RESEARCH ARTICLE

Unified invariant of knots from homological braid action on Verma modules

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Abstract

We re-build the quantum $\mathfrak{sl}(2)$ unified invariant of knots F_∞ from braid groups' action on tensors of Verma modules. It is a two variables series having the particularity of interpolating both families of colored Jones polynomials and ADO polynomials, that is, semisimple and non-semisimple invariants of knots constructed from quantum $\mathfrak{sl}(2)$. We prove this last fact in our context that re-proves (a generalization of) the famous Melvin–Morton–Rozansky conjecture first proved by Bar-Natan and Garoufalidis. We find a symmetry of F_∞ nicely generalizing the well-known one of the Alexander polynomial, ADO polynomials also inherit this symmetry. It implies that quantum $\mathfrak{sl}(2)$ non-semisimple invariants are not detecting knots' orientation. Using the homological definition of Verma modules we express F_∞ as a generating sum of intersection pairing between fixed Lagrangians of configuration spaces of disks. Finally, we give a formula for F_∞ using a generalized notion of determinant, that provides one for the ADO family. It generalizes that for the Alexander invariant.

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Contents

1 | INTRODUCTION

1.1 | Quantum invariants of knots associated with $\mathfrak{sl}(2)$

From a quantum group and its category of finite-dimensional representations, one can construct invariants of knots, links (and ribbon graphs). It is the original construction of Reshetikhin and Turaev [36]. A quantum group could be think as a one parameter deformation of the enveloping algebra of a given semisimple Lie algebra. In the present paper, we only study the knot invariants arising from $U_q\mathfrak{sl}(2)$ which is a standard notation for the quantum group associated with $\mathfrak{sl}(2)$. Such knot invariants could be computed from *braid group* representations that are also part of the theory because they are defined using same finite dim. representations of $U_q\mathfrak{sl}(2)$. It is the approach of the present work. Historically, there are two families of knots that were extracted from the corresponding Reshetikhin–Turaev construction in this $\mathfrak{sl}(2)$-case.

(1) The colored Jones polynomials $\{J_N \in \mathbb{Z}\left[q^{\pm 1}\right], N \in \mathbb{N}\}$ obtained from *standard irreducible* rep. of dim. $N + 1$ of $U_q\mathfrak{sl}(2)$ denoted S_N as input (see, e.g., [32]), by use of Reshetikhin–Turaev construction. They could all be derived from the famous Jones polynomial [21].
(2) The *ADO* polynomials, sometimes called *colored Alexander polynomials*, $\{ADO_r \in \mathbb{C}\left[A^{\pm 1}\right], r \in \mathbb{N}\}$ arising from particular irreducible representations of $U_q\mathfrak{sl}(2)$ when q is evaluated at a root of unity. They were first defined by Akutsu–Deguchi–Ohtsuki [1], but they require a slight modification of the original tool developed by Reshetikhin–Turaev while the philosophy of using $U_q\mathfrak{sl}(2)$ is a constant. The first of the family is the well-known Alexander polynomial denoted $\mathcal{A}$.

The construction of Reshetikhin and Turaev uses the fact that in categories of representations of quantum groups they find inherent tools of the category behaving nicely with *Reidemeister moves*. Namely, there are *R-matrices* allowing to linearly represent braid groups carrying Reidemeister moves for braids, and *Markov traces* allowing to extract knot invariants from braid groups representations hence taking care of the remainder Reidemeister moves. Even though finding these two objects in any context is not trivial (e.g., colored Jones vs. ADO polynomials, where they are differently defined), they are always operators on $U_q\mathfrak{sl}(2)$ modules satisfying nice equation translating Reidemeister moves in an algebraic language. In the end, one obtains powerful topological invariants but their full algebraic flavor makes the topological interpretation of their content difficult and the subject of many conjectures in the field. One of the most famous expectation of topological content is the hyperbolic volume, which is the subject of the *volume conjecture* first stated by Kashaev [22] and relocated in the context of colored Jones polynomials by Murakami

and Murakami in [32]. The question on how to interpret topologically quantum invariants is more generally central.

Two other questions could be addressed to the picture.

- Could we construct knot invariants out of infinite dim. modules of $U_q\mathfrak{sl}(2)$? While Reshetikhin–Turaev construction requires finite dimension.
- Are quantum invariants colored Jones and ADO related or even equivalent? Even though the theory of representations of $U_q\mathfrak{sl}(2)$ is singularly different when q is a root of one (ADO case) than when q is generic (colored Jones case).

These three last questions (the two above and the topological interpretation of the construction of knot invariants) have recently reached new steps by use of the same objects: $U_q\mathfrak{sl}(2)$ Verma modules, which are infinite dim. modules on $U_q\mathfrak{sl}(2)$.

In [38], the second author has constructed a knot invariant denoted F_∞ using as input $U_q\mathfrak{sl}(2)$ Verma modules. The obtained object is a two variable infinite sum converging in the sense that it lives in a nice completion of the ring of Laurent polynomials with two variables $\mathcal{R} := \mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$. By nice, we mean, for example, that F_∞ can be evaluated at q being a root of unity or s being a power of q. Moreover in the first case F_∞ recovers the ADO polynomials and in the second the colored Jones ones. This double interpolation property implies an equivalence between the two families of knots invariants.

In [29], the first author has reconstructed $U_q\mathfrak{sl}(2)$ Verma modules, their tensor products, and the quantum braid group representation upon them from *homology of configuration spaces of points in punctured disks with coefficients in a local ring isomorphic to* $\mathcal{R}$. The action of braid groups on these modules is given by (more or less) homeomorphisms of configuration spaces, using the fact that braid groups are *mapping class groups* (isotopy classes of homeomorphisms) of punctured disks. Hence one can use this purely homological definition of Verma modules and quantum braid group representations avoiding dealing with quantum modules theory, shedding light on the topological content of it.

The present paper studies in details the tools surrounding Verma modules (their tensor product, braid group representations and knot invariant) developed in the two papers [29, 38], more particularly what topological information one could extract out of F_∞.

Next steps could be achieved using $U_q\mathfrak{sl}(2)$ and its modules, for instance, constructions of *topological quantum field theories* (*TQFTs*) which is a categorical construction providing invariants of links and embedded graphs (extending those of knots), 3-manifolds and mapping class groups of surfaces representations (extending those of braids). This was initiated by Reshetikhin and Turaev again [37] and the universal construction of Blanchet–Habegger–Masbaum–Vogel [8]. In the colored Jones context (for which the category of $U_q\mathfrak{sl}(2)$ modules is semisimple), the output is the *Witten–Reshetikhin–Turaev TQFT* (WRT). More recently, Blanchet–Costantino–Geer–Patureau have succeeded in constructing TQFT [7] from the category of modules on $U_q\mathfrak{sl}(2)$ when q is a root of 1 (which is non-semisimple). We call them *non-semisimple TQFTs* and the inherent knot invariant is hence the ADO family. These non-semisimple TQFTs are improvements of WRT because, for example, they detect lens spaces and Dehn twists, but F_∞ shows that at the level of invariants of knots they are the same. The invariant F_∞ is still not defined on links, while colored Jones and ADO families might differ at some point (as at the end one associated TQFT contains more). Notice that the authors have tried hard to generalize F_∞ to links finding systematic and important convergence issues. As the definition of F_∞ is made in the non-semisimple spirit: namely opening the knot along a strand and transforming it to a long knot, one has to make this process independent of the choice of a strand in the case of links, or to make the invariant computable from a full

trace (which corresponds to a fully closed knot). Refer to Figures 4 and 5 for pictures of long knots with an opened strand, or of a partial braid closure (with one strand left opened) that justifies the use of a partial trace in the definition of F_∞. For (renormalized) Jones and ADO families of invariants, well-defined for links, a normalization is applied so to make the computation independent of the choice of a strand. This normalization cannot be generalized in the infinite context of Verma modules that is used here to define F_∞, roughly speaking it corresponds to a normalization by a series and the result is not converging, namely it does not live in a good completion of $\mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$, same for the full trace. Hence, it seems to the authors that it cannot be generalized to links in a way that interpolates both families of invariants (in good completions we expect to be able to evaluate q at roots of unity, for instance). Following the homological approach of Section 4 it means that no homological model can provide a unification of Jones and ADO families in the general case of links (we call a unification something that is a link invariant in a ring that can specialized to the underlying families), as none of them can be obtained out of universal Verma modules. Consequently, the level of closed 3-manifold invariants has no chance to be reached and it is not a surprise: as mentioned WRT invariants (arising from Jones polynomials) do not classify lens spaces while some non-semisimple ones (arising from the ADO family) do. Indeed both families have no chance to be unified in the same way F_∞ does for knots. Nevertheless, F_∞ is related to Habiro's universal invariant [11] for knots in the way it interpolates the whole family of Jones polynomials, for instance, and this universal invariant was successfully generalized to integral homology spheres in [14, 15] by Habiro and Lê. The first author is currently trying to generalize the present unifying invariant for knots to integral homology spheres (which would still be consistent with the mentioned gap at the level of lens spaces), using homology techniques as in Section 4 based on the fact that homological models for quantum representations of mapping class groups were recently generalized to arbitrary genus surfaces in [35].

1.2 | Content: Unified invariant of knots and homological action of braids

In [38], the second author defines an invariant of knots denoted F_∞ which is an element living in some completion $\widehat{\mathcal{R}}^{\widehat{I}}$ of $\mathcal{R} := \mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$. This definition implies the application of a universal invariant constructed by Lawrence and Ohtsuki [23, 24, 34] and widely studied by Habiro [11], on any vector of some quantum $\mathfrak{sl}(2)$ Verma module. In [29], the first author has developed braid group representations on tensor products of these Verma modules with coefficients in $\mathcal{R}$ providing a homological definition arising from local systems on configuration spaces of points in punctured disks. In this paper, we express F_∞ as a *partial trace* of the braid action on tensor products of Verma modules.

Theorem (Theorem 33). *Let V be the universal Verma $\mathcal{R}$-module of $U_q\mathfrak{sl}(2)$. Let $\mathcal{K}$ be a knot such that it is the closure of a braid $\beta \in \mathcal{B}_n$. Then*

$$F_\infty = \mathrm{Tr}_{2,\dots,n}\left(h \circ \beta, V^{\otimes n}\right),$$

where the right term is the partial trace of the action of β on $V^{\otimes n}$ post composed with the (fixed) operator h explicitly defined later on.

We re-prove the following property.

Theorem (Theorem 39). *For an integer $r \in \mathbb{N}^*$ and ζ_{2r} a 2rth root of 1, we have:*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{(A)^{rf} \times \mathrm{ADO}_r(A, \mathcal{K})}{\mathcal{A}_\mathcal{K}(A^{2r})},$$

where f is the framing of the knot, ADO_r *is the rth ADO polynomial [1] and $\mathcal{A}_\mathcal{K}$ the Alexander polynomial of $\mathcal{K}$.*

The latter was proved in [38] but considering Melvin–Morton–Rozansky (MMR) conjecture which is a theorem due to Bar-Natan and Garoufalidis [4]. Here we prove it carefully studying the structure of tensor products of Verma modules when q is a root of 1. Hence, we have re-proved MMR conjecture in a slight generalization, namely an analytic relation between any colored Jones polynomial and the Alexander polynomial.

It is well-known that the Alexander polynomial of a knot is invariant under the change of variable $s \mapsto s^{-1}$. We extend this symmetry to the entire F_∞ and it gives a nice symmetry for the ADO invariants of knots too.

Theorem (Theorem 43, Corollary 44). *For any knot $\mathcal{K}$:*

- $F_\infty(\mathcal{K})$ *is unchanged under* $s \mapsto s^{-1}q^{-2}$,
- $\mathrm{ADO}_r(\mathcal{K})$ *is unchanged under* $s \mapsto s^{-1}\zeta_{2r}^{-2}$.

The second bullet point implies that the non-semisimple $U_q\mathfrak{sl}(2)$ invariant of planar graphs introduced by Costantino–Geer–Patureau in [9] does not detect orientation of knots (Corollary 45).

Using the homological definition from [29] for tensor products of Verma modules, and Poincaré duality in homology, we express F_∞ as the intersection pairing with coefficients in $\mathcal{R}$ between fixed middle dimension homology classes in configuration spaces of points in punctured disks.

Theorem (Theorem 68). *Let $\beta \in \mathcal{B}_n$ be a braid such that its closure is the knot $\mathcal{K}$. Then*

$$F_\infty(\mathcal{K}) = s^{n-1} \sum_{\overline{k}} \left\langle \beta \cdot A''(\overline{k}) \cap B''(\overline{k}) \right\rangle q^{-2\sum k_i},$$

where for any list of $n-1$ integers $\overline{k}$, $A''(\overline{k})$ and $B''(\overline{k})$ are precisely defined middle dimension manifolds of the space of configurations of points in the nth punctured disks. The action of β is naturally defined by homeomorphism of the punctured disk, and $\langle \cdot \cap \cdot \rangle$ is a homological intersection pairing in $\mathcal{R}$ given by Poincaré duality.

The latter means that the right term in the equation, which is an infinite sum of intersection pairing of middle dimension homology classes, lives in $\widehat{\mathcal{R}}^{\widehat{I}}$ and is invariant under Markov moves.

Finally, we express F_∞ using a generalized notion of determinant of matrices called *quantum determinant of right quantum matrices*, defined for matrices with noncommutative entries. This quantum determinant is presented in [10]. The quantum determinant formula resembles the classical one for the Alexander polynomial: it is the quantum determinant of a deformed Burau matrix instead of a regular determinant of the regular Burau matrix. It is stated in Theorem 82, and generalizes formula of Lê and Huynh [17] for colored Jones polynomials.

1.3 | Plan of the paper

In Section 2, we establish the context of the quantum group $U_q\mathfrak{sl}(2)$ and its Verma module. We define the action of braid groups, the splitting into finite dim. levels, and we carefully study the structure while specializing q at roots of one, giving rise to a particular *r-part factorization* (Proposition 23).

In Section 3, we redefine (Theorem 33) the knot invariant F_∞ as a partial trace on braid group representations previously defined after having recalled its former definition from [38]. Using the r-part factorization at roots of unity from previous section, we prove Theorem 39 re-proving the factorization of F_∞ at roots of one, re-proving MMR conjecture. We then prove Theorem 43 providing an Alexander-like symmetry for invariants F_∞ and ADO.

In Section 4, we prove Theorem 68 that expresses F_∞ as a sum of intersection pairing between Lagrangians in configuration spaces of punctured disks. This requires first a precise recall of the homological set-up from [29], that is, the homological definition of Verma modules, their tensor products and the braid action.

In Section 5, we recall the definition of *quantum determinant* for right quantum matrices. We recall the context of paper [16], and finally prove Theorem 82 providing a quantum determinant formula for invariants F_∞ and ADO.

2 | QUANTUM $\mathfrak{sl}(2)$ AND ITS UNIVERSAL VERMA MODULE

We introduce quantum numbers, factorials and binomials.

Definition 1. Let i, k, l, n be integers. We define the following elements of $\mathbb{Z}\left[q^{\pm 1}\right]$:

$$[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}, [k]_q! := \prod_{i=1}^{k} [i]_q, \begin{bmatrix} k \\ l \end{bmatrix}_q := \frac{[k]_q!}{[k-l]_q![l]_q!}; \tag{1}$$

$$\{n\} = q^n - q^{-n} \text{and} \{n\}! = \prod_{i=1}^{n} \{i\}; \tag{2}$$

with the convention $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $n < 0$.

We also fix notation for elements of $\mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$ but using the following notation $q^{\pm\alpha} := s^{\pm 1}$ that will be useful later on.

$$\{\alpha\}_q = q^\alpha - q^{-\alpha}, \{\alpha + k\}_q = q^{\alpha+k} - q^{-\alpha-k}, \{\alpha; n\}_q = \prod_{i=0}^{n-1} \{\alpha - i\}_q, \tag{3}$$

where one can easily deduce how to write them in $\mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$. (To do computation in $\mathbb{C}$ and think of q, α and s as complex numbers, one must fix a logarithm of q.)

