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The Klein-Gordon equation in the cosmic string and rainbow gravity spacetime under the influence of an external magnetic field and Coulomb potential for PDM particles

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Abstract. The Klein-Gordon equation in the cosmic string and rainbow gravity spacetime under the influence of an external magnetic field and Coulomb potential for PDM particles is investigated using the NU method. From solution of the Klein-Gordon system with PDM particles, the energy levels are obtained. The energy levels are numerically calculated as a function of rainbow gravity parameters, as a function of magnetic field parameters, as a function of position-dependent mass parameters. We apply some rainbow functions and the results show that the negative and positive energy levels for rainbow function 1 are symmetrical compared to rainbow function 2.

1. Introduction

The rainbow gravity (RG) model has been an interesting topic of research for many years [1-6]. Rainbow gravity is special relativity deformed from the semi-classical model to general relativity from the RG model. The rainbow gravity (RG) model assumes that the spacetime background is influenced by the energy of particles moving with high energy, so that the metric elements of spacetime depend on energy [5, 7-9, 11-14, 16-20]. In this context, the Planck energy assumes a crucial role within rainbow gravity (RG) by serving as the boundary that delineates classical physics from quantum mechanics across energy scales independent of the speed of light. Consequently, the impacts of rainbow gravity become noticeable, exemplified in assessments like high-energy cosmic ray threshold tests [7, 16-18], observations of TeV photons [19], evaluations of quantum gravity through gamma-ray burst observations [7], and experiments in astrophysics aimed at detecting novel effects stemming from corrections in quantum gravity [20].

The latest research on the gravitational effects of rainbows has been carried out. Among them, thermodynamic properties of black holes [21-25], neutron stars in conditions of dynamic stability [26], Thermodynamics for modified Schwarzschild black holes [27], thermodynamic analysis for massively charged black holes [28], thermodynamics of geometry and machines for black hole heat in rainbow gravity [29], rainbow gravity theory and f(R) used to distort general relativity [30], solution for the initial singularity in a closed cosmology with rainbow modifications [31], and the entropy of black holes [32], black hole singularity elimination in early universe [33], Casimir effect in Einstein's rainbow universe [9], massive Schwarzschild in scalar fields [34], massive Yang-Mills black holes in five dimensions [35].

On the flip side, rainbow gravity has been explored within diverse spacetime contexts regarding the Klein-Gordon (KG) equation, covering spin zero-like mesons, Dirac equations for fermions exhibiting half spin-like characteristics, and the Duffen-Kemmer-Peatiau (DKP) equation for bosons and photons with spin one-like attributes. Among these investigations, Bezzerra et al. [9-10] delved into Landau levels employing the Schrödinger and Klein-Gordon equations, while Bakke and Mota [36-37] examined solutions to the Dirac equation under the influence of the Aharonov-Bohm field. Hosseinpour et al. [5] focused on DKP particles with non-minimal coupling under the gravitational impact of rainbow cosmic strings, Sogut et al. [13] scrutinized the cosmic spacetime of rainbow gravitational strings using photon quantum dynamics, Kangal et al. [14] studied the Klein-Gordon equation in rainbow gravitational spacetime on Godel-type trivial topologies, Montigny et al. [15] explored the Klein-Gordon equation in global monopole spacetime within the framework of rainbow gravity. Recently, Mustafa [6,38] conducted a study on the Klein-Gordon equation in rainbow cosmic string gravitational spacetime, considering the influence of both non-uniform and uniform magnetic fields, with and without the inclusion of position-dependent mass (PDM).

Previous research has not thoroughly integrated the influence of various scalar potentials into the KG equation in the domains of cosmic strings and rainbow gravity. This limitation arises from challenges in solving intricate equations, necessitating reliance on numerical analyses. Consequently, this paper endeavors to broaden the scope of Klein-Gordon studies within cosmic strings and rainbow gravity for particles with position-dependent mass (PDM). This extension involves the consideration of external magnetic fields and Coulomb potentials. The scalar potential under scrutiny in this study is the Coulomb potential, which has been previously examined in the context of gravitational wave propagation [49], the quark model [50], and relativistic quantum mechanics by various authors [46-55].

The structure of this manuscript is laid out as follows. In Section 2, an exploration of the foundational principles of the Klein-Gordon equations within cosmic strings and rainbow gravitational spacetime is undertaken, taking into account the impact of external magnetic fields and Coulomb potential on particles characterized by position-dependent mass (PDM). Section 3 outlines the analysis method employing the NU method, which is utilized for solving second-order differential equations associated with the Klein-Gordon equation within cosmic strings and rainbow gravity. Moving on to Section 4, the outcomes and discussion encompassing two categories of Klein-Gordon equations namely constant mass and PDM particles are presented. Subsequently, the Klein-Gordon equation is applied to rainbow functions 1 and 2 to derive the particle energy, followed by a thorough analysis of the energy results. Finally, Section 5 encapsulates the research findings and draws conclusions based on the conducted study.

