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Painlevé analysis and exact solutions of bosonized $\mathcal{N} = 1$ supersymmetric Burgers equation

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Abstract: Based on the bosonization approach, the $\mathcal{N} = 1$ supersymmetric Burgers (SB) system is transformed to a coupled pure bosonic system. The Painlevé property and the Bäcklund transformations (BT) of the bosonized SB (BSB) system are obtained through standard singularity analysis. Explicit solutions such as the muti-solitary waves and error function waves are provided for the BT. The exact solutions of the BSB system are obtained from the generalized tanh expansion method.

Keywords: supersymmetric Burgers equation; bosonization approach; Painlevé analysis; generalized tanh expansion method

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1 Introduction

Super extensions of classically integrable systems lead to super integrable systems. These have undergone extensive development over the years. There exists an extensive literature devoted to construction of the supersymmetric integrable models, such as Korteweg-de Vries, modified Korteweg-de Vries, Burgers, Sine-Gordon, Kadomtsev-Petviashvili hierarchy and nonlinear Schrödinger equation [1–8]. Supersymmetric systems provide more prolific fields for mathematical and physical researchers. They exhibit the Painlevé property, the Lax representation, an infinite number of conservation laws, the Bäcklund the Darboux transformations, bilinear forms and multi-soliton solutions [9–15, 26]. However to treat the integrable systems with fermions, such as the supersymmetric integrable systems and pure integrable fermionic systems, is much more complicated than to study the integrable pure bosonic systems [17]. It is significantly important to establish a proper bosonization procedure to deal with the supersymmetric systems. Recently, a simple bosonization approach to treat the super integrable systems has been proposed [18, 19]. The method can effectively avoid difficulties caused by intractable anticommuting fermionic fields [18–21]. In this work, we shall use the bosonization approach to the $\mathcal{N} = 1$ SB system.

The paper is organized as follows. In Section 2, the SB system is changed to a system of coupled bosonic equations based on the bosonization approach. The Painlevé property and the BT of the coupled bosonic equations are studied by the standard singularity analysis. Section 3 is devoted to the generalized tanh expansion approach for the coupled bosonic equations. A nonauto-BT theorem is given through the approach. Some novel exact solutions of the BSB system can be expressed with the nonauto-BT theorem. The conclusions and discussions are given in last section.

2 Bosionization of the SB equation and Painlevé property

2.1 Bosonization approach with two fermionic parameters

The SB system reads [5]

$$\Phi_t + \Phi D\Phi_x + \Phi_{xx} = 0, \qquad (1)$$

where $\Phi = \Phi(x, t, \theta)$ is a fermionic superfield depending on the usual variable (x, t) and super spatial variable θ. Expansion of Φ in terms of θ yields $\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t)$. $D = \partial_\theta + \theta\partial_x$ is the covariant derivative. In terms of the component fields, (1) yields to

$$u_t + uu_x + u_{xx} - \xi\xi_{xx} = 0, \qquad (2a)$$

$$\xi_t + \xi_{xx} + \xi u_x = 0. \qquad (2b)$$

It is obvious that (2) includes a commuting u and an anticommuting ξ field. (1) and (2) are invariant under the supersymmetric transformation $\delta_\eta u = \eta\xi_x$, $\delta_\eta\xi = \eta u$, where η is a arbitrary odd parameter. It will degenerate to the usual classical system with vanishing fermionic sector. In order to avoid the difficulties in dealing with the

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anticommutative fermionic field ξ, the component fields ξ and u expand by introducing the two fermionic parameters [18–21]

$$u(x,t) = v + w\zeta_1\zeta_2, \tag{3a}$$
$$\xi(x,t) = p\zeta_1 + q\zeta_2, \tag{3b}$$

where ζ_1 and ζ_2 are two Grassmann parameters, while the coefficients v, p, q and w are four usual real or complex functions with respect to the spacetime variable (x,t). Substituting (3) into the SB system (2), we obtain

$$v_t + v_{xx} + vv_x = 0, \tag{4a}$$
$$p_t + p_{xx} + pv_x = 0, \tag{4b}$$
$$q_t + q_{xx} + qv_x = 0, \tag{4c}$$
$$w_t + w_{xx} + (vw)_x = qp_{xx} - pq_{xx}. \tag{4d}$$

The above approach is just the bosonization procedure for the SB system (2) with two fermionic parameters (2-BSB). (4a) is exactly the usual Burgers equation which has been widely studied. (4b)–(4d) are linear homogeneous in p, q and w, respectively. These pure bosonic systems can be easily solved theoretically. This is just one of the advantages of the bosonization approach.

