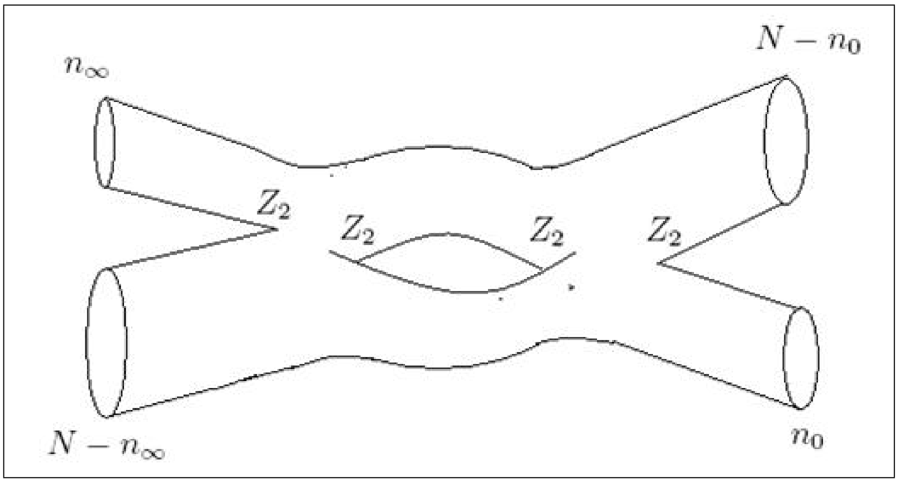

Scattering Amplitudes in Dijkgraaf-Verlinde-Verlinde Matrix String Theory

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Abstract

This master thesis reviews DVV matrix string theory and it is IR limit S_N orbifold sigma model by stuying the scattering amplitudes it defines. The conjectured correspondence between second quantized Type IIA string theory and IR Matrix string theory in the large N limit is discussed and some evidence through explicit calculation of scattering amplitudes in three level is mentioned. General discussion of symmetric product spaces and elliptic genus is included for a clear understanding of orbifold CFT model and is linked to one loop amplitude of string theory. Merely, the outermost aim of this work is calculating the one loop amplitudes in DVV matrix string theory, hence to check if this perturbative picture really matches with the string theory calculations. Some difficulties in this calculation is identified, even though final formula is not achieved the equations are analyzed and simplified for further studies.

Introduction

Since the first idea of string theory appeared in the context of dual models, it has attracted lots of attention because of it's simplicity and geometrical beauty. Also it can be said that it is the most promising theory of unification of fundamental forces. At least now that is what we have as the best explanation till somebody comes up with a simpler and more beautiful theory of quantum gravity. We believe string theory still worth studying.

Since the it apperared the techniques including supersymmetry and geometry was constructed in diverse ways and physicists discovered with five consistent superstring theories in 10D namely Type I, Type IIA ,Type IIB, Heterotic $E_8 \times E_8$ and heterotic SO(32). In 90s physics community got puzzled of having five different theories of a so called unified theory. Taking into account the fact that this theories are connected by duality symmetries in 1995 Horava and Witten proposed a new idea, that is to say they claimed that five theories were the perturbative limits of a unifying 11 dimensional theory: M-theory. By definition M-theory corresponds to the non-perturbative theory of typeIIA string theory up on compactification of the 11th dimension. From another point of view M-theory is believed to give 11D SUGRA as a low energy limit, which is equivalent to typeIIA SUGRA up on compactification on a circle. The puzzle is to find a non-perturbative theory of type IIA strings on the other side of the square 2. On the other hand Polchinski, the same year, proposed theory of D-branes, where strings start and end. The low energy interactions of D-branes is explained by 10D $\mathcal{N} = 1$ SYM and 10D SYM is believed to be the non-perturbative theory of typeIIA strings. In order to recognize the correspondence one should reduce the theory to 2D to obtain $\mathcal{N} = 8$ SYM. However, the correspondence is not manifest, one has to investigate the strong coupling limit of this theory to obtain a perturbative string theory. The story of DVV matrix theory starts here.

In 1996 Banks-Fishler-Seiberg and Suskind proposed a model with D-particle degrees of freedom represented with the matrices of rank N. This theory is believed to be equivalent to light-front gauge formulation of the M-theory. By compactifying the M(atrix) theory on a circle in the 9th direction then using T-duality and last fliping 11th and 9th dimensions

Figure 1: M-Theory

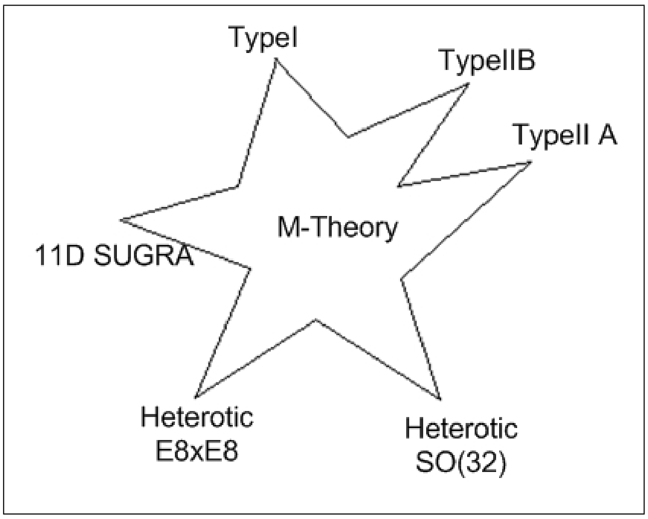
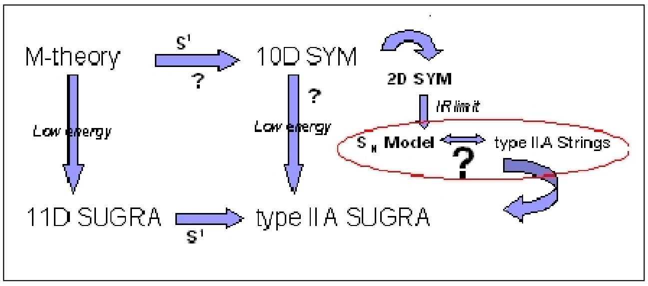


Figure 2: The Big Picture



one can obrain 2D SYM action in the Hamiltonian formulation. This is supposed to be the non-perturbative formulation of the type IIA string theory. In 1997 Dijkgraff Verlinde and Verlinde conjectured that there is an IR fixed point where theory becomes 2D free CFT on $S_N R^8$ orbifold target space. The coordinates of the eigenvalues of the matrices in 2D SYM was identified with the components of the second quantized type IIA strings in Green-Schwarz lightcone formulation. Moreover they proposed that this theory is equivalent to interacting type IIA string theory in the large N limit, by constructing a least irrelevat SUSY and SO(8) invariant vertex that creates different worldsheet topologies.

After DVV conjecture there has been lots of work in various part of this problem. One of them is problem of showing that these two theoris define the same amplitudes, namely the scattering amplitudes of the physical particles are the same for both theories. In the same lines it has been already shown that in three level the amplitudes of the S_N orbifold perturbation theory coincide with the string theory amplitudes by Arutyunov and Frolov. It is also an intersting problem to show this correspondence in loop level. Loop amplitudes constitute very important information about the nature of the string theory such as modulo invariance, we believe this will be enlightening.

Our interest in this thesis lies in the lines of showing the correspondence of IR physics of the 2D SYM and string theory via calculating the one loop partition function and check the modulo invariance. Unfortunately the methods we used turned out to give a very complicated picture even for the simplest loop amplitude. Throughout the thesis the orbifold CFT methods constructed by Dixon et all. is used. This so called Stress Energy Tensor (SET) method provide us with a tool to calculate the CFT correlators on the orbifolds by using the analytical properties. In this respect the DVV vertex effectively acts as joining and splitting three vertex of strings, and this was provided by changing the analytical properties of the world-sheet locally by twist fields. We will mention how one can construct a string theory by using the twist fields formulation. Calculating the loop amplitudes is still an open problem. Merely we tried to analyze the amplitudes in the context of SET method and simplified the picture. We hope this work will shed light on the quest for a final formula.

Table of Contents

Chapter 1

String Perturbation Theory and Scattering Amplitudes

1.1 Introduction

In the following chapter we will try to present the string perturbation theory techniques in order to give some flavour of superstring theory amplitudes. We will start with Polyakov path integral, for it's geometrical manifestation of quantum levels and calculational appropriateness. However, we will also point out the equvalence of the lightcone and Polyakov pictures by the end of the chapter in the context of Riemann Surface theory. Throughout the chapter our main emphasis will be on scattering amplitudes of closed (super)string theory, since DVV matrix model in the strong coupling limit only gives rise to second quantized closed strings[14]. We are going to restate some crucial facts about the tree and one loop amplitudes of bosonic and superstring theories. As reader can evaluate, we can't go into details of superstring theory in this thesis. However, one can consult references [1]-[6] for more detailed background material.

The simplest string theory is the theory of bosonic D=26 free strings on flat background(a theory that can be interpreted as of pure gravity without matter), non-realistic though, for its negative mass square ground state, the tachyons. Generically the theory is anomalous and anomaly cacelation leads to $D = 26$.In the meanwhile, it manifests lorentz covariance after fixing the conformal gauge[2]. We will use this theory for building up general techniques. On the other hand there are several string theories with matter couplings (fermions), and with spacetime supersymmetry such as Type-I, Type-II and Heterotic string Theories. One can calculate amplitudes of these theories paying a little bit more effort using the techniques constructed for bosonic strings.

1.2 Bosonic Closed String Perturbation Theory

Bosonic string theory is simply an embedding of two dimensional world-sheet to 26-dim space-time. In this respect, it is completely geometrical and this embedding can be realized in many inequivalent ways. The physical content of the theory begins with the idea that this embedding must obey some action principle. In complete analogy with relativistic point particle case Nambu and Goto proposed that this must be a minimal area principle (an analog of minimal action principle): area swept by the propagating string must be minimized to obtain classical solution. This works very well for classical case, nevertheless the quantization is intricate. On the other hand, the NG action is shown to be equivalent to an action proposed by Polyakov by using a metric on world-sheet. This action gives back NG action by implementing the equation of motion for worl-sheet metric γ^{ab} ([1]-[6])

$$\mathcal{S}_P[X,\gamma] = \int_M d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu \tag{1.2.1}$$

where M denotes the world-sheet manifold, γ^{ab} is world-sheet metric and γ denotes $Det(\gamma^{ab})$. One can observe that this action has following symmetries:

1- D dimensional Poincaré Invariance:

$$X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\mu(\tau,\sigma) + a^\mu,$$

$$\gamma'_{ab}(\tau,\sigma) = \gamma_{ab}(\tau,\sigma) \tag{1.2.2}$$

2- Diffeomorphism Invariance:

$$X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma),$$

$$\partial_a \sigma'^c \partial_b \sigma'^d \gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma) \tag{1.2.3}$$

for new coordinates $\sigma'^a(\tau,\sigma)$

3- Two-dimensional Weyl invariance (conformal symmetry)

$$X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma),$$

$$\gamma'_{ab}(\tau,\sigma) = exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma) \tag{1.2.4}$$

The conformal symmetry is rather special for string theory (generally for conformal field theories)and mostly missing in higher dimensional extended object actions. The symmetry implies that conformally equivalent metrics corresponds to same embedding in space-time, merely this means physical parameters of the theory is ignorant to the length scale of the strings.

There are two basic facts we want to emphasize at this point. First of all, one can discuss possible extension of this action by the Gauss-Bonnet term:

$$\chi = \frac{1}{4}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}} R \tag{1.2.5}$$

This term satisfies the symmetries up to a boundary term:

$$(-\gamma')^{\frac{1}{2}} R' = (-\gamma)^{\frac{1}{2}}(R - 2\nabla^2\omega)$$

the variation is a total derivative, because $(-\gamma)^{\frac{1}{2}}\nabla_a v^a = \partial((-\gamma)^{\frac{1}{2}} v^a)$, for any v^a. However, this topological term will turn out to be essential when we discuss the string perturbation theory,since it constitutes the grading parameter of perturbation levels. For the time being we just add this term to the action.

The second possible discussion can be made on which boundary condition one can possibly choose. The variation of action 1.2.1 with respect to X^μ is :

$$\delta\mathcal{S}_p = \frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^{\ell} d\sigma (-\gamma)^{\frac{1}{2}} \delta X^\mu \nabla^2 X_\mu - \frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty} d\tau (-\gamma)^{\frac{1}{2}} \delta X^\mu \partial^\sigma X_\mu \Big|_{\sigma=0}^{\sigma=\ell} \tag{1.2.6}$$

vanishing of first term gives the equation of motion $\nabla^2 X^\mu = 0$. Second term is a boundary term which vanishes under different circumstances. One of this gives rise to open string, this is so called Neuman boundary conditions:

$$\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau,\ell) = 0. \tag{1.2.7}$$

Another possibility is periodic (closed string) boundary conditions:

$$\partial^\sigma X^\mu(\tau,0)=\partial^\sigma X^\mu(\tau,\ell) \tag{1.2.8}$$

$$X^\mu(\tau,0)=X^\mu(\tau,\ell) \tag{1.2.9}$$

$$\gamma_{ab}(\tau,0)=\gamma_{ab}(\tau,0) \tag{1.2.10}$$

This is the boundary conditions we are going to use from now on for bosonic strings otherwise the contrary stated, since often we deal with the closed strings in this thesis.

Last two local symmetries above give rise to the gauge degrees of freedom, which one must fix to obtain a physically relevant theory. We will discuss the intricate problem of gauge fixing in the context of Polyakov path integral and Faddev-Popov determinant. We will emphasize the geometrical interpretation in terms of Conformal Killing Vectors (CKV) using complex geometry. Thus, we will use Wick rotated form of ($\tau \to i\tau$)the τ coordinate in order to obtain Euclidean signatured metric in Polyakov action. This is extremely essential in String Perturbation Theory since 2-dim compact connected manifolds are perfectly classified by their genus. (There is no proof of the fact that Lorenzian and Euclidean theories of quantum string has a correspondence a full theory. Nevertheless, it has been shown by explicit calculations that they define same amplitudes[2].)

After giving an introduction to classical string action we deal with the quantum strinf theory. As mentioned there are many ways to attain a physically relevant quantization but may be the most effective one, Polyakov path integral, will be our main concern. The path integral merely is the statistics of random paths weighted by a function of the action. In our context we want to sum up all distinct embedding of the worldsheet to spacetime. The perturbation parameter here is genus such that every order in genus contribute less and less to the perturbative expansion (Observe that this is because of Gauss-Bonnet term χ coupled to Polyakov action by g_s). This naive expectation will be discussed later when we start the intricate theory of defining an appropriate measure for the path integral i.e. moduli parameters and loop amplitudes in sections (1.3) and (1.4).

