A stochastic approach to galactic propagation

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Abstract: We present a newly developed numerical code solving general Fokker-Planck type transport equations by means of stochastic differential equations in four (space and momentum) dimensions and time. Besides propagation the code is capable of describing the full diffusion tensor as well as particle sources and sinks. The approach was to design the code very general and flexible, so that it can be applied to a large variety of physical problems. Adaption to graphics cards within the CUDA framework significantly improves the performance. Here, we apply our code to the propagation of cosmic ray protons in the Galaxy. Special account is devoted to the spiral arm structure and the consequence for turbulence and, thus, for the diffusion process, leading to different diffusion tensors within and outside of the spiral arms.

Keywords: Galactic transport, Stochastic Differential Equations, Fokker-Planck-Equation

1 Introduction

The propagation of cosmic rays (CR) in the galaxy (or the heliosphere) is described by the Fokker-Planck type transport equation

$$\begin{aligned}\frac{\partial f}{\partial t} &= \vec{\nabla}\cdot\left(\mathcal{K}\cdot\vec{\nabla}f - \vec{V}f\right) - \vec{W}\cdot\vec{\nabla}f \\ &\quad + \Omega\frac{\partial f}{\partial q} + \frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2 D_{qq}\frac{\partial f}{\partial q}\right) - Lf + S. \quad (1)\end{aligned}$$

Here, $f = f(\vec{r}, q, t)$ is the distribution function depending on the three spatial coordinates, $\vec{r}$, the momentum q (or alternatively the particles' rigidity or energy), and t denotes the time. The velocity has been split here into a conservative ($\vec{V}$) and a non-conservative ($\vec{W}$) contribution to the transport equation. Further quantities are the (spatial) diffusion tensor, $\mathcal{K}$, the momentum loss term, Ω, the momentum diffusion coefficient, D_{qq}, the linear term, L, describing catastrophic losses and the source term, S. In addition, initial and boundary conditions have to be specified. Stochastic differential equations (SDE) have been used to model the propagation of CR in the heliosphere [8] and recently also to the galaxy [4]. We present a new, versatile code based on SDE to solve Fokker Planck type equations in up to three spatial dimension, momentum and time.

2 Stochastic Differential Equations

A stochastic differential equation describes the time evolution of a stochastic process. For our purposes we can consider this process as the propagation (e.g. [2]) of a pseudo particle (or phase space element) and thus the SDE

$$d\underline{\mathrm{r}} = \underline{\mathrm{A}}dt + \mathcal{B}\cdot d\underline{\mathrm{W}}, \quad (2)$$

as its propagation equation, where dt is the time step. $\underline{\mathrm{r}}$ stands for the four-dimensional vector $(\vec{r}, q)$, and $\underline{\mathrm{A}}$ is a four-dimensional propagation or convection vector, while $\mathcal{B}$ is a 4×4 tensor representing the diffusive part in Eq. (1). The quantity $d\underline{\mathrm{W}}$ is a Wiener process, usually written as

$$d\underline{\mathrm{W}} = \sqrt{dt}\,\underline{\mathrm{h}} \quad (3)$$

with $\underline{\mathrm{h}}$ being a four-dimensional vector of normally (i.e. Gaussian) distributed random numbers h_i, $i = 1, \ldots, 4$ in the range $(-\infty, \infty)$. If, as it is the case in Eq. (1), there is no coupling between spatial and momentum diffusion, the SDE can be split into two parts:

$$d\vec{r} = \vec{A}_r dt + \mathcal{B}_r \cdot d\vec{W}_r \quad (4a)$$

$$dq = A_q dt + B_{qq} dW_q, \quad (4b)$$

where the momentum diffusion is a scalar contribution, while the spatial part consists of a 3×3 tensor (cf. [6]). In this case $\mathcal{B}_r$ has to satisfy the relation $\mathcal{B}_r{}^t\mathcal{B}_r = \mathcal{K} + {}^t\mathcal{K}$, where the symmetry of the second order derivatives of f was employed.

