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Gliner vacuum, self-consistent theory of Ruppeiner geometry for regular black holes

Chen Lan[1,2,a] , **Yan-Gang Miao**[1,b]

[1] School of Physics, Nankai University, 94 Weijin Road, Tianjin 300071, China
[2] Department of Physics, Yantai University, 30 Qingquan Road, Yantai 264005, China

Abstract   In the view of the Gliner vacuum, we remove the deformations in the first law of mechanics for regular black holes, where one part of deformations associated with black hole mass will be absorbed into enthalpy or internal energy, and the other part associated with parameters rather than mass will constitute a natural $V-P$ term. The improved first law of mechanics redisplays its resemblance to the first law of thermodynamic systems, which implies a restored correspondence of the mechanic variables to the thermodynamic ones. In particular, the linear relation between the entropy and horizon area remains unchanged for regular black holes. Based on the modified first law of thermodynamics, we establish a self-consistent theory of Ruppeiner geometry and obtain a universal attractive property for the microstructure of regular black holes. In addition, the repulsive and attractive interactions inside and outside regular black holes are analyzed in detail.

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[a] e-mail: lanchen@nankai.edu.cn

[b] e-mail: miaoyg@nankai.edu.cn (corresponding author)

1 Introduction

It is closely related [1,2] to the *first law of thermodynamics* (1LT) and its corresponding entropy to construct the Ruppeiner geometry for black holes (BHs). As is known, the 1LT is usually deduced from its resemblance to the *first law of mechanics* (1LM). Based on $T \propto \kappa$ calculated from quantum theory [3], one can read from the resemblance the linear relation: $S \propto A$. This is now known as the entropy/area law, where the corresponding entropy is dubbed as the Bekenstein–Hawking entropy (BHE) [4,5]. The 1LT cannot be correctly obtained when the resemblance mentioned above breaks. Furthermore, if the entropy calculated from the path-integral approach [6] or Wald's method [7,8] does not coincide with that from the 1LT, the Ruppeiner geometry based on the 1LT will be unreliable.

A regular black hole (RBH) is such a system that its 1LM is deformed [9], which brings about the breaking of the resemblance between its 1LM and 1LT. From this point of view, a RBH does not have a well-defined 1LT, and then its Ruppeiner geometry is suspect. As the 1LT is considered to be the basis of Ruppeiner geometry and many other research topics, such as the superradiance and area spectrum, etc., the lack of a well-defined 1LT leads to an obstacle for us to establish the Ruppeiner geometry for RBHs.

We take the noncommutative geometry inspired BH [10] as an example of RBHs to show its present issue. Its shape function, $f(r, M, \theta)$, contains two parameters, where one is

BH mass M and the other θ is related to the minimal length of a noncommutative space. If θ is set to be constant, the 1LT takes the form, $dM = TdS$, where T is obtained from the Euclidean path integral near the horizon of this BH, see e.g. Ref. [11]. The entropy calculated from such a 1LT breaks the entropy/area law, $S \neq A/4$. On the other hand, if the entropy of this BH without backreaction obeys [12] the entropy/area law, the 1LT does not hold, i.e., $dM \neq TdS$, when the Hawking temperature is not modified and still calculated by the path-integral approach. In a word, this is the present situation for any RBHs[1] that either the entropy/area law or the 1LT breaks if the Hawking temperature is computed by the formula, $-g'_{tt}(r_{\mathrm{H}})/(4\pi)$.

In this work we persist in adopting the Hawking temperature formula and maintain both the entropy/area law and the 1LT unbroken by applying the radial pressure of Gliner vacuum [13,14] as RBH's pressure. The so-called Gliner vacuum is an extended conception of vacua, which was introduced by Gliner and developed by Dynmnikova, Gurevich and Starobinsky, see the historical review [15] and references within. The Gliner vacuum is different from the traditional vacuum with a vanishing energy–momentum tensor, $T_{\alpha\beta} = 0$. It is defined [16] as a kind of matter that does not allow any preferred reference frame. In the case of a spherically symmetric spacetime, this definition demands [16,17] $T^0_{\ 0} = T^1_{\ 1}$ and provides an infinite set of comoving reference frames based on Petrov's classification [18]. Thus, the Gliner vacuum is anisotropic in a spherically symmetric spacetime.

Due to the pressure of the Gliner vacuum, we can remove the deformations [19,20] in the 1LM for RBHs. In other words, the deformation related to M will be absorbed into enthalpy or internal energy, and the other deformations corresponding to the parameters rather than M constitute a natural $V–P$ term. For RBHs this procedure not only offers a solution to reconstruct the 1LT, consequently a self-consistent theory of Ruppeiner geometry, but also provides new insight into the 1LT related research topics, like the superradiance and area spectrum mentioned above.

This paper is organized as follows. In Sect. 2, we analyze the RBHs that are spherically symmetric and have a single shape function in terms of the Ricci decomposition. Next, we examine in Sect. 3 the deformations of the first law of mechanics in the RBHs. We discuss in Sect. 4 how to remove the deformations in the first law of mechanics for some known RBHs in the view of the Gliner vacuum. In Sect. 5 we establish a self-consistent theory of Ruppeiner geometry for the RBHs with a spherical symmetry and a single shape function by a well-defined 1LT. The results show that all the RBHs are of attractive interaction. It is known that the matters generating the RBHs violate the strong energy condition (SEC) around the center, i.e., they have repulsive

interactions. Therefore, we try in Sect. 6 to explain in terms of the SEC on how the matters with a repulsive interaction create the RBHs with attractive interactions. To give a more intuitive illustration of the interaction structure of RBHs, in Sect. 7, we compare them with Reissner–Nordström black holes (RN BHs) by considering the Raychaudhuri equation. Finally, we give our summary in Sect. 8.

2 Regular black holes

We start with a spherically symmetric metric $g_{\mu\nu} = \mathrm{diag}\{-f(r), f^{-1}(r), r^2, r^2\sin^2\theta\}$ with the shape function,

$$f(r) = 1 - \frac{2M}{r}\sigma(r, M, \alpha_i),\tag{1}$$

where α_i's are parameters rather than mass M. If σ converges to one when r goes to infinity, the corresponding metric is asymptotic to the external spacetime of Schwarzschild BHs. In fact, the asymptotic flatness of our metric only requires $\lim_{r\to\infty} \sigma/r = 0$, i.e., σ is divergent slower than r as $r \to \infty$, which is weaker than the condition of asymptotic to the Schwarzschild spacetime. The requirement of asymptotic to the Schwarzschild BH simply guarantees the attractive nature (strong energy condition) and causality (dominant energy condition) outside the horizon. In addition, for simplicity, we suppose that the BHs depicted by Eq. (1) are of two horizons at most.

It is convenient to apply the Ricci decomposition [21] for our investigations of RBHs,

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + S_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma},\tag{2}$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor, and

$$S_{\alpha\beta\mu\nu} = \frac{R}{12}g_{\alpha[\mu}g_{\nu]\beta},\tag{3}$$

$$E_{\alpha\beta\mu\nu} = \frac{1}{2}\left(g_{\alpha[\mu}Z_{\nu]\beta} - g_{\beta[\mu}Z_{\nu]\alpha}\right),\tag{4}$$

with the traceless tensor,

$$Z_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R.\tag{5}$$

Thus, we rewrite σ in terms of the curvature invariants,[2] R, W, and E, when r is around the center of RBHs,

$$\sigma = \frac{r^3}{24M}\left(R - 2\sqrt{3W} + 3\sqrt{2E}\right),\tag{6}$$

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[1] Their metrics satisfy $g_{tt}g_{rr} = -1$.

