F}(W)$ is the limit of the diagram (3.1).

On the other hand, suppose that $\mathscr{F}(W)$ is the limit of the diagram (3.1). Let $m \in \mathscr{F}(W)$ such that $\rho^{W}_{W_iW}(m) = 0$, for each $i \in I$. This means that $m \in \bigcap \operatorname{Ker}(\rho^{W}_{W_iW})$, and by assumption on $\mathscr{F}(W)$, $\bigcap \operatorname{Ker}(\rho^{W}_{W_iW}) = 0$, whence $m = 0$. Let

$$A := \{(m_i) \in \prod \mathscr{F}(W_iW) \mid \rho^{W_iW}_{W_iW_jW}(m_i) = \rho^{W_jW}_{W_iW_jW}(m_j), \text{ for all } (i,j) \in I \times I\}.$$

It is clear that $A$ is an $\mathsf{SG}$ $R$-submodule of $\prod \mathscr{F}(W_iW)$, which guarantees the existence of only one homogeneous $R-$homomorphism $\beta : A \to \mathscr{F}(W)$ such that $\rho^{W}_{W_iW} \circ \beta = \pi_i$, for all $i \in I$, where $\pi_i$ denotes the usual projection. In this way, it is immediate that $\beta(m_i)$ is the element that satisfies the gluing condition. □

**Definition 3.52.** Let $M$ be an $\mathsf{LSG}$ $R$-module. We define the presheaf $\widehat{M}$ in the following way: for objects, $\widehat{M}(1) = Q_{\kappa_+}(M)$, and for $W \in \mathscr{W} \setminus \{1\}$, $\widehat{M}(W) = Q_W(M)$. Now, for morphisms, if $W \neq 1$ then to the map $V \to W$ we assign it the trivial morphism such that the following diagram commutes

$$\begin{array}{ccc} Q_W(M) & \xrightarrow{\rho^W_V} & Q_V(M) \\ \uparrow & \nearrow & \\ M & & \end{array} \tag{3.2}$$

while for the morphism $W \to 1$ we assign the composition map

$$Q_{\kappa_+}(M) \to Q_W(Q_{\kappa_+}(M)) \to Q_W(M),$$

where the first arrow is the natural map, and the second arrow is precisely the isomorphism obtained in Lemma 3.41.

If $W = S_1 \dots S_n$ and $w \in W$, say $w = s_1 \dots s_n$, then the element $\frac{1}{s_n} \otimes \cdots \otimes \frac{1}{s_1} \otimes m \in Q_W(M)$ will be noted as $\frac{m}{w}$. In particular, $\frac{m}{1}$ stands for $1 \otimes m$ in $Q_S(M)$, for $1 \otimes 1 \otimes m$ in $Q_{ST}(M)$, and so on, which element is meant depends on the module it belongs to.

The proof of the following two lemmas follow the same ideas to those presented in the setting of $\mathbb{N}$-graded rings [174, Lemmas 2 and 3].

**Lemma 3.53.** *Given elements $\frac{m}{w} \in Q_W(M)$ and $a \in R$, there exist $w' \in W$, $b \in R$ such* *that* $w'a = bw$ *and* $a\frac{m}{w} = \frac{bm}{w'} \in Q_W(M)$.

*Proof.* Let $W = S_1 \dots S_n$ and $w = s_1 \dots s_n$. We consider $a_n = a$ and define $a_i$ recursively. More exactly, for an element $a_i$, the Ore's condition guarantees the existence of elements $s_i' \in S_i$ and $a_{i-1} \in R$ such that $s_i' a_i = a_{i-1} s_i$. Hence, $a\frac{m}{w} = \frac{1}{s_n'} \otimes \dots \otimes \frac{1}{s_1'} \otimes a_0 m$. If we define $b = a_0$ and $w' = s_1' \dots s_n'$, then the assertion follows. $\square$

**Lemma 3.54.** *If* $\frac{m}{w} = \frac{n}{1}$ *in* $Q_W(M)$ *for some element* $n \in M$*, then there exist* $\tilde{w} \in W$ *and* $r \in R$ *such that* $\tilde{w} = rw$ *and* $\tilde{w}n = rm$.

*Proof.* Induction on the length $n$ of the word $w = s_1 \dots s_n$. The case $n = 1$ it is clear. Suppose that the assertion holds for any word of length less than $n$. Let $W' = S_1 \dots S_{n-1}$ and $w' = s_1 \dots s_{n-1}$. Then $\frac{1}{s_n} \otimes \frac{m}{w'} = 1 \otimes \frac{n}{1} \in S_n^{-1}(Q_{W'})$, so that there exist elements $a, b \in R$ such that $a\frac{m}{w'} = b\frac{n}{1} = \frac{bn}{1} \in Q_{W'}(M)$ and $as_n = b \in S_n$. By Lemma 3.53 there exist elements $w'' \in W'$ and $c \in R$ with $w''a = cw'$ and $\frac{cm}{w''} = a\frac{m}{w'} = \frac{bn}{1}$. By hypothesis, there exist $w''' \in W'$ and $d \in R$ such that $w''' = dw''$ and $w'''bn = dcm$, so that if we consider $\tilde{w} = w'''b$ and $x = dc$ the assertion follows. $\square$

Let $\{W_i \mid i \in I\}$ be a global cover. The limit of the diagram

$$\begin{array}{ccc} Q_W(Q_{W_i}(M)) & \longrightarrow & Q_W(Q_{W_j}(Q_{W_i}(M))) \\ & \searrow\!\!\!\!\!\nearrow & \\ Q_W(Q_{W_j}(M)) & \longrightarrow & Q_W(Q_{W_i}(Q_{W_j}(M))) \end{array} \qquad (3.3)$$

will be denoted by $\Gamma_W(\widehat{M})$. Notice that due to the universal property of the limit, for the family $\{M \to Q_{W_i} \mid i \in I\}$ there is a unique morphism $\varphi : M \to \Gamma_1(\widehat{M})$. This morphism is of great importance in the following lemma.