2.2 Painlevé property, Bäcklund transformations and explicit solutions for the 2-BSB system

In this section, the Painlevé property and the BT of the 2-BSB system will be studied. If the solution is single valued for every non-characteristic singularity manifold, the equation is proclaimed to possess the Painlevé property [22]. In order to perform Painlevé analysis, the bosonic fields v, p, q and w expand about the singularity manifold $\phi(x,t) = 0$ as:

$$\begin{aligned} v &= \sum_{j=0}^{\infty} v_j\phi^{j-\alpha_1}, \quad p = \sum_{j=0}^{\infty} p_j\phi^{j-\alpha_2}, \\ q &= \sum_{j=0}^{\infty} q_j\phi^{j-\alpha_3}, \quad w = \sum_{j=0}^{\infty} w_j\phi^{j-\alpha_4}, \end{aligned} \tag{5}$$

with the (v_j, p_j, q_j, w_j) being arbitrary functions of (x,t). From the leading order analysis result, all the constants α_1, α_2, α_3 and α_4 are positive integers, i.e., 1, 1, 1, 2 respectively. Consequently, the recursion relations to determine the functions v_j, p_j, q_j and w_j can be obtained. The resonance values of j are given by:

$$j = -1, 0, 0, 0, 2, 3, 3, 3. \tag{6}$$

After detailed calculations, the resonance conditions are satisfied identically because the functions v_j, p_j, q_j and w_j are all determined by eight arbitrary functions ϕ, p_0, q_0, w_0, v_2, p_3, q_3 and w_3. From the above considerations we deduce that the 2-BSB system is really Painlevé integrable.

The standard Painlevé truncated expansion is:

$$\begin{aligned} v &= \frac{v_0}{\phi} + v_1, \qquad p = \frac{p_0}{\phi} + p_1, \\ q &= \frac{q_0}{\phi} + q_1, \qquad w = \frac{w_0}{\phi^2} + \frac{w_1}{\phi} + w_2, \end{aligned} \tag{7}$$

where $v_0 = 2\phi_x$, $w_1 = (\phi_x w_{0,x} - \phi_{xx}w_0)\phi_x^{-2}$. Equation (7) defines a BT, where two pairs of function (v, p, q, w) and (v_1, p_1, q_1, w_2) satisfy (4). The following theorem thus arises.

Theorem 1. If the fields ϕ, p_0, q_0 and w_2 are the solutions of the following Schwarzian 2-BSB system

$$\left(\frac{\phi_t}{\phi_x}\right)_t - \frac{1}{2}\left(\left(\frac{\phi_t}{\phi_x}\right)^2\right)_x + 2\left(\frac{\phi_t}{\phi_x}\right)_{xx} + \{\phi; x\}_x = 0, \tag{8a}$$

$$p_{0,t} + p_{0,xx} - p_{0,x}\frac{2\phi_{xx}}{\phi_x} - p_0\left(\frac{\phi_{xt}}{\phi_x} + \frac{\phi_{xxx}}{\phi_x} - \frac{2\phi_{xx}^2}{\phi_x^2}\right) = 0, \tag{8b}$$

$$q_{0,t} + q_{0,xx} - q_{0,x}\frac{2\phi_{xx}}{\phi_x} - q_0\left(\frac{\phi_{xt}}{\phi_x} + \frac{\phi_{xxx}}{\phi_x} - \frac{2\phi_{xx}^2}{\phi_x^2}\right) = 0, \tag{8c}$$