[2] In fact, W and E can be expressed by the three independent curvature invariants, R, R_2, and K, see Appendix A for the details.

where $W \equiv W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ and $E \equiv E_{\mu\nu\rho\sigma} E^{\mu\nu\rho\sigma}$, and both W and E are non-negative. According to the relations,

$$R_2 \equiv R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} \left(R^2 + 2E \right), \tag{7}$$

$$K \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{1}{6} R^2 + W + E, \tag{8}$$

the contraction of Ricci tensors R_2 and the Kretschmann scalar K are also non-negative. Based on the Ricci decomposition Eq. (2), the shape function then becomes

$$f(r) = 1 - \frac{\lambda(r)}{3} r^2, \quad 4\lambda(r) \equiv R - 2\sqrt{3W} + 3\sqrt{2E}. \tag{9}$$

That a BH is regular means that the independent curvature invariants, R, R_2, and K, are convergent to finite constants everywhere in this BH spacetime. In particular, as $r \to 0^+$ we have

$$R \sim R(0), \quad R_2 \sim R_2(0), \quad K \sim K(0), \tag{10}$$

with $R(0)$, $R_2(0)$, $K(0) = \text{const.} < \infty$, where the zero argument refers to the limit to the center of BHs. We can see from Eqs. (1) and (6) that the finiteness of the shape function requests that σ converges to zero faster than r^3, i.e., $\lim_{r \to 0} \sigma/r^3 = \text{const.} < \infty$, see Appendix B for the detailed analyzes of R and K.

It is worth mentioning that the treatment of RBHs is different from that of singular BHs (SBHs). For the latter, a complete Lagrangian is given, its equations of motion are solved, and thus the metric of SBHs is obtained. For the former, however, the first step is to propose such a metric that the corresponding curvature invariants are finite and geodesics are complete at the singularity of this metric; and the second step is to construct the related Lagrangian that generates this RBH, i.e., to find out the reasonable matter source.

According to the behaviors of σ, the metrics of RBHs can be classified into two types. The first type of RBHs is constrained locally by $0 < \sigma(r, M, \alpha_i) \le 1$, thus the outer horizon r_+ must be upper bounded by the Schwarzschild radius, $r_{\text{Sch}} = 2M$, i.e.,

$$\frac{r_+}{r_{\text{Sch}}} = \sigma(r_+, M, \alpha_i) \le 1. \tag{11}$$

The second type of RBHs is picked out by the local condition, $\sigma(r, M, \alpha_i) > 1$. As the matter sources associated with the metrics in the second type are not clear, we concentrate only on the first type and assume that σ is a monotonically increasing function[3] of r, $d\sigma/dr > 0$, in the remaining of the present work.

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[3] The monotonicity comes from the positivity of energy density of matters, see Eq. (69). Thus, $\sigma(r)$ is a sigmoid function in the range of $r \in [0, \infty)$.

In addition, let us make a brief discussion about the factor $\lambda(r)$ in Eq. (9). It is not difficult to check $\lambda(r) \sim R(0)/4$ in the limit of $r \to 0^+$, because $R \sim O(1)$, $E \sim O(r^2)$, and $W \sim O(r^2)$, i.e. the orders of E and W are higher than that of R. The same result can also be obtained by the method shown in Refs. [22,23], see Appendix B. Then changing the parameters (mass, charge, etc.) in $f(r_+) = 0$, such that r_+ approaches zero, we obtain (see Eq. (108))

$$1 - \frac{R(0)}{12} r_+^2 + O\left(r_+^3 \right) = 0, \tag{12}$$

which implies that the horizon exists only for nonnegative $R(0)$. Meanwhile, the negative R is forbidden by the dominant energy condition [22]. In other words, a RHB is of a dS core rather than an AdS one around the center if $R(0)$ does not vanish. Incidentally, if such a RHB is immersed in an AdS spacetime with the cosmological constant $\tilde{\Lambda}$, the corresponding shape function becomes

$$f(r) = 1 - \frac{\lambda(r)}{3} r^2 - \frac{\tilde{\Lambda}}{3} r^2, \quad \tilde{\Lambda} < 0, \tag{13}$$

which gives rise to the fact that $\lambda(0)$ is larger than $-\tilde{\Lambda}$, otherwise, no horizons exist.

3 Deformations of the first law of mechanics

SBHs, e.g., the Schwarzschild BH and Reissner–Nördstrom (RN) BH, can be regarded as thermodynamic systems because their mechanic laws have a resemblance to the thermodynamic ones. Nevertheless, this resemblance is broken in RBHs. If RBHs are of the shape function of Eq. (1), we make differentiation on the two sides of $r_+ = 2M\sigma(r_+, M, \alpha_i)$ and obtain

$$dr_+ = \frac{2 \left(M \sum_i \partial_{\alpha_i} \sigma d\alpha_i + M \partial_M \sigma dM + \sigma dM \right)}{1 - 2M \partial_{r_+} \sigma}. \tag{14}$$

To construct the 1LM, by substituting Eq. (14) into $\kappa dA/(8\pi)$, where $A = 4\pi r_+^2$ and κ is surface gravity at r_+, we give

$$\frac{\kappa}{8\pi} dA = (1 - \tau) dM + \sum_i \beta_i d\alpha_i + \Delta(\alpha_i), \tag{15}$$

where β_i is the conjugate of α_i, τ defined by

$$\tau \equiv 1 - \frac{r_+}{2M} - M \frac{\partial \sigma(r_+, M, \alpha_i)}{\partial M} \tag{16}$$

means the deformation associated with mass M, and $\Delta(\alpha_i)$ stands for the deformations associated with α_i's. To restore

the resemblance between the mechanic and thermodynamic laws for RBHs, we have to remove reasonably those deformations just mentioned. In the following, we analyze the difficulties that we shall encounter and propose a possible way to overcome them.

3.1 Deformations associated with M and α_i

If one simply applies the traditional replacement for Eq. (15),

$$\kappa \to 2\pi T, \quad A \to 4S, \quad M \to E, \tag{17}$$

one will obtain the formula,

$$T\mathrm{d}S = (1 - \tau)\mathrm{d}E + \sum_i \beta_i \mathrm{d}\alpha_i + \Delta(\alpha_i), \tag{18}$$

which cannot be regarded as the 1TL of an isolated system because of the deformations. In other words, either the RBH is not an isolated system or the correspondence between mechanic and thermodynamic variables Eq. (17) is not appropriate.

The usual attempt is to let the term $\tau \mathrm{d}M$ in Eq. (15) be absorbed into the entropy of RBHs in order to restore the resemblance between the mechanic and thermodynamic laws for RBHs. To this end, we define the conditional entropy describing the entropy of RBHs,

$$
\begin{aligned}
S_c &\equiv \int_{r_{\text{ext}}}^{r_+} \frac{\mathrm{d}M}{T} \\
&= \int_{r_{\text{ext}}}^{r_+} \frac{\mathrm{d}A}{4} + \int_{r_{\text{ext}}}^{r_+} \frac{\tau}{T} \left(\frac{\mathrm{d}M}{\mathrm{d}\tilde{r}_+} \right)_{\alpha_i} \mathrm{d}\tilde{r}_+,
\end{aligned}
\tag{19}
$$

where T is Hawking temperature and r_{ext} denotes the horizon radius of extreme RBHs. In Eq. (19), the first term is just the BHE, while the second one represents the deformation or deviation from the BHE. We can prove that $\tau > 0$ when $0 < \sigma \le 1$, which implies that the deformation in Eq. (19) is positive. Let us analyze two aspects.

- If σ does not contain M, our statement is obviously true due to Eq. (11), that is, $\tau > 0$ when $0 < \sigma \le 1$.
- If σ depends on M explicitly, we make a dimensionless rescaling by M for all variables, such as r_+ being rescaled to $x_+ \equiv r_+/(2M)$, thus the shape function becomes

$$f = 1 - \frac{\sigma(x, \tilde{\alpha}_i)}{x}, \tag{20}$$

where $\tilde{\alpha}_i$ is the dimensionless counterpart of α_i which is rescaled by a power function of M. Then repeating the procedure that was described at the beginning of this

section, we obtain

$$\tau = 1 - x_+ \left[1 - \frac{\partial \sigma(x_+, \tilde{\alpha}_i)}{\partial x_+} \right]. \tag{21}$$

Since the slope of tangent line of function $\sigma(x, \tilde{\alpha}_i)$ at the outer horizon x_+ is not greater than unit when $0 < \sigma \le 1$ and $x_+ < 1$, we verify the above statement.