**Lemma 3.55.** *Let* $\varphi : M \to \Gamma_1(\widehat{M})$ *the morphism described above. Then* $\operatorname{Coker}(\varphi)$ *is* $\kappa_+$*-torsion.*

*Proof.* Let $\xi = \left(\frac{m_i}{w_i}\right)_i \in \Gamma_1(\widehat{M})$ with $w_i \in W_i$ and $\frac{1}{w_i} \otimes 1 \otimes m_i = 1 \otimes \frac{1}{w_j} \otimes m_j \in Q_{W_i}(Q_{W_j}(M))$, for all $i, j$. Fix $j$. Then $\frac{1 \otimes m_i}{w_i} = \frac{\frac{1}{w_j} \otimes m_j}{1} \in Q_{W_i}$, whence by Lemma 3.54 there exist elements $w_i' \in W_i$ and $a_i \in R$ such that $w_i' = a_i w$ and $w_i'(\frac{1}{w_j} \otimes m_j) = a_i(1 \otimes m_i)$, that is, $w_i'(\frac{m_j}{w_j}) = \frac{a_i m_i}{1} \in Q_{W_j}(M)$. By taking $I := \sum_{i \in I} Rw_i'$, it is clear that $I \in \bigcap_i \mathscr{L}(W_i) = \mathscr{L}(\kappa_+)$. In this way, there exist elements $t_j, n_j \in \mathbb{N}$ such that $R_{\geq t_j}^{n_j} \subseteq I$, which shows that $(R_{\geq t_j}^{n_j})\frac{m_j}{w_j}$ is contained in the direct image of the map $M \to Q_{W_j}(M)$, that is, $(R_{\geq t_j}^{n_j})\frac{m_j}{w_j} \subseteq \operatorname{Im}(M \to Q_{W_j}(M))$. If $n := \max\{n_i \mid i \in I\}$ and $t := \max\{t_i \mid i \in I\}$, then it is straightforward to see that this reasoning is true for every element $j$.

Let $a \in R_{\geq t}^n$. Then $a\xi = (\frac{n_i}{1})_i$, for some elements $n_i \in M$ with $1 \otimes 1 \otimes n_i = 1 \otimes 1 \otimes n_j$ in $Q_{W_i}(Q_{W_j}(M))$, for every $i, j$. Fix $i$. Lemma 3.54 guarantees that for each $j$, there exist elements $\tilde{w}_j \in W_i$ and $x_j \in R$ such that $\tilde{w}_j = x_j$ and $\tilde{w}_j\frac{n_j}{1} = x_j\frac{n_i}{1}$. Now, by Remark 3.45, we can find elements $w_i^* \in W_i$ with $w_i^*\frac{n_i}{1} = w_i^*\frac{n_j}{1}$, for all $j$. Hence,

$w_i^* a\xi = \varphi(w_i^* n_i)$. As above, by defining $J = \sum_{i\in I} Rw_i^*$, there exist elements $t'(a)$ and $n'(a)$ (notice that all elements depend on $a$) such that $\left(R_{\geq t'(a)}^{n'(a)}\right) a\xi \subseteq \varphi(M)$. Since $R$ is a left Noetherian ring, $R_{\geq t}^{n}$ is finitely generated, say by the elements $a_1, \dots a_r$. By defining $n' = \max\{n(a_k) \mid 1 \leq k \leq r\}$ and $t' = \max\{t'(a_k) \mid 1 \leq k \leq r\}$, we have $\left(R_{\geq t+t'}^{n+n'}\right) \xi \subseteq \varphi(M)$, i.e., $\mathrm{Coker}(\varphi)$ is $\kappa_+$-torsion, which concludes the proof. □

**Proposition 3.56.** *The presheaf $\widehat{M}$ is a sheaf.*

*Proof.* Fix a global cover $\{W_i \mid i \in I\}$ and let $\varphi : M \to \Gamma_1(\widehat{M})$ be the map established in Lemma 3.55. Let us see $\Gamma_1(\widehat{M}) \cong Q_{\kappa_+}(M) = \widehat{M}(1)$. Since for the family $\{Q_{\kappa_+}(M) \to Q_{W_i} \mid i \in I\}$ the universal property of the limit guarantees the existence of a unique morphism $\phi : Q_{\kappa_+}(M) \to \Gamma_1(\widehat{M})$, we obtain the following commutative diagram

$$\begin{array}{ccccc} & & \Gamma_1(\widehat{M}) & \longrightarrow & Q_{W_i}(M) \longrightarrow Q_{W_j}(Q_{W_i}(M)) \\ & \overset{\varphi}{\nearrow} & \uparrow \phi & \overset{\phi_i}{\nearrow} & \\ M & \longrightarrow & Q_{\kappa_+}(M) & & \end{array} \tag{3.4}$$

It is clear that $\mathrm{Ker}(\phi) \subseteq \bigcap \mathrm{Ker}(\phi_i) = \bigcap \kappa_{W_i}(Q_{\kappa_+}(M)) = \kappa_+(Q_{\kappa_+}(M)) = 0$. On the other hand, since $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\phi)$ and $\Gamma_1(\widehat{M})/\mathrm{Im}(\varphi)$ is $\kappa_+$-torsion (Lemma 3.55), it follows that $\Gamma_1(\widehat{M})/\mathrm{Im}(\phi)$ is $\kappa_+$-torsion also. Besides, if $S_i$ is the last letter of $W_i$, then $Q_{W_i}(M)$ is $\kappa_{S_i}$-torsion-free, and so it is $\kappa_+$-torsion-free. In this way, $\Gamma_1(\widehat{M})$ is the limit of objects that are $\kappa_+$-torsion-free, and it is clear that $\Gamma_1(\widehat{M})$ is $\kappa_+$-torsion-free also. Since we have the short exact sequence

$$0 \to Q_{\kappa_+}(M) \to \Gamma_1(\widehat{M}) \to \Gamma_1(\widehat{M})/\mathrm{Im}(\phi) \to 0,$$

it follows that $Q_{\kappa_+}(M) \cong \Gamma_1(\widehat{M})$ [45, Proposition 3.4].

Finally, by recalling that $Q_W$ is an exact functor that commutes with finite limits, if $W \neq 1$ we have

$$\Gamma_W(\widehat{M}) \cong Q_W(\Gamma_1(\widehat{M})) \cong Q_W(Q_{\kappa_+}(M)) \cong Q_W(M) = \widehat{M}(W).$$

Therefore, by Remark 3.51 we conclude that $\widehat{M}$ is a sheaf. □

Next, we define the notion of affine cover and quasi-coherent sheaf.

**Definition 3.57.** An *affine* cover is a finite subset $\{T_i \mid i \in I\}$ of $\mathcal{O}$ such that $\bigcap_{i\in I} \mathscr{L}(T_i) = \mathscr{L}(\kappa_+)$.