$$w_{2,t} + w_{2,xx} + (v_1w_2)_x + p_1q_{1,xx} - q_1p_{1,xx} = 0. \tag{8d}$$

where $\{\phi; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$ is the Schwarzian derivative of ϕ. The solution (v_1, p_1, q_1, w_2) is related to (ϕ, p_0, q_0, w_0) by:

$$v_1 = -\frac{\phi_t + \phi_{xx}}{\phi_x}, \tag{9a}$$

$$\begin{aligned} p_1 &= \frac{p_0\phi_{xx} - 2p_{0,x}\phi_x - p_0\phi_t}{2\phi_x^2}, \\ q_1 &= \frac{q_0\phi_{xx} - 2q_{0,x}\phi_x - q_0\phi_t}{2\phi_x^2}, \end{aligned} \tag{9b}$$

$$\begin{aligned} w_2 = {} & \frac{1}{2\phi_x^2}w_{0,t} + \frac{1}{2\phi_x^2}w_{0,xx} - \left(\frac{3\phi_{xx}}{2\phi_x^3} + \frac{\phi_t}{2\phi_x^3}\right)w_{0,x} \\ & + \frac{1}{2\phi_x^2}(p_0q_{0,xx} - p_{0,xx}q_0) \\ & + \left(\frac{3\phi_{xx}^2}{2\phi_x^4} + \frac{\phi_{xx}\phi_t}{2\phi_x^4} - \frac{\phi_{xxx}}{2\phi_x^3} - \frac{\phi_{xt}}{2\phi_x^3}\right)w_0 \\ & + \frac{\phi_{xx}}{\phi_x^3}(p_0q_{0,x} - p_{0,x}q_0). \end{aligned} \tag{9c}$$

An auto-BT (7) and a nonauto-BT (9) are obtained with the singularity analysis. It is useful to find new solutions from a simple seed solution. Here, we take the trivial solution $v_1 = p_1 = q_1 = w_2 = 0$ for (4). The solution for

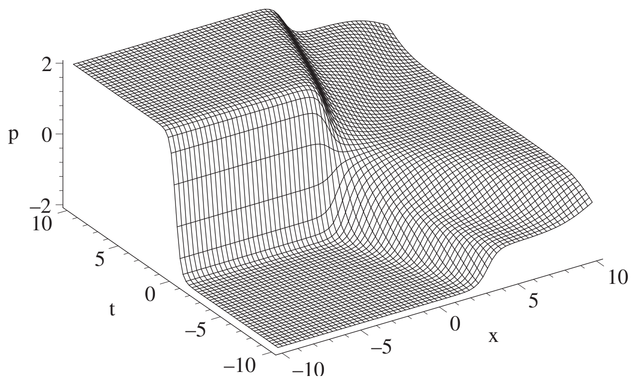

Figure 1: Plot of four-solitary wave fusion for p.

(ϕ, p_0, q_0, w_0) will be obtained by considering (8) and (9). The multi-solitary waves solution ϕ can be written as [23]

$$\phi = 1 + \sum_{i=1}^{n} \exp(k_i x + \omega_i t), \tag{10}$$

where k_i and ω_i satisfy the dispersion relation $\omega_i + k_i^2 = 0$. By substituting (10) into (9b) and combining with (8b) and (8c), the solutions of p_0 and q_0 result:

$$p_0 = q_0 = \sum_{i=1}^{n} k_i \exp(k_i x + \omega_i t). \tag{11}$$

The solution for w_0 can also be solved with (8d) and (9c). We omit the expression of this solution since it is very lengthy. Finally, the solution (v, p, q, w) can be given with the BT (7). Figure 1 shows the solution p with the parameters $n = 4$, $k_1 = -1$, $k_2 = 1$, $k_3 = 2$, $k_4 = -2$.

In addition, the field ϕ possesses an error function solution

$$\phi = \mathrm{erf}\left(\frac{x}{2\sqrt{-t}}\right), \tag{12}$$

where the error function $\mathrm{erf}(x)$ is defined by:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\xi^2) d\xi.$$

The solution p_0, q_0, w_0 can be obtained with the similar procedure.