A typical amplitude can be written in the following form

$$\mathcal{A} = \sum_{over-all-histories} \textit{(relevant physical objects)} \exp(-\mathcal{S}_{classical}(geometry))$$

This clumsy way of explaining the general picture will be clarified in context later. However, no matter what the theory at hand is meaning of "over all histories" and "geometry" is not very clear. Indeed, if we only consider the smooth manifolds ones by restricting scalar fields X to be an embedding to the target space with an arbitrary worldsheet metric

$$X^\mu : \text{2-D World-sheet} \hookrightarrow \text{26-D Target Space}$$

then this is a well behaved connected metrizable surface. Well known examples are the sphere and torus for oriented case, and all other compact connected oriented surfaces of genus-0 and genus-1 are homeomorphic to these respectively. Since we require our fields to be embedding than the surfaces we are dealing with must be diffeomorphic. One can immediately see why we are looking for a diffeomorphism invariance in the action, namely having a diffeomorphism invariant theory provides us with the luxury of specifying our perturbation expansion by genus. On the other hand this symmetry should be factored out from the measure by a choice of the gauge. This was explained by a form of very familiar Faddeev-Popov theory, which appears in the path integrals with gauge symmetries [2-5].

In two dimensions the action manifests Weyl invariance, as well as diffemorphism invariance, so this symmetry also must be factored out from the measure. The a proper amplitude is given by the following integral, for bosonic string theory

$$\mathcal{A} = \sum_{topologies} \int \frac{[\mathfrak{D}X\mathfrak{D}\gamma]}{V_{Diffeo \times Weyl}} << Vertex >> exp\Big(\int_M d\tau d\sigma (-\gamma)^{1/2} (\frac{1}{4\pi\alpha'} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi} R)\Big) \tag{1.2.11}$$

At this point we have to emphasize that this path integral is not well defined because of the Lorentzian signature of worldsheet metric. It can be overcome by Wick rotation $\tau \to -i\tau$. This trick provides us with Euclidean path integral which is in perfect Gaussian form. Only intricate job is definition of the measure for integral to be well behaved and finite in an appropriate measure. (From now on we denote the Euclidean metric by "g_{ab}".)

We haven't yet specified what "vertex" refers to. In fact we replaced it with the expression "physically relevant objects". We will see in details in the following pages that "vertex" is an analog of the vertex operators in quantum field theory. It is a source of any kind of physical excitations. This is hard to imagine without referring to "Hilbert Space Techniques" i.e. second quantized formalism of the creation and annihilation of particles by applying the oscillator modes to ground state. It won't be discussed here, reader can consult the introductory text [1] at this point. However, we will try to justify the idea of vertex in the context of Polyakov path integral.

A general form of a vertex operator can be written as:

$$\mathcal{V}_{k_1 \ldots k_n} = \prod_{i=1}^{n} \int d^2\sigma_i (g(\sigma_i))^{1/2} \mathcal{V}[k_1, ..., k_n; \sigma_1, ..., \sigma_n] \tag{1.2.12}$$

where all local operators are normal ordered.

It is integrated on the world-sheet to obtain a diffeomorphism invariant theory. This creates particles through the oscillatory modes of the string. Using this form, a general amplitude can be expressed as follows:

$$\mathcal{A}_{k_1 \ldots k_n} = \sum_{topologies} \int \frac{[\mathfrak{D}X\mathfrak{D}g]}{V_{Diffeo \times Weyl}} exp\Big(- \mathcal{S}_P - \chi \Big) \prod_{i=1}^{n} \int d^2\sigma_i (g(\sigma_i))^{1/2} \mathcal{V}[k_1, ..., k_n; \sigma_1, ..., \sigma_n] \tag{1.2.13}$$

the diffeomorphism and Lorentz invariant vertex operator of first ground state particles, tachyons with $m^2 = -k^2 = -2$ is given by

$$\mathcal{V}_0[k] = g \int h^{1/2} d^2\sigma : e^{ik.X} : \tag{1.2.14}$$

where "::" denotes the normal ordering and the vertex operator for massless states, namely graviton, photon and dilaton (scalar field)is

$$\mathcal{V}_1[k, S_{\mu,\nu}] = g \int d^2\sigma g^{1/2} \{ g^{ab} S_{\mu\nu} [: \partial_a X^\mu \partial_b X^\nu e^{ik.X} :] + 2\phi R [: e^{ik.X} :] \} \tag{1.2.15}$$

this are the two vertices that we will use throughout the thesis. We will define more vertex operators in bosonic string theory if necessary later.

1.3 Gauge Fixing and Moduli Spaces

In this section we will explain how to fix the gauge and define a well behaved path integral. The fact that $diff \times Weyl$ has infinite volume causes enormous over-counting, which makes integral 1.2.13 ill-defined. However, after fixing the gauge the integral will make sense mathematically. We need to divide the measure by the volume of gauge group by which one can regularize the integral:

$$\mathcal{Z} = \int \frac{[\mathfrak{D}X\mathfrak{D}g]}{V_{gauge}} exp(-\mathcal{S}) \tag{1.3.1}$$

we will realize the required adjustment to integral by gauge fixing and integrating with respect to one gauge slice that only cuts the gauge group ones. We will obtain an appropriate choice by using Faddeev-Popov method. It will be shown explicitly that one cannot fix all the degrees of freedom by choosing a gauge i.e. holomorphic(anti-holomorphic) transformations are still a symmetry of three level amplitudes (amplitudes defined on sphere). (in Canonical Quantization, this gives rise to Virasoro algebra and one should be factored out the corresponding descendent states from the Hilbert space). This additional symmetry will be represented as an integral over two Grassman fields b_{ef} and c^d. This is a very crucial point to be discussed in detail since has a deep geometrical meaning.

We will denote the gauge choice (sometimes called fiducial metric) as $\hat{g}_{ab}$. A very appropriate gauge is flat Euclidean metric:

$$\hat{g}_{ab} = \delta_{ab} \tag{1.3.2}$$

One can only use the diffemorphism degree of freedoms to bring the metric in the following form:

$$\hat{g}_{ab} = exp(2\omega(\sigma))\delta_{ab} \tag{1.3.3}$$

This form of the metric manifests the holomorphic (anti-holomorphic) symmetry. It is convenient here to introduce complex parameters $z = \sigma + i\tau$ such that by the holomorphic transformation $z \to f(z)$ metric transforms as:

$$ds'^2 = exp(2\omega)|\partial_z f|^{-2} dz' d\overline{z}' \tag{1.3.4}$$

making the choice $\omega = ln|\partial_z f|$ leaves the metric invariant. This extra symmetry does not conflict with the Previous counting of degrees of freedom since it has zero volume compared to $diff \times Weyl$

One can discuss the fate of Gauss-Bonnet term as well. The variation with respect to Weyl transformations is

$$(-\gamma')^{\frac{1}{2}} R' = (-\gamma)^{\frac{1}{2}} (R - 2\nabla^2 \omega) \tag{1.3.5}$$

To set $R' = 0$ one must choose ω as the solution of $2\nabla^2\omega = R$. This is always possible at least locally.

Now think of a variation of metric and the fields with following inner products:

$$||\delta g|| = \int d^2\sigma \sqrt{g} g^{ab} g^{cd} \delta g_{ac} \delta g_{cd} \tag{1.3.6}$$

$$||X^\mu|| = \int \sqrt{g} \delta X^\mu \delta X_\mu \tag{1.3.7}$$

observe that this norms are not Weyl invariant. This means that in the quantum theory generically Weyl factor will couple to the fields.

The variation of the metric under infinitesimal diffemorphisms and Weyl rescaling can be decomposed into

$$\delta g_{ab} = \nabla_a \xi_b + \nabla_a \xi_b + 2\Lambda g_{ab} = (\tilde{P}\xi)_{ab} + 2\tilde{\Lambda} g_{ab} \tag{1.3.8}$$

where we redefine P and Λ as:

$$\hat{P}\xi = \nabla_a \xi_b + \nabla_b \xi_a - g_{ab} \nabla^a \xi_a \tag{1.3.9}$$

$$\hat{\Lambda} = g_{ab}(2\Lambda + \nabla^a \xi_a) \tag{1.3.10}$$

New integration measure can be written as

$$\mathfrak{D}g = \mathfrak{D}\hat{P}\xi \mathfrak{D}\tilde{\Lambda} = \mathfrak{D}\xi \mathfrak{D}\Lambda \Big| \frac{\partial(\hat{P}\xi, \tilde{\Lambda})}{\partial(\xi, \Lambda)} \Big| \tag{1.3.11}$$

where the jacobian is

$$\Big|\frac{\partial(\hat{P}\xi,\tilde{\Lambda})}{\partial(\xi,\Lambda)}\Big| = \Big|det\begin{pmatrix} \hat{P} & 0 \\ * & 1 \end{pmatrix}\Big| = |detP| = \sqrt{det|\hat{P}\hat{P}^{\dagger}|} \tag{1.3.12}$$

The Weyl anomaly is cancelled when we set dimension to D=26. This is critical dimension for the Bosonic string. Assuming that the operator $\hat{P}\hat{P}^{\dagger}$ has no zero modes (which will be investigated separately), the path integral can be written as:

$$\mathcal{Z} = \sum_{topologies}\int \mathfrak{D}X^{\mu}\sqrt{det|\hat{P}\hat{P}^{\dagger}|}exp\Big(-S_p(\hat{g},X) - \lambda\chi(g)\Big) \tag{1.3.13}$$

a problem arise when $\hat{P}\hat{P}^{\dagger}$ has zero modes. Those reparametrizations satisfying

$$\hat{P}\xi^* = 0 \tag{1.3.14}$$

are called conformal Killing vector. This is an extra symmetry of the action which cannot be gauged away by fixing the metric since it leaves the metric invariant. This should be fixed separately.

On the other hand another ambiguity arise when :

$$(\hat{P}^{\dagger}t^*)_b = -2\nabla^a t^*_{ab} = 0 \tag{1.3.15}$$

where t^*_{ab} is a symmetric traceless rank two tensor. This corresponds to the deformations of metric which cannot be compensated by reparametrizations or Weyl rescalings. This space is called Teichmuller space or moduli space and parameter spanning it is Teichmuller parameter or modulus .

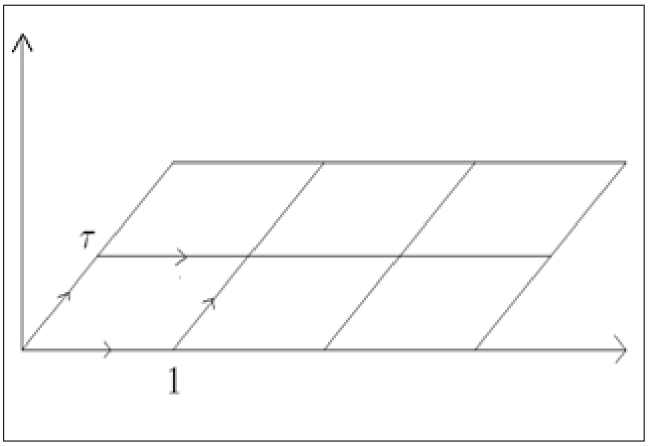
The choice done to fix the extra diffeomorphism symmetry does not effect this metric deformations since we have them transversal to each other:

$$(\xi,\hat{P}^{\dagger}t^*) = (\hat{P}\xi,t^*) = 0 \tag{1.3.16}$$

There is a well known relation between dimension of the Teichmuller space and number of the CKVs. This is given by following relation:

$$\frac{1}{2}(dim(Ker\hat{P}) - dim(Ker\hat{P}^{\dagger})) = 3(g-1) \tag{1.3.17}$$

Figure 1.1: The torus is a cotiant of $\mathbb{C}$ and lattice $\mathbb{L}$, $\mathbb{C}/\mathbb{L}$.



which is a special case of famous Riemann-Roch Theorem. LHS is divided to 2 since we consider complex parameters here, this result can be explicitly stated by the following table:

Table 1.

Genus	number of zeros of $\hat{P}$	number of zeros of $\hat{P}^\dagger$
0	3	0
1	1	1
≥ 2	0	3g-3

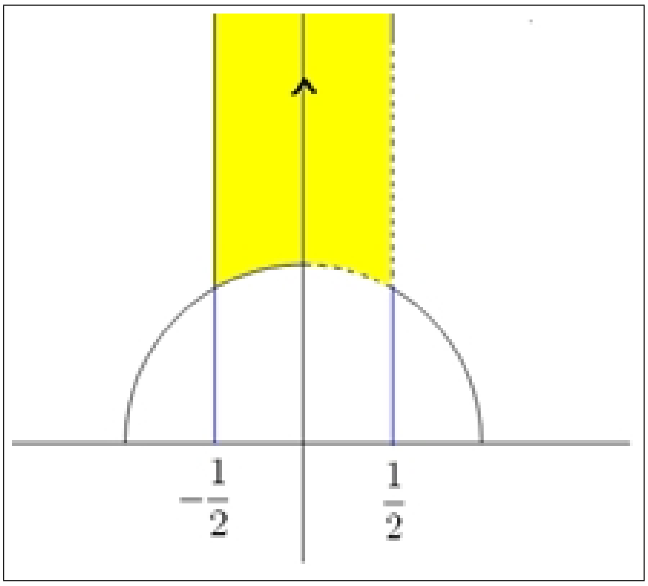
The Teichmuller parameter spanning the metric deformations that cannot be compensated by $diff \times weyl$ has an automorphism group in general. This is the group of area preserving automorphisms. To avoid over counting, we should factor out the automorphism group from moduli space. The domain of moduli parameter is called fundamental region denoted by F_0:

$$\mu_g = \frac{\Gamma_g}{diff \times Weyl} \tag{1.3.18}$$

where Γ_g is space of all metrics.

It is useful at this point to investigate the moduli one loop amplitudes for future purposes. Any kind of one loop diagram is diffemorphic to torus with punctures which correspondes to external states. The torus can be represented by a lattice as in 1.1. On the other hand this representation is far from being unique. First of all, the lattice is translation invariant, we

Figure 1.2: The moduli space of the string one loop amplitude.



can choose origin at an arbitrary point (corrsponding to CKV of loop amplitudes). Second, it is invariant under $PSL(2,\mathbb{Z})$ transformations (modulo invariance). We should mode out this transformations from upper half plane to avoid over-counting. The fundamental domain $H/PSL(2,\mathbb{Z})$ is

$$F_0 = \left[\tau := \tau_1 + i\tau_2 : \tau_2 > 0, -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, |\tau| = 1\right]$$

it looks as in figure 1.2.

1.4 Tree and One Loop Amplitudes of Bosonic Strings

The tree amplitudes are amplitudes on the sphere. Sphere has three holomorphic CKV which spans the automorphism group $SL(2,\mathbb{C})$ and 3 anti-holomorphic CKV which spans $SL(2,\mathbb{C})$, as well (by modding out discreet symmetry $z \to -z$ one gets $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I$). This means we are free to set three of our points to 0,1,∞, which is the convention used in [1]-[6]. Consequently one, two and three point functions are just constants and irrelevant for our purposes.(We just remark that two point function vanishes so the cosmological constant at tree level is vanishing.)First nontrivial amplitude is four point amplitude so we will assume the number of vertices $n \geq 4$.

