



OPEN Relativistic bound state solutions and quantum information theory in D dimensions under exponential-type plus Yukawa potentials

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The bound-state solution of the radial Klein-Gordon equation has been obtained under the interaction of an exponential-type and Yukawa potential functions. The Greene-Aldrich approximation has been used to overcome the centrifugal barrier and enable the analytical solutions of the energy and wave functions in closed form. The momentum space wave function in D dimensions has been constructed using the Fourier transform. The mean values have been conjectured for the position and momentum spaces using two equivalent equations. The effects of the potential parameters on the expectation values and quantum information measurement have been investigated. For the 1D case, the results obey the Heisenberg uncertainty principle, Fisher, Shannon, Onicescu, and the Rényi entropic inequalities. Other information complexities measures, such as Shannon Power, Fisher-Shannon, and Lopez-Ruiz-Mancini-Calbet, have been verified. For the ground state, the 1D momentum expectation value $\langle p^2 \rangle_{00}$ coincides with the 3D $\langle p^2 \rangle_{000}$ values, which is an indication of degeneracy. The total energy of a particle in both 1D and 3D space may be degenerate due to the inter-dimensional degeneracy of the quantum numbers. However, in this present result, the degeneracy in 1D and 3D occurred for fixed quantum states at different momentum intervals. Thus, in 1D, a particle may transit an entire space ($-\infty < p < \infty$) with a certain kinetic energy, which must be equal to its kinetic energy if it moves through the interval $0 < p < \infty$ in 3D space. This may have implications for kinetic energy degeneracy in higher dimensions.

Keywords Radial Klein-Gordon equation, Shannon Entropy, Renyi entropy, Fisher information, Kinetic energy

Quantum mechanical observables are useful for predicting the behaviour of particles confined in potential barriers and wells. The average kinetic and potential energies of particles and molecules may aid in the understanding of the spectroscopic and statistical properties of quantum mechanical systems. The position mean values and the corresponding momentum average values have been used to measure physical quantities in many branches of science¹. For example, the mean value $\langle r^2 \rangle$ has been used to determine the Pauli-Lagevin diamagnetic susceptibility². The expected value for the momentum moments (p^a) may provide useful information on kinetic energy measurement, Compton profile, and the relativistic kinetic energy corrections³. Also, the entropic densities derived from the probability of the wave functions play important roles in determining the Shannon⁴, Tsallis⁵, and Rényi⁶ information entropies, as well as energy exchanges in atomic physics⁷. The matrix elements of

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certain particles, together with their energy levels, have been applied to obtain the strength of oscillators⁸ and the optical properties of quantum dots⁹. The bound-state solutions to the wave equations are vital for describing the properties of quantum systems. A considerable number of potential functions have been used to describe physical systems with the relativistic and nonrelativistic wave equations^{10–20} where only a few permit exact analytical solutions. The solutions of the non-exactly solvable systems may be obtained using approximations^{21,22} as well as numerical methods²³. Gil-Barrera et al.²⁴ investigated one-dimensional quantum information entropies with hyperbolic potentials. They obtained the Shannon entropies for both position and momentum spaces and their results verify the global Shannon entropic sum inequality. Dong et al.²⁵ obtained the bound states of the one-dimensional Schrödinger equation with a squared tangent potential function. They applied the wave function to obtain the position and momentum space expectation values and the Shannon entropies. Their results satisfy the Heisenberg uncertainty and the Shannon sum inequality. Quantum information theoretic measures for one-dimensional systems under different potential wells have also been examined in the literature^{26,27}. In two dimensions, the cylindrical polar wave equation is used to obtain the bound state for any magnetic and principal quantum numbers. The resulting wave function has been applied to obtain quantum information theoretic measures with different potential functions^{28,29}. For the three-dimensional problem, the bound state solutions for any principal, orbital, and magnetic quantum numbers can be obtained by solving the Schrödinger equation in spherical polar coordinates, where the wave function may be separated into the spherical harmonics and the radial component. The radial part of the wave function has been used to investigate quantum information under a three-dimensional infinite Spherical well³⁰. The energy levels of the multi-dimensional eigenvalue problem can be obtained using any preferred analytical method but the analytical solutions of the momentum space wave functions in two and three dimensions may require a rigorous algebraic approach depending on the potential function. Our motivation is drawn from the fact that quantum information theoretic studies for two and three dimensions have been conspicuously neglected due to the difficulty in obtaining the Fourier transforms for the momentum space wave functions.

The objectives of this paper are: To obtain the analytical bound state solutions of the relativistic radial Klein-Gordon equation in closed form with an exponential-type potential plus Yukawa potential function. To obtain expectation values and verify quantum information theoretic complexity measures in multi-dimensional spaces using the nonrelativistic bound state solutions. The organization of the remaining parts of the paper are as follows. In sect. two, the energy levels and wave function of the time-independent radial Klein-Gordon equation (KGE) will be obtained in closed form using the Parametric Nikiforov-Uvarov approach (pNU)³¹. Section three contains the determination of the non-relativistic wave function which will be applied to obtain quantum expectation values and theoretic information complexity measures in D dimensions. The inequalities characterizing the theoretic quantities will be verified for both 1D-3D spaces. In sect. four, we present the discussion of results and the article is concluded in sect. five.

Bound state solution of exponential-type plus Yukawa potential function using pNU method

In this work, we proposed the an exponential potential plus Yukawa potential function given by

$$U(r) = V_0 \left(\frac{1 - qe^{-\alpha(r-R)}}{1 - e^{-\alpha r}} \right)^2 + V_1 \frac{e^{-\alpha r}}{r}. \quad (1)$$

The first term is a special case of the molecular potential³² for $q = 1$, $V_0 = D_e$, $V_1 = 0$. Also, the Yukawa potential³³ can be obtained when $V_0 = 0$. The notations V_i ($i = 0, 1$), α and R are the respective depths of the potential, range and equilibrium distance. The bound state solutions with the individual potentials have been obtained for the wave equations in existing literature^{32–34} using different methods. The bound state solutions have various applications in molecular and nuclear physics.

The time-independent radial KGE can be expressed as^{35,36}

$$\frac{d^2 \psi_{nl}}{dr^2} + \left(\frac{1}{\hbar^2 c^2} (E_{nl}^R - U(r))^2 - \frac{1}{\hbar^2 c^2} (Mc^2 + S(r))^2 - \frac{l(l+1)}{r^2} \right) \psi_{nl} = 0 \quad (2)$$

where E_{nl}^R , M , \hbar and c are the respective relativistic energy, rest mass, reduced Planck's constant and speed of light. The notations $U(r)$ and $S(r)$ are repulsive vector and attractive scalar potentials. In this work, we used the condition for equal vector and scalar potential such that (2) can be reduced further to

$$\frac{d^2 \psi_{nl}}{dr^2} + \left(\epsilon_{nl} - \Lambda U(r) - \frac{l(l+1)}{r^2} \right) \psi_{nl} = 0, \quad (3)$$

where

$$\epsilon_{nl} = \frac{(E_{nl}^R)^2 - M^2 c^4}{\hbar^2 c^2},$$

$$\Lambda = \frac{2(E_{nl}^R + Mc^2)}{\hbar^2 c^2}.$$

The radial Schrödinger equation can be obtained from (3) using the transformation $\frac{(E_{nl}^R - Mc^2)}{\hbar^2 c^2} \rightarrow 2E_{nl}^R$ and $\frac{(E_{nl}^R + Mc^2)}{\hbar^2 c^2} \rightarrow \mu / \hbar^2$. Inserting the potential in Eq. (1) into Eq. (3) gives

$$\frac{d^2 \psi_{nl}}{dr^2} + \left(\epsilon_{nl} - \Lambda \left(V_0 \left(\frac{1 - qe^{-\alpha(r-R)}}{1 - e^{-\alpha r}} \right)^2 + V_1 \frac{e^{-\alpha r}}{r} \right) - \frac{l(l+1)}{r^2} \right) \psi_{nl} = 0. \tag{4}$$

To solve Eq. (4) we adopted the pNU method used for obtaining the bound states of the wave equations with solvable potential functions in quantum mechanics. The details of the method can be found in the literature³¹. Equation (4) is not exactly solvable owing to the centrifugal barrier term. To this end an approximation such as the Greene-Aldrich approximation for small $\alpha \ll 1$ and radial distance²¹ has been used to solve exponential-type potential.

$$\frac{1}{r^2} \sim \left(\frac{\alpha}{1 - e^{-\alpha r}} \right)^2. \tag{5}$$

Using the change of variable, $\omega = e^{-\alpha r}$, Eq. (4) transforms to the standard hypergeometric-type differential equation:

$$\frac{d^2 \psi_{nl}}{d\omega^2} + \frac{1 - \omega}{\omega(1 - \omega)} \frac{d\psi_{nl}}{d\omega} + \left(\frac{-\gamma_1 \omega^2 + \gamma_2 \omega - \gamma_3}{\omega^2(1 - \omega)^2} \right) \psi_{nl} = 0, \tag{6}$$

where

$$\gamma_1 = \frac{1}{\alpha^2} (\Lambda V_0 q^2 e^{2\alpha R} - \Lambda \alpha V_1 - \epsilon_{nl}), \tag{7}$$

$$\gamma_2 = \frac{1}{\alpha^2} (2\Lambda V_0 q e^{\alpha R} - \Lambda \alpha V_1 - 2\epsilon_{nl}), \tag{8}$$

$$\gamma_3 = \frac{1}{\alpha^2} (\alpha^2 l(l+1) + \Lambda V_0 - \epsilon_{nl}). \tag{9}$$

Comparing the parameters associated with the standard hypergeometric equation³¹ with Eq. (6), we obtained the necessary conditions for the energy spectra and the wave function respectively

$$\gamma_3 = \frac{1}{4} \left(n + \frac{1}{2} + \sqrt{\gamma_3 + \gamma_1 - \gamma_2 + \frac{1}{4}} + \frac{\gamma_3 - \gamma_1}{n + \frac{1}{2} + \sqrt{\gamma_3 + \gamma_1 - \gamma_2 + \frac{1}{4}}} \right)^2, \tag{10}$$

$$\psi_{nl}(\omega) = N_{nl} \omega^{\sqrt{\gamma_3}} (1 - \omega)^{\frac{1}{2} + \sqrt{\gamma_3 + \gamma_1 - \gamma_2 + \frac{1}{4}}} P_n^{(2\sqrt{\gamma_3}, 2\sqrt{\gamma_3 + \gamma_1 - \gamma_2 + \frac{1}{4}})}(1 - 2\omega), \tag{11}$$

where,

$P_n^{(a, b)}(1 - 2\omega)$ is the Jacobi polynomial of order n and N_{nl} is the normalization constant.