**Definition 3.58.** A sheaf $\mathscr{F}$ is *quasi-coherent* if there exists an affine cover $\{T_i \mid i \in I\}$, and for each $i \in I$ there exists an $\mathsf{SG}$ $T_i^{-1}R$-module $M_i$ such that for all morphisms $V \to W$

in the category $\underline{\mathscr{W}}$, we have a commutative diagram given by

$$\begin{array}{ccc} \mathscr{F}(T_iW) & \xrightarrow{\rho^{T_iW}_{T_iV}} & \mathscr{F}(T_iV) \\ \downarrow & & \downarrow \\ Q_W(M_i) & \longrightarrow & Q_V(M_i) \end{array} \tag{3.5}$$

where the vertical maps are isomorphisms in $\mathsf{LSG} - R$ and $Q_1(*) := Q_{\kappa_+}(*)$. $\mathscr{F}$ is called *coherent* if moreover all $M_i$ are finitely generated $\mathsf{SG}$ $T_i^{-1}R$-modules.

**Remark 3.59.** Note that the sheaf $\widehat{M}$ is quasi-coherent for each object in the category $\mathsf{LSG} - R$. If $M$ is finitely generated $\mathsf{SG}$ module, then $\widehat{M}$ is coherent.

The proof of the following proposition is analogous to the proof of [174, Theorem 1]. For the completeness of the thesis, we include it here.

**Proposition 3.60.** *If $\mathscr{F}$ is a quasi-coherent sheaf on $\mathscr{W}$ and $\Gamma_*(\mathscr{F})$ denotes its global sections $\mathscr{F}(1)$, then $\mathscr{F}$ is isomorphic to $\widehat{\Gamma_*(\mathscr{F})}$, the sheaf associated to $\Gamma_*(\mathscr{F})$.*

*Proof.* First of all, notice that we can suppose that $M_i$ is $\kappa_+$-closed because if this is not the case, then we can replace it by $Q_{\kappa_+}(M_i)$ and the commutative diagram 3.5 holds. We want to see that $\mathscr{F}(W) \cong Q_W(\Gamma_*(\mathscr{F}))$ for all $W \in \mathscr{W}$. If $W \neq 1$, then by Remark 3.51 and the fact that $Q_W$ commutes with finite limits (recall that $Q_W$ is an exact functor), it follows that $\mathscr{F}(W)$ is the limit of the diagram (3.1), while that $Q_W(\Gamma_*(\mathscr{F}))$ is the limit of the diagram

$$\begin{array}{ccc} Q_W(\mathscr{F}(W_i)) & \longrightarrow & Q_W(\mathscr{F}(W_iW_j)) \\ & \times & \\ Q_W(\mathscr{F}(W_j)) & \longrightarrow & Q_W(\mathscr{F}(W_jW_i)) \end{array} \tag{3.6}$$

Notic that we have the isomorphism $\mathscr{F}(T_i) \cong Q_{\kappa_+}(M_i) = M_i$, and by the diagram 3.5, for every $W$ there exists an isomorphism $\psi_i^W : Q_W(\mathscr{F}(T_i)) \to \mathscr{F}(T_iW)$. If $W = S_1 \ldots S_n$ and $W_t := S_1 \ldots S_t$, then we obtain the following commutative diagram

$$\begin{array}{ccccccccccc} \mathscr{F}(T_i) & \xrightarrow{\rho^{T_i}_{T_iW_1}} & \mathscr{F}(T_iW_1) & \xrightarrow{\rho^{T_iW_1}_{T_iW_2}} & \mathscr{F}(T_iW_2) & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}(T_iW_{n-1}) & \xrightarrow{\rho^{T_iW_{n-1}}_{T_iW}} & \mathscr{F}(T_iW) \\ \downarrow & & \downarrow \psi_i^{W_1} & & \downarrow \psi_i^{W_2} & & & & \downarrow \psi_i^{W_{n-1}} & & \downarrow \psi_i^{W} \\ \mathscr{F}(T_i) & \longrightarrow & Q_{W_1}(\mathscr{F}(T_i)) & \longrightarrow & Q_{W_2}(\mathscr{F}(T_i)) & \longrightarrow & \cdots & \longrightarrow & Q_{W_{n-1}}(\mathscr{F}(T_i)) & \longrightarrow & Q_W(\mathscr{F}(T_i)) \end{array} \tag{3.7}$$

Since $Q_{W_t}(\mathscr{F}(T_i))$ is an $S_t^{-1}R$-module, and so $\mathscr{F}(T_iW_t)$ also is, for an element $s_t \in S_t$ we can multiply by $s_t^{-1}$, whence the commutativity of the diagram above guarantees that

$$\psi_i^W\left(\frac{1}{s_n} \otimes \cdots \otimes \frac{1}{s_1} \otimes m\right) = s_n^{-1}\rho^{T_iW_{n-1}}_{T_iW}(s_{n-1}^{-1}\rho^{T_iW_{n-2}}_{T_iW_{n-1}}(\ldots s_2^{-1}\rho^{T_iW_1}_{T_iW_2}(s_1^{-1}\rho^{T_i}_{T_iW_1}(m))\ldots)). \tag{3.8}$$

On the other hand, we have

$$\mathscr{F}(T_iT_jW) \cong Q_{T_jW}(\mathscr{F}(T_i)) = Q_W(Q_{T_j}(\mathscr{F}(T_i))) \cong Q_W(\mathscr{F}(T_iT_j)).$$

If we write as $\psi_{ij}^W : Q_W(\mathscr{F}(T_iT_j)) \to \mathscr{F}(T_iT_jW)$ the isomorphism above, by using a similar diagram to the above it can be seen that

$$\psi_{ij}^W\left(\frac{1}{s_n}\otimes\cdots\otimes\frac{1}{s_1}\otimes m\right) = s_n^{-1}\rho_{T_iT_jW}^{T_iT_jW_{n-1}}(s_{n-1}^{-1}\rho_{T_iT_jW_{n-1}}^{T_iT_jW_{n-2}}(\dots s_2^{-1}\rho_{T_iT_jW_2}^{T_iT_jW_1}(s_1^{-1}\rho_{T_iT_jW_1}^{T_i}(m))\dots)). \tag{3.9}$$