In what follows, we will define $U_q\mathfrak{sl}(2)$ in Subsection 2.1, then its Verma modules and the associated action of braid groups in Subsection 2.2. We study the structure of this braid group representation while variables are evaluated at some particular value in Subsection 2.3. In the case of q being a root of one, we show that the representation splits into *r-parts* subrepresentations.

2.1 | The algebra $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$

In this section, we define an integral version for the quantized algebra associated with $\mathfrak{sl}(2)$. By integral, we mean as an algebra over the ring of Laurent polynomials in one variable, but first we define the standard algebra $U_q\mathfrak{sl}(2)$ on the rational field.

Definition 2. The algebra $U_q\mathfrak{sl}(2)$ is the algebra over $\mathbb{Q}(q)$ generated by elements E, F and $K^{\pm 1}$, satisfying the following relations:

$$KEK^{-1} = q^2 E, KFK^{-1} = q^{-2}F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \text{ and } KK^{-1} = K^{-1}K = 1.$$

The algebra $U_q\mathfrak{sl}(2)$ is endowed with a coalgebra structure defined by Δ and ϵ as follows:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$
$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1$$

and an antipode defined as follows:

$$S(E) = EK^{-1}, S(F) = -KF, S(K) = K^{-1}, S(K^{-1}) = K.$$

This provides a *Hopf algebra* structure, so that the category of modules over $U_q\mathfrak{sl}(2)$ is monoidal.

We are interested in an integral version that resembles Lusztig version but with only half of *divided powers* for generators. This version is used and introduced in [12, 20, 29, 38] (with subtle differences in the definitions of divided powers for F). Let:

$$F^{(n)} := \frac{(q - q^{-1})^n}{[n]_q!} F^n.$$

Let $\mathcal{R}_0 = \mathbb{Z}\left[q^{\pm 1}\right]$ be the ring of integral Laurent polynomials in the variable q.

Definition 3 (Half integral algebra). Let $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ be the $\mathcal{R}_0$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by $E, K^{\pm 1}$ and $F^{(n)}$ for $n \in \mathbb{N}^*$. We call it a *half integral version* for $U_q\mathfrak{sl}(2)$, the word half to illustrate that we consider only half of divided powers as generators.

Remark 4 (Relations in $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$, [20, (16) (17)]). The relations among generators involving divided powers are the following ones:

$$KF^{(n)}K^{-1} = q^{-2n}F^{(n)}$$

$$\left[E, F^{(n+1)}\right] = F^{(n)}\left(q^{-n}K - q^n K^{-1}\right) \text{ and } F^{(n)}F^{(m)} = \begin{bmatrix} n + m \\ n \end{bmatrix}_q F^{(n+m)}.$$

Together with relations from Definition 2, they complete a presentation of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

$U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ inherits a Hopf algebra structure with a coproduct given by:

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes K + 1 \otimes E, \text{and} \Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-j(n-j)} K^{j-n} F^{(j)} \otimes F^{(n-j)}.$$

Proposition 5 (Poincaré–Birkhoff–Witt basis). *The algebra* $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ *admits the following set as an* $\mathcal{R}_0$-*basis:*

$$\left\{ K^l E^m F^{(n)}, l \in \mathbb{Z}, m, n \in \mathbb{N} \right\}.$$

2.2 | Verma modules and braiding

We define the *Verma modules*. They are infinite-dimensional modules over $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ depending on a parameter. Again we work with an integral version by including the parameter in the ring of Laurent polynomials as a formal variable. Let $\mathcal{R} := \mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$.

Definition 6 (Verma modules for $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$). Let V^s be the Verma module of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$. It is the infinite $\mathcal{R}$-module, generated by the family of vectors $\{v_i, i \in \mathbb{N}\}$, and endowed with an action of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$, generators acting as follows:

$$K \cdot v_j = sq^{-2j}v_j, E \cdot v_j = v_{j-1} \text{and} F^{(n)}v_j = \left(\begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{j+k}) \right) v_{j+n}.$$

Remark 7 (Weight vectors). We will often make implicitly the change of variable $s := q^\alpha$ and denote V^s by V_α. This choice is made to use a practical and usual denomination for eigenvalues of the K action (which is diagonal in the given basis). Namely we say that vector v_j is *of weight* $\alpha - 2j$, as $K \cdot v_j = q^{\alpha-2j}v_j$. The notation with s shows an integral Laurent polynomials structure strictly speaking. In the case $s = q^\alpha$ one can use a simpler notation in the action of $F^{(n)}$:

$$\prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{j+k}) = \{\alpha - j, n\}$$

Definition 8 (*R*-matrix, [20, (21)]). Let $s = q^\alpha$, $t = q^{\alpha'}$. The operator $q^{H \otimes H/2}$ is the following:

$$q^{H \otimes H/2} : \begin{cases} V^s \otimes V^t & \to & V^s \otimes V^t \\ v_i \otimes v_j & \mapsto & q^{(\alpha-2i)(\alpha'-2j)} v_i \otimes v_j \end{cases}.$$

We define the following R-matrix:

$$R : q^{H \otimes H/2} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)},$$

which will be well-defined as an operator on Verma modules in what follows.

We recall the Artin presentation of the braid groups.

Definition 9. Let $n \in \mathbb{N}$. The *braid group* on n strands $\mathcal{B}_n$ is the group generated by $n-1$ elements satisfying the so called *"braid relations"*:

$$\mathcal{B}_n := \left\langle \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \cdots, n-2 \end{array} \right\rangle$$

Proposition 10 [20, Theorem 7]. *Let V^s and V^t be Verma modules of $U_q^{\frac{L}{2}} \mathfrak{sl}(2)$ (with $s = q^\alpha$ and $t = q^{\alpha'}$). Let R be the following operator:*

$$\mathrm{R} : q^{-\alpha\alpha'/2} T \circ R,$$

where T is the twist defined by $T(v \otimes w) = w \otimes v$. Then R provides a braiding for $U_q^{\frac{L}{2}} \mathfrak{sl}(2)$ integral Verma modules. Namely, the morphism:

$$\phi_n : \begin{cases} \mathcal{R}[\mathcal{B}_n] & \to & \underset{\mathcal{R}, U_q^{\frac{L}{2}} \mathfrak{sl}(2)}{\mathrm{End}} \left(V^{s \otimes n} \right) \\ \sigma_i & \mapsto & 1^{\otimes i-1} \otimes \mathrm{R} \otimes 1^{\otimes n-i-1} \end{cases}$$

is an $\mathcal{R}$-algebra morphism. It provides a representation of $\mathcal{B}_n$ such that its action commutes with that of $U_q^{\frac{L}{2}} \mathfrak{sl}(2)$. In the sequel, we will sometime denote $\phi_n(q^\alpha, \cdot)$ to emphasize the dependence in variables.

Remark 11. One can consider a braid action over $V^{s_1} \otimes \cdots \otimes V^{s_n}$ (considering more variables in the ring) such that the morphism ϕ_n is well-defined but becomes an algebra morphism only when restricted to the pure braid group $\mathcal{PB}_n$. These braids indeed define endomorphisms. See [31, appendix] for a detailed explanation on *colored versions*.

Elements of same weight in the tensor product of Verma modules form a subrepresentation of the braid group.

Definition 12 (Sub-weight representations). Let:

$$V_{n,m}(q, q^\alpha) := \mathrm{Span} \left\{ v_{i_1} \otimes \cdots \otimes v_{i_n} \in V_\alpha^{\otimes n} \text{such that} \sum_{k=1}^n i_k = m \right\}$$

be the space of *sub-weight m vectors*. It is stable under the action of braids so that we denote:

$$\varphi_{n,m}(q, q^\alpha, .) : B_n \to \mathrm{End}_{\mathcal{R}}(V_{n,m}),$$

the associated (restricted) representation. When there is no ambiguity on variables, we will write $V_{n,m} := V_{n,m}(q, q^\alpha)$ and $\varphi_{n,m}(\beta) := \varphi_{n,m}(q, q^\alpha, \beta)$.

Remark 13. The stability of sub-weight vectors under braid actions is deduced from the fact that the latter action commutes with that of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ and from the fact that sub-weight vectors are eigenvectors for the K action. Namely,

$$V_{n,m}(q, q^\alpha) = \{v \in V_\alpha^{\otimes n} | Kv = q^{\alpha-2m}v\}. \tag{4}$$

This is for q being a formal variable. At roots of unity (i.e., when q is a root of 1, see next section), as $q^{-2r} = 1$, Equation (4) does not stand. Still the braid action preserves $V_{n,m}$ and this can be seen directly from the terms of the R-matrix preserving the sum of indices of tensors.

2.3 | **Specialization of variables**

Working with the ring $\mathcal{R}$ is particularly comfortable for specialization of variables, that is, giving a complex value to variables q and s. This corresponds to a morphism:

$$\text{spec} : \mathcal{R} \to \mathbb{C}$$

and algebraically speaking, all the data set just presented has to be replaced by:

$$\tilde{\mathcal{U}}_{\text{spec}} := U_q^{\frac{L}{2}}\mathfrak{sl}(2)\otimes_{\text{spec}}\mathbb{C} \text{ and } V^s\otimes_{\text{spec}}\mathbb{C}, \text{ and so on.}$$

This is what we will mean by *specialization*. (We will simply denote $\tilde{\mathcal{U}}$ when the specialization is clear.)

2.3.1 | Specialization to integral weights

We can take a specialization at *integral weights* setting $s = q^\alpha = q^N$ for $N \in \mathbb{Z}$ in the previous formulae and we denote V^N the corresponding Verma module with integral weights. We find a classical sub-module in that case:

Definition 14 (Simple module of dim. N). We denote S_N the module spanned by $\{v_0, \ldots, v_N\}$. It is a sub-module of V^N isomorphic to the highest weight simple module of dim. $N + 1$.

This specialization has a symmetry as shown in the following lemma:

Lemma 15. *For $N \in \mathbb{N}^*$, we have the isomorphism of $\tilde{\mathcal{U}}$ modules:*

$$V^{-N-2} \cong V^N/S_N.$$

Proof. While $(v_i)_{i\in\mathbb{N}}$ is set to be the basis of V^N, $(\overline{v_i})_i$ the basis of the quotient V^N/S_N, we get:

$$E\overline{v_{N+1}} = 0$$

$$E\overline{v_{N+1+i+1}} = \overline{v_{N+1+i}}$$

$$K\overline{v_{N+1+i}} = q^{-N-2-2i}\overline{v_{N+1+i}}$$

$$F^{(n)}\overline{v_{N+1+i}} = \begin{bmatrix} n+N+1+i \\ n \end{bmatrix}_q \{-i-1;n\}_q \overline{v_{N+1+n+i}}.$$

We slightly transform the last equality using:

$$\{n+N+1+i;n\}_q\{-i-1;n\}_q = \{i+n;n\}_q\{-N-2-i;n\}_q,$$

so that:

$$F^{(n)}\overline{v_{N+1+i}} = \begin{bmatrix} n+i \\ n \end{bmatrix}_q \{-N-2-i;n\}_q \overline{v_{N+1+n+i}}.$$

Setting $v_i := \overline{v_{N+1+i}}$ for $i \geqslant 0$, one recognizes precisely the definition of V^{-N-2}. □

2.3.2 | Specialization of q to 1

We treat the case $q = 1$ slightly differently from other roots of unity (see next section). We fix particular notations in this context.

Notation 16. When $q = 1$ we fix:

- $SB_{n,m} := V_{n,m}(1, q^\alpha)$,
- $w_i := v_i$,
- $\psi_{n,m}(q^\alpha, \beta) := \varphi_{n,m}(1, q^\alpha, \beta)$.

(The notation SB refers to the fact that it is isomorphic to a symmetric power of the Burau representation, see next proposition).

A nice property of the $q = 1$ case is that the sub-weight m level representation can be obtained as a symmetric power of the first sub-weight level.

Proposition 17. *Let $\beta \in B_n$, then:*

$$\psi_{n,m}(q^\alpha, \beta) = Sym^m(\psi_{n,1}(q^\alpha, \beta)).$$

Proof. First we need to consider a diagonal change of bases. We set $u_j := j!w_j$. Let $e_k := u_0 \otimes \cdots \otimes u_1 \otimes \cdots \otimes u_0 \in SB_{n,1}$ where the only u_1 is located at the kth position. The family $\{e_k, k = 1, \dots, n\}$ is a basis of $SB_{n,1}$. We can identify higher weight tensors with symmetric powers of the e_k using the one to one following correspondence:

$$u_{j_1} \otimes \cdots \otimes u_{j_n} \longleftrightarrow \prod_{k=1}^n e_k^{j_k}.$$

Now, in the basis e_k, we have:

$$\psi_{n,1}(\sigma_i) = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \begin{matrix} 1-q^{-2\alpha} & q^{-\alpha} \\ q^{-\alpha} & 0 \end{matrix} & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

We compute the symmetric power action in the u_j basis,

$$\text{Sym}^m(\psi_{n,1}(\sigma_i))u_{\bar{j}} = \prod_{k=0}^{n}(\psi_{n,1}(\sigma_i)e_k)^{j_k}$$

$$= e_1^{j_1}\cdots e_{i-1}^{j_{i-1}}((1-q^{-2\alpha})e_i + q^{-\alpha}e_{i+1})^{j_i} \times (q^{-\alpha}e_i)^{j_{i+1}}e_{i+2}^{j_{i+2}}\cdots e_n^{j_n}$$

$$= \sum_{l=0}^{j_i}\binom{j_i}{l}\{\alpha\}^l q^{-(j_i+j_{i+1})\alpha}u_{j_1}\otimes\cdots\otimes u_{j_{i+1}+l}\otimes u_{j_i-l}\otimes\cdots\otimes u_{j_n}$$

If we transpose it back in the basis of the w_j's we get:

$$Sym^m(\psi_{n,1}(\sigma_i))w_{\bar{j}} = \sum_{l=0}^{j_i}\binom{j_{i+1}+l}{l}\{\alpha\}^l q^{-(j_i+j_{i+1})\alpha}w_{j_1}\otimes\cdots\otimes w_{j_{i+1}+l}\otimes w_{j_i-l}\otimes\cdots\otimes w_{j_n}$$

$$= \psi_{n,m}(q^\alpha,\beta)w_{\bar{j}}$$

The last equality is directly checked from the set-up: Defs. 6 and 8 and Proposition 10. □

2.3.3 | Specialization of q to roots of 1: r-part subrepresentations

In this subsection, we set $q = \zeta_{2r}$ that corresponds to a specialization as defined above.

Definition 18. The r-part of a tensor $v = v_{i_1+rj_1}\otimes\cdots\otimes v_{i_n+rj_n} \in V_\alpha^{\otimes n}$ where $i_1,\ldots i_n \leqslant r-1$ is defined by

$$\text{rp}(v) := \sum_{k=0}^{n}j_k.$$

Definition 19. We define subspaces of $V_\alpha^{\otimes n}$

$$V_n^m(\zeta_{2r},q^\alpha) := \text{Span}\langle v|\text{rp}(v) = m\rangle$$

$$V_n^{\leqslant m}(\zeta_{2r},q^\alpha) := \bigoplus_{i=0}^{m}V_n^m(\zeta_{2r},q^\alpha)$$

when there is no ambiguity, we will write $V_n^m := V_n^m(\zeta_{2r},q^\alpha)$ and $V_n^{\leqslant m} := V_n^{\leqslant m}(\zeta_{2r},q^\alpha)$.