Inserting the quantities γ_i ($i = 1, 2, 3$) in Eqs. (7-9) into (10) and (11), the bound state energy of the radial KGE can be written in closed-form

$$\epsilon_{nl} = \Delta_0 - \frac{\alpha^2}{4} \left(n + \Delta_1 + \frac{\Delta_2}{n + \Delta_1} \right)^2, \tag{12}$$

where

$$\begin{aligned} \Delta_0 &= \Lambda V_0 + \alpha^2 l(l+1), \\ \Delta_1 &= \frac{1}{2} + \sqrt{\left(l + \frac{1}{2} \right)^2 + \frac{\Lambda V_0}{\alpha^2} (1 - qe^{\alpha R})^2}, \\ \Delta_2 &= \frac{\Lambda V_0}{\alpha^2} (1 - q^2 e^{\alpha R}) + \frac{\Lambda V_1}{\alpha} + l(l+1). \end{aligned}$$

Using the transformation $\frac{(E_{nl}^R - Mc^2)}{\hbar^2 c^2} \rightarrow 2E_{nl}^R$ and $\frac{(E_{nl}^R + Mc^2)}{\hbar^2 c^2} \rightarrow \mu / \hbar^2$, the non-relativistic energy is obtained as

$$E_{nl}^{NR} = K_0 - \frac{\alpha^2 \hbar^2}{8\mu} \left(n + K_1 + \frac{K_2}{n + K_1} \right)^2, \tag{13}$$

where

$$K_0 = V_0 + \frac{\alpha^2 \hbar^2 l(l+1)}{2\mu},$$

$$K_1 = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{2\mu V_0}{\alpha^2 \hbar^2} (1 - qe^{\alpha R})^2},$$

$$K_2 = \frac{2\mu V_0}{\alpha^2 \hbar^2} (1 - q^2 e^{\alpha R}) + \frac{2\mu V_1}{\alpha \hbar^2} + l(l+1).$$

The nonrelativistic wave function can be written as

$$\psi_{nl}^{NR}(r) = N_{nl} (e^{-\alpha r})^A (1 - e^{-\alpha r})^{\frac{1}{2}+B} P_n^{(2A, 2B)}(1 - 2e^{-\alpha r}), \tag{14}$$

where

$$A = \sqrt{\left(l(l+1) + \frac{2\mu V_0}{\alpha^2 \hbar^2} - \frac{2\mu E_{nl}^{NR}}{\alpha^2 \hbar^2}\right)},$$

$$B = K_1 - \frac{1}{2}.$$

Normalization of the wave function and determination of expectation values in position and momentum spaces

One-dimensional case

The normalization factor for the position space wave function can be obtained starting from the probability of finding a quantum particle in the region $0 < r < \infty$ of space provided the condition is satisfied

$$\int_0^\infty |\psi_{nl}^{NR}(r)|^2 dr = 1. \tag{15}$$

Using the wave function in (14), Eq. (15) turned out as

$$N_{nl}^2 \int_0^\infty (e^{-\alpha r})^{2A} (1 - e^{-\alpha r})^{1+2B} \left(P_n^{(2A, 2B)}(1 - 2e^{-\alpha r})\right)^2 dr = 1. \tag{16}$$

If we let $\omega = e^{-\alpha r}$ then

$$\frac{N_{nl}^2}{\alpha} \int_0^1 \omega^{2A-1} (1 - \omega)^{1+2B} \left(P_n^{(2A, 2B)}(1 - 2\omega)\right)^2 d\omega = 1. \tag{17}$$

The explicit expression for the Jacobi polynomial can be written in two different ways³⁷

$$P_n^{(2A, 2B)}(1 - 2\omega) = \frac{\Gamma(n + 2A + 1)}{\Gamma(n + 2A + 2B + 1)} \sum_{k=0}^n \frac{(\omega)^k (-1)^k \Gamma(n + 2A + 2B + 1 + k)}{k! (n - k)! \Gamma(k + 2A + 1)}, \tag{18}$$

$$P_n^{(2A, 2B)}(1 - 2\omega) = (-1)^n \Gamma(n + 2A + 1) \Gamma(n + 2B + 1) \times \sum_{j=0}^n \frac{(-1)^j (\omega)^{n-j} (1 - \omega)^j}{j! (n - j)! \Gamma(n + 2A - j + 1) \Gamma(2B + j + 1)}. \tag{19}$$

Substituting Eqs. (18) and (19) into (17) gives

$$\begin{aligned} & (-1)^n \frac{\Gamma(n + 2B + 1) (\Gamma(n + 2A + 1))^2}{\Gamma(n + 2A + 2B + 1)} \\ & \times \sum_{j=0}^n \sum_{k=0}^n \frac{(-1)^{j+k} \Gamma(n + 2A + 2B + k + 1)}{j! (n - j)! k! (n - k)! \Gamma(n + 2A - j + 1) \Gamma(2B + j + 1)} I(j, k) = 1 \end{aligned} \tag{20}$$

where

$$I(j, k) = \frac{N_{nl}^2}{\alpha} \int_0^1 \omega^{2A+n+k-j-1} (1 - \omega)^{j+1+2B} d\omega. \tag{21}$$

Using the integral notation for the hypergeometric function^{37,38}

$$\int_0^1 \omega^{A-1}(1-\omega)^{C-A-1}(1-\omega)^{-B} d\omega = \frac{{}_2F_1(A, B, C : 1) \Gamma(A) \Gamma(C-A)}{\Gamma(C)}. \tag{22}$$

Equation (22) can further be reduced to

$$\int_0^1 \omega^{A-1}(1-\omega)^{-B} d\omega = \frac{{}_2F_1(A, B, A+1 : 1)}{A}. \tag{23}$$

The property of the hypergeometric function is given as

$${}_2F_1(A, B, C : 1) = \frac{\Gamma(C) \Gamma(C-A-B)}{\Gamma(C-A) \Gamma(C-B)}, \tag{24}$$

With Eq. (24), the solution of Eq. (21) is obtained as

$$I(j, k) = \frac{N_{nl}^2 (j+2B+1) \Gamma(j+2B+1) \Gamma(n+2A+k-j)}{\alpha (n+2A+2B+k+1) \Gamma(n+2A+2B+k+1)}. \tag{25}$$

Inserting (25) into (20) gives the normalization constant as

$$N_{nl} = \frac{1}{\sqrt{G_{nl}}},$$

where

$$G_{nl} = (-1)^n \frac{\Gamma(n+2B+1) \Gamma(n+2A+1)^2}{\alpha \Gamma(n+2A+2B+1)} \times \sum_{j=0}^n \sum_{k=0}^n \frac{(-1)^{j+k} \Gamma(n+2A+k-j) (2B+j+1)}{j! (n-j)! k! (n-k)! \Gamma(n+2A-j+1) \Gamma(2A+k+1) (n+2A+2B+k+1)}. \tag{26}$$

The momentum space wave function can be obtained from the Fourier transform as follows:

$$\psi_{nl}^{NR}(p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_{nl}^{NR}(r) e^{-ipr} dr. \tag{27}$$

The wave function $\psi_{nl}^{NR}(p)$ is complex and satisfies the ortho-normality condition

$$\int_0^\infty (\psi_n^{NR}(p))^* \psi_{n'}^{NR}(p) dr = \delta_{nn'}, \tag{28}$$

where p and $\delta_{nn'}$ are the respective momentum of a particle and the Kronecker delta.

Using the wave function in (14) with the explicit expression of the Jacobi function in Eq. (18), the momentum space wave function is given as

$$\psi_{nl}^{NR}(p) = \frac{N_{nl}}{\sqrt{2\pi}} \frac{\Gamma(n+2A+1)}{\Gamma(n+2A+2B+1)} \sum_{k=0}^n \frac{(-1)^k \Gamma(n+2A+2B+1+k)}{k! (n-k)! \Gamma(k+2A+1)} \times \int_0^\infty (e^{-\alpha r})^{A+k} (1-e^{-\alpha r})^{\frac{1}{2}+B} e^{-ipr} dr. \tag{29}$$

Solving Eq. (29) analytically is a difficult task but with the help of the Mathematica program, the solution becomes

$$\psi_{nl}^{NR}(p) = \frac{N_{nl} \Gamma(\frac{3}{2}+B) \Gamma(n+2A+1) \Gamma(A+i\frac{p}{\alpha}) H(p)}{\sqrt{2\pi} \alpha \Gamma(1+2A) \Gamma(n+1) \Gamma(n+A+B+\frac{3}{2}+i\frac{p}{\alpha})}, \tag{30}$$

where $H(p)$ is a generalized hypergeometric function given as

$$H(p) = {}_pF_Q \left(\left(-n, n+2A+2B+1, A+i\frac{p}{\alpha} \right), \left(1+2A, A+B+\frac{3}{2}+i\frac{p}{\alpha} \right), 1 \right). \tag{31}$$

Two and three-dimensional cases

The two-dimensional Schrödinger equation can be written as an eigenvalue Eq.

$$H_{nm}\psi_{nm}^{NR}(r, \phi_r) = E_{nm}^{NR}\psi_{nm}^{NR}(r, \phi_r). \quad (32)$$

The 2D Hamiltonian is given in cylindrical polar coordinates as

$$H_{nm} = -\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi_r^2} \right) + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}. \quad (33)$$

The eigenfunction $\psi_{nm}^{NR}(r, \phi_r) = \frac{R_{nm}(r)}{\sqrt{2\pi r}} e^{im\phi_r}$, in the boundary $0 < r < \infty$, $0 < \phi_r < 2\pi$ is the solution of (32) where $\frac{R_{nm}(r)}{\sqrt{r}}$ and $\frac{e^{im\phi_r}}{\sqrt{2\pi}}$ correspond to the respective solutions of the radial and azimuthal components of the 2D Schrödinger equation given by

$$\frac{d^2 R_{nm}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left(E_{nm}^{NR} - V(r) - \frac{\hbar^2}{2\mu} \left(m^2 - \frac{1}{4} \right) \right) R_{nm}(r) = 0, \quad (34)$$

$$\frac{d^2 \Phi(\phi_r)}{d\phi_r^2} + m^2 \Phi(\phi_r) = 0, \quad (35)$$

where E_{nm}^{NR} , $R_{nm}(r)$ and m are the 2D energy, wave function and the magnetic quantum number. The 2D bound states of both the Klein-Gordon and Schrödinger equations are obtained by mapping the 1D problem using the relation $(l + 1/2) \rightarrow m$. The 2D wave function normalization condition can be written as

$$\int_0^\infty \left| \frac{R_{nm}(r)}{\sqrt{r}} \right|^2 r dr \int_0^{2\pi} \frac{e^{im\phi_r}}{\sqrt{2\pi}} \left(\frac{e^{-im\phi_r}}{\sqrt{2\pi}} \right) d\phi_r = 1, \quad (36)$$

where

$$R_{nm}(r) = N_{nm} (e^{-\alpha r})^A (1 - e^{-\alpha r})^{\frac{1}{2}+B} P_n^{(2A, 2B)} (1 - 2e^{-\alpha r}). \quad (37)$$