2. Basic Theory

2.1. The Klein-Gordon equation for particles with position-dependent mass (PDM) in cosmic string and rainbow gravity spacetime, affected by an external magnetic field and Coulomb potential

The metric of cosmic string spacetime (using natural unit $c = \hbar = G = 1$) can be expressed as

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2 \tag{1}$$

From equation (1), Rainbow gravity (RG) takes an energy-dependent form as

$$ds^2 = -\frac{1}{Q_0(y)^2} dt^2 + \frac{1}{Q_1(y)^2}\left(dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2\right);\ y = \frac{E}{E_p} \tag{2}$$

where $\alpha = 1 - 4G\mu$ is the constant representing the deficit angle in conical spacetime, G is is the gravitational constant of Newton's law, μ is the linear mass density of the cosmic string such that $\alpha < 1$, E denotes the particle energy, and $E_p = \sqrt{\frac{\hbar c^5}{G}}$ is the Planck energy. The line elements in equations (1) and (2) have a signature of $(-,\ +,\ +,\ +)$, allowing the expression of the metric tensor $g_{\mu v}$ in equation (2) as:

$$g_{\mu v} = \begin{bmatrix} -\frac{1}{Q_0(y)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{Q_1(y)^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha^2 r^2}{Q_1(y)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{Q_1(y)^2} \end{bmatrix};\ \mu, v = t, r, \varphi, z \tag{3}$$

and

$$\det(g_{\mu v}) = -\frac{\alpha^2 r^2}{Q_0(y)^2 Q_1(y)^6} \tag{4}$$

The inverse of the metric in equation (3) can be formulated as

$$g^{\mu v} = \begin{bmatrix} -Q_0(y)^2 & 0 & 0 & 0 \\ 0 & Q_1(y)^2 & 0 & 0 \\ 0 & 0 & \frac{Q_1(y)^2}{\alpha^2 r^2} & 0 \\ 0 & 0 & 0 & Q_1(y)^2 \end{bmatrix} \tag{5}$$

where $Q_0(y), Q_1(y)$ is rainbow functions. Based on a model from the rainbow theory of gravity, the Planck energy limit E_p is used to separate classical physics and quantum mechanics into invariant energy scales other than the speed of light. Therefore, the relativistic energy-momentum dispersion relationship is improved using the rainbow gravitational influence as follows

$$E^2 Q_0(y)^2 - p^2 c^2 Q_1(y)^2 = m^2 c^4;\ 0 \le (y = E/E_p) \le 1 \tag{6}$$

where mc^2 represents the energy associated with the rest mass. The form of the PDM momentum operator is taken from the von Roos PDM in the nonminimal coupling of the von Roos Schrödinger PDM as follows [39-42]

$$\hat{p}_j(r) = -i\left[\partial_j - \frac{\partial_i f(r)}{4f(r)}\right];\ j = 1, 2, 3, \tag{7}$$

where $f(r)$ is the dimensionless scalar multiplier is positive [5,45].

The Klein-Gordon equation in the cosmic string spacetime under the influence of rainbow gravity in equation (2) for a particle with charge e in the 4-vector potential A_μ (using natural units $c = \hbar = G = 1$) can be expressed as [6]

$$\frac{1}{\sqrt{-g}} D_\mu\left(\sqrt{-g} g^{\mu v} D_v \Psi\right) = m^2 \Psi \tag{8}$$

In this context, $D_\mu = \partial_\mu - ieA_\mu$ signifies the gauge-covariant derivative, $g = det(g_{\mu v})$, and m represents the rest mass energy of the particle as indicated by the KG equation.

Moreover, by incorporating equation (8) into the scalar potential through the replacement of $m \to m + S(r)$, we acquire

$$\frac{1}{\sqrt{-g}} D_\mu\left(\sqrt{-g} g^{\mu v} D_v \Psi\right) = (m + S(r))^2 \Psi \tag{9}$$

For scalar potential, we choose the Coulomb potential as follows [15]

$$S(r) = \frac{\tau}{r} \tag{10}$$

where τ is constant values.

By substituting equation (10) and equation (7) into equation (9) where $D_\mu \to \widetilde{D}_\mu = D_\mu + F_\mu = \partial_\mu + F_\mu - ieA_\mu$, $F_\mu = (0, F_r, 0, 0), F_r = \frac{f'(r)}{4f(r)}$ and $f(\mathbf{r}) = f(r)$ is only radially dependent, so we get

$$\frac{1}{\sqrt{-g}}\left(D_\mu + F_\mu\right)\sqrt{-g}g^{\mu v}\left(D_v - F_v\right)\Psi = \left(m + \frac{\tau}{r}\right)^2\Psi \tag{11}$$

by substituting equation (4) and equation (5) into equation (11), we obtain

$$\left\{-Q_0(y)^2\partial_t^2 + \left[\partial_r^2 + \frac{1}{r}\partial_r - M(r)\right]Q_1(y)^2 + \left(\partial_\varphi - ieA_\varphi\right)^2\frac{Q_1(y)^2}{\alpha^2 r^2} + \partial_z^2 Q_1(y)^2\right\}\Psi(t, r, \varphi, z)$$
$$= \left(m + \frac{\tau}{r}\right)^2\Psi(t, r, \varphi, z) \tag{12}$$

where

$$M(r) = -\frac{3}{16}\left(\frac{f'(r)}{f(r)}\right)^2 + \frac{f''(r)}{4f(r)} + \frac{f'(r)}{4rf(r)} \tag{13}$$

the new wave function is set in equation (12), thus obtained

$$\Psi(t, r, \varphi, z) = exp\left(i\left[L\varphi + k_z z - Et\right]\right)\psi(r) \tag{14}$$

equation (14) is substituted into equation (12), so that we get

$$Q_0(y)^2 E^2\psi(r) - k_z^2 Q_1(y)^2\psi(r) + Q_1(y)^2\left[\partial_r^2 + \frac{1}{r}\partial_r - M(r) - \frac{\left(L - eA_\varphi\right)^2}{\alpha^2 r^2}\right]\psi(r)$$
$$= \left(m + \frac{\tau}{r}\right)^2\psi(r) \tag{15}$$

the following we will consider $A_\varphi = \frac{1}{2}Br$ which in turn produces external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A} = B\hat{z}$. So, equation (15) can be written as

$$\left\{Q_0(y)^2 E^2 - Q_1(y)^2\left(k_z^2 + \frac{\widetilde{B}^2}{4}\right) + Q_1(y)^2\partial_r^2 + Q_1(y)^2\frac{1}{r}\partial_r - Q_1(y)^2 M(r) -\right.$$
$$\left.Q_1(y)^2\frac{\widetilde{L}^2}{r^2} + Q_1(y)^2\frac{\widetilde{L}\widetilde{B}}{r}\right\}\psi(r) = \left(m + \frac{\tau}{r}\right)^2\psi(r) \tag{16}$$

where

$$\widetilde{L} = \frac{L}{\alpha}, \widetilde{B} = \frac{eB}{\alpha} \tag{17}$$

Where L represents the magnetic quantum numbers, and B denotes the external magnetic field. Equation (16) represents the KG equation for particles with PDM in cosmic string and rainbow gravity spacetime, influenced by an external magnetic field and Coulomb potential.