Notice that $\rho_{T_iT_jW_t}^{T_iW_t}$ and $\rho_{T_jT_iW_t}^{T_iW_t}$ are $S_t^{-1}R$-linear for $t = 1, \dots, n$, since both are $R$-homomorphisms between $S_t^{-1}R$-modules. In this way, the expressions (3.8) and (3.9) imply the commutativity of the following two diagrams

$$\begin{array}{ccc} Q_W(\mathscr{F}(T_i)) & \xrightarrow{Q_W\left(\rho_{T_iT_j}^{T_i}\right)} & Q_W(\mathscr{F}(T_iT_j)) \\ \downarrow{\scriptstyle\psi_i} & & \downarrow{\scriptstyle\psi_{ij}} \\ \mathscr{F}(T_iW)) & \xrightarrow{\rho_{T_iT_jW}^{T_iW}} & \mathscr{F}(T_iT_jW) \end{array} \quad \begin{array}{ccc} Q_W(\mathscr{F}(T_i)) & \xrightarrow{Q_W\left(\rho_{T_jT_i}^{T_i}\right)} & Q_W(\mathscr{F}(T_jT_i)) \\ \downarrow{\scriptstyle\psi_i} & & \downarrow{\scriptstyle\psi_{ji}} \\ \mathscr{F}(T_iW) & \xrightarrow{\rho_{T_jT_iW}^{T_iW}} & \mathscr{F}(T_jT_iW) \end{array} \tag{3.10}$$

Hence, it is clear that diagrams (3.51) and (3.6) must to have isomorphic limits, that is, $\mathscr{F} \cong Q_W(\mathscr{F}(\Gamma_*))$. Besides, for a morphism $V \to W$ the map $\rho_V^W$ is determined by the maps $\rho_{T_iV}^{T_iW}$ and $\rho_{T_iT_jV}^{T_iT_jW}$, which shows that the diagram

$$\begin{array}{ccc} Q_W(\Gamma_*(\mathscr{F})) & \longrightarrow & Q_v(\Gamma_*(\mathscr{F})) \\ \downarrow & & \downarrow \\ \mathscr{F}(W) & \xrightarrow{\rho_v^w} & \mathscr{F}(V) \end{array}$$

is commutative.

For $W = 1$ we have to show that $\Gamma_*(\mathscr{F}) \cong Q_{\kappa_+}(\Gamma_*(\mathscr{F}))$. Since $\mathscr{F}(T_i)$ and $\mathscr{F}(T_iT_j)$ are $\kappa_+$-torsion-free ($\mathscr{F}(T_i) \cong M_i$ and $\mathscr{F}(T_iT_j) \cong T_j^{-1}(\mathscr{F}(T_i))$), then $\Gamma_*(\mathscr{F})$ is $\kappa_+$-torsion-free because it is the limit of objects $\kappa_+$-torsion-free. Let us see that $\Gamma_*(\mathscr{F})$ is $\kappa_+$-injective. By [45, Proposition 3.2], it is sufficient to show that for all $I \in \mathscr{L}(\kappa_+)$, every $R$-homomorphism $f : I \to \Gamma_*(\mathscr{F})$ can be extended to a $R$-homomorphism $g : R \to \Gamma_*(\mathscr{F})$.

Since $\mathscr{F}(T_i)$ is $\kappa_+$-injective, the map $\rho_{T_i} \circ f$ can be extended to a map $g_i : R \to \mathscr{F}(T_i)$. If $x_i = g_i(1)$, then $g_i(r) = rx_i$ for every $r \in R$. In particular, for each $a \in I$ we have that $a\rho_{T_iT_j}^{T_i}(x_i) = \rho_{T_iT_j}^{T_i}(\rho_{T_i}(f(a))) = \rho_{T_iT_j}^{T_j}(\rho_{T_j}(f(a))) = \rho_{T_iT_j}^{T_j}(x_j)$, which shows that there exists an element $x \in \Gamma_*(\mathscr{F})$ such that $\rho_{T_i}(x) = x_i$, for every $i$. Notice that the map $g : I \to \Gamma_*(\mathscr{F})$ defined by $g(r) = rx$ extends $f$, so we conclude $\Gamma_*(\mathscr{F}) = Q_{\kappa_+}(\Gamma_*(\mathscr{F}))$. □

**Theorem 3.61.** *The category of quasi-coherent sheaves is equivalent to the category* $(R, \kappa_+) - \mathsf{LSG}$.

*Proof.* Let $\mathscr{F}$ be a quasi-coherent sheaf. From the last part of the proof of the Proposition 3.60, we have that $\Gamma_*(\mathscr{F})$ is an object of $(R, \kappa_+) - \mathsf{LSG}$. Moreover, if $M$ belongs to $(R, \kappa_+) - \mathsf{LSG}$, then $M \cong Q_{\kappa_+}(M)$. In this way, $\widehat{\square}$ and $\Gamma_*\square$ are functors between the category $(R, \kappa_+) - \mathsf{LSG}$ and the category of quasi-coherent sheaves, which are equivalences by Propositions 3.56 and 3.60. ☐

Finally, we arrive to the most important result of the chapter: the *Serre-Artin-Zhang-Verevkin* theorem for semi-graded rings.

**Theorem 3.62. (Serre-Artin-Zhang-Verevkin theorem).** *The category of coherent sheaves is equivalent to* Proj$(R)$*.*

*Proof.* From Theorem 3.61, it is sufficient to show that $M$ belongs to Proj$(R)$ if and only if $\widehat{M}$ is coherent.

We fix a cover $\{T_i \mid i \in I\}$. If $M$ is an element of Proj$(R)$, then there exist $m_1, \dots, m_k \in M$ such that $M/N$ is $\kappa_+$-torsion with $N = \langle m_1, \dots, m_k \rangle^{\mathsf{SG}}$. Let $f_i : M \to T_i^{-1}M$ be the canonical map. It is straightforward to see that $\langle \frac{m_1}{1}, \dots, \frac{m_k}{1} \rangle_R^{\mathsf{SG}} = f_i(N)$, and that the $T_i^{-1}M$-submodule $J_i = \langle \frac{m_1}{1}, \dots, \frac{m_k}{1} \rangle^{\mathsf{SG}}_{T_i^{-1}R}$ satisfies the relation $f_i(N) \subseteq J_i$. Let $m \in M$. Since $M/N$ is $\kappa_+$-torsion, there exists $I \in \mathscr{L}(\kappa_+)$ such that $Im \subseteq N$. By using that $T_i$ is non-trivial, there exists $t_i \in I \cap T_i$, whence $t_i m \in N$, which shows that $\frac{t_i m}{1} \in f_i(N)$. Since $J_i$ is a $T_i^{-1}R$-module and $\frac{t_i m}{1} \in J_i$, then $J_i = T_i^{-1}M$, that is, $T_i^{-1}M$ is finitely generated as an $\mathsf{SG}$ $T_i^{-1}R$-module.

