Quasinormal ringing of acoustic black holes

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Abstract

The quasinormal modes of acoustic black holes in Laval nozzles are discussed. The equation for sound waves in transonic flow is rewritten into a Schrödinger-type equation with a potential barrier, and quasinormal frequencies are calculated by the WKB method. The results of numerical simulations show that the quasinormal modes are excited when the background flow in the nozzle is externally perturbed, as well as in the real black hole case. Also, we discuss how the quasinormal modes change when the outgoing waves are partially reflected at the boundary. It is found that the partially reflected quasinormal modes damp more slowly than the ordinary ones. Using this fact, we propose an experimental setup for detecting the quasinormal ringing of an acoustic black hole efficiently.

1 Introduction

When the geometry around a black hole is slightly perturbed, a characteristic ringdown wave is emitted. This kind of phenomenon is known as *quasinormal ringing*, which is expressed as a superposition of *quasinormal modes*, The central frequencies and the damping times of the quasinormal modes are determined by the geometry around the black hole, and thus the gravitational quasinormal ringing of a black hole is expected to play the important role of connecting gravitational-wave observation to astronomy.

Although quasinormal modes are themselves linear perturbations, they are in many cases excited after a *nonlinear* oscillation of a black hole, such as the black hole formation by the coalescence of the binary neutron stars. Therefore, when we want to study the excitation of the quasinormal modes in such a complicated situation, we have to resort to numerical relativity, which needs extremely powerful computatioal resources.

Here we present an alternative way for studying the quasinormal ringing of black holes by using a transonic fluid flow, called an *acoustic black hole* [2, 5]. In a transonic flow, sound waves can propagate from the subsonic region to the supersonic region, but cannot in the opposite way. Therefore, the sonic point of a transonic flow can be considered as the "event horizon" for sound waves, and the supersonic region as the "black hole region". Furthurmore, it is shown that the wave equation for a sound wave in an inhomogeneous flow is precisely equivalent to the wave equation for a massless scalar field in a curved spacetime [2]. This implies that an acoustic black hole has the quasinormal modes, which makes it possible for us to study the quasinormal ringing of black holes *in laboratories*.

In this paper, we show some results of out numerical simulations to prove that the quasinormal ringing of sounds occurs actually , and propose a feasible way to demonstrate QN ringing in a laboratory. For future experiments in laboratories, we treat one of the most accessible models of acoustic black holes: transonic airflow in a *L*aval nozzle [3]. A Laval nozzle is a wind tunnel that is pinched in the middle, and makes it possible to create a stable transonic flow in a laboratory.

A quasinormal mode is characterized by a complex frequency (called *q*uasinormal frequency) ω_{Q}, or equivalently, a pair of the central frequency $f_c \equiv \mathrm{Re}(\omega_{\mathrm{Q}})/2\pi$ and the quality factor $Q \equiv |\mathrm{Re}(\omega_{\mathrm{Q}})/2\mathrm{Im}(\omega_{\mathrm{Q}})|$. The quality factor is a quantity that is propotional to the number of cycles of the oscillation within the damping time. In experiments, damping oscillation like quasinormal ringing is inevitably buried in noise within a few damping times. Therefore, in order to detect quasinormal ringing efficiently, it is important to design a Laval nozzle which gives large quality factor.

We also discuss the quasinormal modes when the outgoing waves are*p*artially reflected at the boundary. In this situation, the quasinormal modes are found to decay *m*ore slowly than in the case where the reclection at the boundary does not occur.