3. Method of analysis

3.1. The Nikiforov-Uvarov method (NU)

The utilization of the Nikiforov-Uvarov method (NU) [56-57] to solve second-order differential equations is executed in the following manner

$$\psi''(s) + \frac{\widetilde{\varpi}(s)}{\zeta(s)}\psi'(s) + \frac{\widetilde{\zeta}(s)}{\zeta^2}\psi(s) \tag{18}$$

with appropriate coordinate transformations, $s = s(r)$, where $\zeta(s)$ and $\widetilde{\zeta}(s)$ are most polynomial of second order and $\widetilde{\varpi}(s)$ is first-degree polynomial. The structure of the second-order differential equation can be expressed as follows:

$$\frac{\partial^2\psi(s)}{\partial s^2} + \frac{\xi_1 - \xi_2 s}{s(1 - \xi_3 s)}\frac{\partial\psi(s)}{\partial s} + \left[\frac{-\lambda_1 s^2 + \lambda_2 s - \lambda_3}{s^2(1 - \xi_3 s)^2}\right]\psi(s) = 0 \tag{19}$$

In accordance with the NU method, the equations for eigenfunction and energy eigenvalue transform into

$$\psi\left(s\right)=s^{\xi_{12}}\left(1-\xi_3 s\right)^{-\xi_{12}-\frac{\xi_{13}}{\xi_3}}P_n^{\left(\xi_{10}-1,\frac{\xi_{11}}{\xi_3}-\xi_{10}-1\right)}\left(1-2\xi_3 s\right) \tag{20}$$

$$\begin{aligned}&n\xi_2-\left(2n+1\right)\xi_5+\left(2n+1\right)\left(\sqrt{\xi_9}+\xi_3\sqrt{\xi_8}\right)+n\left(n-1\right)\xi_3+\xi_7+2\xi_3\xi_8\\&+2\sqrt{\xi_8\xi_9}=0\end{aligned} \tag{21}$$

where

$$\begin{aligned}&\xi_4=\tfrac{1-\xi_1}{2},\ \xi_5=\tfrac{\left(\xi_2-2\xi_3\right)}{2},\ \xi_6=\xi_5^2+\lambda_1,\ \xi_7=2\xi_4\xi_5-\lambda_2,\ \xi_8=\xi_4^2+\lambda_3,\\&\xi_9=\xi_3\xi_7+\xi_3^2\xi_8+\xi_6,\ \xi_{10}=\xi_1+2\xi_4+2\sqrt{\xi_8},\\&\xi_{11}=\xi_2-2\xi_5+2\left(\sqrt{\xi_9}+\xi_3\sqrt{\xi_8}\right),\ \xi_{12}=\xi_4+\sqrt{\xi_8},\ \xi_{13}=\xi_5-\left(\sqrt{\xi_9}+\xi_3\sqrt{\xi_8}\right)\end{aligned} \tag{22}$$

4. Result and Discussion

Within the framework of spacetime and rainbow gravity, and under the impact of external magnetic fields and Coulomb potentials, we will examine two variants of the Klein-Gordon equation: one involving a constant mass and the other pertaining to particles with position-dependent mass (PDM).

4.1. Klein-Gordon equation for Constant Mass in cosmic string and rainbow gravity spacetime under the influence of an external magnetic field and Coulomb potential

For the Klein-Gordon equation with a constant mass, where $f\left(r\right)=1\Longleftrightarrow M\left(r\right)=0$ equation (16) is simplified to

$$\begin{aligned}&\left\{Q_0\left(y\right)^2E^2-Q_1\left(y\right)^2\left(k_z^2+\tfrac{\widetilde{B}^2}{4}\right)+Q_1\left(y\right)^2\partial_r^2+Q_1\left(y\right)^2\tfrac{1}{r}\partial_r-Q_1\left(y\right)^2\tfrac{\widetilde{L}^2}{r^2}+Q_1\left(y\right)^2\tfrac{\widetilde{L}\widetilde{B}}{r}\right\}\psi\left(r\right)\\&=\left(m+\tfrac{\tau}{r}\right)^2\psi\left(r\right)\end{aligned} \tag{23}$$

we can also rewrite equation (23) is similar to equation (19) as

$$\begin{aligned}&\tfrac{\partial^2\psi(r)}{\partial r^2}+\tfrac{1}{r}\tfrac{\partial\psi(r)}{\partial r}+\tfrac{1}{r^2}\left[-\tfrac{1}{Q_1(y)^2}\left(m^2-Q_0\left(y\right)^2E^2+Q_1\left(y\right)^2\left(k_z^2+\tfrac{\widetilde{B}^2}{4}\right)\right)r^2\right.\\&\left.+\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1\left(y\right)^2-2m\tau\right)r-\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}^2Q_1\left(y\right)^2+\tau^2\right)\right]\psi\left(r\right)=0\end{aligned} \tag{24}$$

by comparing equation (24) with equation (22)