Suppose that every one of the $T_i^{-1}M$ is a finitely generated $\mathsf{SG}$ $T_i^{-1}R$-module. Note that if $T_i^{-1}M = \langle \frac{m_{1,i}}{s_{1,i}}, \dots, \frac{m_{t_i,i}}{s_{t_i,i}} \rangle^{\mathsf{SG}}_{T_i^{-1}R}$, then $T_i^{-1}M = \langle \frac{m_{1,i}}{1}, \dots, \frac{m_{t_i,i}}{1} \rangle^{\mathsf{SG}}_{T_i^{-1}R}$. Since there are finite elements $i$'s, then the union set $\bigcup_{i \in I} \{m_{1,i}, \dots, m_{t_i,i}\}$ is also finite, $\{m_1, \dots, m_k\}$ say. If we define $N = \langle m_1, \dots, m_k \rangle^{\mathsf{SG}}$, it is straightforward to see that $T_i^{-1}N$ is an $\mathsf{SG}$ $T_i^{-1}R$-submodule of $T_i^{-1}M$, whence $T_i^{-1}N = T_i^{-1}M$.

For an element $m \in M$, since $\frac{m}{1} \in T_i^{-1}N$ there exist elements $n_i \in N$ and $t_i \in T_i$ such that $\frac{m}{1} = \frac{n_i}{t_i}$. Hence, there exist $c_i, d_i \in R$ such that $c_i m = d_i n_i$ and $c_i = d_i t_i \in T_i$, whence $c_i m \in N$. By using that $c_i \in T_i$ for each $i \in I$ and that $\{T_i \mid \in I\}$ is an affine cover, it follows that $I = \sum Rc_i \in \mathscr{L}(\kappa_+)$. We conclude that $Im \subseteq N$, and therefore $M/N$ is $\kappa_+$-torsion. ☐

Next, we show that the notion of schematicness in the semi-graded setting generalizes the corresponding concept in the case of connected and $\mathbb{N}$-graded algebras introduced and studied by Van Oystaeyen and Willaert [173, 174, 175, 177, 180].

**Remark 3.63.** Consider a positively graded left Noetherian ring $R$. It is clear that $R_+ = R_{\geq 1}$. Note that if $R$ is generated in degree one, then $R_{\geq t} = (R_+)^t$, which shows that $\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \text{there exists } n \in \mathbb{N} \text{ with } (R_+)^n \subseteq I\}$. On the other hand, the $\mathsf{LSG}$ modules are exactly the same $\mathbb{N}$-graded modules, and the good left Ore sets coincide with the homogeneous left Ore sets. In this way, the notion of schematic ring presented in this thesis generalizes the corresponding notion introduced by Van Oystaeyen and Willaert [174]. Last, but not least, notice that in the $\mathbb{N}$-graded setting, the left noetherianity of $R$

implies that the finitely generated objects of $(R, \kappa_+) - \mathsf{LSG}$ are the Noetherian objects, which shows that Theorem 3.62 generalizes [174, Theorem 3].

Next, we show some examples that illustrate our Theorem 3.62 in the case of non-$\mathbb{N}$-graded rings where [174, Theorem 3] cannot be applied.

**Example 3.64.** (i) Consider the first Weyl algebra $A_1(\Bbbk) = \Bbbk\{x, y\}/\langle yx - xy - 1\rangle$ over a field $\Bbbk$ of $\text{char}(\Bbbk) = p > 0$. It is well-known that $A_1(\Bbbk)$ is a non-$\mathbb{N}$-graded ring, the set $\{x^n y^m \mid n, m \in \mathbb{N}\}$ is $\Bbbk$-basis of $A_1(\Bbbk)$, and $A_1(\Bbbk)$ is a Noetherian ring. Since $x^p, y^p \in Z(A_1(\Bbbk))$, it is clear that $\{x^{pk} \mid k \in \mathbb{N}\}$ and $\{y^{pk} \mid k \in \mathbb{N}\}$ are good left Ore sets. Besides, if $k_1, k_2 \in \mathbb{N}$ then $A_1(\Bbbk)x^{pk_1} + A_1(\Bbbk)y^{pk_2}$ is a two-sided ideal of $A_1(\Bbbk)$ which is a left $\mathsf{SG}$ submodule, whence $A_1(\Bbbk)_{\geq pk_1+pk_2} \subseteq A_1(\Bbbk)x^{pk_1} + A_1(\Bbbk)y^{pk_2}$. Therefore, $A_1(\Bbbk)$ is a schematic algebra and Theorem 3.62 holds.

(ii) In a similar way, it can be shown that the $n$-th Weyl algebra $A_n(\Bbbk)$ is schematicness when $\text{char}(\Bbbk) = p > 0$.

(iii) The well-known Jordan plane $\Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$ is schematic when $\text{char}(\Bbbk) = p > 0$ since the sets $\{x^{pk} \mid k \in \mathbb{N}\}$ and $\{y^{pk} \mid k \in \mathbb{N}\}$ are good left Ore sets.

If $A = \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension over a field $\Bbbk$ with the usual semi-graduation, it is clear that $A$ is an $\mathbb{N}$-graded $\Bbbk$-algebra if and only if it is semi-commutative (Definition 1.26). The next proposition guarantees the schematicness of these extensions (c.f. Proposition 3.8).

**Proposition 3.65.** *If $A = \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ is a semi-commutative skew PBW extension over $\Bbbk$, then $A$ is schematic.*

*Proof.* Let $S_i := \{x_i^k \mid k \in \mathbb{N}\}$, for every $i = 1, \dots, n$. The idea is to show that the $S_i$'s are left Ore sets. Since $A$ is a domain (Proposition 1.31), the cancellability property holds directly.