The momentum space wave function in 2D space can be written via the Fourier transform

$$R_{nm}(p) = \frac{1}{2\pi} \int_0^\infty \frac{R_{nm}(r)}{\sqrt{r}} r dr \int_0^{2\pi} \frac{e^{im\phi_r}}{\sqrt{2\pi}} e^{-i\mathbf{p}\cdot\mathbf{r}} d\phi_r. \quad (38)$$

The solution of the azimuthal integral is straightforward using the integral properties of the Bessel function³⁹

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{im\phi_r}}{\sqrt{2\pi}} e^{-i\mathbf{p}\cdot\mathbf{r}} d\phi_r = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} e^{i(m\phi_r - \cos(\phi_r - \phi_p)pr)} d\phi_r = \frac{(-i)^m}{\sqrt{2\pi}} J_m(pr) e^{im\phi_p}, \quad (39)$$

where.

$J_m(pr)$ is the m -order Bessel function of the first kind.

Substituting Eq. (39) into (38) yields

$$R_{nm}(p) = \frac{(-i)^{|m|} e^{im\phi_p}}{\sqrt{2\pi}} \int_0^\infty \sqrt{r} R_{nm}(r) J_m(pr) dr. \quad (40)$$

Since the wave function contains the Jacobi function, the analytical solution of Eq. (40) is a difficult task to achieve. To this end, we utilized an asymptotic approximation of the Bessel function⁴⁰

$$J_{|m|}(pr) \sim \sqrt{\frac{2}{\pi pr}} \cos \left(pr - \frac{|m|\pi}{2} - \frac{\pi}{4} \right). \quad (41)$$

Substituting Eq. (41) into (40) with $R_{nm}(r)$ gives

$$R_{nm}(p) = \frac{(-i)^{|m|} N_{nm} e^{im\phi_p}}{\sqrt{2\pi}} \int_0^\infty (e^{-\alpha r})^A (1 - e^{-\alpha r})^{\frac{1}{2}+B} P_n^{(2A, 2B)} (1 - 2e^{-\alpha r}) \sqrt{\frac{2}{\pi p}} \cos \left(pr - \frac{|m|\pi}{2} - \frac{\pi}{4} \right) dr. \quad (42)$$

Using the expression in Eq. (18) and evaluating the integral with the help of the Mathematica programme gives the 2D momentum space wave function as

$$R_{nm}(p) = \frac{(-1)^m (i)^{m+1/2} e^{im(\phi_p - \pi/2)} N_{nm} \Gamma(\frac{3}{2} + B) \Gamma(n + 2A + 1)}{2\pi \sqrt{p\alpha} \Gamma(1 + 2A) \Gamma(n + 1)} (e^{im\pi} H_{nm}(p) - H_{nm}^*(p)), \tag{43}$$

where

$$H_{nm}(p) = \frac{\Gamma(A + i\frac{p}{\alpha})}{A + B + \frac{3}{2} + i\frac{p}{\alpha}} \times {}_pF_Q \left(\left(-n, n + 2A + 2B + 1, A + i\frac{p}{\alpha} \right), \left(1 + 2A, A + B + \frac{3}{2} + i\frac{p}{\alpha} \right), 1 \right). \tag{44}$$

For $m = 0$, the momentum space wave function is given by

$$R_{n0}(p) = (i)^{1/2} \frac{N_{n0} \Gamma(\frac{3}{2} + B) \Gamma(n + 2A + 1)}{2\pi \sqrt{p\alpha} \Gamma(1 + 2A) \Gamma(n + 1)} (H_{n0}(p) - H_{n0}^*(p)). \tag{45}$$

For the 3D case, the solution of the eigenfunction of the Schrödinger equation in spherical polar coordinates is given by

$$\Psi_{nlm}(r, \theta_r, \phi_r) = \frac{R_{nl}(r)}{r} Y_{lm}(\theta_r, \phi_r), \tag{46}$$

where $Y_{lm}(\theta, \varphi)$ are the Spherical Harmonics and the solution of the angular part of the Schrödinger equation whereas $R_{nl}(r)$ remains the usual one-dimensional position wave function. The $Y_{lm}(\theta, \varphi)$ is expressed as

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2L + 1)(l - m)!}{4\pi(l + m)!}} P_l^m(\cos(\theta)) e^{im\phi} \tag{47}$$

where the function $P_l^m(\cos(\theta))$ is the associated Legendre polynomial.

The wave function in momentum space is given by the Fourier transform⁴¹

$$\Psi_{nlm}(p, \theta_p, \phi_p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Psi_{nlm}(r, \theta_r, \phi_r) e^{-i\mathbf{p}\cdot\mathbf{r}} d\mathbf{r}. \tag{48}$$

The notation $d\mathbf{r} = (r^2 dr) \sin(\theta) d\theta d\varphi$ is the volume element in position space. The plane-wave expansion for $e^{-i\mathbf{p}\cdot\mathbf{r}}$ is given as⁴²

$$e^{-i\mathbf{p}\cdot\mathbf{r}} = (2\pi)^{3/2} \sum_{l=0}^{\infty} \sum_{m=-l}^l i^{-l} \frac{J_{l+1/2}(pr)}{\sqrt{pr}} Y_{lm}(\theta_p, \phi_p) Y_{lm}^*(\theta_r, \phi_r). \tag{49}$$

Due to axial symmetry, only the $m = 0$ terms would remain such that the plane-wave expansion reduces to

$$e^{-i\mathbf{p}\cdot\mathbf{r}} = (2\pi)^{3/2} Y_{l0}(\theta_p, \phi_p) \sum_{l=0}^{\infty} i^{-l} \frac{J_{l+1/2}(pr)}{\sqrt{pr}} Y_{l0}^*(\theta_r, \phi_r). \tag{50}$$

Inserting Eqs. (46) and (50) into (48) gives

$$\begin{aligned} \Psi_{nlm}(p, \theta_p, \phi_p) &= i^{-l} Y_{lm}(\theta_p, \phi_p) \int_0^\pi \int_0^{2\pi} Y_{l0}(\theta_r, \phi_r) Y_{l0}^*(\theta_r, \phi_r) \sin(\theta) d\theta d\phi \\ &\times \int_0^\infty \frac{R_{nl}(r)}{r} \frac{J_{l+1/2}(pr)}{\sqrt{pr}} r^2 dr \end{aligned} \tag{51}$$

By utilizing Eq. (41) for the Bessel function, and also the orthonormality condition for the Spherical Harmonics, Eq. (51) simplifies to

$$\Psi_{nlm}(p, \theta_p, \phi_p) = \frac{Y_{lm}(\theta_p, \phi_p)}{p} F_{nl}(p), \tag{52}$$

where

$$F_{nl}(p) = i^{-l} \sqrt{\frac{2}{\pi}} \int_0^\infty R_{nl}(r) \cos\left(pr - \frac{(l + \frac{1}{2})\pi}{2} - \frac{\pi}{4}\right) dr. \tag{53}$$

Using the position wave function with the expression for the Jacobi function in (18), the integral in Eq. (53) can be evaluated with the Mathematica programme. The momentum space wave function is then obtained as

$$\Psi_{nlm}(p, \theta_p, \phi_p) = \frac{e^{-i\pi l/2} \Gamma(\frac{3}{2} + B) \Gamma(n + 2A + 1) N_{nl} i^{(1-l)}}{\alpha \sqrt{2\pi} \Gamma(1 + 2A) \Gamma(n + 1)} (e^{i\pi} H_{nl}(p) - H_{nl}^*(p)) Y_{lm}(\theta_p, \phi_p), \tag{54}$$

where

$$H_{nl}(p) = \frac{\Gamma(A + i\frac{l}{\alpha})}{A + B + \frac{3}{2} + i\frac{l}{\alpha}} \times {}_pF_Q\left(\left(-n, n + 2A + 2B + 1, A + i\frac{l}{\alpha}\right), \left(1 + 2A, A + B + \frac{3}{2} + i\frac{l}{\alpha}\right), 1\right). \tag{55}$$

Expectation values and theoretic information measures

The respective 1D position and momentum space expectation values can be given as

$$\langle r^t \rangle = \begin{cases} (-i)^t \int_{-\infty}^{\infty} (R_{n0}(p)) \frac{d^t (R_{n0}(p))^*}{dp^t} dp \\ \int_0^{\infty} (R_{n0}(r))^* r^t R_{n0}(r) dr \end{cases} \tag{56}$$

$$\langle p^t \rangle = \begin{cases} (-i)^t \int_0^{\infty} R_{n0}(r) \frac{d^t (R_{n0}(r))^*}{dr^t} dr \\ \int_{-\infty}^{\infty} (R_{n0}(p))^* p^t R_{n0}(p) dp \end{cases} \tag{57}$$

It is worth stating that the first equations in (56) and (57) containing the derivative, are not suitable to finding the expectation values with $t < 0$ and fractional values of t . In 2D, the expectation values in position and momentum spaces are given by the relations

$$\langle r^t \rangle_{nm} = 2\pi \int_0^{\infty} \left| \frac{R_{nm}(r)}{\sqrt{2\pi}r} \right|^2 r^{t+1} dr, \tag{58}$$

$$\langle p^t \rangle_{nm} = 2\pi \int_0^{\infty} (R_{nm}(p))^* R_{nm}(p) p^{t+1} dp \tag{59a}$$

$$\langle p^2 \rangle_{nm} = -2\pi \int_0^{\infty} R_{nm}(r) \frac{d^2 (R_{nm}(r))^*}{dr^2} r dr. \tag{59b}$$

In 3D space, the mean values in position and momentum spaces are given by

$$\langle r^t \rangle_{nlm} = \int_0^{\infty} r^t \left| \frac{R_{nl}(r)}{r} \right|^2 r^2 dr \int Y_{lm}(\theta_r, \phi_r) Y_{lm}^*(\theta_r, \phi_r) d\Omega, \tag{60}$$

$$\langle p^t \rangle_{nlm} = \int_0^{\infty} p^t (R_{nl}(p))^* R_{nl}(p) p^2 dp \int Y_{lm}(\theta_p, \phi_p) Y_{lm}^*(\theta_p, \phi_p) d\Omega, \tag{61}$$

$$\langle p^2 \rangle_{nlm} = - \int_0^{\infty} R_{nl}(r) \frac{d^2 (R_{nl}(r))^*}{dr^2} r^2 dr \int Y_{lm}(\theta_r, \phi_r) Y_{lm}^*(\theta_r, \phi_r) d\Omega + \langle r^{-2} \rangle_{nlm}, \tag{62a}$$

$$\langle r^2 \rangle_{nlm} = - \int_0^{\infty} R_{nl}(p) \frac{d^2 (R_{nl}(p))^*}{dr^2} p^2 dp \int Y_{lm}(\theta_p, \phi_p) Y_{lm}^*(\theta_p, \phi_p) d\Omega + \langle p^{-2} \rangle_{nlm}, \tag{62b}$$

where $d\Omega = \sin(\theta) d\theta d\phi$ is the differential solid angle.