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K	α		$\omega_{\rm Q}$ $[c_{s0}/L]$	
		(1st WKB)	(2nd WKB)	(3rd WKB)
2	0.1	$2.10 - 0.85i$	$1.78 - 1.01i$	$1.73 - 0.91i$
	0.03	$3.00 - 0.99i$	$2.65 - 1.11i$	$2.61 - 1.02i$
	0.01	$4.00 - 1.20i$	$3.57 - 1.34i$	$3.54 - 1.26i$
	0.003	$5.44 - 1.57i$	$4.86 - 1.76i$	$4.83 - 1.68i$
4	0.1	$3.36 - 1.25i$	$3.09 - 1.36i$	$3.15 - 1.48i$
	0.03	$4.24 - 1.14i$	$4.10 - 1.17i$	$4.12 - 1.23i$
	0.01	$5.02 - 1.10i$	$4.86 - 1.13i$	$4.85 - 1.09i$
	0.003	$5.91 - 1.15i$	$5.69 - 1.20i$	$5.66 - 1.09i$

Table 1: The quasinormal frequency ω_Q of the least-damped mode for different Laval nozzles, calculated by the third order WKB formula [7]. To show the convergence, the first lower order WKB values are also shown.

2 Quasinormal Modes of Acoustic Black Holes in Laval Nozzles

In what follows, we assume that flow in the nozzles is isentropic and one-dimensional. Thus, the wave equation for sound reads

$$\left[\left(\partial_t + \partial_x v\right)\frac{\rho A}{c_s^2}\left(\partial_t + v\partial_x\right) - \partial_x(\rho A \partial_x)\right]\phi = 0 \,, \tag{1}$$

where ρ, v, c_s, A, ϕ are the background density and fluid velocity, the speed of sound, the cross section of the nozzle, and the vecocity potential perturbation. Note that c_s depends on the background state, and is therefore a function of x.

Assuming steady background, the wave equation of sound (1) can be rewritten into the form of Eq. (2):

$$\left[\frac{d^2}{dx^{*2}} + \left(\frac{\omega}{c_{s0}}\right)^2 - V(x^*)\right]H_\omega = 0, \qquad V(x^*) = \frac{1}{g^2}\left[\frac{g}{2}\frac{d^2 g}{dx^{*2}} - \frac{1}{4}\left(\frac{dg}{dx^*}\right)^2\right], \tag{2}$$

where $x^* = c_{s0}\int \frac{dx}{c_s(1-M^2)}$, $M = v/c_s$, $g = \rho A/c_s$, $H_\omega(x^*) = g^{\frac{1}{2}}\ e^{i\omega F(x)}\int e^{i\omega t}\phi(t,x)dt$, and $F(x) = \int \frac{v\,dx}{c_s^2(1-M^2)}$. Here we have also introduced the stagnation speed of sound c_{s0}, which is constant over the isentropic region of the flow.

In this study, we consider a family of Laval nozzles which have the following form:

$$A(x) = \pi\, r(x)^2 \,, \tag{3}$$

$$r(x) = r_\infty - r_\infty(1-\alpha)\exp[-(x/2L)^{2K}] \,, \tag{4}$$

where K is a positive integer and $\alpha \equiv r(0)/r_\infty \in (0,1)$.

Having obtained the form of the potential $V(x^*)$, we can compute quasinormal frequencies by solving the Schrödinger-type wave equation (2) under the outgoing boundary conditions. In this study, we adopt the WKB approach, which was originally proposed by Schutz and Will [6] and has been developed in some works [7, 8].

Table 1 shows the least-damped quasinormal frequency $\omega_{\rm Q}$ for Laval nozzles with different (K, α) up to the 3rd order WKB values. We have found that the WKB method converge well for $K > 1$. It is noted that $f_c \sim c_{s0}/L$ and $1 \lesssim Q \lesssim 3$.