Proposition 20. $V_n^{\leqslant m}$ *is a subrepresentation of braids designed by* $\varphi_n^{\leqslant m}(\beta)$ *(the restriction of* $\varphi_{n,m}$*, with the implicit specialization of variables).*

Proof. First remark that

$$\mathrm{rp}(E^r \otimes F^{(r)}(v_i \otimes v_j)) = \mathrm{rp}(v_i \otimes v_j).$$

Moreover, $F^{(i+rj)} v_{a+ru} = 0$ if $i, a \leqslant r - 1$ and $a + i \geqslant r$, hence

$$\mathrm{rp}(E^n \otimes F^{(n)}(v_i \otimes v_j)) \leqslant \mathrm{rp}(v_i \otimes v_j).$$

Thus, $V_n^{\leqslant m}$ is invariant under the action of the R matrix and its inverse (for the inverse, see, e.g., [38, Proposition 6]). $\qquad\square$

This allows us to have another subrepresentation via projection maps.

Proposition 21. *Let* $\rho_n^m : V_n^{\leqslant m} \to V_n^m$ *the canonical projection map, then* V_n^m *is endowed with a representation of* $\mathcal{B}_n$ *using the projection of the general action:*

$$\varphi_n^m := \rho_n^m \circ \varphi_n^{\leqslant m}|_{V_n^m}.$$

Proof. As $V_n^{\leqslant m-1}$ is a subrepresentation, if $v \in V_n^{\leqslant m-1}$ we have $\varphi_n^{\leqslant m-1}(\beta_1)v \in V_n^{\leqslant m-1}$ and hence $\rho_n^m \circ \varphi_n^{\leqslant m}(\beta_1)v = \rho_n^m \circ \varphi_n^{\leqslant m-1}(\beta_1)v = 0$.
  This means the following:

$$\rho_n^m \circ \varphi_n^{\leqslant m}(\beta_1) \circ \rho_n^m \circ \varphi_n^{\leqslant m}(\beta_2)|_{V_n^m} = \rho_n^m \circ \varphi_n^{\leqslant m}(\beta_1) \circ \varphi_n^{\leqslant m}(\beta_2)|_{V_n^m}.$$

Finally, $\varphi_n^m(\beta_1 \beta_2) = \varphi_n^m(\beta_1) \circ \varphi_n^m(\beta_2)$. $\qquad\square$

Remark 22. As braid group representation, $V_n^0 \cong V_n^{\leqslant 0}$ (meaning $\varphi_n^0 = \varphi_n^{\leqslant 0}$).

Now we state the main result of this section that is the factorization to r-part subrepresentations. Recall the Frobenius map $F_r : \mathbb{Z}[q^\alpha] \to \mathbb{Z}[q^\alpha]$ that sends $q^\alpha \mapsto q^{r\alpha}$ ($s \to s^r$ in the language of Laurent polynomials).

Proposition 23. *The isomorphism*

$$\Phi : \begin{cases} V_n^m & \to & V_n^0 \otimes F_r(\mathrm{SB}_{n,m}) \\ v_{\overline{i+rj}} & \mapsto & v_{\overline{i}} \otimes F_r(w_{\overline{j}}) \end{cases},$$

where $\overline{i + rj} = (i_1 + rj_1, \dots, i_n + rj_n)$ with $i_1, \dots, i_n \leqslant r - 1$, is a braid group representation isomorphism. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
V_n^m & \xrightarrow{\quad \varphi_n^m \quad} & V_n^m \\[4pt]
\Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi} \\[4pt]
V_n^0 \otimes F_r(\mathrm{SB}_{n,m}) & \xrightarrow{\varphi_n^0 \otimes (F_r \circ \psi_{n,m})} & V_n^0 \otimes F_r(\mathrm{SB}_{n,m})
\end{array}
$$

Proof. Using [38, Lemma 26], we can factorize the action of the R matrix as follows. Let $0 \leqslant a, b, i \leqslant r - 1$ such that $0 \leqslant a + i \leqslant r - 1$ and $0 \leqslant b - i \leqslant r - 1$, we have:

$$
q^{\frac{H \otimes H}{2}}(q^{\frac{(i+rj)(i+rj-1)}{2}} E^{i+rj} \otimes F^{(i+rj)}).v_{b+rv} \otimes v_{a+ru} = q^{\frac{\alpha^2}{2}} q^{\frac{(i+rj)(i+rj-1)}{2}} \begin{bmatrix} i + rj + a + ru \\ i + rj \end{bmatrix}_q
$$

$$
\times \{\alpha - a - ru; i + rj\}_q q^{-(a+ru+b+rv)\alpha}
$$

$$
\times q^{2(a+ru+i+rj)(b+rv-i-rj)} v_{b+rv-i-rj} \otimes v_{a+ru+i+rj}
$$

$$
= q^{\frac{\alpha^2}{2}} q^{\frac{i(i-1)}{2}} \begin{bmatrix} a + i \\ i \end{bmatrix}_q
$$

$$
\times \{\alpha - a; i\}_q q^{-(a+b)\alpha} q^{2(a+i)(b-i)} v_{b-i} \otimes v_{a+i}
$$

$$
\otimes F_r\left(\binom{u+j}{j} \{\alpha\}^j q^{-(u+v)\alpha} w_{v-j} \otimes w_{u+j} \right).
$$

Hence, we have

$$
\Phi\left(\rho_2^{u+v}(R.v_{b+rv} \otimes v_{a+ru}) \right) = (R.v_b \otimes v_a) \otimes F_r(R.w_v \otimes w_u).
$$

Finally,

$$
\Phi\left(\varphi_n^m(\beta).v_{\overline{i+rj}} \right) = \varphi_n^0(\beta).v_{\overline{i}} \otimes F_r\left(\psi_{n,m}(\beta).w_{\overline{j}} \right)
$$

$\square$

Example 24. Figure 1 illustrates the weight level pyramid at $n = 2$ and $q = \zeta_6$ where we denote

$$
v_{a,b} = v_a \otimes v_b.
$$

The blue square delimits generators of V_n^0, the red squares those of V_n^1, and so on. Each square corresponds to a tensor in the pyramid at $q = 1$ as shown in Figure 2. Families of colored squares are stable under the braid action φ_n^m where m correspond to a color. The union of a colored family plus higher colored family in the pyramid are stable under the whole quantum braid action φ_n (e.g., the union of red and blue vectors from Figure 1 is stable under $\mathcal{B}_n$ action).
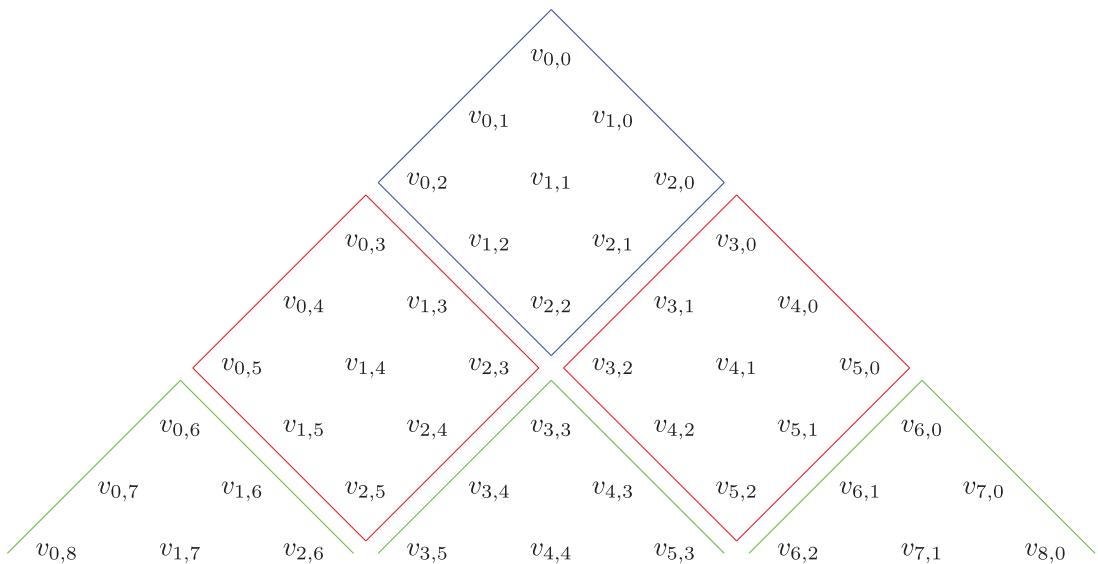
FIGURE 1 Weight level pyramid factorization at root of unity.


FIGURE 2 Weight level pyramid at $q = 1$.

3 │ UNIFIED INVARIANT OF KNOTS FROM QUANTUM BRAID REPRESENTATIONS

We want to define the knot invariant F_∞ from [38] from braid group representations on tensor products of Verma modules that are defined above. We need a completion of the ring $\mathcal{R}$ as F_∞ will be some series living in this completion. We start with definitions for this ring and for the invariant in Subsection 3.1. Then (Subsection 3.2) we can define (Theorem 33) F_∞ from the braid action on tensors of Verma modules. In Subsection 3.3, we use the r-part factorization of the braid action at roots of unity to prove the factorization of F_∞ at roots of unity that recovers ADO polynomials (Theorem 39). In Subsection 3.4, we prove Theorem 43 that shows a symmetry in variables for F_∞ resembling that of the Alexander polynomial. As a corollary, we obtain that ADO polynomials inherit this symmetry relating them more closely to the Alexander polynomial.

3.1 │ Ring completion and unified invariant

We recall $\mathcal{R} = \mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$, we will construct a completion of that ring. For the sake of simplicity, we will denote $q^{\alpha} := s$ as explained before.

(a) Positive crossing.

(b) Negative crossing.
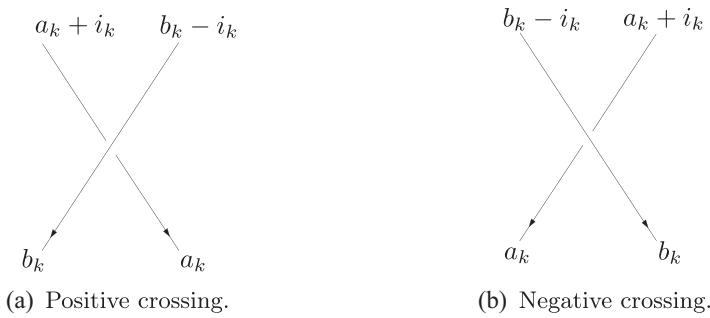
FIGURE 3 The two possibilities for the kth crossing in D.

Definition 25. Let I_n be the ideal of $\mathcal{R}$ generated by the following set $\{\{\alpha + l; n\}_q, l \in \mathbb{Z}\}$.

We then have a projective system:

$$\hat{I} : I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots$$

From which we can define the completion of $\mathcal{R}$ as a projective limit.

Definition 26. Let $\widehat{\mathcal{R}}^{\hat{I}} = \varprojlim_n \frac{R}{I_n} = \{(a_n)_{n \in \mathbb{N}^*} \in \prod_{i=1}^{\infty} \frac{R}{I_n} | p_n(a_{n+1}) = a_n\}$ where $p_n : \frac{R}{I_{n+1}} \to \frac{R}{I_n}$ is the projection map.

Remark 27.

- If $b_0 \in R$ and $b_n \in I_{n-1}$ for $n \geqslant 1$, the partial sums $\sum_{i=0}^{N} b_n$ converge in $\widehat{\mathcal{R}}^{\hat{I}}$ as N goes to infinity.

- We denote the limit $\sum_{i=0}^{+\infty} b_n := (\overline{\sum_{i=0}^{N} b_n})_{N \in \mathbb{N}^*}$.

- Conversely, if $a = (\overline{a_N})_{N \in \mathbb{N}^*} \in \widehat{\mathcal{R}}^{\hat{I}}$, let $a_n \in R$ be any representative of $\overline{a_n}$ in R, then $a = \sum_{i=0}^{+\infty} b_n$ where $b_0 = a_1$ and $b_n = a_{n+1} - a_n$ for $n \in \mathbb{N}^*$.

The completion $\widehat{\mathcal{R}}^{\hat{I}}$ contains $\mathcal{R}$:

Proposition 28. *The canonical projection maps induce an injective map* $\mathcal{R} \hookrightarrow \widehat{\mathcal{R}}^{\hat{I}}$

Proof. See [38, Proposition 17]. $\qquad\square$

We now recall how the unified invariant $F_\infty(q, q^\alpha, \mathcal{K})$ is defined using states diagrams of the knot, which is the subject of [38].

For any knot seen as a (1,1)-tangle, take a diagram D and $\bar{i} = (i_1, \ldots, i_N) \in \mathbb{N}^N$ where N is the number of crossings of D.

Label the top and bottom strands 0 and starting from the bottom strand, label the strand after the kth crossing encountered with the rule described in Figure 3. The resulting labeled diagram is called a *state diagram* of D, we denote it $D_{\bar{i}}$.

(a) The trefoil knot. (b) The figure eight knot.
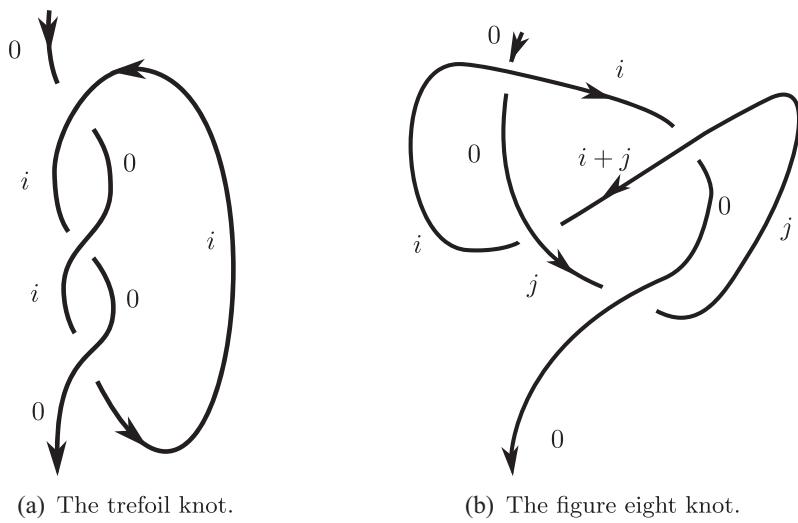
F I G U R E 4   Examples of state diagrams to compute the invariants.

Let $D_{\vec{i}}$ be a state diagram of D, we define:

$$
D(i_1, \dots, i_N) = \left(\prod_{j=1}^{S} q^{\mp(\alpha - 2\epsilon_j)} \right) \prod_{k \in \text{pos}} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha - a_k; i_k\}_q
$$

$$
\times q^{-(a_k + b_k)\alpha} q^{2(a_k + i_k)(b_k - i_k)} \prod_{k \in \text{neg}} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q
$$

$$
\times \{\alpha - a_k; i_k\}_q \, q^{(a_k + b_k)\alpha} q^{-2a_k b_k},
$$

where:

- f is the writhe of D,
- $\text{neg} \cup \text{pos} = [|1, N|]$ and $k \in \text{pos}$ if the kth crossing of D is positive, else $k \in \text{neg}$,
- a_k, b_k are the strands' labels at the kth crossing of the state diagram (see Figure 3),
- S is the number of $\smile + \frown$ appearing in the diagram, and ϵ_j the strand label at the jth $\smile$ or $\frown$, the $\mp$ sign is negative if $\smile$ and positive if $\frown$.

Remark 29. Notice that in [38], the definition of these numbers come with a term $q^{\frac{-f\alpha^2}{2}}$ in front, that is removed here. It comes from the fact that in the present paper we remove a quadratic term in the R-matrix multiplying it by $q^{-\alpha\alpha'/2}$, see the definition of R in Proposition 10, so that both corrections make the following remark consistent: $D(i_1, \dots, i_N)$ is the scalar one obtains by considering only the $E^{i_k} \otimes F^{(i_k)}$ term in the R-matrix action of the kth crossing of D. In the present paper, this quadratic term is also removed from the definition of ADO polynomials, so that later in Theorem 39 we obtain the same interpolation as in [38, Theorem 57].