If the BHE were replaced by the conditional entropy, we would remove the deformation associated with M but we would no longer maintain the linear relation between S_c and the horizon area as a price. Furthermore, the deformation with respect to α_i remains. As a matter of fact, due to the lack of resemblance between RBHs and the traditional thermodynamic systems, the introduction of S_c does not work well in removing the deformations associated with M and α_i. More importantly, S_c is not an independent variable because its definition is subject to the 1TL.

3.2 Proposal for removing deformations

Starting with the classical action and observing the partition function at the zero-loop approximation, we find that the Einstein–Hilbert action of RBHs with the shape function Eq. (1) contributes one part of entropy and the Gibbons–Hawking–York surface term provides the other part of contributions to the entropy of RBHs, and that the combination of the two contributions recovers the linear relation $S \propto A$. Let us give the derivation. The Einstein–Hilbert action takes the form,

$$I_{\text{EH}} = \frac{2M - r_+ \left[1 + 2M\sigma'(r_+) \right]}{4T}, \tag{22}$$

where σ converges to unit as $r \to \infty$ and the prime denotes the derivative with respect to the radial coordinate, and the Gibbons–Hawking–York surface term is

$$I_{\text{GHY}} = -\frac{M}{2T}, \tag{23}$$

where the flat spacetime is selected as background reference [24]. As a result, the total action reads

$$I_{\text{tot}} = I_{\text{EH}} + I_{\text{GHY}} = -\pi r_+^2 + \beta \frac{r_+}{2}, \quad \beta \equiv \frac{1}{T}, \tag{24}$$

with which the entropy can be computed by

$$S = \beta \frac{\partial I_{\text{tot}}}{\partial \beta} - I_{\text{tot}} = \pi r_+^2. \tag{25}$$

That is to say, the semiclassical approach prefers that the entropy of RBHs is $A/4$ rather than S_c. The same result can

also be obtained by the Wald method [24]. The situation of RBHs is similar to the case of dielectric in an external electric field [25] where the variation of internal energy does not alter the entropy of a dielectric system. That is to say, the deformation associated with mass M does not depend on the thermodynamic state of RBHs, thus it should not affect the entropy of RBHs.

In conclusion, our proposal is to maintain the linear relation, $S \propto A$, and simultaneously to remove the deformations mentioned in the above subsection, that is, to establish the resemblance between the mechanic and thermodynamic laws for RBHs in terms of the pressure of Gliner vacuum [13,14] which is dealt with as the pressure of RBHs. Thanks to the pressure of the Gliner vacuum, the deformation associated with M will be absorbed into the enthalpy or internal energy and the deformation associated with α_i will constitute a naturally $V-P$ term. In this way, all deformations disappear and the other terms are well defined in Eq. (15), which gives rise to a normal 1LT in the formulation.

We shall see that our proposal works well for some known RBH models as examples. It is necessary for us to choose models because we need the formulations of shape functions, however, we shall see that our proposal is valid for all RBHs with a single shape function depicted by Eq. (1).

4 Pressure and reconstruction of the first law of mechanics

It is widely known that the negative cosmological constant can be interpreted [26] as thermal pressure of BHs, while the positive cosmological constant has a problem of thermal equilibrium [27], which is caused by the existence of a cosmological horizon. No matter whether the cosmological constant is negative or positive, the essence of this idea is to regard the pressure of *vacuum* as the pressure of BHs. Nevertheless, a RBH has its own vacuum even though it is not involved in an additional AdS (or dS) term in metric. This seed goes back to Sakharov and Gliner [13,28], who re-explained the vacuum as spacetime filled with vacuum. Such a vacuum is currently dubbed the Gliner vacuum. It will be our key to reconstruct the 1LM for RBHs. Next, we introduce the pressure of the Gliner vacuum in the following models.

4.1 Model I

The first model is generated [16] by an anisotropic vacuum and its shape function reads

$$f(r) = 1 - \frac{2M}{r}\sigma(r, M, \Lambda),$$

$$\sigma(r, M, \Lambda) = 1 - \exp\left(-\frac{\Lambda}{6M}r^3\right). \tag{26}$$

Here Λ is a cosmological constant. The asymptotic flatness at infinity requires that Λ be positive. Moreover, $f(r)$ approaches to de Sitter space, i.e., $f(r) \sim 1 - \Lambda r^2/3 + O(r^5)$, as $r \to 0$. When r becomes large, the term $\exp[-\Lambda r^3/(6M)]$ goes to zero, thus $f(r)$ is asymptotic to the Schwarzschild black hole. Since $\sigma \leq 1$ for $r \in \mathbb{R}^+$, the outer horizon r_+ is restricted by $r_{\text{ext}} \leq r_+ \leq 2M$ when $\Lambda > 9.28/(4M^2)$. The extreme radius is $r_{\text{ext}} \approx 0.85 \times 2M$. For $\Lambda < 0$, there is no real solution for $f(r_+) = 0$. Therefore, this model has no cosmological horizons, and thus does not suffer from the problem of thermal equilibrium.

The Smarr formula was obtained from the total mass represented by the Komar integral which can be separated [29] into two parts. The first part is a surface integral over the horizon, and gives $\kappa A/(4\pi)$; while the second one, including a deviation from the first law, is a volume integral with one boundary at spatial infinity and the other at event horizon [9,30]. Combining the two parts, one obtains the Smarr formula,

$$M = \frac{\kappa A}{4\pi} + \epsilon M + \frac{1}{2}\epsilon \Lambda r_+^3, \tag{27}$$

where $\epsilon \equiv \exp\left[-\Lambda r_+^3/(6M)\right] > 0$. This Smarr formula suggests an extended phase space. However, if one applied $P = -\Lambda/(8\pi)$ as thermal pressure [26], the first law of mechanics would be deformed, i.e.,

$$\frac{\kappa}{8\pi}dA = (1 - \tau)dM - \epsilon V dP, \tag{28}$$

where $\tau \equiv \epsilon [1 - \ln(\epsilon)]$, and $V \equiv 4\pi r_+^3/3$ is the thermal volume inside the horizon. Further, we find that the 1-form $\kappa dA/(8\pi) + V dP$ does not satisfy the integrable condition, i.e., it cannot be written as a total derivative of any functions.

By introducing the radial pressure of the Gliner vacuum at the outer horizon,

$$P_+ = \left.\frac{G^r{}_r}{8\pi}\right|_{r=r_+} = -\epsilon\frac{\Lambda}{8\pi}, \tag{29}$$

which is negative, as the pressure for Model I, where $G^r{}_r$ is r–r component of Einstein tensor $G_{\mu\nu}$, we note that the last term of Eq. (27) is nothing else but $-3V P_+$. As a result, following the way in Ref. [20] and introducing *enthalpy*,

$$H = (1 - \tau)M, \tag{30}$$

we can reconstruct the 1LM from Eq. (28),

$$\frac{\kappa}{8\pi}dA = dH - V dP_+, \tag{31}$$

and write the corresponding 1LT,

$$dH = T dS + V dP_+, \tag{32}$$

with

$$T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4}. \tag{33}$$

The deformations of Eq. (28) are completely removed in Eq. (31).

Alternatively, Eq. (31) can be rewritten as

$$\frac{\kappa}{8\pi} dA = dU + P_+ dV, \tag{34}$$

with *total internal energy*

$$U = (1 - \epsilon)M = H - V P_+, \tag{35}$$

which suggests that the deviation from M in Eq. (27) should be absorbed into internal energy. The other compelling reason to think of $(1 - \epsilon)M$ as the energy is that it can be calculated by the integration of energy density ρ over the whole space inside the RBH, i.e.,

$$(1 - \epsilon)M = \int_0^{r_+} \int_0^{2\pi} \int_0^{\pi} \sqrt{-g} \rho \, dr d\phi d\theta = \frac{r_+}{2}, \tag{36}$$

where ρ is defined by $\rho = -G_0^0/(8\pi) = \Lambda \exp\left[-\Lambda r^3/(6M)\right]/(8\pi)$ according to Einstein's equation, $G_\nu^\mu = 8\pi T_\nu^\mu$. In other words, M is the total energy filled in the whole space of the RBH, while ϵM is the energy outside the RBH.