3 Numerical Simulations

In the case of astrophysical black holes, quasinormal ringing occurs when a black hole is formed or when a test particle falls into a black hole. Hence, let us consider analogous situations for our acoustic black holes: (i) *acoustic black hole formation* and (ii) *weak-shock infall.* In simulations of type (i), the initial

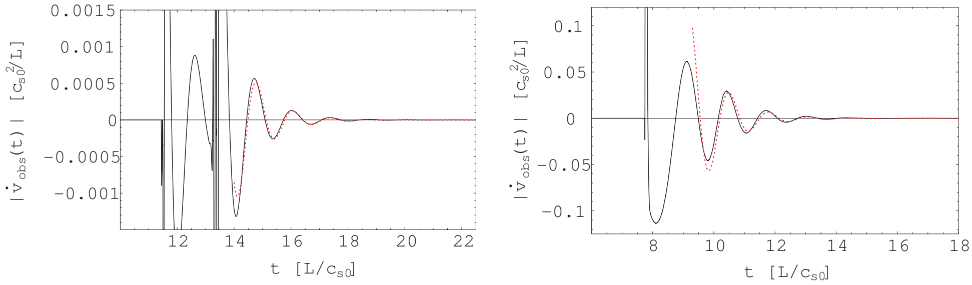

Figure 1: The numerical waveforms $\dot{v}(t)$ obtained by numerical simulations of acoustic black hole formation (left) and weak-shock infall (right). In both simulations, the nozzle parameters are set to $(K, \alpha) = (4, 0.01)$, which give the peak of the potential $V(x^*)$ at $x = 0.632$, and the observer is located at $x = 2.0$. Each waveform is compared to the analytic waveform of the damping oscillaton (dotted line) with the 3rd WKB value of the least-damped quasinormal mode frequency, $\omega_{\rm Q} = 3.54 - 1.26i$.

state of the flow in the nozzle is set to be steady and homogeneous. At time $t = 0$, a sufficiently large pressure difference is set between both ends of the nozzle, and the fluid in the nozzle starts to flow. In type (ii), the initial state is set to stationary transonic flow. At $t = 0$, the upsteream pressure is slightly raised, and a weak shock starts to fall into the sonic horizon. In both cases, we observe sound waves emitted from the potential barrier at a fixed position until the flow sufficiently settles down into a stationary transonic state.

In Fig. 1(a), the waveform observed in a numerical simulation of type (i) is plotted. For $t \lesssim 11L/c_{s0}$, nonlinear oscillation dominates, which corresponds to the "merger phase" oscillation in real black hole formation. At $t \simeq 11L/c_{s0}$, the background state begins to settle down into the statonary transonic state, and the oscillation enters into the ringdown phase. In this phase, the numerical waveform agrees in good accuracy with the analytic waveform of the least-damped quasinormal mode obtained by the third order WKB analysis. We have also performed some simulations of type (ii), and obtained the similar results, except that the nonlinear phase does not exist in the case (see Fig. 1(b)).

4 Effect of Reflection at the Boundary

It turns out from our numerical simulations that the quality factor for the quasinormal ringing of our acoustic black holes is typically $1 \lesssim Q \lesssim 2$. This means that the ringing oscillates by only a few cycles before it is substantially buried in noise. Thus, if one wants to observe it in experiments, this feature is quite unfavorable.

Now let us consider that the upstream tank has a finite length and the outgoing waves are *partially reflected* at the boundary wall of the tank. The existence of the reflection at the boundary effectively generalize the boundary condition for quasinormal modes. Assuming the boundary is located at $x = x_c$, the generalized boundary conditions are

$$H_\omega(x^*) \sim e^{-i\frac{\omega}{c_{s0}}x^*}, \qquad x^* \to -\infty, \tag{5}$$

$$H_\omega(x^*) \sim e^{+i\frac{\omega}{c_{s0}}(x^*-x_c^*)} + \mathcal{R}_\omega\, e^{-i\frac{\omega}{c_{s0}}(x^*-x_c^*)}, \qquad x^* \to x_c^*, \tag{6}$$

where $\mathcal{R}_\omega \in [-1, 1]$ is the reflection coefficient of the boundary wall. Sicne the solutions of the wave equation (2) with these boundary conditions are in a narrow sense no more the quasinormal modes, we shall refer to them as the *Partially Reflected QuasioNormal Modes* (PRQNMs).