$$\lambda_1=\tfrac{1}{Q_1(y)^2}\left(m^2-Q_0\left(y\right)^2E^2+Q_1\left(y\right)^2\left(k_z^2+\tfrac{\widetilde{B}^2}{4}\right)\right) \tag{24a}$$

$$\lambda_2=\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1\left(y\right)^2-2m\tau\right) \tag{24b}$$

$$\lambda_3=\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}^2Q_1\left(y\right)^2+\tau^2\right) \tag{24c}$$

$$\begin{aligned}&\xi_1=1,\ \xi_2=\xi_3=\xi_4=\xi_5=0,\ \xi_6=\xi_5^2+\lambda_1=\lambda_1,\ \xi_7=2\xi_4\xi_5-\lambda_2=-\lambda_2,\\&\xi_8=\xi_4^2+\lambda_3=\lambda_3,\ \xi_9=\xi_3\xi_7+\xi_3^2\xi_8+\xi_6=\lambda_1\end{aligned} \tag{24d}$$

by substituting equations (24a)-(24d) into equation (21), we obtain

$$\begin{aligned}&\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1\left(y\right)^2-2m\tau\right)=\sqrt{\tfrac{1}{Q_1(y)^2}\left(m^2-Q_0\left(y\right)^2E^2+Q_1\left(y\right)^2\left(k_z^2+\tfrac{\widetilde{B}^2}{4}\right)\right)}\\&\left(2n+1+2\sqrt{\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}^2Q_1\left(y\right)^2+\tau^2\right)}\right)\end{aligned} \tag{25}$$

Here, n represents the radial quantum numbers. Let us now explore various rainbow functions to examine how rainbow gravity influences particles with PDM through the KG equation.

4.1.1. The set of rainbow functions 1 By setting rainbow functions $Q_0(y)=1$ and $Q_1(y)=\sqrt{1-\epsilon y^2}$, $y=\frac{E}{E_p}$ [43-44] where ϵ is dimensionless constant used to describe the loop effect of quantum gravity in space-time geometry into equation (25), so that the energy equation is obtained

$$\left((\omega-\rho E^2)^2-\widetilde{d}^2\left(1-\delta E^2\right)\left(q-\sigma E^2\right)-4\left(W-\varrho E^2\right)\left(q-\sigma E^2\right)\right)^2-16\widetilde{d}^2\left(q-\sigma E^2\right)^2 \left(1-\delta E^2\right)\left(W-\varrho E^2\right)=0 \tag{26}$$

where

$$\omega=\widetilde{L}\widetilde{B}-2m\tau,\ \rho=\widetilde{L}\widetilde{B}\delta,\ \ q=m^2+k_z^2+\frac{\widetilde{B}^2}{4},\ \sigma=1+\left(k_z^2+\frac{\widetilde{B}^2}{4}\right)\delta,\ \delta=\frac{\epsilon}{E_p^2},$$
$$\widetilde{d}=2n+1,\ W=\widetilde{L}^2+\tau^2,\ \varrho=\widetilde{L}^2\delta \tag{27}$$

the energy levels of equation (26) are numerically calculated using Matlab 2022 software as a function of rainbow gravity parameters (δ) and as a function of magnetic field parameters (B).

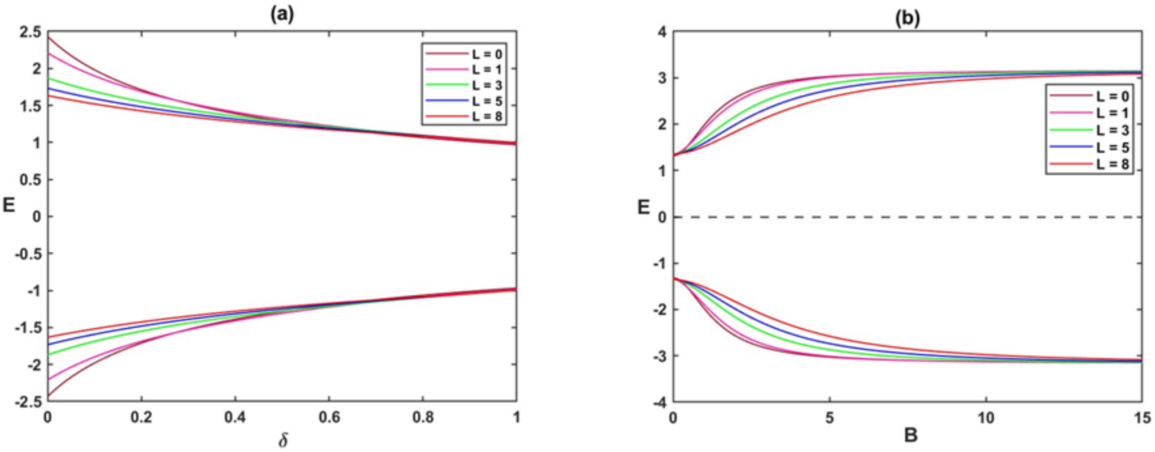


Figure 1. (a) The plot of energy against $\delta=\frac{\epsilon}{E_p^2}$ for $\alpha=\frac{1}{4}$, $m=e=k_z=1$, $\tau=1$, $B=1, n=2, L=0,\ 1,\ 3,\ 5,\ 8$, and (b) energy against B for $\alpha=\frac{1}{4}$, $m=e=k_z=1$, $\tau=1$, $\delta=0.1$, $n=2, L=0,\ 1,\ 3,\ 5,\ 8$.

In Figure 1(a), it is depicted that the clustering of negative/positive energy levels varies for distinct magnetic quantum numbers L as the parameter δ rises from zero. Furthermore, Figure 1(a) illustrates that as $\delta\to 1$, the negative/positive energy levels merge and converge across different magnetic quantum numbers L.