The knowledge of the wave functions is vital in studying the quantum information theoretic measures using the entropic densities. The Rényi entropies in D -dimensional spaces for the position and momentum coordinates are given by the equations⁴³

$$R_g(\rho) = \frac{1}{1-g} \ln \left(\int_{\mathbb{R}^D} (\rho(\mathbf{r}))^g d\mathbf{r} \right), \tag{63}$$

$$R_g(\gamma) = \frac{1}{1-g} \ln \left(\int_{\mathbb{R}^D} (\gamma(\mathbf{p}))^g d\mathbf{p} \right), \quad (64)$$

where $g \in [0, \infty]$ is a real number. The Rényi entropies increase with a decrease in g but decrease as g increases. An increment in the Rényi entropy is an indication of delocalization or the reduction of information for a given system^{39,44}. When the Rényi entropy decreases then we have more information on the system or localization is prominent. Also, the sum of the Rényi entropies for a system should satisfy global inequality⁴⁴

$$R(T) = R_f(\gamma) + R_g(\rho) \geq -\frac{D}{2} \left(\frac{1}{1-f} \ln \frac{f}{\pi} + \frac{1}{1-g} \ln \frac{g}{\pi} \right), \quad (65)$$

where D is the dimension number and the parameters f and g satisfy the linear constraint⁴⁴.

$$\frac{1}{f} + \frac{1}{g} = 2, \quad 0.5 \leq f < 1. \quad (66)$$

When the index parameter $g = 1$, the Rényi entropies reduce to the Shannon entropies⁴⁴

$$S(\rho) = - \int_{\mathbb{R}^D} \rho(\mathbf{r}) \ln \rho(\mathbf{r}) d\mathbf{r} \quad (67)$$

$$S(\gamma) = - \int_{\mathbb{R}^D} \gamma(\mathbf{p}) \ln \gamma(\mathbf{p}) d\mathbf{p}. \quad (68)$$

The Shannon entropic sum obeys the Beckner Bialynicky Birula Mycielski (BBM)⁴⁵ inequality

$$S(T) = S(\rho) + S(\gamma) \geq D(1 + \ln \pi). \quad (69)$$

The Fisher information is a local measure of the spread of the probability distribution which can be represented by the gradient of the probability densities in position and momentum spaces⁴⁶.

$$I(\rho) = \int_{\mathbb{R}^D} \frac{|\nabla \rho(\mathbf{r})|^2}{\rho(\mathbf{r})} d\mathbf{r}, \quad (70)$$

$$I(\gamma) = \int_{\mathbb{R}^D} \frac{|\nabla \gamma(\mathbf{p})|^2}{\gamma(\mathbf{p})} d\mathbf{p}, \quad (71)$$

where $\rho(\mathbf{r}) = |\Psi(\mathbf{r})|^2$ and $\gamma(\mathbf{p}) = |\Psi(\mathbf{p})|^2$ are the respective probability densities in position and momentum spaces. The notation ∇ is the gradient operator of a particle. In spherical coordinates, it is given as

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \quad (72)$$

The higher the fisher information, the more localized the probability density which implies also less uncertainty. The Fisher information obeys the inequality $I(\rho) I(\gamma) \geq 4D^2$. The Onicescu information is the square of the probability densities which measure the distance of the system from uniform equilibrium⁴⁷. The Onicescu information in position and momentum spaces entropies can be expressed as⁴⁸

$$O(\rho) = \int_{\mathbb{R}^D} |\rho(\mathbf{r})|^2 d\mathbf{r}, \quad (73)$$

$$O(\gamma) = \int_{\mathbb{R}^D} |\gamma(\mathbf{p})|^2 d\mathbf{p}. \quad (74)$$

The total Onicescu information obeys the inequality⁴⁹ $O(\rho) O(\gamma) \leq \frac{1}{(2\pi)^D}$. The smaller the value of $O(\rho) O(\gamma)$, the more information we have about the system^{47,50} and vice versa.

Using the information-theoretic quantities, other complex measures can be obtained. The Shannon entropic power in position and momentum spaces are given as⁵¹

$$J(\rho) = \frac{1}{2\pi e} e^{\frac{2S(\rho)}{D}}, \quad (75)$$

$$J(\gamma) = \frac{1}{2\pi e} e^{\frac{2S(\gamma)}{D}}. \quad (76)$$

The Shannon entropic power for a quantum system should satisfy the uncertainty relation

$$J(\rho)J(\gamma) \geq \frac{1}{4}. \quad (77)$$

The Fisher-Shannon complexity measure obeys the uncertainty relations.

$I(\rho)J(\rho) \geq D$ and $I(\gamma)J(\gamma) \geq D$. Another information complexity measure is the Lopez-Ruiz-Mancini-Calbet (C_{LMC}) defined as⁵²

$$C_{LMC}(\rho) = O(\rho) e^{S(\rho)}, \quad (78)$$

$$C_{LMC}(\gamma) = O(\gamma) e^{S(\gamma)}. \quad (79)$$

The LMC complexity obeys the bound $C_{LMC} \geq 1$.

Recently, the LMC-Rényi complexity of the D -dimensional hydrogenic system has been studied and applied to quasi-spherical and highly excited Rydberg states⁵³. The LMC-Rényi complexity is given as^{54,55}.

$$\bar{C}_{LMC}(\rho) = e^{\frac{1}{D}(R_g(\rho) - R_f(\rho))}, \quad 0 < g < f < \infty. \quad (80)$$

$$\bar{C}_{LMC}(\gamma) = e^{\frac{1}{D}(R_g(\gamma) - R_f(\gamma))}, \quad 0 < g < f < \infty. \quad (81)$$

The LMC-Rényi complexity is dimensionless⁵⁵ and bounded from below as $\bar{C}_{LMC}(\gamma) \geq 1$ provided $g < f$.

Discussion of results

The analytical bound state solutions of the radial Klein-Gordon equation have been obtained under an exponential potential plus Yukawa potential via the parametric Nikiforov-Uvarov approach. Using a Greene-Aldrich approximation for the centrifugal barrier, the energy spectra and the wave function were obtained in closed form. The Schrödinger equation is obtained as a special case of the Klein-Gordon equation using an appropriate transformation. Using the non-relativistic wave functions in position and momentum representations, the ground state expectation values and information complexity measures have been investigated with arbitrary parameters ($R = 1.2$, $V_0 = 0.05$, $V_1 = 0.44$, $q = 0.1$) of the potential and natural units $\mu = 0.5$, $\hbar = 1$. The 1D uncertainty in position space (Δr) is observed to decrease as α increases (Fig. 1a) However, in Fig. 1b, the momentum space uncertainty increases with α . The uncertainty product ($\Delta p \times \Delta r$) increases monotonically, while satisfying the Heisenberg uncertainty inequality. In Fig. 2, the r -space Fisher information increases, while the p -space Fisher information decreases with increasing α . This implies better localization of the radial probability density and a more diffuse momentum probability density function with increasing α . This further implies a decrease in the position uncertainty and an increase in momentum uncertainty as α increases. The Fisher product as shown in Fig. 2c shows the same trend as the uncertainty product in Fig. 1c and obeys the inequality $I(\rho) \times I(\gamma) \geq 4$. In Fig. 3, the position Shannon entropy is observed to increase as α increases, while the momentum Shannon entropy decreases (Fig. 3a, b). This is in line with the Fisher information result since it implies a narrowing of the radial probability density with increasing α and a broadening of the momentum probability density. The Shannon entropy sum shows an identical trend to the Fisher product where the result verifies the BBM inequality for the 1D system (see Fig. 3c). Similarly, the Rényi entropy in Fig. 4 shows a similar trend to the Shannon entropy where the Rényi sum is found to be bounded above the lowest value thus satisfying the relations in Eq. (65). In Fig. 5, the radial Onicescu information is observed to decrease monotonically, while the momentum Onicescu information increases. This is, as expected an opposite trend to the Shannon entropy since the Shannon entropy and Onicescu information describe a mutually opposite form of probability density⁴⁷. It is observed that the total Onicescu information decreases sharply as α increases (Fig. 5c), implying an increase in the information content of the system. In Fig. 6, the Shannon power shows a similar trend to the Fisher information. In Fig. 7 (a), the product of the Fisher information and Shannon power for both r - and p -space increase slowly above the lowest bound as α increases. Figure 7b, shows LMC complexity in r -space and p -space increasing with an increase in α . In Fig. 7 (c), the position and the momentum space LMC-Rényi complexities increase with an increase in α . In all the cases, the lower bound of the complexities is satisfied as expected.

The 2D eigenenergy solutions can be obtained by mapping the quantum numbers ($m = l + 1/2$). However, to obtain the momentum space wave function, the Fourier transform for the cylindrical polar coordinate has been applied in Eq. (38). Evaluating the Fourier transform is a difficult task owing to the nature of the position space wave function. In this case, we applied an asymptotic approximation of the Bessel function in Eq. (41) inside the Fourier integral to obtain an approximate solution of the 2D momentum wave function. In Fig. 8 (a-d) we graphed the exact and the asymptotic Bessel function of the first kind for the magnetic quantum numbers ($m = 0 - 3$). It can be seen that the ground state $m = 0$, is a fair approximation.