Fix $i$. Let $x_i^k \in S_i$ and $p = \sum_{j=1}^{m} a_j X^{\alpha_j} \in A$, with $a_j \in \Bbbk$ and $\alpha_j = (\alpha_{1,j}, \dots, \alpha_{n,j}) \in \mathbb{N}^n$, for each $0 \leq j \leq m$. If we consider $e_i := (0, \dots, 1, \dots 0)$, $\Delta(\alpha_j) := \prod_{t<i} d_{t,i}^{\alpha_{t,j}}$, $\overline{\Delta}(\alpha_j) := \prod_{t>i} d_{t,i}^{\alpha_{t,j}}$, and $c_j := \Delta(\alpha_j)(\overline{\Delta}(\alpha_j))^{-1}$, then we have

$$
\begin{aligned}
x_i^k p &= x_i^k \left(\sum_{j=1}^{m} a_j x^{\alpha_j}\right) \\
&= \sum_{j=1}^{m} a_j (\Delta(\alpha_j))^k x^{\alpha_j + ke_i}.
\end{aligned}
$$

Now, by defining $q := \sum_{j=1}^{m} a_j c_j^k x^{\alpha_j}$, we obtain

$$
\begin{aligned}
qx_i^k &= \left(\sum_{j=1}^{m} a_j c_j^k x^{\alpha_j}\right) x_i^k \\
&= \sum_{j=1}^{m} a_j c_j^k (\Delta(\alpha_j))^k x^{\alpha_j + ke_i} \\
&= \sum_{j=1}^{m} a_j (\Delta(\alpha_j))^k x^{\alpha_j + ke_i},
\end{aligned}
$$

whence $x_i^k p = qx_i^k$, which shows that $S_i$ is a left Ore set. It is cleat that the set $S_i$ satisfies the schematicness condition. □

For the skew PBW extensions that are non-$\mathbb{N}$-graded rings, Proposition 3.66 establishes sufficient conditions to guarantee their schematicness.

**Proposition 3.66.** *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension over a left Noetherian ring $R$ with the usual semi-graduation, that is,* $\deg(x_i) = 1$ *and* $\deg(r) = 0$*, for every $i$ and each $r \in R$. If for every $i$ there exists $m_i \geq 1$ such that $x_i^{m_i} \in Z(A)$, then $A$ is schematic.*

*Proof.* From [41, Theorem 3.1.5] we know that $A$ is left Noetherian. Since $x_i^{m_i} \in Z(A)$, it follows that $\{x_i^{m_i m} \mid m \in \mathbb{N}\}$ is a non-trivial good left Ore Set for every $i$. Let us see that these sets satisfy the schematicness condition. Let $t_i \in \mathbb{N}$. Then $\sum_{i=1}^{n} Rx_i^{m_i t_i}$ is a two-sided ideal and an $\mathsf{SG}$ submodule of $A$. Besides, if $t := \sum m_i t_i$ then $\bigoplus_{m \geq t} R_m \subseteq \sum_{i=1}^{n} Rx_i^{m_i t_i}$, whence $R_{\geq t} \subseteq \sum_{i=1}^{n} Rx_i^{m_i t_i}$. □

Examples 3.67 and 3.68 show that the theory presented by Lezama about Serre-Artin-Zhang-Verevkin theorem and the one developed in this thesis are independent.

**Example 3.67.** Proposition 3.66 guarantees that if $R$ is a left Noetherian noncommutative ring, then $A = R[x]$ is schematic, and so Theorem 3.62 holds for $A$. Notice that this result cannot be obtained from the the theory developed by Lezama [96, 99] because does not satisfy Lezama's assumption (C4) that says that $A_0 = R$ is a commutative ring. Notice that in the particular case of the $\Bbbk$-algebra $R = M_n(\Bbbk)$, since $R$ is not connected it does not satisfy the definition of schematicness given by Van Oystaeyen and Willaert (Definition 3.6), and it is not a *finitely semi-graded algebra* in the sense of Lezama [99, Definition 2.4]. However, from our point of view, the algebra is schematic and Theorem 3.62 holds.

**Example 3.68.** Consider $A$ as the 3-dimensional skew polynomial algebra subject to the relations

$$yz = zy, \quad xz = zx, \quad \text{and} \quad yx = xy - z.$$

This algebra satisfies the Serre-Artin-Zhang-Verevkin theorem [41, Example 18.5.15 (v)] following the ideas presented by Lezama [96, 99]

It is straightforward to see that the following relations hold:

$$y^n x = xy^n - ny^{n-1}z, \quad \text{and} \quad yx^n = x^n y - nx^{n-1}z, \quad n > 0.$$

If $\operatorname{char}(\Bbbk) = p > 0$, then $x^p, y^p, z \in Z(A)$, and so Proposition 3.66 implies that $A$ is schematic.

Next, consider the case $\operatorname{char}(\Bbbk) = 0$. Let us see that $A''_n = \{az^n \mid a \in \Bbbk, n \in \mathbb{N}\}$. With this aim, consider $\alpha \in A''_n$. Then $\alpha$ is a homogeneous element of degree $n$, and so

$$\alpha = \sum_{i+j\leq n} a_{i,j} x^i y^j z^{n-i-j}.$$

Since that

$$\begin{aligned}
\alpha x &= \sum_{i+j\leq n} a_{i,j} x^i y^j x z^{n-i-j} \\
&= \sum_{i+j\leq n} a_{i,j} x^i (xy^j - jy^{j-1z}) z^{n-i-j} \\
&= \sum_{i+j\leq n} a_{i,j} x^{i+1} y^j z^{n-i-j} - \sum_{i+j\leq n} j a_{i,j} x^i y^{j-1} z^{n-i-j+1},
\end{aligned}$$

the element $\alpha x$ is homogeneous of degree $n+1$, whence $ja_{i,j} = 0$, for each $i, j$. In a similar way, for the element $y\alpha$ we obtain $ia_{i,j} = 0$, for each $i, j$. These facts imply that the only non-zero coefficient is precisely $a_{0,0}$, and so $\alpha = a_{0,0}z^n$. This shows that $A''_n = \{az^n \mid a \in \Bbbk, n \in \mathbb{N}\}$.

Now, let us prove that $A$ is not schematic. Since $z \in Z(A)$, then $S = \{az^k \mid a \in \Bbbk^*, k \in \mathbb{N}\}$ is a good left Ore set. Note that for all $m \in \mathbb{N}$, $x^m \in A_{\geq m} \backslash Rz$, whence $S$ does not satisfy the schematicness condition. Besides, due to the reasoning above, it is clear that $S$ contains any other good left Ore of $A$, and so if $S$ does not satisfy the schematicness condition, then no other set will.