3 Generalized tanh function expansion method of 2-BSB system

The truncated Painlevé expansion and the tanh expansion method can be used to find exact solutions of the partially solvable nonlinear models, thus, the methods are valid for all integrable systems and even non-integrable models [21]. The methods have been established to find abundant interaction solutions among solitons and other types of nonlinear waves [21, 24, 25]. According to the generalized tanh function expansion method, the expansion solution has the form

$$v = v_0 + v_1 \tanh(f), \tag{13a}$$

$$p = p_0 + p_1 \tanh(f), \tag{13b}$$

$$q = q_0 + q_1 \tanh(f), \tag{13c}$$

$$w = w_0 + w_1 \tanh(f) + w_2 \tanh^2(f). \tag{13d}$$

where $v_0, v_1, p_0, p_1, q_0, q_1, w_0, w_1, w_2$ and f are functions of (x, t) and should be determined later. By substituting (13) into the 2-BSB system (4), we can prove the following nonauto-BT theorem after some detail calculations.

Theorem 2 (Nonauto-BT theorem). If $(f,\ g,\ h,\ n)$ is a solution of

$$\left(\frac{f_t}{f_x}\right)_t - \frac{1}{2}\left[\left(\frac{f_t}{f_x}\right)^2\right]_x + 2\left(\frac{f_t}{f_x}\right)_{xx} + \{f; x\}_x - 4f_{xx}f_x = 0, \tag{14a}$$

$$g_t + g_{xx} + \frac{2f_{xx}}{f_x} g_x - \left(\frac{f_{xxx}}{f_x} + \frac{f_{xt}}{f_x} - \frac{2f_{xx}^2}{f_x^2}\right) g = 0, \tag{14b}$$

$$h_t + h_{xx} + \frac{2f_{xx}}{f_x} h_x - \left(\frac{f_{xxx}}{f_x} + \frac{f_{xt}}{f_x} - \frac{2f_{xx}^2}{f_x^2}\right) h = 0, \tag{14c}$$

$$n_t + n_{xxx} - n_{xx}\frac{4f_{xx} + f_t}{f_x} - n_t \frac{f_{xx}}{f_x} + n_x\left(\frac{9f_{xx}^2 + 3f_{xx}f_t}{f_x^2} - \frac{2f_{xt} + 4f_{xxx}}{f_x} - 2f_x^2\right) + (hg_{xx} - gh_{xx})_x \tag{14d}$$

$$+ n\left(\frac{f_t^2 f_{xx}}{2f_x^3} - \frac{f_t f_{xx}^2}{f_x^3} + \frac{6f_{xx}f_{xxx} + f_{xx}f_{xt} - f_t f_{xt}}{f_x^2} + \frac{f_{tt} - f_{xxxx}}{2f_x} - 4f_x f_{xx}\right) - 2f_x(fw_0)_x + (gh_x - hg_x)\left(\frac{2f_{xxx} - f_{xt}}{f_x} + \frac{2f_{xx}f_t - 4f_{xx}^2}{f_x^2}\right) + \frac{f_t - 3f_{xx}}{f_x}(hg_{xx} - gh_{xx}) = 0, \tag{14e}$$

where $w_0 = -n + \frac{n_t + n_{xx}}{2f_x^2} - n_x \frac{f_t + f_{xx}}{2f_x^3} + n\frac{f_{xx}f_t + 3f_{xx} - f_{xt}f_x - f_{xxx}f_x}{2f_x^4} - \frac{(gh_x - g_x h)(f_t + 2f_{xx})}{2f_x^3} + \frac{gh_{xx} - g_{xx}h}{2f_x^2}$. Then, the field $(v,\ p,\ q,\ w)$ with:

$$v = -\frac{f_{xx}}{2f_x} - \frac{f_t}{f_x} + 2f_x \tanh(f), \tag{15a}$$

$$p = \frac{f_{xx}}{2f_x^2} g - \frac{g_x}{f_x} - \frac{f_t}{2f_x^2} g + g \tanh(f), \tag{15b}$$