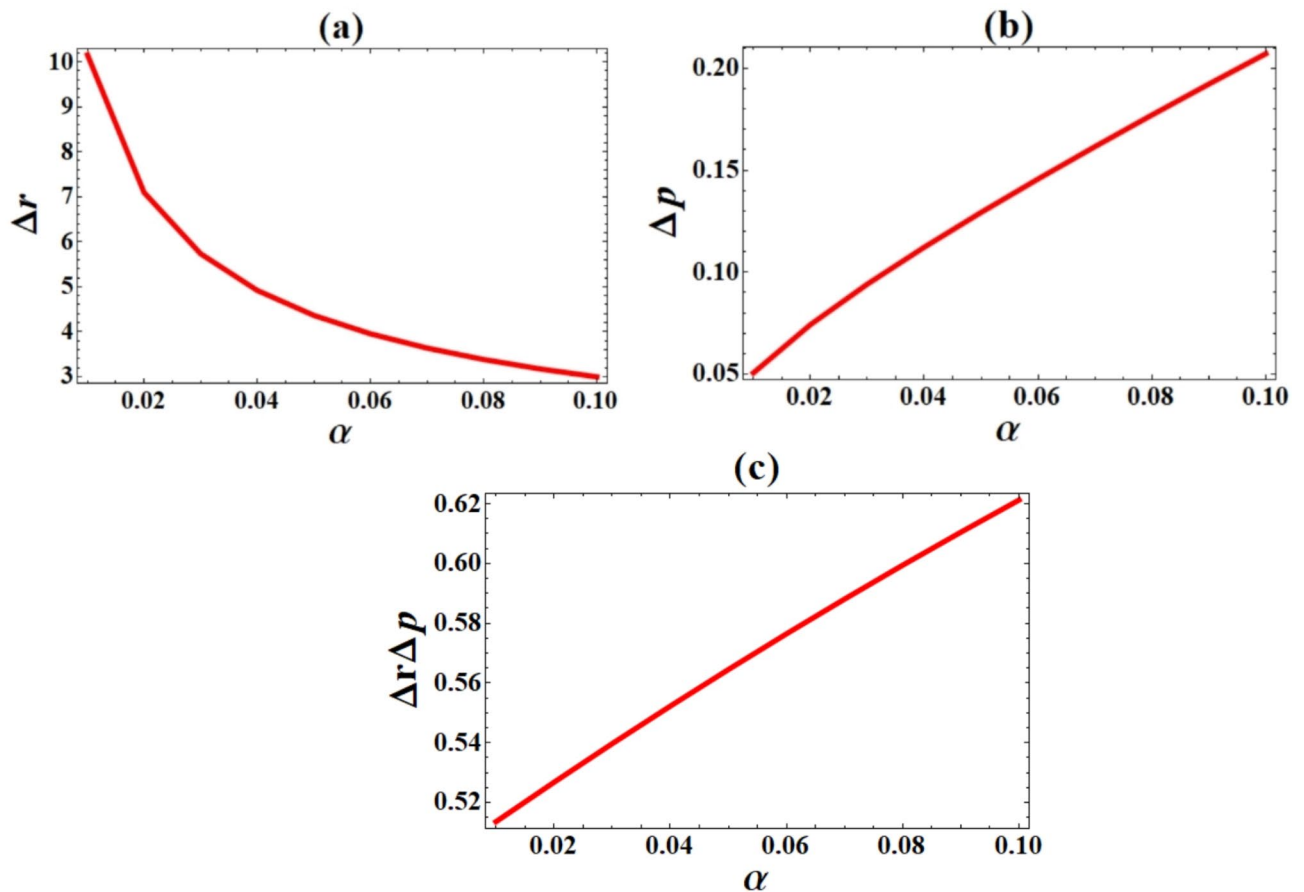


Fig. 1. (a-c). Uncertainties for Δr , Δp and Heisenberg uncertainty product.

Using the eigenfunctions, the ground state 2D mean values for $\langle p^2 \rangle_{00}$ and $\langle r^2 \rangle_{00}$ have been obtained numerically in Table 1. The momentum space expectation value obtained with the asymptotic approximation (Eq. (59a)) is fairly accurate for small α compared to the results obtained from the (Eq. (59b)) in radial representation. The expectation value for $\langle p^2 \rangle$ increases with an increase in the screening parameter while $\langle r^2 \rangle$ decreases as the screening parameter increases. Consequently, the 2D Fisher information in position space increases as α increases but in momentum representation, the Fisher entropy decreases with increasing α parameter. The Fisher product obeys the inequality $I(\rho)I(\gamma) \geq 16$. Also, the Heisenberg-type product $\langle p^2 \rangle_{00} \times \langle r^2 \rangle_{00}$ is bounded above the lowest bound. In Table 2, the 3D expectation values $\langle p^2 \rangle_{000}$ and $\langle p \rangle_{000}$ increase with the screening parameter but $\langle p^{-2} \rangle_{000}$ and $\langle p^{-1} \rangle_{000}$ decrease as the screening parameter increases. The mean values of $\langle p^2 \rangle_{000}$ and $\langle p^{-1} \rangle_{000}$ play a vital role in accessing the kinetic energy of a particle and the Compton profile. The 3D Fisher information in position space increases with α , while in momentum coordinate, it decreases as α increases. The Fisher product obtained in Table 2 is greater than the lowest bound and satisfies the inequality $I(\rho)I(\gamma) \geq 36$. In Table 3, the Shannon entropy in position space decreases as the screening parameter increases but in the momentum space, the Shannon entropy increases steadily. The Shannon sum is bounded above the minimum value and obeys the global inequality $S(\rho) + S(\gamma) \geq 6.43419$. The Rényi entropies variations with the screening parameter follow a similar trend as the Shannon entropies. The Rényi sums are bounded above the lowest bound. In Table 4, the Onicescu information and Shannon entropic power follow a similar trend where the entropies in position space decrease with the screening parameter but increase with the increase in the screening parameter for momentum coordinate. The Shannon entropic power and the Fisher-Shannon Complexity measures obey the inequality $J(\rho)J(\gamma) \geq \frac{1}{4}$ and $I(\rho)J(\rho) \geq 3$. The total Onicescu information entropy obeys the inequality $O(\rho)O(\gamma) \leq \frac{1}{(2\pi)^3}$. However, as α become large, the total Onicescu information is bounded above the minimum value. It is worth stating that the 1D expectation values for $\langle p^{\pm z} \rangle$ ($z = \text{odd integers}$) vanish due to symmetry but are finite for 2D and 3D spaces.

Conclusion

The bound state solutions of the radial Klein-Gordon equation have been obtained under an exponential-type and Yukawa potential functions using the parametric Nikiforov-Uvarov approach. The Greene-Aldrich approximation was used to replace the centrifugal term to allow for the analytical solution of the energy and wave function in closed form. The Schrödinger equation bound states were obtained as a special case by using

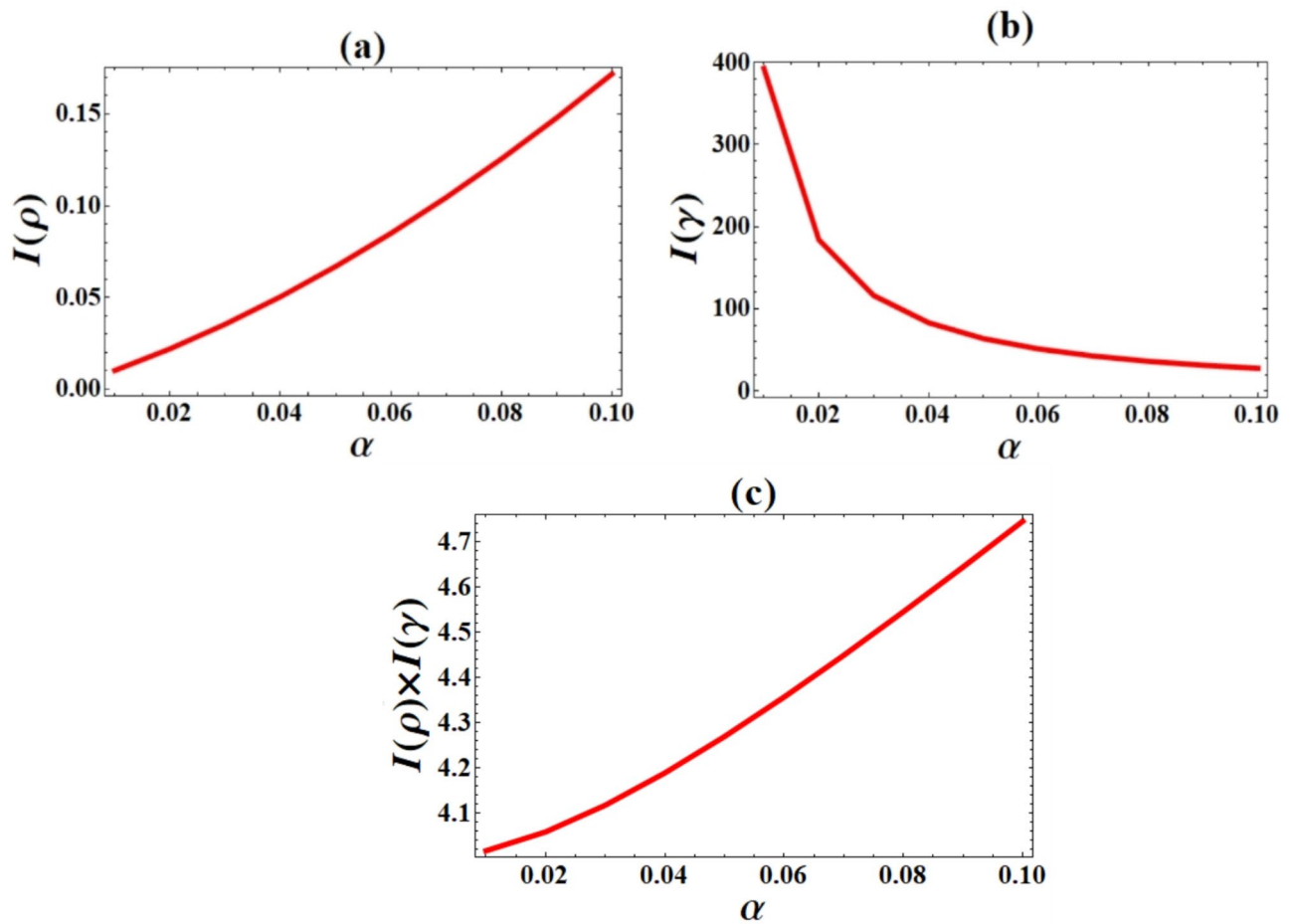


Fig. 2. (a-c). Fisher information in position and momentum coordinates and Fisher uncertainty product.

an appropriate transformation. The wave functions have been utilized to obtain expectation values and entropic densities in both position and momentum in D -dimensional spaces. In addition, the information measurements for the system are analogous to the Heisenberg uncertainty principle and satisfy the entropic inequalities. Also, the 1D momentum expectation value $\langle p^2 \rangle_{00}$ coincides with the 3D $\langle p^2 \rangle_{000}$ values which is an indication of degeneracy. The total energy of a particle in both 1D and 3D space may degenerate due to the inter-dimensional degeneracy of the quantum numbers. However, in this present result, the degeneracy in 1D and 3D occurred for fixed quantum states at different momentum intervals. Thus in 1D, a particle may transit an entire space ($-\infty < p < \infty$) with a certain kinetic energy which must be equal to the kinetic energy of the same particle which moves through the interval $0 < p < \infty$ in 3D space.