In contrast to traditional finite-difference or finite-volume schemes, SDEs can be integrated forward or backward in time, depending on the actual physical application, e.g. the form of the particle sources, S. While the SDE is the same for both directions of integration, the corresponding transport equation changes. For the integration forward in time, the transport equation has to be written in a fully conservative form, while that for the time-backward integration has to be fully non-conservative. For the general form, e.g. also applicable to curvilinear coordinates, the resulting forms of transport equation (1) read:

- Time-forward equation:

$$\begin{aligned} \frac{\partial f}{\partial t} = \quad & \vec{\nabla} \cdot \left(\vec{\nabla} \cdot \left[{}^{t}\mathcal{K} f \right] \right) \\ - \quad & \vec{\nabla} \cdot \left(\left[(\vec{\nabla} \cdot {}^{t}\mathcal{K}) + \vec{U} \right] f \right) \\ + \quad & \frac{\partial^2}{\partial q^2} \left(D_{qq} f \right) \\ - \quad & \frac{\partial}{\partial q} \left(\left[\frac{\partial D_{qq}}{\partial q} - \hat{\Omega} \right] f \right) \\ - \quad & L_F f + S \end{aligned} \tag{5}$$

- Time-backward equation:

$$\begin{aligned} \frac{\partial f}{\partial t} = \quad & \mathcal{K} : (\vec{\nabla} \otimes \vec{\nabla} f) \\ + \quad & \left((\vec{\nabla} \cdot \mathcal{K}) - \vec{U} \right) \cdot \vec{\nabla} f \\ + \quad & D_{qq} \frac{\partial^2 f}{\partial q^2} \\ + \quad & \left(\frac{\partial D_{qq}}{\partial q} + \hat{\Omega} \right) \frac{\partial f}{\partial q} \\ - \quad & L_B f + S \end{aligned} \tag{6}$$

For technical details like the numerical realization of the source (S) and loss (L) terms or the binning process to obtain the distribution function from the SDE solutions (trajectories) we refer to [3].

3 Galactic propagation

As a first test for our new code, we compare the results of our code with those obtained by [1]. In this case, Eq. (1) reduces to

$$\frac{\partial f}{\partial t} - S \quad = \quad \vec{\nabla} \cdot \left(k(q) \vec{\nabla} f \right) - \frac{f}{T} \tag{7}$$

where k is the coefficient of spatial diffusionand T the time-scale of catastrophic losses. As in [1], we approximate continuous losses as catastrophic losses:

$$T_{\mathrm{cont}}(p) \quad = \quad \frac{p}{B}. \tag{8}$$

Where B is the rate of momentum loss.

As in [1], we consider a cylindrical diffusion volume of radius $R = 15\,\mathrm{kpc}$ and height $2H = 4\,\mathrm{kpc}$ (see e.g. [5] and references therein). For the distribution of the interstellar gas, we assume a distribution of the interstellar gas that only depends on the distance from the Galactic disc $\omega \propto n_0/\cosh(z\,h_g)$ with $n_0 = 1.24\,\mathrm{cm}^{-3}$, h_g=30 kpc^{-1} The diffusion coefficient

$$k \quad = \quad \begin{cases} k_0 \left(\dfrac{q}{q_0} \right)^{0.6} & \text{for } q > q_0 \\ k_0 \left(\dfrac{q}{q_0} \right)^{-0.48} & \text{for } q < q_0 \end{cases} \tag{9}$$

(q is here the particle rigidity) with $k_0 = 0.027\mathrm{kpc}^2\mathrm{Myr}^{-1}$ and q_0 =4 GV/c is also assumed to be independent of r and φ.

The galactic CR proton density was calculated assuming the CR originate in SN 130001 events, which cluster in the spiral arms, where we adopted the geometry of the spiral structure from [7]. Each source was modeled as a sphere of radius of 20 pc with lightcurve and spectrum

$$S \quad = \quad a_0\,(t - t_i) \exp\left(1 - a_0\,(t - t_i)\right) \Theta\,(t - t_i)\, q^{-2.1}, \tag{10}$$

where t_i is the time of explosion of the ith supernova. For the sake of simplicity, as in [1], we here keep the spiral arm pattern fixed and calculate the CR proton flux along the Sun's path around the Galaxy.

4 Results

We calculated the CR density for different times at one position inside the spiral arm, using the time-backward method. For each point, a total of 20000 pseudo particles was traced. The same source distribution as in [1] was used. In Fig. 1, we compare the temporal variation of the density of 1 GeV CR protons inside a spiral arm computed as described in [1] to results obtained with our new code. As is apparent in Fig. 1, the SDE solution follows the solution of [1], but does not fully reproduce it. This discrepancy is expected, as the SDE code treats the CR sources as spheres,

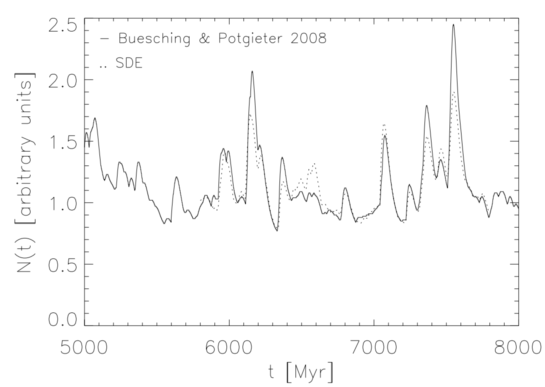

Figure 1: Temporal variation of the density of 1 GeV CR protons inside a spiral arm. Solid line: calculated as described in [1], dotted line: new SDE code.

whereas in [1] the sources are defined on single points in the z grid, but are more extended in r, ϕ directions owing to the Fourier Bessel ansatz ([1] give a resolution of 75 pc in galactocentric radius r and azimuth φ at the position of the Sun).