In Figure 1(b), it is illustrated that the negative/positive energy levels merge and converge as $B\to 0$ for various magnetic quantum numbers L. Additionally, Figure 1(b) demonstrates the clustering of negative/positive energy levels within the range of $B=0$ to $B=5$, followed by their merging and convergence as $B\gg 1$ for different magnetic quantum numbers L.

4.1.2. The set of rainbow functions 2 By setting rainbow functions $Q_0(y)=1$ and $Q_1(y)=\sqrt{1-\epsilon y}$, $y=\frac{E}{E_p}$ [43-44] where ϵ is dimensionless constant used to describe the loop effect of quantum gravity in space-time geometry into equation (25), so that the energy equation is obtained

$$\left((\omega-2\rho E)^2-\widetilde{d}^2\left(1-2\varsigma E\right)\left(q-2\sigma E-E^2\right)-4\left(W-2\varrho E\right)\left(q-2\sigma E-E^2\right)\right)^2 -16\widetilde{d}^2\left(q-2\sigma E-E^2\right)^2\left(1-2\varsigma E\right)\left(W-2\varrho E\right)=0 \tag{28}$$

where

$$\omega=\widetilde{L}\widetilde{B}-2m\tau,\ \rho=\widetilde{L}\widetilde{B}\varsigma,\ q=m^2+k_z^2+\frac{\widetilde{B}^2}{4},\ \sigma=\varsigma\left(k_z^2+\frac{\widetilde{B}^2}{4}\right),\ \varsigma=\frac{\epsilon}{2E_p},$$
$$\widetilde{d}=2n+1,\ W=\widetilde{L}^2+\tau^2,\ \varrho=\widetilde{L}^2\varsigma \tag{29}$$

the energy levels of equation (28) are numerically calculated using Matlab 2022 software as a function of rainbow gravity parameters (ς) and as a function of magnetic field parameters (B).

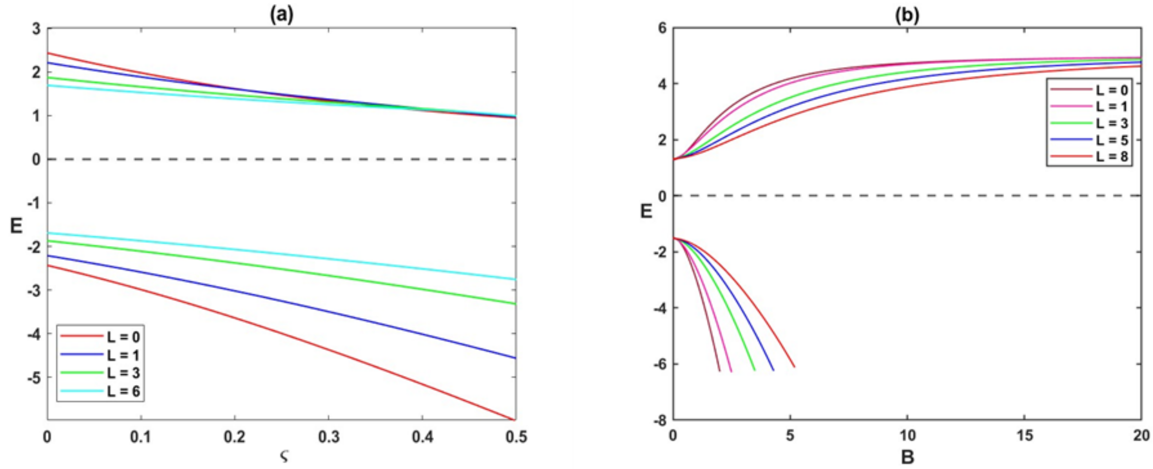


Figure 2. (a) The plot of energy against $\varsigma=\frac{\epsilon}{2E_p}$ for $\alpha=\frac{1}{4},\ m=e=k_z=1,\ \tau=1,$ $B=1, n=2, L=0,\ 1,\ 3,\ 6,$ and (b) energy against B for $\alpha=\frac{1}{4},\ m=e=k_z=1,\ \tau=1,$ $\varsigma=0.1, n=2, L=0,\ 1,\ 3,\ 5,\ 8.$

In Figure 2(a), it is illustrated that the negative/positive energy levels exhibit clustering as $\varsigma\to 0$ for different magnetic quantum numbers L. The positive energy levels merge and converge, whereas the negative energy levels split when $\varsigma\gg 1$ for various magnetic quantum numbers L.

Figure 2(b) showcases the merging of negative/positive energy levels as $B\to 0$ for various magnetic quantum numbers L. Additionally, Figure 2(b) depicts the merging and convergence of positive energy levels, along with the splitting of negative energy levels, as $B\gg 1$ for different magnetic quantum numbers L.

4.2. Klein-Gordon equation for PDM in cosmic string and rainbow gravity spacetime under the influence of an external magnetic field and Coulomb potential