Thus, the internal energy of the RBH should be the energy enclosed in the event horizon, and the corresponding 1LT can be cast as follows:

$$dU = T dS - P_+ dV, \tag{37}$$

where every term is well-defined. As we expected, the resemblance between the mechanic and thermodynamic laws for this RBH is restored, see Eqs. (31) and (32) or Eqs. (34) and (37), and simultaneously the entropy is just the Bekenstein–Hawking entropy obtained by path-integral and Wald's method [24].

4.2 Model II

Now let us turn to the second model, Model I immersed in AdS spacetime, which can be established if the other cosmological constant $\widetilde{\Lambda}$ is introduced into Model I,

$$f(r) = 1 - \frac{2M}{r}\sigma(r, M, \Lambda) - \frac{\widetilde{\Lambda}}{3}r^2,$$
$$\sigma(r, M, \Lambda) = 1 - \exp\left(-\frac{\Lambda}{6M}r^3\right). \tag{38}$$

There is no doubt that such a metric is a solution of Einstein's equations with a cosmological constant term [16]. The homogeneity of the universe remains unchanged if Λ is treated as a local character, i.e., Λ dominates only the inside (and around a certain range) of this RBH. Meanwhile, to bypass the problem of thermal equilibrium [31], we demand $\widetilde{\Lambda} < 0$. The Smarr formula can be calculated in the same way as that mentioned above,

$$M = \frac{\kappa A}{4\pi} + \epsilon M + \frac{1}{2}\epsilon\Lambda r_+^3 + \frac{1}{3}\widetilde{\Lambda}r_+^3, \tag{39}$$

where ϵ has the same form as that of Model I, but the energy density now is calculated by $\rho = -(G_0^0 + \widetilde{\Lambda})/(8\pi)$ because of $G_\nu^\mu + \widetilde{\Lambda}g_\nu^\mu = 8\pi T_\nu^\mu$. We note that the deformations still exit even if the AdS constant $\widetilde{\Lambda}$ is regarded as pressure. If we regard the radial pressure of the Gliner vacuum as the pressure, we can obtain the same 1LM as Eq. (31), where the *enthalpy* is defined in Eq. (30) and the Gliner pressure takes the form,

$$P_+ = -\frac{1}{8\pi}\left(\epsilon\Lambda + \widetilde{\Lambda}\right). \tag{40}$$

The competition of two cosmological constants appears in the pressure. As we have noted in Sect. 2, $\epsilon\Lambda$ should be larger than $-\widetilde{\Lambda}$, otherwise there will be no horizons. As a result, the Gliner pressure in Model II is negative, and this pressure is different from the one simply introduced from the AdS cosmological constants.

4.3 Other models

The models in the above two subsections give a heuristic evidence that a well-defined 1LM can be reconstructed without any deformations and the traditional area law, $S = A/4$, still holds when we introduce the radial pressure of Gliner vacuum. It will be seen in this subsection that some other models, including those with Gliner vacuum or vacuum-like mediums, also support this statement.

(i) Model in Ref. [10]. This model is also called noncommutative geometry inspired black hole. The shape function takes the form,

$$f(r) = 1 - \frac{4M}{\sqrt{\pi} r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right). \tag{41}$$

The Smarr formula is

$$M = \frac{A\kappa}{4\pi} + \epsilon M + \frac{Mr_+^3 e^{-\frac{r_+^2}{4\theta}}}{2\sqrt{\pi}\theta^{3/2}}, \tag{42}$$

where in the second term on the right-hand side ϵ equals

$$\epsilon = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \frac{r_+^2}{4\theta}\right), \tag{43}$$

which coincides with the result given by Eq. (36); the third term corresponds to $-3VP_+$, and the radial pressure of the Gliner vacuum at the outer horizon reads

$$P_+ = -\frac{Me^{-\frac{r_+^2}{4\theta}}}{8\pi^{3/2}\theta^{3/2}}. \tag{44}$$

We can also give the 1LM Eq. (31) with the enthalpy and internal energy,

$$H = \frac{r_+}{2} + \frac{r_+^4}{12\theta r_+ - 12\sqrt{\pi}\,\theta^{3/2} \exp\left(\frac{r_+^2}{4\theta}\right) \mathrm{erf}\left(\frac{r_+}{2\sqrt{\theta}}\right)}, \tag{45a}$$

$$U = M\left[\mathrm{erf}\left(\frac{r_+}{2\sqrt{\theta}}\right) - \frac{r_+ e^{-\frac{r_+^2}{4\theta}}}{\sqrt{\pi}\sqrt{\theta}}\right] = \frac{r_+}{2}. \tag{45b}$$

(ii) Model in Ref. [32]. The shape function is

$$f(r) = 1 - \frac{2Mr}{r^2 + 6M/(r\Lambda)}, \tag{46}$$

where Λ is supposed to be positive. The Smarr formula has the form,

$$M = \frac{\kappa A}{4\pi} + \frac{6M^2}{\Lambda r_+^3 + 6M} + \frac{18\Lambda M^2 r_+^3}{\left(\Lambda r_+^3 + 6M\right)^2}, \tag{47}$$

where the second term on the right-hand side stands for ϵM with $\epsilon = 6M/(6M + \Lambda r_+^3)$, which coincides with the result given by Eq. (36); the third term corresponds to $-3VP_+$, and the radial pressure of the Gliner vacuum at the outer horizon equals

$$P_+ = -\frac{9\Lambda M^2}{2\pi\left(6M + \Lambda r_+^3\right)^2}. \tag{48}$$

The same procedure leads to the 1LM Eq. (31) with the enthalpy and internal energy,

$$H = \frac{r_+^2}{4M}, \quad U = (1 - \epsilon)M = \frac{r_+}{2}. \tag{49}$$

(iii) Model in Ref. [33]. This model is usually called Bardeen's black hole. The shape function is

$$f(r) = 1 - \frac{2Mr^2}{\left(b^2 + r^2\right)^{3/2}}, \tag{50}$$

where b is a magnetic charge. The Smarr formula then reads

$$M = \frac{A\kappa}{4\pi} + M\left[1 - \frac{r_+^3}{\left(b^2 + r_+^2\right)^{3/2}}\right] + \frac{3b^2 M r_+^3}{\left(b^2 + r_+^2\right)^{5/2}}, \tag{51}$$

where we have used $\epsilon = 1 - r_+^3/\left(b^2 + r_+^2\right)^{3/2}$. In this model, the pressure of the Gliner vacuum takes the form,

$$P_+ = -\frac{3b^2 M}{4\pi\left(b^2 + r_+^2\right)^{5/2}}. \tag{52}$$

In the 1LM the enthalpy and the internal energy equal

$$H = \frac{r_+^3}{2\left(b^2 + r_+^2\right)}, \quad U = \frac{Mr_+^3}{\left(b^2 + r_+^2\right)^{3/2}} = \frac{r_+}{2}. \tag{53}$$

(iv) Model in Ref. [34]. This model is generated by a vacuum-like medium. The shape function takes the form,

$$f(r) = 1 - \frac{2M}{r}\exp\left(-\frac{q^2}{2Mr}\right), \tag{54}$$

where q stands for electric charge. The Smarr formula is

$$M = \frac{A\kappa}{4\pi} + \left(1 - e^{-\frac{q^2}{2Mr_+}}\right)M + \frac{q^2}{2r_+}e^{-\frac{q^2}{2Mr_+}}, \tag{55}$$

where ϵ and P_+ read

$$\epsilon = 1 - e^{-\frac{q^2}{2Mr_+}}, \quad P_+ = -\frac{q^2}{8\pi r_+^4}e^{-\frac{q^2}{2Mr_+}}. \tag{56}$$

The enthalpy and internal energy are

$$H = \frac{r_+}{2} - \frac{r_+}{6}\ln\left(\frac{2M}{r_+}\right), \quad U = Me^{-\frac{q^2}{2Mr_+}} = \frac{r_+}{2}. \tag{57}$$

For the above four RBHs, we emphasize that their internal energies calculated by Eq. (36) equal $r_+/2$. This is not a coincidence. In fact, Eq. (36) gives exactly $r_+/2$ for all spherically symmetric RBHs with the shape function Eq. (1).

