

RENORMALIZED VOLUME AND AREA

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Abstract

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In this dissertation we look at some problems on renormalized volume and area. We develop a Chern-Gauss-Bonnet formula for the renormalized area of a hypersurface within a 5-dimensional Poincaré-Einstein manifold. This is followed by an investigation into renormalized area in the Singular Yamabe setting. Lastly, we look at the renormalized volume of a minimally bounded region within a 4-dimensional Poincaré-Einstein manifold.

Tá an obair seo tiomnaithe do mo cara is fearr san domhain leathan iomlain a raibh i gcónaí ag tabhairt tacaíocht agus spreragadh dom, mo seanmháthair Cairméal.

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SYMBOLS

(X, g_+)	Poincaré-Einstein Manifold
(Y, h_+)	Properly embedded submanifold of X
M	Asymptotic boundary of X
Σ	Asymptotic boundary of Y
r	Special defining function for M
Y_ϵ	$Y \cap \{r > \epsilon\}$
Σ_ϵ	∂Y_ϵ
\hat{r}	$r _Y$
$k(\epsilon)$	$h_+ _{\Sigma_\epsilon}$
μ_Y	Inward pointing normal to Y with respect to g_+
B	Second fundamental form of \bar{Y} with respect μ_Y
ν_r	h_+ -inward unit normal to Σ_r in Y
L_r	Second fundamental form of Σ_r in Y_r
M_ϵ	$X \cap \{r = \epsilon\}$

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¹The notation changes slightly in each chapter but this is outlined as we go.

CHAPTER 1

INTRODUCTION

The renormalized volume of an even-dimensional Poincaré-Einstein manifold (X^{n+1}, g_+) is among its most important global invariants. Introduced in [22] (see also [16]), it is defined by taking the order-zero term in the expansion in ϵ of the quantity $\text{Vol}_{g_+}(X \cap \{r > \epsilon\})$, where r is what is known as a special defining function.

Let X^{n+1} be the interior of a compact manifold with boundary, \bar{X} . Then we say a metric g_+ on $\overset{\circ}{X}$ is conformally compact if $\bar{g} := r^2 g_+$ extends as a metric to \bar{X} where r is a defining function for $M := \partial X$ i.e.

$$\begin{aligned} M &= r^{-1}(0) \\ r &> 0 \quad \text{on } \overset{\circ}{X} \\ dr &\neq 0 \quad \text{on } M. \end{aligned}$$

A special defining function r , is a defining function that also locally equals the $r^2 g_+$ -distance to M ; i.e. $|dr|_{\bar{g}}^2 \equiv 1$ on a neighbourhood of M . If a conformally compact manifold X^{n+1} also satisfies the Einstein condition $\text{Ric}(g_+) = -ng_+$ then we say that X is a Poincaré-Einstein manifold.

For a motivating example we consider the case $(X^{n+1}, g_+) = (\mathbb{B}^{n+1}, \text{hyperbolic metric})$

$$g_+ = \frac{4|dx|^2}{(1 - |x|^2)^2}.$$

We will find a special defining function for $\partial\mathbb{B} = \mathbb{S}^n$. Observe

$$|dr|_{\bar{g}}^2 = 1 \iff |d(\log \frac{1}{r})|_{g_+}^2 = 1$$

which is the eikonal equation for $\log \frac{1}{r}$ in the metric g_+ . The distance function $d(x, 0)$ = (hyperbolic distance from x to 0) satisfies the eikonal equation and also $d(x, 0) \rightarrow \infty$ as $|x| \rightarrow 1$ and $\log \frac{1}{r} \rightarrow \infty$ as $|x| \rightarrow 1$ so by taking

$$\log \frac{1}{r} = d(x, 0)$$

we get $r(x) = e^{-d(x,0)}$. It is a fact of hyperbolic geometry that

$$d(x, 0) = \log \left(\frac{1 + |x|}{1 - |x|} \right),$$

hence $r = \frac{1-|x|}{1+|x|}$ is a defining function for the boundary of the hyperbolic ball. In particular, it is special in the sense that r corresponds to the $r^2 g_+$ -distance to $M = \partial X$. It follows that

$$\bar{g} = r^2 g_+ = \frac{4|dx|^2}{(1 + |x|)^4} \tag{1.1}$$

and furthermore

$$\bar{g}|_{\mathbb{S}^n} = \frac{1}{4}(\text{Standard metric on } \mathbb{S}^n).$$

Writing $(dx^i)^2$ in terms of spherical coordinates we see

$$|dx|^2 = \begin{bmatrix} |x|^2(\text{std metric on } \mathbb{S}^n) & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore

$$|dx|^2 = |x|^2(\text{std metric on } \mathbb{S}^n) + (d|x|)^2. \tag{1.2}$$

We note that

$$\begin{aligned} |x|^2 &= \frac{1-r^2}{(1+r)^2} \\ \frac{1}{(1+|x|)^4} &= \frac{(1+r)^4}{16} \\ d|x| &= \frac{-2dr}{(1+r)^2}. \end{aligned}$$

Plugging (1.2) into (1.1) and applying the above substitutions gives

$$\bar{g} = (1-r^2)^2 g_0 + dr^2. \tag{1.3}$$

Note that we've written $\bar{g} = g_r + (dr)^2$, where g_r is a 1-parameter family of metrics on \mathbb{S}^n , $g_0 = \frac{1}{4}$ (Standard metric on \mathbb{S}^n) and r measures the \bar{g} -distance to \mathbb{S}^n .

We can compute the renormalized volume directly in our model case $(X^{n+1}, g_+) = (\mathbb{B}^{n+1}, (\text{hyperbolic metric}))$:

For the sake of concreteness let $n = 3$. We know

$$\text{Vol}_{g_+}(r > \epsilon) = \int_{\epsilon}^1 \int_{\mathbb{S}^3} \frac{\sqrt{\det g_+}}{\sqrt{\det g_0}} d\nu_{g_0} dr$$

$$\begin{aligned} \sqrt{\det g_+} &= r^{-4} \sqrt{\det \bar{g}} \\ &= r^{-4} \sqrt{\det[(1-r^2)^2 g_0 + dr^2]} \\ &= r^{-4} (1-r^2)^3 \sqrt{\det g_0}. \end{aligned}$$

Therefore we may write

$$\begin{aligned} \text{Vol}_{g_+}(r > \epsilon) &= \int_{\epsilon}^1 \int_{\mathbb{S}^3} \frac{\sqrt{\det g_+}}{\sqrt{\det g_0}} d\nu_{g_0} dr \\ &= \frac{\text{Area}(\mathbb{S}^3)}{8} \int_{\epsilon}^1 r^{-4} (1-r^2)^3 dr. \end{aligned}$$

Now applying integration by parts gives

$$\int_{\epsilon}^1 r^{-4}(1-r^2)^3 dr = \frac{\epsilon^{-3}}{3}(1-\epsilon^2)^3 - 2 \int_{\epsilon}^1 r^{-2}(1-r^2)^2 dr$$

$$\int_{\epsilon}^1 r^{-2}(1-r^2)^2 dr = \epsilon^{-1}(1-\epsilon^2)^2 - 4 \int_{\epsilon}^1 1-r^2 dr.$$

The boundary terms for IBP contribute no constant with respect to ϵ , it follows that the constant term in the expansion of $\text{Vol}_{g_+}(r > \epsilon)$ is given by

$$V = \frac{\text{Area}(\mathbb{S}^3)}{3} = \frac{4\pi^2}{3}. \quad (1.4)$$

A priori it is not clear that V is independent of r , but as we said before, if r is a special defining function on an even dimensional Poincaré-Einstein manifold then the order-zero term in the expansion in ϵ of the quantity $\text{Vol}_{g_+}(r > \epsilon)$ only depends on g_+ .

It turns out that, given any metric g_{∞} conformal to $r^2 g_+|_M$ where r is any defining function for M , we can find a unique special defining function \hat{r} such that $\hat{r}^2 g_+|_M = g_{\infty}$. The conformal class $[g_+]_{\infty} := [\bar{g}]_{TM}$ is an invariant of g_+ called its conformal infinity. We now state a lemma of Graham [20] that puts the previous remarks about special defining functions into technical terms. We follow this with a discussion of how this generalises (1.3).

Lemma 1.1. Given $g_{\infty} \in [g_+]_{\infty}$ there exists a unique defining function r such that $|dr|_{\bar{g}} \equiv 1$ in a neighbourhood of M in \bar{X} , such that $g_{\infty} = r^2 g_+|_{TM}$ ($\bar{g} = r^2 g_+$).

This gives a natural isomorphism of sets:

{Special defining functions for M } =

{Defining functions r for M such that $|dr|_{\bar{g}} \equiv 1$ in a neighbourhood of M } $\cong [g_+]_{\infty}$.

A defining function determines for some ϵ , an identification of $M \times [0, \epsilon)$ with a neighbourhood of M in \overline{X} : $(p, t) \in M \times [0, \epsilon)$ corresponds to $\phi(p, t)$, where ϕ is the flow of $\nabla_{\bar{g}}r$. If r is a special defining function, then $r(\phi(p, t)) = t$. So we can think about the t coordinate as just being r and $\nabla_{\bar{g}}r$ is orthogonal to the slices $M \times \{t\}$. The metric \bar{g} then takes the form $\bar{g} = g_r + dr^2$ where g_r is a 1-parameter family of metrics on M .

Conformally transforming the Riemann tensor gives:

$$R_{ijkl}^{g_+} = -|dr|_{\bar{g}}^2(g_{ik}g_{jl} - g_{il}g_{jk}) + O_{ijkl}^{\bar{g}}(r^{-3}). \quad (1.5)$$

Note we are using the convention that $g_{ij} = (g_+)_{ij}$. It follows that g_+ has asymptotic sectional curvature equal to $-|dr|_{\bar{g}}^2|_M$; it can be checked that this function is an invariant of g_+ . If we assume that g_+ satisfies the Einstein condition $Ric(g_+) = -ng_+$, then contracting (1.5) gives $|dr|_{\bar{g}}^2|_M \equiv 1$; which means that g_+ is asymptotically hyperbolic (we will sometimes say AH for short).

Using the Einstein condition, one can compute what's known as the Fefferman-Graham [12] expansion for g_r . When $n + 1$ is even, this takes the form:

$$g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + g^{(n-1)}r^{n-1} + g^{(n)}r^n + O(r^{n+1})$$

where $g^{(i)}$ are tensors on M and are formally determined for $0 \leq j \leq n - 1$ and $g^{(n)}$ is trace-free with respect to any metric in $[\bar{g}]_M$.

One of the basic theorems regarding renormalized volume in dimension four is Anderson's [3], [7] Gauss-Bonnet theorem, which states that

$$4\pi^2\chi(X^4) = 3V + \frac{1}{8} \int_X |W_{g_+}|_{g_+}^2 dv_{g_+}. \quad (1.6)$$

Here W_{g_+} is the Weyl tensor of g_+ and, since $|W_{g_+}|_{g_+}^2$ is a pointwise conformal invariant of weight -4, the integral is guaranteed to converge despite the infinite volume of (X, g_+) . In his paper, Anderson notes that, given a Poincaré-Einstein manifold (X^4, g_+) , (1.6) implies the following rigidity result:

$$V(g_+) \leq \frac{4\pi^2}{3} \chi(X^4) \tag{1.7}$$

with equality if and only if g_+ is hyperbolic.

Graham and Witten [19] outlined an analogous theory on even dimensional minimal submanifolds Y of Poincaré-Einstein manifolds X such that the boundary of Y extends to the boundary of X i.e. Y is properly embedded in X . Given such a submanifold Y of a Poincaré-Einstein manifold (X, g_+) and given a choice of special defining function r for ∂X , we can carry out a procedure in regards to the constant term in the expansion of $\text{Area}_{g_+|_Y}(Y \cap r > \epsilon)$. It turns out that this is an invariant of Y which we refer to as the renormalized area. Alexakis and Mazzeo [1] obtained an explicit formula for the renormalized area of a minimal 2 dimensional minimal submanifold Y of a Poincaré-Einstein manifold:

$$A(Y) = -2\pi\chi(Y) - \frac{1}{2} \int_Y |II|^2 dA + \int_Y W_{1212} dA_{h_+}, \tag{1.8}$$

Where II is the second fundamental form of Y and W_{1212} is the Weyl tensor of g_+ evaluated on an orthonormal frame for Y .

A nice relation we have on 4-dimensional manifolds is the Chern-Gauss-Bonnet formula. We'll use this to give an example of a renormalized area calculation:

If we let $(X^5, g_+) = (\mathbb{B}^5, \text{hyperbolic metric})$ and let Y be the minimal hypersurface given by $\{x^5 = 0\}$. The Chern-Gauss-Bonnet formula applied to $Y \cap \{r > \epsilon\}$ gives

us

$$8\chi(Y \cap \{r > \epsilon\})\pi^2 = \int_{Y \cap \{r > \epsilon\}} \frac{|W|_Y^2}{4} + \frac{1}{24}R_Y^2 - \frac{|E|^2}{2}d\nu_{h_+} + \oint_{\partial Y_\epsilon} Sd\nu_{h_\epsilon} \quad (1.9)$$

where W is the Weyl tensor, R_Y is scalar curvature, E is the trace-free Ricci tensor and S is a boundary term, since Y itself is hyperbolic $W_Y = E = 0$ and $R_Y = -12$.

So we have

$$8\chi(Y \cap \{r > \epsilon\})\pi^2 = \int_{Y \cap \{r > \epsilon\}} 6d\nu_{h_+} + \oint_{\partial Y_\epsilon} Sd\nu_{h_\epsilon}. \quad (1.10)$$

Note that $\chi(Y \cap \{r > \epsilon\}) = \chi(Y)$ for small enough ϵ since Y can be deformation retracted onto $Y \cap \{r > \epsilon\}$ via the flow of $\nabla_{r^2g_+}r$, where r is a special defining function. It turns out that for this example $\oint_{\partial Y_\epsilon} Sd\nu_{h_\epsilon}$ has no constant term as $\epsilon \rightarrow 0$. Expanding the RHS of (1.9) and equating constant terms gives us

$$8\chi(Y)\pi^2 = 6A(Y) \quad (1.11)$$

which gives us the formula

$$A(Y) = \frac{4\pi^2}{3}.$$

In this thesis we dedicate a chapter to computing the renormalized area of a properly embedded 4-dimensional minimal hypersurface Y of a Poincaré-Einstein manifold, X and we obtain an explicit formula.

Theorem 1.2. Let (Y, h_+) be a 4-dimensional properly embedded minimal hypersurface of a Poincaré-Einstein manifold (X, g_+) . Then the renormalized area of Y , $A(Y)$, is given by

$$6A(Y) = - \int_Y \frac{|W|_{h_+}^2}{4} dA_{h_+} + \int_Y \frac{|E|_{h_+}^2}{2} dA_+ - \int_Y \frac{|B|^4}{24} dA_+ \quad (1.12)$$

$$- f.p. \int_{Y_\epsilon} |B|^2 dA_+ + 8\pi^2 \chi(Y)$$

Where (once r is chosen so that ∂Y is minimal in ∂X)

$$f.p. \int_{Y_\epsilon} |B|^2 dA_+ = \lim_{\epsilon \rightarrow 0} \left[\int_{Y_\epsilon} |B|^2 dA_+ - \int_\Sigma |II|_k^2 ds_k \epsilon^{-1} \right].$$

where B is the second fundamental form of Y , II is the second fundamental form of ∂Y in ∂X , E is the trace-free Ricci tensor of Y and W is the Weyl tensor of Y .

Recently there has been work in answering the question: can the Einstein condition in the above results be relaxed to constant scalar curvature? A useful tool for analysing this question comes from a classical problem in geometric analysis: The Yamabe problem. In 1974 Loewner and Nirenberg [26] introduced what became known as the singular Yamabe problem. We can formulate this as follows:

Let (\bar{X}^{n+1}, \bar{g}) be a smooth compact Riemannian manifold with boundary. Then there exists a defining function u (which is differentiable up to order $n + 1$ at least) for ∂X such that the metric $u^{-2}\bar{g}$ on $\overset{\circ}{X}$ has constant scalar curvature. This metric on $\overset{\circ}{X}$ is called a Singular Yamabe metric.

Examined by Graham in [17] and by Gover and Waldron in [15] it turns out that, choosing a special defining function, the constant term of the expansion in ϵ of $\text{Vol}_{u^{-2}\bar{g}}(r > \epsilon)$ will depend on this choice. However, the expansion has a log term coefficient that is given by an integral of local curvature terms on M and is an invariant of $(M, [\bar{g}|_M])$, it has since been referred to in the literature as the Graham-Witten

energy. Both of the above mentioned papers answered a question posed by Gover and Waldron in [14] and showed that this coefficient actually generalises the Willmore energy of a surface to higher dimensions in a certain sense. Recently, in [15], Gover and Waldron showed, among other things, that there exists an analogous invariant on minimal submanifolds of Singular Yamabe manifolds. In Chapter 3 we will look at an alternate approach to showing that the Graham-Witten energy of a properly embedded submanifold in this setting is an invariant. We also compute this invariant for a low dimensional case.

In Chapter 4 we will be considering a 3-dimensional properly embedded minimal hypersurface of a Poincaré-Einstein space, Y . Y splits X into two pieces, one of which we will label X^+ . We will be concerned, not with the renormalized area of Y , but with the renormalized *volume* V_+^+ of X^+ , which we may define as the constant term in the expansion $\text{Vol}_{g_+}(\{x \in X^+ : r(x) > \varepsilon\})$, with r a geodesic (i.e. special) defining function. It is not immediately obvious that this quantity is independent of the choice of r : the proof in the global case depends strongly on the product decomposition $[0, \delta)_r \times M$ of a collar neighborhood of M in X , but generically there is no such decomposition of a collar neighborhood of $M^+ = M \cap X^+$ in X^+ . One could prove using rather more elaborate versions of the arguments of [16] that V_+^+ is invariant in this context, but our interest is in a Gauss-Bonnet formula, and so we approach the result by a somewhat different path, as described below.

We note that renormalized volume of regions in AH spaces divided in two by hypersurfaces was considered in [15] using quite different techniques. The authors showed that a volume could be defined in quite general circumstances – in particular, not assuming the Einstein or minimality conditions – but did not show that it is well-defined independent of all choices in the four-dimensional Einstein case.

Let $N \subset \mathring{X}$ be any hypersurface, and let $h = g_+|_{TN}$ be the induced metric on N .

Define an extrinsic curvature quantity \mathcal{C}_N on N by the formula

$$\mathcal{C}_N = \mathring{L}_N^{\alpha\beta} R_{\alpha\beta}^{g_+} - \mathring{L}_N^{\alpha\beta} R_{\alpha\beta}^h + \frac{1}{3} H_N |\mathring{L}_N|_h^2 - \frac{1}{3} \text{tr}_h \mathring{L}_N^3.$$

Here L_N is the second fundamental form of N and \mathring{L}_N its tracefree part, while $H_N = h^{\alpha\beta} L_{\alpha\beta}$ is its mean curvature. The curvature terms appearing are the Ricci tensors of the respective metrics, and α, β are indices on TN . It is easy to show (and will be shown within) that \mathcal{C}_N is a pointwise conformal invariant of weight -3 ; indeed, in the notation of [6], $\mathcal{C}_N = -\frac{1}{2}\mathcal{L}_4 - \frac{1}{3}\mathcal{L}_5$.

The first main result of Chapter 4 is the following.

Theorem 1.3. Let (X^4, g_+) be an asymptotically hyperbolic space satisfying the Einstein condition $\text{Ric}(g_+) = -3g_+$, with conformal infinity $(M^3, [\bar{h}])$. Let Y^3 be a complete minimal hypersurface dividing X into two pieces X^+ and X^- such that $X^+ \cap X^- = Y$ and such that $Y \cap M = \Sigma^2 \neq \emptyset$. Let r be a fixed geodesic defining function for M , and let V_+^+ be the constant term in the expansion

$$\text{vol}_{g_+}(\{x \in X^+ : r(x) > \varepsilon\}) = c_0 \varepsilon^{-3} + c_2 \varepsilon^{-1} + V_+^+ + o(1).$$

Let $\tilde{h} = g_+|_{TY}$. Then

$$\pi^2(4\chi(X^+) - \chi(\Sigma^2)) = 3V_+^+ + \frac{1}{8} \int_{\mathring{X}^+} |W_{g_+}|_{g_+}^2 dv_{g_+} + \int_Y \mathcal{C}_Y dv_{\tilde{h}}. \quad (1.13)$$

One then immediately obtains

Corollary 1.4. The renormalized volume V_+^+ is independent of the choice of geodesic defining function r , and it satisfies (1.13).

A natural question about the newly defined renormalized volume is how it changes if Y is varied through minimal surfaces in X . The second main result of chapter 4 is as follows.

Theorem 1.5. Let $X, M, Y, \Sigma, X^+, g_+, \bar{h}$, and V_+^+ be as in Theorem 1.3. Suppose that $\mathcal{F} : (-\varepsilon, \varepsilon)_t \times Y \rightarrow X$ is a C^3 variation of Y through minimal surfaces in X , so that $\mathcal{F}(t, \Sigma) \subset M$ for all t . Let $\tilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon, \varepsilon) \times \Sigma}$. Define $\tilde{f} \in C^\infty(\Sigma)$ by $\tilde{f} = \left\langle \frac{d}{dt} \Big|_{t=0} \tilde{\mathcal{F}}, \bar{\nu}_M \right\rangle$, where $\bar{\nu}_M$ is the inward-pointing normal vector to Σ in

M^+ with respect to \bar{h} . Define $f \in C^\infty(\mathring{Y})$ by $f = \langle \frac{d}{dt} \Big|_{t=0} \mathcal{F}, \mu_Y \rangle_{g_+}$, where μ_Y is the (X^+, g_+) -inward unit normal vector along Y . Let r be a geodesic defining function near M . Then

$$\frac{d}{dt} \Big|_{t=0} V_+^+ = \frac{1}{2} \oint_{\Sigma} \tilde{f} g^{(3)}(\bar{\nu}_M, \bar{\nu}_M) dv_{\bar{k}} - \frac{1}{3} f.p. \int_{\mathring{Y}} f |\mathring{L}_Y|_h^2 dv_{\bar{h}},$$

where $\bar{k} = \bar{h}|_{T\Sigma}$, $\tilde{h} = g_+|_{TY}$, $g^{(3)}$ is the nonlocal term in the expansion in r of g_+ , and $f.p. \int_{\mathring{Y}} f |\mathring{L}_Y|_h^2 dv_{\bar{h}}$ denotes the zeroth-order part, in ε , of $\int_{Y \cap \{r > \varepsilon\}} f |\mathring{L}_Y|_h^2 dv_{\bar{h}}$.

For discussion of the nonlocal term $g^{(3)}$, see (4.3).

The above theorem is stated for variations of Y through minimal surfaces, whose existence in general we do not assert. However, one can broaden the definition of V_+^+ to any dividing hypersurface by using (1.13). In that case, Theorem 1.5 remains valid for any variation of Y that preserves minimality to first order; see section 4.4, where we also explain why C^3 -regularity of such a variation is in general optimal.

In considering the existence problem for the variation of Y , the required boundary data would be the induced variation of Σ , so another natural question is whether the derivative \dot{V}_+^+ only depends on the induced normal variation \tilde{f} . For example, suppose there are two variations of Y through minimal surfaces that induce the same variation of Σ ; do the derivatives of V_+^+ with respect to these variations agree? The answer is yes, at least if $|\mathring{L}_Y|_h^2 \leq 3$ everywhere; see Lemma 4.1.

These theorems may be interpreted physically within the AdS/CFT correspondence of high-energy and condensed matter physics. To do so, we assume that $(M^3, [\bar{h}])$ is a spacelike slice within a static four-dimensional conformal field theory Ω ; and that (X^4, g_+) is an Einstein spacelike slice within a static asymptotically anti-de-Sitter Einstein five-dimensional spacetime Z with conformal infinity Ω . The surface Σ is then known as an entangling surface between M^+ and M^- , and Y is the so-called Ryu-Takayanagi surface extending Σ . According to the ‘‘volume = complexity’’ conjecture ([28, 5, 11, 4, 23]), then, V_+^+ encodes the algorithmic complexity of the quantum state of M^+ . The above theorems can then be interpreted as giving

formulae for this complexity and for its derivative as the entangling surface Σ is varied continuously, so long as Y also varies continuously. (As demonstrated in [5], the latter will not always be the case.)

The assumption that X and its five-dimensional ambient Lorentzian manifold Z are both Einstein, of course, is rather restrictive. In general physical situations, one might expect that the Ricci tensor of X includes some extrinsic terms. But even if so, these would have well-defined asymptotics due to the asymptotically AdS condition on Z , and it would be straightforward, if tedious, to carry out our calculation the same way in that context.

In section 4.2, we introduce our setting and notation. In section 4.3, we prove Theorem 1.3; and in section 4.4, we prove Theorem 1.5. ¹

¹The notation changes slightly in each chapter but this is outlined as we go.

CHAPTER 2

RENORMALIZED AREA OF A MINIMAL HYPERSURFACE IN A 5-DIMENSIONAL POINCARÉ-EINSTEIN SPACE

Let (X^5, g_+) be a Poincaré-Einstein manifold, let Y^4 be a minimal hypersurface that extends smoothly (enough) to the boundary of X i.e. $\partial Y \subset \partial X$ and ∂Y is smooth (enough). Let g_∞ be a representative for the conformal infinity of g_+ such that ∂Y is minimal in ∂X with respect to g_∞ and let r be a special defining function for ∂X such that $\bar{g} := r^2 g_+$ satisfies $\bar{g}|_{\partial X} = g_\infty$. This r exists since it can be obtained by solving to the following equation: given $g_0 \in [g_+]_\infty$ let s be a function on M such that

$$(H(g_0|_{\partial Y}) - (\nabla_{g_0} \log s)^\perp) = 0$$

where H is mean curvature of ∂Y as a submanifold of M . By the conformal transformation law for mean curvature, $s^2 g_0$ would have 0 mean curvature on ∂Y as a submanifold of M . Then by Graham's lemma there exists a unique special defining function r giving us

$$r^2 g_+|_M = s^2 g_0$$

We put a bar over a metric dependant quantity to indicate that it is taken with respect to the compactified metric. Some objects we will decorate with a subscript

or superscript of the the manifold they are taken with respect to; in this context

$$X = (X, g_+),$$

$$\bar{X} = (\bar{X}, \bar{g}),$$

$$Y = (Y, h_+),$$

$$\bar{Y} = (\bar{Y}, \bar{h}).$$

In calculations involving indices we use $g_{ij} := (g_+)_{ij}$. Now if we take coordinates $\{x^\alpha\}$ on Y and extend the chart induced basis to X by the unit normal μ_Y , letting the index N correspond to μ_Y and letting latin indices run over the submanifold and μ_Y , we see

$$\begin{aligned} -20 &= R_X \\ &= g^{ij} g^{kl} R_{ikjl} \\ &= 2g^{ij} R_{iNjN}^X + g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\beta\delta}^X \\ &= 2Ric_X(\mu_Y, \mu_Y) + g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\beta\delta}^X \\ &= -8g(\mu_Y, \mu_Y) + g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\beta\delta}^X \\ &= -8 + g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\beta\delta}^X, \end{aligned}$$

but the Gauss equation gives us

$$R_{\alpha\gamma\beta\delta}^X = R_{\alpha\gamma\beta\delta}^Y + B_{\alpha\delta}B_{\gamma\beta} - B_{\alpha\beta}B_{\gamma\delta}. \quad (2.1)$$

Therefore,

$$\begin{aligned}
-20 &= -8 + g^{\alpha\beta} g^{\gamma\delta} R_{\alpha\gamma\beta\delta} \\
&= -8 + g^{\alpha\beta} g^{\gamma\delta} [R_{\alpha\gamma\beta\delta}^Y + B_{\alpha\delta} B_{\gamma\beta} - B_{\alpha\beta} B_{\gamma\delta}] \\
&= -8 + R_Y + |B|_{h_+}^2 - [tr_h B]^2 \\
&= -8 + R_Y + |B|_{h_+}^2.
\end{aligned}$$

This give us

$$-12 = R_Y + |B|_h^2, \quad (2.2)$$

isolating the curvature term and squaring both sides gives us

$$(R_Y)^2 = |B|_h^4 + 24|B|_h^2 + 144. \quad (2.3)$$

By [8] Chern-Gauss-Bonnet gives us

$$8\pi^2 \chi(Y_\epsilon) = \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4} + 4\sigma_2(h_+^{-1}P_{h_+})dA_{h_+} + \int_{r=\epsilon} S_r ds_{k_\epsilon}, \quad (2.4)$$

where

$$S_r = R^Y H_r - 2Ric^Y(\nu_r, \nu_r)H_r - 2R_Y{}^d{}_{adb}L_r^{ab} + \frac{2}{3}H_r^3 - 2H_r|L_r|_{h_+}^2 + \frac{4}{3}tr(L_r^3), \quad (2.5)$$

where $\{a, b, c, d\}$ correspond to coordinates on ∂Y . $\sigma_2(h_+^{-1}P_{h_+})(p)$ is the second symmetric function of the endomorphism of $T_p Y$ given by $V \rightarrow P_\alpha^\beta V^\alpha$ and where P_{h_+} is the Schouten tensor of h_+ ;

$$P_{h_+} = \frac{1}{2} \left(Ric^{h_+} - \frac{R^{h_+}}{6} h_+ \right).$$

Where L_ϵ is the second fundamental form of Σ_ϵ within Y_ϵ with respect to the inward

pointing unit-normal and H_ϵ is the trace of L_ϵ with respect to k_ϵ . We will use (2.3) and (2.4) along with

$$|Ric|_{h_+}^2 = |E|_{h_+}^2 + \frac{R_Y^2}{4}, \quad (2.6)$$

and

$$4\sigma_2(h_+^{-1}P_{h_+}) = -\frac{|Ric_Y|_{h_+}^2}{2} + \frac{R_Y^2}{6} \quad (2.7)$$

in the pursuit of a formula for the renormalized area of Y ; $A(Y)$.