For the case PDM KG-particles with the dimensionless scalar multiplier is positive in the form of $f(r)=exp(4\eta r)$ in equation (13), so that $M(r)=\eta^2+\frac{\eta}{r}$. Equation (16) can be written as

$$\Big\{Q_0(y)^2E^2-Q_1(y)^2\left(Q_z^2+\frac{\widetilde{B}^2}{4}\right)-m^2+Q_1(y)^2\partial_r^2+Q_1(y)^2\frac{1}{r}\partial_r-Q_1(y)^2\eta^2-Q_1(y)^2\eta$$
$$+Q_1(y)^2\frac{\widetilde{L}\widetilde{B}}{r}-2m\frac{\tau}{r}-Q_1(y)^2\frac{\widetilde{L}^2}{r^2}-\frac{\tau^2}{r^2}\Big\}\psi(r)=0 \tag{30}$$

we can also rewrite equation (30) is similar to equation (19) as

$$\frac{\partial^2\psi(r)}{\partial r^2}+\frac{1}{r}\frac{\partial\psi(r)}{\partial r}+\frac{1}{r^2}\Big[-\frac{1}{Q_1(y)^2}\left(m^2-Q_0(y)^2E^2+Q_1(y)^2\left(k_z^2+\frac{\widetilde{B}^2}{4}\right)+Q_1(y)^2\eta^2\right)r^2$$
$$+\frac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1(y)^2-2m\tau-Q_1(y)^2\eta\right)r-\frac{1}{Q_1(y)^2}\left(\widetilde{L}^2Q_1(y)^2+\tau^2\right)\Big]\psi(r)=0 \tag{31}$$

by comparing equation (31) with equation (22)

$$\lambda_1 = \tfrac{1}{Q_1(y)^2}\left(m^2 - Q_0\left(y\right)^2 E^2 + Q_1\left(y\right)^2\left(k_z^2 + \tfrac{\widetilde{B}^2}{4}\right) + Q_1\left(y\right)^2\eta^2\right) \tag{32a}$$

$$\lambda_2 = \tfrac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1\left(y\right)^2 - 2m\tau - Q_1\left(y\right)^2\eta\right) \tag{32b}$$

$$\lambda_3 = \tfrac{1}{Q_1(y)^2}\left(\widetilde{L}^2 Q_1\left(y\right)^2 + \tau^2\right) \tag{32c}$$

$$\begin{aligned}&\xi_1 = 1,\, \xi_2 = \xi_3 = \xi_4 = \xi_5 = 0,\, \xi_6 = \xi_5^2 + \lambda_1 = \lambda_1,\, \xi_7 = 2\xi_4\xi_5 - \lambda_2 = -\lambda_2,\\ &\xi_8 = \xi_4^2 + \lambda_3 = \lambda_3,\, \xi_9 = \xi_3\xi_7 + \xi_3^2\xi_8 + \xi_6 = \lambda_1\end{aligned} \tag{32d}$$

by substituting equations (32a)-(32d) into equation (21), we obtain

$$\begin{aligned}&\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}\widetilde{B}Q_1\left(y\right)^2 - 2m\tau - Q_1\left(y\right)^2\eta\right) =\\ &\sqrt{\tfrac{1}{Q_1(y)^2}\left(m^2 - Q_0\left(y\right)^2 E^2 + Q_1\left(y\right)^2\left(k_z^2 + \tfrac{\widetilde{B}^2}{4}\right) + Q_1\left(y\right)^2\eta^2\right)}\\ &\left(2n + 1 + 2\sqrt{\tfrac{1}{Q_1(y)^2}\left(\widetilde{L}^2 Q_1\left(y\right)^2 + \tau^2\right)}\right)\end{aligned} \tag{33}$$

Here, n represents the radial quantum numbers. Let us now explore various rainbow functions to examine how rainbow gravity influences particles with PDM through the KG equation.

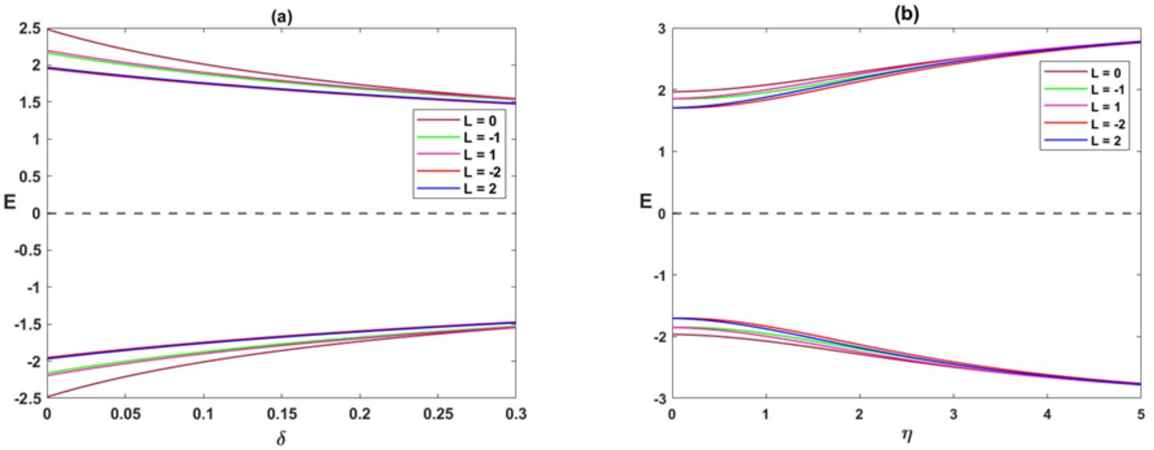
4.2.1. The set of rainbow functions 1 By setting rainbow functions $Q_0\left(y\right) = 1$ and $Q_1\left(y\right) = \sqrt{1 - \epsilon y^2}$, $y = \frac{E}{E_p}$ [43-44] where ϵ is dimensionless constant used to describe the loop effect of quantum gravity in space-time geometry into equation (33), so that the energy equation is obtained

$$\begin{aligned}&\left(\left(\omega - \rho E^2\right)^2 - \widetilde{d}^2\left(1 - \delta E^2\right)\left(q - \sigma E^2\right) - 4\left(W - \varrho E^2\right)\left(q - \sigma E^2\right)\right)^2 - 16\widetilde{d}^2\left(q - \sigma E^2\right)^2\\ &\left(1 - \delta E^2\right)\left(W - \varrho E^2\right) = 0\end{aligned} \tag{34}$$