An interesting property noted in Ref. [20] is that the Φ-Q term in Reissner–Nördstrom (RN) BHs can be absorbed into V-P_+ term, such that $\kappa dA/(8\pi) = dM - \Phi dQ$ becomes Eq. (31). This result implies that the Gliner vacuum provides a unified treatment for both RBHs and SBHs in the establishment of 1LM. In this way, the enthalpy and internal energy of RN BHs take the form,

$$H = \frac{r_+}{2} - \frac{Q^2}{6r_+}, \quad U = \frac{r_+}{2}. \tag{58}$$

When $Q \to 0$, the enthalpy of RN BHs reduces to that of Schwarzschild BHs, $H = r_+/2 = M$, which coincides with its internal energy. For RN-AdS BHs, one has $H = U - Q^2/(6r_+) - \Lambda r_+^3/6$ with $U = r_+/2$. It is worth emphasizing that the internal energies of SBHs, such as the Schwarzschild BHs, RN BHs, and RN-AdS BHs we just mentioned, are calculated from the 1TL but not from Eq. (36). The reason is that the energy density of SBHs is singular or vanishing at $r = 0$, which leads to divergence of Eq. (36) or makes the integration trivial. In order to make Eq. (36) suitable for SBHs, we introduce a cutoff of the radial coordinate for the SBHs whose energy densities are singular, such that the integration is regularized. Taking the RN BHs as an example, the regularized integration of internal energy is

$$ U = \int_{r_0}^{r_+} \int_0^{2\pi} \int_0^\pi \sqrt{-g}\rho \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta, \quad \rho = \frac{Q^2}{8\pi r^4}, \qquad (59) $$

where r_0 is the cutoff. Since $U = r_+/2$, we can fix $r_0 = Q^2/(2M)$. Meanwhile, we can prove $r_0 < r_-$, i.e.,

$$ Q^2/(2M) < M - \sqrt{M^2 - Q^2}, \qquad (60) $$

which implies that the cutoff is located inside the inner horizon. As to the Schwarzschild BHs, one can use the interior metric of Schwarzschild BHs [35] and suppose that the radius of the matter surface equals $2M$, thus one can derive the energy density,

$$ \rho(r) = \frac{3}{32\pi M^2} \left[\mathcal{H}\left(\frac{r}{2M}\right) - \mathcal{H}\left(\frac{r}{2M} - 1\right) \right], \qquad (61) $$

where $\mathcal{H}(\cdot)$ is the Heaviside function. The energy in the volume $V = 4\pi r_+^2/3$ takes the value $U = r_+/2 = M$.

Now we summarize the features of the new pressure we have proposed in Models I and II and also applied to the other four models. At first, the Gliner pressure is not universal, i.e., it has different values for different models, which is similar to the Hawking temperature that is horizon dependent. Next, there are no conceptional contradictions between the pressure and the vacuum or vacuum-like matter that generates RBHs, i.e., it is widely known that the RBHs are generated by the vacuum or vacuum-like matter with negative pressure rather than the AdS constant with positive pressure. At last, we find the following relation,

$$ |P_+| \propto R(0), \qquad (62) $$

where $R(0)$ can be explained as the average energy of a vacuum. This gives rise to our conclusion that the new pressure we proposed, i.e., the radial pressure of the Gliner vacuum originates from the average energy of the Gliner vacuum, which is consistent with the case for SBHs where the pressure comes simply from the average energy of (AdS) vacuum. In

summary, by introducing the Gliner pressure as the pressure of RBHs we avoid such a contradiction that a RBH generated by the matter (vacuum) with negative pressure has positive pressure because this contradiction is physically unacceptable.

5 Ruppeiner geometry of regular black holes

Having the reconstructed 1LT, see Eq. (32), at hand, we apply the Gibbs energy, $G = H - TS$, as the starting point and calculate the line element by following Ref. [36],

$$ \mathrm{d}l^2 = \frac{1}{k_B} \left[\frac{C_{P_+}}{T^2} \mathrm{d}T^2 + \frac{1}{T} \left(\frac{\partial V}{\partial P_+}\right)_T \mathrm{d}P_+^2 \right], \qquad (63) $$

where k_B is Boltzmann constant and the capacity at constant pressure is defined as usual,

$$ C_{P_+} = T \left(\frac{\partial S}{\partial T}\right)_{P_+}. \qquad (64) $$

Considering the metric with the shape function, Eq. (1), we find

$$ P_+ = -\frac{M\sigma'(r_+)}{4\pi r_+^2}, \quad T = \frac{1}{4\pi r_+} - \frac{M\sigma'(r_+)}{2\pi r_+}, \qquad (65) $$

and then obtain the equation of state by combining the above two formulas,

$$ P_+ = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}, \qquad (66) $$

whose dependence on parameters, such as mass, charge, etc., is hidden in the outer horizon. Based on this equation of state, we derive the thermodynamic curvature for a general spherically symmetric RBH with Eq. (1) as its shape function,

$$ \frac{\mathcal{R}}{2\pi T^2} = y\left(\sqrt{1-y} - 3\right) - 4\left(\sqrt{1-y} - 1\right), \qquad (67) $$

where y defined by $y \equiv 2P_+/(\pi T^2)$ is dimensionless. Since $P_+ < 0$, we deduce $\mathcal{R}/T^2 < 0$. That is to say, the interaction of all spherical symmetric RBHs with a single shape function is attractive from the microscopic perspective. Nevertheless, it is known that RBHs are generated by matters which are of repulsive interaction around the center, thus a natural problem is how a repulsive matter forms an attractive black hole. Let us give a quantitative analysis in terms of the strong energy condition (SEC) of RBHs.

6 Repulsive and attractive interactions inside and outside regular black holes

Instead of analyzing the energy–momentum tensor of matter T^μ_ν, we concentrate on the diagonalized Einstein tensor,

$$G^\mu_\nu = \text{diag}\left\{-\frac{2M\sigma'}{r^2}, -\frac{2M\sigma'}{r^2}, -\frac{M\sigma''}{r}, -\frac{M\sigma''}{r}\right\}, \quad (68)$$

because G^μ_ν and T^μ_ν are equivalent due to the Einstein equation $G^\mu_\nu = 8\pi T^\mu_\nu$. The energy density ρ and pressures ρ_r and $\rho_\perp$ can then be obtained,

$$\rho = \frac{M\sigma'}{4\pi r^2}, \quad p_r = -\frac{M\sigma'}{4\pi r^2}, \quad p_\perp = -\frac{M\sigma''}{8\pi r}. \quad (69)$$

Moreover, in order to discuss the SEC, we introduce parameter γ,

$$\gamma \equiv \rho + p_r + 2p_\perp = -\frac{M\sigma''}{4\pi r}, \quad (70)$$

whose sign indicates the attractive ($\gamma > 0$) or repulsive ($\gamma < 0$) interaction. This can be understood clearly from Raychaudhuri's equation [37]. Moreover, if the expansion, rotation, and shear terms in Raychaudhuri's equation are neglected when compared with the variation of expansion, one has

$$\frac{d\xi}{d\tau} = -4\pi\gamma, \quad (71)$$

where ξ denotes the expansion of geodesics and τ affine parameter. Based on this equation, one can determine that the gravity is attractive ($\gamma > 0$) or repulsive ($\gamma < 0$).

Further, since $\sigma \sim r^n$ with $n \geq 3$ as $r \to 0$, see Appendix B for the details, we can deduce that $\sigma'' > 0$ and $\gamma < 0$ around $r = 0$, namely, the matters generating RBHs are of repulsive interaction, which is also known as the violation of SEC [14]. Meanwhile, there is a special point arising from $\sigma''(r_*) = 0$, where r_* can be regarded as the point of phase transitions. This point is special because the matters generating RBHs are of repulsive interaction in the range of $0 < r < r_*$, while they are of attractive interaction in the range of $r > r_*$, i.e., r_* separates the two phases of RBHs.