Claim 2.1.

$$|d\hat{r}|_{\hat{h}}^2 = 1 + O(r^5), \quad (2.8)$$

$$\bar{S}_r = O(r), \quad (2.9)$$

$$\partial_{\hat{r}} R^{\bar{Y}}(\bar{\nu}_r, \bar{\nu}_r)|_{r=0} = 0, \quad (2.10)$$

$$\partial_{\hat{r}} R^{\bar{Y}}|_{r=0} = 0, \quad (2.11)$$

$$\int_Y |E|_{h_+}^2 dA_+ < \infty, \quad (2.12)$$

$$\int_Y |B|_{h_+}^4 dA_+ < \infty. \quad (2.13)$$

Proof. Now we need to describe specific coordinates that will be used throughout the rest of this paper. Let $\{x^\mu, x^\lambda\}$ be coordinates on a neighbourhood U of M that intersects Σ non-trivially, in particular choose coordinates such that $\Sigma \cap U = \{x^4 = 0\}$ and $|\partial_4|_{g_\infty} \equiv 1$. Extend these coordinates to a local system on X via the flow of $\nabla_{\bar{g}} r$. Because r is a special defining function for M we know that this coordinate extension is $\{x^i, r\}$. We let $0 \leq i, j \leq 4$, $1 \leq \mu, \lambda \leq 4$ and $1 \leq a, b \leq 3$ we let r correspond to the index 0 and $0 \leq \alpha, \beta \leq 3$. Now we let $z: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a function such that locally $Y = \{x^4 = z(x^a, r)\}$, this induces a natural local parametrization for Y ,

$$\phi(x^a, r) := (x^a, z(x^a, r), r). \quad (2.14)$$

Observations: Let $\hat{r} = r|_Y$, $= (\phi^{-1})^0$ then $\phi_*(\partial_r) = \partial_{\hat{r}}$. Also $\{x^a|_{r=\epsilon}\}$ give coordinates

on Σ_ϵ via $\phi|_{r=\epsilon}$. ϕ induces coordinates $\{x^\alpha|_Y\}$ on Y , the induced coordinate basis on Y is given by $\phi_*(\partial_\alpha)$, from now on, when we write a Greek index on an induced metric, we will mean this basis i.e.

$$h_{\alpha\beta} := g(\phi_*(\partial_\alpha), \phi_*(\partial_\beta)). \quad (2.15)$$

Now we make a note about notation, whenever an index attached to an object which depends on the ambient metric appears with a hat, i.e. $g_{\hat{\alpha}\hat{\beta}}$ we mean to refer to the object viewed with respect to the basis $\{\phi_*(\partial_\alpha) = \partial_{\hat{\alpha}}, \phi_*(\partial_r) = \partial_{\hat{r}}\}$; i.e. $g_{\hat{\alpha}\hat{\beta}} = h_{\alpha\beta}$. We let the indices N, \bar{N} correspond to $\mu_Y, \bar{\mu}_Y$ and we let n, \bar{n} correspond to $\nu_r, \bar{\nu}_r$. It is straightforward to compute

$$\phi_*(\partial_\alpha) = \partial_\alpha + z_\alpha \partial_4$$

$$h_{\alpha\beta} = g_{\hat{\alpha}\hat{\beta}} = g_{\alpha\beta} + g_{4\alpha}z_\beta + g_{4\beta}z_\alpha + g_{44}z_\alpha z_\beta.$$

From the minimal surface equation (see [21]), one can derive

$$z(x^a, r) = \frac{H_{(\Sigma \hookrightarrow M)}(x^a)}{6} r^2 + z^{(4)}(x^a) \frac{r^4}{4} + O(r^5),$$

where $H_{(\Sigma \hookrightarrow M)}(x^a)$ is the mean curvature of Σ in M . Our choice of special defining function gives $H_{(\Sigma \hookrightarrow M)} \equiv 0$ and therefore

$$z = O(r^4).$$

It follows that

$$\bar{h}_{\alpha\beta} = \bar{g}_{\alpha\beta} + O(r^5) \quad (2.16)$$

and

$$\bar{h}^{\alpha\beta} = \bar{g}^{\alpha\beta} + O(r^5). \quad (2.17)$$

Now,

$$|dr|_{\bar{h}}^2 = \bar{h}^{\alpha\beta} \hat{r}_{\alpha} \hat{r}_{\beta} = \bar{h}^{rr} = 1 + O(r^5). \quad (2.18)$$

So we get the first part of our claim. Now if we look at (2.5) with respect to the compactified metric, we have

$$\bar{S}_r = R^{\bar{Y}} \bar{H}_r - 2\text{Ric}^{\bar{Y}}(\bar{\nu}_r, \bar{\nu}_r) \bar{H}_r - 2R_{\bar{Y}^d}{}^{ab} \bar{L}_r^{ab} + \frac{2}{3} \bar{H}_r^3 - 2\bar{H}_r |\bar{L}_r|_{\bar{h}}^2 + \frac{4}{3} \text{tr}(\bar{L}_r^3). \quad (2.19)$$

All of the terms involving \bar{H}^r and \bar{L}^r vanish on the boundary since the second fundamental form of Σ_ϵ within Y vanishes, to see this consider the following: Recall that $\bar{\nu}_\epsilon$ is the inward pointing unit-normal to ∂Y_ϵ within Y_ϵ . Note: $\hat{r} = r|_Y$.

$$\bar{\nu}_\epsilon = \frac{\nabla_{\bar{h}} \hat{r}}{|\nabla_{\bar{h}} \hat{r}|} = \frac{\nabla_{\bar{h}} \hat{r}}{\sqrt{\bar{h}^{rr}}} = \partial_{\hat{r}} + O^\alpha(r^5) \partial_\alpha$$

$$\begin{aligned} \bar{L}_{ab}^\epsilon &= \bar{h}(\nabla_{\phi_*(\partial_a)}^{\bar{Y}} \phi_*(\partial_b), \bar{\nu}_\epsilon) \\ &= \Gamma_{ab}^\alpha(\bar{h}) \bar{h}_{\alpha r} 1 + O(r^5) \\ &= O(r) \end{aligned} \quad (2.20)$$

$$\Gamma_{ab}^r(\bar{h}) = \frac{\bar{h}^{rd}}{2} \left\{ \partial_{\hat{a}} \bar{h}_{bd} + \partial_{\hat{b}} \bar{h}_{ad} - \partial_{\hat{d}} \bar{h}_{ab} \right\} + \frac{\bar{h}^{rr}}{2} \left\{ \partial_{\hat{a}} \bar{h}_{br} + \partial_{\hat{b}} \bar{h}_{ar} - \partial_{\hat{r}} \bar{h}_{ab} \right\} = O(r) \quad (2.21)$$

hence by (2.21) and (2.18)

$$\bar{L}_{ab}^r = O(r). \quad (2.22)$$

This implies that $\bar{S}_r = O(r)$. Next we will compute μ_Y for use later.

$$\mu_Y = \frac{\nabla_{g_+} w}{|\nabla_{g_+} w|_{g_+}} \quad (2.23)$$

where $w = x^4 - z(x^a, r)$.

$$\nabla_{g_+} w = g^{ij} \partial_i w \partial_j = r^2 \partial_4 - r^2 z_r \partial_r - [g^{a\mu} \partial_a z + O^\mu(r^4)] \partial_\mu \quad (2.24)$$

$$|\nabla_{g_+} w|_{g_+} = r(1 + O(r^2)), \quad (2.25)$$

it follows that

$$\frac{\nabla_{g_+} w}{|\nabla_{g_+} w|_{g_+}} = \frac{r \partial_4}{1 + O(r^2)} - \frac{r z_r \partial_r}{1 + O(r^2)} - \frac{[r \bar{g}^{a\mu} \partial_a z + O^\mu(r^3)] \partial_\mu}{1 + O(r^2)} \quad (2.26)$$

Therefore

$$\mu_Y = r(1 + O(r^2)) \partial_4 + O(r^4) \partial_r + O^a(r^3) \partial_a \quad (2.27)$$

$$\bar{\mu}_Y = (1 + O(r^2)) \partial_4 + O(r^3) \partial_r + O^a(r^2) \partial_a. \quad (2.28)$$

Note that $\bar{\mu}_Y = \partial_4 \in TM$ when $r = 0$. Now we apply the Gauss equation to the compactified Riemann tensors

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}^{\bar{X}} = R_{\alpha r \beta r}^{\bar{Y}} + \bar{B}_{\alpha r} \bar{B}_{\beta r} - \bar{B}_{\alpha\beta} \bar{B}_{rr}.$$

This then gives us

$$R_{\hat{r}\hat{r}}^{\bar{X}} - R_{\bar{N}\hat{r}\bar{N}\hat{r}}^{\bar{X}} = R_{rr}^{\bar{Y}} + \bar{B}_{rr}^2 - \bar{H}_r \bar{B}_{rr},$$

but

$$\bar{B}_{\alpha\beta} = -\frac{1}{2} \bar{\mu}_Y(\bar{g}_{\hat{\alpha}\hat{\beta}}).$$

Hence

$$\bar{B}_{rr} = -\frac{1}{2}\bar{\mu}_Y(1 + O(r^5)) = O(r^2).$$

$$R_{\bar{N}\hat{r}\bar{N}}^{\bar{X}\hat{\alpha}} = \partial_{\hat{r}}\bar{\Gamma}_{\bar{N}\bar{N}}^{\hat{\alpha}} - \partial_{\bar{N}}\bar{\Gamma}_{\hat{r}\bar{N}}^{\hat{\alpha}} + \bar{\Gamma}_{\bar{N}\bar{N}}^{\hat{\beta}}\bar{\Gamma}_{\hat{\beta}\hat{r}}^{\hat{\alpha}} - \bar{\Gamma}_{\hat{r}\bar{N}}^{\hat{\gamma}}\bar{\Gamma}_{\hat{\gamma}\bar{N}}^{\hat{\alpha}} = O(r^2) \quad (2.29)$$

The above equality holds since

$$\bar{\Gamma}_{\bar{N}\bar{N}}^{\hat{\alpha}} = \frac{\bar{g}^{\hat{\alpha}\hat{\beta}}}{2}\{2\partial_{\bar{N}}\bar{g}_{\bar{N}\hat{\beta}} - \partial_{\hat{\beta}}\bar{g}_{\bar{N}\bar{N}}\} = 0 \quad (2.30)$$

$$\bar{\Gamma}_{\hat{r}\bar{N}}^{\hat{\alpha}} = \frac{\bar{g}^{\hat{\alpha}\hat{\beta}}}{2}\{\partial_{\hat{r}}\bar{g}_{\bar{N}\hat{\beta}} + \partial_{\bar{N}}\bar{g}_{\hat{r}\hat{\beta}} - \partial_{\hat{\beta}}\bar{g}_{\hat{r}\bar{N}}\} = O(r^5) \quad (2.31)$$

$$\bar{\Gamma}_{\bar{N}\hat{r}}^{\hat{\alpha}} = \frac{\bar{g}^{\hat{\alpha}\hat{\gamma}}}{2}\{\partial_{\bar{N}}\bar{g}_{\hat{r}\hat{\gamma}} + \partial_{\hat{r}}\bar{g}_{\bar{N}\hat{\gamma}} - \partial_{\hat{\gamma}}\bar{g}_{\bar{N}\hat{r}}\} = O(r^2). \quad (2.32)$$

and the first derivative of \bar{g} and \bar{g}^{-1} with respect to r vanish on M . Hence,

$$R_{\bar{N}\hat{r}\bar{N}\hat{r}}^{\bar{X}} = O(r^2). \quad (2.33)$$

Therefore

$$R_{\hat{r}\hat{r}}^{\bar{X}} = R_{rr}^{\bar{Y}} + O(r^2), \quad (2.34)$$

but

$$R_{\hat{r}\hat{r}}^{\bar{X}} = Ric^{\bar{X}}(\partial_r + z_r\partial_4, \partial_r + z_r\partial_r) = R_{rr}^{\bar{X}} + O(r^2).$$

So we may write

$$R_{rr}^{\bar{Y}} = R_{rr}^{\bar{X}} + O(r^2)$$

but we know from [20] that conformally transforming the Einstein condition

$$Ric(g_+) + 4g_+ = 0$$

gives

$$r\bar{g}''_{ij} - 3\bar{g}'_{ij} - \bar{g}^{kl}\bar{g}'_{kl}\bar{g}_{ij} - r\bar{g}^{kl}\bar{g}'_{kl}\bar{g}'_{ij} - 2r\text{Ric}_{ij}^{\bar{X}} = 0$$

so we can say that the first derivative of $R_{ij}^{\bar{X}}$ vanishes at $r = 0$ and therefore the first derivative of $R^{\bar{X}}$ does also. Combined with (2.34); this proves (3) and (4). Now we prove (5): we first compute a Christofel symbol that we will use later on.

$$\Gamma_{\alpha\gamma}^r(\bar{h}) = \frac{\bar{h}^{r\beta}}{2} \{ \partial_{\bar{\alpha}} \bar{h}_{\gamma\beta} + \partial_{\bar{\gamma}} \bar{h}_{\alpha\beta} - \partial_{\bar{\beta}} \bar{h}_{\alpha\gamma} \} = O(r) \quad (2.35)$$

$|E(h)|_{h_+}^2 dA_{h_+} = |E(h)|_{\bar{h}}^2 dA_{\bar{h}}$, so we need only to show that $|E(h)|_{\bar{h}} \in L^\infty(Y)$, to see this, consider the following:

Let $\tilde{E}_{\alpha\beta}(h) := \text{Ric}_{\alpha\beta}(h) + 3h_{\alpha\beta}$, then

$$\begin{aligned} \tilde{E}_{\alpha\beta}(h) &= \text{Ric}_{\alpha\beta}(h) + 3h_{\alpha\beta} \\ &= \text{Ric}_{\alpha\beta}(r^{-2}\bar{h}) + 3r^{-2}\bar{h}_{\alpha\beta} \\ &= \text{Ric}_{\alpha\beta}(\bar{h}) - 2(\nabla_{\bar{\alpha}}^{\bar{h}} \nabla_{\bar{\beta}}^{\bar{h}}(-\log \hat{r}) - \nabla_{\bar{\alpha}}^{\bar{h}} \log \hat{r} \nabla_{\bar{\beta}}^{\bar{h}} \log \hat{r}) \\ &\quad - (\Delta^{\bar{h}} \log \hat{r} + |d \log \hat{r}|_{\bar{h}}^2) \bar{h}_{\alpha\beta} + 3\hat{r}^{-2} \bar{h}_{\alpha\beta} \\ &= R_{\alpha\beta}(\bar{h}) + 2\left(\nabla_{\bar{\alpha}}^{\bar{h}} \frac{\partial_{\bar{\beta}} \hat{r}}{r} - \frac{\partial_{\bar{\alpha}} \hat{r} \partial_{\bar{\beta}} \hat{r}}{r^2}\right) \\ &\quad - \left(\bar{h}^{\delta\gamma} \nabla_{\bar{\delta}}^{\bar{h}} \frac{\partial_{\bar{\gamma}} \hat{r}}{r} + \bar{h}^{\delta\gamma} \frac{\partial_{\bar{\delta}} \hat{r} \partial_{\bar{\gamma}} \hat{r}}{r^2}\right) \bar{h}_{\alpha\beta} + 3r^{-2} \bar{h}_{\alpha\beta} \\ &= R_{\alpha\beta}(\bar{h}) + 2\left(\frac{\partial_{\bar{\alpha}} \partial_{\bar{\beta}} \hat{r}}{r} - \Gamma_{\alpha\beta}^{\delta}(\bar{h}) \frac{\partial_{\bar{\delta}} \hat{r}}{r}\right) \\ &\quad - \left(\bar{h}^{\delta\gamma} \frac{\partial_{\bar{\delta}} \partial_{\bar{\gamma}} \hat{r}}{r} - \bar{h}^{\delta\gamma} \Gamma_{\delta\gamma}^r(\bar{h}) \frac{1}{r} - \bar{h}^{\delta\gamma} \frac{\partial_{\bar{\delta}} \hat{r} \partial_{\bar{\gamma}} \hat{r}}{r^2} + \bar{h}^{\delta\gamma} \frac{\partial_{\bar{\delta}} \hat{r} \partial_{\bar{\gamma}} \hat{r}}{r^2}\right) \bar{h}_{\alpha\beta} + 3r^{-2} \bar{h}_{\alpha\beta} \\ &= R_{\alpha\beta}(\bar{h}) - \frac{2\Gamma_{\alpha\beta}^r(\bar{h})}{r} - \frac{3\bar{h}^{rr} \bar{h}_{\alpha\beta}}{r^2} + \frac{\Gamma_{\delta\gamma}^r(\bar{h}) \bar{h}^{\delta\gamma} \bar{h}_{\alpha\beta}}{r} + \frac{3\bar{h}_{\alpha\beta}}{r^2} \end{aligned}$$

but $Ric_{\alpha\beta}(\bar{h}) = O(1)$ and by (2.21)

$$\frac{\Gamma_{\alpha\beta}^r}{r}(\bar{h}) = O(1).$$

So

$$\tilde{E}_{\alpha\beta}(h_+) = O(1) - \frac{3\bar{h}^{rr}\bar{h}_{\alpha\beta}}{r^2} + \frac{3\bar{h}_{\alpha\beta}}{r^2} \quad (2.36)$$

but also

$$\begin{aligned} \frac{3\bar{h}^{rr}\bar{h}_{\alpha\beta}}{r^2} &= \frac{3(1 + O(r^5))\bar{h}_{\alpha\beta}}{r^2} \\ &= \frac{3\bar{h}_{\alpha\beta}}{r^2} + O(r^3). \end{aligned} \quad (2.37)$$

Applying (3.1) to (2.36) gives us

$$\tilde{E}_{\alpha\beta}(h_+) = O(1). \quad (2.38)$$

Now observe that $tf_{h_+}\tilde{E}_{\alpha\beta}(h) = E_{\alpha\beta}(h)$, so we get that $E_{\alpha\beta}(h) = O(1)$, which gives us (5). We know by the conformal transformation law for the second fundamental form that

$$B = \frac{1}{r}\bar{B} + \frac{\bar{\mu}_Y(r)}{r^2}\bar{g}. \quad (2.39)$$

Therefore

$$\begin{aligned} |B|_{h_+}^2 &= \left[\bar{B}_{\alpha\beta} + \frac{\bar{\mu}_Y(r)}{r}\bar{g}_{\hat{\alpha}\hat{\beta}}\right] \left[\bar{B}_{\gamma\delta} + \frac{\bar{\mu}_Y(r)}{r}\bar{g}_{\hat{\gamma}\hat{\delta}}\right] \frac{h^{\alpha\gamma}h^{\beta\delta}}{r^2} \\ &= r^2|\bar{B}|_{\bar{h}}^2 + O(r^3). \end{aligned}$$

It is clear then that $\int_Y |B|_{h_+}^2 dA_+$ may diverge but $\int_Y |B|_{h_+}^4 dA_+ < \infty$. \square

Recall (2.4):

$$8\pi^2\chi(Y_\epsilon) = \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4} + 4\sigma_2(h^{-1}P)dA_{h_+} + \int_{r=\epsilon} Sds_{k_\epsilon}.$$

By the same principle applied to the compactified metric we get:

$$8\pi^2\chi(Y_\epsilon) = \int_{r>\epsilon} \frac{|\bar{W}|_{\bar{h}}^2}{4} + 4\sigma_2(\bar{h}^{-1}\bar{P})dA_{\bar{h}} + \int_{r=\epsilon} \bar{S}ds_{\bar{k}_\epsilon}, \quad (2.40)$$

where

$$\bar{S}_r = R^{\bar{Y}}\bar{H}_r - 2R^{\bar{Y}}(\bar{\nu}_r, \bar{\nu}_r)\bar{H}_r - 2R_{\bar{Y}}{}^d{}_{ab}\bar{L}_r^{ab} + \frac{2}{3}\bar{H}_r^3 - 2\bar{H}_r|\bar{L}_r|^2 + \frac{4}{3}tr(\bar{L}_r^3) \quad (2.41)$$

and

$$4\sigma_2(\bar{h}^{-1}\bar{P}) = \frac{R_{\bar{Y}}^2}{6} - \frac{|Ric^{\bar{Y}}|_{\bar{h}}^2}{2}. \quad (2.42)$$

We will now use the following identity

$$|Ric|_{h_+}^2 = |E|_{h_+}^2 + \frac{R_Y^2}{4} \quad (2.43)$$

and the following formula:

Lemma 2.2.

$$4\sigma_2(\bar{h}^{-1}\bar{P}) = 4\hat{r}^{-4}\sigma_2(h^{-1}P) + 2\nabla_{\hat{h}}^{\hat{\alpha}}(\hat{r}^{-3}|d\hat{r}|_{\hat{h}}^2\hat{r}_{\hat{\alpha}} - \frac{\hat{r}_{\hat{\alpha}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\hat{\beta}}\hat{r}^{\hat{\beta}} + \hat{r}^{-1}R_{\hat{\alpha}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{\alpha}}) \quad (2.44)$$

(The Greek index here corresponds to covariant differentiation and $\Delta_{\bar{h}} = \nabla_{\hat{\alpha}}^{\bar{h}}\nabla_{\hat{\alpha}}^{\bar{h}}$).

Proof.

$$\sigma_2(h^{-1}P) = \frac{1}{2}([tr_h P]^2 - |P|_{h_+}^2).$$

Let $\varphi = -\log \hat{r}$. Then the conformal transformation formula for P gives us

$$P_{\alpha\beta} = \bar{P}_{\alpha\beta} - \varphi_{\alpha\beta} + \varphi_{\alpha}\varphi_{\beta} - \frac{1}{2}|d\varphi|_{\frac{2}{h}\bar{h}}^2$$

$$P_{\alpha\beta} = \bar{P}_{\alpha\beta} - \frac{r_{\hat{\alpha}\hat{\beta}}}{\hat{r}} - \frac{1}{2\hat{r}^2}|d\hat{r}|_{\frac{2}{h}\bar{h}}^2.$$

Taking the trace of both sides gives

$$tr_h P = \hat{r}^2 tr_{\bar{h}} \bar{P} - \hat{r} \Delta_{\bar{h}} \hat{r} - 2\hat{r}^2 |d\varphi|_{\frac{2}{h}}^2.$$

This allows us to compute the conformal change of $\sigma_2(h^{-1}P)$

$$\begin{aligned} 4\sigma_2(h^{-1}P) &= 2([tr_h P]^2 - |P|_{h^+}^2) = \\ &= 2([\hat{r}^2 [tr_{\bar{h}} \bar{P}] - \hat{r} \Delta_{\bar{h}} \hat{r} - 2|d\hat{r}|_{\frac{2}{h}}^2]^2 \\ &\quad - \hat{r}^4 [\bar{P}_{\alpha\beta} - \frac{\hat{r}_{\alpha\beta}}{\hat{r}} - \frac{1}{2\hat{r}^2} |d\hat{r}|_{\frac{2}{h}\bar{h}}^2] [\bar{P}^{\alpha\beta} - \frac{\hat{r}^{\alpha\beta}}{\hat{r}} - \frac{1}{2\hat{r}^2} |d\hat{r}|_{\frac{2}{h}\bar{h}}^2]) \\ &= 2([\hat{r}^4 [tr_{\bar{h}} \bar{P}]^2 - 2\hat{r}^3 tr_{\bar{h}} \bar{P} \Delta_{\bar{h}} \hat{r} - 4\hat{r}^2 tr_{\bar{h}} \bar{P} |d\hat{r}|_{\frac{2}{h}}^2 + \hat{r}^2 (\Delta_{\bar{h}} \hat{r})^2 + 4\hat{r} \Delta_{\bar{h}} \hat{r} |d\hat{r}|_{\frac{2}{h}}^2 + 4|d\hat{r}|_{\frac{4}{h}}^4] \\ &\quad - \hat{r}^4 [|P|^2 - \frac{2\hat{r}^{\alpha\beta}}{\hat{r}} \bar{P}_{\alpha\beta} - \frac{tr_{\bar{h}} \bar{P}}{\hat{r}^2} |d\hat{r}|_{\frac{2}{h}}^2 + \frac{1}{\hat{r}^2} |\nabla_{\bar{h}}^2 \hat{r}|_{\frac{2}{h}}^2 + \frac{\Delta_{\bar{h}} \hat{r}}{\hat{r}^3} |d\hat{r}|_{\frac{2}{h}}^2 + \frac{1}{\hat{r}^4} |d\hat{r}|_{\frac{4}{h}}^4]) \\ &= 2\hat{r}^4 [[tr_{\bar{h}} \bar{P}]^2 - |\bar{P}|_{\frac{2}{h}}^2] - 4\hat{r}^3 tr_{\bar{h}} \bar{P} \Delta_{\bar{h}} \hat{r} - 6\hat{r}^2 tr_{\bar{h}} \bar{P} |d\hat{r}|_{\frac{2}{h}}^2 \\ &\quad + 2\hat{r}^2 (\Delta_{\bar{h}} \hat{r})^2 + 6\hat{r} (\Delta_{\bar{h}} \hat{r}) |d\hat{r}|_{\frac{2}{h}}^2 + 8|d\hat{r}|_{\frac{4}{h}}^4 + 4\hat{r}^{\alpha\beta} \bar{P}_{\alpha\beta} \hat{r}^3 + 2\hat{r}^2 |\nabla_{\bar{h}}^2 \hat{r}|_{\frac{2}{h}}^2 - 2|d\hat{r}|_{\frac{4}{h}}^4] \\ &= 4\hat{r}^4 \sigma_2(\bar{h}^{-1} \bar{P}) + 6|d\hat{r}|_{\frac{2}{h}}^2 + 6\hat{r} (\Delta_{\bar{h}} \hat{r}) |d\hat{r}|_{\frac{2}{h}}^2 + 2\hat{r}^2 [(\Delta_{\bar{h}} \hat{r})^2 - |\nabla_{\bar{h}}^2 \hat{r}|_{\frac{2}{h}}^2 - 3tr_{\bar{h}} \bar{P} |d\hat{r}|_{\frac{2}{h}}^2] \\ &\quad - 4\hat{r}^3 [\bar{P}_{\alpha\beta} \hat{r}^{\alpha\beta} - tr_{\bar{h}} \bar{P} \Delta_{\bar{h}} \hat{r}]. \end{aligned}$$