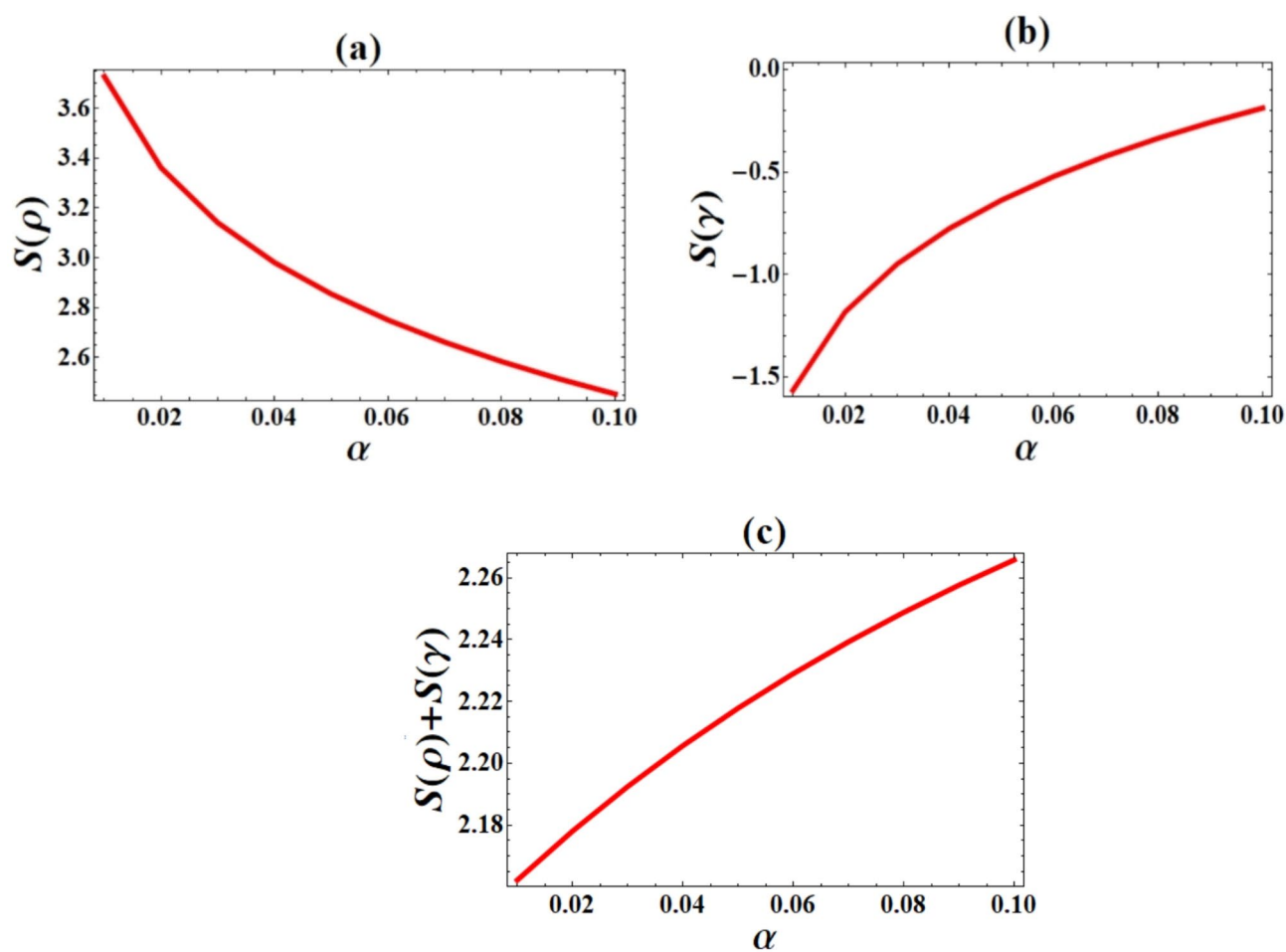


Fig. 3. (a-c). Shannon entropies in position and momentum coordinates and Shannon sum.

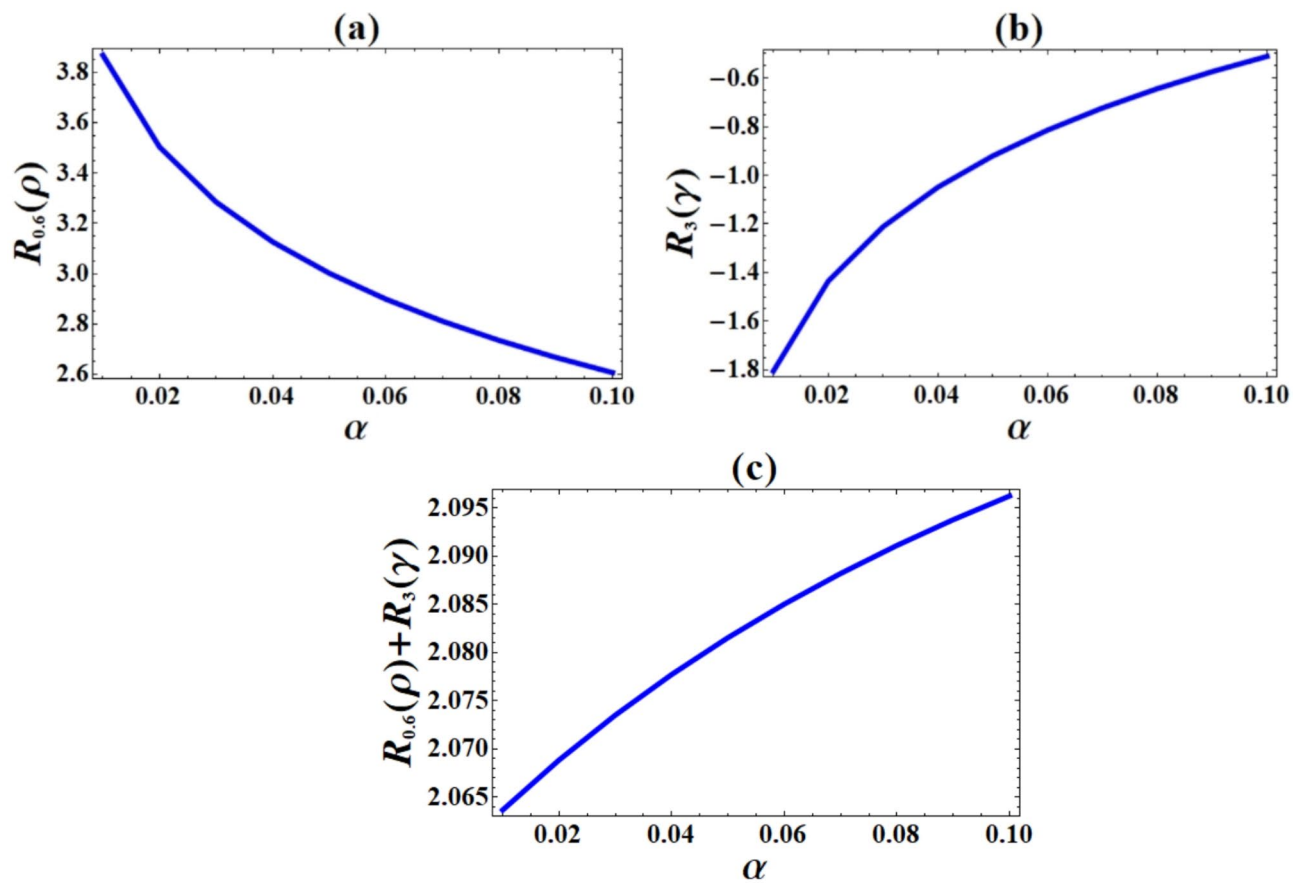


Fig. 4. (a-c). Rényi entropies in position and momentum coordinates and Rényi sum.