Now let us estimate the position of r_*. At first, we note that Eq. (70) can be regarded as Newton's equation of a one-dimensional particle with mass M, i.e.,

$$-\Phi' = M\sigma'', \quad \Phi' \equiv 4\pi r\gamma, \quad (72)$$

where Φ is "potential" and can be solved analytically,

$$\Phi = -M\sigma' + \Phi_0. \quad (73)$$

Here Φ_0 is an integration constant and $\Phi - \Phi_0 < 0$ in $r \in (0, \infty)$ because σ is a monotone increasing function of r. Then, considering the asymptotic behaviors of σ at $r = 0$ and $r \to \infty$, we obtain

$$\lim_{r \to 0} \sigma' = 0, \quad \lim_{r \to \infty} \sigma' = 0, \quad (74)$$

which implies that Φ is a potential well with one global minimum at $r = r_*$. The reason is that σ is a sigmoidal function and thus its first derivative is bell shaped. In other words, we have $\sigma'(r_*) > \sigma'(r_{\text{ext}}) = 1/(2M)$, where $\sigma'(r_{\text{ext}}) = 1/(2M)$ comes from the combination of $T(r_{\text{ext}}) = 0$ and $f(r_{\text{ext}}) = 0$, but we still cannot determine whether $r_* > r_{\text{ext}}$ or $r_* < r_{\text{ext}}$, where r_{ext} denotes the radius of extreme RBHs. At last, we know that the temperature of a RBH is nonnegative and vanishes at $r = r_{\text{ext}}$, from which we can deduce $T'(r_{\text{ext}}) > 0$, i.e.,

$$\sigma''(r_{\text{ext}}) < \frac{2\sigma'(r_{\text{ext}})}{r_{\text{ext}}} - \frac{2\sigma(r_{\text{ext}})}{r_{\text{ext}}^2}, \quad (75)$$

then applying $\sigma'(r_{\text{ext}}) = 1/(2M)$ and $\sigma(r_{\text{ext}}) = r_{\text{ext}}/(2M)$ to replace $\sigma'(r_{\text{ext}})$ and $\sigma(r_{\text{ext}})$, we arrive at

$$\sigma''(r_{\text{ext}}) < 0, \quad (76)$$

which helps us rule out $r_* > r_{\text{ext}}$.

In summary, we have $r_* < r_{\text{ext}}$, i.e., the point of phase transitions should be located inside the extreme horizon, which gives us an explanation of how a repulsive matter forms an attractive black hole. The whole physical picture should be like this: Along the radial coordinate, the matter first shows the repulsive interaction around $r = 0$; when r passes through the phase transition point r_*, the repulsive interaction becomes the attractive one. Since thermodynamics describes a BH as a quantum system from the outside, the interaction of RBHs should reflect the attractive nature. Here we have explained how a repulsive matter forms a black hole with attractive interaction. In addition, we note from Eq. (99),

$$\gamma \propto -\left(R - \sqrt{2E}\right), \quad (77)$$

which implies that the sign of γ depends on the competition between two scalar curvatures R and E outside RBHs. When $R < \sqrt{2E}$, the interaction is attractive, while $R > \sqrt{2E}$ means a repulsive interaction. The balance $R = \sqrt{2E}$ corresponds to the phase transition point r_*.

7 Comparison with Reissner–Nordström black holes

To give a more intuitive illustration of the interaction structure of RBHs, we compare RBHs with RN BH by considering

the full version of the Raychaudhuri equation because a RN BH also has the attractive interaction outside its horizon and repulsive one in the vicinity of $r = 0$ [38,39].

According to our formula Eq. (67) for the thermodynamic curvature of a general RBH with the spherical symmetry, a RN BH has a negative thermodynamic curvature as well, i.e. an attractive interaction outside its horizon, but its local structure of interaction in the vicinity of $r = 0$ is different from that of RBHs.

To show the difference, we start with the *ingoing* radial geodesics on a general spherically symmetric metric with single shape function Eq. (1), the tangent vector field of those geodesics is

$$u_\alpha = (-1, u_r, 0, 0), \tag{78}$$

where

$$u_r = -\frac{1}{f}\sqrt{\frac{2M}{r}\sigma(r)}. \tag{79}$$

The main element in the Raychaudhuri equation is a B-tensor, which is defined as the gradient of the tangent vector, $B_{\alpha\beta} := \nabla_\beta u_\alpha$, thus the expansion scalar ξ can be expressed by the trace of $B_{\alpha\beta}$, $\xi = \mathrm{Tr}\, B_{\alpha\beta}$. In our case, the trace reads

$$\xi = -\sqrt{\frac{M}{2r^3\sigma}}\left(r\sigma' + 3\sigma\right), \tag{80}$$

and it is negative definite if σ is a monotonically non-decreasing function of r. The general Raychaudhuri equation in our notation is

$$\frac{d\xi}{d\tau} = \Theta(r), \qquad \Theta(r) := -B_{\alpha\beta}B^{\alpha\beta} - 4\pi\gamma, \tag{81}$$

where

$$B_{\alpha\beta}B^{\alpha\beta} = \frac{M}{2r^3\sigma}\left[r^2(\sigma')^2 - 2r\sigma\sigma' + 9\sigma^2\right]. \tag{82}$$

For a RN BN, we write its $\sigma(r)$ from Eq. (1),

$$\sigma_{\mathrm{RN}}(r) = 1 - \frac{Q^2}{2Mr}, \tag{83}$$

and then compute the corresponding Θ,

$$\Theta_{\mathrm{RN}}(r) = -\frac{9\Delta^2 + 2\Delta Q^2 + Q^4}{4\Delta r^4}, \tag{84}$$

where $\Delta = 2Mr - Q^2$. When expanding it around $r = 0$,

$$\Theta_{\mathrm{RN}}(r) \sim \frac{2Q^2}{r^4} + O\left(\frac{1}{r^3}\right), \tag{85}$$

we can see that it is divergent as r approaches to zero. Moreover, We note that this quantity is not positive definite and there exist two phases,

$$\Theta_{\mathrm{RN}} < 0, \quad \text{when } r > Q^2/(2M), \tag{86a}$$
$$\Theta_{\mathrm{RN}} > 0, \quad \text{when } r < Q^2/(2M), \tag{86b}$$

where the critical point $r_0 = Q^2/(2M)$ is located inside the inner horizon because of Eq. (60). In other words, the variation of expansion scalar with respect to the affine parameter is negative in the first phase Eq. (86a), while it becomes positive when the geodesics cross the critical point r_0 into the second phase Eq. (86b). Finally, the SEC of RN BHs holds in the whole domain,

$$\gamma_{\mathrm{RN}} = \frac{Q^2}{4\pi r^4} > 0. \tag{87}$$

It is consistent with the result obtained from the weak-field approximation [39], i.e., the RN BH reveals a repulsive interaction in the vicinity of $r = 0$ although its SEC holds everywhere.

The situation of RBHs is different. Let us take the Bardeen BH as an example. Its Θ is strictly negative, i.e., it has only one phase compared with the RN BH,

$$\Theta_{\mathrm{B}} = -\frac{3Mr^2\left(10b^2 + 3r^2\right)}{2\left(b^2 + r^2\right)^{7/2}} < 0, \tag{88}$$

and the variation of expansion scalar with respect to the affine parameter vanishes at the center of Bardeen BHs, which is obvious from the expansion around $r = 0$,

$$\Theta_{\mathrm{B}} \sim -\frac{15Mr^2}{b^5} + O\left(r^3\right). \tag{89}$$

In fact, for all RBHs, if σ has the power expansion like Eq. (108) at $r = 0$, then one has

$$\Theta = 4a_1 Mr + \left(10a_2 - \frac{a_1^2}{2a_0}\right)Mr^2 + O\left(r^3\right), \tag{90}$$

where a_i's are abbreviate notations of the coefficients of Eq. (108). In other words, the variation of expansion scalar is vanishing at $r = 0$. Oppositely, the variation of expansion scalar for BHs with a singularity at their centers is always divergent.

For the Bardeen BH, we have

$$\gamma_{\mathrm{B}} = \frac{3Mb^2\left(3r^2 - 2b^2\right)}{4\pi\left(b^2 + r^2\right)^{7/2}}, \tag{91}$$

which means that the SEC of Bardeen BHs does not hold when $r < \sqrt{2/3}\,b$. This is different from the situation of

RN BHs, which is obvious from the comparison between Eqs. (91) and (87).