Under the third order WKB approximation, the frequencies of PRQNMs $\omega_{\rm PQ}$ are given by the solutions of simultaneous equations

$$\frac{(2\pi)^{1/2}}{R(\nu)^2\Gamma(-\nu)} = \mathcal{R}_\omega \exp[i\pi\nu + 2i\frac{\omega}{c_{s0}}\Delta_\omega], \tag{7}$$

$$\left(\frac{\omega}{c_{s0}}\right)^2 = V(x_0^*) + \left(-2V''(x_0^*)\right)^{\frac{1}{2}}\tilde{\Lambda}(\nu) - i\left(\nu + \frac{1}{2}\right)\left(-2V''(x_0^*)\right)^{\frac{1}{2}}\left(1 + \tilde{\Omega}(\nu)\right), \tag{8}$$

$\mathcal{R}$	$\omega_{\rm PQ}$ $[c_{s0}/L]$	$\mathcal{R}$	$\omega_{\rm PQ}$ $[c_{s0}/L]$
0	$3.54 - 1.26i$	0	$3.54 - 1.26i$
10^{-4}	$3.46 - 1.20i$	-10^{-4}	$3.64 - 1.22i$
10^{-3}	$3.39 - 1.01i$	-10^{-3}	$3.72 - 1.01i$
0.01	$3.35 - 0.73i$	-0.01	$3.76 - 0.73i$
0.1	$3.34 - 0.41i$	-0.1	$3.76 - 0.42i$
1.0	$3.38 - 0.05i$	-1.0	$3.72 - 0.14i$

Table 2: The least-damped PRQNMs $\omega_{\rm PQ}$ for nozzle parameters $(K, \alpha) = (2, 0.01)$ and different values of $\mathcal{R}$, calculated by using the 3rd WKB formulae, Eqs. (7) and (8). The position of the half mirror, or contact surface, is set to $x_c = 3.0$.

where $\Gamma(-\nu)$ is the Gamma function, and Δ_ω is the distance between the half mirror and the classical turning point on the potential barrier. The functions $R(\nu)$, $\tilde{\Lambda}(\nu)$ and $\tilde{\Omega}(\nu)$ are shown in [7]. Note that, for $\mathcal{R}_\omega \equiv 0$, Eq. (7) gives $\nu = 0, 1, 2, ...$, which results the ordinary quasinormal mode case.

Table 2 shows how the frequency of the least-damped PRQNM $\omega_{\rm PQ}$ deviates from the ordinary quasinormal frequency $\omega_{\rm PQ}$ depending on $\mathcal{R}$. It is clear that $\mathrm{Im}(\omega_{\rm PQ})$ decreases drastically as $\mathcal{R}$ increases, although $\mathrm{Re}(\omega_{\rm Q})$ does not vary greatly with $\mathcal{R}$. This means that the reflection at the boundary *enhances* the quality factor of the quasinormal ringing. This fact is intuitively understandable if one notes that $\mathrm{Im}(\omega_{\rm Q})$ represents the energy dissipation rate, while $\mathrm{Re}(\omega_{\rm Q})$ characterizes the curvature of the background geometry. A partially-reflecting boundary wall suppress the dissipation of wave energy to infinity, but does not directly deform the background geometry.

5 Conclusion

We have analyzed the quasinormal modes of transonic fluid flow in Laval nozzles by WKB calculations and numerical simulations, and shown that the quasinormal ringing does arise in responce to some external perturbation. We have also argued the effect of a boundary wall which reflects the outgoing waves on quasinormal modes, and found that the existence of such a wall enhances the damping time of the ringing.

The results of our numerical study suggests a effective experimental setup for detecting the quasinormal ringing of transonic airflow in a laboratory. First, prepare a Laval nozzle and an air tank which is filled with sufficiently high-pressure air, and connect them with a shock tube. Then, remove a diaphragm in the shock tube to generate a transonic flow in the nozzle, and observe the sound waves coming out of the nozzle until the background flow sufficiently settles down. In this situation, the damping time of the quasinormal ringing will be enhanced in the upstream air tank, which will make the detection of the ringdown wave easier.

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