

MACH'S PRINCIPLE: EXACT FRAME-DRAGGING BY ENERGY CURRENTS IN THE UNIVERSE

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We show that the dragging of axis directions of local inertial frames by a weighted average of the energy currents in the universe (Mach's postulate) is exact for all linear perturbations of all Friedmann-Robertson-Walker universes.

1 Mach's Principle

1.1 The Observational Fact: 'Mach zero'

The time-evolution of local inertial axes, i.e. the local *non-rotating frame* is *experimentally* determined by the spin axes of *gyroscopes*, as in inertial guidance systems in airplanes and satellites. This is true both in Newtonian physics (Foucault 1852) and in General Relativity.

It is an *observational fact* within present-day accuracy that the *spin axes of gyroscopes* do not precess *relative to quasars*. This observational fact has been named 'Mach zero', where 'zero' designates that this fact is not yet Mach's principle, it is just the observational starting point. — There is an extremely small dragging effect by the rotating Earth on the spin axes of gyroscopes, the Lense - Thirring effect, which makes the spin axes of gyroscopes precess relative to quasars by 43 milli-arc-sec per year. It is hoped that one will be able to detect this effect by further analysis of the data which have been taken by Gravity Probe B.

1.2 The Question

What physical cause explains the observational fact 'Mach zero' ? Equivalently: What physical cause determines the time-evolution of gyroscope axes ? In the words of John A. Wheeler: *Who gives the marching orders* to the spin axes of gyroscopes, i.e. to inertial axes ?

1.3 Mach's Postulate

An answer to this fundamental question was formulated by Ernst Mach in his postulate (1883) that inertial axes (i.e. the spin axes of gyroscopes) *exactly follow* an average of the motion of the *masses in the universe*: Mach postulated *exact frame dragging* of inertial axes by the motion of cosmological masses, *not merely a little bit* of frame-dragging as in Lense-Thirring effect.

Mach did not know, *what mechanism*, what new force could do the job, he merely stated: 'the laws of motion could be conceived ...'. Mach also asked: "*What share* has every mass in the determination of direction ... in the law of inertia? No definite answer can be given by our experiences."

1.4 Our Results

We have shown that *exact dragging* of inertial axis directions, i.e. Mach's Principle, follows from Cosmological General Relativity for *general, linear perturbations* of FRW backgrounds with $K = (\pm 1, 0)$. This also holds for FRW backgrounds with arbitrarily small energy density and pressure compared to ρ_{crit} (Milne limit of FRW universe).

These results have been demonstrated for the first time in our paper¹ for $K = 0$, and in our paper² for $K = (\pm 1, 0)$.

2 Theoretical Results and Tools

2.1 Cosmological Vorticity Perturbations

The vector sector of cosmological perturbations is the sector of vorticity perturbations. Two important theorems for the vorticity sector are needed to understand the following summary:

1. The *slicing* of space-time in slices Σ_t of fixed time is *unique*. The *lapse* function (elapsed time between slices) and g_{00} are *unperturbed*.
2. The *intrinsic geometry* of each slice Σ_t , i.e. of 3-space, remains *unperturbed*.

The *coordinate choice uniquely adapted* to our 3-geometry is comoving Cartesian coordinates for FRW with $K = 0$, resp. comoving spherical coordinates for $K = (\pm 1, 0)$. Hence the only quantity referring to vorticity perturbations is the *shift* 3-vector β^i (resp $\beta_i = g_{0i}$):

$$ds^2 = -dt^2 + a^2[d\chi^2 + R_{\text{com}}^2(d\theta^2 + \sin^2\theta d\phi^2)] + 2\beta_i dx^i dt, \quad (1)$$

$$R_{\text{com}} = (\chi, \sin\chi, \sinh\chi). \quad (2)$$

2.2 Gravitomagnetism

The general *operational definitions* of the gravitomagnetic and gravitoelectric fields are given via measurements by *FIDOs* (*Fiducial Observers*) with *LONBs* (*Local Ortho-Normal Bases*), where LONB components are denoted by *hats* over indices.

Gravitoelectric field $\vec{E}_g \equiv \vec{g}$:

$$\frac{d}{dt} p_i^g \equiv m E_i^g \quad \text{free-falling quasistatic test particle.} \quad (3)$$

Gravitomagnetic field \vec{B}_g :

$$\Omega_i^{\text{gyro}} \equiv -\frac{1}{2} B_i^g \quad \text{precession of gyro comoving with FIDO.} \quad (4)$$

Gravitomagnetic vector potential \vec{A}_g : Because all 3-scalars must be unperturbed in the vector sector, $\text{div } \vec{A}_g \equiv 0$, and \vec{A}_g is uniquely determined by \vec{B}_g ,

$$\vec{B}_g =: \text{curl } \vec{A}_g \Rightarrow \vec{A}_g = \vec{\beta} \equiv \text{shift vector.} \quad (5)$$

Our choice of FIDOs: Our FIDOs are at fixed values of the spatial coordinates x^i , and the spatial axes are fixed in the direction of our coordinate basis vectors.

2.3 Einstein's G_k^0 Equation: The Momentum Constraint

New result: The momentum constraint is *form-identical* for all three FRW background geometries, $K = (0, \pm 1)$:

$$(-\Delta + \mu^2) \vec{A}_g = -16\pi G_N \vec{J}_g, \quad (6)$$

where $(\mu/2)^2 \equiv -(dH/dt) \equiv (H\text{-dot radius})^{-2}$, and $\vec{J}_g = \text{energy current density} = \text{momentum density}$. Since the source in Eq. (6) is the momentum density, this equation is called the '*momentum constraint*'.

The momentum constraint is an *elliptic equation*, i.e. there are *no partial time-derivatives* of perturbations, although the momentum constraint refers to *time-dependent* gravitomagnetism.