where

$$\begin{aligned}&\omega = \widetilde{L}\widetilde{B} - 2m\tau - \eta,\ \rho = \widetilde{L}\widetilde{B}\delta - \eta\delta,\ \ q = m^2 + k_z^2 + \tfrac{\widetilde{B}^2}{4} + \eta^2,\ \sigma = 1 + \left(k_z^2 + \tfrac{\widetilde{B}^2}{4}\right)\delta + \eta^2\delta,\\ &\delta = \tfrac{\epsilon}{E_p^2},\, \widetilde{d} = 2n + 1,\, W = \ \widetilde{\ell}^2 + \tau^2,\, \varrho = \ \widetilde{\ell}^2\delta\end{aligned} \tag{35}$$

the energy levels of equation (34) are numerically calculated using Matlab 2022 software as a function of rainbow gravity parameters (δ), as a function of position-dependent of mass (PDM) parameters (η), and as a function of magnetic field parameters (B).



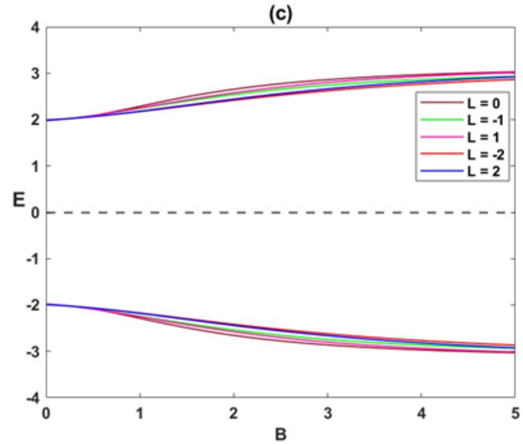


Figure 3. (a) The plot of energy against $\delta = \frac{\epsilon}{E_p^2}$ for $\alpha = \frac{1}{4}$, $m = e = k_z = 1$, $\tau = 1$, $B = 1, n = 1, \eta = 0.5, L = 0, \pm 1, \pm 2$, (b) energy against η for $\alpha = \frac{1}{4}$, $m = e = k_z = 1$, $\tau = 1$, $B = 1$, $\delta = 0.1$, $n = 1$, $L = 0$, ± 1, ± 2, and (c) energy against B for $\alpha = \frac{1}{4}$, $m = e = k_z = 1$, $\tau = 1$, $\delta = 0.1$, $\eta = 2$, $n = 1$, $L = 0$, ± 1, ± 2.

In Figure 3(a), it is depicted that the negative/positive energy levels cluster for different magnetic quantum numbers L when $\delta \to 0$. Additionally, Figure 3(a) illustrates the merging and convergence of negative/positive energy levels when $\delta \gg 1$ for different magnetic quantum numbers L.

Figure 3(b) shows that the negative/positive degenerate energy level corresponding to $L = \pm\, L$ occurs at the same point $\eta = 0$ and breaks down as the value of the parameter η increases. Furthermore, Figure 3(b) illustrates the merging and convergence of negative/positive energy levels as $\eta \gg 1$ for different magnetic quantum numbers L.

Figure 3(c) demonstrates the merging and convergence of negative/positive energy levels as $B \to 0$ for different magnetic quantum numbers L. Moreover, Figure 3(c) illustrates the splitting of negative/positive energy levels when $B \gg 1$ for different magnetic quantum numbers L.

4.2.2. The set of rainbow functions 2 By setting rainbow functions $Q_0(y) = 1$ and $Q_1(y) = \sqrt{1 - \epsilon y}$, $y = \frac{E}{E_p}$ [43-44] where ϵ is dimensionless constant used to describe the loop effect of quantum gravity in space-time geometry into equation (33), so that the energy equation is obtained

$$\left((\omega - 2\rho E)^2 - \widetilde{d}^2 (1 - 2\varsigma E)(q - 2\sigma E - E^2) - 4(W - 2\varrho E)(q - 2\sigma E - E^2)\right)^2$$
$$-16\widetilde{d}^2 (q - 2\sigma E - E^2)^2 (1 - 2\varsigma E)(W - 2\varrho E) = 0 \tag{36}$$

where

$$\omega = \widetilde{\ell}\widetilde{B} - 2m\tau - \eta,\ \rho = \widetilde{\ell}\widetilde{B}\varsigma - \eta\varsigma,\ q = m^2 + k_z^2 + \frac{\widetilde{B}^2}{4} + \eta^2,$$
$$\sigma = \varsigma\left(k_z^2 + \frac{\widetilde{B}^2}{4}\right) + \eta^2\varsigma,\ \varsigma = \frac{\epsilon}{2E_p},\ \widetilde{d} = 2n + 1,\ W = \widetilde{\ell}^2 + \tau^2,\ \varrho = \widetilde{\ell}^2\varsigma \tag{37}$$

the energy levels of equation (36) are numerically calculated using Matlab 2022 software as a function of rainbow gravity parameters (ς), as a function of position-dependent mass (PDM) parameters (η), and as a function of magnetic field parameters (B).