$$q = \frac{f_{xx}}{2f_x^2} h - \frac{h_x}{f_x} - \frac{f_t}{2f_x^2} h + h \tanh(f), \tag{15c}$$

$$w = \frac{n_t}{2f_x^2} + \frac{n_{xx}}{2f_x^2} - \left(\frac{3f_{xx}}{2f_x^3} - \frac{f_t}{2f_x^3}\right)n_x + \left(\frac{f_{xx}f_t}{2f_x^4} + \frac{3f_{xx}^2}{2f_x^4}\right.$$
$$\left. - \frac{f_{xt}}{2f_x^3} - \frac{f_{xxx}}{2f_x^3} - 1\right)n + \frac{1}{2f_x^2}(gh_{xx} - hg_{xx}) \quad (15d)$$
$$+ \frac{f_{xx}}{f_x^3}(hg_x - gh_x) + \frac{f_t}{2f_x^3}(h_x g - g_x h)$$
$$+ \left(\frac{f_{xx}n}{f_x^2} - \frac{n_x}{f_x}\right)\tanh(f) + n\tanh^2(f), \quad (15e)$$

is a solution of the 2-BSB system (4).

Some nontrivial solutions of the 2-BSB system emanate from some quite trivial solutions of (14). Here we list two examples. A quite trivial solution of (14) has the form

$$f = k_0x + \omega_0 t + l_0, \qquad g = k_1 x + l_1,$$
$$h = k_2 x + l_2, \qquad n = k_3 x + \omega_3 t + l_3, \quad (16)$$

where k_0, k_1, k_2, k_3, l_0, l_1, l_2, l_3, ω_0 and ω_3 are all the free constants. The soliton solution of 2-BSB system has in the following form using the line solution (16) and the nonauto-BT theorem:

$$v = -\frac{\omega_0}{k_0} + 2k_0\tanh(f), \quad (17a)$$
$$p = 2g\tanh(f) - \frac{\omega_0 k_1 x + \omega_0 x_1 + 2k_1k_0}{2k_0^2}, \quad (17b)$$
$$q = 2h\tanh(f) - \frac{\omega_0 k_2 x + \omega_0 x_2 + 2k_2k_0}{2k_0^2}, \quad (17c)$$
$$w = n\tanh^2(f) - \frac{k_3}{k_0}\tanh(f) - n$$
$$- \frac{\omega_0 k_3 + \omega_3 k_0 + \omega_0 k_1 x_2 - \omega_0 k_2 x_1}{2k_0^3}. \quad (17d)$$

Though the soliton solution (17) is a traveling wave in the space time (x, t) for the boson field v, it is not a traveling wave for other boson fields p, q and w, then the superfiled Φ of SB system is not a traveling wave except for the case g, h and n being constants, i.e., $k_1 = k_2 = k_3 = 0$. Besides, we assume the traveling wave solutions f, g, h and n read

$$f = F(\eta), \qquad g = G(\eta),$$
$$h = H(\eta), \qquad n = N(\eta), \qquad \eta = kx + \omega t + l. \quad (18)$$