Therefore

$$\begin{aligned}
4\hat{r}^{-4}\sigma_2(h^{-1}P) = & \tag{2.45} \\
& 4\sigma_2(\bar{h}^{-1}\bar{P}) + 6\hat{r}^{-4}|d\hat{r}|_{\bar{h}}^2 + 6\hat{r}^{-3}(\Delta_{\bar{h}}\hat{r})|d\hat{r}|_{\bar{h}}^2 \\
& + 2\hat{r}^{-2}[(\Delta_{\bar{h}}\hat{r})^2 - |\nabla_{\bar{h}}\hat{r}|_{\bar{h}}^2 - 3tr_{\bar{h}}\bar{P}|d\hat{r}|_{\bar{h}}^2] \\
& - 4\hat{r}^{-1}[\bar{P}_{\alpha\beta}\hat{r}^{\alpha\beta} - tr_{\bar{h}}\bar{P}\Delta_{\bar{h}}\hat{r}].
\end{aligned}$$

Expanding the second term of (2.44) now gives the lemma. \square

Plugging (2.43) into (2.42) gives us:

$$4\sigma_2(h^{-1}P) = -\frac{|E|^2}{2} + \frac{R_Y^2}{24} \tag{2.46}$$

plugging (2.46) into (2.44) gives

$$\begin{aligned}
4\sigma_2(\bar{h}^{-1}\bar{P}) = & \hat{r}^{-4}\left(-\frac{|E|_{\bar{h}}^2}{2} + \frac{R_Y^2}{24}\right) \tag{2.47} \\
& + 2\nabla_{\bar{h}}^{\hat{\alpha}}(\hat{r}^{-3}|d\hat{r}|_{\bar{h}}^2\hat{r}_{\hat{\alpha}} - \frac{\hat{r}_{\hat{\alpha}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\beta}\hat{r}^{\hat{\beta}} + \hat{r}^{-1}R_{\hat{\alpha}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{\alpha}})
\end{aligned}$$

and using the above equality to re-write $4\sigma_2(\bar{h}^{-1}\bar{P})$ in (2.40) we get:

$$\begin{aligned}
8\pi^2\chi(Y_\epsilon) = & \int_{r>\epsilon} \frac{|\bar{W}|_{\bar{h}}^2}{4}dA_{\bar{h}} + \int_{r>\epsilon} r^{-4}\left(\frac{R_Y^2}{24} - \frac{|E|^2}{2}\right)dA_{\bar{h}} \\
& + 2 \int_{r>\epsilon} \nabla_{\bar{h}}^{\hat{\alpha}}(\hat{r}^{-3}|d\hat{r}|_{\bar{h}}^2\hat{r}_{\hat{\alpha}} - \frac{\hat{r}_{\hat{\alpha}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\hat{\beta}}\hat{r}^{\hat{\beta}} + \hat{r}^{-1}R_{\hat{\alpha}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{\alpha}})dA_{\bar{h}} \\
& + \int_{r=\epsilon} \bar{S}_r ds_{\bar{k}_\epsilon}.
\end{aligned}$$

Using the conformal invariance of the Weyl term and using $r^{-4}dA_{\bar{h}} = dA_+$ yields:

$$\begin{aligned} 8\pi^2\chi(Y_\epsilon) &= \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4} - \frac{|E|_{h_+}^2}{2} + \frac{R_Y^2}{24}dA_{h_+} \\ &+ \int_{r>\epsilon} 2\bar{\nabla}^{\hat{\alpha}}(\hat{r}^{-3}|d\hat{r}|_{\bar{h}}^2\hat{r}_{\hat{\alpha}} - \frac{\hat{r}_{\hat{\alpha}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\hat{\beta}}\hat{r}^{\hat{\beta}} + \hat{r}^{-1}R_{\hat{\alpha}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{\alpha}})dA_{\bar{h}} \\ &+ \int_{r=\epsilon} \bar{S}_r ds_{\bar{k}_\epsilon}. \end{aligned}$$

Using integration by parts on the second integral yields

$$\begin{aligned} 8\pi^2\chi(Y_\epsilon) &= \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4} - \frac{|E|_{h_+}^2}{2} + \frac{R_Y^2}{24}dA_{h_+} \tag{2.48} \\ &- \int_{r=\epsilon} 2(\hat{r}^{-3}|d\hat{r}|_{\bar{h}}^2\hat{r}_{\hat{0}} - \frac{r_{\hat{0}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\hat{0}}\hat{r}^{\hat{\alpha}} + \hat{r}^{-1}R_{\hat{0}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{0}})ds_{\bar{k}(\epsilon)} \\ &+ O(r^2) + \int_{r=\epsilon} \bar{S}_r ds_{\bar{k}_\epsilon}. \end{aligned}$$

Letting

$$\mathcal{B}_r := \bar{S}_r - 2(\hat{r}^{-3}|d\hat{r}|_{\bar{h}}^2\hat{r}_{\hat{0}} - \frac{\hat{r}_{\hat{0}}\Delta_{\bar{h}}\hat{r}}{\hat{r}^2} + \hat{r}^{-2}\hat{r}_{\hat{\alpha}\hat{0}}\hat{r}^{\hat{\alpha}} + \hat{r}^{-1}R_{\hat{0}\hat{\beta}}^{\bar{Y}}\hat{r}^{\hat{\beta}} - \frac{1}{2}\hat{r}^{-1}R^{\bar{Y}}\hat{r}_{\hat{0}})$$

allows us to write

$$8\pi^2\chi(Y_\epsilon) = \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4} - \frac{|E|_{h_+}^2}{2} + \frac{R_Y^2}{24}dA_{h_+} + \int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon}.$$

Now we plug (2.3) in for R_Y^2 and get

$$\begin{aligned} 8\pi^2\chi(Y_\epsilon) &= \int_{r>\epsilon} \frac{|W|_{h_+}^2}{4}dA_{h_+} - \int_{r>\epsilon} \frac{|E|_{h_+}^2}{2}dA_+ + 6 \int_{r>\epsilon} dA_+ + \int_{r>\epsilon} \frac{|B|_{h_+}^4}{24} + |B|_{h_+}^2 dA_+ \\ &+ \int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon}. \end{aligned}$$

By Claim 1.1 we have that trace-free Ricci term and the $|B|^4$ term converge.

$\lim_{\epsilon \rightarrow 0} \int_{Y_\epsilon} |W|_{h_+}^2 dA_+$ clearly converges since $|W|_{h_+}^2$ is a conformal invariant of weight -4 . $\chi(Y_\epsilon) = \chi(Y)$ for ϵ small enough since we can use the flow of $\nabla_{\bar{g}} r$ to deformation retract Y onto Y_ϵ .

So we get that

$$\lim_{\epsilon \rightarrow 0} \left(6 \int_{r>\epsilon} dA_+ + |B|_{h_+}^2 dA_+ + \int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon} \right) \text{ exists.}$$

In the following claim we analyze the individual terms in the above limit and we compute their divergent parts.

Claim 2.3. We have the following equations:

$$6 \int_{r>\epsilon} dA_+ = 2 \int_{\Sigma} ds_{k_\infty} \epsilon^{-3} - \int_{\Sigma} \frac{3}{4} k_\infty^{ab} k_\infty^{cd} R_{abcd}^M ds_{k_\infty} \epsilon^{-1} + 6A(Y) + O(\epsilon), \quad (2.49)$$

$$\int_{r=\epsilon} \mathcal{B}_r ds_\epsilon = -2 \int_{\Sigma} ds_{k_\infty} \epsilon^{-3} + \int_{\Sigma} k_\infty^{ab} k_\infty^{cd} R_{abcd}^{\bar{Y}} - \frac{1}{4} k_\infty^{ab} k_\infty^{cd} R_{abcd}^M ds_{k_\infty} \epsilon^{-1} + O(\epsilon), \quad (2.50)$$

$$\int_{r>\epsilon} |B|_{h_+}^2 dA_+ = \int_{\Sigma} |II|_{k_\infty}^2 ds_{k_\infty} \epsilon^{-1} + O(1), \quad (2.51)$$

where II is the second fundamental form of Σ in M .

Proof. For the duration of this proof we let $k = k_\infty$, first consider the following computations that we use later in the proof: We know from the Fefferman-Graham expansion that

$$\bar{g} = g_\infty - \frac{1}{2} P(g_\infty) r^2 + O(r^4)$$

now because $\bar{h}_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ agree up to order 5, we can say

$$\bar{h}_{ab}''|_{r=0} = -2P_{ab}(g_\infty).$$

Therefore we can say

$$\begin{aligned}
k^{ab}\bar{h}''_{ab}|_{r=0} &= -2k^{ab}P_{ab}(g_\infty) = k^{ab}\left[\frac{R^M k_{ab}}{6} - R_{ab}^M\right] \\
&= \frac{R^M}{2} - k^{ab}R_{ab}^M = \frac{R^M}{2} - \frac{2k^{ab}k^{cd}R_{acbd}^M}{2} - k^{ab}R_{a4b4}^M
\end{aligned} \tag{2.52}$$

but

$$R^M = k^{ab}k^{cd}R_{acbd}^M + 2k^{ab}R_{a4b4}^M$$

so we get

$$tr_k \bar{h}'' = -\frac{k^{ab}k^{cd}R_{acbd}^M}{2}. \tag{2.53}$$

Another preliminary calculation we need is the following

$$\begin{aligned}
\Delta_{\bar{h}}\hat{r} &= \bar{h}^{\alpha\beta}\nabla_{\alpha}\bar{h}\nabla_{\beta}\bar{h}\hat{r} \\
&= \bar{h}^{\alpha\beta}\nabla_{\alpha}\bar{h}\partial_{\beta}\hat{r} \\
&= -\bar{h}^{\alpha\beta}\Gamma_{\alpha\beta}^{\gamma}(\bar{h})\hat{r}_{\hat{\gamma}} \\
&= -\bar{h}^{\alpha\beta}\Gamma_{\alpha\beta}^r(\bar{h}) \\
&= -\bar{h}^{\alpha\beta}\frac{\bar{h}^{r\gamma}}{2}\{\partial_{\hat{\alpha}}\bar{h}_{\beta\gamma} + \partial_{\hat{\beta}}\bar{h}_{\alpha\gamma} - \partial_{\hat{\gamma}}\bar{h}_{\alpha\beta}\} \\
&= \frac{\bar{h}^{\alpha\beta}}{2}\{\bar{h}'_{\alpha\beta}\} + O(r^3),
\end{aligned}$$

with the last equality following from (2.17). Also

$$\hat{r}^{\beta}\hat{r}_{\alpha\beta} = \bar{h}^{\beta\gamma}\nabla_{\hat{\gamma}}\bar{h}\hat{r}_{\hat{\beta}}\nabla_{\hat{\alpha}}\bar{h}\hat{r}_{\hat{\alpha}} \tag{2.54}$$

$$= \bar{h}^{\beta r}\left[-\Gamma_{r\alpha}^{\delta}\hat{r}_{\hat{\delta}}\right] + O(r^5) \tag{2.55}$$

$$= O(r^4) \tag{2.56}$$

therefore

$$\hat{r}^{\hat{\beta}}\hat{r}_{\hat{0}\hat{\beta}}\hat{r}^{-2} = O(r^2)$$

and so this term contributes nothing to the constant or divergent part of the expansion of $\int_{\Sigma_\epsilon} \mathcal{B} ds_{k(\epsilon)}$. Now because the third derivative of \bar{h} vanishes on Σ , we get

$$\Delta_{\bar{h}} \hat{r} = \frac{tr_k \bar{h}''|_{r=0}}{2} r + O(r^3). \quad (2.57)$$

Therefore

$$\frac{\Delta_{\bar{h}}(\hat{r})}{\hat{r}^2} = \frac{tr_k(\bar{h}''|_{r=0})}{2\hat{r}} = -\frac{k^{ab}k^{cd}R_{acbd}^M}{4\hat{r}} + O(r). \quad (2.58)$$

We also note that

$$\frac{2R_{00}^{\bar{Y}} - R^{\bar{Y}}}{2\hat{r}} = -\frac{k^{ab}k^{cd}R_{acbd}^{\bar{Y}}}{2\hat{r}} + O(r). \quad (2.59)$$

Now we look at the expansion of the area function, $Ar(\epsilon) = Area_{h_+}(Y_\epsilon)$.

$$\begin{aligned} Ar(\epsilon) &= \int_{r>\epsilon} dA_+ = \int_{r>\epsilon} r^{-4} dA_{\bar{h}} \quad (2.60) \\ &= Ar(r_0) + \int_\epsilon^{r_0} r^{-4} \int_\Sigma \left[1 + \frac{\partial_r|_{r=0}^2(\sqrt{det\bar{h}})r^2}{2\sqrt{detk}} + O(r^4) \right] ds_k dr \\ &= \int_\Sigma ds_k \frac{1}{3\epsilon^3} + \int_\Sigma \frac{tr_k(\bar{k}'')}{4} ds_k \frac{1}{\epsilon} + A(Y) + O(\epsilon). \end{aligned}$$

Now by (2.53) the second coefficient of the expansion for $6 \int_{r>\epsilon} dA_+$ is:

$$\frac{3tr_k \bar{h}''|_{r=0}}{2} = -\frac{3k^{ab}k^{cd}R_{acbd}^M}{4}.$$

Now we look at $\int_{r=\epsilon} \mathcal{B}_r ds$:

$$\int_{r=\epsilon} 2(\hat{r}^{-3} |d\hat{r}|_{\bar{h}}^2 \hat{r}_0 - \frac{\hat{r}_0 \Delta_{\bar{h}} \hat{r}}{\hat{r}^2} + \hat{r}^{-2} \hat{r}_{0\hat{\beta}} \hat{r}^{\hat{\beta}} + \hat{r}^{-1} R_{0\hat{\beta}}^{\bar{Y}} \hat{r}^{\hat{\beta}} - \frac{1}{2} \hat{r}^{-1} R^{\bar{Y}} \hat{r}_0) ds_{\bar{k}_\epsilon} + \int_{r=\epsilon} \bar{S}_r ds_{\bar{k}_\epsilon}. \quad (2.61)$$

First we note

$$\frac{\partial \sqrt{\det \bar{k}}}{\partial r} = \frac{\sqrt{\det \bar{k}}}{2} [\bar{k}^{ab} \bar{k}'_{ab}] \quad (2.62)$$

$$\frac{\partial^2 \sqrt{\det \bar{k}}}{\partial r^2} = \frac{\sqrt{\det \bar{k}}}{4} [\bar{k}^{ab} \bar{k}'_{ab}]^2 + \frac{\sqrt{\det \bar{k}}}{2} [(\bar{k}^{ab})' \bar{k}'_{ab} + \bar{k}^{ab} \bar{k}''_{ab}] \quad (2.63)$$

$$\left. \frac{\partial^3 \sqrt{\det \bar{k}}}{\partial r^3} \right|_{r=0} = 0 \quad (2.64)$$

(2.62) vanishes on the boundary since $\bar{k}' = 0$ on the boundary. Now, by (2.10), (2.11), (2.59) and the previous calculation

$$2 \int_{r=\epsilon} [\hat{r}^{-1} R_{\hat{\alpha}\hat{\beta}}^{\bar{Y}} \hat{r}^{\hat{\beta}} - \frac{1}{2} \hat{r}^{-1} R^{\bar{Y}} \hat{r}_{\hat{\alpha}}] ds_{\bar{k}_\epsilon} = - \int_{\Sigma} k^{ab} k^{cd} R_{abcd}^{\bar{Y}} ds_{k_\infty} \frac{1}{\epsilon} + O(\epsilon)$$

contributes no constant term to the expansion of $\int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon}$. By (2.8), (2.62) and (2.64) it follows that

$$\begin{aligned} 2 \int_{r=\epsilon} \hat{r}^{-3} |d\hat{r}|_h^2 ds_{\bar{k}_\epsilon} &= 2 \int_{\Sigma} ds_{k_\infty} \frac{1}{\epsilon^3} - \int_{\Sigma} \frac{\text{tr}_k(\bar{k}'')}{2} ds_{k_\infty} \frac{1}{\epsilon} + O(\epsilon^2) \\ &= 2 \int_{\Sigma} ds_{k_\infty} \frac{1}{\epsilon^3} - \int_{\Sigma} \frac{k^{ab} k^{cd} R_{abcd}^M}{4} ds_{k_\infty} \frac{1}{\epsilon} + O(\epsilon^2) \end{aligned}$$

Recalling (2.58) we now deduce

$$\int_{r=\epsilon} \mathcal{B}_r ds = -2 \int_{\Sigma} ds_{k_\infty} \frac{1}{\epsilon^3} + \int_{\Sigma} -k^{ab} k^{cd} R_{abcd}^M + k^{ab} k^{cd} R_{abcd}^{\bar{Y}} ds_{k_\infty} \frac{1}{\epsilon} + O(\epsilon).$$

Notice that by the Gauss curvature equation and the fact that Σ is minimal in M and also totally geodesic in \bar{Y} by (2.22), we have

$$k^{ab} k^{cd} R_{abcd}^M - k^{ab} k^{cd} R_{abcd}^{\bar{Y}} = |II|_k^2.$$

Therefore

$$\int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon} = -2 \int_{\Sigma} ds_k \frac{1}{\epsilon^3} - \int_{\Sigma} |II|_k^2 ds_k \frac{1}{\epsilon} + O(\epsilon).$$

The third part of the claim now follows also since we now that

$$\lim_{\epsilon \rightarrow 0} \left[6 \int_{r>\epsilon} dA_+ + \int_{r>\epsilon} |B|_{h_+}^2 dA_+ + \int_{r=\epsilon} \mathcal{B}_r ds_{\bar{k}_\epsilon} \right] < \infty.$$

□

Corollary 2.4. Now we get the formula for the renormalised area:

$$\begin{aligned} 6A(Y) = & - \int_Y \frac{|W|_{h_+}^2}{4} dA_{h_+} + \int_Y \frac{|E|_{h_+}^2}{2} dA_+ - \int_Y \frac{|B|^4}{24} dA_+ \\ & - f.p. \int_{Y_\epsilon} |B|^2 dA_+ + 8\pi^2 \chi(Y) \end{aligned} \quad (2.65)$$

Where

$$f.p. \int_{Y_\epsilon} |B|^2 dA_+ = \lim_{\epsilon \rightarrow 0} \left[\int_{Y_\epsilon} |B|^2 dA_+ - \int_{\Sigma} |II|_k^2 ds_k \epsilon^{-1} \right].$$

CHAPTER 3

THE SINGULAR YAMABE CASE

3.1 Renormalized Area Expansion and the Graham-Witten Energy

First we make note of some preliminary facts; let $(x^{i'}, x^i)$ be coordinates on $(\mathbb{R}^l \times \mathbb{R}^k, g)$ where g is some smooth metric. Then if Y^k is a minimal submanifold given by $\{x^{i'} = z^{i'}(x^i)\}$ (i.e. $Y = \text{graph}(z)$) and $h := g|_Y$, the following equations hold:

$$h_{ij} = g_{ij} + g_{i'i} z_{,j}^{i'} + g_{i'j} z_{,i}^{j'} + g_{i'j'} z_{,i}^{i'} z_{,j}^{j'} \quad (3.1)$$

$$\left[\partial_j + \frac{1}{2} (\log(\det h))_{,j} \right] \left[h^{ij} \left(g_{ik'} + g_{i'k'} z_{,i}^{i'} \right) \right] - \frac{1}{2} h^{ij} \left[g_{ij,k'} + 2g_{ii',k'} z_{,j}^{i'} + g_{i'j',k'} z_{,i}^{i'} z_{,j}^{j'} \right] = 0, \quad (3.2)$$

$\forall k'$

note: we are abusing notation in the expression for h_{ij} , the indices on h correspond to the basis for TY induced by pushing forward $\{\partial_{x^i}\}$ via the standard parametrization for Y , induced by z , whereas the indices on g correspond to the basis $\{\partial_{x^{i'}}, \partial_{x^i}\}$.

Let (\bar{X}, \bar{g}) be a smooth compact Riemannian Manifold of dimension $n + 1$ with boundary $\partial \bar{X} = M$. Let $X = \text{int}(\bar{X})$, let u be a defining function for M solving the singular Yamabe problem described in the introduction i.e. such that (X, g_+) has $R_{g_+} = -n(n + 1)$ where $g_+ = u^{-2} \bar{g}$. Now, the normal exponential map of M gives us a diffeomorphism of $M \times [0, \delta)$ onto some neighborhood of M in \bar{X} . Letting r be the

\bar{g} distance to the boundary, we see that we get the splitting;

$$\bar{g} = dr^2 + g_r$$

for some one-parameter family of metrics, g_r on M . Now let Y^{k+1} be a submanifold of (X, g_+) where $0 \leq k \leq n - 1$. Assume that Y extends smoothly to \bar{X} and set $\Sigma = Y \cap M$. Locally, near a point p of Σ , we may choose (by the local immersion theorem) coordinates $\{x^{\alpha'}, x^\alpha\}$ $1 \leq \alpha \leq k$ and $1 \leq \alpha' \leq n - k$, so that $\Sigma = \{x^{\alpha'} = 0\}$ and as a consequence, we also get $\partial_{x^\alpha} \perp_{\bar{g}} \partial_{x^{\alpha'}}$. Under the identification discussed above, we may extend $\{x^\alpha, x^{\alpha'}\}$ to a neighbourhood of p in \bar{X} , together with r , this gives us a local coordinate system at p . Now we assume that Y is a minimal submanifold of (X, g_+) and that Y can be expressed in our local coordinate system as $\{z^{\alpha'} = z^{\alpha'}(x^\alpha, r)\}$ for some smooth function z .

Let $\bar{h} = \bar{g}|_Y$, then by (3.1):

$$\begin{aligned}\bar{h}_{\alpha\beta} &= \bar{g}_{\alpha\beta} + \bar{g}_{\alpha'\alpha} z_{,\beta}^{\alpha'} + \bar{g}_{\alpha'\beta} z_{,\alpha}^{\alpha'} + \bar{g}_{\alpha'\beta'} z_{,\alpha}^{\alpha'} z_{,\beta}^{\beta'} \\ \bar{h}_{\alpha r} &= \bar{g}_{\alpha'\alpha} z_{,r}^{\alpha'} + \bar{g}_{\alpha'\beta'} z_{,\alpha}^{\alpha'} z_{,r}^{\beta'} \\ \bar{h}_{rr} &= 1 + \bar{g}_{\alpha'\beta'} z_{,r}^{\alpha'} z_{,r}^{\beta'}\end{aligned}$$

And by using (3.2) in the case of our submanifold $\{x^{\alpha'} = z^{\alpha'}(x^\alpha, r)\}$, using the

equations $g_+ = u^{-2}\bar{g}$, $h_+ = u^{-2}\bar{h}$ and defining $L := \log(\det(\bar{h}))$ we get;

$$\begin{aligned}
& \left[\partial_\beta - (k+1)\frac{u_\beta}{u} + \frac{1}{2}L_\beta \right] \left[\bar{h}^{r\beta} \bar{g}_{\alpha'\gamma'} z_{,r}^{\alpha'} + \bar{h}^{\alpha\beta} (\bar{g}_{\alpha\gamma'} + \bar{g}_{\alpha'\gamma'} z_{,\alpha}^{\alpha'}) \right] \\
& + \left[\partial_r - (k+1)\frac{u_r}{u} + \frac{1}{2}L_r \right] \left[\bar{h}^{rr} \bar{g}_{\alpha'\gamma'} z_{,r}^{\alpha'} + \bar{h}^{\alpha r} (\bar{g}_{\alpha\gamma'} + \bar{g}_{\alpha'\gamma'} z_{,\alpha}^{\alpha'}) \right] \\
& - \frac{1}{2} \bar{h}^{\alpha\beta} \left[\bar{g}_{\alpha\beta,\gamma'} + 2\bar{g}_{\alpha\alpha',\gamma'} z_{,\beta}^{\alpha'} + \bar{g}_{\alpha'\beta',\gamma'} z_{,\alpha}^{\alpha'} z_{,\beta}^{\beta'} \right] \\
& - \bar{h}^{\alpha r} \left[\bar{g}_{\alpha\alpha',\gamma'} z_{,r}^{\alpha'} + \bar{g}_{\alpha'\beta',\gamma'} z_{,\alpha}^{\alpha'} z_{,r}^{\beta'} \right] \\
& - \frac{1}{2} \bar{h}^{rr} \left[\bar{g}_{\alpha'\beta',\gamma'} z_{,r}^{\alpha'} z_{,r}^{\beta'} \right] \\
& + \frac{u_{\gamma'}}{u} \bar{h}^{\alpha\beta} \left[\bar{g}_{\alpha\beta} + 2\bar{g}_{\alpha\alpha'} z_{,\beta}^{\alpha'} + \bar{g}_{\alpha'\beta',\alpha} z_{,\beta}^{\beta'} \right] \\
& + 2 \frac{u_{\gamma'}}{u} \bar{h}^{\alpha r} \left[\bar{g}_{\alpha\alpha'} z_{,r}^{\alpha'} + \bar{g}_{\alpha'\beta',\alpha} z_{,r}^{\beta'} \right] \\
& + \frac{u_{\gamma'}}{u} \bar{h}^{rr} \left[1 + \bar{g}_{\alpha'\beta',\alpha} z_{,r}^{\alpha'} z_{,r}^{\beta'} \right] = 0, \quad \forall \gamma'
\end{aligned}$$

note: we don't bother with a comma when we differentiate u or L as there is no ambiguity.