Example 30. See Figure 4 for some examples of state diagrams.

Definition 31 [38, Definition 20]. We define the knot invariant:

$$F_\infty(q, A, \mathcal{K}) := \sum_{\bar{i}=0}^{+\infty} D(i_1, \dots, i_N).$$

One advantage of removing the quadratic term (Remark 29) is that it makes F_∞ an element of $\widehat{\mathcal{R}}^{\widehat{I}}$, while the version of [38, Definition 20] is in $q^{\frac{f\alpha^2}{2}} \widehat{\mathcal{R}}^{\widehat{I}}$.

3.2 | Unified invariant from the action of braids on Verma modules

We recall the definition of the braid group representation on tensor products of Verma modules:

$$\varphi_n(q^\alpha, .) : B_n \to \mathrm{End}((V^\alpha)^{\otimes n}).$$

The notion of *partial trace* is used to compute knot invariants out of finite-dimensional quantum braid representation. We extend this notion to infinite-dimensional modules in the case of Verma modules.

Definition 32 (Partial trace on Verma modules). Let $\beta \in B_n$ whose closure is a knot,

$$\mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta)) := \sum_{\bar{j} \in \mathbb{N}_0^n} \left[((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))v_{\bar{j}}\right]_{v_{\bar{j}}} \in \widehat{\mathcal{R}}^{\widehat{I}},$$

where:

- $\mathbb{N}_0^n := \{(0, j_2, \dots, j_n) \in \mathbb{N}^n\}$,
- $\left[((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))v_{\bar{j}}\right]_{v_{\bar{j}}} \in \mathbb{Z}[q^\pm, q^{\pm\alpha}]$ is the projection of $(1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)v_{\bar{j}}$ on $v_{\bar{j}}$.

It is called *partial trace* inherited from the standard notion of partial trace on tensor products of vector spaces, see Subsection 3.4.

Let $\mathcal{K}$ be a long knot, $\beta \in B_n$ whose closure is $\mathcal{K}$ and D^β be the diagram associated with $\mathcal{K}$ seen as the closure of β. The general picture is the following.

The following result redefines the unifying invariant as a partial trace on braid representations.

Theorem 33. *Let $\mathcal{K}$ be a knot in S^3 and $\beta \in B_n$ a braid whose closure is $\mathcal{K}$, then we have*

$$F_\infty(q, q^\alpha, \mathcal{K}) = \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)) \in \widehat{\mathcal{R}}^{\widehat{I}}$$

Proof. We denote $\mu_2(\bar{i}), \dots, \mu_n(\bar{i})$ the labels of the closing strands of the state diagram $D_{\bar{i}}^\beta$. We let $\mu(\bar{i}) = (0, \mu_2(\bar{i}), \dots, \mu_n(\bar{i}))$. We can then write

$$\left[((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))v_{\bar{j}}\right]_{v_{\bar{j}}} = \sum_{\substack{\bar{i}=0 \\ \mu(\bar{i})=\bar{j}}}^{+\infty} D(i_1, \dots, i_N),$$
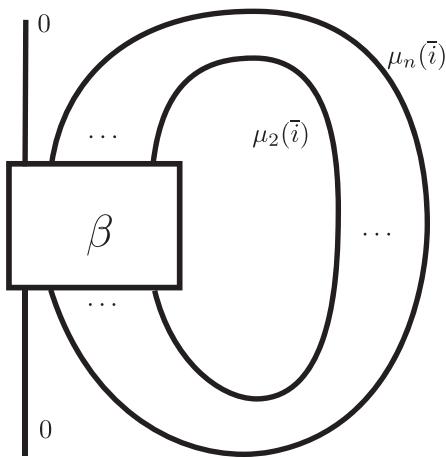
FIGURE 5 State diagram of a braid partial closure.

where $[((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))v_{\bar{j}}]_{v_{\bar{j}}} \in \mathbb{Z}[q^\pm, q^{\pm\alpha}]$ is the projection of $(1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)v_{\bar{j}}$ on $v_{\bar{j}}$. Hence,

$$\sum_{\bar{j}=0}^{+\infty}[((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))v_{\bar{j}}]_{v_{\bar{j}}} = \sum_{\bar{i}=0}^{+\infty} D(i_1, \dots, i_N).$$

Finally, comparing with the definition of F_∞ from diagrams (Definition 31), one recognizes the same formula, so that

$$\mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta)) = F_\infty(q, q^\alpha, \mathcal{K}),$$

which concludes the proof. □

Using Theorem 33, we can then write the unified invariant using the decomposition of ϕ_n by weight subrepresentations (Definition 12).

Corollary 34. *Let $\mathcal{K}$ be a knot in S^3 and $\beta \in B_n$ a braid whose closure is $\mathcal{K}$, then we have*

$$F_\infty(q, q^\alpha, \mathcal{K}) = \sum_m \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_{n,m}(\beta)).$$

We briefly recall relations of F_∞ with *colored Jones invariants*.

Notation 35. There are two versions of interest for colored Jones polynomials of knots, the original one (corresponding to a trace on quantum representations of braid groups) and the *normalized one* (corresponding to a partial trace on quantum representations of braid groups).

- The Nth colored Jones polynomial of a knot is defined as follows.

$$\mathrm{J}_K(N) = q^{-w(\beta)} \sum_{m=0}^{N} \mathrm{Tr}\left(\varphi_n(\beta), S_N^{\otimes n}\right),$$

where $\beta \in \mathcal{B}_n$ is an nth strands braid whose closure is the knot K, and $w(\beta)$ is its *writhe*. $\mathrm{Tr}\left(\varphi_n(\beta), S_N^{\otimes n}\right)$ means the trace of the braid action $\phi_n(\beta)$ restricted to $S_N^{\otimes n} \subset \left(V^N\right)^{\otimes n}$. The reader must be careful as $S_N^{\otimes n}$ is not stable under the braid action in general, but it is for braids whose closures are knots (see [30, Definition 3.14]).

- The Nth normalized colored Jones polynomial of a knot is defined as follows (see [38, Corollary 53]):

$$\mathrm{J}'_K(N)(q^2) = F_\infty(q, q^N, \mathcal{K})$$

They are related by $\mathrm{J}_K(N) = [N]_q \, \mathrm{J}'_K(N)$ (see [17, section 1.1.4] for the relation between these two in the context of Verma modules, in particular trace vs. partial trace).

3.3 | At roots of unity: factorization of the unified invariant

Now we can finally factorize the unified invariant at roots of unity using braid representations. The result is already given in [38], it uses a conjecture of MMR proved by Bar-Natan–Garoufalidis rather than a structural study of braid representations on Verma modules. Here, we give another proof of the result, using braid group representations. Hence, it re-proves MMR conjecture and moreover a generalization of it. This subsection assumes $q = \zeta_{2r}$. We refer to Subsection 2.3.3 for notations of submodules in this case.

First, we recall how we can obtain *ADO polynomials* (sometime called *colored Alexander invariants*) with the 0 r-part representation. Namely, ADO polynomials were formerly defined by Akutsu–Deguchi–Ohtsuki in [1] using matrix associated with braids (it should resemble bellow's formula). Then using state sum formula in [33]. More recently using $U_q \mathfrak{sl}(2)$ representation theory at roots of unity (similar to the present context, but with slightly different conventions, see Corollary 45) in [9, section 2.2]. The following proposition could be considered as a definition for the present work (we relate it to other definitions in the proof).

Proposition 36.

$$ADO_r(q^\alpha, \mathcal{K}) = \mathrm{Tr}_{2,\dots,n}((1 \otimes (K^{1-r})^{\otimes n-1})\varphi_n^0(\beta))$$

Proof. It is the same proof as that of Theorem 33. We use the definition of ADO polynomials as quantum invariants and the state sum formula of [38, Proposition 13], set $q = \zeta_{2r}$, and use truncated R matrix $R_r = q^{H \otimes H/2} \sum_{n=0}^{r-1} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$ and K^{r-1} as a pivotal element. $\square$

Now we state a factorization result at roots of unity. It is a corollary of Propositions 23 and 36.

Corollary 37.

$$\mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n^m(\beta)) = ADO_r(q^\alpha, \mathcal{K}) \times F_r(\mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\psi_{n,m}(\beta))$$

where F_r is the Frobenius map that sends q^α to $q^{r\alpha}$.

Moreover, we can use MacMahon Master Theorem to prove the following proposition.

Proposition 38. *For $\beta \in B_n$ whose closure is a knot $\mathcal{K}$, then:*

$$\sum_m \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\psi_{n,m}(\beta)) = \frac{q^{f\alpha}}{A_{\mathcal{K}}(q^{2\alpha})},$$

where $A_{\mathcal{K}}$ is the Alexander polynomial.

Proof. Using MacMahon Master Theorem, we have

$$\sum_m [\mathrm{Sym}^m(\psi_{n,1}(\beta))v_{\overline{i}}]_{v_{\overline{i}}} t_1^{i_1} \otimes \cdots \otimes t_n^{i_n} = \frac{1}{\det\left(I_n - \left(\begin{smallmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{smallmatrix}\right)\psi_{n,1}(\beta)\right)}.$$

Now if one takes $t_1 = 0$ and $t_i = 1$ for $i \neq 1$, we have the following equality:

$$\sum_m \mathrm{Tr}_{2,\dots,n}(\mathrm{Sym}^m(\psi_{n,1}(\beta))) = \frac{1}{\det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)\psi_{n,1}(\beta))}.$$

As $\psi_{n,1}(\beta)$ is the unreduced Burau representation $B(t)$ in the basis $f_k = q^{-k\alpha}e_k$ setting $t = q^{-2\alpha}$, and as we are taking a $(n-1) \times (n-1)$ minor of $I_n - B(t)$, we obtain:

$$det(I_n - \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)\psi_{n,1}(\beta) = q^{(n-1+f)\alpha}A_{\mathcal{K}}(q^{2\alpha}). \qquad \square$$

Now,

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(q^\alpha, \beta))$$

$$= \sum_m \mathrm{Tr}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n^m(q^\alpha, \beta))$$

and using Corollary 37 and Proposition 38, we recover the factorization theorem:

Theorem 39 (Factorization of F_∞ at roots of unity). *For a knot $\mathcal{K}$ and an integer $r \in \mathbb{N}^*$, we have the following factorization in $\widehat{\mathcal{R}}^{\widehat{I}}$:*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{(A)^{rf} \times ADO_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})},$$

where f is the framing of the knot. (We have named the variable A instead of s used to define the Verma modules. It is more standard when working with Alexander-like invariants.)

We recall the MMR conjecture, which is a theorem of Bar-Natan and Garoufalidis.

Theorem 40 (Bar-Natan, Garoufalidis [4]). *For a knot $\mathcal{K}$, the following equality in holds in $\mathbb{Q}[[h]]$:*

$$\lim_{n \to \infty} J'_{\mathcal{K}}(n)(e^{h/n}) = \frac{1}{A_{\mathcal{K}}(e^h)}$$

in the sense that, $\forall m \in \mathbb{N}$,

$$\lim_{n \to \infty} \mathrm{coeff}\left(J'_{\mathcal{K}}(n)(e^{h/n}), h^m\right) = \mathrm{coeff}\left(\frac{1}{A_{\mathcal{K}}(e^h)}, h^m\right)$$

where, for any analytic function f, $\mathrm{coeff}(f(h), h^m) = \frac{1}{m!} \frac{d^m}{dh^m} f(h)|_{h=0}$, and $\mathrm{J}'_{\mathcal{K}}(n)$ is the nth colored Jones polynomial.

Re-proof of MMR conjecture. *Let $\mathcal{K}$ be a 0 framed knot. From the unified invariant,*

- *on one hand, we recover the colored Jones polynomials (see [38, Corollary 59])*

$$F_\infty(q, q^N, \mathcal{K}) = \mathrm{J}'_{\mathcal{K}}(N)(q^2),$$

- *on the other hand, using Theorem 39 at $r = 1$, we get*

$$F_\infty(1, A, \mathcal{K}) = \frac{1}{A_{\mathcal{K}}(A^2)}.$$

Using the identification $q = e^h$ and $q^\alpha = e^{\alpha h}$, we have an injective map (see [13, Propositions 6.8 and 6.9])

$$\widehat{\mathcal{R}}^{\widehat{I}} \to \mathbb{Q}[\alpha][[h]].$$

Hence, as elements in $\mathbb{Q}[\alpha][[h]]$, we have the following limit (in the sense defined in Theorem 40)

$$\lim_{n\to\infty} F_\infty(q^{\frac{1}{n}}, q, \mathcal{K}) = F_\infty(1, q, \mathcal{K}),$$

so that

$$\lim_{n\to\infty} \mathrm{J}'_{\mathcal{K}}(n)(e^{\frac{2h}{n}}) = F_\infty(q^{\frac{1}{n}}, q, \mathcal{K})$$

$$= F_\infty(1, q, \mathcal{K})$$

$$= \frac{1}{A_{\mathcal{K}}(e^{2h})}$$

which re-proves Theorem 40. $\qquad\square$

Remark 41. There is also an alternative proof for MMR conjecture by Ito [19, Corollary 3.3]. Ito also makes use of MacMahon Master Theorem to make the Alexander polynomial appear. One important difference is that we use the unified invariant formulae, defined as elements of $\widehat{\mathcal{R}}^{\widehat{I}}$ and thus, well-defined as elements of $\mathbb{Q}[\alpha][[h]]$ alongside evaluation maps. In Ito's paper, it is not clear how the right-hand side of the main formula of Theorem 3.1 is h-adic (and it would mean that $\mathrm{Tr}(L_{n,m}(\beta))$ should be h-adically small as m grows to infinity)

The invariant F_∞ thus interpolates both families of ADO polynomials and colored Jones polynomials. In [32, Theorem 2.1], a first relation between ADO polynomials and Jones polynomials evaluated at an appropriate root of 1 is provided. Simply evaluating F_∞ now generalizes this to an infinite set of relations:

$$ADO_r(\zeta_{2r}^N, \mathcal{K}) = \mathrm{J}'_{\mathcal{K}}(N)(\zeta_r)\big(= F_\infty(\zeta_{2r}, \zeta_{2r}^N, \mathcal{K})\big) \tag{5}$$

(see also [38, Remark 58]), where N is any positive integer. We make the remark here that to establish this one has to notice that $\mathcal{A}_{\mathcal{K}}(1) = 1$. This is recalling that the Alexander polynomial

is usually well-defined only up to multiplication by $\pm A^k$ (for some integer k) and that one of its basic property is that its value at 1 is always ± 1 (as it corresponds to the torsion of the untwisted H_1 of the knot complement). The Alexander polynomial $\mathcal{A}_{\mathcal{K}}$ related to F_{∞} as above is then the one that is 1 on the unknot and that is 1 when $A = 1$ (see [38], the discussion before Theorem 60).

Moreover, F_{∞} is the unique function interpolating one or the other family, in the following sense.