A general metric is conformally equivalent to the standard metric on the sphere i.e.

$$g_{ab} = \frac{2\delta_{ab}}{1+|z|^2} \tag{1.4.1}$$

this is the result of the fact that the moduli space is trivial i.e. all degrees of freedom of metric is canceled by $diff \times Weyl$ (Check Table-1).

For instance a general n-point three amplitude for tachyonic vertices can be given without considering conventions above as follows

$$<< \mathcal{V}(k_1)...\mathcal{V}(k_n) >>= (2\pi)^{26}\delta(k_{tot}) \prod_{j=1}^{n} \int \frac{2d^2 z_j}{1+|z_i|^2} exp\Big[-\frac{1}{2}\sum_{i,j=1}^{n} k_i.k_j \mathcal{G}(z_i, z_j)\Big] \tag{1.4.2}$$

However, there is another subtlety here: the zero modes of the fields X^μ are making the integral 1.2.13 divergent. We integrated out this modes. As a result we got a δ-function in 1.4.2 which takes care of the conservation of momentum. This is the case since total on-shell momentum must be zero. We should consider the propagator equation by excluding the zero modes as follows

$$\nabla^2 \mathcal{G}(z.z') = 4\pi\delta(z, z') - 4\pi.C \tag{1.4.3}$$

one can check that the right propagator can be given as

$$\mathcal{G}(z, z') = -ln\frac{|z-z'|}{(1+|z|^2)(1+|z'|^2)} \tag{1.4.4}$$

up on using this in (1.33) and considering $k_i^2 = 2$ the general tachyonic amplitude becomes

$$<< \mathcal{V}(k_1)...\mathcal{V}(k_n) >>= (2\pi)^{26}\delta(k) \prod_{j=1}^{n} \int 2d^2 z_j \prod_{i<j} |z_i - z_j|^{2k_i.k_j} exp\Big[-\sum_{i=1}^{n} \mathcal{G}(z_i, z_i)\Big] \tag{1.4.5}$$

the singulary that arises from the coinciding points should be regularized by $PSL(2, \mathbb{C})$invariant way. This requires setting $\mathcal{G}(z_i, z_i)$ to a constant λ independent of z_i. The remaining amplitude is $PSL(2, \mathbb{C})$ invariant and thus divergent due to the infinite volume of $Ker\hat{P}$. If we fix three points and cancel the volume arising from CKV, we can get a well defined tree amplitude

$$< \mathcal{V}(k_1)...\mathcal{V}(k_n) >= c(2\pi)^{26}\delta(k)\prod_{j=1}^{n-3}\int d^2z_j \prod_{1\leq i<j\leq n-1}|z_i - z_j|^{2k_i.k_j} \tag{1.4.6}$$

by setting n=4 we obtain well known Virasoro-Shapiro amplitude

$$< \mathcal{V}(k_1)\mathcal{V}(k_2)\mathcal{V}(k_3)\mathcal{V}(k_4) >= c(2\pi)^{26}\delta(k)\int d^2z|z|^{2k_1.k_2}|1-z|^{2k_1.k_3} \tag{1.4.7}$$

The one loop case is completely different since there is only one CKV (i.e.translations) and one moduli which is $PSL(2,\mathbb{Z})$ invariant(it is the largest area preserving diffeomrophism group of torus). As discussed in previous section by moding out $PSL(2,\mathbb{Z})$ we obtain fundamental region

$$F_0 = \Big[\tau := \tau_1 + i\tau_2 : \tau_2 > 0, \frac{-1}{2} \leq \tau_1 \leq \frac{1}{2}, |\tau| = 1\Big] \tag{1.4.8}$$

n-point one loop scattering amplitude is given by the following integral in [6]:

$$< \mathcal{V}(k_1)...\mathcal{V}(k_n) >= \int_{F_0}\frac{d^2\tau}{\tau_2^2}\sqrt{det(\hat{P}^\dagger\hat{P})}\Big[\frac{4\pi^2}{\tau_2}det(\nabla^2_{\hat{g}})\Big] << \mathcal{V}(k_1)...\mathcal{V}(k_n) >> \frac{1}{Vol(Ker\hat{P})} \tag{1.4.9}$$

the determinants are evaluated in [6] and given by

$$det(\hat{P}^\dagger\hat{P}) = det(2\nabla^2_{\hat{g}}) = \frac{1}{2}det(\nabla^2_{\hat{g}}) \tag{1.4.10}$$

$$det(\nabla^2_{\hat{g}}) = {\tau_2}^2|\eta(\tau)|^4 \tag{1.4.11}$$

where Dedekind eta function η is defined in appendix A. Moreover one finds the $Vol(Ker\hat{P}) = 2\tau_2$, so that the Weyl-Peterson measure is

$$d\mu_{wp} = \frac{d^2\tau}{\tau_2^2}\sqrt{det(\hat{P}^\dagger\hat{P})}\Big[\frac{4\pi^2}{\tau_2}det(\nabla^2_{\hat{g}})\Big]\frac{1}{Vol(Ker\hat{P})} = \frac{d^2\tau}{8\pi^2\tau^2}\frac{1}{(4\pi^2\tau_2)^{12}}|\eta(\tau)|^{-48} \tag{1.4.12}$$

with the help of transformation law of η function it is easy to check that this measure is $PSL(2,\mathbb{Z})$ invariant.

The propagator on torus is a logarithm of doubly periodic function which has simple zero. It shown in [6] that it has the following form:

$$\mathcal{G}(z,z';\tau) = -ln\Big|\frac{\theta_1(z,z';\tau)}{\theta_1'(0;\tau)}\Big|^2 - \frac{\pi}{2\tau_2}(z-\bar{z}-z'+\bar{z}')^2 \quad (1.4.13)$$

The one loop amplitude is then

$$< \mathcal{V}(k_1)...\mathcal{V}(k_n) >= \delta(k)\int_{F_0}\frac{d^2\tau}{2\tau^2}\frac{1}{(\tau_2)^{12}}|\eta(\tau)|^{-48}\prod_{j=1}^{n}\int d^2z_j\prod_{i<j}\mathcal{F}(z_i,z_j)^{k_i.k_j} \quad (1.4.14)$$

where function $\mathcal{F}$ is given as

$$\mathcal{F}(z_i,z_j) = \Big|\frac{\theta_1(z,z';\tau)}{\theta_1'(0;\tau)}\Big|^2 exp\Big(\frac{\pi}{2\tau_2}(z-\bar{z}-z'+\bar{z}')^2\Big) \quad (1.4.15)$$

We pay special attention to vacuum amplitude indeed since during research the calculation of 4-p loop amplitudes turned out to be technically complicated we will focus on calculation of one loop vacuum amplitude in the third chapter. The scalar partition function can be derived from the path integral just given by the argument in Weyl-Petersen measure. However, we will follow approach of [2], for technical reasons. The following calculation gives a special case of elliptic genus, which is discussed in the second chapter.

The partition function is defined as follows

$$< 1 >_{T^2}(\tau) = \mathcal{Z}(\tau) = Tr[\exp 2\pi i\tau_1 P - 2\tau_2 H] \quad (1.4.16)$$

here physical reasoning is that we have two first class constraints that govern the symmetry of the theory, namely level matching condition $P = L_0 - \bar{L}_0$ generating rigid σ transformations and hamiltonian $H = L_0 + \bar{L}_0 - \frac{1}{24}(c+\bar{c})$ generating world-sheet time τ (don't mix it up with modulus τ) translations. We basically identify the ends which leads to taking the trace. Such a trace weighted by the free energy of the system is called partition function as in statistical mechanics.

$$\mathcal{Z}(\tau) = (q\bar{q})^{-d/24}tr[q^{L_0}\bar{q}^{\bar{L}_0}] \quad (1.4.17)$$

with $q = \exp(2i\pi\tau)$(this is the τ of complex structure.) and $\bar{q}$ is complex conjugate.

The trace breaks up into sums over occupation numbers $N_{\mu n}$ and $\tilde{N}_{\mu n}$ in the zero spin sector of string spectrum and integral over k_ν. The partition function is then given by

$$\mathcal{Z}(\tau) = V_d(q\bar{q})^{-d/24} \int \frac{d^d k}{2\pi} \exp\left(-\pi\tau_2\alpha' k^2\right) \prod_{\mu,n} \sum_{N_{\mu n},\tilde{N}_{\mu n}} q^{nN_{\mu n}} \bar{q}^{n\tilde{N}_{\mu n}} \tag{1.4.18}$$

here V_d the volume of the space-time and basically comes from the normalization of the momentum integral $V_d(2\pi)^d \int d^d k$. The various sums are geometric,

$$\sum_{N=0}^{\infty} q^{nN} = \frac{1}{1-q^n} \tag{1.4.19}$$

so we obtain

$$\mathcal{Z}(\tau) = iV_d \mathcal{Z}_X(\tau)^d \tag{1.4.20}$$

where

$$\mathcal{Z}_X = (4\pi^2\alpha'\tau_2)^{-1/2} |\eta(\tau)|^2 \tag{1.4.21}$$

with

$$\eta(\tau) = q^{1/4} \prod_{n=1}^{\infty} (1-q^n) \tag{1.4.22}$$

one can immediately observe that this is $SL(2,Z)$ invariant.

By considering CFT of the ghost states on torus the gauge fixed vacuum amplitude is given by $d = (D-2) = 24$.

1.5 Fermionic Strings and Supersymmetry

In this section we will try to present some crucial aspects of the Superstrings and Heterotic Strings. As stated before DVV matrix theory gives rise to only closed strings so we will especially focus on the theories with closed strings such as TypeII. Furthermore, we will try to do the same treatment, as in section 3, to discuss amplitudes and moduli space of superstrings.

Following is the generalization of bosonic Polyakov action to a supersymmetric action, via introduction of "vielbein" $g^{\alpha\beta} e^a_\alpha e^b_\beta = \eta^{ab}$ (with Greek curved incices and Latin flat indices)in order to define world-sheet spinors:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma e\Big(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha(\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi_\mu\Big) \qquad (1.5.1)$$

We denoted the determinant of "vielbein" by e. Metric $g^{\alpha\beta}$ is function of vielbein e^a_α and ρ^α are gamma matrices in two dimensions.This action has five local symmetries i.e. local supersymmetry, Weyl invariance, Super-Weyl invariance, local Lorentz symmetry, local diffeomorphisms [1-6]. We have a similar situation as bosonic strings: one has to choose a gauge to avoid over-counting. By an appropriate choice of gauge, superconformal gauge, $e^a_\alpha = e^\phi \delta^a_\alpha$ and $\chi_\alpha = \rho_\alpha \lambda$ supersymmetric action simplifies to following CFT action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big(\partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\Big) \qquad (1.5.2)$$

with corresponding equation of motion for X^μ and ψ^μ

$$\partial_\alpha \partial^\alpha X^\mu = 0 \qquad (1.5.3)$$

$$\rho^\alpha \partial_\alpha \psi^\mu = 0 \qquad (1.5.4)$$

for closed strings this corresponds to right and left moving solutions $X^\mu = X^\mu_+(z) + X^\mu_-(\bar{z})$ and $\psi^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi^\mu_-(\bar{z}) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \psi^\mu_+(z)$ and we are left with two conformal killing equations

$$(P\xi)_{\alpha\beta} = 0 \qquad (1.5.5)$$

$$\Pi(\epsilon)_\beta = \rho^\alpha \rho_\beta \partial_\alpha \epsilon = 0 \qquad (1.5.6)$$

where ϵ is a local two dimensional Majorana spinor (real spinor in other words).

The zero modes of the adjoint of this operators corresponds to metric degrees of freedom that cannot be removed by reparametrization+Weyl.

$$(P^\dagger \tau)_{\alpha\beta} = 0 \qquad (1.5.7)$$

$$(\Pi^\dagger \sigma)_{\alpha\beta} = 0 \qquad (1.5.8)$$

which are moduli and supermoduli respectively.

This theory has an symmetry algebra that is analogous to Virasoro algebra which takes care of ruling out unphysical states. The constraints that govern the algebra is $T_{\alpha\beta} = 0$ and $G_\alpha = 0$, stress energy tensor generating reparametrizations $\xi(z)$ and and supercurrent generating supersymmetry transformations $\epsilon(z)$ respectively. The cancellation of the Weyl anomaly in super Virasoro algebra requires to set $D = 10$ and $a = \frac{1}{2}$ of quantum superstrings.

Moreover Lorentz invariance allows following boundary conditions for closed superstrings:

$$\psi_+^\mu(e^{2\pi i}z) = \pm\psi_+^\mu(z) \tag{1.5.9}$$

$$\psi_-^\mu(e^{-2\pi i}\bar{z}) = \pm\psi_-^\mu(\bar{z}) \tag{1.5.10}$$

We will use the standard terminology

- Periodic boundary conditions which gives rise to space time fermions are called "Ramond" denoted by R
- Anti-periodic boundary conditions which gives rise to spacetime bosons are called "Neveu-Schwarz" denoted by NS.

The above construction considers world-sheet supersymmetry, on the other hand it is proved by Green and Schwarz that this corresponds to space-time supersymmetry in light-cone gauge. This can be shown by bosonization and re-fermionization of the spinors of SO(8) thanks to the triality symmetry of irrediucible representations such as $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ [1]. Moreover, the full supersymmetric action cannot have arbitrary number of susy's. The most number one gets is $\mathcal{N} = 2$ to restore κ symmetry (which is a local susy). One can than get actions with $\mathcal{N} = 1$ supersymmetry by letting one of spinors θ to be zero. A general superstring action is then [1]

$$S_{susy} = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{ab}\Pi_a\Pi_b + \frac{1}{\pi}\int d^2\{-i\epsilon\partial_a X^\mu(\bar{\theta}^1\Gamma^\mu\partial_b\theta^1 - \bar{\theta}^2\Gamma^\mu\partial_b\theta^2) + \epsilon^{ab}\bar{\theta}^1\Gamma^\mu\partial_a\bar{\theta}^2\theta^2\Gamma^\mu\partial_a\theta^2\} \tag{1.5.11}$$

where the canonical derivative term is $\Pi_a = \partial_a X^\mu - i\bar{\theta}_A\Gamma^\mu\partial_a\theta^A$ with two Majorana-Weyl spinors in 10D, θ^1 and θ^2. This action has the following simplified form in light cone gauge:

$$S_{LC} = -\frac{1}{2\pi}\int d^2\sigma(\partial^\alpha X^i\partial_\alpha X^i - i\bar{S}^a\rho^\alpha\partial_\alpha S^a) \qquad (1.5.12)$$

where $\sqrt{p^+}\theta^A \to S^{Aa}$, which is the symbol chosen for the eight surviving components after the gauge choice, that is to say S is a spinor of SO(8)group in either $\mathbf{8}_s$ or $\mathbf{8}_c$ representation.