Now if we multiply the equation above by u we get $\mathcal{M}(z) = 0$, where;

$$\mathcal{M}(z)_{\gamma'} = [u\partial_\beta - (k+1)u_\beta + \frac{u}{2}L_\beta] [\bar{h}^{r\beta} \bar{g}_{\alpha'\gamma'} z_r^{\alpha'} + \bar{h}^{\alpha\beta} (\bar{g}_{\alpha\gamma'} + \bar{g}_{\alpha'\gamma'} z_{,\alpha}^{\alpha'})] \quad (3.3)$$

$$+ [u\partial_r - (k+1)u_r + \frac{u}{2}L_r] [\bar{h}^{rr} \bar{g}_{\alpha'\gamma'} z_r^{\alpha'} + \bar{h}^{\alpha r} (\bar{g}_{\alpha\gamma'} + \bar{g}_{\alpha'\gamma'} z_{,\alpha}^{\alpha'})] \quad (3.4)$$

$$- \frac{u}{2} \bar{h}^{\alpha\beta} [\bar{g}_{\alpha\beta,\gamma'} + 2\bar{g}_{\alpha\alpha',\gamma'} z_{,\beta}^{\alpha'} + \bar{g}_{\alpha'\beta',\gamma'} z_{,\alpha}^{\alpha'} z_{,\beta}^{\beta'}] \quad (3.5)$$

$$- u \bar{h}^{\alpha r} [\bar{g}_{\alpha\alpha',\gamma'} z_r^{\alpha'} + \bar{g}_{\alpha'\beta',\gamma'} z_{,\alpha}^{\alpha'} z_r^{\beta'}] \quad (3.6)$$

$$- \frac{u}{2} \bar{h}^{rr} [\bar{g}_{\alpha'\beta',\gamma'} z_r^{\alpha'} z_r^{\beta'}] \quad (3.7)$$

$$+ u_{\gamma'} \bar{h}^{\alpha\beta} [\bar{g}_{\alpha\beta} + 2\bar{g}_{\alpha\alpha'} z_{,\beta}^{\alpha'} + \bar{g}_{\alpha'\beta',\alpha} z_{,\beta}^{\beta'}] \quad (3.8)$$

$$+ 2u_{\gamma'} \bar{h}^{\alpha r} [\bar{g}_{\alpha\alpha'} z_r^{\alpha'} + \bar{g}_{\alpha'\beta',\alpha} z_{,\alpha}^{\alpha'} z_r^{\beta'}] \quad (3.9)$$

$$+ u_{\gamma'} \bar{h}^{rr} [1 + \bar{g}_{\alpha'\beta',\alpha} z_r^{\alpha'} z_r^{\beta'}] \quad (3.10)$$

Notice that if we let $r = 0$, every line vanishes except for the second term of line (2), we get

$$\begin{aligned} (k+1)u_r \bar{h}^{rr} \bar{g}_{\alpha'\gamma'} z_r^{\alpha'} \Big|_{r=0} &= 0 \quad \forall \gamma' \\ \implies (k+1)\bar{g}_{\alpha'\gamma'} z_r^{\alpha'} &= 0 \quad \forall \gamma' \\ \implies z_r^{\beta'} &= 0 \quad \forall \beta' \end{aligned}$$

Now if we observe that most of the terms of $\mathcal{M}(z)_{\gamma'}$ are $O(r^2)$ we see

$$\begin{aligned}
& [u\partial_r - (k+1)u_r](\bar{h}^{\bar{r}r}\bar{g}_{\alpha'\gamma'}z_{,r}^{\alpha'}) - \frac{u}{2}\bar{h}^{\alpha\beta}\bar{g}_{\alpha\beta,\gamma'} = O(r^2) \\
\implies & u\bar{h}^{\bar{r}r}\bar{g}_{\alpha'\gamma'}z_{,rr}^{\alpha'} - (k+1)u_r\bar{h}^{\bar{r}r}\bar{g}_{\alpha'\gamma'}z_{,r}^{\alpha'} - \frac{u}{2}\bar{h}^{\alpha\beta}\bar{g}_{\alpha\beta,\gamma'} = O(r^2) \\
\implies & \bar{g}_{\alpha'\gamma'}z_{,rr}^{\alpha'} - (k+1)\bar{g}_{\alpha'\gamma'}z_{,r}^{\alpha'} - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta,\gamma'} = O(r^2) \\
& \implies k\bar{g}_{\alpha'\gamma'}z_{,rr}^{\alpha'}|_{r=0} = -\frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta,\gamma'} \\
& \implies kz_{,rr}^{\beta'}|_{r=0} = -\frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\alpha\beta,\gamma'}\bar{g}^{\gamma'\beta'}
\end{aligned}$$

Now for convenience in some future computations, we write;

$$\mathcal{M}(z)_{\gamma'} = A_\beta B^\beta(z)_{\gamma'} + A_r B^r(z)_{\gamma'} - \frac{u}{a}E(z)_{,\gamma'} + u_{\gamma'}E(z)$$

where;

$$\begin{aligned}
A_i &= u\partial_i - (k+1)u_i + \frac{u}{2}L_i \\
B^i(z)_{\gamma'} &= \bar{h}^{\bar{r}i}\bar{g}_{\alpha'\gamma'}z_{,r}^{\alpha'} + \bar{h}^{\alpha r}(\bar{g}_{\alpha\gamma'} + \bar{g}_{\alpha'\gamma'}z_{,\alpha}^{\alpha'}) \\
E(z) &= \bar{h}^{\alpha\beta}[\bar{g}_{\alpha\beta} + 2\bar{g}_{\alpha\alpha'}z_{\beta'}^{\alpha'} + \bar{g}_{\alpha'\beta'}z_{,\alpha}^{\alpha'}z_{,\beta}^{\beta'}] \\
&+ 2\bar{h}^{\alpha r}[\bar{g}_{\alpha\alpha'}z_{,r}^{\alpha'} + \bar{g}_{\alpha'\beta'}z_{,\alpha}^{\alpha'}z_{,r}^{\beta'}] \\
&+ \bar{h}^{\bar{r}r}[1 + \bar{g}_{\alpha'\beta'}z_{,\bar{r}}^{\alpha'}z_{,r}^{\beta'}]
\end{aligned}$$

Now if we write $z = v + wr^m$, we can say

$$\begin{aligned}
0 &= \mathcal{M}(z)_{\gamma'} = \mathcal{M}(v + wr^m)_{\gamma'} \\
&= A_\beta B^\beta(v + wr^m) + A_r B^r(v + wr^m) - \frac{u}{2} E(v + wr^m)_{,\gamma'} + u_{\gamma'} E(v + wr^m) \\
&= A_\beta [B^\beta(v) + O(r^m)] + A_r [B^r(v) + \bar{h}^{rr} \bar{g}_{\alpha'\gamma'} w^{\alpha'} m r^{m-1} + O(r^m)] \\
&\quad - \frac{u}{2} E(v)_{\gamma'} - \frac{u}{2} E(wr^m)_{\gamma'} + \frac{u}{2} \bar{h}^{\alpha\beta} \bar{g}_{\alpha\beta,\gamma'} + O(r^m) \\
&\quad + u_{\gamma'} E(v) + u_{\gamma'} E(wr^m) - u_{\gamma'} [\bar{h}^{\alpha\beta} \bar{g}_{\alpha\beta} + \bar{h}^{rr}] + O(r^m) \\
&= \mathcal{M}(v)_{\gamma'} + [u\partial_r - (k+1)u_r] (\bar{h}^{rr} \bar{g}_{\alpha'\gamma'} w^{\alpha'} m r^{m-1}) + O(r^m) \\
&= \mathcal{M}(v)_{\gamma'} + [\bar{g}_{\alpha'\gamma'} w^{\alpha'} m(m-1)r^{m-1}] - (k+1) [\bar{g}_{\alpha'\gamma'} w^{\alpha'} m r^{m-1}] + O(r^m) \\
&= \mathcal{M}(v)_{\gamma'} + m(m-k-2) \bar{g}_{\alpha'\gamma'} w^{\alpha'} r^{m-1} + O(r^m)
\end{aligned}$$

This implies that $\mathcal{M}(v)_{\gamma'}$ is $O(r^{m-1})$ and (provided $k+2 < m$)

$$w^{\alpha'}(0) \bar{g}_{\alpha'\gamma'}(0) = - \left. \frac{\mathcal{M}(v)_{\gamma'}}{m(m-k-2)r^{m-1}} \right|_{r=0}$$

Hence;

$$w^{\alpha'}(0) = -\bar{g}^{\alpha'\gamma'}(0) \left[\left. \frac{\mathcal{M}(v)_{\gamma'}}{m(m-k-2)r^{m-1}} \right|_{r=0} \right] \quad \forall m \leq k+1$$

KEY: Fixing a point $p \in M$, this allows us to compute the $k+1 = \dim(Y)$ degree Taylor polynomial for $[r \mapsto z(p, r)]$ centered at $r = 0$.

Now we turn our attention to the derivatives wrt r of $\sqrt{\det \bar{h}}$:

$$\begin{aligned} \frac{\partial \sqrt{\det \bar{h}}}{\partial r} &= \frac{\sqrt{\det \bar{h}}}{2} [\bar{h}^{\alpha\beta} \bar{h}'_{\alpha\beta}] \\ \frac{\partial^2 \sqrt{\det \bar{h}}}{\partial r^2} &= \frac{\sqrt{\det \bar{h}}}{4} [\bar{h}^{\alpha\beta} \bar{h}'_{\alpha\beta}]^2 + \frac{\sqrt{\det \bar{h}}}{2} [(\bar{h}^{\alpha\beta})' \bar{h}'_{\alpha\beta} + \bar{h}^{\alpha\beta} \bar{h}''_{\alpha\beta} + \bar{h}''_{rr}] \\ &\cdot \\ &\cdot \\ &\cdot \\ \frac{\partial^{(k+1)} \sqrt{\det \bar{h}}}{\partial r^{k+1}} &= \sqrt{\det \bar{h}} * \text{Polynomial}(\bar{h}_{ab}, \bar{h}^{ab}, \bar{h}'_{ab}, (\bar{h}^{ab})', \dots, \bar{h}_{ab}^{(k)}, (\bar{h}^{ab})^{(k)}, \bar{h}_{ab}^{(k+1)}) \end{aligned}$$

[where a,b run over the greek indices and r]

Now recall the formulae:

$$\begin{aligned} \bar{h}_{\alpha\beta} &= \bar{g}_{\alpha\beta} + 2\bar{g}_{\alpha'(\alpha} z_{\beta)}^{\alpha'} + \bar{g}_{\alpha'\beta'} z_{\alpha}^{\alpha'} z_{\beta}^{\beta'} \\ \bar{h}_{\alpha r} &= \bar{g}_{\alpha'\alpha} z_{,r}^{\alpha'} + \bar{g}_{\alpha'\beta'} z_{,\alpha}^{\alpha'} z_{,r}^{\beta'} \\ \bar{h}_{rr} &= 1 + \bar{g}_{\alpha'\beta'} z_{,r}^{\alpha'} z_{,r}^{\beta'} \end{aligned}$$

In light of the previous facts we uncovered about z , it's clear that $\bar{h}_{\alpha\beta}$ is formally determined up to order $k+1$, using the fact that $\bar{g}_{\alpha'\alpha}$ is $O(r)$ as $r \rightarrow 0$ and z is $O(r^2)$, we see that $\bar{h}_{\alpha r}$ and \bar{h}_{rr} are also formally determined up to (and including) order $k+1$.

Therefore, it follows that the $(k+1)$ st Taylor polynomial of $\sqrt{\det \bar{h}}$ is formally determined.

Now we know that we can write;

$$u = r + r^2\varphi = r + u^{(2)}r^2 + u^{(3)}r^3 + \dots + u^{(n+1)}r^{(n+1)} + \mathcal{L}r^{(n+1)} \log(r) + O(r^{n+2})$$

$$\implies 1 + r\varphi = 1 + u^{(2)}r + u^{(3)}r^2 + \dots + u^{(n+1)}r^n + \mathcal{L}r^n \log(r) + O(r^{n+1})$$

Now we write

$$\begin{aligned} \sqrt{deth}d\nu_h &= u^{-(k+1)}\sqrt{deth}dxdr = r^{-(k+1)}(1 + r\varphi)^{-(k+1)}\sqrt{deth}dxdr = \\ &= r^{-(k+1)}(1 + r\varphi)^{-(k+1)}\frac{\sqrt{deth}}{\sqrt{deth_0}}\sqrt{deth_0}dxdr = r^{-(k+1)}(1 + r\varphi)^{-(k+1)}\frac{\sqrt{deth}}{\sqrt{deth_0}}d\nu_Ndr \end{aligned}$$

Now we investigate the term $(1 + r\varphi)^{-(k+1)}$, we know we can write

$$\begin{aligned} (1 + r\varphi)^{-(k+1)} &= \frac{1}{(1 + r\varphi)^{k+1}} \\ &= \frac{1}{1 - [1 - (1 + r\varphi)^{k+1}]} \\ &= \sum_{l=0}^{\infty} [1 - (1 + r\varphi)^{k+1}]^l \quad (\text{shrinking the collar nbhd if necessary}) \\ &= \sum_{l=0}^{\infty} [1 - (1 + r\varphi)^{k+1}]^l \\ &= \sum_{l=0}^{\infty} [1 - (1 + u^{(2)}r + u^{(3)}r^2 + \dots + u^{(n+1)}r^n + \mathcal{L}r^n \log(r) + O(r^{n+1}))^{k+1}]^l \\ &= q(r) + O(r^n \log r) \end{aligned}$$

where $q(r)$ is an n degree polynomial in r , whose coefficients are formally determined.

Therefore, we may write

$$(1 + r\varphi)^{-(k+1)} \frac{\sqrt{\text{deth}}}{\sqrt{\text{deth}_0}} = \nu^{(0)} + \nu^{(1)}r + \dots + \nu^{(k)}r^k + \nu^{(k+1)}r^{k+1} + O(r^{k+1} \log(r)),$$

where ν^i is formally determined for $0 \leq i \leq k+1$.

\implies

$$r^{-(k+1)}(1 + r\varphi)^{-(k+1)} \frac{\sqrt{\text{deth}}}{\sqrt{\text{deth}_0}} = \nu^{(0)}r^{-(k+1)} + \nu^{(1)}r^{-k} + \dots + \nu^{(k)}r^{-1} + \nu^{(k+1)} + O(\log r)$$

Now we look at the asymptotics of the function $\bar{A}(\epsilon) := Ar_{g_+}(Y \cap \{r > \epsilon\})$

$$\begin{aligned} \bar{A}(\epsilon) &= \int_{r>\epsilon} \sqrt{\text{deth}} d\nu_h \\ &= \int_{\epsilon_0 > r > \epsilon} \nu^{(0)}r^{-(k+1)} + \nu^{(1)}r^{-k} + \dots + \nu^{(k)}r^{-1} + \nu^{(k+1)} + O(\log r) d\nu_h + \bar{A}(\epsilon_0) \\ &= \int_{\epsilon}^{\epsilon_0} \int_N \nu^{(0)}r^{-(k+1)} + \nu^{(1)}r^{-k} + \dots + \nu^{(k)}r^{-1} + \nu^{(k+1)} + O(\log r) d\nu_N dr + \bar{A}(\epsilon_0) \\ &= \int_N \frac{\nu^{(0)}}{k} d\nu_N \epsilon^{-k} + \int_N \frac{\nu^{(1)}}{k-1} d\nu_N \epsilon^{1-k} + \dots + \int_N \nu^{(k)} d\nu_N \log \left[\frac{1}{\epsilon} \right] + A + o(1) \end{aligned}$$

Claim 3.1. The log coefficient of the above expansion for $\bar{A}(\epsilon)$ is a conformal invariant of \bar{g} .

Proof. Let $\hat{g} = e^{2\omega}\bar{g}$ for some smooth function ω . Let \hat{r} be the \hat{g} -distance to M . We may write $r = \hat{r}b(x, \hat{r})$ for some smooth non-vanishing function b . (shrinking our collar nbhd if necessary). Notice that, for $p \in M$, $\partial_{\hat{r}} = C(p)\partial_r$ on M . Where $C > 0$, hence $b(x(p), 0) = C(p) > 0$. And observe that;

$$\partial_{\hat{r}} \hat{r} b(x, \hat{r}) \Big|_{\hat{r}=0} = b(x, 0) = C > 0.$$

Therefore we may shrink our collar nbhd if necessary to ensure that $\hat{r}b(x, \hat{r})$ is strictly increasing wrt \hat{r} . Now if we define $\hat{\epsilon} := \epsilon b(x, \epsilon)$, it follows by the previous considera-

tions that $\hat{r} > \epsilon \iff r > \hat{\epsilon}$. Now we will compare the asymptotic expansions $\bar{A}(\epsilon)$ and $\hat{\bar{A}}(\epsilon)$ that we get from \bar{g} and $\hat{\bar{g}}$ and show that the difference has no log term, giving us the claim.

Now

$$\begin{aligned}
Area_{h_+}(\{r > \epsilon\}) - Area_{h_+}(\{\hat{r} > \epsilon\}) &= Area_{h_+}(\{r > \epsilon\}) - Area_{h_+}(\{r > \hat{\epsilon}\}) = \\
&\int_N \int_{\epsilon}^{\hat{\epsilon}} r^{-(k+1)} (1+r\varphi)^{-(k+1)} \frac{\sqrt{deth}}{\sqrt{deth_0}} dr d\nu_N = \\
&\int_N \int_{\epsilon}^{\hat{\epsilon}} \nu^{(0)} r^{-(k+1)} + \nu^{(1)} r^{-k} + \nu^{(2)} r^{(1-k)} + \dots + \nu^{(k)} r^{-1} + \nu^{(k+1)} + O(\log r) dr d\nu_N = \\
&\sum_{0 \leq l \leq k-1} \int_N \frac{\nu^{(l)}(x)}{l-k} \epsilon^{l-k} [b(x, \epsilon)^{l-k} - 1] d\nu_N + \int_N \nu^{(k)}(x) \log(b(x, \epsilon)) d\nu_N + Constant + o(1)
\end{aligned}$$

Which clearly has no $\log(\epsilon)$ term as ϵ goes to 0. \square

3.2 Computation of Graham-Witten Energy for a Selected Case

In this section, we will calculate the invariant $\int_N \nu^{(k)} d\nu_N$ for the case where $k+1 = 2$.

$$(1+r\varphi)^{-(k+1)} \frac{\sqrt{deth}}{\sqrt{deth_0}} = \nu^{(0)} + \nu^{(1)} r + \nu^{(2)} r^2 + \dots + \nu^{(k)} r^k + \nu^{(k+1)} r^{k+1} + O(r^{k+1} \log(r)),$$

Case: $k+1=2$

$$(1+r\varphi)^{-(k+1)} = \sum_{l=0}^{\infty} [1 - (1+u^{(2)}r + u^{(3)}r^2 + \dots + u^{(n+1)}r^n + \mathcal{L}r^{n+1} \log(r) + O(r^{n+1}))^2]^l =$$

$$\sum_{l=0}^{\infty} [1 - (1 + u^{(2)}r + u^{(3)}r^2 + \dots + u^{(n+1)}r^n + \mathcal{L}r^{n+1} \log(r) + O(r^{n+1}))^2]^l =$$

$$\sum_{l=0}^{\infty} [-(2u^{(2)}r + O(r^2))]^l = 1 - 2u^{(2)}r + O(r^2)$$

$$\frac{\sqrt{deth\bar{h}}}{\sqrt{deth_0}} = 1 + \frac{1}{2}[\bar{h}^{\alpha\beta}\bar{h}'_{\alpha\beta}]r + O(r^2)$$

$$(1 + r\varphi)^{-(k+1)} \frac{\sqrt{deth\bar{h}}}{\sqrt{deth_0}} = [1 - 2u^{(2)}r][1 + \frac{1}{2}[\bar{g}^{\alpha\beta}\bar{g}'_{\alpha\beta}]r] = 1 - 2u^{(2)}r + \frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}'_{\alpha\beta}r$$

$$- u^{(2)}[\bar{g}^{\alpha\beta}\bar{g}'_{\alpha\beta}]r^2 + O(r^2)$$

Now note that

$$\bar{h}'_{\alpha\beta}(0) = \partial_r \bar{g}_{\alpha\beta} \Big|_{r=0} = \partial_r \langle \partial_\alpha, \partial_\beta \rangle = \langle \nabla_r \partial_\alpha, \partial_\beta \rangle + \langle \partial_\alpha, \nabla_r \partial_\beta \rangle = \langle \nabla_\alpha \partial_r, \partial_\beta \rangle + \langle \partial_\alpha, \nabla_\beta \partial_r \rangle$$

$$\text{But } \langle \partial_\beta, \partial_r \rangle = 0 \implies \partial_r \langle \partial_\beta, \partial_r \rangle = 0$$

$\implies \langle \nabla_\alpha \partial_r, \partial_\beta \rangle = -\langle \partial_r, \nabla_\alpha \partial_\beta \rangle = -L_{\alpha\beta} = -B^r_{\alpha\beta}$, where L is the second fundamental form of M and B is the second fundamental form of N corresponding to ∂_r .

\therefore

$$\bar{h}'_{\alpha\beta}(0) = -2L_{\alpha\beta} = -2B_{\alpha\beta}^r$$

Also,

$$u^{(2)} = \varphi(0) = \frac{1}{4n} \bar{h}^{ij} \bar{h}'_{ij} \Big|_{r=0} = \frac{1}{4n} \bar{g}^{ij} \bar{g}'_{ij} = -\frac{1}{2n} H_M$$

where H_M is the mean curvature of M .

Hence,

$$\nu^{(k)} = \frac{H_M}{n} - H_N^r$$

and so the Graham-Witten energy is given by

$$\int_N \frac{H_M}{n} - H_N^r d\nu_N$$

CHAPTER 4

RENORMALIZED VOLUME OF MINIMALLY BOUNDED REGIONS

4.1 Acknowledgement of Collaboration

In this section we cover joint work from a paper with my adviser, Matthew Gursky, and collaborator Stephen McKeown.

4.2 Setting and Notation

Recall that an asymptotically hyperbolic (AH) manifold is a compact manifold X^{n+1} with boundary M^n , equipped on the interior $\overset{\circ}{X}$ with a metric g_+ such that, for any defining function φ for M , the metric $\bar{g} = \varphi^2 g_+$ extends to a Riemannian metric on $X = \bar{X}$; and such that, in addition, $|d\varphi|_{\bar{g}} = 1$ along M . The optimal regularity of \bar{g} is in general a delicate question, but in the context of this chapter (i.e., X is four dimensional) by a result of Chruściel- Delay-Lee-Skinner [10] we may assume that there is a compactification such that \bar{g} is smooth up to the boundary. The canonical example of an AH manifold is hyperbolic space itself, where X is the unit ball \mathbb{B}^{n+1} , and the metric is $g_H = \frac{4|dx|^2}{(1-|x|^2)^2}$. Given an AH metric, the metric $\bar{h} = \bar{g}|_{TM}$ is a metric on M , but is not well defined since the choice of φ is arbitrary. However, the conformal class $[\bar{h}]$ is well defined, and is called the *conformal infinity*.

A defining function r for M is called geodesic if $|dr|_{r^2 g_+} = 1$ on a neighborhood of M . Such a function induces a diffeomorphism

$$\psi : [0, \varepsilon)_r \times M \hookrightarrow X \tag{4.1}$$

onto a neighborhood of M in X such that

$$\psi^* g_+ = \frac{dr^2 + \bar{h}_r}{r^2}, \quad (4.2)$$

where \bar{h}_r is a one-parameter family of metrics on M . A lemma of Graham-Lee ([18]) states that geodesic defining functions are in one-to-one correspondence with the representatives \bar{h} of $[\bar{h}]$, according to the correspondence $\bar{h}_0 = \bar{h}$. The form (4.2) is called the geodesic normal form corresponding to $\bar{h} = \bar{h}_0$. We may assume that any geodesic compactification of X is smooth ([10]).

An AH metric is called Einstein (or AHE) if it satisfies as well the condition $\text{Ric}(g) + ng = 0$. We will be concerned exclusively with four-dimensional AHE spaces, i.e. the case $n = 3$. In this case, it is known ([12, 13, 16]) that in geodesic normal form, \bar{h}_r has the expansion

$$\bar{h}_r = \bar{h} - r^2 P^{\bar{h}} + r^3 g^{(3)} + O(r^4), \quad (4.3)$$

where $\text{tr}_{\bar{h}} g^{(3)} = 0$ and where $P^{\bar{h}}$ is the Schouten tensor of \bar{h} , given by

$$P^{\bar{h}}_{\mu\nu} = R^{\bar{h}}_{\mu\nu} - \frac{1}{4} R_{\bar{h}} \bar{h}_{\mu\nu}. \quad (4.4)$$

Apart from the trace condition, the tensor $g^{(3)}$ is not locally determined by the geometry of (M^3, \bar{h}) .

The renormalized volume of (X, g_+) is defined as follows ([22, 16]). Choose a metric $\bar{h} \in [\bar{h}]$, and let r be the corresponding geodesic defining function. Then the set $\{r > \varepsilon\}$ has volume

$$\text{vol}_{g_+}(\{r > \varepsilon\}) = c_0 \varepsilon^{-3} + c_2 \varepsilon^{-1} + V_+ + o(1). \quad (4.5)$$

The renormalized volume is V_+ , and it is independent of the choice of \bar{h} (that is, of r).

In our setting of interest, there exists as well an orientable minimal surface $Y^3 \subset X$, intersecting M transversely in a closed two-manifold $\Sigma^2 = Y \cap M$, and dividing X into two connected pieces X^+ and X^- such that $Y = X^+ \cap X^-$. We write $M^+ = X^+ \cap M$ and $M^- = X^- \cap M$, so that $\Sigma = M^+ \cap M^-$. The assignment of the signs $+$ and $-$ is arbitrary, and corresponds to a choice of unit normal vector field on Y .

We now introduce the notations we will use. We let (X^4, M^3, g_+) be an AHE space, and $Y^3 \subset X$ a minimal surface as above. We will let $[\bar{h}]$ be the conformal infinity, and corresponding to the metric \bar{h} will be the geodesic defining function r . The compactified metric is $\bar{g} = r^2 g_+$. Furthermore, X^+, M^+ , and Σ^2 will be as above. For $\varepsilon > 0$, we let $X_\varepsilon = \{r > \varepsilon\}$, with $X_\varepsilon^+ = X^+ \cap X_\varepsilon$. We set $Y_\varepsilon = \overline{Y \cap X_\varepsilon}$ and $M_\varepsilon = \{r = \varepsilon\}$. Similarly we set $M_\varepsilon^+ = X^+ \cap M_\varepsilon$. Finally, $\Sigma_\varepsilon = Y \cap M_\varepsilon^+$.

Next, there are a number of metrics to name. We let $h_\varepsilon = g_+|_{TM_\varepsilon}$, while $\bar{h}_\varepsilon = \varepsilon^2 h_\varepsilon = \bar{g}|_{TM_\varepsilon}$. We let $\tilde{h} = g_+|_{TY}$, while $\bar{\tilde{h}} = r^2 \tilde{h} = \bar{g}|_{TY}$. We let $\bar{k} = \bar{g}|_{T\Sigma}$, while $k_\varepsilon = g_+|_{T\Sigma_\varepsilon}$ and $\bar{k}_\varepsilon = r^2 k_\varepsilon = \varepsilon^2 k_\varepsilon$. The decorations of ε will sometimes change position as needed; for example, we will write $h_{\mu\nu}^\varepsilon$, but $h_\varepsilon^{\mu\nu}$.

Now, near $\Sigma \subset M$, we can uniquely solve the eikonal equation and find $w \in C^\infty(M)$ such that $|dw|_{\bar{h}}^2 = 1$ near Σ , $w|_\Sigma = 0$, and $w \geq 0$ on M^+ . The metric \bar{h} then takes the form $\bar{h} = dw^2 + \bar{k}_w$, with \bar{k}_w a one-parameter family of metrics on Σ . Near any point $p \in \Sigma$, we can choose coordinates x^1, x^2 on a neighborhood of p in Σ ; then by the flow of $\text{grad}_{\bar{h}} w$ on M^+ , the system $(x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of p in M . Finally, by the flow of $\text{grad}_{\bar{g}} r$, the system $(r = x^0, x^1, x^2, x^3 = w)$ extends to a coordinate system on a neighborhood of

p in X . Now, we will regard Y as given by a function

$$w = u(r, x^1, x^2), \quad (4.6)$$

where $u(0, x^1, x^2) \equiv 0$. This is the same convention as in [19]. In fact, we may regard a neighborhood of Σ in this way as a product $[0, \varepsilon)_r \times \Sigma \times (-\varepsilon, \varepsilon)_w$; when using this product identification, we will use ζ to refer to a point of Σ , so that a generic point may be written (r, ζ, w) .

When using index notation locally, we will let $0 \leq i, j \leq 3$ be indices on TX ; $1 \leq \mu, \nu \leq 3$ be indices on TM ; and $1 \leq a, b \leq 2$ be indices on $T\Sigma$. We also let $0 \leq \alpha, \beta \leq 2$, which we will use when discussing TY .

Turning to extrinsic geometry, we let $\bar{\mu}_{M_\varepsilon}, \bar{\mu}_Y$ be the X^+ -inward unit \bar{g} -normal to the given hypersurface; the unbarred versions will refer to the unit normal with respect to g_+ . We let $\bar{\nu}_{M_\varepsilon}$ be the \bar{g} -unit normal to Σ_ε that is directed into M^+ , and $\bar{\nu}_{Y_\varepsilon}$, similarly, the Y_ε -inward \bar{g} -unit normal to Σ_ε . We let $\bar{L}_{M_\varepsilon}, \bar{L}_Y$ be the second fundamental forms of the indicated hypersurfaces with respect to the inward unit normals $\bar{\mu}_{M_\varepsilon}$ and $\bar{\mu}_Y$, and computed with respect to \bar{g} . Thus, for example,

$$\bar{L}_Y(A, B) = -\langle \nabla_A^{\bar{g}} \bar{\mu}_Y, B \rangle.$$

The tracefree parts are denoted $\overset{\circ}{\bar{L}}_{M_\varepsilon}$, etc. In all of these, we will sometimes write the hypersurface in the upper position, should it be convenient to do so to place covariant indices; similarly, an unbarred L will refer to the second fundamental form with respect to g_+ instead of \bar{g} . We let $\bar{H}_{M_\varepsilon} = \bar{h}_\varepsilon^{\mu\nu} \bar{L}_{\mu\nu}^{M_\varepsilon}$ be the mean curvature of M_ε with respect to \bar{g} (or, if we omit the ε , that of M); similarly for \bar{H}_Y , while H_{M_ε} and H_Y are the same quantities with respect to g_+ (recall we assume $H_Y \equiv 0$). We let $\bar{\bar{I}}_{Y_\varepsilon}$ be the second fundamental form of Σ_ε viewed as a hypersurface of Y_ε with respect to $\bar{\bar{h}}$, while $\bar{\bar{I}}_{M_\varepsilon}$ is the same for Σ viewed as a hypersurface in M_ε with respect

to \bar{h}_ε . The traces of these (i.e., the mean curvatures of Σ_ε viewed as a hypersurface of the respective three-manifold) we denote $\bar{\eta}_{Y_\varepsilon}, \bar{\eta}_{M_\varepsilon}$. Again, the unbarred versions are with respect to the unbarred metrics \tilde{h} and h_ε . We also let $\bar{\eta}_M$ be the mean curvature of $(\Sigma, \bar{k}) \subset (M, \bar{h})$.