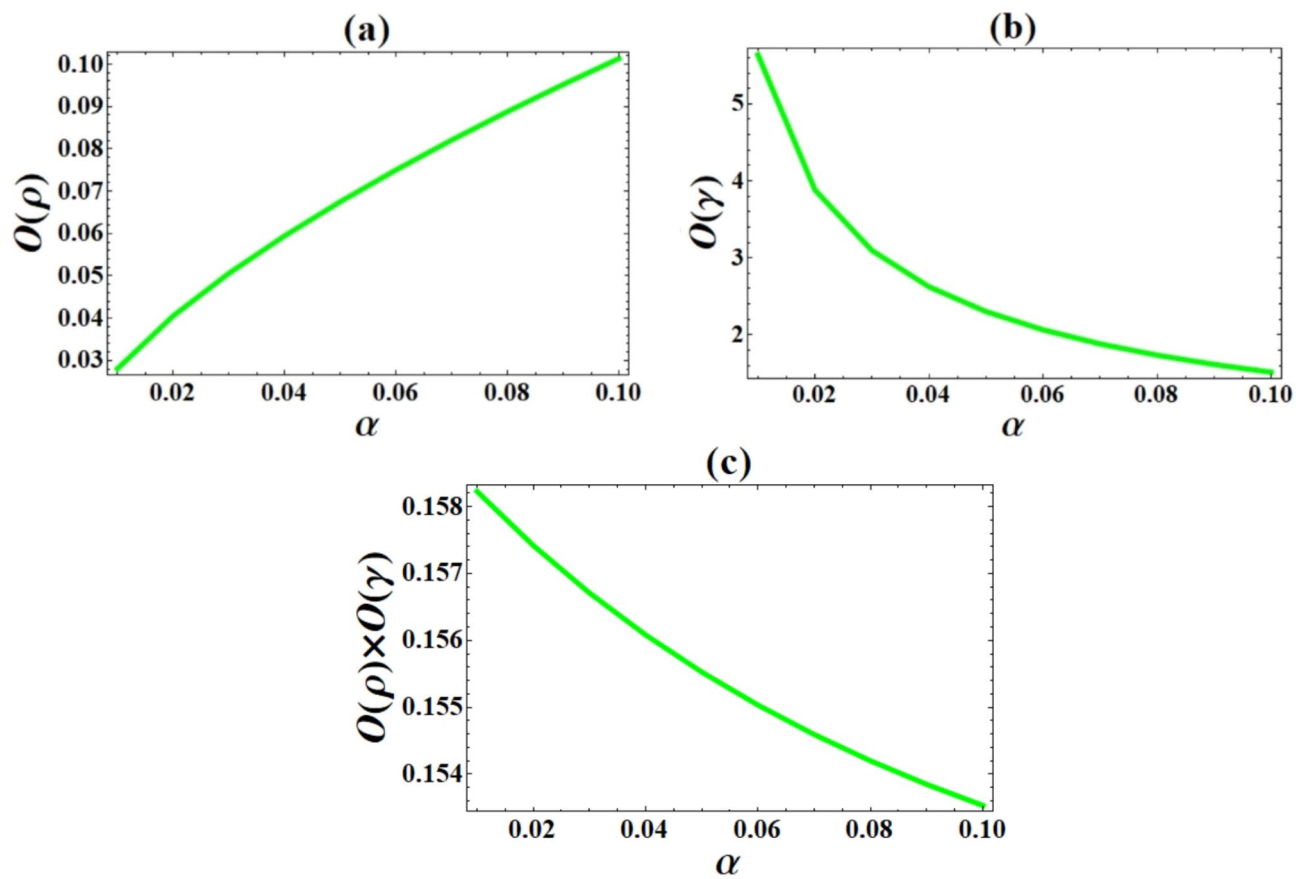


Fig. 5. (a-c). Onicescu information entropies in position and momentum coordinates and total Onicescu information entropies.

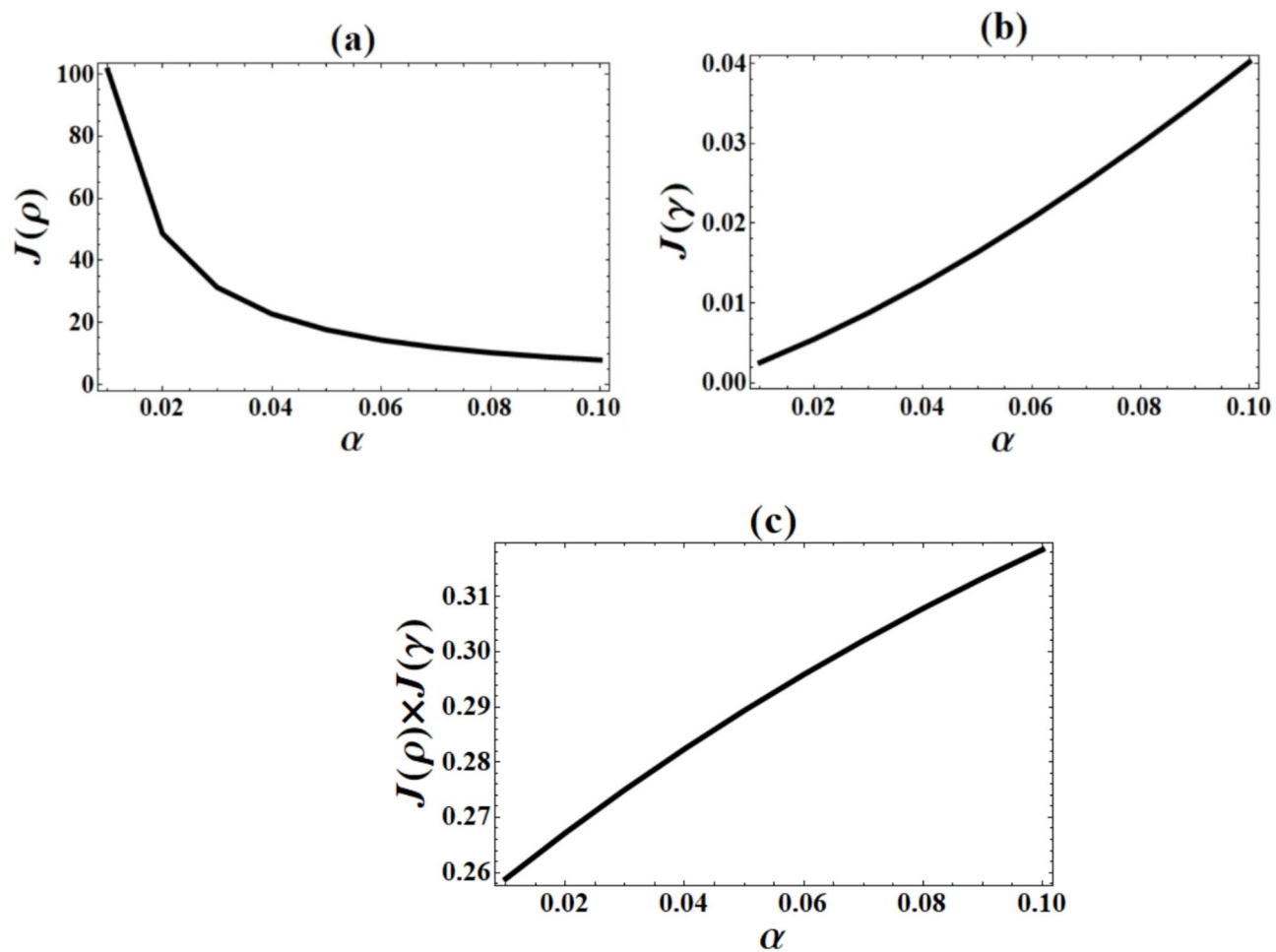


Fig. 6. (a-c). Shannon power complexities measures in position and momentum spaces.

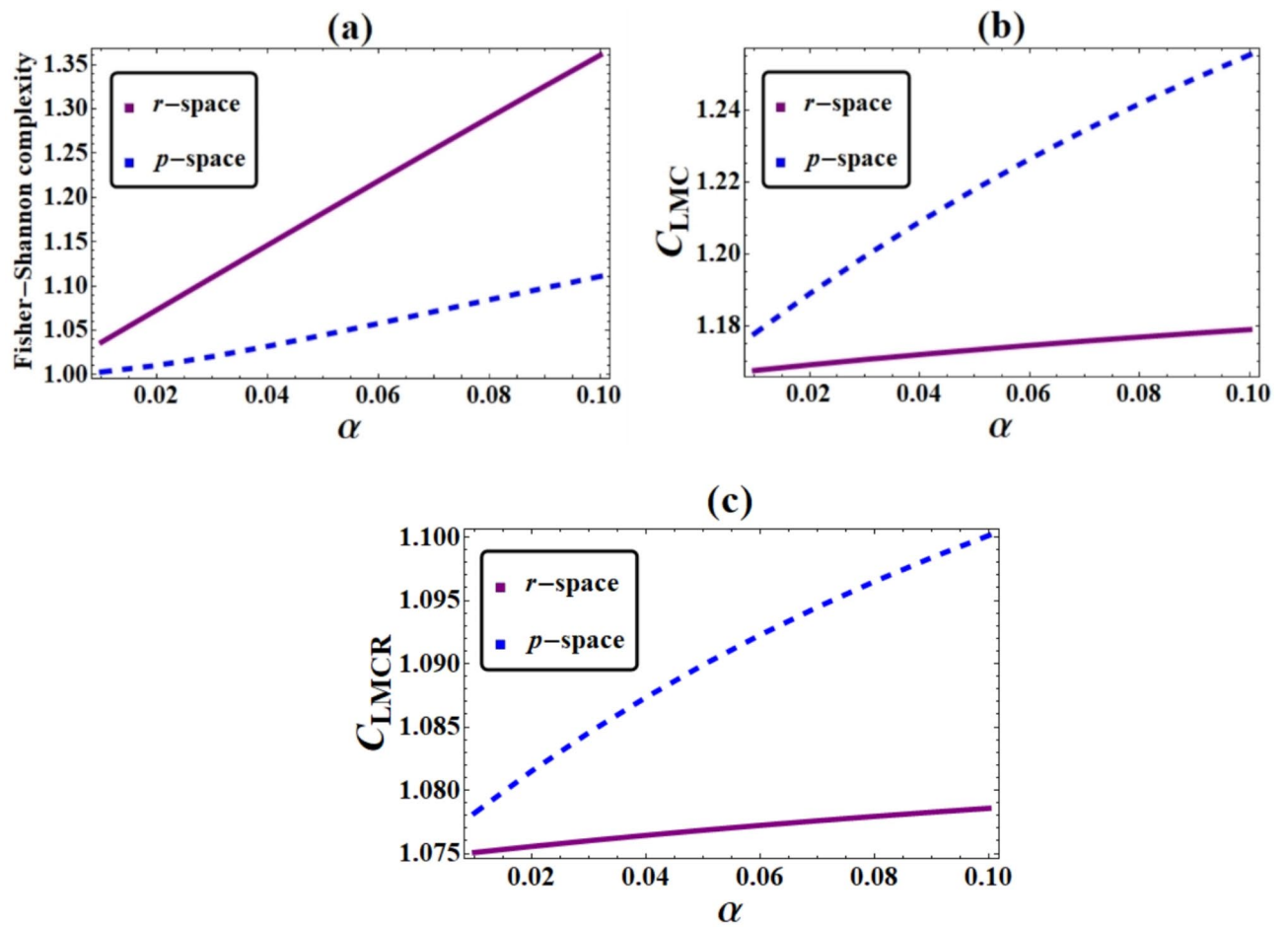


Fig. 7. (a). Fisher-Shannon complexities measures in position and momentum spaces (b) Lopez-Ruiz-Mancini-Calbet (C_{LMC}). (c) Lopez-Ruiz-Mancini-Calbet-Rényi (C_{LMCR}) complexities measures.