Proposition 42 (Unicity property of F_{∞}). *Let $\mathcal{K}$ be a knot, $F_{\infty}(q, A, \mathcal{K})$ is the only element in $\widehat{\mathcal{R}^{\widehat{I}}}$ interpolating colored Jones polynomials or ADO over Alexander elements at an infinite number of values.*

In other word, if $u(q, A) \in \widehat{\mathcal{R}^{\widehat{I}}}$ is such that, for an infinite number of r or N in $\mathbb{N}^*$, we have:

$$u(\zeta_{2r}, A) = \frac{(A)^{rf} \times ADO_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})}$$

or

$$u(q, q^N) = J'_{\mathcal{K}}(N)(q^2)$$

then, we have the equality:

$$u(q, A) = F_{\infty}(q, A, \mathcal{K}).$$

Proof. The map $f : \mathbb{Q}[\alpha][[h]] \to \prod_{k \in \mathbb{N}} \mathbb{Q}[[h]]$, $x \mapsto (f_k(x))_{k \in \mathbb{N}}$ from [38, section 4.4] is injective. In fact, for any infinite subset $J \in \mathbb{N}^*$, $f_J : \mathbb{Q}[\alpha][[h]] \to \prod_{k \in J} \mathbb{Q}[[h]]$, $x \mapsto (f_k(x))_{k \in J}$ is injective. Thus, if for an infinite number of $N \in \mathbb{N}^*$:

$$u(q, q^N) = J'_{\mathcal{K}}(N)(q^2),$$

then

$$u(q, A) = F_{\infty}(q, A, \mathcal{K}).$$

Moreover, for any $N \in \mathbb{N}^*$, $ADO_r(\zeta_{2r}^N, \mathcal{K}) = J'_{\mathcal{K}}(N)(\zeta_r)$ (see (5)). Hence, if for an infinite number of $r \in \mathbb{N}^*$:

$$u(\zeta_{2r}, A) = \frac{(A)^{rf} \times ADO_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})},$$

then

$$u(\zeta_{2r}, \zeta_{2r}^N) = J'_{\mathcal{K}}(N)(\zeta_r)$$

and as $J'_{\mathcal{K}}(N)(q)$ is a Laurent polynomial, by knowing an infinite number of its evaluation, we have:

$$u(q, q^N) = J'_{\mathcal{K}}(N)(q^2),$$

so that $u(q, A) = F_{\infty}(q, A, \mathcal{K})$ (using the first part of the proof). $\square$

In [32], Murakami and Murakami have relocated the *volume conjecture* of Kashaev [22] in the context of colored Jones and ADO polynomials. It can be reformulated as a limit of evaluations of F_∞.

Conjecture 1 (Volume conjecture, [22, 32]). *The following equality would hold for any hyperbolic knot* $\mathcal{K}$,

$$2\pi \lim_{N \to \infty} \frac{\mathrm{Log}(F_\infty(e^{\frac{i2\pi}{N}}, 1, \mathcal{K}))}{N} = \mathrm{Vol}(S^3 \backslash \mathcal{K}),$$

where $\mathrm{Vol}(S^3 \backslash \mathcal{K})$ *is the hyperbolic volume of the complement of the knot.*

3.4 | Generalization of Alexander polynomials' symmetry

To prove a symmetry for the ADO invariants, we must change a bit how we use the partial trace. Throughout the paper, we have set the first element in the tensor products to be v_0. In fact, we can define the partial trace using the natural definition on tensor products:

$$\widetilde{\mathrm{Tr}}_{2,\dots,n} : \mathrm{End}\left((V^\alpha)^{\otimes n}\right) \to \mathrm{End}(V^\alpha)$$

and we have, by definition:

$$\widetilde{\mathrm{Tr}}_{2,\dots,n}(f).v_0 = \mathrm{Tr}_{2,\dots,n}(f)v_0.$$

As V^α is absolutely simple, the partial trace $\widetilde{\mathrm{Tr}}_{2,\dots,n}(f)$ is scalar, allowing us to identify to its value $\mathrm{Tr}_{2,\dots,n}(f)$. In other words,

$$\widetilde{\mathrm{Tr}}_{2,\dots,n}(f).w = \mathrm{Tr}_{2,\dots,n}(f)w$$

for any $w \in V^\alpha$. Combining this fact with Lemma 15, one can get a symmetry for the unified invariant.

Theorem 43 (An Alexander-like symmetry for F_∞). *Let* $\mathcal{K}$ *be a 0 framed knot,*

$$F_\infty(q, q^\alpha, \mathcal{K}) = F_\infty(q, q^{-\alpha-2}, \mathcal{K}).$$

In other words, F_∞ *is not sensitive to* $s \mapsto s^{-1}q^{-2}$.

Proof. Using Theorem 33 at V^N, we have the identity

$$\widetilde{\mathrm{Tr}}_{2\dots n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta))v_0 = F_\infty(q, q^N, \mathcal{K})v_0$$

and, as the partial trace $\widetilde{\mathrm{Tr}}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta))$ is scalar, we get:

$$\widetilde{\mathrm{Tr}}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta))v_{N+1} = F_\infty(q, q^N, \mathcal{K})v_{N+1}.$$

But using Lemma 15, we also have in V^N/S_N:

$$\widetilde{\mathrm{Tr}}_{2,\dots,n}((1 \otimes K^{\otimes n-1})\varphi_n(\beta))\overline{v_{N+1}} = F_\infty(q, q^{-N-2}, \mathcal{K})\overline{v_{N+1}}.$$

Thus, for all $N \in \mathbb{N}^*$, we have

$$F_\infty(q, q^N, \mathcal{K}) = F_\infty(q, q^{-N-2}, \mathcal{K}).$$

Using Proposition 42, we have the equality at formal weight q^α:

$$F_\infty(q, q^\alpha, \mathcal{K}) = F_\infty(q, q^{-\alpha-2}, \mathcal{K}). \qquad \square$$

Corollary 44 (Colored Alexander symmetry). *Let $\mathcal{K}$ be a 0 framed knot,*

$$ADO_r(A, \mathcal{K}) = ADO_r(A^{-1}\zeta_{2r}^{-2}, \mathcal{K}).$$

Proof. At $q = \zeta_{2r}$, we have the factorization:

$$F_\infty(\zeta_{2r}, q^\alpha, \mathcal{K}) = \frac{A^{rf} \times ADO_r(A, \mathcal{K})}{A_\mathcal{K}(A^{2r})}$$

and as

$$A_\mathcal{K}(A^{2r}) = A_\mathcal{K}(A^{-2r})$$

one obtains the desired identity. $\qquad \square$

In [9, section 2.2], Costantino, Geer, and Patureau define an invariant of trivalent graphs denoted N_r constructed also from the theory of $U_q\mathfrak{sl}(2)$ at $q = \zeta_{2r}$. For knots it is the ADO polynomial, more precisely there is a simple change of variable (coming from the fact that they take the variable to be the middle weight instead of here being the highest weight):

$$N_r(q^\alpha, \mathcal{K}) = ADO_r\left(q^{\alpha+1-r}, \mathcal{K}\right).$$

Corollary 45. *The $U_q\mathfrak{sl}(2)$ non-semisimple invariant N_r is not sensitive to orientation for knots.*

Proof. From [9, section 2.2, (1)], we know that if $\mathcal{K}^{-1}$ is the knot $\mathcal{K}$ with reversed orientation, then:

$$N_r(q^\alpha, \mathcal{K}^{-1}) = N_r(q^{-\alpha}, \mathcal{K}).$$

Re-expressing N_r as ADO and using the colored Alexander symmetry from previous corollary, one deduces directly the invariance under reverse of orientation. $\qquad \square$

Remark 46. Authors do not know how much latter invariance generalizes to next objects (e.g., links, graphs) for the non-semisimple invariant N_r.

4 | UNIFIED INVARIANT FROM HOMOLOGY OF CONFIGURATION SPACES

This section first re-defines the tensor products of Verma modules and the action of braid groups upon them by homeomorphisms using homology of configuration spaces of points in punctured

disks (Subsection 4.1). This is another point of view independent of quantum groups theory of representation, that was established in [29]. This homological interpretation of quantum Verma tensors as Lagrangians of configuration spaces is the key point for Theorem 68 expressing F_∞ as an intersection pairing between such Lagrangians. Subsection 4.2 presents two families of manifolds in configuration spaces defining dual homology classes regarding the Poincaré duality. In Subsection 4.3, we prove Theorem 68 and we discuss its consequences.

4.1 | A homological definition for $U_q \mathfrak{sl}(2)$ Verma modules

Definition 47. Let $r \in \mathbb{N}$, $n \in \mathbb{N}$, D be the unit disk, and $\{w_1, \dots, w_n\} \in D^n$ points chosen on the real line in the interior of D. Let $D_n = D \setminus \{w_1, \dots, w_n\}$ be the unit disk with n punctures. Let:

$$\mathrm{Conf}_r(D_n) := \left\{ (z_1, \cdots, z_r) \in (D_n)^r \text{ such that } z_i \neq z_j \forall i, j \right\}$$

be the configuration space of points in the punctured disk D_n. We define the following space:

$$X_r(w_1, \dots, w_n) := \mathrm{Conf}_r(D_n) \Big/ \mathfrak{S}_r \tag{6}$$

to be the space of *unordered* configurations of r points inside D_n, where the permutation group $\mathfrak{S}_r$ acts by permutation on coordinates.

When no confusion arises in what follows, we omit the dependence in $w_1, \dots, w_n$ to simplify notations. All the following computations rely on a choice of base point that we fix from now on.

Definition 48 (Base point). Let $\boldsymbol{\xi}^r = \{\xi_1, \dots, \xi_r\}$ be the base point of X_r chosen so that $\xi_i \in \partial D_n$ ($\forall i$) as in the following picture:

.
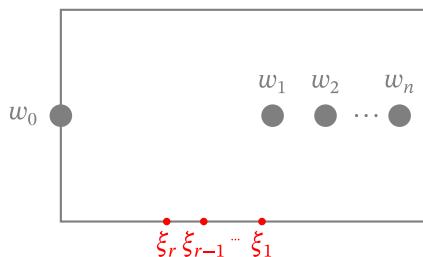
We have illustrated the unit disk (as a square) with the punctures $w_1, \dots, w_n$, we have add another point w_0 on the boundary that will be used later on, and also the base point just defined.

We give a presentation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ as a braid subgroup (*the mixed braid group*).

Remark 49 [29, Remark 2.2]. The group $\pi_1(X, \boldsymbol{\xi}^r)$ is isomorphic to the subgroup of $\mathcal{B}_{r+n}$ generated by:

$$\langle \sigma_1, \dots, \sigma_{r-1}, B_{r,1}, \dots, B_{r,n} \rangle$$

where the σ_i $(i = 1, \dots, r-1)$ are the first standard generators of $\mathcal{B}_{r+n}$, and $B_{r,k}$ (for $k = 1, \dots, n$) is the following pure braid:

$$B_{r,k} = \sigma_r \cdots \sigma_{r+k-2} \sigma_{r+k-1}^2 \sigma_{r+k-2}^{-1} \cdots \sigma_r^{-1}.$$

See [29, Example 2.3] for a picture that illustrates the correspondence between above generators and braids. It will help the reader understanding the following definition of a local system.

Definition 50 (Local ring $\mathcal{R}_r$.). We define the following morphism:

$$\rho_r : \begin{cases} \mathbb{Z}\left[\pi_1(X_r, \boldsymbol{\xi}^r)\right] & \to & \mathcal{R} := \mathbb{Z}\left[s^{\pm 1}, t^{\pm 1}\right] \\ \sigma_i & \mapsto & t \\ B_{r,k} & \mapsto & s^2. \end{cases}$$

In what follows, we will use the notation $q^\alpha := s$. Using this notation, the morphism becomes:

$$\rho_r : \begin{cases} \mathbb{Z}\left[\pi_1(X_r, \boldsymbol{\xi}^r)\right] & \to & \mathcal{R} := \mathbb{Z}\left[q^{\pm \alpha}, t^{\pm 1}\right] \\ \sigma_i & \mapsto & t \\ B_{r,k} & \mapsto & q^{2\alpha}. \end{cases}$$

(We may sometimes omit the dependence in $(w_1, \dots, w_n)$.) The data set $(\rho_r, \mathcal{R})$ will be re-united under the notation $\mathcal{R}_r$ and named local ring of coefficients.

Definition 51 [29, Definition 2.6]. Let $r \in \mathbb{N}$, and let $w_0 = -1$ be the leftmost point in the boundary of D_n (see the picture in Definition 48), we define the following set:

$$X_r^-(w_1, \cdots, w_n) = \{\{z_1, \cdots, z_r\} \in X_r(w_1, \cdots, w_n) \text{such that} \exists i, z_i = w_0\}.$$

We let H^{lf} designate the homology of locally finite chains, and we use the following notation for relative homology modules with local coefficients in the ring $\mathcal{R}$:

$$\mathcal{H}_r^{\mathrm{rel}} := \mathrm{H}_r^{\mathrm{lf}}\left(X_r, X_r^-; \mathcal{R}_r\right).$$

See next remark for precisions on such construction.

Remark 52. We recall how this homology modules are constructed, namely we work with the following homology theories:

- the *locally finite* version of the singular homology, for which we consider locally finite infinite linear combination of singular simplices, (see [29, appendix]);
- the homology of the pair (X_r, X_r^-);
- the local ring $\mathcal{R}_r$. Let ρ_r be the morphism from Definition 50. This can be seen as the homology associated with the chain complex $C_\bullet(\widehat{X_r})$ where $\widehat{X_r}$ is the covering naturally associated with the kernel of ρ_r which is naturally endowed with an action of $\mathcal{R}$ by deck transformation as ρ_r is surjective (hence the deck transformation group of $\widehat{X_r}$ is generated by t and s).

We define classes in $\mathcal{H}_r^{\mathrm{rel}}$. We refer the reader to [29] for further details on these constructions.

Definition 53 (Multi-arc diagrams). Let $(k_0, \ldots, k_{n-1})$ such that $\sum k_i = r$. we define $A'(k_0, \ldots, k_{n-1})$ to be the following diagram:
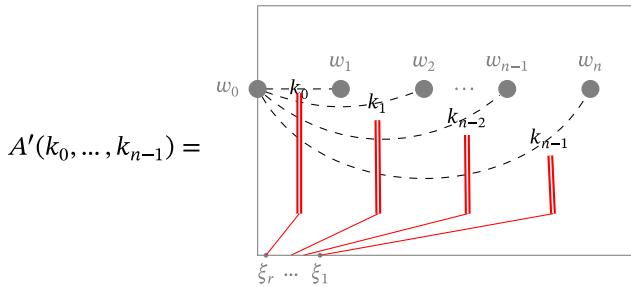
Remark 54. These above diagrams are denoted A' because there will be slightly different versions for them later on and denoted A.

For $A'(k_0, \ldots, k_{n-1})$ defined above, let:

$$\phi_i : I_i \to D_n$$

be the embedding of the dashed black arc number i indexed by k_{i-1}, where I_i is a copy of the unit interval. Let Δ^k be the standard (open) k simplex:

$$\Delta^k = \{0 < t_1 < \cdots < t_k < 1\}$$

for $k \in \mathbb{N}$. For all i, we consider the map $\phi^{k_{i-1}}$:

$$\phi^{k_{i-1}} : \begin{cases} \Delta^{k_{i-1}} & \to & X_{k_{i-1}} \\ (t_1, \ldots, t_{k_{i-1}}) & \mapsto & \{\phi_i(t_1), \ldots, \phi_i(t_{k_{i-1}})\}, \end{cases}$$

which is a singular locally finite (k_{i-1})-chain and moreover a cycle in $X_{k_{i-1}}$ because locally finite homology of an open ball is one-dimensional and concentrated in the ambient dimension [29, appendix].

To get a class in the homology with $\mathcal{R}$ coefficients, one may choose a lift of the chain to the cover $\widehat{X_r}$ associated with the morphism ρ_r. We do so using the red handles of $A'(k_0, \ldots, k_{n-1})$ (the union of red paths) with which is naturally associated a path:

$$\mathbf{h} = \{h_1, \ldots, h_r\} : I \to X_r$$

joining the base point $\boldsymbol{\xi}$ and (a point in) the r-chain assigned to the union of dashed arcs. At the cover level $(\widehat{X_r})$ there is a unique lift $\widehat{\mathbf{h}}$ of $\mathbf{h}$ that starts at $\widehat{\boldsymbol{\xi}}$, *a choice of lift of the base point to $\widehat{X_r}$ that we fix from now on*. It is the unique lift property applied to lift based paths to a covering space. The lift $\widehat{A}(k_0, \ldots, k_{n-1})$ of $A(k_0, \ldots, k_{n-1})$ passing by $\widehat{\boldsymbol{\xi}}(1)$ defines a cycle in $C_r^{\text{rel}-}$, and we still call (by abuse of notation) $A'(k_0, \ldots, k_{n-1})$ the associated class in $\mathcal{H}_r^{\text{rel}}$ as we will only use this class out of the original object.