We define a smooth function $\theta_0^\varepsilon \in C^\infty(\Sigma_\varepsilon)$ to be the angle, at each point, between Y and M_ε ; that is, $\cos(\theta_0^\varepsilon) = -\langle \bar{\mu}_Y, \bar{\mu}_{M_\varepsilon} \rangle$. If the ε is omitted, then it denotes the angle between M and Y at a point of Σ . Since θ_0^ε is manifestly a conformal invariant, we do not distinguish between barred and unbarred versions.

If A is a vector or tensor field, we write $A = O_{\bar{g}}(\varphi)$, for φ a function, whenever $|A|_{\bar{g}} = O(\varphi)$.

4.3 The Gauss-Bonnet Formula

We now prove Theorem 1.3. We do so by using a form of the Gauss-Bonnet formula that has good conformal invariance properties, which allows us to compute using \bar{g} instead of g_+ .

Proof of Theorem 1.3. Let (X, M, g_+) be an AHE space with conformal infinity $[\bar{h}]$, and let Y be as in the previous section. Let $\bar{h} \in [\bar{h}]$, and let r be the corresponding geodesic defining function. Let $\varepsilon > 0$. Then $\overline{X_\varepsilon^+}$ is a four-manifold with codimension-two corner Σ_ε , and boundary hypersurfaces M_ε^+ and Y_ε (see section 4.2 for all notation). The Gauss-Bonnet theorem for Riemannian manifolds with corners (in this case X_ε^+), proven first in [2] (and see [9]), can be rewritten in the following conformally useful way ([27], building on [6]).

$$\begin{aligned}
4\pi^2\chi(X_\varepsilon^+) &= \int_{X_\varepsilon^+} \left(\frac{1}{8}|W_{g_+}|_{g_+}^2 + \frac{1}{2}Q_{g_+} \right) dv_{g_+} + \int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y) dv_{\bar{h}} \\
&\quad + \int_{M_\varepsilon^+} (\mathcal{L}_{M_\varepsilon} + T_{M_\varepsilon}) dv_h + \oint_{\Sigma_\varepsilon} (U_{\Sigma_\varepsilon} + G_{\Sigma_\varepsilon}) dv_{k_\varepsilon}.
\end{aligned} \tag{4.7}$$

Here, W_{g_+} is the Weyl tensor of g_+ , and the norm in question is its two-tensor norm

$W^{ijkl}W_{ijkl}$. Meanwhile, Q_{g_+} is the Q -curvature of g_+ , defined for any metric g by

$$Q_g = -\frac{1}{6}\Delta_g R_g + \frac{1}{6}R_g^2 - \frac{1}{2}R_g^{ij}R_{ij}^g.$$

Here, the Laplacian is a negative operator and the curvatures are respectively the scalar and Ricci curvatures of g . For any metric g , the quantity $|W_g|_g^2 dv_g$ is a pointwise conformal invariant of weight zero. Under a conformal transformation $\tilde{g} = e^{2\omega}g$, the Q curvature transforms as

$$e^{4\omega}Q_{\tilde{g}} = Q_g + P_4^g\omega,$$

where P_4^g is the *Paneitz operator* associated to g ; we will not use the Paneitz operator and so omit it here.

We give the definition of \mathcal{L}_N and T_N , due to [6], for an arbitrary boundary hypersurface (N^3, h) embedded in a four-manifold endowed with metric g . The definition is

$$\mathcal{L}_N = \mathring{L}_N^{\mu\nu}R_{\mu\nu}^g - 2\mathring{L}_N^{\mu\nu}R_{\mu\nu}^h + \frac{2}{3}H_N|\mathring{L}_N|_h^2 - \text{tr}_h \mathring{L}_N^3, \quad (4.8)$$

where L_N and H_N are the second fundamental form and the mean curvature as before, and μ, ν are indices on TN . Similarly, the T -curvature is defined by

$$\begin{aligned} T_N = & -\frac{1}{12}\mu(R_g) - \mathring{L}_N^{\mu\nu}R_{\mu\nu}^g + \mathring{L}_N^{\mu\nu}R_{\mu\nu}^h - \frac{1}{2}H_N|\mathring{L}_N|_h^2 + \frac{2}{3}\text{tr}_h \mathring{L}_N^3 \\ & + \frac{1}{6}R_h H_N - \frac{1}{27}H_N^3 - \frac{1}{3}\Delta_h H_N, \end{aligned} \quad (4.9)$$

where μ is the inward-pointing unit normal to N . Under the conformal change $\tilde{g} = e^{2\omega}g$, this transforms according to the equation

$$e^{3\omega}\tilde{T}_N = T_N + P_3^g\omega, \quad (4.10)$$

where $P_3^g : C^\infty(X) \rightarrow C^\infty(N)$ is the conformally covariant boundary operator

$$\begin{aligned} P_3^g f = & \frac{1}{2} \mu \Delta_g f - \Delta_h \mu(f) - H_N \Delta_h f - \mathring{L}_N^{\mu\nu} \nabla_\mu^h \nabla_\nu^h f - \frac{1}{3} H_N^\mu f_\mu \\ & + \left(\frac{1}{6} R_g - \frac{1}{2} R_h - \frac{1}{2} |\mathring{L}_N|_h^2 + \frac{1}{3} H_N^2 \right) \mu(f). \end{aligned} \quad (4.11)$$

Next we turn to the corner quantities. For a corner (Ξ, k) that forms the intersection between two boundary hypersurfaces N and S making angle $\theta_0 \in C^\infty(\Xi)$, G is defined by

$$G_\Xi = \frac{1}{2} \cot(\theta_0) (|\mathring{I}I_N|_k^2 + |\mathring{I}I_S|_k^2) - \csc(\theta_0) \mathring{I}I_{ab}^N \mathring{I}I_S^{ab}, \quad (4.12)$$

where II , etc., are as in section 4.2. The G curvature is a pointwise conformal invariant of weight -2 (when the ambient metric on the four-manifold is changed conformally). Next, U_Ξ is defined by

$$U_\Xi = (\pi - \theta_0) K_\Xi - \frac{1}{4} \cot(\theta_0) (\eta_N^2 + \eta_S^2) + \frac{1}{2} \csc(\theta_0) \eta_N \eta_S - \frac{1}{3} (\nu_N H_N + \nu_S H_S). \quad (4.13)$$

Here, K_Ξ is the Gaussian curvature of Ξ , and the other quantities are defined analogously to those in the previous section. Under a global conformal change $\tilde{g} = e^{2\omega} g$, U transforms according to the equation

$$e^{2\omega} \tilde{U}_\Xi = U_\Xi + P_2^g \omega, \quad (4.14)$$

where $P_2^g : C^\infty(X) \rightarrow C^\infty(\Xi)$ is the conformally covariant operator

$$\begin{aligned} P_2^g f = & (\theta_0 - \pi) \Delta_k f + \nu_N \mu_N f + \nu_S \mu_S f \\ & + \cot(\theta_0) (\eta_N \nu_N f + \eta_S \nu_S f) - \csc(\theta_0) (\eta_S \nu_N f + \eta_N \nu_S f) \\ & + \frac{1}{3} (H_N \nu_N f + H_S \nu_S f). \end{aligned} \quad (4.15)$$

We now analyze formula (4.7) in the context of our space (X_ε^+, g_+) . Because $|W_{g_+}|_{g_+}^2 dv_{g_+}$ is a pointwise conformal invariant of weight zero, its integral converges as $\varepsilon \rightarrow 0$ to $\int_{X^+} |W_{\bar{g}}|_{\bar{g}}^2 dv_{\bar{g}}$, which in particular is finite.

In our setting, $R_{ij}^{g_+} = -3g_{ij}^+$ and $R_{g_+} \equiv -12$, so $\Delta_{g_+} R_{g_+} \equiv 0$ and $Q_{g_+} \equiv 6$. The integral of $\frac{1}{2}Q_{g_+}$ therefore is simply the integral of 3, so the second integral over X_+ becomes simply $3 \text{vol}_{g_+}(\{r > \varepsilon\} \cap X^+)$, which is the same quantity considered in (4.5), except that the latter is over all of X instead of X^+ . To compute the contribution from this integral, we consider four different regions of X . First, let $r_0 > 0$ be small – sufficiently small, in particular, that the geodesic normal form (4.2) holds for $r < 2r_0$, and that the region $\mathcal{U} = \{r < 2r_0, -2r_0 < w < 2r_0\}$ has the decomposition $[0, 2r_0) \times \Sigma \times (-2r_0, 2r_0)$, with $|u(r, \zeta)| < \frac{1}{2}r_0$ on \mathcal{U} . Having chosen r_0 , we will leave it fixed for all time.

The first region of interest to us is then $A = \{p \in X^+ : r(p) \geq r_0\}$. (This set does not depend on ε , which we assume is smaller than r_0 .) Next, we want to capture the points near the boundary M_ε^+ . The obvious set to consider is $B_\varepsilon = (\varepsilon, r_0) \times M^+$. The problem is that this may omit points that are contained in X^+ or include points contained in X^- , because Y is given not by $w = 0$ but by $w = u(r, \zeta)$, where u may be positive or negative away from Σ . To address this, we need to add the volume of the omitted points, C_ε , and subtract the volume of the over-included points D_ε , viz.,

$$X_\varepsilon^+ = (A \cup B_\varepsilon \cup C_\varepsilon) \setminus D_\varepsilon.$$

To proceed, we analyze the volume form dv_{g_+} . First, at all points, we have $dv_{g_+} = r^{-4} dv_{\bar{g}}$. Near M , we can write

$$dv_{\bar{g}} = dv_{\bar{h}_r} dr$$

using the normal-form identification (4.1). Now in local coordinates (r, x^1, x^2, x^3)

near M , we may write

$$dv_{\bar{h}_r} = \sqrt{\frac{\det(\bar{h}_r)}{\det(\bar{h})}} dv_{\bar{h}}.$$

As shown for example in [16], we have the expansion

$$\sqrt{\frac{\det(\bar{h}_r)}{\det(\bar{h})}} = 1 + v^{(2)}r^2 + v^{(4)}r^4 + O(r^5),$$

where $v^{(2)}, v^{(4)} \in C^\infty(M)$ are the so-called *renormalized volume coefficients*. Either by direct computation using (4.3) or by using equation (4.5) and the equation at the top of the same page of ([17]) (remembering that M is totally geodesic with respect to \bar{g} and that the singular Yamabe metric for \bar{g} is g_+), we may show that $v^{(2)} = -\frac{1}{8}R_{\bar{h}}$. Thus,

$$\begin{aligned} dv_{g_+} &= r^{-4} \left(1 - \frac{1}{8}r^2 R_{\bar{h}} + O(r^4) \right) dv_{\bar{h}} dr \\ &= \left(r^{-4} - \frac{1}{8}r^{-2} R_{\bar{h}} + O(1) \right) dv_{\bar{h}} dr. \end{aligned}$$

We next derive an expression for $dv_{\bar{g}}$ (and thus dv_{g_+}) near Σ . Since $\bar{h} = dw^2 + \bar{k}_w$ near Σ , we have

$$\begin{aligned} dv_{\bar{h}} &= \sqrt{\frac{\det(\bar{k}_w)}{\det(\bar{k})}} dv_{\bar{k}} dw \\ &= (1 + O(w)) dv_{\bar{k}} dw. \end{aligned}$$

Hence, near Σ , we have

$$dv_{g_+} = \left(r^{-4} - \frac{1}{8}r^{-2} R_{\bar{h}} + O(1) \right) (1 + O(w)) dv_{\bar{k}} dw dr.$$

We then have

$$\begin{aligned}
\text{vol}_{g_+}(X_\varepsilon^+) &= \text{vol}_{g_+}(A) + \text{vol}_{g_+}(B_\varepsilon) + \text{vol}_{g_+}(C_\varepsilon) - \text{vol}_{g_+}(D_\varepsilon) \\
&= \text{vol}_{g_+}(A) + \int_{M^+} \int_\varepsilon^{r_0} \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}} \\
&\quad - \oint_\Sigma \int_\varepsilon^{r_0} \int_0^{u(r, \zeta)} (r^{-4} + O(r^{-2})) (1 + O(w)) dw dr dv_{\bar{k}}(\zeta), \quad (4.16)
\end{aligned}$$

where the last integral represents $\text{vol}_{g_+}(C_\varepsilon) - \text{vol}_{g_+}(D_\varepsilon)$. Now, by equations (2.13) and (2.14) in [19],

$$u(r, \zeta) = \frac{1}{4} r^2 \bar{\eta}_M(\zeta) + r^4 \log(r) v(\zeta) + O(r^4), \quad (4.17)$$

where $\bar{\eta}_M$ is the mean curvature of Σ viewed as a hypersurface of (M, \bar{h}) and $v \in C^\infty(\Sigma)$. Thus, we find

$$\begin{aligned}
3 \text{vol}_{g_+}(X_\varepsilon^+) &= 3 \text{vol}_{g_+}(A) + 3 \int_M \int_\varepsilon^{r_0} \left(r^{-4} - \frac{1}{8} r^{-2} R_{\bar{h}} + O(1) \right) dr dv_{\bar{h}} \\
&\quad - 3 \oint_\Sigma \int_\varepsilon^{r_0} \left(\frac{1}{4} r^{-2} \bar{\eta}_M + v \log(r) + O(1) \right) dr dv_{\bar{k}} \\
&= \varepsilon^{-3} \text{vol}_{\bar{h}}(M^+) - \varepsilon^{-1} \left(\frac{3}{8} \int_{M^+} R_{\bar{h}} dv_{\bar{h}} + \frac{3}{4} \oint_\Sigma \bar{\eta}_M dv_{\bar{k}} \right) \\
&\quad + 3V_+^+ + o(1). \quad (4.18)
\end{aligned}$$

Here V_+^+ is the collection of all the order-zero terms in ε in the volume expansion, and is defined to be the renormalized volume; of course, we have not shown so far that V_+^+ is independent of the choice of $\bar{h} \in [\bar{h}]$ (or equivalently, of r).

Since (as we saw above) $Q_{g_+} = 6$, the above right-hand side is thus the integral $\int_{X^+} \frac{1}{2} Q_{g_+} dv_{g_+}$. We next turn to the boundary integrals over Y_ε and M_ε , beginning with Y_ε . We will analyze \mathcal{L}_Y and T_Y with respect to the metric g_+ ; of course, since \mathcal{L}_Y is a pointwise conformal invariant, it is automatic that the integral of \mathcal{L}_Y

over Y_ε will converge as $\varepsilon \rightarrow 0$. Now, because g_+ is Einstein and Y is minimal in (X, g_+) , the first and third terms in (4.8) vanish in this case. Thus, we get simply $\mathcal{L}_Y = -2\mathring{L}_Y^{\alpha\beta} R_{\alpha\beta}^{\tilde{h}} - \text{tr}_{\tilde{h}} \mathring{L}_Y^3$.

Next turning to T_Y , we again compute with respect to the ambient metric g_+ , i.e., with respect to the non-compactified setting. Again, due to the Einstein condition of g_+ and the minimal condition on Y , the first, second, fourth, sixth, seventh, and eighth terms of (4.9) vanish, so we get

$$\begin{aligned} T_Y &= \mathring{L}^{\alpha\beta} R_{\alpha\beta}^{\tilde{h}} + \frac{2}{3} \text{tr}_{\tilde{h}} \mathring{L}_Y^3 \\ &= -\frac{1}{2} \mathcal{L}_Y + \frac{1}{6} \text{tr}_{\tilde{h}} \mathring{L}_Y^3. \end{aligned}$$

Now, \mathcal{L}_Y and $\text{tr}_{\tilde{h}} \mathring{L}_Y^3$ are both pointwise conformal invariants of weight -3 , so we have exhibited T_Y itself as such a pointwise conformal invariant. We define

$$\mathcal{C}_Y = \frac{1}{2} \mathcal{L}_Y + \frac{1}{6} \text{tr}_{\tilde{h}} \mathring{L}_Y^3.$$

This is a pointwise conformal invariant, and the upshot of the above remarks is that

$$\int_{Y_\varepsilon} (\mathcal{L}_Y + T_Y) dv_{\tilde{h}} = \int_{Y_\varepsilon} \mathcal{C}_Y dv_{\tilde{h}} = \int_Y \mathcal{C}_Y dv_{\tilde{h}} + O(\varepsilon). \quad (4.19)$$

We now turn to the integral over M_ε^+ in (4.7). Here, we will compute \bar{T}_{M_ε} and $\bar{\mathcal{L}}_{M_\varepsilon}$, the extrinsic curvature quantities with respect to the *compactified* metrics \bar{g} and \bar{h}_ε ; then we will compute the transformation to g_+, h_ε using equation (4.10), which in particular implies that

$$\int_{M_\varepsilon^+} (\mathcal{L}_M + T_M) dv_{g_+} = \int_{M_\varepsilon^+} (\bar{\mathcal{L}}_M + \bar{T}_M + P_3^{\bar{g}}(-\log r)) dv_{\bar{g}}.$$

Our goal is thus to compute the right-hand side of this equation. We begin by

computing some basic quantities. Recalling that $\bar{g} = dr^2 + \bar{h}_r$ and $M_\varepsilon = \{r = \varepsilon\}$, we find that

$$\bar{L}_{M_\varepsilon} = -\frac{1}{2}\partial_r \bar{h}_r|_{r=\varepsilon} = \varepsilon P^{\bar{h}} + O(\varepsilon^2),$$

where $P^{\bar{h}}$ is the Schouten tensor of \bar{h} , and we have used (4.3). Thus,

$$\bar{H}_{M_\varepsilon} = \varepsilon(P_{\bar{h}}^\mu)_\mu + O(\varepsilon^3) = \frac{1}{4}\varepsilon R_{\bar{h}} + O(\varepsilon^3). \quad (4.20)$$

The reason the error is $O(\varepsilon^3)$ is that the r^3 term in the expansion of \bar{h}_r is trace-free.

We also have

$$\overset{\circ}{\bar{L}}_{M_\varepsilon} = \varepsilon \overset{\circ}{P}^{\bar{h}} + O(\varepsilon^2).$$

We next wish to compute $R_{\bar{g}}$ on M_ε . To do this, we use the fact that $R_{g_+} \equiv -12$ and that $g_+ = r^{-2}\bar{g}$. Thus, we will use the conformal transformation formula for scalar curvature. Let $\omega = -\log(r)$. It will be useful to record that

$$\Delta_{\bar{g}}\omega = r^{-2} + \frac{1}{4}R_{\bar{h}} + O(r^2), \quad (4.21)$$

which follows easily from (4.3). Thus, from the conformal change formula, we find

$$\begin{aligned} -12 &= r^2(R_{\bar{g}} - 6\Delta_{\bar{g}}\omega - 6|d\omega|_{\bar{g}}^2) \\ &= r^2\left(R_{\bar{g}} - 6r^{-2} - \frac{3}{2}R_{\bar{h}} - 6r^{-2} + O(r^2)\right), \end{aligned}$$

whence

$$R_{\bar{g}} = \frac{3}{2}R_{\bar{h}} + O(r^2).$$

We next compute the tracefree tangential Ricci tensor $\overset{\circ}{R}_{\mu\nu}^{\bar{g}}$. We will use again the same technique of conformal transformation and the fact that $\text{Ric}(g_+) = -3g_+$. We

first find using (4.3) that

$$\nabla_{\mu}^{\bar{g}} \nabla_{\nu}^{\bar{g}} \omega = P_{\mu\nu}^{\bar{h}} + O(r).$$

It then follows from the equation

$$R_{\mu\nu}^{g+} = R_{\mu\nu}^{\bar{g}} - 2\nabla_{\mu}^{\bar{g}} \nabla_{\nu}^{\bar{g}} \omega + 2\omega_{\mu}\omega_{\nu} - (\Delta_{\bar{g}}\omega - 2|d\omega|_{\bar{g}}^2)\bar{g}_{\mu\nu}$$

that

$$\mathring{R}_{\mu\nu}^{\bar{g}} = 2\mathring{P}_{\mu\nu}^{\bar{h}} + O(r).$$

We are ready to analyze the curvature integrands on M_{ε} . First, we easily find using (4.8) and the above that

$$\bar{\mathcal{L}}_{M_{\varepsilon}} = O(\varepsilon),$$

where the first-order contribution is from the first two terms of (4.8), and the last two terms provide contributions of order $O(\varepsilon^3)$. Next, we compute $\bar{T}_{M_{\varepsilon}}$, recalling that $\bar{\mu}_{M_{\varepsilon}} = \frac{\partial}{\partial r}$. Then it again follows from the above computations that

$$\bar{T}_{M_{\varepsilon}} = O(\varepsilon).$$

The lowest-order contributions come once again from the first three terms of (4.9), as well as the sixth.

We next turn to computing $P_3^{\bar{g}}(\omega) = -P_3^{\bar{g}}(\log(r))$ for $P_3^{\bar{g}}$ associated to M_{ε} . First, observe that $\omega|_{M_{\varepsilon}} \equiv -\log(\varepsilon)$, and $\bar{\mu}_{M_{\varepsilon}}(\omega) \equiv \frac{1}{\varepsilon}$. Thus, all tangential derivatives of both quantities vanish, which means the second through fifth terms of (4.11) vanish. Thus, only the first and last remain. It follows from (4.21) that

$$\frac{1}{2}\bar{\mu}_{M_{\varepsilon}}\Delta_{\bar{g}}\omega = -\varepsilon^{-3} + O(\varepsilon).$$

Next, using again the facts that $R_{\bar{g}} = \frac{3}{2}R_{\bar{h}} + O(r^2)$ and our above calculations, we

find that the last term of (4.11) simplifies to

$$\left(\frac{1}{6}R_{\bar{g}} - \frac{1}{2}R_{\bar{h}_\varepsilon} - \frac{1}{2}|\overset{\circ}{L}_{M_\varepsilon}|_{\bar{h}_\varepsilon}^2 + \frac{1}{3}\bar{H}_{M_\varepsilon}^2 \right) \bar{\mu}(-\log(r)) = \frac{1}{4}\varepsilon^{-1} + O(\varepsilon).$$

Now, we wish to perform the integral over M_ε^+ , not M_ε . Just as for the interior integral, the simplest approach will be first to compute the integral over $\{\varepsilon\} \times M^+$, and then subtract or add whatever was missed near the corner due to turning of Y away from Σ . First, we observe that from our above computations, it is clear that

$$\int_{M_\varepsilon^+} (\bar{T}_{M_\varepsilon} + \bar{\mathcal{L}}_{M_\varepsilon} + P_3^{\bar{g}}(-\log(r))) dv_{\bar{h}_\varepsilon} = \int_{M_\varepsilon^+} P_3^{\bar{g}}(-\log(r)) dv_{\bar{h}_\varepsilon} + O(\varepsilon).$$

We may focus therefore only on contributions from $P_3^{\bar{g}}(-\log(r))$. We write

$$\begin{aligned} \int_{M_\varepsilon^+} P_3^{\bar{g}}(\omega) dv_{\bar{h}_\varepsilon} &= \int_{\{\varepsilon\} \times M^+} P_3^{\bar{g}}(\omega) dv_{\bar{h}_\varepsilon} \\ &\quad - \oint_{\Sigma} \int_0^{u(\varepsilon, \zeta)} P_3^{\bar{g}}(\omega) (1 + O(w)) dw dv_{\bar{k}}(\zeta). \end{aligned}$$

(Compare (4.16).) We compute the first term first. Recall that $dv_{\bar{h}_\varepsilon} = (1 - \frac{1}{8}\varepsilon^2 R_{\bar{h}} + O(\varepsilon^4)) dv_{\bar{h}}$. Thus,

$$\begin{aligned} \int_{\{\varepsilon\} \times M^+} P_3^{\bar{g}}(\omega) &= \int_{M^+} \left(-\varepsilon^{-3} + \frac{1}{4}\varepsilon^{-1} R_{\bar{h}} + O(\varepsilon) \right) \left(1 - \frac{1}{8}\varepsilon^2 R_{\bar{h}} + O(\varepsilon^4) \right) dv_{\bar{h}} \\ &= -\varepsilon^{-3} \text{vol}_{\bar{h}}(M^+) + \frac{3}{8}\varepsilon^{-1} \int_{M^+} R_{\bar{h}} dv_{\bar{h}} + O(\varepsilon). \end{aligned}$$

As for the corner integral, we find using (4.17)

$$\begin{aligned} \oint_{\Sigma} \int_0^{u(\varepsilon, \zeta)} P_3^{\bar{g}}(\omega) (1 + O(w)) dw dv_{\bar{k}}(\zeta) &= \oint_{\Sigma} (-\varepsilon^{-3} + O(\varepsilon^{-1})) \\ &\quad \cdot \left(\frac{1}{4}\varepsilon^2 \bar{\eta}_M + O(\varepsilon^4 \log(\varepsilon)) \right) dv_{\bar{k}} \\ &= -\frac{1}{4}\varepsilon^{-1} \oint_{\Sigma} \bar{\eta}_M dv_{\bar{k}} + O(\varepsilon \log \varepsilon). \end{aligned}$$

Thus, we have found that

$$\begin{aligned} \int_{M_\varepsilon^+} (T_M + \mathcal{L}_M) dv_{g_+} &= -\varepsilon^{-3} \text{vol}_{\bar{h}}(M^+) \\ &+ \varepsilon^{-1} \left(\frac{3}{8} \int_{M^+} R_{\bar{h}} dv_{\bar{h}} + \frac{1}{4} \oint_{\Sigma} \bar{\eta}_M dv_{\bar{k}} \right) + o(1). \end{aligned} \quad (4.22)$$

We are finally ready to evaluate the corner terms U_{Σ_ε} and G_{Σ_ε} in (4.7). Just as for M_ε^+ , our strategy will be to evaluate first with respect to \bar{g} , and then use the conformal transformation formula (4.14) and the pointwise conformal invariance of G . Thus, we will find

$$\oint_{\Sigma_\varepsilon} (G_k + U_k) dv_k = \oint_{\Sigma_\varepsilon} (\bar{G}_{\Sigma_\varepsilon} + \bar{U}_{\Sigma_\varepsilon} + P_2^{\bar{g}}(-\log r)) dv_{\bar{k}_\varepsilon}.$$

To begin, we wish to estimate θ_0^ε , which enters the formulas for U, G , and P_2 . To do this, we find normal vectors $\bar{\mu}_{M_\varepsilon}$ and $\bar{\mu}_Y$. The first is easy: $\bar{\mu}_{M_\varepsilon} = \frac{\partial}{\partial r}$. For the second, we observe that, for ε small, we can write Y as the zero level set of $F = w - u(r, \zeta)$ (where, again, $\zeta \in \Sigma$). Now,

$$\begin{aligned} \text{grad}_{\bar{g}} F &= (1 + O(r^2)) \frac{\partial}{\partial w} - \frac{\partial u}{\partial r} \frac{\partial}{\partial r} - \bar{k}^{ab} \frac{\partial u}{\partial x^a} \frac{\partial}{\partial x^b} + O^i(r^3 \log(r)) \partial_i \\ &= (1 + O(r^2)) \frac{\partial}{\partial w} - \frac{1}{2} r \bar{\eta}_M \frac{\partial}{\partial r} - \frac{1}{4} r^2 \bar{k}^{ab} \frac{\partial \bar{\eta}_M}{\partial x^a} \frac{\partial}{\partial x^b} + O_{\bar{g}}(r^3 \log(r)). \end{aligned}$$

Since $\left| \frac{\partial}{\partial w} \right|_{\bar{g}} = 1 + O(r^2)$, we have

$$|\text{grad}_{\bar{g}} F|_{\bar{g}} = 1 + O(r^2).$$

Consequently,

$$\bar{\mu}_Y = \frac{\text{grad}_{\bar{g}} F}{|\text{grad}_{\bar{g}} F|_{\bar{g}}} = (1 + O(r^2)) \frac{\partial}{\partial w} - \left(\frac{1}{2} r \bar{\eta}_M + O(r^3 \log(r)) \right) \frac{\partial}{\partial r} + O_{\bar{g}}(r^2). \quad (4.23)$$

Thus,

$$\cos(\theta_0^\varepsilon) = -\langle \bar{\mu}_{M_\varepsilon}, \bar{\mu}_Y \rangle = \frac{1}{2} \varepsilon \bar{\eta}_M + O(\varepsilon^3 \log(\varepsilon)).$$

Next we wish to estimate the second fundamental form $\overline{II}_{Y_\varepsilon}$ of Σ_ε viewed as a submanifold of Y_ε . To do this, we first want to know the inward-pointing unit normal vector $\bar{\nu}_{Y_\varepsilon}$ to Σ_ε in Y_ε . By inspection, we can see that

$$V = \frac{\partial}{\partial r} - \frac{\partial F}{\partial r} \frac{\text{grad}_{\bar{g}} F}{|dF|_{\bar{g}}^2}$$

is normal to Σ_ε and tangent to Y_ε , so

$$\bar{\nu}_{Y_\varepsilon} = \frac{V}{|V|_{\bar{g}}} = (1 + O(\varepsilon^2)) \frac{\partial}{\partial r} + \frac{1}{2} \varepsilon \bar{\eta}_M \frac{\partial}{\partial w} + O(\varepsilon^3 \log \varepsilon). \quad (4.24)$$

Now, a local frame for $T\Sigma_\varepsilon$ is given by $\{X_1, X_2\}$, where

$$X_a = \frac{\partial}{\partial x^a} - \frac{\partial F}{\partial x^a} \frac{\partial}{\partial w}.$$

Since $\nabla_{\partial_r}^{\bar{g}} \partial_r = O^i(\varepsilon) \partial_i$ (which is easy to check), we may conclude that $\langle \nabla_{X_a}^{\bar{g}} \bar{\nu}_{Y_\varepsilon}, X_b \rangle_{\bar{g}} = O(\varepsilon)$. Thus, by Weingarten's equation,

$$|\overline{II}_{Y_\varepsilon}|_{\bar{g}} = O(\varepsilon).$$

It now follows that $\overline{G}_{\Sigma_\varepsilon} = O(\varepsilon)$: the first term in (4.12) because $\cot(\theta_0^\varepsilon) = O(\varepsilon)$, and the second because of the estimate on $\overline{II}_{Y_\varepsilon}$.