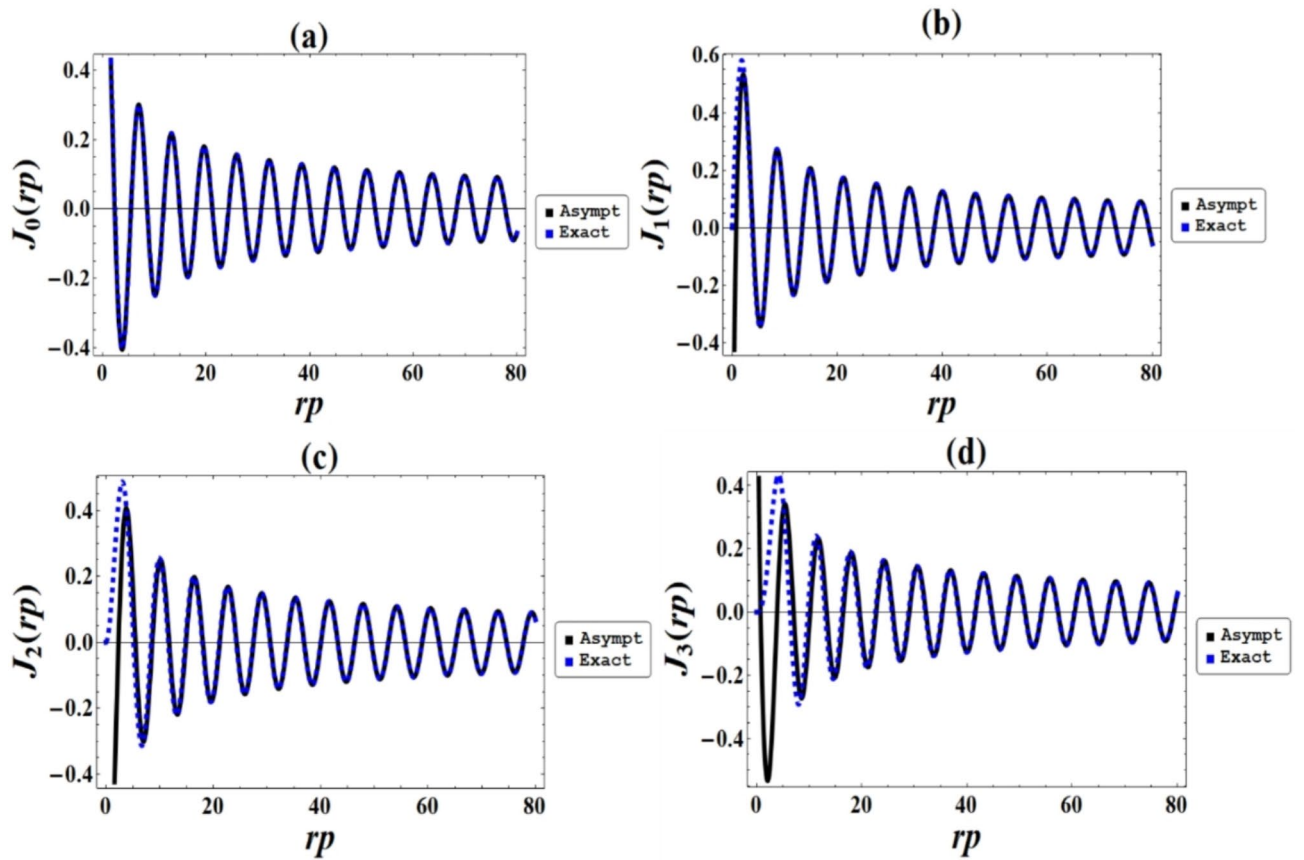


Fig. 8. (a-d). Variations of exact and asymptotic Bessel function of the First Kind with position for magnetic quantum numbers ($m = 0 - 3$).

α	$\langle p^2 \rangle_{00}$ Eq. (59a) Approx.	$\langle p^2 \rangle_{00}$ Eq. (59b)	$\langle r^2 \rangle_{00}$	$\langle p^2 \rangle_{00} \times \langle r^2 \rangle_{00} \geq 1$	$I(\rho)$	$I(\gamma)$	$I(\rho) I(\gamma) \geq 16$
0.01	0.002541326	0.002493106	4306.091	10.73554	0.009972424	17224.36284	171.7686499
0.02	0.005416421	0.005238612	1123.629	5.886256	0.020954447	4494.515626	94.18008973
0.03	0.008594540	0.008217203	520.585	4.277753	0.032868811	2082.340161	68.44404455
0.04	0.012050080	0.011410387	304.9067	3.479103	0.04564155	1219.626988	55.66566564
0.05	0.015759959	0.014800603	202.9748	3.004149	0.059202413	811.8990658	48.06638395
0.06	0.019702921	0.018371273	146.4707	2.690853	0.073485090	585.8826394	43.05363872
0.07	0.023859253	0.022106823	111.7205	2.469785	0.088427292	446.8819182	39.51655791
0.08	0.028210634	0.025992683	88.72661	2.306243	0.103970733	354.9064311	36.89988165
0.09	0.032740034	0.030015260	72.66152	2.180954	0.120061041	290.6460934	34.89527259
0.10	0.037431630	0.034161906	60.95576	2.082365	0.136647626	243.8230578	33.31784190

Table 1. Expectation values and Fisher information entropies for 2D space. $n = 0, m = 0, R = 1.2, V_0 = 0.05, V_1 = 0.44, \mu = 0.5, \hbar = 1, q = 0.1$.

α	$\langle p^2 \rangle_{000}$	$\langle p \rangle_{000}$	$\langle p^{-1} \rangle_{000}$	$\langle p^{-2} \rangle_{000}$	$I(\rho)$	$I(\gamma)$	$I(\rho) I(\gamma) \geq 36$
0.01	0.002557668	0.041048813	40.29226729	2951.243435	0.010230672	17225.98229	176.2333747
0.02	0.005511269	0.061135394	24.49862082	1006.471427	0.022045076	4496.239144	99.11993364
0.03	0.008860647	0.078472725	18.15693933	536.3398048	0.035442588	2084.154132	73.86781623
0.04	0.012603270	0.094563538	14.63353808	343.8326318	0.05041308	1221.518272	61.58049837
0.05	0.016734504	0.109921196	12.36364785	244.1424676	0.066938016	813.8550516	54.47784247
0.06	0.021247872	0.124773449	10.76962459	185.0247736	0.084991488	587.8912648	49.96575338
0.07	0.026135322	0.139234858	9.584729196	146.7095098	0.104541288	448.9316680	46.93189480
0.08	0.031387480	0.153367334	8.667544367	120.2624856	0.12554992	356.9863285	44.81960499
0.09	0.036993894	0.167205962	7.935648847	101.1275684	0.147975576	292.7456878	43.31921177
0.10	0.042943275	0.180771438	7.337560203	86.76722227	0.17177310	245.9324094	42.24457235

Table 2. Momentum expectation values and Fisher information theory in 3D space. $n = 0, m = 0, l = 0, R = 1.2, V_0 = 0.05, V_1 = 0.44, \mu = 0.5, \hbar = 1, q = 0.1$.

α	$R_{0.6}(p)$	$R_3(y)$	$R_{0.6}(p) + R_3(y) \geq 6.173744$	$S(\rho)$	$S(\gamma)$	$S(T) \geq 6.434190$
0.01	14.76874936	-8.415297133	6.353452231	14.57817873	-6.911968880	7.666209847
0.02	13.06233682	-6.808638692	6.253698132	12.82435864	-5.643282433	7.181076212
0.03	12.07605317	-5.861777412	6.214275759	11.79475897	-4.856932613	6.937826357
0.04	11.38499951	-5.189378532	6.195620979	11.06402118	-4.275872265	6.788148914
0.05	10.85592632	-4.669577504	6.186348820	10.49850146	-3.810191189	6.688310271
0.06	10.42941595	-4.247517107	6.181898847	10.03844178	-3.419013050	6.619428725
0.07	10.07372514	-3.893610348	6.180114794	9.651803320	-3.080509593	6.571293727
0.08	9.769869444	-3.590001096	6.179868347	9.319343666	-2.781742896	6.537600770
0.09	9.505575272	-3.325047789	6.180527482	9.028562946	-2.514390161	6.514172785
0.10	9.272440305	-3.090719539	6.181720766	8.770859478	-2.272719558	6.498139920

Table 3. 3D Shannon and Rényi entropies in position and momentum representations. $S(T) = S(\rho) + S(\gamma)$. $n = 0, m = 0, l = 0, R = 1.2, V_0 = 0.05, V_1 = 0.44, \mu = 0.5, \hbar = 1, q = 0.1$.

α	$O(\rho) \times 10^{-5}$	$O(\gamma)$	$O(\rho) O(\gamma) \leq 0.004031$	$J(\rho)$	$J(\gamma)$	$J(\rho) J(\gamma) \geq \frac{1}{4}$	$I(\rho) J(\rho) \geq 3$	$I(\gamma) J(\gamma) \geq 3$
0.01	0.056587745	3185.648256	0.00180268652	973.5106372	0.000583856	0.568389948	9.959668018	10.05749172
0.02	0.339492454	692.6050885	0.00235134201	302.3832835	0.001360279	0.411325671	6.666062465	6.116140289
0.03	0.985333836	276.7308571	0.00272672277	152.2152414	0.002297733	0.349750036	5.394902087	4.788830456
0.04	2.117181098	142.8486960	0.00302436559	93.51642161	0.003384806	0.316534980	4.714450844	4.134602829
0.05	3.849779582	85.18750654	0.00327953123	64.14357350	0.004617026	0.296152540	4.293643549	3.757589852
0.06	6.289377919	55.78804894	0.00350872088	47.20111194	0.005992665	0.282860440	4.011692739	3.523035258
0.07	9.532860530	39.02895037	0.00372057541	36.47614482	0.007509772	0.273927543	3.813263160	3.371374621
0.08	13.66699482	28.68186233	0.00391994864	29.22483060	0.009164918	0.267843180	3.669175144	3.271750475
0.09	18.76792223	21.89743744	0.00410970498	24.07478188	0.010953052	0.263692329	3.562479716	3.206458634
0.10	24.90129872	17.23443644	0.00429159850	20.27447256	0.012867849	0.260888843	3.482609003	3.164621000

Table 4. 3D Onicescu information, Shannon entropic power and Fisher-Shannon Complexity measures. $n = 0, m = 0, l = 0, R = 1.2, V_0 = 0.05, V_1 = 0.44, \mu = 0.5, \hbar = 1, q = 0.1$.

Data availability

All data reported in this article were obtained from the analytical equations discussed within the article.

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Declarations

Competing interests

The authors declare no competing interests.

Additional information

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