By considering variety of supercharges and boundary conditions one can construct different theories, namely TypeI, TypeII and Heterotic strings with different chiral symmetries. Space-time superstring theory is shown to be equivalent to world-sheet theory by a simple transformation of the algebras. Similar to its world-sheet cousin, we have anomalous dimension D=10 for space-time superstring theory. Reader is directed to [1]-[6] for further discussions.

In D=10 theories with open string are called TypeI and irrelevant since our purposes for MST construction is only formulated for closed string theories. The closed superstring theories with opposite chirality of fermionic coordinates is called TypeIIA and the one with the same chirality is TypeIIB. However, the choice of two different theories for two chiralities opens the path for Heterotic strings, which is merely consist of a chiral typeII theory and a bosonic string theory of opposite chirality.

We will only present the TypeIIA theory in details in chepter 2 while discussing the symmetric productas and strings since in the original paper [12], DVV conjeture states that the 'IR limit of 2d SYM action' is equivalent to 'second quantized LC typeIIA string theory'. Type IIA theory has 16 supercharges with 8 leftgoing and 8 rightgoing. The eqautions of motion for the fields in light-cone gauge are:

$$\partial_+\partial_- X^i = 0 \qquad (1.5.13)$$

$$\partial_+ S^a = 0 \qquad (1.5.14)$$

$$\partial_- S^{\dot{a}} = 0 \qquad (1.5.15)$$

where i ,a and $\dot{a}$ are indices of $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ respectively (the spin(8) gamma matrices are discussed in Appendix E).

1.6 Tree and One Loop Amplitudes of Superstrings

This section is devoted to amplitudes of Type IIA strings and discuss their relevance to our approach.

The four point tree amplitude of the type II theory is calculated in [6] as

$$< V(\epsilon_1, k_1)V(\epsilon_2, k_2)V(\epsilon_3, k_3)V(\epsilon_4, k_4) >= (2\pi)^{10}\delta(k)g^4 \int d^2z^1 d^2\theta_2 |z^{12}|^{-s}|z_1 - 1|^u e^{\wp_4^{\zeta}+\wp_4^{\bar{\zeta}}}$$

$$= \pi(2\pi)^{10}\delta(k)g^4 \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1+s/2)\Gamma(1+t/2)\Gamma(1+u/2)} \epsilon^{1\bar{1}}\epsilon^{2\bar{2}}\epsilon^{3\bar{3}}\epsilon^{4\bar{4}} K_{1234}K_{\bar{1}\bar{2}\bar{3}\bar{4}} \quad (1.6.1)$$

where K_{1234}s are kinematical factors given as

$$K_{1234} = (st\eta_{13}\eta_{24} - su\eta_{14}\eta_{23} - tu\eta_{12}\eta_{34}) - s(k_1^4k_3^2\eta_{24} + k_2^3k_4^1\eta_{13} - k_1^3k_4^2\eta_{23} - k_2^4k_3^1\eta_{14})$$

$$+t(k_2^1k_4^3\eta_{13} + k_3^4k_1^2\eta_{24} - k_2^4k_1^3\eta_{34} - k_3^1k_4^2\eta_{12}) - u(k_1^2k_4^3\eta_{23} + k_3^4k_2^1\eta_{14} - k_1^4k_2^3\eta_{34} - k_3^2k_4^1\eta_{12}) \quad (1.6.2)$$

by superconformal invariance the zero, one and two point functions of superstring all vanish [6].

On the other hand, the one loop four point amplitude is given by as follows [6]

$$< V(\epsilon_1, k_1)...V(\epsilon_4, k_4) >= g^4\delta(k)A_1 \times \epsilon^{1\bar{1}}\epsilon^{2\bar{2}}\epsilon^{3\bar{3}}\epsilon^{4\bar{4}} K_{1234}K_{\bar{1}\bar{2}\bar{3}\bar{4}} \quad (1.6.3)$$

where reduced amplitude is given by

$$A_1 = \int_{\mu_1} \frac{d^2\tau}{2\tau_2^2}\frac{1}{(\tau_2)^4} \int d^2z_1 d^2z_2 d^2z_3 d^2z_4 \times |F_{12}F_{34}|^{-s/2}|F_{23}F_{14}|^{-t/2}|F_{13}F_{24}|^{-u/2} \quad (1.6.4)$$

where F is defined before for bosonic case, this will be investigated for superstrings in the Appendix .

The zero point loop amplitude of the type IIA string is given by [5]:

$$Z^{IIA} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{1}{2}\sum_{\bar{a}\bar{b}}^{1}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}\frac{\theta^4[^a_b]\bar{\theta}^4[^{\bar{a}}_{\bar{b}}]}{\eta^4\bar{\eta}^4} \tag{1.6.5}$$

where $\theta[^a_b]$ defined in Appendix A. It is not hard to prove that this function is modulo invariant.

1.7 Riemann Surfaces and Light-cone Perturbation Theory

In this section we will introduce the light-cone string perturbation theory, from the perspective of branched coverings of Riemann surfaces.

Historically there have been two functional integral approaches developed to treat the loop expansion in string theory. The interacting string picture[1], which was developed first, is closely related to the physical picture of strings propagating in spacetime and undergoing occasional interactions. In contrast, the Polyakov approach [2] involves a sum over geometrical surfaces and hence the physical picture of string propagation is more obscured. A long-standing question has been whether these two formalisms are in fact equivalent.

Basically the amplitudes in both approaches are consist of integrals of a measure over some finite-dim space. Hence equality is established if we (a)demonstrate the equivalence of the integration regions (b)demonstrate the equivalence of the integration measures. This issue was fully treated in [7]

A typical amplitude in Polyakov approach is in the form

$$A_p = \int_{\mu_n^g}[dm]\frac{det <\mu_\alpha,\phi_\beta>}{det(\phi_\alpha,\phi_\beta)}(det' P_1^\dagger P_1)^{1/2}\Big(\frac{2\pi}{Vol}det'\Delta\Big)^{-13} < V_1...V_n > \tag{1.7.1}$$

which is calculated by Polyakov action $\frac{1}{2}\int_M d^2\sigma\sqrt{h}h^{ab}\partial_a X^\mu\partial_b X_\mu$ In the interacting string picture the analogous amplitude is

$$A_1(1,...,n) = \int[ds]\Big(\frac{2\pi}{Vol}det'\Delta\Big)^{-12} < W_1...W_n > \tag{1.7.2}$$

which is calculated by light-cone action $\frac{1}{2}\int d^2\sigma[(\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2]$

The equivalence lies within the Abelian differentials that completely fixes the topology of the string propagation on Riemann surfaces. One can say the above equivalence is an equivalence of Mandelstam diagrams and Riemann surfaces, via a simple cut and paste operation of the Riemann sheets one can construct any the Riemann surfaces corresponding the genus expansion of amplitudes.

In [7] the equivalence of the measures was demonstrated. We will try to give a physical account of the light-cone diagrams. First of all the Abelian differential dX^+ has bunch of zeros and poles. The zeros corresponds to the interaction points and poles are external string states

Assume a local coordinate w

$$dX^+ = dw \sim (z - z_0)dz \rightarrow (w - w_0) \sim (z - z_0)^2 \tag{1.7.3}$$

and the poles are in the form

$$dw \sim p_i^+ \frac{dz}{z - z_i} \rightarrow (w - w_i) \sim p_i^+ log(z - z_i) \tag{1.7.4}$$

higher order zeros corresponds to higher order interactions and length of the external states is given by light-cone momentum p_i^+.

The story of matrix strings starts over here since this picture is in complete analogy with the string 'bit'construction of symmetric products, which has various lengths of external states labelled by the number of the 'bits'(nothing but of p_i^+ on each cycle (i)) and there are internal local fixed points where interactions occur. The next chapter is about this construction of string theory in terms of symmetric products and and topology changing vertex.

Chapter 2

DVV Matrix String theory

In this chapter we provide a brief introduction to Matrix String Theory and DVV conjecture of string interactions and orbifold CFT. We will first review the derivation of the 2D SYM action by reduction of 10D SYM. Afterwards, we will present some crucial issues of the DVV matrix string theory. The matrix theory construction lies within the orbifold CFT techniques, so we will also review the orbifold CFT. By the end of the chapter we will talk about symmetric product orbifolds and second quantized strings.

The investigation for matrix model of string theory is related to quest for a consistent string limit of M-theory, which is considered as ultimate unifying theory of all the forces in the nature. The five consistent superstring theories, which are related by duality transformations are believed to be different limits of M-theory corresponding different sectors in perturbation theory. The low energy limit of M-theory is believed to be 11 dimensional SUGRA leads to Type IIA SUGRA up on compactification on a circle in the light cone frame. Still mysterious, M-theory attracts lots of attention and the main idea of DVV matrix theory can be embedded in this quest for a consistent and practical interpretation of M-theory in the context of M(atrix) theories.

2.1 An introduction to Matrix Theories

To comlete the M-theory picture given in the figure 2 there proposed a dimensional reduction scheme. The low-energy physics of N Dirichlet p-branes living in flat space is described in static gauge by the dimensional reduction to p + 1 dimensions of $\mathcal{N} = 1$ SYM in 10D. This constitutes the basis for the BFSS theory(the reduction to 0+1 SYM theory of $N \times N$ matices which was copnjectured to give light front formulation of the M-theory). However we will not discuss BFSS theory here, on the other hand we will review the reduction from 10D to 2D which eventually gives the super Yang-Mills action with 16 supercharges. In other words this section will be devoted to fill in some of the details of this theory in ten dimensions, and describe explicitly the dimensionally reduced theory in the case of 1-branes, or strings. The ten-dimensional $U(N)$ super Yang-Mills theory has the action [28]

$$S = \int d^{10}\xi \Big(-\frac{1}{4} Tr F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} Tr \bar{\psi} \Gamma^{\mu} D_{\mu} \psi \Big) \tag{2.1.1}$$

where the field strength is

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i g_{YM} [A_{\mu}, A_{\nu}] \tag{2.1.2}$$

is the curvature of a U(N) hermitian gauge field A_{μ}. The fields A_{μ} and ψ are both in the adjoint representation of $U(N)$ and carry adjoint indices which we will generally suppress. The covariant derivative D_{μ} of ψ is given by

$$\partial_{\mu} \psi - i g_{YM} [A_{\mu}, \psi] \tag{2.1.3}$$

where g_{YM} is the Yang-Mills coupling constant. ψ is a 16-component Majorana-Weyl spinor of SO(9,1).

The action 2.1.1 is invariant under the supersymmettry transformations

$$\delta A_{\mu} = \frac{i}{2} \bar{\epsilon} \Gamma_{\mu} \psi \tag{2.1.4}$$

$$\delta \psi = -\frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon \tag{2.1.5}$$

where ϵ is a majorana-Weyl spinor. Thus the theory has 16 independent charges. There are 8 bosonic and 8 fermionic degrees of freedom after imposing the Dirac equation.

It is always possible to rescale the fields of SYM action so that coupling constant only appears as an overall multiplicative constant. We find the following action up on absorbing the coupling to ψ and A_{μ},

$$S = \frac{1}{4 g_{YM}^2} \int d^{10}\xi \Big(-\frac{1}{4} Tr F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} Tr \bar{\psi} \Gamma^{\mu} D_{\mu} \psi \Big) \tag{2.1.6}$$

where the covariant derivative is given by

$$D_\mu = \partial_\mu - iA_\mu \tag{2.1.7}$$

The ten-dimensional super Yang-Mills theory described can be used to construct a super Yang-Mills theory in p+1 dimensions with 16 supercharges by the simple process of dimensional reduction. This is done by assuming that all fields are independent of coordinates p + 1, . . . , 9. After dimensional reduction, the 10D field A_μ decomposes into a (p + 1)-dimensional gauge field and 9 - p adjoint scalar fields X^a and we have also fermionic degrees of freedom Θ^α which transforms. The action of the dimensionally reduced theory takes the form

$$S = \frac{1}{4g_{SYM}^2}\int d^{p+1}\xi\ Tr\Big(- F_{\alpha\beta}F^{\alpha\beta} - 2(D_\alpha X^a)^2 + [X^a, X^b]^2 + \Theta^T\Gamma_i D^i\Theta + \Theta^T\Gamma^i[X^i, \Theta]\Big) \tag{2.1.8}$$

As discussed in [28] this is precisely the action describing the low energy dynamics of N coincident Dirichlet p-branes in static gauge (although there the fields X and A_μ are normalized by the factor $X \to X/(2\pi\alpha')$. The field A_α is the gauge field on the D-brane world-volume, and the fields X^a describe transverse fluctuations of the D-branes.

The classical vacuum corresponds to a static solution of the equations of motion where the potential energy of the system is minimized. This occurs when the curvature $F_{\alpha\beta}$ and the fermion fields vanish, and in addition the fields X^a are covariantly constant and commute with one another. When the fields X^a all commute with one another at each point in the (p + 1)-dimensional world-volume of the branes, the fields can be simultaneously diagonalized by a gauge transformation, so that we have

$$X^a = diag(x_1^a, ..., x_N^a) \tag{2.1.9}$$

In such a configuration, the N diagonal elements of the matrix X^a can be associated with the positions of the N distinct D-branes in the a-th transverse direction [28]. In accord with this identification, one can easily verify that the masses of the fields corresponding to off-diagonal matrix elements are precisely given by the distances between the corresponding branes. From this discussion, we see that the moduli space of classical vacua for the (p+1)-dimensional field theory arising from dimensional reduction of 10D SYM is given by

$$\frac{(R^{9-p})^N}{S_N} \tag{2.1.10}$$

The factors of R correspond to positions of the N D-branes in the (9 - p)- dimensional transverse space. The symmetry group S_N is the residual Weyl symmetry of the gauge group. In the D-brane language this corresponds to a permutation symmetry acting on the D-branes, indicating that the D-branes should be treated as indistinguishable objects.

After a general discussion of the reduction procedure and its physical relevance lets we emphasize that the action 2.1.8 up on rescaling the bosonic and fermionic degrees of freedom and identifying the g_{SYM} with g we obtain the following action:

$$S_{SYM} = \frac{1}{2\pi}\int d^2\sigma Tr((D_\mu X^i)^2 + \Theta^T\Gamma_i D^i\Theta + \frac{1}{g^2}F_{\mu\nu}^2 - g^2[X^i, X^j]^2 + g\Theta^T\gamma_i[X^i, \Theta]) \tag{2.1.11}$$

this is the starting point of the matrix string theory. In fact this action also can be derived from M(atrix) theory Hamiltonian via compactification on S^1 with radious R_9. This is also another way to approach the fact that up on compactification on a circle M-theory must give a non-perturbative theory of strings.