Substituting (18) into (14), we have

$$F_1''' - \frac{4F_1''F_1'}{F_1} - 4F_1'F_1^2 + \frac{3F_1'^3}{F_1^2} = 0, \qquad F_1 = F', \quad (19a)$$
$$G'' + \left(\frac{\omega}{k^2} - \frac{2F_1'}{F_1}\right)G' + \left(\frac{2F_1'^2}{F_1^2} - \frac{F_1''}{F_1} - \frac{\omega}{k^2}\frac{F_1'}{F_1}\right)G = 0, \quad (19b)$$
$$H'' + \left(\frac{\omega}{k^2} - \frac{2F_1'}{F_1}\right)H' + \left(\frac{2F_1'^2}{F_1^2} - \frac{F_1''}{F_1} - \frac{\omega}{k^2}\frac{F_1'}{F_1}\right)H = 0, \quad (19c)$$
$$N_{xxxx} - N_{xxx}\frac{8F_1'}{F_1} + N_{xx}\left(\frac{33F_1'^2}{F_1^2} - \frac{10F_1''}{F_1} - 4F_1^2\right)$$
$$- N_x\left(\frac{2F_1'''}{F_1} + \frac{66F_1'^3}{F_1^3} - \frac{40F_1'F_1''}{F_1^2} + 8F_1F_1'\right)$$
$$+ N\left(\frac{54F_1'^4}{F_1^4} - \frac{52F_1'^2F_1''}{F_1^3} + \frac{4F_1'F_1'''}{F_1^2} + \frac{6F_1''^2}{F_1^2} + 8F_1'^2\right)$$
$$+ (G''H' - G'H'')\left(\frac{12F_1'}{F_1} - \frac{2\omega}{k^2}\right)$$
$$+ (G'H - GH')\left(\frac{6F_1'^3}{F_1^3} + \frac{6F_1'F_1''}{F_1^2} - \frac{3F_1'''}{F_1} - \frac{2\omega}{k^2}\frac{F_1''}{F_1} - \frac{\omega^2}{k^4}\frac{F_1'}{F_1}\right)$$
$$+ (G'''H - GH''')\frac{4F_1'}{F_1} \quad (19d)$$
$$+ (GH'' - G''H)\left(\frac{27F_1'^2}{2F_1^2} - \frac{F_1''}{F_1} - \frac{2\omega}{k^2}\frac{F_1'}{F_1} - 2F_1^2 - \frac{\omega^2}{2k^4}\right)$$
$$+ 4(G'H''' - G'''H') = 0. \quad (19e)$$

The traveling wave solution of (19a) thus reads

$$F_1 = \pm\text{arctanh}\frac{\exp\left(\mp\frac{\sqrt{C_1+16C_2^2}}{2}(\eta + C_3)\right) + 32C_2}{8\sqrt{C_1 + 16C_2^2}}, \quad (20)$$

with C_1, C_2 and C_3 are arbitrary constants. The corresponding solution of the 2-BSB system reads

$$v = 2kF_1\tanh(f) - k\frac{F_1'}{F_1} - \frac{\omega}{k}, \quad (21a)$$
$$p = G\tanh(f) - \frac{G'}{F_1} - \frac{\omega}{2k^2}\frac{G}{F_1} - \frac{GF_1'}{2F_1^2}, \quad (21b)$$
$$q = H\tanh(f) - \frac{G'}{F_1} - \frac{\omega}{2k^2}\frac{H}{F_1} - \frac{HF_1'}{2F_1^2}, \quad (21c)$$
$$w = N\tanh^2(f) + \left(N\frac{F_1'}{F_1^2} - \frac{N'}{F_1}\right)\tanh(f) + \frac{N''}{2F_1^2} - N'\frac{3F_1'}{2F_1^3}$$
$$+ N\left(\frac{3F_1'}{2F_1^4} - \frac{F_1''}{2F_1^3} - 1\right)$$
$$+ \frac{F_1'(G'H - GH')}{F_1^3} + \frac{GH'' - HG''}{2F_1^2} + \frac{\omega}{2k^2}\frac{GH' - HG'}{F_1^2}. \quad (21d)$$

4 Conclusions

In summary, we have presented the bosonization procedure for the SB equation. The SB equation is simplified to the Burgers equation with three linear differential equations. The 2-BSB system is shown to possess Painlevé property. An auto-BT and a nonauto-BT are constructed by truncating the standard Painlevé expansion. We have found explicit solutions such as muti-solitary waves and error function waves by using the BT theorem. In addition, some exact solutions of the 2-BSB system are obtained from the generalized tanh expansion method. All these solutions are obtained via the bosonization procedure and are different from those obtained via other methods such as the Hirota bilinear method [26, 27]. We believe that there are still many interesting applications for the bosonization approach. For instance, (i) the exploration of the $N \geq 2$ supersymmetric systems with the bosonization approach, (ii) the extension of the bosonization approach to deal with a Clifford algebra in place of a Grassmann one. This will be focus of future work.

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