Definition 55 (Multi-arcs (first version)). Following the above construction, we naturally assign a class $A'(k_0, \ldots, k_{n-1}) \in \mathcal{H}_r^{\text{rel}}$ with any n-tuple such that $\sum k_i = r$. This class is called a *multi-arc*.

Now we state a proposition that clarifies the structure of the homology as $\mathcal{R}$-modules.

Proposition 56 (Multi-arcs generate the homology, [29, Proposition 3.6]). *Let $r \in \mathbb{N}$, the homology of the pair (X_r, X_r^-) has the following structure.*

- *The module $\mathcal{H}_r^{\text{rel}}$ is free over $\mathcal{R}$.*
- *The set of multi-arcs:*

$$\{A'(k_0, \cdots, k_{n-1}) \text{ such that } \sum k_i = r\}$$

  yields a basis of $\mathcal{H}_r^{\text{rel}}$.
- *The module $\mathcal{H}_r^{\text{rel}}$ is the only nonvanishing module of $\text{H}_{\bullet}^{\text{lf}}(X_r, X_r^-; \mathcal{R})$.*

The braid group was earlier defined (Definition 9) using its so called Artin presentation. Here we give another definition, relying on topological objects.

Definition 57. The braid group on n strands is the mapping class group of D_n.
$$\mathcal{B}_n = \text{Mod}(D_n) = \overset{+}{\text{Homeo}}(D_n, \partial D) / \text{Homeo}_0(D_n, \partial D),$$

namely the group of isotopy classes of homeomorphisms of the unit disk: preserving the orientation, the set of punctures, and being the identity on the boundary.

Remark 58. This definition is isomorphic to the Artin presentation of the braid group (Definition 9) by sending generator σ_i to the isotopy class of the half Dehn twist swapping punctures w_i and w_{i+1}.

Lemma 59 (Lawrence representations). *For all $r, n \in \mathbb{N}$, the modules $\mathcal{H}_r^{\text{rel}}$ are endowed with an action of the braid group $\mathcal{B}_n$.*

Idea of the construction. *It is Lawrence construction of braid groups representations [25]. See [29, Lemma 6.33] for this precise lemma. The representations are constructed as follows (sketch of proof).*

- *Let S_i be the Dehn twist associated with the standard Artin generator σ_i of $\mathcal{B}_n$, for $i \in \{1, \dots, n-1\}$ (see Remark 58).*
- *The homeomorphism S_i extends to X_r coordinate by coordinate. Namely, extended S_i is the map that sends a configuration $\{z_1, \dots, z_r\}$ to $\{S_i(z_1), \dots, S_i(z_r)\}$, as S_i is a homeomorphism, so is its extension.*
- *The action of S_i on X_r naturally lifts to $\mathcal{H}_r^{\text{rel}}$ (it is the heart of [29, Lemma 6.33] and of Lawrence's work).*
- *By defining the action of σ_i on $\mathcal{H}_r^{\text{rel}}$ by that of S_i one obtains a well-defined (and multiplicative) action of $\mathcal{B}_n$ on $\mathcal{H}_r^{\text{rel}}$. It is well-defined as braids are homeomorphisms considered up to isotopy while we study their homological action.*

The above representations are often called Lawrence(-like) representations.

We can now recall the main result from [29] relating these homological representations with Verma modules representations defined in Subsection 2.2.

Theorem 60 [29, Theorem 2,3]. *The isomorphism of $\mathcal{R}$-modules:*

$$\begin{cases} \mathcal{H} := \bigoplus_{m \in \mathbb{N}} \mathcal{H}_m^{\text{rel}} & \rightarrow & V_\alpha^{\otimes n} = \bigoplus_{m \in \mathbb{N}} V_{n,m} \\ A(k_0, \dots, k_{n-1}) & \mapsto & v_{k_0} \otimes \cdots \otimes v_{k_{n-1}}. \end{cases}$$

is $\mathcal{B}_n$ equivariant. In the above isomorphism, the following vectors are involved:

$$A(k_0, \dots, k_{n-1}) := q^{\alpha \sum_{i=1}^{n-1} i k_i} A'(k_0, \dots, k_{n-1})$$

for any $(k_0, \dots k_{n-1}) \in \mathbb{N}^n$ (see [29, Definition 6.16]). The identification of rings $\mathcal{R}$ is made by considering $q^{-2} = -t$ (the variable s being the same on both sides).

Remark 61.

- A diagonal term $q^{-\alpha\left(\sum_{i=1}^{n-1} i k_i\right)}$ normalizes vectors A [29, Definition 6.16].

- The isomorphism from the above theorem also respects the $U_q^{\frac{L}{2}} \mathfrak{sl}(2)$ action that is defined on Verma modules in Section 2.2. In the sense that there is an action of $U_q^{\frac{L}{2}} \mathfrak{sl}(2)$ defined on $\mathcal{H}$, see [29, Theorem 1].

We will use this isomorphism relating quantum braid representations with homology so for interpreting the partial trace defining F_∞ in terms of homological intersections.

4.2 | Homological duality

4.2.1 | Multi-arcs: Another version

We recall that for $(k_0, \dots, k_{n-1})$ such that $\sum k_i = m$, there is a multi-arc $A'(k_0, \dots, k_{n-1})$ defining a vector in $\mathcal{H}_m^{\mathrm{rel}}$, and so that the whole family yields a basis. We draw such an element but with a slightly different drawing that better fits with the knot invariant we are seeking.

Definition 62 (Multi-arcs (second version)). For $(k_0, \dots, k_{n-1}) \in \mathbb{N}^n$ such that $\sum k_i = r$, we define the following diagram.
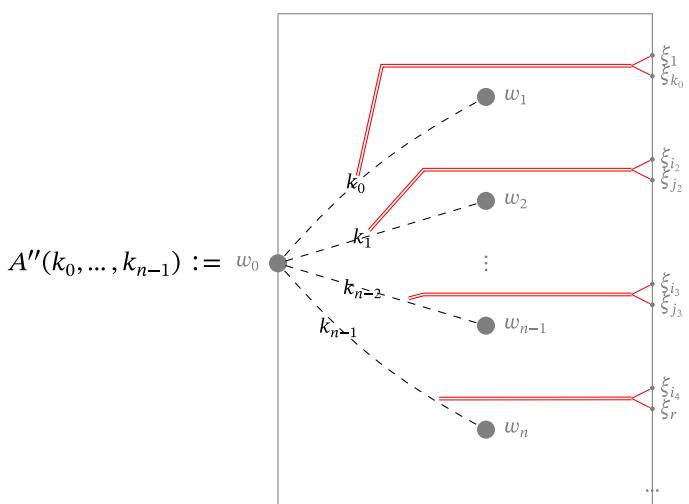
As diagrams from Definition 54 naturally defines classes in $\mathcal{H}_r^{\mathrm{rel}}$ (see Definition 55, natural process explained above it), same natural process associates classes in $\mathcal{H}_r^{\mathrm{rel}}$ with the above $A''(k_0, \dots, k_{n-1})$. We use latter notation to designate the homology class also.

We have three families of diagrams corresponding to homology classes. They are related diagonally as follows.

Proposition 63. *In $\mathcal{H}_r^{\mathrm{rel}}$, the following relations hold.*

$$A(k_0, \dots, k_{n-1}) = q^{\sum_{i=1}^{n-1} i k_i \alpha} A'(k_0, \dots, k_{n-1}), \tag{7}$$

$$A'(k_0, \dots, k_{n-1}) = (-t)^{\frac{r(r-1)}{2}} q^{2\alpha \sum_{i=0}^{n-1} (n-i)k_i} A''(k_0, \dots, k_{n-1}) \tag{8}$$

for all $(k_0, \dots, k_{n-1})$ *such that* $\sum k_i = r$*. Finally,*

$$A(k_0, \dots, k_{n-1}) = (-t)^{\frac{r(r-1)}{2}} q^{\alpha 2nr} q^{-\alpha \sum_{i=0}^{n-1} i k_i} A''(k_0, \dots, k_{n-1}). \tag{9}$$

Proof. The first equality of the proposition was already considered in [29] and was recalled in Theorem 60. The second one follows from the following equalities:
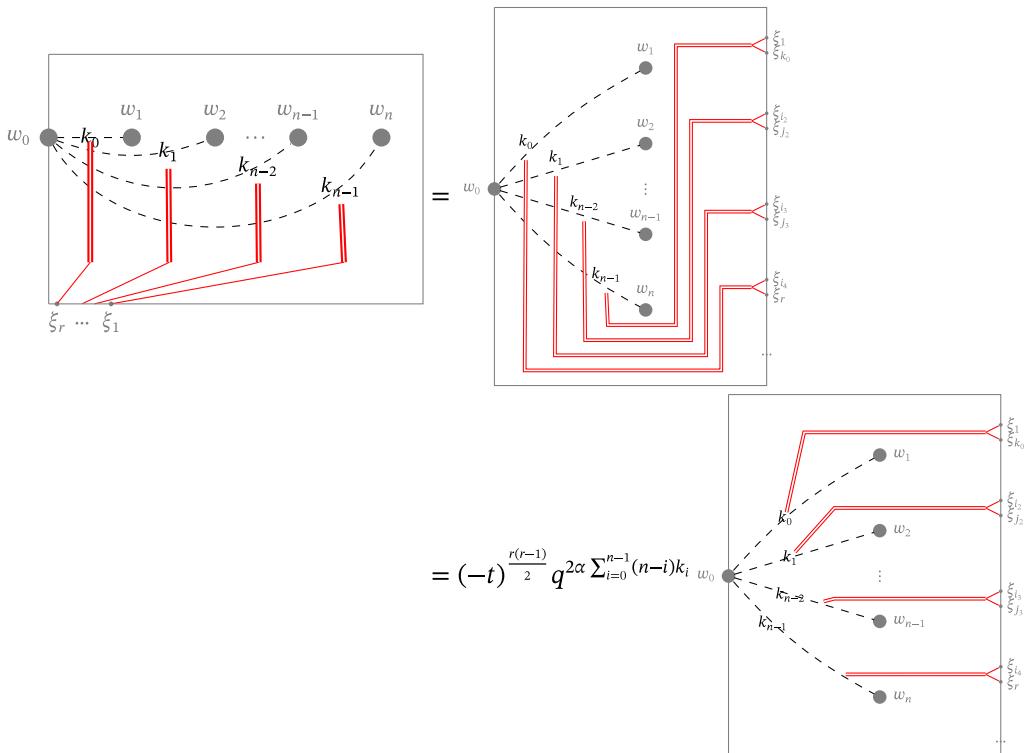
The first equality comes from an isotopy of the disk, the second one comes from the application of the *handle rule* ([29, Remark 4.1], see details in following Remark 64). Then one recognizes

leftmost diagram to be $A(k_0, \ldots, k_{n-1})$ and last one to be $A''(k_0, \ldots, k_{n-1})$. Finally,

$$A(k_0, \ldots, k_{n-1}) = (-t)^{\frac{r(r-1)}{2}} q^{\alpha \sum_{i=0}^{n-1}(2n-i)k_i} A''(k_0, \ldots, k_{n-1})$$

$$= (-t)^{\frac{r(r-1)}{2}} q^{\alpha\left(2nr - \sum_{i=0}^{n-1} ik_i\right)} A''(k_0, \ldots, k_{n-1})$$

provides last relation. $\qquad\square$

Remark 64 (Handle rule). We give more details on the handle rule applied once in the proof of the previous proposition. The handle rule [29, Remark 4.1] states:
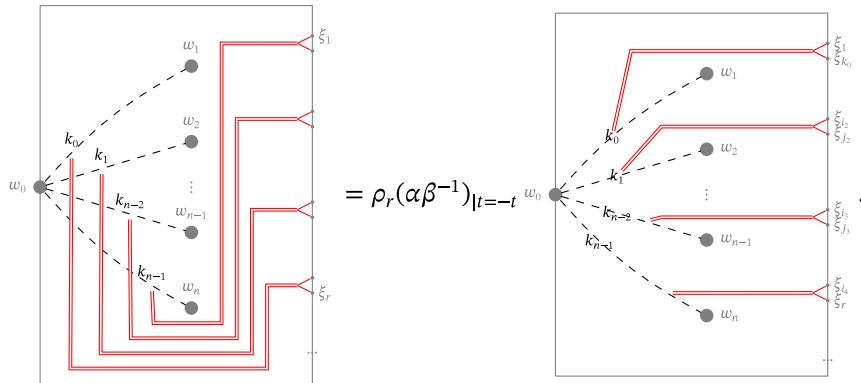
where α is the path in X_r corresponding to the (red)-handle on the left and β to that on the right, and ρ_r the representation of $\pi_1(X_r)$ recalled in Definition 50. We evaluate ρ_r at $t = -t$, see [29, Remark 4.2], because in diagrams the permutation of the red strands implies a permutation of embeddings of configurations. Hence, the homology class must be multiplied by the sign of the permutation (i.e., the power of t in $\rho_r(\alpha\beta^{-1})$) corresponding to the induced change of orientation.

In the present case, the path $\alpha\beta^{-1}$ is drawn below.
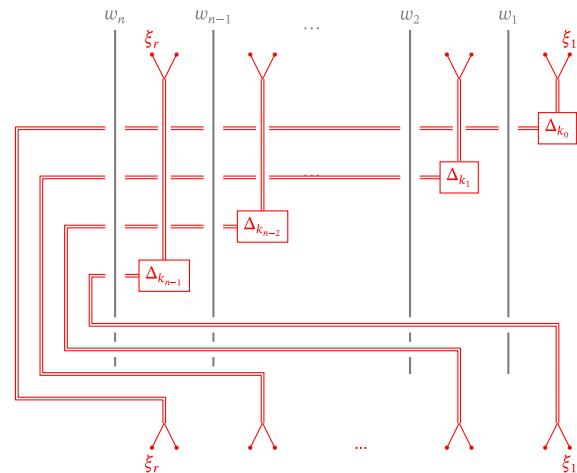
Every little red tube means parallel red paths not crossing with each other. Crossings involving these tubes are materialized using Δ boxes inside which the following happens:
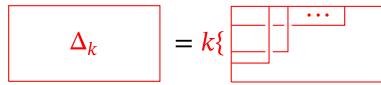

Then $\rho_r(\alpha\beta^{-1})_{|t=-t} = (-t)^a q^{\alpha b}$ such that a is the sum (with signs) of red–red crossings, and b is the total winding number of red strands around gray ones. The reader should pay attention to the fact that in [29] braids a read from top to bottom as in the present work we do the converse.

4.2.2 | Barcodes

We now define homology classes in $H_r\left(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r\right)$ that are usually called *barcodes*.