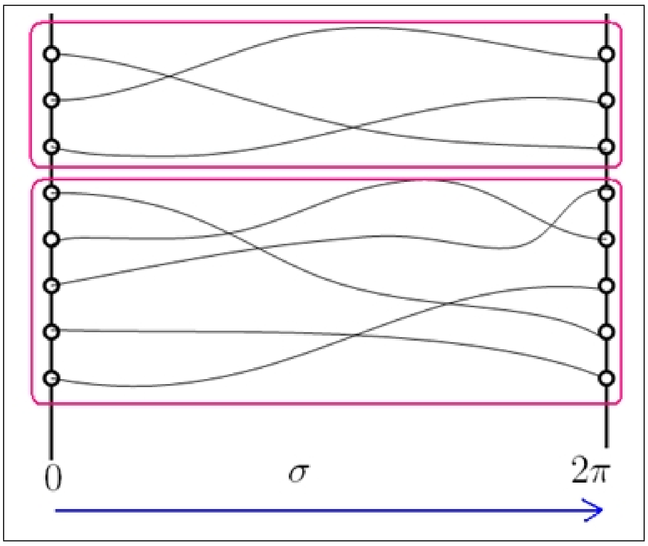
2.2 Matrix String Theory and S_N Orbifold CFT

Simply, DVV matrix theory is a super-Yang-Mills theory with 16 supercharges in 1+1 dimensions, whose dynamics is governed by the following action [12]:

$$S_{SYM} = \frac{1}{2\pi}\int d^2\sigma Tr((D_\mu X^i)^2 + \Theta^T\Gamma_i D^i\Theta + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2}[X^i, X^j]^2 + \frac{1}{g_s}\Theta^T\gamma_i[X^i, \Theta]) \tag{2.2.1}$$

here 8 bosonic fields X^i are $N \times N$ matrices, as are the 8 fermionic fields Θ_L^α and $\Theta_R^{\dot{\alpha}}$. The fields X^i, Θ_L^α and $\Theta_R^{\dot{\alpha}}$ transform respectively in $\mathbf{8}_v$ vector, and $\mathbf{8}_s$ and $\mathbf{8}_c$ spinor representations of SO(8) R-symmetry group of transversal rotations. The two dimensional world-sheet is taken to be a cylinder and parameterized by coordinates (σ, τ) with σ is on circle. Fermions are taken in Ramond sector and there is no specific projection on particular fermion number[12-13].

Figure 2.1: The spectral flow of eigenvalues. The cycles of length (3) and (5) are indicated by the red squares.



The eigenvalues of the matrix coordinates X^i are identified as the coordinates of TypeIIA string theory. The action 2.2.1 has an IR fixed point and as theory flows to that point it recovers the complete string spectrum of TypeIIA theory. Correspondence based on equivalence between second-quantized string theory and S_N orbifold sigma-model, which we will investigate in details in the following pages.

The commutator terms in 2.2.1 dominates the action as g_s goes to 0 (or equivalently $g \to \infty$), which means there is no dynamical term left. For having a non-trivial dynamics one has to assume that the bracket terms vanish. On the other hand gauge field strength F decouples in the IR limit, since the gauge field is boken to $T = U(1)^k$, with k number of strings[16]. The degrees of freedom is given by the solution to Laplace equation on torus:

$$g^2\Delta(A) = 0 \tag{2.2.2}$$

this must be identically zero as $g \to \infty$ which means we are only left with quantum mechanics of finite degrees of freedom in strong coupling limit so as the term $S = \frac{1}{g^2}\int(F_{\mu\nu})^2$ vanishes in the strong coupling limit.

The argument above justifies the fact that the conformal field theory that describes the

IR limit is the $\mathcal{N} = 8$ SCFT on the orbifold target space

$$S^N\mathbf{R} = (\mathbf{R}^8)^N/S_N \tag{2.2.3}$$

In $g_s = 0$ limit the fields X and Θ commute. Consequently, one can diagonalize both fields simultaneously. One can then write X matrix coordinates as

$$X^i = U(\sigma)x^iU^{-1}(\sigma) \tag{2.2.4}$$

with $U \epsilon U(N)$ and x^i is a diagonal matrix with eigen values $x^i_1, ..., x^i_N$. That is to say x^i takes values in the Cartan sub-algebra of $U(N)$. The only gauge freedom left is Weyl group which is isomorphic to symmetric group S_N, which describes the model with Green-Schwarz light cone coordinates $x^i_I, \theta^\alpha_I, \theta^{\dot\alpha}_I$ with $I = 1, 2....N$. The theory that corresponds to $N \to \infty$ is described by these eigenvalues which includes twisted sector because of the different orientation of the full gauge group $U(N)$ at $\sigma = 0$ and $\sigma = 2\pi$. This two different groups are isomorphic up to a Weyl group element. By using the fact that matrix X is periodic we have following identification:

$$U(\sigma + 2\pi)x^i(2\pi)U^{-1}(\sigma + 2\pi) = U(\sigma + 2\pi)gx^i(0)g^{-1}U^{-1}(\sigma + 2\pi)$$

$$x^i(\sigma + 2\pi) = gx^i(\sigma)g^{-1} \tag{2.2.5}$$

with g ϵ Weyl group of U(N), the symmetric group S_N. It is illustrated in the figure 2.1

In full propagation picture the figure 2.1 is interpreted as a snapshot of the string propagation at fixed τ. The action of S_N provides us with the cycles consists of different numbers of subset of N objects,(N_n)(where n is the length of the cycle and N_n stands for the number of repetitions of this cycle), at each value of τ where there is no coincidence. These cycles establishes the many free string picture (second quantized TypeIIA string theory). On the other hand, the coinciding fields in this diagram is very important since those will constitute the interaction of strings. For the time being we only consider the free string picture and try to understand the Hilbert space structure.

The Hilbert space of the S_N orbifold SFT is compsed into twisted sectors:

$$H(S^N\mathbf{R}^8) = \bigoplus_{partitions\{N_n\}} H_{\{N_n\}} \tag{2.2.6}$$

Here we note that conjugacy classes characterize the structure of the Hilbert space. An overall conjugation in terms of an arbitrary element of S_N leaves picture the same. So what we really need here separate conjugacy classes consists of cycles of (n) with N_n copies of the same cycle (Appendix C).

$$\sum_n nN_n = N \tag{2.2.7}$$

A general conjugacy class $[g]$ is decomposed as

$$[g] = (1)^{N_1}(2)^{N_2}...(s)^{N_s} \tag{2.2.8}$$

In each twisted sector one must keep only the states invariant under the centralizer subgroup C_g of g

$$C_g = \prod_{n=1}^{s} S_{N_n} \times \mathbf{Z}_n^{N_n} \tag{2.2.9}$$

where each S_{N_n} permutes N_n identical cycles while $\mathbf{Z}_n$ acts within one cycle. The corresponding Hilbert space is

$$H_{\{N_n\}} = \bigotimes S^{N_n} H_n \tag{2.2.10}$$

where

$$S^N H = (H \otimes ... \otimes H)^{S_N} \tag{2.2.11}$$

The space H_n denotes the single string Hilbert space on $\mathbf{R}^8 \times S^1$ space with winding number n. This model can be explained by a sigma model of coordinate fields x_I^i with following boundary conditions in Z_n invariant sector

$$x_I^i(\sigma + 2\pi) = x_{I+1}^i(\sigma) \quad mod(n), \quad I \quad \epsilon \quad (1, ...n) \tag{2.2.12}$$

we can construct a string by gluing together these fields, defined on interval $0 \leq \sigma \leq 2n\pi$. To define whole theory on the $[0, 2\pi]$ interval we transform $\sigma \to n_k\sigma$ in each Z_{n_k}invariant twisted sector. In terms of canonically normalized single string L_0^i operator the total L_0^{tot} is given as

$$L_0^{tot} = \sum_i \frac{L_0^i}{n_i} \tag{2.2.13}$$

one can see this from $L_0 - \bar{L}_0 = \frac{\partial}{\partial \sigma}$ transforms as $L_0^i - \bar{L}_0^i = \frac{1}{n_i}\frac{\partial}{\partial \sigma}$

We normalize the total light-cone momentum $p_+^{tot} = 1$ and the fraction of the momentum for each cycle is

$$p_+^{tot} = \frac{n_i}{N} \tag{2.2.14}$$

The heuristic philosophy of the theory "only long strings of S_N orbifold sigma model survive in the $N \to \infty$ limit" can be seen here since only states that corresponds to the finite fraction of the total momentum survives. Another crucial point is no winding modes are allowed in this limit since non-zero winding modes become infinitely massive. This can be also understood as interpreting $N \to \infty$ flow as the world-sheet radius R goes to infinity, since in the mass term contribution of is proportional to Rm_i, mode m_i must vanish. In this respect the level matching condition $L_0^i - \bar{L}_0^i = n_i m_i \to 0$ implies that the mass shell condition is

$$p_-^{tot} = N L_0^{tot} \tag{2.2.15}$$

and one can recover the mass-shell condition for individual string as

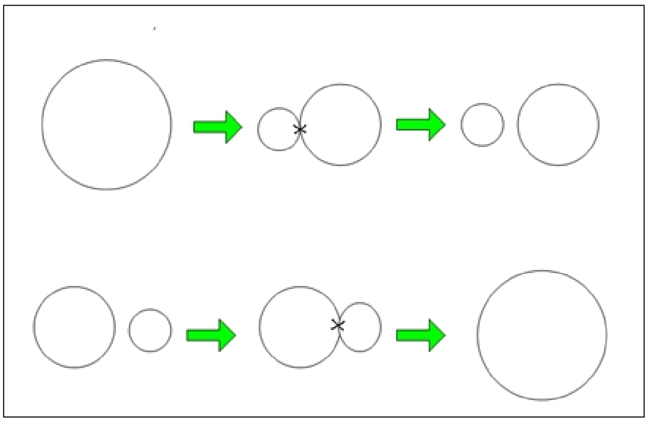
$$p_-^i = \frac{N L_0^i}{n_i} \tag{2.2.16}$$

combining this with (2.13) gives $p_-^i p_+^i = L_0^i$, considering all strings have the same string tension α'=1.

2.3 DVV Interaction Vertex

The free theory above does not have any interesting dynamics since there is no terms governing the joining-splitting process. The first irrelevant operator correction around IR fixed

Figure 2.2: The cycles join or split by transposition of the elements. The string interaction vertex.



point constitutes such a vertex. This is a non-conformal perturbative correction for small value of g_s. This vertex must be SO(8) and Susy invariant. On the other hand it should couple to the free action by a power of $g_s = \frac{1}{gL}$ that will make overall correction scales as $\frac{1}{L}$ with h_i conformal dimension of V_i.

$$S_{DVV} = S_{SCFT} + g_s^{h_i - 2} \int d^2\sigma V_i \tag{2.3.1}$$

As mentioned above the fundamental structure that forms the strings are cycles in the symmetric group. We need an effective operator that breaks a big cycles into two small ones or combine two small cycles into a big one as in figure 2.2. The mechanism lies beneath the recovering of the U(2) symmetry at the coincidence points of the eigenvalues of matrix X, where fixed points of the orbifold lyes. Around the the point the symmetry $U(1)_i \times U(1)_j$ recovered as $U(2)_{ij}$, which is broken to the Weyl group Z_2 at large N limit. Observe that in a cycle of 2.2.8 exchange of two object breaks the cycle and exchange of two objects from two small cycles fuse the cycles.(Check Appendix C for the properties of symmetric group.)

The least irrelevant mighty vertex conjectured by Dijkgraaf, Verlinde and Verlinde in [12] is

$$V_{int} = \sum_{I<J} \int d^2 z (\tau^i \Sigma^i \otimes \bar{\tau}^j \bar{\Sigma}^j)_{IJ} \tag{2.3.2}$$

where τ and Σ are excited twist fields given by the relations

$$x_{-}^{i}(z).\sigma(0) \sim z^{-\frac{1}{2}}\tau^{i}(0) \tag{2.3.3}$$

$$\theta_{-}^{\alpha}(z).\Sigma^{i}(0) \sim z^{-\frac{1}{2}}\gamma_{\alpha\dot{\alpha}}^{i}\Sigma^{\dot{\alpha}}(0) \tag{2.3.4}$$

$$\theta_{-}^{\alpha}(z).\Sigma^{\dot{\alpha}}(0) \sim z^{-\frac{1}{2}}\gamma_{\alpha\dot{\alpha}}^{i}\Sigma^{i}(0) \tag{2.3.5}$$

the SO(8) is satisfied by space-time contractions and SUSY invariance is satisfied because variation is a full derivative

$$[G_{-\frac{1}{2}}^{\dot{\alpha}}, \tau^{i}\Sigma^{i}] = \partial_{z}(\sigma\Sigma^{\dot{\alpha}}) \tag{2.3.6}$$

from the relations 2.3.3, 2.3.4 and 2.3.5 one can check that the conformal dimension of the single twist fields τ and σ are 1 and $\frac{1}{2}$ and consequently the conformal weight of the full V_{int} is $(\frac{3}{2}, \frac{3}{2})$. Corresponding conformal dimension of the coupling constant is -1, hence interaction will scale linear in g_s as required.[13]

2.4 Orbifold Conformal Field Theory

In this section we will review the CFT on orbifold backgrounds developed by Dixon et all. [21]. The main discussion will follow Dixon et all.[21] and Arutyunov and Frolov [22]. We will first explain the operator product expansions (OPE)of twist fields and the geometry of orbifolds. Then we will show the simplest loop calculation, the correlator of four Z_2 operator on sphere, by using stress energy tensor method. With some remarks about the bosonization and cocyles of the fermionic twist fields, we will finish the section.

Orbifold is defined as quotient of a smooth manifold M and action of discrete group S.

$$\Omega = M/S \tag{2.4.1}$$

which has a set of fixed points. A fixed point is simply $S \times \{p\} \rightarrow \{p\}$. In our case the orbifold geometry is given by the embedding

$$X : \text{2-d World Sheet} \hookrightarrow \text{orbifold space-time} \tag{2.4.2}$$

For most of our purposes we will just have a quotient of Euclidean space R^d/S. This construction will be reviewed in due paragraphs.

The main elements of conformal field theory of orbifolds are twist fields. Twist fields are located on the fixed point set of the orbifold which merely acts as a brunch cut on the world-sheet where in the vicinity, the field $X(z, \bar{z})$ is multi-valued. A full rotation around the fixed points changes the Riemann sheet the string propagates.

Twist basically can be perceived as an analogue of spin field of fermions. For instance, in the Ramond sector chiral field $\psi(e^{2\pi i}z) = -\psi(z)$ is generated by a spin field S which is merely fermionic vertex operator.

$$\psi(z)S(0) \sim z^{-1/2}\tilde{S}(0) \tag{2.4.3}$$

with $\tilde{S}$ is excited spin field located at zero. The conformal dimension is $h_s = 1/16$.