Our new approach: For the source we have used the *LONB components* $\vec{J}_k^\varepsilon = T_k^0$, which is a *measurable input*, and which needs *no prior knowledge* of g_{0i} , which is the output. Einstein had emphasized that the coordinate-basis components T_k^0 are *not* a directly measurable input: '*If you have $T_{\mu\nu}$ and not a metric, the statement that matter by itself determines the metric is meaningless.*'

New result: The momentum constraint for time-dependent gravitomagnetism for all three FRW background geometries has the same form as Ampère's law for stationary magnetism, except for the term $\mu^2 \vec{A}_g$, which causes a *Yukawa suppression* beyond the *H-dot radius*. There are no curvature terms in Eq. (6).

2.4 The Laplacian on Vector Fields in Riemannian 3-Spaces

The Laplacian Δ acting on vector fields in Eq. (6) is the *de Rham - Hodge Laplacian*, which mathematicians simply call 'the Laplacian', and which differs from ∇^2 , which mathematicians call the '*rough Laplacian*'. Unfortunately all publications on cosmological vector perturbations up to ours have used the '*rough Laplacian*' ∇^2 . The difference between the two operators is given by the *Weitzenböck formula*:

$$(\Delta - \nabla^2) \vec{A} = -(2K/a_c^2) \vec{A}, \quad (7)$$

where $K = (\pm 1, 0)$ is the curvature index for the FRW background, and a_c is its curvature radius. For vorticity fields (divergence zero) the de Rham - Hodge Laplacian is defined by

$$(\Delta \vec{a})_\mu = -(\text{curl curl } \vec{a})_\mu = -(*d*d\vec{a})_\mu, \quad (8)$$

where we have given both the notation of elementary vector calculus and the notation of calculus of differential forms with $d \equiv$ exterior derivative and $*$ \equiv Hodge dual.

The de Rham - Hodge Laplacian on vector fields is *singled out* by the following properties:

1. If all sources (curl and div) are zero \Rightarrow the de Rham - Hodge Laplacian gives zero.
2. The de Rham - Hodge Laplacian *commutes* with curl, div, grad.
3. The *identities* of vector calculus in Euclidean 3-space (familiar from Classical Electrodynamics) remain true in Riemannian 3-spaces for the Hodge - de Rham Laplacian.
4. The *action principle* for Ampère magnetism in Riemannian 3-spaces directly produces Ampère's equation for \vec{A} with the Hodge - de Rham Laplacian and *without curvature terms*.
5. For electromagnetism in curved space-time the *equivalence principle forbids curvature terms* in equations with the Hodge - de Rham Laplacian.

Every one of these properties does not hold for the '*rough*' Laplacian ∇^2 .

3 The Bottom Lines

3.1 The Solution of the Momentum Constraint

Cosmological gravitomagnetism on a background of *open* FRW universes gives *identical expressions* for $K = (0, -1)$:

$$\begin{aligned}\vec{B}_g(P) &= -2\vec{\Omega}_{\text{gyro}}(P) = \\ &= -4G_N \int d(\text{vol}_Q) [\vec{n}_{PQ} \times \vec{J}_\varepsilon(Q)] Y_\mu(r_{PQ})\end{aligned}\quad (9)$$

$$Y_\mu(r) = \frac{-d}{dr} \left[\frac{1}{R} \exp(-\mu r) \right] = \text{Yukawa force}, \quad (10)$$

where r = radial distance, and $2\pi R$ = circumference of the great circle through Q and centered at P . The solution Eq. (9) is analogous to Ampère's solution for stationary magnetism, but Eq. (9) is valid for *time-dependent* gravito-magnetodynamics, and it has a *Yukawa suppression*.

There is a fundamental difference between our solution Eq. (9) for cosmological gravitomagnetism and the corresponding solutions in other theories, Ampère's magnetism, electromagnetism in Minkowski space, and General Relativity in the solar system: Our solution for Cosmological General Relativity is manifestly *form-invariant* when going to globally rotating frames. In contrast, the other theories give *solutions*, which are *not* form-invariant when going to globally rotating frames.

If the background is a *closed* FRW universe, one simply makes the following replacement in Eqs. (9, 10):

$$\exp(-\mu r) \Rightarrow \sinh^{-1}(\mu\pi) \sinh[\mu(\pi - r)]. \quad (11)$$

3.2 Exact Dragging of Inertial Axes

From *symmetry* under rotations and reflections one concludes: The precession of a gyroscope can only be acted on by the component of the matter velocity field with $J^P = 1^+$ relative to the gyroscope. This component of the velocity field is equivalent to a rigid rotation of matter with angular velocity $\vec{\Omega}_{\text{matter}}(r)$.

From Eq. (9) one concludes that inertial axes, i.e. the spin axes of *gyroscopes*, *exactly follow* the *weighted average* of the energy currents of cosmic *matter*,

$$\vec{\Omega}_{\text{gyro}} = \langle \vec{\Omega}_{\text{matter}} \rangle \equiv \int_0^\infty dr \vec{\Omega}_{\text{matter}}(r) W(r) \quad (12)$$

$$W(r) = \frac{1}{3} 16\pi G_N (\rho + p) R^3 Y_\mu(r), \quad (13)$$

for perturbations of open FRW universes. The *weight function* $W(r)$ is *normalized to unity*,

$$\int_0^\infty dr W(r) = 1, \quad (14)$$

as it must be for a proper averaging weight function in any problem. — For perturbations of a closed FRW universe one again makes the replacement of Eq. (11).

References

1. C. Schmid, Phys. Rev. D **74**, 044031 (2006).
2. C. Schmid, arXiv:0801.2907v1 [astro-ph].