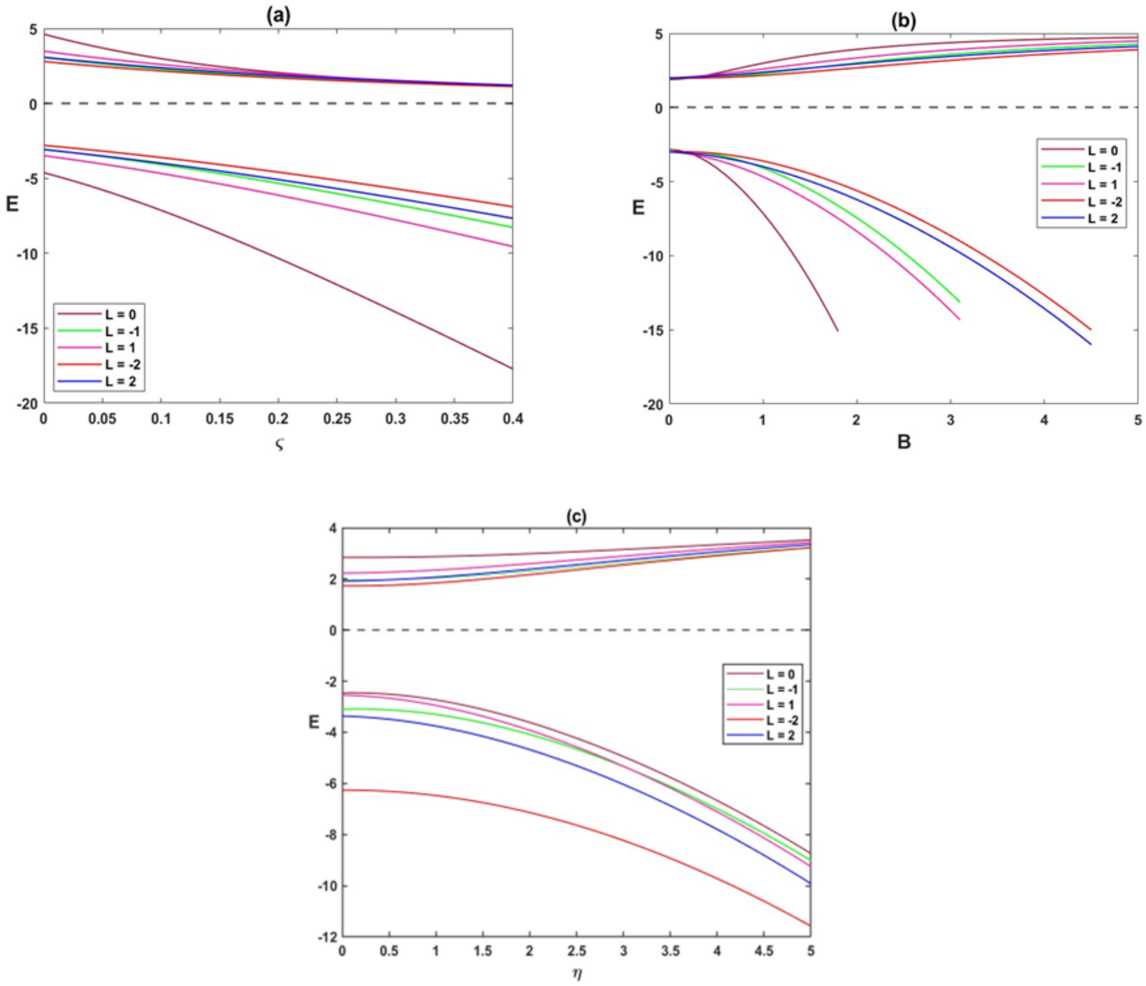


Figure 4. (a) The plot of energy against $\varsigma = \frac{\epsilon}{2E_p}$ for $\alpha = \frac{1}{8}$, $m = e = k_z = 1$, $\tau = 1$, $B = 1$, $n = 1$, $\eta = 2$, $L = 0,\ \pm1,\ \pm2$, (b) energy against B for $\alpha = \frac{1}{8}$, $m = e = k_z = 1$, $\tau = 1$, $n = 1$, $\varsigma = 0.1$, $\eta = 2$, $L = 0,\ \pm1,\ \pm2$, and (c) energy against η for $\alpha = \frac{1}{8}$, $m = e = k_z = 1$, $\tau = 1$, $B = 1$, $\varsigma = 0.1$, $n = 1$, $L = 0,\ \pm1,\ \pm2$.

In Figure 4(a), it is depicted that the negative/positive energy levels cluster around $L = 0$ as $\varsigma \to 0$. Furthermore, Figure 4(a) illustrates the merging and convergence of positive energy levels, along with the splitting of negative energy levels, when $\varsigma \gg 1$ for different magnetic quantum numbers L.

Figure 4(b) illustrates the merging and convergence of negative/positive energy levels as $B \to 0$ for different magnetic quantum numbers L. Additionally, Figure 4(b) demonstrates the splitting of negative/positive energy levels when $B \gg 1$ for different magnetic quantum numbers L.

In Figure 4(c), it is demonstrated that the negative/positive energy levels cluster as $\eta \to 0$ for different magnetic quantum numbers L. Figure 4(c) shows that the negative/positive energy levels break down when $\eta \gg 1$ for different magnetic quantum numbers L.

5. Conclusion

The Klein-Gordon equations within cosmic strings and the gravitational spacetime of rainbow, affected by external magnetic fields and Coulomb potentials for PDM particles, are resolved through the NU method. The energy equation is derived using this method, and the Klein-Gordon equation is subject to the application of the rainbow function. Numerical calculations are performed to determine energy levels, varying with rainbow gravity parameters, magnetic field parameters, and position-dependent mass parameters. Energy levels are calculated numerically as a function of rainbow gravity parameters, as a function of magnetic field parameters, as a function of position-dependent mass parameters. The results show that the negative and positive energy levels for rainbow function 1 $Q_0(y) = 1$ and $Q_1(y) = \sqrt{1-\epsilon y^2}$, $y = \frac{E}{E_p}$ are symmetrically compared to rainbow function 2 $Q_0(y) = 1$ and $Q_1(y) = \sqrt{1-\epsilon y}$, $y = \frac{E}{E_p}$ (constant mass and PDM).

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