For bosonic CFT which are consist of Id. operator,the complex scalar field X and its exponential e^{ipX} and anti-holomorphic piece, twisting of the field in the vicinity of a certain point w is given by the following OPE for the simplest twist field analogue of the above equation

$$\partial X \ \sigma(w, \bar{w}) \sim (z - w)^{-1/2}\tau(w, \bar{w}) \tag{2.4.4}$$

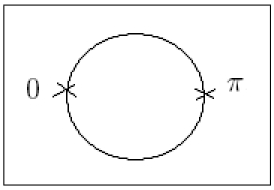
the action of this twist field is just reflection with respect to the origin. The field τ is the excited twist field which has holomorphic conformal dimension $h_\tau = h_\sigma + 1/2$ where antiholomorphic dimension did not change.

The simplest orbifold geometry is the identification $X \sim X + 2\pi R$ and $X \sim -X$, which has the fixed points X=0 and $X = \pi R$. Basically there is two twist fields, one on each fixed point of the orbifold. The classical ground state is basically a static oscillating string attached to either fixed points.

The complex field X behaves around a fixed point of above kind, Z_2 twist field, as

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{\pi i}X(z, \bar{z}) = -X(z, \bar{z}) \tag{2.4.5}$$

Figure 2.3: The orbifold S^1/Z_2.



in general in the vicinity of the twist fields this relation is

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{2\pi ik/N}X(z,\bar{z}) \tag{2.4.6}$$

which is just corresponding action of the Z_N twist field. This can be summarized in the operator formalism with more concrete terms as follows

$$\partial X(z)\sigma_+(w,\bar{w}) \sim (z-w)^{-(1-k/N)}\tau_+(w,\bar{w}) + ...$$

$$\bar{\partial} X(z)\sigma_+(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-k/N}\tau'_+(w,\bar{w}) + ...$$

$$\partial X(\bar{z})\sigma_+(w,\bar{w}) \sim (z-w)^{-k/N}\tilde{\tau}'_+(w,\bar{w}) + ...$$

$$\bar{\partial} X(\bar{z})\sigma_+(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-(1-k/N)}\tilde{\tau}_+(w,\bar{w}) + ... \tag{2.4.7}$$

and similar equations for σ_-. The power is determined by the phase condition (2.24) and the integer power is determined by the fact that twisted vacuum is the highest weight state. One can give the oscillator mode expansion of the holomorphic and anti holomorphic twisted fields but in fact this is practically useless, because the twist fields generically does not have a mode expansion.

$$\partial X(z) = \sum_{m=-\infty}^{\infty} \alpha_{m-k/N} z^{-m-1+k/N} \tag{2.4.8}$$

$$\partial\bar{X}(z) = \sum_{m=-\infty}^{\infty} \bar{\alpha}_{m-k/N} z^{-m-1-k/N} \tag{2.4.9}$$

the first excited states are

$$\alpha_{1-k/N}|\sigma> \tag{2.4.10}$$

$$\bar{\alpha}_{k/N}|\sigma> \tag{2.4.11}$$

This is a bit tricky for the fermionic theory since it includes issues such as bosonization and cocyles. Let's start with we have a free fermion theory in 2d, with Majorana fermions. One can define two chiral theories which is equivalent to full theory: $\psi = \psi_1 + i\psi_2$ and $\bar{\psi} = \psi_1 - i\psi_2$ where we used just the components of Majorana spinor. The bosonization of the fermionic theory can be defined,for the equivalence of these two chiral theories,as

$$\bar{\psi}\psi = 2i\partial H \tag{2.4.12}$$

$$\sqrt{\frac{1}{2}} i\psi = exp(iH) \tag{2.4.13}$$

$$\sqrt{\frac{1}{2}} i\bar{\psi} = exp(-iH) \tag{2.4.14}$$

The Z_n spin fields are simply $S_{\pm} = exp(\pm kH/N)$ with the conformal dimension $h_s = \frac{1}{2}(\frac{k}{N})^2$. This transformations gives the right OPEs up to cocyles and non-singular terms which is relevant for conformal field theories.

Turning back to the main scope of this section we will spend few words on orbifold CFT correlation functions. Our ultimate goal is to calculate the correlation functions in a CFT, so we will consider the correlation function of twist fields

$$F(z_1, ..., z_n) =< \sigma(z_1)....\sigma(z_n) > \tag{2.4.15}$$

this correlation function cannot be calculated by conventional methods since one cannot use Wick's theorem since the twist fields are not expressed in terms of oscillator modes. To avoid this complication, we will use a method called stress-energy tensor (SET) method, which is nothing but using the Ward identities to extract the information about the correlation function around the twist field locations. Main element of the method is the following correlator:

$$g(z,w) = \frac{-\frac{1}{2} < \partial X(z)\partial\bar{X}(w)\sigma(z_1)....\sigma(z_n) >}{< \sigma(z_1)....\sigma(z_n) >} \tag{2.4.16}$$

from this we obtain

$$<< T(z) >>= lim_{z\to w}\Big[\frac{-\frac{1}{2} < \partial X(z)\partial\bar{X}(w)\sigma(z_1)....\sigma(z_n) >}{< \sigma(z_1)....\sigma(z_n) >} - \frac{1}{(z-w)^2}\Big] \tag{2.4.17}$$

then considering the Ward identity

$$< T(z)F(z_1,...,.z_n) >= \sum_{j=1}^{n}[\frac{h_i}{(z-z_j)^2} + \frac{\partial_j}{z-z_i}] < F(z_1,...,.z_n) > \tag{2.4.18}$$

consequently

$$Res_{z_i} << T(z) >>= \partial_i ln(< F >) \tag{2.4.19}$$

if one can calculate LHS of 2.4.19 independently 2.43 provides a system of differential equations

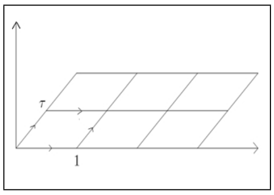
$$\partial_j ln(< F(z_1,...,.z_n) >) = \Gamma(z_1,...,.z_n) \tag{2.4.20}$$

for $j = 1,....,n$. For the calculation of LHS arguments of analytic properties of the correlation function i.e. action of local twist fields and global monodromy conditions will be enough as discussed in Appendix B.

The correlation function **??** is only non-zero when the total action of twist fields is trivial.This is a selection rule analogous to fermion selection rule i.e. odd number off fermions has vanishing correlation function. This selection rule has more concrete implication, when travelled around a closed loops that encircles zero effective twist the integral of the dX vanishes

$$0 = \Delta_{C_I} X_{qu} = \oint_{C_I} dz\partial X + \oint_{C_I} d\bar{z}\bar{\partial}X \tag{2.4.21}$$

Figure 2.4: The torus is represented by a lattice.



this constitutes the missing information to calculate the Green function from the analytic properties. The demonstration of the method is given in the following pages, the calculation of four Z_2 twist correlator.

-Four Z2 correlator calculation.

For the sake of concreteness we apply SET method to the simplest nontrivial correlator, four Z_2 field. First version is a restatement of the calculation in Dixon et.all.[21] and second version is a simple application of the method given in Arutyunov and Frolov[22], just slightly different but more useful for our construction and will be given in the 3rd apter.

$$< \sigma_2(z_1)\sigma_2(z_2)\sigma_2(z_3)\sigma_2(z_4) > \tag{2.4.22}$$

by using SL(2,Z) invariance we can transform three of the points to 0,1 and ∞ and only moduli left is cross ratio x.

$$Z(x) = \lim_{z_\infty \to \infty} |z_\infty|^{1/4} < \sigma_2(z_\infty)\sigma_2(1)\sigma_2(x)\sigma_2(0) > \tag{2.4.23}$$

by using the uniformization technique such that

$$\partial X_{cl} = \lim_{z_\infty \to \infty} |z_\infty|^{1/4} < \partial X(z)\sigma_2(z_\infty)\sigma_2(1)\sigma_2(x)\sigma_2(0) >$$

$$= \frac{const}{[(z - z_\infty)(z-1)(z-x)z]^{1/2}} \tag{2.4.24}$$

the differential equation $dt = dz\, \partial X$ defines a classical elliptic function , the Weierstrass function $\wp(t)$

$$z(t) = \frac{\wp(t) - e_1}{e_2 - e_1} \tag{2.4.25}$$

$$x = \frac{e_3 - e_1}{e_2 - e_1} \tag{2.4.26}$$

where $e_1 + e_2 + e_3 = 0$. Elliptic functions are defined on the torus which can be represented by the following parallelogram with the fixed point locations.

On the other hand the modulus of the torus is given by

$$x = \left(\frac{\theta_3(\tau)}{\theta_4(\tau)}\right)^4 \tag{2.4.27}$$

Full Green's function on torus is parity odd $X(-t) = -X(t)$, so that holomorphic field ∂X is invariant under reflection (sheet interchange). The Green function

$$g_\tau(t,t') = g(t,t') + g(t,-t') \tag{2.4.28}$$

$$g(t,t') = -\frac{1}{2} < \partial X_{qu}(t) \partial X_{qu}(t') > \tag{2.4.29}$$

we will also need

$$h(\bar{t},t') = -\frac{1}{2} < \bar{\partial} X_{qu}(\bar{t}) \partial X_{qu}(t') > \tag{2.4.30}$$

The full Green's function has vanishing integrals around the closed loops, on z-plane: $\Delta_C X_{qu} = 0$, which corresponds to the fundamental cycles on torus. Applied to above definitions we obtain the following conditions:

$$0 = \int_0^1 dt g(t,t') + \int_0^1 d\bar{t} h(\bar{t},t')$$

$$0 = \int_0^\tau dt g(t,t') + \int_0^\tau d\bar{t} h(\bar{t},t') \tag{2.4.31}$$

by implementing this conditions one obtains the Green's function

$$g(t,t') = \frac{1}{2}\wp(t-t') - \frac{1}{2}\int_0^1 dt\ \wp(t) - \frac{\pi}{2Im\tau} \tag{2.4.32}$$

where

$$I = \int_0^1 dt\ \wp(t) = (2\pi i)^2 \Big[\frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{nu^n}{1-u^n}\Big] \tag{2.4.33}$$

stress energy tensor on the torus is then

$$T(t) = \lim_{t \to t'} [g(t,t') - \frac{1}{2}\frac{1}{(t-t')^2}] \tag{2.4.34}$$

However, we need stress energy tensor on the sphere so one should use the following transformation rule

$$T(t) = \Big(\frac{dz}{dt}\Big)^2 T(z) + \frac{1}{12}c\Big(\frac{z'''}{z'} - \frac{3}{2}\Big(\frac{z''}{z'}\Big)^2\Big) \tag{2.4.35}$$

by using the appropriate transformation rules for Weierstrass $\wp$ function (appendix-A)one can easily obtain

$$< T(z) >= \Big(\frac{e_2 - e_1}{\wp'(t)}\Big)^2 \Big[\wp(2t) - I - \frac{\pi}{Im\tau}\Big] \tag{2.4.36}$$

by expanding this around z=x

$$< T(z) >= \frac{1}{16}\frac{1}{(z-x)^2} + \frac{1}{z-x}\Big\{\frac{1}{8}(\frac{1}{x} + \frac{1}{x-1}) + \frac{[-2(e_2-e_1)x - 2e_1 - I - \pi(Im\tau)^{-1}]}{4(e_2-e_1)x(x-1)}\Big\} \tag{2.4.37}$$

and observing that

$$\frac{1}{4(e_2-e_1)x(x-1)} = \frac{1}{4(\pi)^2 x\theta_2^4} = \frac{1}{4\pi i}\frac{d\tau}{dx}$$

one can derive the promised differential equation for the correlation function.

$$\partial_x ln Z_{qu} = -\frac{1}{24}\Big(\frac{1}{x} + \frac{1}{x-1}\Big) - \frac{1}{4\pi i}\Big[I + \frac{\pi}{Im\tau}\Big]\frac{d\tau}{dx} \tag{2.4.38}$$

which integrates to

$$Z_{qu}(x) = const.|x(1-x)|^{-1/2}(Im\tau)^{-1/2}|u|^{-1/2}\prod_{n=1}^{\infty}|1-u^n|^{-2} \tag{2.4.39}$$

where $u = exp(2\pi i\tau)$.

****************-**Fermionic correlators**

2.5 Symmetric Products, Elliptic Genera and String Theory

The symmetric orbifolds constitute the main object of the matrix string theory. We gave an introduction to the orbifoldization via identifying the symmetric copies of the manifold X in section 2.2 for a special case of R^8. In the meanwhile, the general theory of symmetric product orbifolds, elliptic genus and relation to second quantized strings is worthwhile to present since we believe this will help to better understand the quantization of the S_N orbifold theories and the corresponding perturbation theory.

In [14], Dijkgraaf et all. proved the conjectured explicit form of the elliptic genus,namely partition function of Hilbert space structure, first using modular forms then expressed in terms of Hecke operators, where they calculated free energy of second quantized string theory in the hamiltonian picture. We will try to get the same result from the Lagrangian picture by using SET method and DVV matrix theory in the 3rd chapter. The equivalence of this two pictures is discussed in [16] in details.

The single string picture of loop amplitudes given in the sections 1.3 and 1.6, provides some insight to approach the many string problem. On the other hand, in Lagrangian construction the trace operation of Hamiltonian picture is naturally given by boundary conditions via matching the incoming and outgoing strings and modulus is given by the invariant cross ratio of the twist field locations. Our actual goal is to calculate partition function of the single string propagating on symmetric product orbifold and compare it with the partition function of second quantized DLCQ string theory.

Let X be a smooth manifold, a symmetric orbifold is, as defined for a special case in section (2.1),

$$S_N X = (X)^N / S_N \tag{2.5.1}$$

where S_N acts as permutation of N chosen points on product manifold $X \times ... \times X$. This is trivial symmetrization operation on vector spaces to construct symmetric tensors.

$$Sym(T) = \frac{1}{n!}[T_{abc..} + \text{permutation of indices}] \tag{2.5.2}$$

the main claim in [16] is the equivalence of symmetric orbifold and DLCQ theories, can be summarized in the following theorem

Theorem 1. ***Theorem:*** *The discreet light cone quantization of a free scalar field on space-time $M = (R \times S^1) \times X$ with total longitudinal momentum $p^+ = N$ is given by the quantum mechanics on the orbifold symmetric product $S^N X$,*

$$H^{QFT}(X) = H^{QM}_{orb}(SX) \tag{2.5.3}$$

Furthermore the lightcone Hamiltonian p^- is identified by the non-relativistic quantum mechanics Hamiltonian H.

This result can be extended to susy version [16].Moreover the result can be possibly interpreted as a string theory by just identifying the cycles in the symmetric group as coordinates of the strings i.e.

$$\text{quantum mechanics on SX } \rightarrow \text{quantum field theory on X}$$
$$\text{conformal field theory on} SX \rightarrow \text{second quantized string theory on X}$$

in more specific terms we want to identify the DLCQ string theory on $(R \times S^1) \times X$ with SCFT on SX.