We next turn to $\overline{U}_{\Sigma_\varepsilon}$. The second and third terms in (4.13) are $O(\varepsilon)$ for the same reason. Turning to the fourth term, $\bar{\nu}_M \overline{H}_{M_\varepsilon} = O(\varepsilon)$ by (4.20). To compute $\bar{\nu}_{Y_\varepsilon} \overline{H}_Y$, we first compute \overline{H}_Y using the conformal change formula. Recall that $H_Y \equiv 0$. Then again taking $\omega = -\log r$, we find from the conformal transformation formula

$H_Y = e^{-\omega}(\overline{H}_Y - 3\overline{\mu}_Y(\omega))$ that

$$0 = r(\overline{H}_Y - \frac{3}{2}\overline{\eta}_M + O(r^2 \log(r))),$$

whence

$$\overline{H}_Y = \frac{3}{2}\overline{\eta}_M + O(r^2 \log(r)).$$

Thus, $\overline{\nu}_{Y_\varepsilon}\overline{H}_Y = O(\varepsilon \log(\varepsilon))$; so since $\theta_0^\varepsilon = \frac{\pi}{2} + O(\varepsilon)$, we have

$$\overline{U}_{\Sigma_\varepsilon} = \frac{\pi}{2}K_{\bar{k}} + O(\varepsilon \log \varepsilon).$$

Consequently,

$$\oint_{\Sigma_\varepsilon} (\overline{G}_{\Sigma_\varepsilon} + \overline{U}_{\Sigma_\varepsilon}) dv_{\bar{k}_\varepsilon} = \pi^2 \chi(\Sigma) + O(\varepsilon \log \varepsilon).$$

We still need to compute the integral of $P_2^{\bar{g}}(-\log r)$. First, still letting $\omega = -\log r$, observe that $\omega|_{M_\varepsilon} \equiv -\log \varepsilon$ and that $\overline{\mu}_M \omega \equiv -\frac{1}{\varepsilon}$. Thus, the first and second terms of (4.15) in $P_2^{\bar{g}}(\omega)$ vanish identically, as do the terms $\overline{\eta}_{M_\varepsilon} \overline{\nu}_{M_\varepsilon} \omega$, $\overline{\eta}_{Y_\varepsilon} \overline{\nu}_{M_\varepsilon} \omega$, and $\overline{H}_{M_\varepsilon} \overline{\nu}_{M_\varepsilon} \omega$.

Now, the third term takes the form

$$\begin{aligned} \overline{\nu}_{Y_\varepsilon} \overline{\mu}_Y \omega &= \overline{\nu}_{Y_\varepsilon} \left(\frac{1}{2} \overline{\eta}_M + O(r^2 \log(r)) \right) \\ &= \frac{1}{4} \varepsilon \overline{\eta}_M \partial_w \overline{\eta}_M + O(\varepsilon \log \varepsilon) \\ &= O(\varepsilon \log \varepsilon). \end{aligned}$$

Next, $\overline{\nu}_{Y_\varepsilon} \omega = -\frac{1}{\varepsilon} + O(\varepsilon)$, so $\cot(\theta_0^\varepsilon) \overline{\eta}_{Y_\varepsilon} \overline{\nu}_{Y_\varepsilon} \omega = O(\varepsilon)$.

On the other hand, $-\csc(\theta_0^\varepsilon) \overline{\eta}_{M_\varepsilon} \overline{\nu}_{Y_\varepsilon} \omega = \varepsilon^{-1} \overline{\eta}_M + O(\varepsilon)$, since $\overline{\eta}_{M_\varepsilon} = \overline{\eta}_M + O(\varepsilon^2)$ and $\csc(\theta_0^\varepsilon) = 1 + O(\varepsilon^2)$.

Finally,

$$\frac{1}{3}\overline{H}_Y\bar{\nu}_{Y_\varepsilon}\omega = -\frac{1}{2}\varepsilon^{-1}\bar{\eta}_M + O(\varepsilon \log \varepsilon).$$

Adding together all these terms, we therefore find that $P_2^{\bar{g}}\omega = \frac{1}{2}\varepsilon^{-1}\bar{\eta}_M + O(\varepsilon \log \varepsilon)$.

Thus,

$$\oint_{\Sigma_\varepsilon} (\overline{G}_\varepsilon + \overline{U}_\varepsilon + P_2^{\bar{g}}(-\log r)) dv_{\bar{k}_\varepsilon} = \frac{1}{2}\varepsilon^{-1} \oint_{\Sigma} \bar{\eta}_M dv_{\bar{k}} + \pi^2\chi(\Sigma) + O(\varepsilon \log \varepsilon). \quad (4.25)$$

Combining (4.7), (4.18), (4.19), (4.22), and (4.25), we find

$$\pi^2(4\chi(X_\varepsilon^+) - \chi(\Sigma)) = 3V_+^+ + \frac{1}{8} \int_{X_\varepsilon^+} |W_{g_+}|_{g_+}^2 dv_{g_+} + \int_{Y_\varepsilon} \mathcal{C}_Y dv_{\bar{h}} + O(\varepsilon \log \varepsilon).$$

Letting $\varepsilon \rightarrow 0$ yields the result. □

4.4 Variation of Renormalized Volume

In this section we give a proof of Theorem 1.5. Since this will require extensive calculations we begin by establishing some new notational conventions.

In addition to using the coordinate system (r, x^1, x^2, w) , it will be convenient to use the system $(x^{\bar{0}}, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}) = (r, x^1, x^2, w - u)$, where u is as in (4.6). We will still use $0 \leq i, j \leq 3$ to refer to coordinate fields on X , but will use $0 \leq \tilde{\alpha}, \tilde{\beta} \leq 2$ to refer to the coordinate fields tangent to Y . It will also be useful on the interior $\overset{\circ}{X}$ to let $x^{\hat{n}}$ be the g_+ -distance to $\overset{\circ}{Y}$, so that $\frac{\partial}{\partial x^{\hat{n}}} = \mu_Y$ is the g_+ -unit inward normal vector to $\overset{\circ}{Y}$. The system $(r, x^1, x^2, x^{\hat{n}})$ is clearly another coordinate system near $\overset{\circ}{Y}$, and the corresponding coordinate vector fields tangent to Y are the same.

As in the introduction, suppose $\mathcal{F} : (-\varepsilon, \varepsilon)_t \times Y \rightarrow X$ is a C^3 variation of Y through minimal surfaces in X such that $\mathcal{F}(t, \Sigma) \subset M$ for all t . For each $t \in (-\varepsilon, \varepsilon)$, $\mathcal{F}_t(Y) = Y^t$ splits X into two disjoint sets, X_t^+ , X_t^- and we can make our choice of X_t^+ consistent by fixing a point $p \in X_0^+$ and requiring that $p \in X_t^+$ for t in a possibly

smaller time interval $t \in (-\delta, \delta)$. Let $V_+^+(t) = V_+^+(X_t^+)$. We will also use the notation $V_+^+(\mathcal{F}_t(Y))$. Our goal is to use the formula (1.13) to compute a formula for the first variation, \dot{V}_+^+ .

Before proceeding we recall that strictly speaking, the formula for V_+^+ given by (1.13) only holds for minimal Y . However, as we remarked in the introduction, one can use this formula to define V_+^+ for any dividing hypersurface, in particular for $Y^t = \mathcal{F}_t(Y)$, where \mathcal{F}_t is a general variation of Y .

We begin by showing that we can make two simplifying assumptions about the variation \mathcal{F} . First, we show that it suffices to consider normal variations of Y . We then show that we can weaken the assumption that $Y^t = \mathcal{F}_t(Y)$ is minimal for each t , and only assume that minimality is preserved infinitesimally.

To see why it suffices to consider normal variations, let Z be the variation field of \mathcal{F} :

$$\left. \frac{d}{dt} \mathcal{F}_t \right|_{t=0} = Z.$$

We may uniquely write $Z = Z^\perp + Z^\top$, where the two fields are respectively orthogonal to Y (with respect to any compactification \bar{g}) and tangential to Y . Clearly, it does not matter which compactification is chosen when defining Z^\perp , since orthogonality is a conformally invariant notion.

We claim that \dot{V}_+^+ only depends on Z^\perp . By Theorem 9.34 of [25] and the fact that $\mathcal{F}_t(\Sigma) \subset M$ for all t , there is a smooth flow $\mathcal{F}^\top : (-\varepsilon, \varepsilon) \times Y \rightarrow X$ generated by Z^\top , such that $\mathcal{F}_t^\top(Y) = Y$. Therefore,

$$\left. \frac{d}{dt} V_+^+(\mathcal{F}_t^\top(Y)) \right|_{t=0} = 0.$$

Also, if $\mathcal{F}^\perp : (-\varepsilon, \varepsilon) \times Y \rightarrow X$ is any normal variation of Y such that $\left. \frac{d}{dt} \mathcal{F}_t^\perp \right|_{t=0} = Z^\perp$,

then by linearity of the derivative

$$\begin{aligned} \left. \frac{d}{dt} V_+^+(\mathcal{F}_t(Y)) \right|_{t=0} &= \left. \frac{d}{dt} V_+^+(\mathcal{F}_t^\top(Y)) \right|_{t=0} + \left. \frac{d}{dt} V_+^+(\mathcal{F}_t^\perp(Y)) \right|_{t=0} \\ &= \left. \frac{d}{dt} V_+^+(\mathcal{F}_t^\perp(Y)) \right|_{t=0}. \end{aligned}$$

Therefore, the derivative only depends on Z^\perp , as claimed.

The same argument shows that if H_{Y^t} is the mean curvature of $\mathcal{F}_t(Y)$ and $H_{Y^t}^\perp$ is the mean curvature of $\mathcal{F}_t^\perp(Y)$, then

$$0 = \left. \frac{d}{dt} H_{Y^t} \right|_{t=0} = \left. \frac{d}{dt} H_{Y^t}^\perp \right|_{t=0}. \quad (4.26)$$

The upshot is that it suffices to consider normal variations \mathcal{F} of Y with $Y^t = \mathcal{F}_t(Y)$ minimal to first order; i.e., such that

$$\left. \frac{d}{dt} H_{Y^t} \right|_{t=0} = 0. \quad (4.27)$$

Let $\mathcal{F} : (-\varepsilon, \varepsilon) \times Y \rightarrow X$, be a C^3 normal variation satisfying (4.27). As in the statement of Theorem 1.5, we let $f = \langle \mu_Y, \left. \frac{d}{dt} \right|_{t=0} \mathcal{F} \rangle_{g_+}$, where μ_Y is the (X^+, g_+) -inward unit normal vector along Y . Since \mathcal{F} is normal, we can write

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_t = f \mu_Y. \quad (4.28)$$

Also, let $\tilde{\mathcal{F}} = \mathcal{F}|_{(-\varepsilon, \varepsilon) \times \Sigma}$. Then $\tilde{\mathcal{F}}$ determines $\tilde{f} \in C^\infty(\Sigma)$ given by

$$\tilde{f} = \left\langle \left. \frac{d}{dt} \right|_{t=0} \tilde{\mathcal{F}}, \bar{\nu}_M \right\rangle, \quad (4.29)$$

where $\bar{\nu}_M$ is the inward-pointing normal vector to Σ in M^+ with respect to \bar{h} .

From now on, to simplify notation we will let primes denote $\left. \frac{d}{dt} \right|_{t=0}$. By the formu-

las (4.74), (4.81), and (4.82) in the appendix, the variations of the induced metric, second fundamental form, and mean curvature of Y are given by

$$\begin{aligned}\tilde{h}'_{\tilde{\alpha}\tilde{\beta}} &= -2fL_{\tilde{\alpha}\tilde{\beta}}, \\ L'_{\tilde{\alpha}\tilde{\beta}} &= \nabla_{\tilde{\alpha}}^{\tilde{h}}\nabla_{\tilde{\beta}}^{\tilde{h}}f - \tilde{h}^{\gamma\delta}L_{\tilde{\alpha}\tilde{\gamma}}L_{\tilde{\beta}\tilde{\delta}}f + R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}^{g^+}f, \\ H' &= \Delta_{\tilde{h}}f + (|L_Y|_{\tilde{h}}^2 - 3)f.\end{aligned}\tag{4.30}$$

By (4.26), $H' = 0$, so the last formula above implies that f must satisfy

$$\Delta_{\tilde{h}}f = (3 - |L_Y|_{\tilde{h}}^2)f.\tag{4.31}$$

Lemma 4.1. $f \in C^\infty(\mathring{Y})$ has an asymptotic expansion of the form

$$f = r^{-1}\tilde{f} + o(1),\tag{4.32}$$

where $\tilde{f} \in C^\infty(\Sigma)$ is given by (4.29).

Conversely, if $|L_Y|_{\tilde{h}}^2 \leq 3$ on \mathring{Y} , then given $\tilde{f} \in C^\infty(\Sigma)$, there is a unique solution f to (4.31) satisfying the expansion (4.32).

Proof. We first observe that near M ,

$$|L_Y|_{\tilde{h}}^2 = O(r^2).\tag{4.33}$$

This follows from (4.62) below, but it can also be seen by using the fact that L_Y is trace-free (since Y is minimal), and the the trace-free second fundamental form is a conformal invariant (of weight 1). Using (4.33), it is easy to see that the indicial roots of the operator

$$\mathcal{P} = \Delta_{\tilde{h}} - (3 - |L_Y|_{\tilde{h}}^2)$$

are -1 and 3 . It follows that f has an expansion of the form

$$f = r^{-1}f_{-1} + O(1),$$

for some $f_{-1} \in C^\infty(\Sigma)$. However, using the expansion of the metric \tilde{h} near M in (4.3), we have $h^{\bar{0}\bar{0}} = 1 + O(r^2)$, and using this it is easy to see that

$$f - r^{-1}f_{-1} = o(1).$$

as in (4.32). Since $\mu_Y = r\bar{\mu}_Y$, (4.28) implies

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_t &= f\mu_Y \\ &= [r^{-1}f_{-1} + o(1)] r\bar{\mu}_Y \\ &= f_{-1}\bar{\mu}_Y + o(r), \end{aligned}$$

and it follows from (4.29) and the definition of $\tilde{\mathcal{F}}$ that $f_{-1} = \tilde{f}$.

Conversely, given \tilde{f} , if we let

$$f_{-1} = r^{-1}\tilde{f}$$

then $\mathcal{P}f_{-1} = O(1)$. It then follows from standard arguments (see [24]) that there is a unique solution of $\mathcal{P}f = 0$ with $f = r^{-1}f_{-1} + O(1)$. Again using the expansion of the metric it is readily checked that $f = r^{-1}\tilde{f} + o(1)$.

□

Remark 4.2. Although $f \in C^\infty(\mathring{Y})$, since the indicial roots of the equation satisfied by f are -1 and 3 , the expansion of f must in general be expected to have a term $r^3 \log r$, so $rf \in C^{3,\alpha}(\bar{Y})$, and optimal regularity of \mathcal{F} is C^3 .

Proof of Theorem 1.5. The statement of Theorem 1.5 consists of two claims: the

formula for the derivative of V_+^+ , and the assertion that \tilde{f} determines f . Since the latter follows from the uniqueness claim in Lemma 4.1, to complete the proof of the theorem we just need to carry out the calculation of \dot{V}_+^+ .

By Theorem 1.3,

$$3V_+^+(X_t) = \pi^2(4\chi(X_t^+) - \chi(\partial Y^t)) - \frac{1}{8} \int_{X_t^+} |W_{g_+}|_{g_+}^2 dv_{g_+} - \int_{Y^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t}.$$

We let $\tilde{h}_t = g_+|_{T\tilde{Y}_t}$. For $\varepsilon > 0$ small, recall that $X_\varepsilon = \{x \in X : r(x) > \varepsilon\}$. We let $Y_\varepsilon^t = Y^t \cap X_\varepsilon$, and define

$$3V_\varepsilon(t) = \pi^2(4\chi(X_t^+ \cap X_\varepsilon) - \chi(\partial Y_\varepsilon^t)) - \frac{1}{8} \int_{X_t^+ \cap X_\varepsilon} |W_{g_+}|_{g_+}^2 dv_{g_+} - \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t}.$$

Then

$$3 \frac{d}{dt} V_\varepsilon(t) \Big|_{t=0} = -\frac{1}{8} \frac{d}{dt} \int_{X_t^+ \cap X_\varepsilon} |W_{g_+}|_{g_+}^2 dv_{g_+} \Big|_{t=0} - \frac{d}{dt} \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t} \Big|_{t=0}.$$

For the first integral,

$$-\frac{1}{8} \frac{d}{dt} \Big|_{t=0} \int_{X_t^+ \cap X_\varepsilon} |W_{g_+}|_{g_+}^2 dv_{g_+} = \frac{1}{8} \int_{Y_\varepsilon} |W_{g_+}|_{g_+}^2 f dv_{\tilde{h}}. \quad (4.34)$$

To analyze the second integral, we let $dv_{\tilde{h}_t}^\varepsilon = \psi dv_{\tilde{h}_t}$, where $\psi = \theta(r - \varepsilon)$, with θ the

Heaviside function. Then

$$\begin{aligned}
\frac{d}{dt} \Big|_{t=0} \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t} &= \frac{d}{dt} \Big|_{t=0} \int_{Y^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t}^\varepsilon \\
&= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \left[\int_Y (\mathcal{C}_{Y^\tau} \circ \mathcal{F}_\tau)(\psi \circ \mathcal{F}_\tau)(\mathcal{F}_\tau^* dv_{\tilde{h}_\tau} - dv_{\tilde{h}}) \right. \\
&\quad + \int_Y (\mathcal{C}_{Y^\tau} \circ \mathcal{F}_\tau - \mathcal{C}_Y)(\psi \circ \mathcal{F}_\tau) dv_{\tilde{h}} \\
&\quad \left. + \int_Y \mathcal{C}_Y(\psi \circ \mathcal{F}_\tau - \psi) dv_{\tilde{h}} \right] \\
&= \int_{Y_\varepsilon} \mathcal{C}_Y \left(\frac{d}{dt} dv_{\tilde{h}_t} \Big|_{t=0} \right) \\
&\quad + \int_{Y_\varepsilon} \frac{d}{dt} \mathcal{C}_Y^t \Big|_{t=0} dv_{\tilde{h}} + \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_Y \mathcal{C}_Y(\psi \circ \mathcal{F}_\tau - \psi) dv_{\tilde{h}}.
\end{aligned}$$

Now by the Implicit Function Theorem, the equation $r(\mathcal{F}(t, r, \zeta)) = \varepsilon$ can be written $r = \xi(t, \zeta)$ for some smooth $\xi : (-\varepsilon, \varepsilon) \times \Sigma \rightarrow \mathbb{R}$. Writing $dv_{\tilde{h}} = \eta r^{-3} dr dv_{\tilde{k}_\varepsilon}$ for some smooth correction factor η that is one on Σ_ε , we may use the fundamental theorem of calculus to write the last term as

$$\begin{aligned}
\lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_Y \mathcal{C}_Y(\psi \circ \mathcal{F}_\tau - \psi) dv_{\tilde{h}} &= - \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{\Sigma_\varepsilon} \int_\varepsilon^{\xi(\tau, \zeta)} \mathcal{C}_Y(r, \zeta) \eta(r, \zeta) r^{-3} dr dv_{\tilde{k}_\varepsilon}(\zeta) \\
&= - \int_{\Sigma_\varepsilon} \frac{d}{dt} \Big|_{t=0} \int_\varepsilon^{\xi(t, \zeta)} \mathcal{C}_Y(r, \zeta) \eta(r, \zeta) r^{-3} dr dv_{\tilde{k}_\varepsilon}(\zeta) \\
&= - \int_{\Sigma_\varepsilon} \mathcal{C}_Y(\varepsilon, \zeta) \varepsilon^{-3} \frac{\partial \xi}{\partial t} \Big|_{t=0} dv_{\tilde{k}_\varepsilon}(\zeta) \\
&= \int_{\Sigma_\varepsilon} \mathcal{C}_Y \varepsilon^{-1} dr (f \mu_Y) dv_{k_\varepsilon} \\
&= \int_{\Sigma_\varepsilon} \mathcal{C}_Y \langle r \partial_r, f \mu_Y \rangle_{g_+} dv_{k_\varepsilon}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt} \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t} \Big|_{t=0} &= \int_{Y_\varepsilon} \left(\frac{d}{dt} \mathcal{C}_{Y^t} \Big|_{t=0} \right) dv_{\tilde{h}} + \int_{Y_\varepsilon} \mathcal{C}_Y \left(\frac{d}{dt} dv_{\tilde{h}_t} \Big|_{t=0} \right) \\
&\quad + \int_{\Sigma_\varepsilon} \mathcal{C}_Y \langle r \partial_r, f \mu_Y \rangle_{g_+} dv_{k_\varepsilon}.
\end{aligned} \tag{4.35}$$

We dispose of the last term with

Claim 4.3.

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} \mathcal{C}_Y \langle r \partial_r, f \mu_Y \rangle_{g_+} dv_{k_\varepsilon} = 0.$$

Proof. We know that $\mu_Y = r \bar{\mu}_Y$ and that $\mathcal{C}_Y^{g_+} = r^3 \mathcal{C}_Y^{\bar{g}}$. We also know from (4.23) that

$$\langle r \partial_r, \bar{\mu}_Y \rangle_{\bar{g}} = O(\varepsilon^2).$$

So we get

$$\mathcal{C}_Y^{g_+} \langle r \partial_r, \mu_Y \rangle_{g_+} = r^3 \mathcal{C}_Y^{\bar{g}} \langle r \partial_r, \mu_Y \rangle_{g_+} = O(\varepsilon^4).$$

Therefore, taking into account the asymptotics of f , we get

$$\int_{\Sigma_\varepsilon} \mathcal{C}_Y \langle r \partial_r, f \mu_Y \rangle_{g_+} dv_{k_\varepsilon} = O(\varepsilon). \quad (4.36)$$

□

By (4.27) and the formula for the variation of the volume form (4.83) in the appendix we have

$$\frac{d}{dt} dv_{\tilde{h}_t} \Big|_{t=0} = H_Y dv_{\tilde{h}} = 0, \quad (4.37)$$

since Y is minimal. The minimality of Y to first order also implies $H_{Y^t} = O(t^2)$.

Since g_+ is Einstein, the formula for \mathcal{C}_{Y^t} thus simplifies to

$$\mathcal{C}_{Y^t} = -(L_{Y^t})^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}_t} - \frac{1}{3} \text{tr}_{\tilde{h}_t} (L_{Y^t})^3 + O(t^2), \quad (4.38)$$

where L_{Y^t} is the second fundamental form of Y^t with respect to μ_Y and $R^{\tilde{h}_t}$ is the

Ricci tensor of \tilde{h}_t . Combining (4.35), (4.36), (4.37) and (4.38) we obtain

$$\begin{aligned} \frac{d}{dt} \int_{Y_\varepsilon^t} \mathcal{C}_{Y^t} dv_{\tilde{h}_t} \Big|_{t=0} &= - \int_{Y_\varepsilon} \frac{d}{dt} ((L^{Y^t})^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}_t}) \Big|_{t=0} dv_{\tilde{h}} \\ &\quad - \frac{1}{3} \int_{Y_\varepsilon} \frac{d}{dt} \text{tr}_{\tilde{h}_t} (L^{Y^t})^3 \Big|_{t=0} dv_{\tilde{h}} + O(\varepsilon). \end{aligned}$$

We intend to apply integration by parts on the integrand of this expression to write quantities in terms of boundary integrals on Σ . We first write the integrands in terms of geometric quantities on Y .