8 Summary

Starting from the idea of the Gliner vacuum, we apply the approach given in Ref. [20] to remove the deformations in the 1LM for RBHs. In addition, we provide a possible explanation for the deformation of the mass term. The new 1LM redisplays the resemblance between RBHs and traditional thermodynamic systems. In other words, all the variables in the new 1LM have their thermodynamic counterparts, in particular, the area law, $S \propto A$, is recovered. Based on the reconstructed 1LM, we give a self-consistent theory of Ruppeiner geometry, and show that all RBHs with the spherical symmetry and one single shape function should have an attractive interaction in the range of $r \geq r_{ext} > r_*$ from the microscopic perspective. This shows that our analyses of interactions inside and outside RBHs are consistent with the Ruppeiner thermodynamic geometry we established in Sect. 5. Furthermore, the new 1LM offers a universal treatment for both RBHs and SBHs. However, the local properties of the interaction structures in the vicinity of $r = 0$ between RBHs and SBHs are different. When r goes to zero, the expansion scalar of RBHs maintains unchanged due to no singularities while that of SBHs blows up. Finally, the explanation is given on how a repulsive matter forms a RBH with an attractive interaction. Our result may shed light on solving the related problems in superradiance and area spectrum for RBHs.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical paper without experimental data.]

Appendix A: The representation of the shape function via curvature invariants

We start with the curvature invariants that are expressed [34] by the shape function, Eq. (1),

$$R = \frac{2M}{r^2}\left(2\sigma' + r\sigma''\right), \tag{92}$$

$$R_2 = \frac{2M^2}{r^4}\left[4(\sigma')^2 + r^2(\sigma'')^2\right], \tag{93}$$

$$K = \frac{4M^2}{r^6}\left\{4\left[3\sigma^2 - 4r\sigma\sigma' + 2r^2(\sigma')^2\right] + 4r^2\left(\sigma - r\sigma'\right)\sigma'' + r^4(\sigma'')^2\right\}. \tag{94}$$

Although these equations are derived from a RBH, they are valid for all spherically symmetric BHs with a single shape function. We note that R, R_2, and K contain σ and its first and second derivatives with respect to r. Thus, we can solve these three algebraic equations and express σ and its derivatives in terms of the curvature invariants. By ignoring redundant roots,[4] we obtain

$$\sigma = \frac{r^3}{24M}\left(R \pm 2\sqrt{3K + R^2 - 6R_2} + 3\sqrt{4R_2 - R^2}\right),$$
$$\sigma' = \frac{r^2}{8M}\left(R + \sqrt{4R_2 - R^2}\right),$$
$$\sigma'' = \frac{r}{4M}\left(R - \sqrt{4R_2 - R^2}\right), \tag{95}$$

where the different signs in σ correspond to two regions separated by the line $r^2\sigma'' + 6\sigma = 4r\sigma'$ in the parameter space. The plus sign depicts the region $r^2\sigma'' + 6\sigma > 4r\sigma'$, while the minus one $r^2\sigma'' + 6\sigma < 4r\sigma'$ is more closer to the center. On the other side, the Riemann tensor breaks down into three parts in terms of the Ricci decomposition [21,40], $R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + S_{\mu\nu\rho\sigma} + E_{\mu\nu\rho\sigma}$, where $W_{\mu\nu\rho\sigma}$ is traceless part called the Weyl tensor, $S_{\mu\nu\rho\sigma}$ is scalar part, and $E_{\mu\nu\rho\sigma}$ is semi-traceless part. Moreover, we have the following relationships,

$$E \equiv E_{\mu\nu\rho\sigma}E^{\mu\nu\rho\sigma} = 2R_2 - \frac{R^2}{2},$$
$$W \equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = K - 2R_2 + \frac{R^2}{3}. \tag{96}$$

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[4] There are four roots originally, two of them are removed by the weak or null energy condition, $r\sigma'' \leq 2\sigma'$.

Then applying these relations to replace R_2 and K in Eq. (95), we obtain

$$\sigma = \frac{r^3}{24M} \left(R + 3\sqrt{2E} \pm 2\sqrt{3W} \right), \qquad (97)$$

$$\sigma' = \frac{r^2}{8M} \left(R + \sqrt{2E} \right), \qquad (98)$$

$$\sigma'' = \frac{r}{4M} \left(R - \sqrt{2E} \right). \qquad (99)$$

We can also express the Hawking temperature and Gliner pressure in terms of the curvature invariants,

$$T = \frac{r}{24\pi} \left(\sqrt{3W} - R \right) \Big|_{r=r_+},$$
$$P_+ = -\frac{1}{32\pi} \left(\sqrt{2E} + R \right) \Big|_{r=r_+}, \qquad (100)$$

where the case of the minus sign before $\sqrt{3W}$ in T has been ruled out due to the positivity of temperature. The zero point of T as a function of r_+ corresponds to the solution of the algebraic equation $\sqrt{3W} = R$. In other words, $\sqrt{3W} = R$ signifies the ground state of BH configurations.

Appendix B: The asymptotic behavior of σ around $r = 0$

Using Eq. (1), we compute the scalar curvature,

$$R(r) = \frac{2M \left(2\sigma' + r\sigma'' \right)}{r^2}. \qquad (101)$$

If $R(r)$ is finite around $r = 0$, its Taylor expansion has the form,

$$R(r) = \sum_{n=0}^{\infty} \frac{r^n R^{(n)}(0)}{n!}. \qquad (102)$$

By solving the ordinary differential equation, Eq. (101), and using the above Taylor expansion, we obtain a general solution for σ,

$$\tilde{\sigma} = c_1 + \frac{c_2}{r} + r^3 \sum_{n=0}^{\infty} \frac{r^n R^{(n)}(0)}{2M(n+3)(n+4)n!}, \qquad (103)$$

where c_1 and c_2 are two integration constants. Substituting Eq. (103) into the Kretschmann scalar, we can separate the scalar into two parts, where one is finite and the other divergent at $r = 0$,

$$K = K^{\text{fin}} + K^{\text{div}}, \qquad (104)$$

with

$$K^{\text{fin}} = 4 \sum_{n=0}^{\infty} \frac{r^n R^{(n)}(0)}{(n+3)(n+4)n!}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(m^2 + 7m + 6 \right) r^m R^{(m)}(0)}{(m+3)(m+4)m!}$$
$$+ 4 \sum_{m=0}^{\infty} \frac{m(2m+3)r^m R^{(m)}(0)}{(m+3)(m+4)m!}$$
$$\times \sum_{n=0}^{\infty} \frac{nr^n R^{(n)}(0)}{(n+3)(n+4)n!}$$
$$+ r^2 \left(\sum_{n=0}^{\infty} \frac{(n-1)nr^n R^{(n)}(0)}{(n+3)(n+4)n!} \right)^2, \qquad (105)$$

and

$$K^{\text{div}} = \frac{8c_2 M}{r^3} \sum_{n=0}^{\infty} \frac{n(n+1)r^n R^{(n)}(0)}{(n+3)(n+4)n!}$$
$$+ \frac{8c_1 M}{r^4} \sum_{n=0}^{\infty} \frac{n(3n+5)r^n R^{(n)}(0)}{(n+3)(n+4)n!}$$
$$+ \frac{192 c_1 c_2 M^2}{r^7} + \frac{48 c_2^2 M^2}{r^6} + \frac{224 c_1^2 M^2}{r^8}. \qquad (106)$$

A regular black hole implies $K^{\text{div}} = 0$, i.e., $c_1 = 0 = c_2$. As a result, we derive the asymptotic behavior of the Kretschmann scalar,

$$K = K^{\text{fin}} \sim \frac{[R(0)]^2}{6} + \frac{R(0)R'(0)}{3}r + O(r^2), \qquad (107)$$

and the asymptotic behavior of σ around $r = 0$,

$$\sigma = r^3 \sum_{n=0}^{\infty} \frac{r^n R^{(n)}(0)}{2M(n+3)(n+4)n!} \sim \frac{r^3 R(0)}{24M}$$
$$+ \frac{r^4 R'(0)}{40M} + O(r^5). \qquad (108)$$