Following [16], as a concrete example we present TypeII superstring in lightcone gauge. We make the lightcone decomposition of coordinates (x^+, x^-, x^i). The physical degrees of freedom are then

$$x : \Sigma \rightarrow R^8 \tag{2.5.4}$$

The model has 16 supercharges (8 left and 8 right moving) and carries $spin(8)$ R-symmetry. With the fields x and θ action of the first quantized sigma model is simply following CFT

$$S = \int d^2\sigma (\frac{1}{2}\partial x^i \bar{\partial} x^i + \theta^a \bar{\partial} \theta^a + \bar{\theta}^{\dot{a}} \partial \bar{\theta}^{\dot{a}}) \tag{2.5.5}$$

the model has Hilbert space that is of the form

$$H = L^2(R^8) \otimes V \otimes F \otimes \bar{F} \tag{2.5.6}$$

we recognize the familiar components here: the zero modes $L^2(R^8)$ describes quantum mechanics of the center of mass. The fermionic zero modes gives 16×16 dimensional vector space of ground states

$$V = (V_0 \oplus S^-) \otimes (V_0 \oplus S^+) \tag{2.5.7}$$

this space forms the representation of Clifford algebra $Cliff(S^+) \otimes Cliff(S^-)$ generated by the fermion zero modes. Finally, the Fock space F of non-zero-modes is given by

$$F_q = \bigotimes_{n>0} \Big(\bigwedge_{q^n} S^- \otimes S_{q^n} V \Big) \tag{2.5.8}$$

with a similar expression for $\bar{F}$.

In the lightcone gauge the coordinate x_+ is given by

$$x_+(\sigma, \tau) = p^+ \tau \tag{2.5.9}$$

for fixed longitudinal momentum $p^+ > 0$, whereas x^- is determined by the constraints

$$\partial x^- = \frac{1}{p^+}(\partial x)^2, \quad \bar{\partial} x^- = \frac{1}{p^+}(\bar{\partial} x)^2 \tag{2.5.10}$$

The Hilbert space of physical states of a single string with longitudinal momentum p^+ is given by the CFT Hilbert space H restricted to states with zero world sheet momentum, the level matching condition

$$P = L_0 - \bar{L}_0 = 0 \tag{2.5.11}$$

the light-cone energy is determined by the mass-shell relation

$$p^- = \frac{1}{p^+}(L_0 + \bar{L}_0) = \frac{1}{p^+} H \tag{2.5.12}$$

One can also compactify the null coordinate x^- by identifying $x^- \sim x^- + 2\pi R$. This has two consequences. First, the light cone momentum is not free anymore but constrained with

$p^+ = n/R$ for $n \epsilon Z_{>0}$ to keep zero mode wave function single valued. Second, string will decompose in different sectors according to number of wrapping around the compact null direction, namely winding number is given by

$$w^- = \int dx^- = 2\pi m R, \quad m \epsilon Z. \tag{2.5.13}$$

However, using the constraints 2.5.10 we obtain

$$w^- = \frac{2\pi}{p^+}(L_0 - \bar{L}_0) = \frac{2\pi R}{n}(L_0 - \bar{L}_0) \tag{2.5.14}$$

So, in order m to be an integer, we see that the CFT Hilbert space must now be restricted to H^n consisting of all states that satisfied the modified level-matching condition

$$P = L_0 - \bar{L}_0 = 0(mod\, n) \tag{2.5.15}$$

This is exactly equivalent to the Z_n invariant sectors of symmetric orbifold Hilbert space, where we have $L_0^i - \bar{L}_0^i = n_i m_i$ as well. This motivates the correspondence between free second quantized type IIA string spectrum and S_N CFT defined on :

$$S^N R^8 = R^{8N}/S_N \tag{2.5.16}$$

Indeed, in this correspondence we have:

$$p^+ = N, \quad p^- = H = L_0 + \bar{L}_0, \quad w_- = P = L_0 - \bar{L}_0 \tag{2.5.17}$$

This gives the following form for the second quantized Fock space

$$F_p = \bigotimes_{n>0} S_{p^n} H^{(n)} \tag{2.5.18}$$

this is both the Hilbert space of the free string theory and of the sigma model on $S_p R^8$. So we can identify their partition functions:

$$Z^{string}(R^8; p, q, \bar{q}) = Z^{SCFT}(S_p R^8; q, \bar{q}), \tag{2.5.19}$$

Elliptic Genus

The above correspondence is realized in Dijkgraaf et all.[14] by demonstrating the equivalence between one loop amplitudes of SUSY sigma model on S_N and second quantized string theory on $M \times S^1$. Consequently we will review the proof of the identity and its stringy interpretation

$$\sum_{N=0}^{\infty} p^N \chi(S^N X; q, y) = \prod_{n>0, m\geq 0, l} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}} \tag{2.5.20}$$

where the coefficient $c(m, l)$ is given via the expansion

$$\chi(X; q, y) = \sum_{m\geq 0, l} c(m, l) q^m y^l \tag{2.5.21}$$

proof of this result lies within standard results of orbifold conformal field theory of Dixon et all.[21] and generalizes the orbifold Euler number. The central idea behind the proof of the above identity is that partition function of single string on $S^N X$ decomposes into several distinct topological sectors, corresponding to various ways one string wounds around $S^N X \times S^1$ can be disentangled into different separate strings that wind one or more times around $X \times S^1$. This realizes the correspondence. It is useful to think a single string on symmetric orbifold $S^N X \times S^1$ as a map that associates to each point on the S^1 N points in X. By following the trace of this N points as we go around the S^1 we obtain a collection of strings on $M \times S^1$ with total winding number N. Since all permutations of N points on X corresponds to the same points on $S^N X$ string can reconnect in different ways corresponding the conjugacy classes of g ϵ S_N. The decomposition of [g] into irreducible cycles of length (n) corresponds to decomposition with several strings of winding number n. The combinatorial description of conjugacy classes as well as the appropriate symmetrizations of the wave functions, are both naturally accounted for second quantized string theory.[14]

Elliptic genus of $S^N X$ can now be computed in Hamiltonian picture by taking the trace over Hilbert space in various twisted sectors. We introduce following notation for the partition function:

$$\chi(\mathcal{H}; q, y) = Tr_{\mathcal{H}}((-1)^F y^{F_L} q^H) \tag{2.5.22}$$

for every sub-Hilbert space H of supersymmetric sigma model. We note the following rules :

$$\chi(\mathcal{H} \oplus \mathcal{H}') = \chi(\mathcal{H}) + \chi(\mathcal{H}')$$

$$\chi(\mathcal{H} \otimes \mathcal{H}') = \chi(\mathcal{H}).\chi(\mathcal{H}') \tag{2.5.23}$$

The elliptic genus of the twisted sector is H_n is given by

$$\chi(\mathcal{H}_n; q, y) = \chi(\mathcal{H}; q^{1/n}, y) = \sum_{m \geq 0, l} c(m, l) q^{m/n} y^l \tag{2.5.24}$$

this is the left moving partition sum of the single string with winding n on $X \times S^1$. This can be related to the string with winding number n by rescaling $q \to q^{1/n}$

The projection on the Z_n invariant sub-space is

$$\chi(\mathcal{H}_n^{Z_n}; q, y) = \sum_{m \geq 0, l} c(nm, l) q^m y^l \tag{2.5.25}$$

the following result is given in [14], if $\chi(H; q, y)$ has the expansion

$$\chi(\mathcal{H}; q; y) = \sum_{m,l} d(m, l) q^m y^l \tag{2.5.26}$$

then we have the following expansion for the partition function of symmetrized product of Hilbert spaces

$$\sum_{N \geq 0} p^N \chi(S^N \mathcal{H}; q, y) = \prod_{m,l} \frac{1}{(1 - p q^m y^l)^{d(m,l)}} \tag{2.5.27}$$

the proof of the main identity follows from combination of the two main results above. The Hilbert space of symmetric orbifold is decomposed as

$$\mathcal{H}(s^N X) = \bigoplus_{\sum n N_n = N} \bigotimes_{n > 0} S^{N_n} \mathcal{H}_n^{Z_n} \tag{2.5.28}$$

with this form of Hilbert space $H(S^N X)$, we find for the partition function

$$\sum_{N \geq 0} p^N \chi(S^N X; q, y) = \sum_{N \geq 0} p^N \sum_{\sum n N_n = N} \prod_{n > 0} \chi(S^{N_n} \mathcal{H}_n^{Z_n}; q, y) = \prod_n \sum_{N \geq 0} p^{nN} \chi(S^N \mathcal{H}_n^{Z_n}; q, y) \tag{2.5.29}$$

by applying the result 0.00 we obtain the desired identity

$$\sum_{N\geq 0} p^N \chi(S^N X; q, y) = \prod_{n>0;m\geq 0,l} \frac{1}{(1-p^n q^m y^l)^{d(mn,l)}} \tag{2.5.30}$$

Chapter 3

Tree and Loop Amplitudes in Matrix String Theory

In this chapter, we will apply the methods presented in the second chapter to a concrete example, namely calculation of four point scattering amplitudes by using SET method. The amplitudes are first calculated and showed to be equal to Virasoro amplitude of Bosonic strings and Veniziano amplitude of type IIA string theory by Arutyunov and Frolov [22-23].Later this work was extended to all physical particles in [24] . This result is a strong evidence for the conjectured correspondence of S_N orbifold sigma models corrected by DVV vertex and second quantized interacting string theory. Our main goal is to understand and calculate the possible loop amplitudes from the former and show that it is equivalent to later.

3.1 S_N Orbifold CFT and Perturbation Theory

In this section we will present the main results of DVV construction in details and try to construct a recipe to calculate the string amplitudes of wanted order in terms of the perturbation theory around IR fixed point. As first irrelevant term in the expansion around the fixed point S_N Orbifold CFT is corrected by DVV vertex as stated in the section (2.2.2). Coupling constant g_s scales as inverse length so as $R \to \infty$ we recover second quantized string theory as conjectured and justified by Dijkgraff,Verlinde and Verlinde in [12], also demonstrated in [13]. The main elements of a perturbation theory, namely Hilbert space and perturbative expansion with respect to coupling constant will be main concern of this section. We will use the Lagrangian approach mentioned throughout sections section (2.2.1)-(2.2.4). To give the flavour of the DVV theory we start with bosonic model with 24 dimensions and bosonic twist fields.

The action of bosonic R^{24} sigma model is

$$S = \frac{1}{2\pi} \int d\tau d\sigma (\partial_\tau X_I^i \partial_\tau X_I^i - \partial_\sigma X_I^i \partial_\sigma X_I^i) \tag{3.1.1}$$

where $0 \leq \sigma \leq 2\pi$, i=1,...,D; I=1,2....,N and the fields take values in $S^N R^D \equiv (R^D)^N / S_N$ with the boundary conditions

$$X^i(\sigma + 2\pi)\sigma_g(0) = gX^i(\sigma)\sigma_g(0) \tag{3.1.2}$$

where g belongs to permutation group S_N. The Hilbert space is consist of sectors corresponding to conjugacy classes $[g]$.

$$H(S^N R^8) = \bigoplus_{[g]} H_{[g]} \tag{3.1.3}$$

where the conjugacy classes are consist of partitions $\{N_n\}$, can be represented as

$$[g] = (1)^{N_1}(2)^{N_2}...(s)^{N_s} \tag{3.1.4}$$

where (l) is a cycle in [g] and N_n is the repetition number of a cycle of the same length. The centralizer group of any element in $[g]$ is isomorphic to

$$C_g = \prod_{n=1}^{s} S_{N_n} \;\times\; Z_n^{N_n} \tag{3.1.5}$$

where S_{N_n} permutates identical n_n cycles and $Z^{N_n} n$ acts within a cycle of length (n). The Hilbert space decomposes into

$$H_{N_n} = \bigotimes_{n=1}^{s} S^{N_n} H_n \tag{3.1.6}$$

where the Z_n invariant subsector H_n is spanned by the following conformal fields constructed by gluing the fields in a Z_n invariant sector. This expression is periodic around a circle of length 2π:

$$Y_{n_\alpha}^i(\sigma, \tau) = \frac{1}{\sqrt{n}} \sum_{I \,\epsilon\, (n_\alpha)} X_I^i(\sigma, \tau) \tag{3.1.7}$$

Considering the fields

$$X(ze^{2i\pi}, \bar{z}e^{-2i\pi}) = h^{-1}g_c hX(z, \bar{z}) \tag{3.1.8}$$

with $g_c \, \epsilon \, [g]$ we define the following field which have trivial monodromy

$$Y_{n_\alpha}[h](\sigma, \tau) = \frac{1}{\sqrt{n}} \sum_{I \, \epsilon \, (n_\alpha)} (hX)^i_I(\sigma, \tau) \tag{3.1.9}$$

Consequently an S_N invariant twist vertex is defined as follows:

$$\sigma_{[g]}[\{k_\alpha\}](z, \bar{z}) = \frac{1}{N!} \sum_{h \epsilon S_N} : e^{i \frac{1}{\sqrt{n_\alpha}} \sum_i k^i_\alpha Y^i_\alpha} : \sigma_{h^{-1}g_c h}(z, \bar{z}) \tag{3.1.10}$$

the conformal dimension of the twist fields are given by

$$< \sigma_n | T(z) | \sigma_n >= \frac{\Delta_n}{z^2} \tag{3.1.11}$$

in [22] the dimension for Z_n twist fields were calculated and given as $\Delta_n = \frac{d}{24}(n - \frac{1}{n})$. Consequently the conformal dimension of a Z_n invariant vertex is

$$\Delta_n[k] = \Delta_{(n)} + \frac{k^2}{8n} = \frac{D}{24}(n - \frac{1}{n}) + \frac{k^2}{8n} \tag{3.1.12}$$

so with the help of formula $\Delta_g = \sum_{n=1}^{s} N_n \Delta_{(n)}$ given in [21], one finds

$$\Delta_g[\{k_\alpha\}] = \frac{D}{24}(N - \sum_{n=1}^{s} \frac{N_n}{n}) + \sum_\alpha \frac{k_\alpha}{8n_\alpha} \tag{3.1.13}$$

One can also check that this vertex is invariant under exchange of momenta k_α corresponding the cycles of (n_α) of same length.

The DVV interaction vertex [13] is simply corresponds to transposition of the elements I and J by the action of group element $g_{IJ} = 1 - E_{II} - E_{JJ} + E_{IJ} + E_{JI}$, where E_{IJ} are matrix unities.

The twist fields σ_g have following OPE

$$\sigma_{g_1}(z, \bar{z}) = \sigma_{g_2}(0) = \frac{1}{|z|^{2\Delta_{g_1} + 2\Delta_{g_2} - 2\Delta_{g_1 g_2}}} (C^{g_1 g_2}_{g_1, g_2} \sigma_{g_1 g_2}(0) + C^{g_2 g_1}_{g_1, g_2} \sigma_{g_2 g_1}(0) + ... \tag{3.1.14}$$

here we have two leading terms since there are two different ways to trace the path around z and 0. It is clear that $g_1 g_2$ and $g_2 g_1$ belongs to same conjugacy class and hence, $\Delta_{g_1 g_2} = \Delta g_2 g_1$.