Finally, Proposition 3.69 presents necessary conditions to assert the schematicness of skew PBW extensions with two indeterminates.

**Proposition 3.69.** *Let $A = \sigma(\Bbbk)\langle x, y\rangle$ be a skew PBW extension over $\Bbbk$ defined by the relation*

$$yx = dxy + ex + fy + g, \quad d, e, f, g \in \Bbbk. \tag{3.11}$$

*$A$ is schematic if and only if one of the following cases holds:*

(1) *$yx = dxy$ (quantum plane).*

(2) *$yx = xy + g$ with $\operatorname{char}(\Bbbk) = p > 0$.*

(3) *$yx = dxy + g$ with $d \neq 1$ and $d^p = 1$ for some $p \in \mathbb{N}$.*

*Proof.* We divide the proof into four parts.

(a) Let $P := dx + f$, $Q := ex + g$, $\overline{P} := dy + e$ and $\overline{Q} := fy + g$. Notice that the binomial theorem holds for $P$ and $\overline{P}$, that is,

$$P^i = \sum_{k=0}^{i} \binom{i}{k} d^{i-k} f^k x^{i-k}, \quad \text{and} \quad \overline{P}^i = \sum_{k=0}^{i} \binom{i}{k} d^{i-k} e^k y^{i-k}, \text{ for all } i > 0,$$

and that $yx = Py + Q = x\overline{P} + \overline{Q}$.

Let us see some relations of commutativity between $x$ and $y$.

For $n \geq 1$, the following identities

$$yx^n = P^n y + \sum_{i=0}^{n-1} P^{n-1-i} x^i Q, \text{ and} \tag{3.12}$$

$$y^n x = x\overline{P}^n + \sum_{i=0}^{n-1} \overline{P}^{n-1-i} y^i \overline{Q} \tag{3.13}$$

hold.

The case $n = 1$ is clear. Suppose that the assertion holds for $n$. Then

$$\begin{aligned} yx^{n+1} &= \left( P^n y + \sum_{i=0}^{n-1} P^{n-1-i} x^i Q \right) x \\ &= P^n(Py + Q) + \sum_{i=0}^{n-1} P^{n-1-i} x^{i+1} Q \\ &= P^{n+1} y + (P^n + \sum_{i=0}^{n-1} P^{n-1-i} x^{i+1}) Q \\ &= P^{n+1} y + \sum_{i=0}^{n} P^{n-i} x^i Q, \end{aligned}$$

which concludes the proof. In a similar way, we can prove the other equality.

(b) For $n > 0$, we write $\Delta_n := \sum_{i=0}^{n-1} d^i$.

Let us see that if $\xi = ax^n$ (resp. $\xi = ay^n$) belongs to $R''_n$ with $a \neq 0$, then $f = 0$ (resp. $e = 0$). In the case $Q \neq 0$ (resp. $\overline{Q} \neq 0$), it follows that $\Delta_n = 0$.

The equalities

$$\begin{aligned} y\xi &= ayx^n \\ &= a\left(P^n y + \sum_{i=0}^{n-1} P^{n-1-i}x^iQ\right) \\ &= a\left(\sum_{k=0}^{n}\binom{n}{k}d^{n-k}f^kx^{n-k}y + \sum_{i=0}^{n-1} P^{n-1-i}x^iQ\right), \end{aligned}$$

show that the element $y\xi$ is homogeneous of degree $n+1$, and that the monomials having $y$ satisfy that if $k \neq 0$ then $a\binom{n}{k}d^{n-k}f^k = 0$. In particular, if $k = n$, then $af^n = 0$, whence $f = 0$.

Now, with respect with the other monomials, it is clear that these form a polynomial element of degree less than $n+1$, which shows that

$$a\sum_{i=0}^{n-1} P^{n-1-i}x^iQ = 0.$$

Since $f = 0$, $P = dx$, and so

$$0 = a\sum_{i=0}^{n-1} P^{n-1-i}x^iQ = a\left(\sum_{i=0}^{n-1} d^{n-1-i}\right)x^{n-1}Q.$$

Thus, if $Q \neq 0$ then

$$0 = \sum_{i=0}^{n-1} d^{n-1-i} = \sum_{i=0}^{n-1} d^i = \Delta_n.$$

The proof for the case $ay^n$ is analogous.

(c) Let $n \geq 1$ and

$$\xi = \sum_{i=0}^{n} a_ix^iy^{n-i} \in R''_n.$$

Let us show that if $\overline{Q} \neq 0$ (resp. $Q \neq 0$) and $\xi \neq a_nx^n$ (resp. $\xi \neq a_0y^n$), then $e = 0$ (resp. $f = 0$) and $\Delta_k = 0$ for some $0 \leq k \leq n$.

Note that $\xi x$ is a homogeneous element of degree $n+1$. We have the equalities given by

$$\begin{aligned}
\xi x &= a_n x^{n+1} + \sum_{i=0}^{n-1} a_i x^i y^{n-i} x \\
&= a_n x^{n+1} + \sum_{i=0}^{n-1} a_i x^i \left( x\overline{P}^{n-i} + \sum_{j=0}^{n-1-i} \overline{P}^{n-1-i-j} y^j \overline{Q} \right) \\
&= a_n x^{n+1} + \sum_{i=0}^{n-1} \left( a_i x^{i+1}\overline{P}^{n-i} + a_i x^i \sum_{j=0}^{n-1-i} \overline{P}^{n-1-i-j} y^j \overline{Q} \right).
\end{aligned} \tag{3.14}$$

Suppose that there exists $0 \leq i \leq n-1$ such that $a_i \neq 0$, and let $t := \min\{0 \leq i \leq n \mid a_i \neq 0\}$. Then

$$\xi x = a_n x^{n+1} + \sum_{i=t}^{n-1} \left( a_i x^{i+1}\overline{P}^{n-i} + a_i x^i \sum_{j=0}^{n-1-i} \overline{P}^{n-1-i-j} y^j \overline{Q} \right).$$

Notice that the lower exponent of $x$ appears when $i = t$, and we have that

$$a_t x^t \sum_{j=0}^{n-1-t} \overline{P}^{n-1-t-j} y^j \overline{Q}$$

is a polynomial element of degree less than $n+1$ that have no another terms of $\xi x$, whence necessarily this polynomial has to be the zero element. Since $a_t \neq 0$, it follows that

$$\left( \sum_{j=0}^{n-1-t} \overline{P}^{n-1-t-j} y^j \right) \overline{Q} = 0.$$