Definition 65 (Barcodes). We fix notation for the following diagrams:
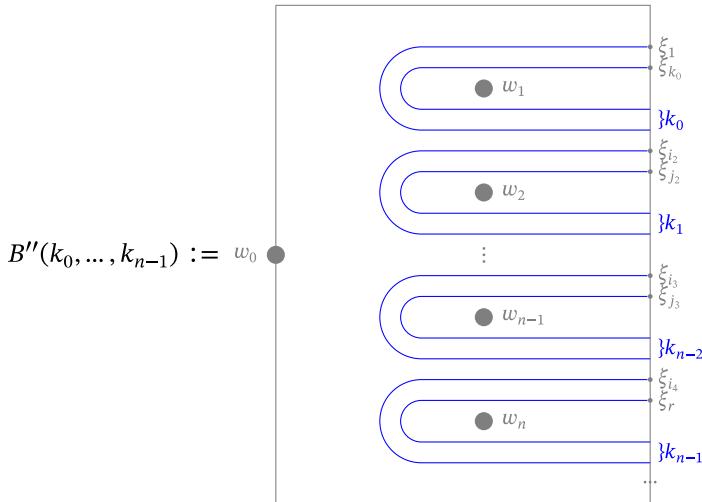
where $\sum k_i = r$. We naturally assign a class in $H_r\left(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r\right)$ with the above diagram according to the following process.

- The union of blue arcs well defines an embedding:

$$\Phi : I^r \to X_r,$$

where I is the unit interval.
- As ends of blue arcs are lying in $\partial X_r \setminus X_r^-$, the hypercube Φ defines a homology class in $H_r\left(X_r, \partial X_r \setminus X_r^-; \mathbb{Z}\right)$.
- It remains to choose a lift of Φ to the cover $\widehat{X_r}$ so to work in the local ring set-up. We do so using the fact that the image of Φ contains the base point ξ, so that we choose the only lift of Φ containing our choice of lift for the base point $\widehat{\xi}$.

We still denote $B''(k_0, \dots, k_{n-1})$ the resulting element of $H_r\left(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r\right)$.

Proposition 66. *We have the following:*

$$\left\langle A''(k_0, \dots, k_{n-1}) \cap B''(k'_0, \dots, k'_{n-1}) \right\rangle = \delta_{(k_0, \dots, k_{n-1}),(k'_0, \dots, k'_{n-1})},$$

where δ is the Kronecker (list) symbol, and $\langle \cdot \cap \cdot \rangle$ is the intersection pairing arising from the Poincaré–Lefschetz duality:

$$\mathcal{H}_r^{\mathrm{rel}} \times \mathrm{H}_r\left(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r\right) \to \mathcal{R}.$$

Proof. We put diagrams associated with $A''(k_0, \dots, k_{n-1})$ and $B''(k'_0, \dots, k'_{n-1})$ all together in one picture.
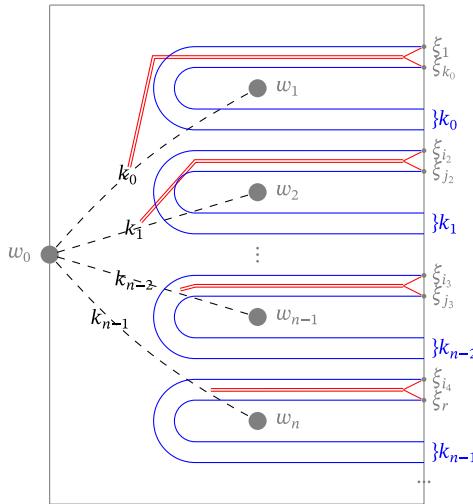
An intersection point is a configuration lying on both manifolds, namely such a configuration has one point on each blue arc, and k of them on a dashed arc indexed by k. The only way there is an intersection is: $k_0 = k'_0, \dots, k_{n-1} = k'_{n-1}$ that explains the Kronecker symbol in the formula. When we are in this equality case, we call $\boldsymbol{p} := \{p_1, \dots, p_r\}$ the single intersection configuration, and it remains to prove that the intersection is equal to 1 at this configuration. We recall that the Poincaré–Lefschetz duality gives an intersection pairing:

$$\mathcal{H}_r^{\mathrm{rel}} \times \mathrm{H}_r\left(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r\right) \to \mathcal{R},$$

see [30, Lemma 4.1]. This pairing is given by graded intersection, where each intersection point contributes for a sign (that of the intersection) times a monomial in $\mathcal{R}$. Let $\widehat{A''}(k_0, \dots, k_{n-1})$ and $\widehat{B''}(k_0, \dots, k_{n-1})$ be the lifts of the corresponding manifolds chosen using the red handle, respectively, the one that contains $\hat{\boldsymbol{\xi}}^r$. In our case, the only monomial $m_{\boldsymbol{p}}$ involved could be computed by defining the following loop in X_r, by composing paths.

- First the path going from $\{\xi_1, \dots, \xi_r\}$ to $A''(k_0, \dots, k_{n-1})$ following red handles.
- Then joining $\{p_1, \dots, p_r\}$ going along $A''(k_0, \dots, k_{n-1})$.
- Then going back to $\boldsymbol{\xi}^r$ running along $B''(k_0, \dots, k_{n-1})$.

This composition of paths yields a loop denoted $\gamma_{\boldsymbol{p}}$ of X_r based at $\boldsymbol{\xi}$. By considering one of its lift to $\widehat{X_r}$, one can check that it relates $\widehat{\boldsymbol{\xi}}$ and $m_{\boldsymbol{p}}\widehat{\boldsymbol{\xi}}$. The explanation of this fact is exactly the same as the

one given before [31, Lemma 3.11] that is adapted from [5, section 3.1]. Knowing this, we directly conclude:

$$m_{\boldsymbol{p}} = \rho_r(\gamma_{\boldsymbol{p}}),$$

and moreover that:

$$m_{\boldsymbol{p}} = \rho_r(\gamma \boldsymbol{p}) = 1.$$

One can check that the braid given by $\gamma_{\boldsymbol{p}}$ seen as an element of $\pi_1(X_r, \boldsymbol{\xi})$ (following the model [29, Remark 2.2]) is trivial. This ensures the above equality by the definition of ρ_r (Definition 50). $\square$

Remark 67 (Homological dual bases). The above theorem says that sets $\{A''(k_0, \cdots, k_{n-1}) \text{such that} \sum k_i = r\}$ and $\{B''(k_0, \cdots, k_{n-1}) \text{such that} \sum k_i = r\}$ are dual bases of $\mathcal{H}_r^{\mathrm{rel}}$, respectively, $\mathrm{H}_r(X_r, \partial X_r \setminus X_r^-; \mathcal{R}_r)$ (the one to one correspondence being given by the canonical indexing).

4.3 | Unified invariant from homological intersection

Theorem 68 (The unified invariant from homological intersection pairing). *Let* $\beta \in \mathcal{B}_n$ *a braid such that its closure is the knot* $\mathcal{K}$. *Then, letting* $t = -q^{-2}$:

$$F_\infty(\mathcal{K}) = s^{n-1} \sum_{\overline{k} \in \mathbb{N}_0^n} \left\langle \beta \cdot A''(0, k_1, \ldots, k_{n-1}) \cap B''(0, k_1, \ldots, k_{n-1}) \right\rangle q^{-2 \sum k_i},$$

where the action of β *is that from Lemma 59. The latter means that the right term in the equation, which is an infinite sum of intersection pairing of middle dimension homology classes, lives in* $\widehat{\mathcal{R}}^{\widehat{I}}$ *and is invariant under Markov moves.*

Proof. The main tool is Theorem 60 which shows that (under $t = -q^{-2}$) by sending $v_{i_1} \otimes \cdots \otimes v_{i_{n-1}}$ to $A(i_0, \ldots, i_{n-1})$ (for any integers $i_0, \ldots, i_{n-1}$) then matrices for the quantum action and the homological actions of β are strictly identical. The partial trace formula from Theorem 33 is the same replacing the v_i's by the A vectors from the homological side. Then the partial trace of any endomorphism f of $\mathcal{H}$ could be expressed as follows:

$$\mathrm{Tr}_{2,\ldots,n}(f) = \sum_{\overline{k} \in \mathbb{N}_0^n} \langle f(A(0, k_1, \ldots, k_{n-1})), A_*(0, k_1, \ldots, k_{n-1}) \rangle$$

where A_* means the dual family of A regarding the Poincaré–Lefschetz duality studied in Proposition 66. As the change of bases from A's to A'''s is diagonal (see Proposition 63), we can replace A and A_* in the above formula by the A'''s and its dual family, namely the B'' as it was proved in Proposition 66. Now the f we wish to consider here is $(1 \otimes K^{\otimes n-1}) \circ \phi_n(\beta)$. One notices that $(1 \otimes K^{\otimes n-1})$ on the image by $\phi_n(\beta)$ of any $A''(0, k_1, \ldots, k_{n-1})$ is the multiplication by $s^{n-1} q^{-2 \sum k_i}$ that concludes the proof. $\square$

The fact that F_∞ interpolates ADO invariants and colored Jones polynomials by some specialization, implies the above theorem at the corresponding specialization gives infinite sum from

which one can extract these invariants out of Lagrangian intersections. Moreover, infinite sums are not crucial.

Corollary 69. *Let $\beta \in \mathcal{B}_n$ a braid such that its closure is the knot $\mathcal{K}$.*

(1) *Let $J'_\mathcal{K}(N)$ be the Nth colored Jones polynomial (normalized) and* spec *the specialization morphism sending s to q^N, then:*

$$J'_\mathcal{K}(N) = q^{N(n-1)} \sum_{\substack{\bar{k} \text{ such that} \\ \sum k_i < N, \forall i}} \mathrm{spec}\left(\left\langle \beta \cdot A''(0, k_1, \cdots, k_{n-1}) \cap B''(0, k_1, \cdots, k_{n-1}) \right\rangle q^{-2 \sum k_i} \right)$$

(2) *Let ADO_r be the rth ADO polynomial and* spec *the specialization morphism sending q to ζ_{2r}, then:*

$$\mathrm{ADO}_r(\mathcal{K}) = s^{n-1} \sum_{\substack{\bar{k} \text{ such that} \\ \forall i, k_i < r}} \mathrm{spec}\left(\left\langle \beta \cdot A''(0, k_1, \cdots, k_{n-1}) \cap B''(0, k_1, \cdots, k_{n-1}) \right\rangle q^{-2 \sum k_i} \right)$$

Proof. The proof is the same as that of the previous theorem. The sum is truncated directly as only first weight levels are necessary to compute them, see:

(1) [38, Lemma 51] for the colored Jones case;
(2) Proposition 36 for the ADO case.

$\square$

Remark 70 (Normalized colored Jones). As a partial trace is involved in the above formula for the colored Jones polynomial, we are dealing with the normalized version (being 1 on the unknot) (see Notation 35), which is a different version as that in [30]. It explains the difference of the homology classes involved in the sums from here and the mentioned paper.

Example 71 (The trefoil knot). We illustrate the fact that the homological formula from Theorem 68 gives an independent algorithm of computation arising from homological computation by computing F_∞ with this formula and comparing with the expression given in [38, section 5]. To do the computation, we use the element $A'(0, k) \in \mathcal{H}$ (instead of A'' in the formula, for clarity of diagrams) for $k \in \mathbb{N}$, the disk is considered with two punctures (w_1, w_2) as we need a braid with two strands to get the trefoil knot as a braid closure (namely the closure of $\sigma_1^{-3} \in \mathcal{B}_2$, we use inverse twists for simplicity of diagrams). We use notation $s = q^\alpha$.


We recall that this element have to be paired with the dual class of $A'(0, k)$ (and then summed over $k \in \mathbb{N}$) so to obtain F_∞. The class $B'(0, k)$ with the following diagram is this dual class (i.e.,

$A'(0, k)$ and $B'(0, k)$ have pairing 1, see the proof of Proposition 66):

$$B'(0, k) :=$$
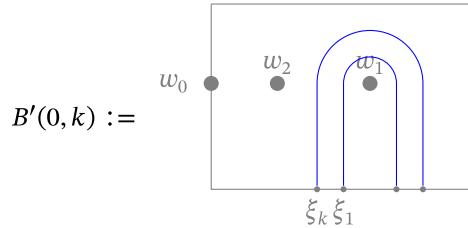
We simplify the diagram of $\sigma_1^{-3}(A'(0, k))$ using rules from [29, section 4] to simplify the computation of the intersection.
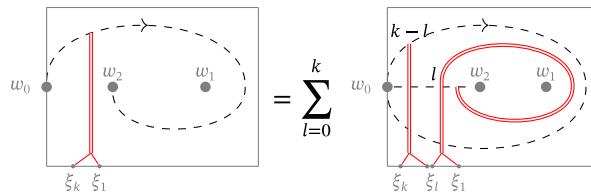
This is similar to [29, Example 4.6]. In the right-hand sum, when ever l is not 0, the manifold associated with the diagram has no intersection with $B'(0, k)$ (an intersection point is a k-tuple, one on each blue line of $B'(0, k)$ and respecting the indices of $A'(0, k)$). Hence, the only diagram having nontrivial intersection with $B'(0, k)$ is when $l = 0$ so that:

$$\left\langle \sigma_1^{-3} A'(0, k) \cap B'(0, k) \right\rangle = \sum_{l=0}^{k} \left\langle \quad \cap B'(0, k) \right\rangle$$

$$= \left\langle \quad \cap B'(0, k) \right\rangle$$

We use the *handle rule* [29, Remark 4.2] (see also in the proof of Proposition 63), to rearrange the red handle:

$$= (-1)^k (-t)^{\frac{k(k-1)}{2}} \left(q^{2\alpha} \right)^{2k}$$

The coefficient showing up is the image by ρ_k of the loop in X_k defined as the composition of the red path on the left with that on the right. The $(-1)^k$ appears because we have reversed the

orientation of the dashed arc. Then,

$$\left\langle \sigma_1^{-3} A'(0,k) \cap B'(0,k) \right\rangle = \left\langle (-1)^k (-t)^{\frac{k(k-1)}{2}} \left(q^{2\alpha}\right)^{2k} \qquad \cap B'(0,k) \right\rangle$$

$$= (-1)^k (-t)^{\frac{k(k-1)}{2}} \left(q^{2\alpha}\right)^{2k} \left\langle \qquad \cap B'(0,k) \right\rangle$$

$$= (-1)^k (-t)^{\frac{k(k-1)}{2}} \left(q^{2\alpha}\right)^{2k} \left\langle \prod_{i=0}^{k-1} (1 - q^{-2\alpha}(-t)^{-i}) A'(0,k) \cap B'(0,k) \right\rangle$$
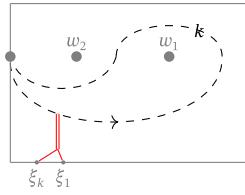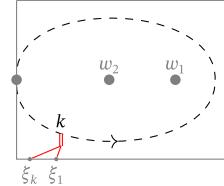
where in the second equality, we have again applied [29, Example 4.6] dividing the k indexed arc into two arcs passing by w_0, but again, we have kept the only term of the hypothetical sum that has nontrivial intersection with $B'(0,k)$. Last equality is straightforward from [29, Proposition 7.4]. Finally, to compute F_∞ one has to do the identification $-t = q^{-2}$, and:

$$F_\infty(\sigma_1^{-3}) = \sum_{k \in \mathbb{N}} \langle A'(0,k) \cap B'(0,k) \rangle q^{\alpha - 2k}$$

$$= \sum_{k \in \mathbb{N}} q^{\alpha - 2k} q^{-k(k-1)} q^{4\alpha k} (-1)^k \prod_{i=0}^{k-1} (1 - q^{-2\alpha} q^{2i})$$

$$= \sum_{k \in \mathbb{N}} q^{\alpha - 2k} q^{-k(k-1)} q^{4\alpha k} (-1)^k q^{-\alpha k} q^{\frac{k(k-1)}{2}} \prod_{i=0}^{k-1} (q^\alpha q^{-i} - q^{-\alpha} q^i)$$

$$= \sum_{k \in \mathbb{N}} q^{\alpha - 2k} q^{3\alpha k} q^{-\frac{k(k-1)}{2}} (-1)^k \prod_{i=0}^{k-1} (q^\alpha q^{-i} - q^{-\alpha} q^i).$$

One recovers precisely the formula for F_∞ for the trefoil knot given in [38, section 5], with zero framing. The identification is under the change $q \to q^{-1}$ as we chose σ_1^{-3} for the trefoil instead of σ_1^3.