Define

$$\begin{aligned} A &= (L_{Y^t})^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}_t} \\ B &= \text{tr}_{\tilde{h}_t} (L_{Y^t})^3. \end{aligned}$$

Differentiating A gives

$$\begin{aligned} A' &= \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\delta}} R_{\tilde{\gamma}\tilde{\delta}}^{\tilde{h}} \nabla_{\tilde{\alpha}}^Y \nabla_{\tilde{\beta}}^{\tilde{h}} f + 3f(L^2)^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} + f \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\delta}} R_{\tilde{\gamma}\tilde{\delta}}^{\tilde{h}} R_{\tilde{\alpha}\tilde{\eta}\tilde{\beta}\tilde{\eta}}^{g+} \\ &\quad + \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\delta}} L_{\tilde{\gamma}\tilde{\delta}} (R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}})'. \end{aligned} \quad (4.39)$$

A standard formula for the variation of the Ricci tensor (see e.g. [29]) gives us

$$\begin{aligned} (R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}})' &= -\frac{1}{2} [\Delta_{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}} - \nabla_{\tilde{\alpha}}^{\tilde{h}} (\tilde{\delta}_{\tilde{\beta}} \tilde{h}') - \nabla_{\tilde{\beta}}^{\tilde{h}} (\tilde{\delta}_{\tilde{\alpha}} \tilde{h}') + \nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} (\text{tr}_{\tilde{h}} \tilde{h}')] \\ &\quad - \tilde{h}^{\tilde{\gamma}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} R_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}}^{\tilde{h}} \tilde{h}'_{\tilde{\eta}\tilde{\zeta}} + \frac{1}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R_{\tilde{\alpha}\tilde{\eta}}^{\tilde{h}} \tilde{h}'_{\tilde{\beta}\tilde{\zeta}} + \frac{1}{2} \tilde{h}^{\tilde{\eta}\tilde{\zeta}} R_{\tilde{\beta}\tilde{\eta}}^{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\zeta}}. \end{aligned} \quad (4.40)$$

Here $\tilde{\delta}$ is the divergence with respect to \tilde{h} . Now, by (4.30), $\Delta_{\tilde{h}} \tilde{h}'_{\tilde{\alpha}\tilde{\beta}} = -2fL_{\tilde{\alpha}\tilde{\beta}}$. By the same equation,

$$\text{tr}_{\tilde{h}} \tilde{h}' = 0.$$

Taking the divergence of both sides above gives us

$$\begin{aligned}
\tilde{\delta}_{\tilde{\beta}}\tilde{h}' &= \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla_{\tilde{\gamma}}\tilde{h}'_{\tilde{\alpha}\tilde{\beta}} \\
&= -\nabla_{\tilde{h}}^{\tilde{\alpha}}(2fL_{\tilde{\alpha}\tilde{\beta}}) \\
&= -2f\nabla_{\tilde{h}}^{\tilde{\alpha}}L_{\tilde{\alpha}\tilde{\beta}} - 2L_{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{h}}^{\tilde{\alpha}}f.
\end{aligned} \tag{4.41}$$

Now by Codazzi, we have

$$R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{h}}^{g_+} = \nabla_{\tilde{\beta}}^{\tilde{h}}L_{\tilde{\alpha}\tilde{\gamma}} - \nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\beta}\tilde{\gamma}}$$

along Y . Contracting $\tilde{\alpha}$ and $\tilde{\gamma}$ and using the Einstein condition on g_+ along with the fact that Y is minimal gives

$$0 = R_{\tilde{\beta}\tilde{h}}^{g_+} = -\tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\beta}\tilde{\gamma}} + \tilde{h}^{\tilde{\alpha}\tilde{\gamma}}\nabla_{\tilde{\beta}}^{\tilde{h}}L_{\tilde{\alpha}\tilde{\gamma}} = -\nabla_{\tilde{h}}^{\tilde{\gamma}}L_{\tilde{\beta}\tilde{\gamma}} + \nabla_{\tilde{\beta}}^{\tilde{h}}H = -\nabla_{\tilde{h}}^{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\alpha}}.$$

Hence

$$\nabla_{\tilde{h}}^{\tilde{\alpha}}L_{\tilde{\beta}\tilde{\alpha}} = 0 \tag{4.42}$$

and

$$\tilde{\delta}_{\tilde{\beta}}\tilde{h}' = -2L_{\tilde{\beta}\tilde{\gamma}}\nabla^{\tilde{\gamma}}f.$$

Turning to the fifth term of (4.40), we consider the Riemann tensor on Y . As the dimension of Y is three, it follows that the Weyl tensor of \tilde{h} vanishes, giving us

$$\begin{aligned}
R_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}}^{\tilde{h}} &= \tilde{h}_{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\gamma}\tilde{\delta}}^{\tilde{h}} - \tilde{h}_{\tilde{\alpha}\tilde{\delta}}R_{\tilde{\beta}\tilde{\gamma}}^{\tilde{h}} - \tilde{h}_{\tilde{\beta}\tilde{\gamma}}R_{\tilde{\alpha}\tilde{\delta}}^{\tilde{h}} + \tilde{h}_{\tilde{\gamma}\tilde{\delta}}R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} - \frac{1}{2}R^{\tilde{h}}\tilde{h}_{\tilde{\alpha}\tilde{\beta}}\tilde{h}_{\tilde{\gamma}\tilde{\delta}} \\
&\quad + \frac{1}{2}R^{\tilde{h}}\tilde{h}_{\tilde{\alpha}\tilde{\delta}}\tilde{h}_{\tilde{\beta}\tilde{\gamma}}.
\end{aligned}$$

Thus,

$$-\tilde{h}^{\tilde{\gamma}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}}^{\tilde{h}}\tilde{h}'_{\tilde{\eta}\tilde{\zeta}} = -R_{\tilde{h}}^{\tilde{\zeta}\tilde{\eta}}\tilde{h}'_{\tilde{\eta}\tilde{\zeta}}\tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\beta}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} + \tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\beta}\tilde{\zeta}} - \frac{1}{2}R^{\tilde{h}}\tilde{h}'_{\tilde{\alpha}\tilde{\beta}}.$$

So we can write the last three terms of (4.40) as

$$\begin{aligned} & -\tilde{h}^{\tilde{\gamma}\tilde{\eta}}\tilde{h}^{\tilde{\delta}\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\gamma}\tilde{\beta}\tilde{\delta}}^{\tilde{h}}\tilde{h}'_{\tilde{\eta}\tilde{\zeta}} + \frac{1}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\beta}\tilde{\zeta}} + \frac{1}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\beta}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} = \\ & -R_{\tilde{\eta}\tilde{\zeta}}^{\tilde{h}}(\tilde{h}^{\tilde{\eta}\tilde{\zeta}})'\tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \frac{3}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\beta}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\alpha}\tilde{\zeta}} + \frac{3}{2}\tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\eta}}^{\tilde{h}}\tilde{h}'_{\tilde{\beta}\tilde{\zeta}} - \frac{1}{2}R^{\tilde{h}}\tilde{h}'_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

Therefore we have found

$$\begin{aligned} (R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}})' &= \Delta^{\tilde{h}}(fL_{\tilde{\alpha}\tilde{\beta}}) - \nabla_{\tilde{\alpha}}^{\tilde{h}}(L_{\tilde{\beta}\tilde{\gamma}}\nabla^{\tilde{\gamma}}f) - \nabla_{\tilde{\beta}}^{\tilde{h}}(L_{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{\gamma}}f) + 2f(R_{\tilde{\eta}\tilde{\zeta}}^{\tilde{h}}L^{\tilde{\eta}\tilde{\zeta}})\tilde{h}_{\tilde{\alpha}\tilde{\beta}} \\ & - 3fL_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\beta}\tilde{\gamma}}^{\tilde{h}} - 3fL_{\tilde{\beta}}^{\tilde{\gamma}}R_{\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}} + fR^{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

This then lets us write down an expression for $\langle L, (\text{Ric}^{\tilde{h}})' \rangle_{\tilde{h}}$:

$$L^{\tilde{\alpha}\tilde{\beta}}(R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}})' = L^{\tilde{\alpha}\tilde{\beta}}\Delta_{\tilde{h}}(fL_{\tilde{\alpha}\tilde{\beta}}) - 2L^{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{\beta}}^{\tilde{h}}(L_{\tilde{\alpha}\tilde{\gamma}}\nabla^{\tilde{\gamma}}f) - 6fL^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} + fR^{\tilde{h}}|L|^2;$$

hence

$$\begin{aligned} A' &= R_{\tilde{h}}^{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{\alpha}}^{\tilde{h}}\nabla_{\tilde{\beta}}^{\tilde{h}}f - 3f(L^2)^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} + fR_{\tilde{h}}^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\eta}\tilde{\beta}\tilde{\eta}}^{g+} \\ & + L^{\tilde{\alpha}\tilde{\beta}}\Delta_{\tilde{h}}(fL_{\tilde{\alpha}\tilde{\beta}}) - 2L^{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{\alpha}}^{\tilde{h}}(L_{\tilde{\beta}\tilde{\gamma}}\nabla_{\tilde{h}}^{\tilde{\gamma}}f) + fR^{\tilde{h}}|L|^2. \end{aligned}$$

Rather more straightforwardly, we may now write B' as

$$\begin{aligned}
B' &= (\operatorname{tr} L^3)' = 3(\tilde{h}^{\tilde{\alpha}\tilde{\gamma}})' \tilde{h}^{\tilde{\beta}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} L_{\tilde{\alpha}\tilde{\beta}} L_{\tilde{\gamma}\tilde{\delta}} L_{\tilde{\eta}\tilde{\zeta}} + 3\tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} L'_{\tilde{\alpha}\tilde{\beta}} L_{\tilde{\gamma}\tilde{\delta}} L_{\tilde{\eta}\tilde{\zeta}} \\
&= 6f|L^2|_{\tilde{h}}^2 + 3\tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\eta}} \tilde{h}^{\tilde{\delta}\tilde{\zeta}} [\nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f - L_{\tilde{\alpha}\tilde{\beta}}^2 f \\
&\quad + (R_{\tilde{\alpha}\tilde{\eta}\tilde{\beta}\tilde{\eta}}^{g+} L_{\tilde{\alpha}}^{\tilde{\gamma}} L_{\tilde{\beta}\tilde{\gamma}}) f] L_{\tilde{\gamma}\tilde{\delta}} L_{\tilde{\eta}\tilde{\zeta}} \\
&= 3f|L^2|_{\tilde{h}}^2 + 3(\nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f)(L^2)^{\tilde{\alpha}\tilde{\beta}} + 3fR_{\tilde{\alpha}\tilde{\eta}\tilde{\beta}\tilde{\eta}}^{g+} (L^2)^{\tilde{\alpha}\tilde{\beta}}.
\end{aligned}$$

It will also be useful to record that, by Gauss's equation and the Einstein condition,

$$R_{\tilde{\eta}\tilde{\zeta}}^{\tilde{h}} = -3\tilde{h}_{\tilde{\eta}\tilde{\zeta}} - R_{\tilde{\eta}\tilde{n}\tilde{\zeta}\tilde{n}}^{g+} - (L^2)_{\tilde{\eta}\tilde{\zeta}}. \quad (4.43)$$

It then also follows that

$$R_{\tilde{h}} = -6 - |L|^2. \quad (4.44)$$

We now focus on rewriting four terms in A' and B' to make them amenable to integration by parts. We thus make the following definitions:

$$\begin{aligned}
D_1 &= \int_{Y_\varepsilon} R_{\tilde{h}}^{\tilde{\alpha}\tilde{\beta}} \nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}} \\
D_2 &= \int_{Y_\varepsilon} L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}}(f L_{\tilde{\alpha}\tilde{\beta}}) dv_{\tilde{h}} \\
D_3 &= - \int_{Y_\varepsilon} 2L^{\tilde{\alpha}\tilde{\beta}} \nabla_{\tilde{\alpha}}^{\tilde{h}} (L_{\tilde{\beta}\tilde{\gamma}} \nabla_{\tilde{h}}^{\tilde{\gamma}} f) dv_{\tilde{h}} \\
D_4 &= \int_{Y_\varepsilon} 3(L^2)^{\tilde{\alpha}\tilde{\beta}} \nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}}.
\end{aligned}$$

We will write each of the above terms as an integral over Y_ε plus an integral over Σ_ε .

Recall that ν_{Y_ε} is the inward pointing \tilde{h} unit-normal vector field to Σ_ε in Y_ε . We find

$$\begin{aligned}
D_1 &= \int_{Y_\varepsilon} \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\delta}} R_{\tilde{\gamma}\tilde{\delta}}^{\tilde{h}} \nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}} \\
&= - \int_{Y_\varepsilon} \tilde{h}^{\tilde{\alpha}\tilde{\gamma}} \tilde{h}^{\tilde{\beta}\tilde{\delta}} \nabla_{\tilde{\alpha}}^{\tilde{h}} R_{\tilde{\gamma}\tilde{\delta}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} \nu_{Y_\varepsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f dv_{k_\varepsilon} \\
&= - \int_{Y_\varepsilon} \frac{1}{2} (\nabla_{\tilde{h}}^{\tilde{\alpha}} R_{\tilde{h}}) \nabla_{\tilde{\alpha}}^{\tilde{h}} f dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} \nu_{Y_\varepsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \frac{1}{2} R_{\tilde{h}} \Delta_{\tilde{h}} f dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon} \left[\frac{1}{2} R_{\tilde{h}} \nu_{Y_\varepsilon}(f) - \text{Ric}_{\tilde{h}}(\nu_{Y_\varepsilon}, \nabla^{\tilde{h}} f) \right] dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \left(\frac{6 + |L|^2}{2} \right) (|L|^2 - 3) f dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} \left[\text{Ric}_{\tilde{h}}(\nu_{Y_\varepsilon}, \nabla^{\tilde{h}} f) - \frac{1}{2} R_{\tilde{h}} \nu_{Y_\varepsilon}(f) \right] dv_{k_\varepsilon} \\
&= \int_{Y_\varepsilon} \left(\frac{|L|^4}{2} + \frac{3|L|^2}{2} - 9 \right) f dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} \left(\text{Ric}_{\tilde{h}} - \frac{1}{2} R_{\tilde{h}} \tilde{h} \right) \left(\nabla^{\tilde{h}} f, \nu_{Y_\varepsilon} \right) dv_{k_\varepsilon}.
\end{aligned}$$

Next,

$$\begin{aligned}
D_2 &= \int_{Y_\varepsilon} L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}}(f L_{\tilde{\alpha}\tilde{\beta}}) dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} \left[|L|^2 \Delta_{\tilde{h}} f + f L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} + 2L^{\tilde{\alpha}\tilde{\beta}} \nabla_{\tilde{h}}^{\tilde{\gamma}} f \nabla_{\tilde{\gamma}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} \right] dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} \left[|L|^2 \Delta_{\tilde{h}} f + L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}}(L_{\tilde{\alpha}\tilde{\beta}}) f + \langle \nabla^{\tilde{h}} f, \nabla^{\tilde{h}} |L|^2 \rangle \right] dv_{\tilde{h}} \\
&= \int_{Y_\varepsilon} f L^{\tilde{\alpha}\tilde{\beta}} \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} |L|^2 \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon}.
\end{aligned}$$

We now recall Simon's identity, which in this setting reads

$$\Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} = 2R_{\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}} L_{\tilde{\gamma}\tilde{\beta}}^{\tilde{\gamma}} + L_{\tilde{\alpha}}^{\tilde{\delta}} R_{\tilde{\delta}\tilde{\beta}}^{\tilde{h}} - \langle L, \text{Ric}_{\tilde{h}} \rangle \tilde{h}_{\tilde{\alpha}\tilde{\beta}} - L_{\tilde{\alpha}}^{\tilde{\gamma}} R_{\tilde{\gamma}\tilde{h}\tilde{\beta}\tilde{h}}^{g^+} - \frac{1}{2} R^{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}}. \quad (4.45)$$

We provide a derivation for the reader's convenience, as there are many versions in

different conventions in the literature. By Codazzi we know

$$R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} = \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\gamma}\tilde{\beta}} - \nabla_{\tilde{\gamma}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}},$$

so we may write

$$\begin{aligned} \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{\tilde{h}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} &= \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{\tilde{h}} \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\gamma}\tilde{\beta}} - \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{\tilde{h}} \nabla_{\tilde{\gamma}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}} \\ &= \nabla_{\tilde{h}}^{\tilde{\gamma}} \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\gamma}\tilde{\beta}} - \Delta_{\tilde{h}} L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned} \quad (4.46)$$

Now, letting Γ_{ij}^k be the Christoffel symbols of g_+ , we observe the following:

$$\begin{aligned} \nabla_{\tilde{\delta}}^{\tilde{h}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} &= \nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} - \Gamma_{\tilde{\delta}\tilde{\gamma}}^{\tilde{n}} R_{n\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} - \Gamma_{\tilde{\delta}\tilde{\alpha}}^{\tilde{n}} R_{\tilde{\gamma}n\tilde{\beta}\tilde{n}}^{g_+} - \Gamma_{\tilde{\delta}\tilde{\beta}}^{\tilde{n}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{n}\tilde{n}}^{g_+} \\ &= \nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} + L_{\tilde{\delta}\tilde{\gamma}} R_{\tilde{n}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} + L_{\tilde{\delta}\tilde{\alpha}} R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g_+}. \end{aligned} \quad (4.47)$$

Therefore we get

$$\tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{\tilde{h}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} = \tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} + L^{\tilde{\gamma}}_{\tilde{\alpha}} R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g_+}.$$

The second Bianchi identity gives

$$\nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} + \nabla_{\tilde{\beta}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{n}\tilde{\delta}}^{g_+} + \nabla_{\tilde{n}} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\beta}}^{g_+} = 0.$$

Contracting on $\tilde{\delta}$ and $\tilde{\gamma}$ yields

$$\tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} + \nabla_{\tilde{\beta}}^{g_+} R_{\tilde{\alpha}\tilde{n}}^{g_+} + \nabla_{\tilde{n}}^{g_+} R_{\tilde{\alpha}\tilde{\beta}}^{g_+} = 0.$$

By the Einstein condition, the last two terms of the above equation vanish, so

$$\tilde{h}^{\tilde{\delta}\tilde{\gamma}} \nabla_{\tilde{\delta}}^{g_+} R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} = 0.$$

Thus we may write

$$\tilde{h}^{\delta\tilde{\gamma}}\nabla_{\tilde{\delta}}^{\tilde{h}}R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} = L_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g_+}, \quad (4.48)$$

which gives us

$$\begin{aligned} \Delta_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} &= \tilde{h}^{\tilde{\gamma}\tilde{\delta}}\nabla_{\tilde{\delta}}^{\tilde{h}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} - \tilde{h}^{\tilde{\delta}\tilde{\gamma}}\nabla_{\tilde{\delta}}^{\tilde{h}}R_{\tilde{\gamma}\tilde{\alpha}\tilde{\beta}\tilde{n}}^{g_+} \\ &= \tilde{h}^{\tilde{\gamma}\tilde{\delta}}\nabla_{\tilde{\delta}}^{\tilde{h}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} - L_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g_+}. \end{aligned}$$

Now we want to commute the covariant derivatives in the first term on the right-hand side of this equation. By the Ricci identity,

$$\nabla_{\tilde{\delta}}^{\tilde{h}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} - \nabla_{\tilde{\alpha}}^{\tilde{h}}\nabla_{\tilde{\delta}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} = R_{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}}{}^{\tilde{\eta}}L_{\tilde{\eta}\tilde{\beta}} + R_{\tilde{\delta}\tilde{\alpha}\tilde{\beta}}^{\tilde{h}}{}^{\tilde{\eta}}L_{\tilde{\eta}\tilde{\gamma}}.$$

Contracting $\tilde{\delta}$ and $\tilde{\gamma}$ and using (4.42) gives

$$\tilde{h}^{\tilde{\delta}\tilde{\gamma}}\nabla_{\tilde{\delta}}^{\tilde{h}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} = \tilde{h}^{\tilde{\delta}\tilde{\gamma}}R_{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}}L_{\tilde{\eta}\tilde{\beta}}{}^{\tilde{\eta}} + R_{\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}}L_{\tilde{\beta}}{}^{\tilde{\gamma}}. \quad (4.49)$$

Now the second term on the right-hand side of (4.49) is

$$\begin{aligned} \tilde{h}^{\tilde{\eta}\tilde{\zeta}}\tilde{h}^{\tilde{\gamma}\tilde{\delta}}R_{\tilde{\delta}\tilde{\alpha}\tilde{\beta}\tilde{\zeta}}^{\tilde{h}}L_{\tilde{\eta}\tilde{\gamma}} &= \tilde{h}^{\tilde{\eta}\tilde{\zeta}}\tilde{h}^{\tilde{\gamma}\tilde{\delta}}L_{\tilde{\eta}\tilde{\gamma}}[\tilde{h}_{\tilde{\delta}\tilde{\beta}}\tilde{h}_{\tilde{\alpha}\tilde{\zeta}} - \tilde{h}_{\tilde{\delta}\tilde{\zeta}}\tilde{h}_{\tilde{\alpha}\tilde{\beta}} - \tilde{h}_{\tilde{\alpha}\tilde{\beta}}\tilde{h}_{\tilde{\delta}\tilde{\zeta}} + \tilde{h}_{\tilde{\alpha}\tilde{\zeta}}\tilde{h}_{\tilde{\delta}\tilde{\beta}} \\ &\quad - \frac{1}{2}R_{\tilde{h}}\tilde{h}_{\tilde{\delta}\tilde{\beta}}\tilde{h}_{\tilde{\alpha}\tilde{\zeta}} + \frac{1}{2}R_{\tilde{h}}\tilde{h}_{\tilde{\delta}\tilde{\zeta}}\tilde{h}_{\tilde{\alpha}\tilde{\beta}}] \\ &= \tilde{h}^{\tilde{\eta}\tilde{\zeta}}L_{\tilde{\beta}\tilde{\eta}}R_{\tilde{\alpha}\tilde{\zeta}}^{\tilde{h}} - L^{\tilde{\delta}\tilde{\zeta}}R_{\tilde{\delta}\tilde{\zeta}}^{\tilde{h}}\tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \tilde{h}^{\tilde{\gamma}\tilde{\delta}}L_{\tilde{\gamma}\tilde{\alpha}}R_{\tilde{\delta}\tilde{\beta}}^{\tilde{h}} - L_{\tilde{\beta}\tilde{\alpha}}\frac{1}{2}R_{\tilde{h}} \\ &= \tilde{h}^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\beta}\tilde{\eta}} - \langle L, \text{Ric}_{\tilde{h}} \rangle \tilde{h}_{\tilde{\alpha}\tilde{\beta}} + L^{\tilde{\delta}}{}_{\tilde{\alpha}}R_{\tilde{\delta}\tilde{\beta}}^{\tilde{h}} - \frac{1}{2}R^{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

This yields (4.45).

We express each term in Simon's identity in terms of L and Riem_{g_+} using (4.43),

which allows us to write the first term on the right-hand side of (4.45) as

$$\begin{aligned} 2L_{\tilde{\beta}}^{\tilde{\zeta}}R_{\tilde{\alpha}\tilde{\zeta}}^{\tilde{h}} &= 2L_{\tilde{\beta}}^{\tilde{\zeta}}[-3\tilde{h}_{\tilde{\alpha}\tilde{\zeta}} - R_{\tilde{\alpha}\tilde{n}\tilde{\gamma}\tilde{n}}^{g+} - (L^2)_{\tilde{\alpha}\tilde{\zeta}}] \\ &= -6L_{\tilde{\alpha}\tilde{\beta}} - 2R_{\tilde{\alpha}\tilde{n}\tilde{\zeta}\tilde{n}}^{g+}L_{\tilde{\beta}}^{\tilde{\zeta}} - 2(L^3)_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

Then (4.43) again allows us to write

$$-\langle L, \text{Ric}_{\tilde{h}} \rangle = L^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\eta}\tilde{n}\tilde{\zeta}\tilde{n}}^{g+} + \text{tr}_{\tilde{h}}(L^3),$$

which gives us for the third term of (4.45)

$$-\langle L, \text{Ric}_{\tilde{h}} \rangle \tilde{h}_{\tilde{\alpha}\tilde{\beta}} = L^{\tilde{\eta}\tilde{\zeta}}R_{\tilde{\eta}\tilde{n}\tilde{\zeta}\tilde{n}}^{g+} \tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \text{tr}_{\tilde{h}}(L^3) \tilde{h}_{\tilde{\alpha}\tilde{\beta}}.$$

Using (4.44) gives us that the last term on the right-hand side of (4.45) may be written as

$$\begin{aligned} -\frac{1}{2}R_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} &= -\frac{1}{2}L_{\tilde{\alpha}\tilde{\beta}}[-6 - |L|^2] \\ &= 3L_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}|L|^2L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

Putting the above formulas into (4.45) gives

$$\begin{aligned} \Delta_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} &= \tilde{h}^{\tilde{\gamma}\tilde{\delta}}\nabla_{\tilde{\delta}}^{\tilde{h}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\gamma}\tilde{\beta}} - L_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \\ &= \tilde{h}^{\tilde{\delta}\tilde{\gamma}}R_{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}}^{g+}L_{\tilde{\eta}\tilde{\beta}}^{\tilde{\eta}} + \tilde{h}^{\tilde{\delta}\tilde{\gamma}}R_{\tilde{\delta}\tilde{\alpha}\tilde{\gamma}}^{g+}L_{\tilde{\eta}\tilde{\gamma}}^{\tilde{\eta}} - L_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \\ &= 2R_{\tilde{\alpha}\tilde{\gamma}}^{\tilde{h}}L_{\tilde{\beta}}^{\tilde{\gamma}} + L_{\tilde{\alpha}}^{\tilde{\delta}}R_{\tilde{\delta}\tilde{\beta}}^{\tilde{h}} - \langle L, \text{Ric}_{\tilde{h}} \rangle \tilde{h}_{\tilde{\alpha}\tilde{\beta}} - L_{\tilde{\alpha}}^{\tilde{\gamma}}R_{\tilde{\gamma}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} - \frac{1}{2}R_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} \\ &= -6L_{\tilde{\alpha}\tilde{\beta}} - 2R_{\tilde{\alpha}\tilde{n}\tilde{\gamma}\tilde{n}}^{g+}L_{\tilde{\beta}}^{\tilde{\gamma}} - 3(L^3)_{\tilde{\alpha}\tilde{\beta}} - 2R_{\tilde{\beta}\tilde{n}\tilde{\gamma}\tilde{n}}^{g+}L_{\tilde{\alpha}}^{\tilde{\gamma}} \\ &\quad + L^{\tilde{\gamma}\tilde{\delta}}R_{\tilde{\gamma}\tilde{n}\tilde{\delta}\tilde{n}}^{g+} \tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \text{tr}(L^3)\tilde{h}_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}|L|^2L_{\tilde{\alpha}\tilde{\beta}}. \end{aligned}$$

Therefore,

$$L^{\tilde{\alpha}\tilde{\beta}}\Delta_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}} = -4(L^2)^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}^{g+} - 6|L|^2 - 3|L^2|^2 + \frac{1}{2}|L|^4.$$

This allows us to write

$$\begin{aligned} & \int_{Y_\varepsilon} fL^{\tilde{\alpha}\tilde{\beta}}\Delta_{\tilde{h}}L_{\tilde{\alpha}\tilde{\beta}}dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} |L|^2\nu_{Y_\varepsilon}(f)dv_{k_\varepsilon} = \\ & \int_{Y_\varepsilon} \left[-4f(L^2)^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}^{g+} - 6f|L|^2 - 3f|L^2|^2 + \frac{1}{2}f|L|^4 \right] dv_{\tilde{h}} \\ & - \oint_{\Sigma_\varepsilon} |L|^2\nu_{Y_\varepsilon}(f)dv_{k_\varepsilon}. \end{aligned}$$

Therefore,

$$\begin{aligned} D_2 &= \int_{Y_\varepsilon} \left[-4f(L^2)^{\tilde{\alpha}\tilde{\beta}}R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}^{g+} - 6f|L|^2 - 3f|L^2|^2 + \frac{1}{2}f|L|^4 \right] dv_{\tilde{h}} \\ & - \oint_{\Sigma_\varepsilon} |L|^2\nu_{Y_\varepsilon}(f)dv_{k_\varepsilon}. \end{aligned}$$

Meanwhile, more straightforwardly,

$$\begin{aligned} D_3 &= - \int_{Y_\varepsilon} 2L^{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{\alpha}}^{\tilde{h}}(L_{\tilde{\beta}\tilde{\gamma}}\nabla_{\tilde{h}}^{\tilde{\gamma}}f)dv_{\tilde{h}} \\ &= 2 \int_{Y_\varepsilon} \left[L^{\tilde{\alpha}\tilde{\beta}}\nabla_{\tilde{\alpha}}^{\tilde{h}}L_{\tilde{\beta}\tilde{\gamma}}\nabla_{\tilde{h}}^{\tilde{\gamma}}f + (L^2)^{\tilde{\alpha}\tilde{\gamma}}\nabla_{\tilde{\alpha}}^{\tilde{h}}\nabla_{\tilde{\gamma}}^{\tilde{h}}f \right] dv_{\tilde{h}} \\ &= - \oint_{\Sigma_\varepsilon} 2L^2(\nabla^{\tilde{h}}f, \nu_{Y_\varepsilon})dv_{k_\varepsilon}. \end{aligned} \tag{4.50}$$

Now using integration by parts and applying (4.42) we see

$$\begin{aligned} D_4 &= \int_{Y_\varepsilon} 3(L^2)^{\tilde{\alpha}\tilde{\beta}} \nabla_{\tilde{\alpha}}^{\tilde{h}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}} \\ &= - \int_{Y_\varepsilon} 3L^{\tilde{\alpha}\tilde{\gamma}} \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\gamma}}^{\tilde{\beta}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{\tilde{h}} - \oint_{\Sigma_\varepsilon} 3L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}\tilde{\beta}} \nu_{Y_\varepsilon}^{\tilde{\alpha}} \nabla_{\tilde{\beta}}^{\tilde{h}} f dv_{k_\varepsilon}. \end{aligned}$$

Next we consider the following:

$$\begin{aligned} -3L^{\tilde{\alpha}\tilde{\gamma}} \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}} \nabla_{\tilde{\gamma}}^{\tilde{h}} f &= -3L^{\tilde{\alpha}\tilde{\gamma}} (\nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}} - \nabla_{\tilde{\beta}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\gamma}}) \nabla_{\tilde{h}}^{\tilde{\beta}} f - 3L^{\tilde{\alpha}\tilde{\gamma}} \nabla_{\tilde{\beta}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\gamma}} \nabla_{\tilde{h}}^{\tilde{\beta}} f \\ &= -3L^{\tilde{\alpha}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}^{g+} \nabla_{\tilde{h}}^{\tilde{\beta}} f - \frac{3}{2} \nabla_{\tilde{\beta}} |L|^2 \nabla_{\tilde{h}}^{\tilde{\beta}} f. \end{aligned} \quad (4.51)$$

Then (4.48) allows us to write

$$3L^{\tilde{\alpha}\tilde{\gamma}} \nabla_{\tilde{h}}^{\tilde{\beta}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}^{g+} f = 3f L^{\tilde{\alpha}\tilde{\gamma}} L_{\tilde{\alpha}}^{\tilde{\beta}} R_{\tilde{\beta}\hat{n}\tilde{\gamma}\hat{n}}^{g+}. \quad (4.52)$$