By using that $\overline{Q} \neq 0$, we have $\sum_{j=0}^{n-1-t} \overline{P}^{n-1-t-j} y^j = 0$. Hence,

$$\begin{aligned}
0 &= \sum_{j=0}^{n-1-t} \overline{P}^{n-1-t-j} y^j \\
&= \sum_{j=0}^{n-1-t} \left( \sum_{k=0}^{n-1-t-j} \binom{n-1-t-j}{k} d^{n-1-t-j-k} e^k y^{n-1-t-j-k} \right) y^j \\
&= \sum_{j=0}^{n-1-t} \sum_{k=0}^{n-1-t-j} \binom{n-1-t-j}{k} d^{n-1-t-j-k} e^k y^{n-1-t-k}.
\end{aligned} \tag{3.15}$$

The coefficient of the monomial $y^0$ is obtained when $j = 0$ and $k = n-1-t$, which implies that

$$\binom{n-1-t}{n-1-t} d^{n-1-t-0-(n-1-t)} e^{n-t-1} = e^{n-t-1} = 0,$$

whence $n-1 \neq t$ and $e = 0$, and by replacing in the expression (3.15), it follows that

$$0 = \sum_{j=0}^{n-1-t} d^{n-1-t-j} y^{n-1-t},$$

and so

$$0 = \sum_{j=0}^{n-1-t} d^{n-1-t-j} = \sum_{j=0}^{n-1-t} d^j = \Delta_{n-t}.$$

The condition that there exists $n > 0$ such that $\Delta_n = 0$ is recursive, so will call it *Condition U*. It is straightforward to see that this condition is satisfied if and only if one of the following conditions hold:

- $d = 1$ and $\text{char}(\Bbbk) = p > 0$.
- $d \neq 1$ and there exists $p > 0$ such that $d^p = 1$.

(d) With the analysis above, we can determine the schematicness of the skew PBW extensions defined by relation (3.11).

First of all, note that if $d = 0$, Part (c) implies that $A'' = \Bbbk$ since the *Condition U* does not hold. Thus, the skew PBW extension $A$ is not schematic. From now on, consider $d \neq 0$. It is clear that the case $e = f = g = 0$ shows that $A$ is schematic. Let us see what happens if one of these three elements is non-zero and the *Condition U* does not hold.

Let $\xi \in A''_n$ with $n > 1$. If $g \neq 0$ then $Q \neq 0 \neq \overline{Q}$, and by Part (c), we have $\xi = a_n x^n = a_0 y^n$, whence $\xi = 0$, and so $A'' = \Bbbk$, which shows that $A$ is not schematic. If $e \neq 0$, then $Q \neq 0$, and Part (c) implies that $\xi = a_0 y^n$, whence $\xi = 0$ (Part (b)). In this case, $A$ is not schematic. Similarly, if $f \neq 0$ then $A$ is not schematic.

Let us see the case where *Condition U* holds (with the less value of $p$ satisfying this condition), and two of the three elements $e, f, g$ being non-zero. If $e \neq 0 \neq f$, then Part (c) implies that $\xi = 0$, whence $A$ is not schematic. Now, if $e \neq 0 \neq g$ and $f = 0$, it follows that $x^p \in Z(R)$. On the other hand, $Q \neq 0 \neq \overline{Q}$ and so Part (c) shows that $\xi = a_n x^n$ with $p \mid n$. In this way, $A'' = \{ax^{pk} \mid a \in \Bbbk, k \in \mathbb{N}\}$, and hence $S = \{ax^{pk} \mid a \in \Bbbk^*, k \in \mathbb{N}\}$ is the greatest left Ore set, and since $S$ does not satisfy the condition of schematicness (due to the powers of $y$), it is clear that $A$ is not schematic. Analogously, one can check that if $f \neq 0 \neq g$ and $e = 0$, then $A$ is not schematic.

Now, let us see the situation where *Condition U* is satisfied and only one element is non-zero. If $e \neq 0$ and $f = 0 = g$, then $\overline{Q} = 0$, and so equation 3.14 can be written as

$$\xi x = a_n x^{n+1} + \sum_{i=0}^{n-1} \sum_{k=0}^{n-i} \binom{n-i}{k} a_i d^{n-i-k} e^k x^{i+1} y^{n-i-k}.$$

Note that every value of $i$ corresponds to only one power of $x$, and when $k \neq 0$ the degree of $x^{i+1}y^{n-i-k}$ is less than $n+1$. These facts show that $\binom{n-i}{k}a_i d^{n-i-k}e^k = 0$, for each $0 \leq i \leq n-1$ and all $0 < k \leq n-i$. In particular, if $k=1$ then $a_i = 0$, and so $\xi = a_n x^n$. Part (b) implies that $p \mid n$, whence $A$ is not schematic by the same reason as above in the case $e \neq 0 \neq g$ and $f = 0$. Analogously, if $f \neq 0$ and $e = 0 = g$, it follows that $A$ is not schematic.

Finally, if the *Condition U* holds and $g \neq 0$ with $e = 0 = f$, then it is straightforward to see that $x^p, y^p \in Z(R)$, whence $A$ is schematic.

□

**Remark 3.70.** Proposition 3.69 shows that there are Ore extensions over schematic rings that are not schematic. This agrees with Proposition 3.8.

## 3.4 Future work

In this chapter, we defined the notion of schematic ring in the context of semi-graded objects and illustrated our Theorem 3.62 with some non-$\mathbb{N}$-graded algebras. With the aim of obtaining new examples of schematic algebras in this more general setting, it is of interest to generalize the criterion formulated by Van Oystayen and Willaert [177, Lemma 2] that says that if $R$ is an $\mathbb{N}$-graded $\Bbbk$-algebra such that its center $Z(R)$ is Noetherian and such that $R$ is a finitely generated $Z(R)$-module, then $R$ is schematic (Proposition 3.11). The research on its schematicness will be crucial for another families of noncommutative rings.

Now, having in mind that Van Oystayen and Willaert [175, 177, 180] investigated several topics of schematic algebras such as Čech cohomology and schematic dimension, a natural task is to study these topics in the context of semi-graded rings and the skew Ore polynomials of higher order.

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