References

1. G. Ruppeiner, Riemannian geometry in thermodynamic fluctuation theory. Rev. Mod. Phys. **67**, 605–659 (1995). https://doi.org/10.1103/RevModPhys.67.605. [Erratum: Rev. Mod. Phys. **68**, 313–313 (1996)]
2. S.-W. Wei, Y.-X. Liu, R.B. Mann, Repulsive interactions and universal properties of charged anti-de Sitter black hole microstructures. Phys. Rev. Lett. **123**(7), 071103 (2019). https://doi.org/10.1103/PhysRevLett.123.071103. arXiv:1906.10840
3. S.W. Hawking, Particle creation by black holes. Commun. Math. Phys. **43**(3), 199–220 (1975). https://doi.org/10.1007/BF02345020. [Erratum: Commun. Math. Phys. **46**, 206 (1976)]
4. J.D. Bekenstein, Black holes and entropy. Phys. Rev. D **7**, 2333–2346 (1973). https://doi.org/10.1103/PhysRevD.7.2333

5. S.W. Hawking, Gravitational radiation from colliding black holes. Phys. Rev. Lett. **26**, 1344–1346 (1971). https://doi.org/10.1103/PhysRevLett.26.1344

6. G.W. Gibbons, S.W. Hawking, Action integrals and partition functions in quantum gravity. Phys. Rev. D **15**, 2752–2756 (1977). https://doi.org/10.1103/PhysRevD.15.2752

7. R.M. Wald, Black hole entropy is the Noether charge. Phys. Rev. D **48**, R3427–R3431 (1993). https://doi.org/10.1103/PhysRevD.48.R3427. arXiv:gr-qc/9307038

8. R. Brustein, D. Gorbonos, M. Hadad, Wald's entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling. Phys. Rev. D **79**, 044025 (2009). https://doi.org/10.1103/PhysRevD.79.044025. arXiv:0712.3206

9. Y. Zhang, S. Gao, First law and Smarr formula of black hole mechanics in nonlinear gauge theories. Class. Quantum Gravity **35**(14), 145007 (2016). https://doi.org/10.1088/1361-6382/aac9d4. arXiv:1610.01237

10. P. Nicolini, A. Smailagic, E. Spallucci, Noncommutative geometry inspired Schwarzschild black hole. Phys. Lett. B **632**(4), 547–551 (2006). https://doi.org/10.1016/j.physletb.2005.11.004. arXiv:gr-qc/0510112

11. A. Dabholkar, S. Nampuri, Quantum black holes. Lect. Notes Phys. **851**, 165–232 (2012). https://doi.org/10.1007/978-3-642-25947-0_5. arXiv:1208.4814

12. R. Banerjee, B.R. Majhi, S. Samanta, Noncommutative black hole thermodynamics. Phys. Rev. D **77**, 124035 (2008). https://doi.org/10.1103/PhysRevD.77.124035. arXiv:0801.3583

13. E.B. Gliner, Algebraic properties of the energy–momentum tensor and vacuum-like states of matter. Sov. Phys. JETP **22**, 378 (1966)

14. S. Ansoldi, Spherical black holes with regular center: a review of existing models including a recent realization with Gaussian sources. In *Conference on Black Holes and Naked Singularities* (2008). arXiv:0802.0330

15. A.S. Silbergleit, A.D. Chernin, *Why Does the Universe Expand? (A Tribute to E.B. Gliner)* (Springer International Publishing, Cham, 2017), pp. 59–70. https://doi.org/10.1007/978-3-319-57538-4_6

16. I. Dymnikova, Vacuum nonsingular black hole. Gen. Relativ. Gravit. **24**, 235–242 (1992). https://doi.org/10.1007/BF00760226

17. E. Elizalde, S.R. Hildebrandt, The family of regular interiors for nonrotating black holes with T0(0) = T1(1). Phys. Rev. D **65**, 124024 (2002). https://doi.org/10.1103/PhysRevD.65.124024. arXiv:gr-qc/0202102

18. A.Z. Petrov, *New Methods in General Theory of Relativity* (URSS, Moscow, 2019). (**in Russian**)

19. M.-S. Ma, R. Zhao, Corrected form of the first law of thermodynamics for regular black holes. Class. Quantum Gravity **31**, 245014 (2014). https://doi.org/10.1088/0264-9381/31/24/245014. arXiv:1411.0833

20. M. Azreg-Aïnou, Black hole thermodynamics: no inconsistency via the inclusion of the missing $P - V$ terms. Phys. Rev. D **91**, 064049 (2015). https://doi.org/10.1103/PhysRevD.91.064049. arXiv:1411.2386

21. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972)

22. G. Melgarejo, E. Contreras, P. Bargueño, Regular black holes with exotic topologies. Phys. Dark Universe **30**, 100709 (2020). https://doi.org/10.1016/j.dark.2020.100709

23. P. Bargueño, Some global, analytical and topological properties of regular black holes. Phys. Rev. D **102**(10), 104028 (2020). https://doi.org/10.1103/PhysRevD.102.104028. arXiv:2008.02680

24. C. Lan, Y.-G. Miao, Entropy and topology of regular black holes (2021). arXiv:2105.00218

25. L.D. Landau, E.M. Lifshits, *Electrodynamics of Continuous Media. Course of Theoretical Physics*, vol. 8 (Fizmatlit, Moscow, 2005)

26. D. Kastor, S. Ray, J. Traschen, Enthalpy and the mechanics of AdS black holes. Class. Quantum Gravity **26**(19), 195011 (2009). https://doi.org/10.1088/0264-9381/26/19/195011. arXiv:0904.2765

27. B.P. Dolan, The cosmological constant and black-hole thermodynamic potentials. Class. Quantum Gravity **28**(12), 125020 (2011). https://doi.org/10.1088/0264-9381/28/12/125020. arXiv:1008.5023

28. A.D. Sakharov, The initial stage of an expanding Universe and the appearance of a nonuniform distribution of matter. Sov. Phys. JETP **22**, 241 (1966)

29. J.M. Bardeen, B. Carter, S.W. Hawking, The four laws of black hole mechanics. Commun. Math. Phys. **31**(2), 161–170 (1973). https://doi.org/10.1007/BF01645742

30. L. Gulin, I. Smolić, Generalizations of the Smarr formula for black holes with nonlinear electromagnetic fields. Class. Quantum Gravity **35**(2), 025015 (2017). https://doi.org/10.1088/1361-6382/aa9dfd. arXiv:1710.04660

31. I. Dymnikova, B. Soltysek, Spherically symmetric space-time with two cosmological constants. Gen. Relativ. Gravit. **30**, 1775–1793 (1998). https://doi.org/10.1023/A:1026619228583

32. S.A. Hayward, Formation and evaporation of nonsingular black holes. Phys. Rev. Lett. **96**, 031103 (2006). https://doi.org/10.1103/PhysRevLett.96.031103. arXiv:gr-qc/0506126

33. J.M. Bardeen, Non-singular general-relativistic gravitational collapse, in *Conference Proceedings of GR5, Tbilisi, USSR*, vol. 174 (1968)

34. L. Balart, E.C. Vagenas, Regular black holes with a nonlinear electrodynamics source. Phys. Rev. D **90**, 124045 (2014). https://doi.org/10.1103/PhysRevD.90.124045. arXiv:1408.0306

35. K. Schwarzschild, On the gravitational field of a sphere of incompressible fluid according to Einstein's theory. Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **424–434**, 1916 (1916). arXiv:physics/9912033

36. S.-W. Wei, Y.-X. Liu, R.B. Mann, Ruppeiner geometry, phase transitions, and the microstructure of charged AdS black holes. Phys. Rev. D **100**(12), 124033 (2019). https://doi.org/10.1103/PhysRevD.100.124033. arXiv:1909.03887

37. S.M. Carroll, *Spacetime and Geometry* (Cambridge University Press, Cambridge, 2019)

38. J.W. Maluf, Repulsive gravity near naked singularities and point massive particles. Gen. Relativ. Gravit. **46**, 1734 (2014). https://doi.org/10.1007/s10714-014-1734-y. arXiv:1401.0741

39. Y.C. Ong, Space-time singularities and cosmic censorship conjecture: a review with some thoughts. Int. J. Mod. Phys. A **35**(14), 14 (2020). https://doi.org/10.1142/S0217751X20300070. arXiv:2005.07032

40. R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984). https://doi.org/10.7208/chicago/9780226870373.001.0001