Theorem 68 expresses F_∞ from intersections of middle dimension submanifolds of configuration spaces, sometimes called Lagrangians. Such interpretations for quantum invariants was initiated by Lawrence, and then Bigelow for the Jones polynomial [6, 26], then in a more quantum way colored Jones and ADO polynomials were formulated in the same spirit in, for example, [3, 30], respectively, [2, 18]. The present theorem should interpolate all these formulae (sometimes under a simple change of *dual bases*, corresponding to changing the manifolds to pair). Moreover, there is the uniqueness property of interpolation (Proposition 42), that we recall in the following remark.

Remark 72.

- The fact that F_∞ is the only two variables element interpolating both colored Jones polynomials or ADO polynomials (Proposition 42), could be interpreted as the only intersection pairing computed from manifolds in abelian covers of configuration spaces of disks interpolating both families. Moreover, it is a knot invariant.
- In [30], the second author has also showed that colored Jones polynomials compute some Lefschetz numbers. This is because colored Jones polynomials could be computed from a full trace on homological representations of braids, not only with a partial trace. With some study of the structure of homology modules, the trace formula then satisfies the Lefschetz formula. Unfortunately, authors have tried to interpret F_∞ as a full trace on homological braid action, and only found convergence problems seeming to be essential.

5 | UNIFIED INVARIANT FROM A QUANTUM DETERMINANT

This section is inspired by the paper [16] where Huynh and Lê compute the colored Jones polynomials from $U_q \mathfrak{sl}(2)$ Verma modules. By some assimilation of tensor products of Verma modules with some quantum plane, they succeed in giving a formula for colored Jones polynomials involving a *quantum determinant* for quantum matrices by use of the quantum MacMahon Master Theorem [10]. The matrix associated with a given braid from which one computes this quantum determinant are *deformed Burau matrices* because while abelianizing the entries one recovers usual Burau matrices. We follow [16, section 0.1] to state the theorem and we will give a direct proof involving their theorem and some interpolation argument.

5.1 | Deformed Burau matrix

We recall that $\mathcal{R}_0 := \mathbb{Z}\left[q^{\pm 1}\right]$. On the polynomial ring $\mathcal{R}_0\left[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}\right]$ we define operators.

Definition 73. Let $\hat{x}, \hat{y}, \hat{u}$ and τ_x, τ_y, τ_u be operator acting on $\mathcal{R}_0\left[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}\right]$ as follows:

$$\hat{x}f(x,y,u) = xf(x,y,u), \tau_x f(x,y,u) = f(qx,y,u)$$

the reader can guess definitions of the four remainder operators. Let $x_1, x_2 \in \{x, y, u\}$ then:

$$x_1 \tau_{x_2} = q^{\delta_{x_1,x_2}} \tau_{x_2} x_1,$$

namely operators $\hat{\ }$ and τ q-commute if they involve the same variable, commute otherwise. Operators $\hat{\ }$ commute one with each other, so do operators τ.

From these operators we define other ones:

$$a_+ := (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1} \quad b_+ := \hat{u}^2 \quad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1} \tag{10}$$

$$a_- := (\tau_y - \hat{x}^{-1})\tau_x^{-1}\tau_u \quad b_- := \hat{u}^2 \quad c_- = \hat{y}^{-1}\tau_x^{-1}\tau_u \tag{11}$$

and two matrices:

$$S_+ := \begin{pmatrix} a_+ & b_+ \\ c_+ & 0 \end{pmatrix} \quad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}. \tag{12}$$

We add more variables, for a fixed index $j \in \mathbb{N}$, we define operators $a_{j,\pm}, b_{j,\pm}, c_{j,\pm}$ acting on $\mathcal{R}_0 \left[x_j^{\pm 1}, y_j^{\pm 1}, u_j^{\pm 1} \right]$ as $a_\pm, b_\pm, c_\pm$ (resp.) do and trivially on any $\mathcal{R}_0 \left[x_i^{\pm 1}, y_i^{\pm 1}, u_i^{\pm 1} \right]$ if $i \neq j$. We extend as well definitions of $S_{j,+}$ (resp., $S_{j,-}$) as those of S_+ (resp., S_-) involving a_j, b_j, c_j.

Definition 74 (Deformed Burau matrix). Let $\beta := \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k} \in \mathcal{B}_n$ be a braid written as a product of Artin generators. We define its *deformed Burau matrix* as follows:

$$\rho(\beta) := \prod_{j=1}^k A_k,$$

where $A_j := Id_{i_j-1} \oplus S_{j,\epsilon_j} \oplus Id_{n-i_j-1}$. Entries of $\rho(\beta)$ are operators acting on $\mathcal{P}_k := \bigotimes_{j=1}^k \mathcal{R}_0 \left[x_j^{\pm 1}, y_j^{\pm 1}, u_j^{\pm 1} \right]$.

Definition 75 (Evaluation of operators). Let P be a polynomial in operators $a_\pm, b_\pm, c_\pm$ (with maybe indices) with coefficients in $\mathcal{R}_0$. The evaluation $\mathcal{E}(P) \in \mathcal{R} := \mathbb{Z} \left[q^{\pm 1}, s^{\pm 1} \right]$ is defined to be the application of P to the constant function $1 \in \mathcal{P}_k$ then substituting u_j by 1 and x_j, y_j by the formal variable s for all $j = 1, \ldots, k$.

The following lemma is part of [17, Lemma 1.4]:

Lemma 76. *Let $d, r, s \in \mathbb{N}$, we have:*

$$\mathcal{E}(b_+^s c_+^r a_+^d) = q^{-rd} s^r (1 - sq^{-r})_{q^{-1}}^d$$

$$\mathcal{E}(b_-^s c_-^r a_-^d) = s^{-r} (1 - s^{-1} q^r)_q^d,$$

where $(1 - x)_q^d = \prod_{i=0}^{d-1} (1 - xq^i)$.

We hence have a convergent series in $\widehat{\mathcal{R}}^{\widehat{I}}$ when evaluating $a_\pm$ series operators with $\mathcal{E}$.

Definition 77 (Evaluation of series operators). Let P be a polynomial in operators $b_\pm, c_\pm$ and a series in $a_\pm$ (with maybe indices) with coefficients in $\mathcal{R}_0$. The evaluation $\mathcal{E}(P) \in \widehat{\mathcal{R}}^{\widehat{I}}$ is defined to be the application of P to the constant function $1 \in \mathcal{P}_k$ then substituting u_j by 1 and x_j, y_j by the formal variable s for all $j = 1, \ldots, k$.

Remark 78. In [16], the framework is the Jones polynomials, meaning that there is an evaluation at q^n along side $\mathcal{E}$. As we try to do the same for the unified invariant, we now must be cautious because operators power series will appear in the formula. Fortunately, as we will see Lemma 81, they are $a_\pm$ power series operators, lying in $\widehat{\mathcal{R}}^{\widehat{I}}$ after evaluating with $\mathcal{E}$.

5.2 │ Quantum determinant

Definition 79 (Right quantum matrices). A 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be *right quantum* if:

$$ac = qca \quad bd = qdb \quad ad = da + qcb - q^{-1}bc.$$

An $m \times m$ matrix is right quantum if any of its 2×2-submatrix is.

Definition 80 (Quantum determinant). Let $A = (a_{ij})$ be a right quantum matrix. Its quantum determinant is defined as follows:

$$\det_q(A) := \sum_{\pi \in \mathfrak{S}_m} (-q)^{\text{inv}(\pi)} a_{\pi 1,1} a_{\pi 2,2} \cdots a_{\pi m,m},$$

where $\text{inv}(\pi)$ is the number of inversions.

$$\widetilde{\det_q}(Id - A) := 1 - C, \text{with} C := \sum_{J \subset \{1, \cdots, m\}} (-1)^{|J|-1} \det_q(A_J),$$

where A_J is the $J \times J$ submatrix of A (which is right-quantum).

Question 1. Does there exist a right quantum change of basis changing the deformed Burau matrices (Definition 74) into deformed *reduced* Burau matrices? The latter would clearly be related to the reduction of braid representation that exists on the quantum side (Definition 12), and with the fact that we remove one row and column in determinant formulae in the next section. It could be related with the fact that we use a partial trace to define F_∞.

5.3 │ Unified invariant from a quantum determinant

Lemma 81. *Let $\beta \in \mathcal{B}_m$ be a braid which standard closure is a knot. The operator $\dfrac{1}{\widetilde{\det_q}(Id - q\rho'(\beta))}$ is a series in $a_\pm$.*

Proof. We first study symmetric powers of the deformed Burau matrices of Artin generators A_j. Let $(x_i)_{1 \leqslant i \leqslant m}$ span an m dimensional quantum algebra, meaning that $x_j x_i = q x_i x_j$ for $i < j$. For a right quantum matrix $A = (a_{ij})$ let $X_i = \sum_{j=1}^m a_{ij} x_j$ and let $G(j_1, \dots, j_m)$ be the coefficient of $x_1^{j_1} \dots x_1^{j_m}$ in $\prod_{i=1}^m X_i^{j_i}$.

Recall that A_k is the deformed Burau matrix of the Artin generator σ_k. Let $\sum_{i=1}^m j_i = N$,

$$\begin{aligned}
\text{Sym}^N(A_i) \prod_{k=1}^m (x_k)^{j_k} &= \prod_{k=1}^m (A_i x_k)^{j_k} \\
&= x_1^{j_1} \cdots x_{i-1}^{j_{i-1}} (a_+ x_i + b_+ x_{i+1})^{j_i} \times (c_+ x_i)^{j_{i+1}} x_{i+2}^{j_{i+2}} \cdots x_m^{j_m} \\
&= \sum_{l=0}^{j_i} \binom{j_i}{l}_{q^{-1}} q^{(j_i-l)j_{i+1}} a_+^l b_+^{j_i-l} c_+^{j_{i+1}} x_1^{j_1} \dots x_i^{j_{i+1}+l} x_{i+1}^{j_i-l} \dots x_m^{j_m}
\end{aligned}$$

Hence, each Artin generator will induce a sum at the level of the action, the index l of the sum is called *state* and the sum is called *state sum*. Recall that $\beta = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$ is such that the induced permutation $\pi(\beta)$ is a derangement (while dealing with braids closing to knots). The element $G(0, j_2, \ldots, j_m)$ is a k-states sum that verifies

$$j_{\pi(\beta)(i)} = j_i + \text{linear combination of states with coeff } 1 \text{ or } -1.$$

As $\pi(\beta)$ is a derangement, j_i is a linear combination of states with coeff 1 or -1. Now if $\sum_{i=2}^{m} j_i = N$, there is always a state l that verifies $l \geqslant \frac{N}{mk}$. By use of the quantum MacMahon Master Theorem [10], we know that

$$\frac{1}{\widetilde{\det}_q(Id - q\rho'(\beta))} = \sum_{j_2, \ldots, j_m} G(0, j_2, \ldots, j_m).$$

Hence, $\frac{1}{\widetilde{\det}_q(Id - q\rho'(\beta))}$ is a series in $a_{\pm}$. $\qquad\square$

Theorem 82. *Let $\beta \in \mathcal{B}_m$ be a braid which standard closure is a knot. One remarks that $\rho(\beta)$ is right quantum. Let $\rho'(\beta)$ be obtained from $\rho(\beta)$ by removing first row and column. Then*

$$F_{\infty}(\hat{\beta}) = s^{(w(\beta)-m+1)/2} \mathcal{E}\left(\frac{1}{\widetilde{\det}_q(Id - q\rho'(\gamma))} \right),$$

where $w(\beta)$ is the writhe of the braid.

Proof. Let $\mathcal{E}_N$ be the evaluation corresponding to the substitution $s = q^{N-1}$. Then, for any $N \in \mathbb{N}$:

$$\mathcal{E}_N\left(s^{(w(\beta)-m+1)/2} \mathcal{E}\left(\frac{1}{\widetilde{\det}_q(Id - q\rho'(\gamma))} \right) \right) = J'_{\hat{\beta}}(N),$$

where $J'_{\hat{\beta}}(N)$ is the open Nth-colored Jones polynomial of $\hat{\beta}$, this is [16, Theorem 1]. We conclude using the unique element interpolating colored Jones polynomials property, Proposition 42. $\qquad\square$

Remark 83. The entire proof of Theorem 1 from [16] adapts to F_{∞} almost step by step and word by word, although here we have preferred to use a stronger and concise argument. The proof from [16] explains in details the relations between the operators $a_{\pm}, b_{\pm}, c_{\pm}$ and the braiding of Verma modules, so that it is important for the understanding of the formula.

Corollary 84 (ADO polynomials from quantum determinant). *Let $\beta \in \mathcal{B}_m$ be a braid which standard closure is a knot, and $\mathcal{E}_{\zeta_{2r}}$ be the composition of $\mathcal{E}$ with the substitution $q = \zeta_{2r}$. Then:*

$$\text{ADO}_r(\hat{\beta}) = s^{(w(\beta)-m+1)/2} F_r\big(\mathcal{E}\big(\det \big(Id - \rho'(\gamma)\big) \big) \big) \mathcal{E}_{\zeta_{2r}}\left(\frac{1}{\widetilde{\det}_q(Id - q\rho'(\gamma))} \right)$$

where $F_r : \mathbb{Z}[s^{\pm 1}] \to \mathbb{Z}[s^{\pm 1}]$, $s \mapsto s^r$.

Proof. Straightforward consequence of Theorems 82 and 39, and the fact that $\mathcal{E}\big(\det \big(Id - \rho'(\gamma)\big) \big)$ is the Alexander polynomial [16, Theorem 1(b)]. $\qquad\square$

Remark 85 (L^2-torsion and volume conjecture). Let $\beta \in \mathcal{B}_n$ be a braid: it acts naturally on the free group F_n on n generators $z_1, \ldots, z_n$. Let $\psi(\beta) := \left(\frac{\partial \beta(z_i)}{\partial z_j} \right)_{i,j}$ be the matrix where partial derivatives arise from Fox differential calculus. Latter matrix has coefficients in $\mathbb{Z}[F_n]$ but it is well-known that by sending them to $\mathbb{Z}[\mathbb{Z}]$ we obtain the Burau matrix of β which is a presentation of the Alexander torsion module of the closure of β (hence one could extract the Alexander polynomial of the closure of β out of its determinant). To relate the matrix $\psi(\beta)$ (with noncommutative entries) with some torsion module, one has to consider the L^2 *torsion* [27]. To compute such an invariant, the regular determinant has to be replaced by the *Fuglede–Kadison* determinant denoted $\det_{\pi_1}$ in [17]. Taking the Fuglede–Kadison determinant of such a matrix $\psi(\beta)$ then computes the L^2 *torsion of the complement of β's closure* (see, e.g., [27, Theorem 4.9]). An important theorem of Lück and Schick [28] relates the L^2-torsion of a manifold with its (eventual) hyperbolic volume. It is summed up in our context in [17, Proposition 0.2] as follows:

$$\exp\left(-\frac{\mathrm{Vol}(K)}{6\pi} \right) = \frac{1}{1 - \det_{\pi_1}(\psi'(\beta))}, \tag{13}$$

where in $\psi'(\beta)$ one line and one column has been removed from $\psi(\beta)$, K is (a knot) the closure of β and Vol its (possible) hyperbolic volume. Equation (13) resembles that from Theorem 82 for F_∞: one has removed one line and column before taking a generalized notion of determinant for matrices with noncommutative entries in the same way. Moreover, the matrices involved are equal the Burau matrix in both cases whenever entries are abelianized. F_∞ is conjectured to contain logarithmically the hyperbolic volume of the knot complement in a nice analytic way (Conjecture 1). The way the Burau matrix is *quantized* in both cases and the relation between Fuglede–Kadison determinant and quantum determinant are still to be clarified.

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