Fig. 2 shows the spectra by [1] and those of the new SDE code in the left and right column, respectively, where the upper panels show the spectra in the inarm regions, while the lower ones depict the spectra in the interarm regions. Both codes show quite similar results, differences occur mainly due to the different treatments of the sources: while the code used by [1] described the sources as flat disks with a high 80 pc and a diameter of 160 pc, the new SDE code is capable of representing them as spheres with a radius of 30 pc, highlightening the importance of a realistic representation of nearby sources for a suitable modelling of the local interstellar spectrum.

5 Conclusions

We presented a newly developed code that solves general Fokker-Planck type transport equations by means of stochastic differential equations in up to four (space and momentum) dimensions and time. First results reproducing the calculations of [1] show reasonable agreement as expected. The differences in the results are expected due to the different numerical treatment of the point like sources in the two calculations. Besides scalar propagation the code is capable of describing the full diffusion tensor as well as particle sources and sinks.

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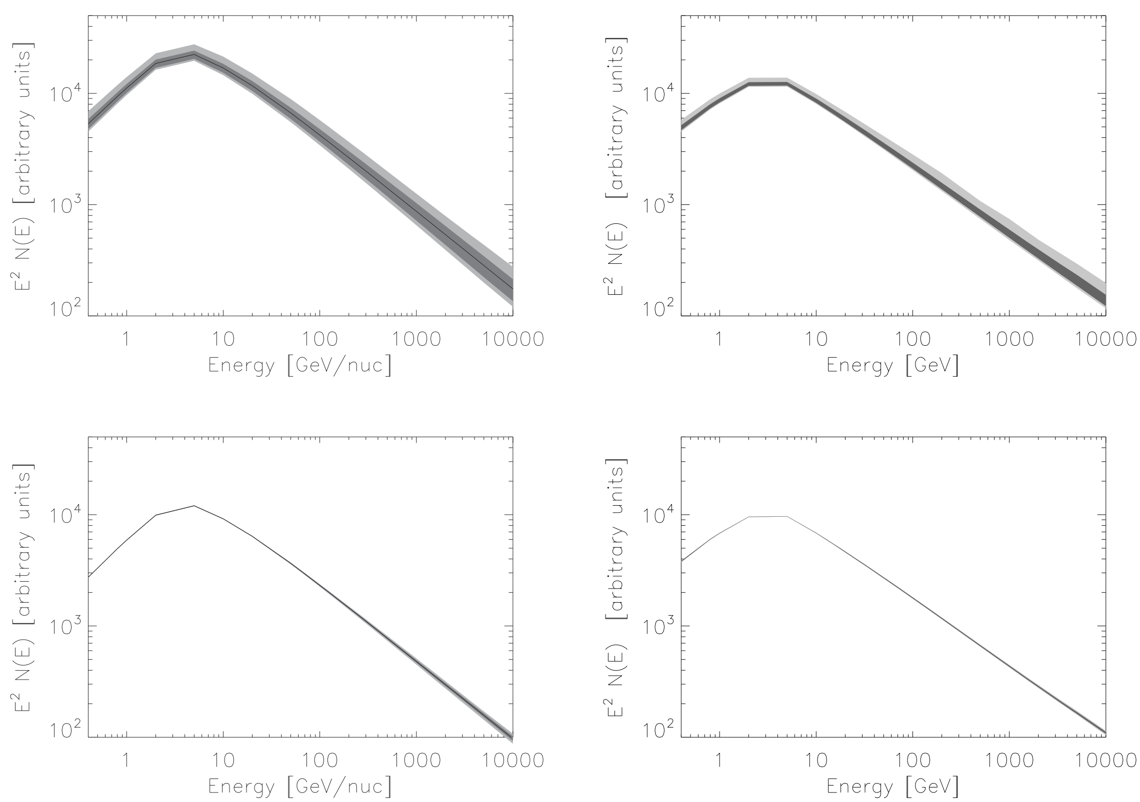

Figure 2: Comparison between the results of [1] (left column) and that obtained with the new code (right column). The upper and lower panels show the spectral variation in the inarm and interarm regions, respectively.