We also observe that

$$\begin{aligned} 3f \nabla_{\tilde{h}}^{\tilde{\beta}} L^{\tilde{\alpha}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}}^{g+} &= \frac{3}{2} f (\nabla_{\tilde{\beta}}^{\tilde{h}} L_{\tilde{\alpha}\tilde{\gamma}} - \nabla_{\tilde{\alpha}}^{\tilde{h}} L_{\tilde{\beta}\tilde{\gamma}}) R_{g+}^{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}} \\ &= \frac{3}{2} f R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}^{g+} R_{g+}^{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\hat{n}} \\ &= -\frac{3}{2} f W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}^{g+} W_{g+}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\hat{n}}. \end{aligned} \quad (4.53)$$

So if we apply integration by parts to D_4 , use (4.51), apply integration by parts again

(to both terms on the RHS of (4.51)) and then use (4.52) and (4.53) we get

$$\begin{aligned}
D_4 &= \int_{Y_\varepsilon} \left[3f(L^2)^{\tilde{\beta}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g+} - \frac{3}{2}fW_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g+} W_{g+}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}} + \frac{3}{2}|L|^2\Delta_{\tilde{h}}f \right] dv_{\tilde{h}} \\
&\quad - \oint_{\Sigma_\varepsilon} \left[-3fL^{\tilde{\alpha}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\tilde{n}}^{g+} \nu_{Y_\varepsilon}^{\tilde{\beta}} - \frac{3}{2}|L|^2\nu_{Y_\varepsilon}(f) \right. \\
&\quad \left. + 3L^2(\nabla^{\tilde{h}}f, \nu_{Y_\varepsilon}) \right] dv_{k_\varepsilon}.
\end{aligned}$$

Now we want to compute $\int_{Y_\varepsilon} C' dv_{\tilde{h}} = -\int_{Y_\varepsilon} A' dv_{\tilde{h}} - \frac{1}{3}\int_{Y_\varepsilon} B' dv_{\tilde{h}}$. Using our expressions for D_1, D_2, D_3 and D_4 and gathering together all of the terms that appear as integrals over Y_ε we get:

$$\begin{aligned}
I_Y &= - \int_{Y_\varepsilon} \left[\left(\frac{3|L|^2}{2} + \frac{|L|^4}{2} - 9 \right) f - 4f(L^2)^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} - 6f|L|^2 \right. \\
&\quad - 3f(L^2)^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} + fR_{\tilde{h}}^{\tilde{\gamma}\tilde{\delta}} R_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} + fR_{\tilde{h}}|L|^2 \\
&\quad - 3f|L|^2 + \frac{1}{2}f|L|^4 + f(L^2)^{\tilde{\alpha}\tilde{\beta}} R_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} - \frac{1}{2}fW_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g+} W_{g+}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}} \\
&\quad \left. + \frac{1}{2}|L|^2\Delta_{\tilde{h}}f + f|L|^2 + fR_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} (L^2)^{\tilde{\alpha}\tilde{\beta}} \right] dv_{\tilde{h}}. \tag{4.54}
\end{aligned}$$

Next, we apply (4.44), (4.43), and

$$R_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} = W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} - \tilde{h}_{\tilde{\alpha}\tilde{\beta}}, \tag{4.55}$$

which implies

$$R_{\tilde{\alpha}\tilde{\beta}}^{\tilde{h}} = -L_{\tilde{\alpha}\tilde{\beta}}^2 - 2\tilde{h}_{\tilde{\alpha}\tilde{\beta}} - W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+};$$

and (4.54) becomes

$$\begin{aligned}
I_Y &= - \int_{Y_\epsilon} \left[\frac{3|L|^2}{2} f + \frac{|L|^4}{2} f - 9f - f(L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} f - W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} \right. \\
&\quad + f|L|^2 + 6f - 4f(L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} + 4f|L|^2 - 6f|L|^2 \\
&\quad - 3f|L^2|^2 + \frac{1}{2}f|L|^4 - |L|^4 f - 6|L|^2 f + 3|L^2|^2 f + 6|L|^2 f \\
&\quad + 3(L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} f + 2|L^2|^2 f + (L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} f - |L|^2 f \\
&\quad + (L^2)^{\tilde{\alpha}\tilde{\beta}} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} f - |L|^2 f - \frac{1}{2} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} + \frac{3}{2}|L|^2 f \\
&\quad \left. - \frac{1}{2}|L|^4 f \right] dv_{\tilde{h}} \\
&= - \int_{Y_\epsilon} \left[-3f - \frac{1}{2} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} f - W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} f \right] dv_{\tilde{h}}.
\end{aligned}$$

We may simplify this helpfully:

Claim 4.4.

$$\frac{1}{2} W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} + W_{\tilde{\alpha}\tilde{\beta}\hat{n}}^{g_+} W_{g_+}^{\tilde{\alpha}\tilde{\beta}\hat{n}} = \frac{1}{8} |W_{g_+}|_{g_+}^2. \quad (4.56)$$

Proof. Observe that

$$\begin{aligned}
|W_{g_+}|_{g_+}^2 &= W_{ijkl}^{g_+} W_{ijkl}^{g_+} \\
&= 4W_{\hat{n}\alpha\beta\gamma}^{g_+} W_{\hat{n}\alpha\beta\gamma}^{g_+} + 4W_{\hat{n}\alpha\hat{n}\beta}^{g_+} W_{\hat{n}\alpha\hat{n}\beta}^{g_+} + 4W_{\alpha\beta\alpha\delta}^{g_+} W_{\alpha\beta\alpha\delta}^{g_+} \\
&= 4W_{\hat{n}\alpha\beta\gamma}^{g_+} W_{\hat{n}\alpha\beta\gamma}^{g_+} + 4W_{\hat{n}\alpha\hat{n}\beta}^{g_+} W_{\hat{n}\alpha\hat{n}\beta}^{g_+} + 4W_{\alpha\beta}^{g_+\alpha} W_{\alpha}^{g_+\beta\alpha\delta}.
\end{aligned}$$

Note that

$$W_{\alpha\beta}^{g_+\alpha}{}_{\delta} = -W_{\hat{n}\beta}^{g_+\hat{n}}{}_{\delta}.$$

Therefore

$$\begin{aligned} 4W_{\hat{n}\alpha\hat{n}\beta}^{g_+}W_{g_+}^{\hat{n}\alpha\hat{n}\beta} + 4W_{\alpha\beta}^{g_+}{}_{\delta}W_{\alpha}^{g_+\beta\alpha\delta} &= 4W_{\hat{n}\alpha\hat{n}\beta}^{g_+}W_{g_+}^{\hat{n}\alpha\hat{n}\beta} + 4W_{\hat{n}\beta}^{g_+}{}_{\delta}W_{\hat{n}}^{g_+\beta\hat{n}\delta} \\ &= 8W_{\hat{n}\alpha\hat{n}\beta}^{g_+}W_{g_+}^{\hat{n}\alpha\hat{n}\beta}, \end{aligned}$$

which gives

$$|W_{g_+}|_{g_+}^2 = 4W_{\hat{n}\alpha\beta\gamma}^{g_+}W_{g_+}^{\hat{n}\alpha\beta\gamma} + 8W_{\hat{n}\alpha\hat{n}\beta}^{g_+}W_{g_+}^{\hat{n}\alpha\hat{n}\beta}.$$

□

It follows from this and (4.31) that (4.54) is equal to

$$I_Y = \int_{Y_\varepsilon} \left[|L|^2 f + \frac{1}{8} |W_{g_+}|_{g_+}^2 f \right] dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon} \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon}. \quad (4.57)$$

Gathering the boundary terms from D_1 , D_2 , D_3 and D_4 and the normal derivative term on the above line we get

$$\begin{aligned} \oint_{\Sigma_\varepsilon} \left[\left(\frac{1}{2} R_{\tilde{h}} \tilde{h} - \text{Ric}_{\tilde{h}} \right) (\nabla^{\tilde{h}} f, \nu_{Y_\varepsilon}) - |L|^2 \nu_{Y_\varepsilon}(f) + 2L^2 (\nabla^{\tilde{h}} f, \nu_{Y_\varepsilon}) \right. \\ \left. + f L^{\tilde{\alpha}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\tilde{n}}^{g_+} \nu_{Y_\varepsilon}^{\tilde{\beta}} + \frac{1}{2} |L|^2 \nu_\varepsilon(f) - L^2 (\nabla f, \nu_\varepsilon) + \nu_{Y_\varepsilon}(f) \right] dv_{k_\varepsilon}. \end{aligned} \quad (4.58)$$

Now we use (4.55) to re-write $-L^2 (\nabla^{\tilde{h}} f, \nu_\varepsilon)$, and we use $R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g_+} = W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g_+}$ to re-write $f L^{\tilde{\alpha}\tilde{\gamma}} R_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}\tilde{n}}^{g_+} \nu_\varepsilon^{\tilde{\beta}}$, giving us

$$\oint_{\Sigma_\varepsilon} \left[-|L|^2 \nu_{Y_\varepsilon}(f) + 2L^2 (\nabla f, \nu_{Y_\varepsilon}) - f L^{\tilde{\alpha}\tilde{\gamma}} W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{n}}^{g_+} \nu_{Y_\varepsilon}^{\tilde{\beta}} + W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g_+} f^{\tilde{\alpha}} \nu_\varepsilon^{\tilde{\beta}} \right] dv_{k_\varepsilon}.$$

Combining this with (4.34), (4.35) and (4.57) gives us

$$\begin{aligned} 3\frac{d}{dt}V_\epsilon(t)|_{t=0} &= \int_{Y_\epsilon} -|L|^2 f dv_{\tilde{h}} + \oint_{\Sigma_\epsilon} \left[-W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \nu_{Y_\epsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f + |L|^2 \nu_{Y_\epsilon}(f) \right. \\ &\quad \left. - 2L_{\tilde{\alpha}\tilde{\gamma}} L^{\tilde{\gamma}\tilde{\beta}} \nu_{Y_\epsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f + L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{g+} f \nu_{Y_\epsilon}^{\tilde{\alpha}} \right] dv_{k_\epsilon} + O(\epsilon). \end{aligned} \quad (4.59)$$

Next we will examine the asymptotics of the term $-W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \nu_{Y_\epsilon}^{\tilde{\alpha}} f^{\tilde{\beta}}$. Now, it follows from (4.3), (4.4), and the second-last equation on the bottom of page 52 of [13] that

$$W_{0\mu 0\nu}^{\bar{g}} = O(r).$$

Moreover, from the first equation on p. 53 of the same book, we may conclude that

$$W_{0\mu 0\nu}^{\bar{g}} = -\frac{3}{2}r g_{\mu\nu}^{(3)},$$

with $g^{(3)}$ as in (4.3). By the conformal change formula for the Weyl tensor, therefore, we find

$$W_{\mu 0\nu 0}^{g+} = -\frac{3}{2}r^{-1} g_{\mu\nu}^{(3)} + O_{\bar{g}}(1). \quad (4.60)$$

Now by (4.32),

$$\begin{aligned} -W_{g_+}(\nu_{Y_\epsilon}, \mu_Y, \nabla^{\tilde{h}} f, \mu_Y) &= -r^3 W_{g_+}(\bar{\nu}_\epsilon, \bar{\mu}_Y, \nabla^Y f, \bar{\mu}_Y) + O(r^4) \\ &= -r^3 W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \bar{\nu}_{Y_\epsilon}^{\tilde{\alpha}} f^{\tilde{\beta}} + O(r^4) \\ &= -r^5 W_{\tilde{\alpha}\tilde{n}\tilde{\beta}\tilde{n}}^{g+} \bar{\nu}_{Y_\epsilon}^{\tilde{\alpha}} \bar{g}^{\tilde{\beta}\tilde{\gamma}} \partial_{\tilde{\gamma}} f + O(r^4) \\ &= r^3 W_{0\bar{n}0\bar{n}} \bar{\nu}_{Y_\epsilon}^0 \tilde{f} + O(r^4), \end{aligned}$$

where \bar{n} corresponds to $\bar{\mu}_Y$. Taking (4.23), (4.24), (4.60), and (4.32), we see that the

first corner term of (4.59) may be written

$$\oint_{\Sigma_\varepsilon} -W_{g^+}(\nu_{Y_\varepsilon}, \mu_Y, \nabla^{\tilde{h}} f, \mu_Y) dv_{k_\varepsilon} = \oint_{\Sigma} \frac{3}{2} g^{(3)}(\bar{\nu}_M, \bar{\nu}_M) \tilde{f} dv_{\tilde{k}} + O(\varepsilon). \quad (4.61)$$

We now simplify the remaining terms of (4.59):

Claim 4.5.

$$\begin{aligned} & \int_{Y_\varepsilon} -|L|^2 f dv_{\tilde{h}} + \oint_{\Sigma_\varepsilon} \left[W_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}^{g^+} \nu_{Y_\varepsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f - |L|^2 \nu_{Y_\varepsilon}(f) + 2L_{\tilde{\alpha}\tilde{\gamma}} L_{\tilde{\beta}}^{\tilde{\gamma}} \nu_{Y_\varepsilon}^{\tilde{\alpha}} \nabla_{\tilde{h}}^{\tilde{\beta}} f \right. \\ & \quad \left. - L^{\tilde{\gamma}\tilde{\delta}} W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\beta}}^{g^+} f \nu_{Y_\varepsilon}^{\tilde{\alpha}} \right] dv_{k_\varepsilon} \\ & = f.p. \int_{\tilde{Y}} -|L|^2 f dv_{\tilde{h}} + O(\varepsilon \log \varepsilon). \end{aligned}$$

Proof. Observe that

$$L_{\tilde{\alpha}\tilde{\beta}} = \frac{\bar{L}_{\tilde{\alpha}\tilde{\beta}}}{r} + \frac{\bar{\mu}_Y(r)}{r^2} \bar{g}_{\tilde{\alpha}\tilde{\beta}}. \quad (4.62)$$

Now

$$\bar{\mu}_Y = (1 + O(r^2)) \partial_w - \left(\frac{\bar{\eta}_M r}{2} + O(r^3 \log r) \right) \partial_r + O^a(r^2) \partial_a. \quad (4.63)$$

Therefore

$$\bar{\mu}_Y(r) = -\frac{1}{2} [\bar{\eta}_M r + O(r^3 \log r)]. \quad (4.64)$$

Now using the fact that $\bar{g}_{\tilde{\alpha}\tilde{\beta}} = \bar{g}_{\alpha\beta} + O(r^2)$ we may write

$$\bar{L}_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{2} \bar{\mu}_Y \bar{g}_{\alpha\beta} + O(r^2) = -\frac{1}{2} \partial_w \bar{g}_{\alpha\beta} + O(r^2). \quad (4.65)$$

Therefore we may write

$$L_{\tilde{\alpha}\tilde{\beta}} = -\frac{\partial_w \bar{g}_{\alpha\beta}}{2r} - \frac{\bar{\eta}_M \bar{g}_{\alpha\beta}}{2r} + O(r \log r). \quad (4.66)$$

Hence

$$\begin{aligned}
|L|_h^2 \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon} &= \varepsilon^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[\frac{\partial_w \bar{g}_{\alpha\beta}}{2\varepsilon} + \frac{\bar{\eta}_M \bar{g}_{\alpha\beta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] \cdot \\
&\quad \cdot \left[\frac{\partial_w \bar{g}_{\gamma\delta}}{2\varepsilon} + \frac{\bar{\eta}_M \bar{g}_{\gamma\delta}}{2\varepsilon} + O(\varepsilon \log \varepsilon) \right] \left[\tilde{f} \varepsilon^{-1} + O(\varepsilon) \right] dv_{k_\varepsilon} \\
&= [\varepsilon^{-1} |\bar{I}\bar{I}_M|_k^2 \tilde{f} + O(\varepsilon \log \varepsilon)] dv_{\tilde{k}_\varepsilon},
\end{aligned} \tag{4.67}$$

so we may write

$$\oint_{\Sigma_\varepsilon} |L|_h^2 \nu_{Y_\varepsilon}(f) dv_{k_\varepsilon} = \oint_{\Sigma} |\bar{I}\bar{I}_M|_k^2 \tilde{f} dv_k \varepsilon^{-1} + O(\varepsilon \log \varepsilon).$$

Now,

$$\begin{aligned}
|L|_h^2 f dv_{\tilde{h}} &= r^4 \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} \left[\frac{\partial_w \bar{g}_{\alpha\beta}}{2r} + \frac{\bar{\eta}_M \bar{g}_{\alpha\beta}}{2r} + O(r \log r) \right] \cdot \\
&\quad \cdot \left[\frac{\partial_w \bar{g}_{\gamma\delta}}{2r} + \frac{\bar{\eta}_M \bar{g}_{\gamma\delta}}{2r} + O(r \log r) \right] \left[\tilde{f} r^{-1} + O(r) \right] dv_{\tilde{h}} \\
&= [r^{-2} |\bar{I}\bar{I}_M|_k^2 \tilde{f} + O(\log r)] dv_{\tilde{h}},
\end{aligned} \tag{4.68}$$

so

$$\int_{Y_\varepsilon} |L|_h^2 f dv_{\tilde{h}} = C + \int_\varepsilon^{r_0} \oint_{\Sigma} |L|_h^2 f dv_k dr \tag{4.69}$$

$$= C + \int_\varepsilon^{r_0} \oint_{\Sigma} r^{-2} |\bar{I}\bar{I}_M|_k^2 \tilde{f} + O(\log r) dv_k dr \tag{4.70}$$

$$= C' - \varepsilon^{-1} \oint_{\Sigma} |\bar{I}\bar{I}_M|_k^2 \tilde{f} dv_k + O(\varepsilon \log \varepsilon) \tag{4.71}$$

for some constants C and C' and $r_0 > 0$ chosen small enough. Observe that

$$C' = f.p. \int_{\hat{Y}} |L|^2 f dv_{\tilde{h}}.$$

By (4.66) we can write

$$L_{\tilde{\alpha}\tilde{\delta}} = O(1).$$

Now we can write

$$\begin{aligned} L_{\tilde{\alpha}\tilde{\gamma}}L_{\tilde{\beta}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{\alpha}}\nabla^{\tilde{\beta}}f &= L_{\tilde{a}\tilde{\gamma}}L_{\tilde{b}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{a}}f^{\tilde{b}} + L_{\tilde{0}\tilde{\gamma}}L_{\tilde{b}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{0}}f^{\tilde{b}} + L_{\tilde{a}\tilde{\gamma}}L_{\tilde{0}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{a}}f^{\tilde{0}} + L_{\tilde{0}\tilde{\gamma}}L_{\tilde{0}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{0}}f^{\tilde{0}}. \\ &= O(r^3) \end{aligned}$$

It follows that

$$L_{\tilde{\alpha}\tilde{\gamma}}L_{\tilde{\beta}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{\alpha}}\nabla^{\tilde{\beta}}fdv_k = O(\varepsilon)dv_{\tilde{k}},$$

so

$$\oint_{\Sigma_{\varepsilon}} L_{\tilde{\alpha}\tilde{\gamma}}L_{\tilde{\beta}}^{\tilde{\gamma}}\nu_{\tilde{\epsilon}}^{\tilde{\alpha}}\nabla^{\tilde{\beta}}fdv_k = O(\varepsilon).$$

Now we turn our attention to the term $L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{g+}f\nu_{\tilde{\epsilon}}^{\alpha}$. First observe that

$$W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{g+} = rW_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{g+}$$

and

$$W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{g+} = \frac{W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{n}}^{\bar{g}}}{r^2},$$

where \hat{n} corresponds to $\bar{\mu}_Y$. Now

$$\begin{aligned} W_{\tilde{\gamma}\tilde{0}\tilde{\delta}\tilde{n}}^{\bar{g}} &= R_{\tilde{\gamma}\tilde{0}\tilde{\delta}\tilde{n}}^{\bar{g}} \\ &= \nabla_{\tilde{0}}^{\bar{g}}L_{\tilde{\gamma}\tilde{\delta}}^{\bar{g}} - \nabla_{\tilde{\gamma}}^{\bar{g}}L_{\tilde{0}\tilde{\delta}}^{\bar{g}} \\ &= \partial_{\tilde{0}}\bar{L}_{\tilde{\gamma}\tilde{\delta}} - \Gamma_{\tilde{0}\tilde{\gamma}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{\delta}} - \Gamma_{\tilde{0}\tilde{\delta}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{\gamma}} - (\partial_{\tilde{\gamma}}\bar{L}_{\tilde{0}\tilde{\delta}} - \Gamma_{\tilde{0}\tilde{\gamma}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{\delta}} - \Gamma_{\tilde{\gamma}\tilde{\delta}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{0}}) \\ &= \partial_{\tilde{0}}\bar{L}_{\tilde{\gamma}\tilde{\delta}} - \Gamma_{\tilde{0}\tilde{\delta}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{\gamma}} - (\partial_{\tilde{\gamma}}\bar{L}_{\tilde{0}\tilde{\delta}} - \Gamma_{\tilde{\gamma}\tilde{\delta}}^{\tilde{\beta}}\bar{L}_{\tilde{\beta}\tilde{0}}) \\ &= O(r). \end{aligned}$$

This gives us

$$L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{0}} = O(r^2)$$

and

$$\begin{aligned} L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{\alpha}} &= L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{0}\tilde{\delta}\tilde{\hat{n}}}f\nu_\varepsilon^{\tilde{0}} + L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{b}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{b}} \\ &= L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{0}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{0}} + O(r^2) \\ &= O(r^2). \end{aligned}$$

Therefore we may write

$$L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{\alpha}}dv_k = O(\varepsilon)dv_{\tilde{k}}.$$

We then get that

$$\oint_{\Sigma_\varepsilon} L^{\tilde{\gamma}\tilde{\delta}}W_{\tilde{\gamma}\tilde{\alpha}\tilde{\delta}\tilde{\hat{n}}}^{g_+}f\nu_\varepsilon^{\tilde{\alpha}}dv_k = O(\varepsilon).$$

This proves the claim. □

Combining Claim 4.5 with (4.59) and (4.61) and letting $\varepsilon \rightarrow 0$ yields the theorem. □

4.5 Appendix

In this appendix we give a brief summary of the formulas needed in the proof of Theorem 1.5, based on notes provided by Nicholas Edelen. Although they are all standard, due to differences in notation and convention we have decided to present a summary of the calculations.

Let (X, g) be a Riemannian manifold of dimension $n + 1$, and ∇ denote the Riemannian connection. Let Y be a smooth manifold of dimension n , and consider a one-parameter family of smooth immersions $\mathcal{F} : (-\varepsilon, \varepsilon) \times Y \rightarrow X$. Let $h = (\mathcal{F}_t)^*g$

be the induced metric on Y , and ∇^Y the corresponding connection.

Let V denote the variation field of \mathcal{F}_t :

$$V = \left. \frac{d}{dt} \mathcal{F}_t \right|_{t=0}.$$

Eventually we will assume that \mathcal{F}_t is a normal variation; i.e., if ν is a choice of unit to Y then there is a function $f \in C^\infty(Y)$ such that $V = f\nu$.

Let $\{x^1, \dots, x^n\}$ be local coordinates near a point $0 \in Y$. They induce coordinates on $\mathcal{F}_t(Y)$ defined via $(t, x^1, \dots, x^n) \mapsto \mathcal{F}_t(x^1, \dots, x^n)$, and we have the corresponding coordinate vector fields $\{\partial_1, \dots, \partial_n\}$, along with $\partial_t = V$. Let

$$h_{\alpha\beta}(t, x) = g_{\mathcal{F}_t(Y)}(\partial_\alpha, \partial_\beta).$$

Then

$$\begin{aligned} h'_{\alpha\beta} &= \left. \frac{\partial}{\partial t} h_{\alpha\beta} \right|_{t=0} \\ &= g(\nabla_{\partial_t} \partial_\alpha, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_t} \partial_\beta) \\ &= g(\nabla_{\partial_\alpha} V, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta} V). \end{aligned}$$

If $V = f\nu$, then this becomes

$$h'_{\alpha\beta} = fg(\nabla_{\partial_\alpha} \nu, \partial_\beta) + g(\partial_\alpha, \nabla_{\partial_\beta} \nu). \quad (4.72)$$

Given a choice of normal ν our definition of the second fundamental form of Y is

$$L(\partial_\alpha, \partial_\beta) = g(\nu, \nabla_{\partial_\alpha} \partial_\beta) = -g(\nabla_{\partial_\alpha} \nu, \partial_\beta). \quad (4.73)$$

Therefore, by (4.72) we conclude

$$h'_{\alpha\beta} = -2fL_{\alpha\beta}. \quad (4.74)$$

By the standard formula for the inverse, this implies

$$(h^{\alpha\beta})' = 2fL^\alpha_\gamma L^{\beta\gamma}. \quad (4.75)$$

By our definition of second fundamental form,

$$\begin{aligned} L'_{\alpha\beta} &= \left. \frac{\partial}{\partial t} L_{\alpha\beta} \right|_{t=0} \\ &= g(\nabla_{\partial_t} \nu, \nabla_{\partial_\alpha} \partial_\beta) + g(\nu, \nabla_{\partial_t} \nabla_{\partial_\alpha} \partial_\beta). \end{aligned} \quad (4.76)$$

The first term on the right is easily seen to vanish, since $0 = \partial_t g(\nu, \nu) = 2g(\nabla_{\partial_t} \nu, \nu)$ implies that

$$g(\nabla_{\partial_t} \nu, \nabla_{\partial_\alpha} \partial_\beta) = -L_{\alpha\beta} g(\nabla_{\partial_t} \nu, \nu) = 0. \quad (4.77)$$

For the second term, we commute derivatives to get

$$\begin{aligned} g(\nu, \nabla_{\partial_t} \nabla_{\partial_\alpha} \partial_\beta) &= g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_t} \partial_\beta) + R(V, \partial_\alpha, \partial_\beta, \nu) \\ &= g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\beta} V) + R(V, \partial_\alpha, \partial_\beta, \nu), \end{aligned} \quad (4.78)$$

where R is the curvature tensor of g . If $V = f\nu$ then by (4.77) and (4.78), (4.76)

simplifies to

$$\begin{aligned}
L'_{\alpha\beta} &= g(\nu, \nabla_{\partial_t} \nabla_{\partial_\alpha} \partial_\beta) \\
&= g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\beta} (f\nu)) + fR(\nu, \partial_\alpha, \partial_\beta, \nu) \\
&= \nabla_\alpha^Y \nabla_\beta^Y f + g(\nu, \partial_\alpha f \nabla_{\partial_\beta} \nu + \partial_\beta f \nabla_{\partial_\alpha} \nu + f \nabla_{\partial_\alpha} \nabla_{\partial_\beta} \nu) + fR(\nu, \partial_\alpha, \partial_\beta, \nu) \\
&= \nabla_\alpha^Y \nabla_\beta^Y f + fg(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\beta} \nu) + fR(\nu, \partial_\alpha, \partial_\beta, \nu),
\end{aligned} \tag{4.79}$$

where in the last line we used the fact that $\partial_\alpha g(\nu, \nu) = 0$. Using this fact again we also find

$$g(\nu, \nabla_{\partial_\alpha} \nabla_{\partial_\beta} \nu) = -g(\nabla_{\partial_\alpha} \nu, \nabla_{\partial_\beta} \nu). \tag{4.80}$$

It follows from the definition of the second fundamental form that

$$\nabla_{\partial_\alpha} \nu = -L_\alpha^\gamma \partial_\gamma,$$

hence

$$-g(\nabla_{\partial_\alpha} \nu, \nabla_{\partial_\beta} \nu) = -L_\alpha^\gamma L_{\beta\gamma}.$$

Substituting this into (4.80) and combining with (4.79), we arrive at

$$L'_{\alpha\beta} = \nabla_\alpha^Y \nabla_\beta^Y f - f L_\alpha^\gamma L_{\beta\gamma} + fR(\nu, \partial_\alpha, \partial_\beta, \nu). \tag{4.81}$$

For the variation of the mean curvature $H = h^{\alpha\beta} L_{\alpha\beta}$ we use (4.75) and (4.81) to obtain

$$H' = \Delta_Y f + (|L|^2 + \text{Ric}(\nu, \nu))f. \tag{4.82}$$

Finally, using the standard formula for the derivative of the volume form, we have

$$(dv_h)' = -fH dv_h. \tag{4.83}$$

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