Therefore the twist field σ_{IJ} acting on the highest weight state $\sigma_g(0)|0>$ creates the states $\sigma_{g_{IJ}g}(0)|0>$ and $\sigma_{gg_{IJ}}|0>$. An arbitrary element g has decomposition $(n_1)(n_2)...(n_k)$ and describes a configuration with k strings. Joining and spliting operations of the strings by the action of g_{IJ} is given by the properties of the symmetric group, stated in Appendix-C, and can be summarized as follows, without loss of generality:

$$(n_1)(n_2)...(n_k) \rightarrow \begin{cases} (n_1)^{(1)}(n_1)^{(2)}(n_2)...(n_k) & \text{if I and J } \epsilon\ (n_1)\text{:SPLITTING} \\ (n_1 + n_2)...(n_k) & \text{if I } \epsilon(n_1) \text{ and J } \epsilon(n_2)\text{:JOINING} \end{cases}$$

consequently the Lorentz invariant bosonic DVV vertex can be written in the following form

$$V_{int} = -\frac{\lambda N}{2\pi} \sum_{IJ} \int d^2 z |z| \sigma_{IJ}(z, \bar{z}) \tag{3.1.15}$$

with the Z_2 twist field defined by the action above with conformal dimension $(\frac{3}{2}, \frac{3}{2})$ and a running coupling constant λ which scales with characteristic length as l^{-1}. This term in principle breaks the conformal invariance of the action and defines an expansion around the IR fixed point.

The perturbation theory to a desired order is easy to define now, since the action of Z_2 fields constitute the tool to change the topology of world-sheet and provide the corresponding levels in perturbation theory as in Polyakov picture. Issues about the moduli space of the matrix string will be discussed separately and left to the 4th chapter.

3.2 Three Amplitudes from S_N Orbifold Sigma Models

In this section we will cover a concrete application of the SET method by applying it in calculation of the tree amplitudes of the bosonic S_N R^{24} orbifold sigma model. We will not go into details most of the section is review of [22]. Throughout the calculation of the bosonic tree amplitude one has to calculate following type of correlators, where $g_0 = (n_0)(N - n_0)$,

$g_\infty = (n_\infty)(N - n_\infty)$ and the full vertex is as defined in equation 3.1.10 The g_{IJ} and g_{KL} are vertices that transpose the elements IJ and KL.

$$G(u, \bar{u}) =< \sigma_{g_\infty}[k_1, k_2](\infty)\sigma_{IJ}(1)\sigma_{KL}(x)\sigma_{g_0}(0)[k_3, k_4] > \tag{3.2.1}$$

The main difficulty is one cant apply Wick's theorem to this correlator since we have a Riemann surface of nontrivial topology with local and global monodromies. Only way to use Wick's theorem is to uniformize the background on which the string propagates. This acts technically as considering a mapping from a Riemann surface R to original surface with cuts S that contains all the local information so that topology is made manifest. For calculation of the following propagator (by using the SET construction to extract $G(u, \bar{u})$ from 3.2.1)

$$G^{ij}_{MS}(z, w) = \frac{\langle \partial X^i_M(z)\partial X^j_S(s)\sigma_\infty[k_1, k_2](\infty)\sigma_{IJ}(1)\sigma_{KL}(x)\sigma_0(0)[k_3, k_4]\rangle}{\langle \sigma_\infty[k_1, k_2](\infty)\sigma_{IJ}(1)\sigma_{KL}(x)\sigma_0(0)[k_3, k_4]\rangle} \tag{3.2.2}$$

The indices M and S refers to the correlators defined in different pieces of the Riemann sphere (for tecnicalities consult [23]) and ij is the space index. This function is multivalued on original z-sphere. One uses the unique holomorphic map from z-sphere with twist fields to t-sphere manifesting the nontrivial monodromies around ∞, 1, u, 0.

$$z(t) = \frac{t^{n_0}(t - t_0)^{N - n_0}}{(t - t_\infty)^{N - n_\infty}} \frac{(t_1 - t_\infty)^{N - n_\infty}}{t_1^{n_0}(t_1 - t_0)^{N - n_0}} \tag{3.2.3}$$

the uniqueness is guarantied by the unique representation theorem of meromorphic functions (Appendix A). After using this trick the Wick theorem is applicable as long as one chooses the definite Riemann surface, namely one of the N roots of the 3.2.3. The parametrization of the t-sphere with the only modulus namely the Z_2 twist field location x is as follows

$$t_0 = x - 1 \tag{3.2.4}$$

$$t_\infty = x - \frac{(N - n_\infty)x}{(N - n_0)x + n_0} \tag{3.2.5}$$

$$t_1 = \frac{N - n_\infty - n_0}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N - n_\infty)x}{n_\infty((N - n_0)x + n_0)} \tag{3.2.6}$$

and the reparametrization of the map in terms of the $t_2 = x$ is

$$u=u(x)=(n_0-n_\infty)^{(n_0-n_\infty)}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\Big(\frac{N-n_0}{N-n_\infty}\Big)^{N-n_\infty}\Big(\frac{x+\frac{n_0}{N-n_0}}{x-1}\Big)^{N}\times\Big(\frac{x-\frac{N-n_0-n_\infty}{N-n_0}}{x}\Big)^{N-n_0-n_\infty}$$
$$\Big(x-\frac{n_0}{n_0-n_\infty}\Big)^{n_0-n_\infty} \tag{3.2.7}$$

this map has $2(N-n_0)$ solutions which is the same as the number of the distinct correlators in calculation of [22]. Then the corresponding Green's function is

$$G^{ij}_{MS}=-\delta^{ij}\frac{t'_M(z)t'_S(w)}{(t_M(z)-t_S(w))^2}-\sum_{A,B=1}^{4}\frac{k^i_Ak^j_Bt'_M(z)t'_S(w)}{(t_M(z)-\Omega_A)(t_S(w)-\Omega_B)} \tag{3.2.8}$$

where the total momentum is conserved $k_1+k_2+k_3+k_4=0$ and Ω_A corresponds to the twist locations on t-sphere namely $\infty, t_\infty, t_0, 0$. This function simplifies by using the fact that $\Omega_3=\infty$ term vanishes.

one can obtain the stress energy tensor by implementing it to the formula

$$T(z)=-\frac{1}{2}\sum_{i=1}^{D}\sum_{I=1}^{N}\Big(\partial X^i_I(z)\partial X^i_I(w)+\frac{1}{(z-w)^2}\Big) \tag{3.2.9}$$

and obtain

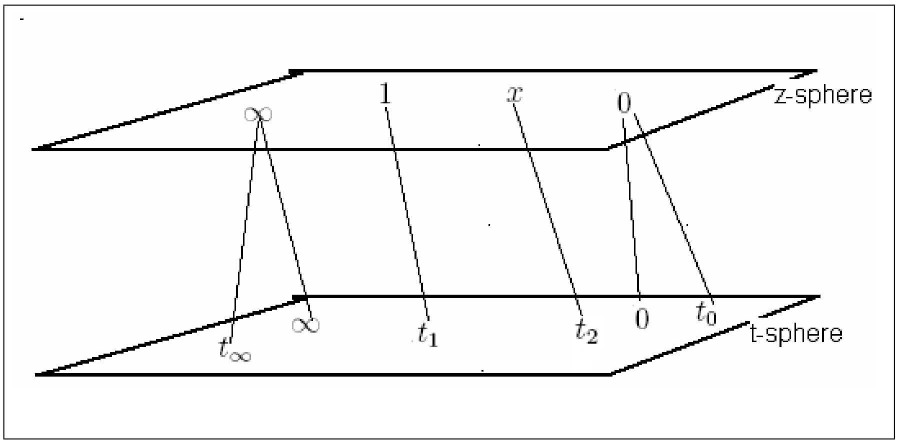
$$\langle\langle T(z)\rangle\rangle=\sum_M\frac{D}{12}\Big(\big(\frac{t''_M(z)}{t'_M(z)}\big)'-\frac{1}{2}\big(\frac{t''_M(z)}{t'_M(z)}\big)\Big)+\sum_{A,B;M}\frac{k^i_Ak^j_B(t'_M(z))^2}{(t_M(z)-\Omega_A)(t_M(z)-\Omega_B)} \tag{3.2.10}$$

by investigating the short distance behavior around the twist field one can get the differential equation for correlator 3.2.1 solution is easily obtained by direct integration

$$<G>=Res_x<T(z)> \tag{3.2.11}$$

the homogeneous solution is given by with an undetermined multiplicative constant $C(g_0,g_\infty)$. Rest of the calculation is technicality. For the solution and the matching of the constant in different regions check original papers [22-24]. We dint go into technical details, however we note that the Virasoro amplitude of bosonic strings was reproduced by the techniques mentioned above.

Figure 3.1: N times covering of z-sphere by t-sphere.



3.3 Calculation of Loop Amplitudes by SET method

In this section we will tried to justify that one loop amplitude calculated from S_N orbifold theory is corresponds to the string vacuum amplitude to correspond the conjectured equivalence between MST and typeIIA strings. This calculation was not demonstrated before so it will be original part of this thesis. The first part is even independently formulated is a result mentioned in [20], we justify this result here by using Rieamnn-Hurwitz formula.

The crucial part of the calculation lies within the theory of Riemann surfaces i.e. uniformization theory, which is stated in the Appendix A. We will not prove the mathematical results here but direct the reader to references [27],[32] and [33]. In papers by Arutyunov and Frolov [22-24] we see the simplest version of the uniformization, a z-sphere is mapped to t-sphere to make the correlation function 3.2.1 single valued in order to use Wick's theorem. The idea is that correlation function shows right short distance behavior according to the twist field action. The topology of the complicated Riemann surface is than made manifest via uniformization. We can consider the original surface as N Riemann sheets connected by certain branch points, which are the fixed points of the orbifold model in another words, corresponding the twist field locations. The Riemann surface of the previous chapter has the following structure on z-sphere:

1. there are two strings at infinity corresponding to branch points of order n_∞ and $N - n_\infty$
2. there are two strings at zero corresponding to branch points of order n_0 and $N_{-}n0$

3. there are two branch points of order two that corresponds to interaction points: at $z = x$ and at $z = 1$

Fields X^μ are maps from worldsheet to orbifold geometry, where they manifest the above multi-valued behavior in the vicinity of the branch points. In the construction of previous section world-sheet is z-sphere and components of the holomorphic one form ∂X^μ are holomorphic functions on the surface parametrized with z-coordinate. On the orbifold geometry, with certain fixed points located at ∞, 1, 0 and x of the type mentioned above. The question is then, can we possibly make this fields single valued mapping the orbifold to another surface. This is possible in our case since global monodromy is trivial on the orbifold we defined by the S_N invariant fields, that is to say the ∂X^μ is single valued on the circle surrounding all the twist field locations[21]. We denote the uniformization function of order N (unremified covering map) as π:

$$\pi : S' \longrightarrow S \tag{3.3.1}$$

and let g' and g denote the genus of the surfaces S' and S respectively. According to Riemann-Hurwitz formula [32], we have the following identity:

$$(g' - 1) = N(g - 1) + \frac{1}{2}B \tag{3.3.2}$$

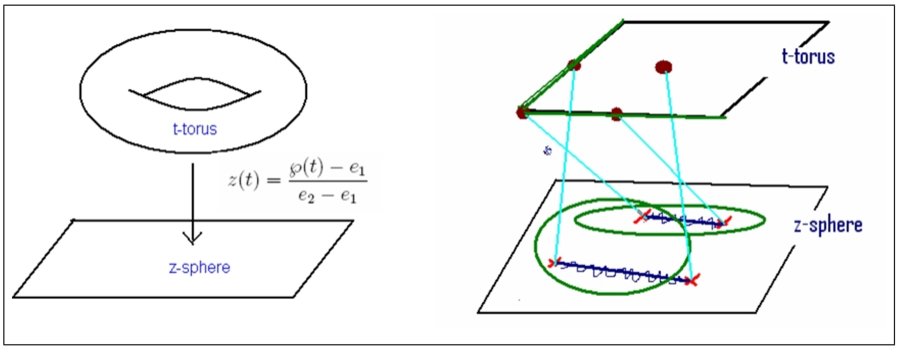
where B is the ramification number simply defined by total number of the missing solutions of the mapping $\pi(z) = z'$. On a branch point this is just (order of branch point-1). We calculate below the B number for the map z(t) in the previous section:

$$\begin{aligned} B_{z(t)} = (N - n_\infty - 1) + (n_\infty - 1) + (N - n_0 - 1) + (n_0 - 1) + (2 - 1) + (2 - 1) \\ = 2(N - 1) \end{aligned}$$

by using this in (3.2.2) and using g'=0 we obtain that $g' - 1 = N(-1) + (N - 1)$ which implies g'=0. This means topology of the orbifold is equivalent to a sphere. This is the reason we call the amplitude given by 3.2.1 a 'tree amplitude'.

We propose the following construction as a 'loop amplitude' of the S_N orbifold σ-model:

Figure 3.2: Mapping from torus to sphere in $<\sigma_2\sigma_2\sigma_2\sigma_2>$ calculation.



I- The external strings at 0 and ∞ has the following behavior on surface S' in terms of coordinates t defined on the surface:

$$\pi(t) \sim c_i^0 \times (t - t_i^0)^{n_i} \tag{3.3.3}$$

$$\frac{1}{\pi(t)} \sim c_j^\infty (t - t_j^{infty})^{n_j} \tag{3.3.4}$$

we will see that according to theorem 4.1.5 of appendix A this completely fixes the function up to multiplicative constants. On the other hand, it is very complicated to parameterize the surfaces with genus more than 1, we will give explicit map only for genus one case. The construction is still valid given the parametrization of the surface.

II- We must get internally the interaction points from this map which looks like

$$\pi(t) - \pi(t_k) \sim (t - t_k)^2 \tag{3.3.5}$$

III- The only way to create a genus is two adjacent $\mathbb{Z}_2$ twist fields which will divide the incoming string into two and reconnect at a different location. This is very crucial and non-trivial point since the fundamental cycles of the surface S' must corresponds to some cycles on the z-sphere that surrounds the twist fields. This is what happens in the four $\mathbb{Z}_2$ calculation. The first cycle corresponds to $\mathcal{A}$ and the other one corresponds to $\mathcal{B}$ cycles on the torus:

the corresponding map is :

$$z(t) = \frac{\wp(t) - e_1}{e_2 - e_1} \tag{3.3.6}$$

In the meanwhile a generic map (which corresponds to a map between z-sphere and t-sphere) for tree amplitude of S_N model with arbitrary number of incoming and outgoing strings is given uniquely up to a constant considering the condition **I** above by:

$$f(t) = C(t_i^0, t_j^\infty; \xi_0, \xi_\infty, \xi_1) \frac{\prod_{i=1}^{\mathbf{s}_0} (t - t_i^0)^{n_i^0}}{\prod_{j=1}^{\mathbf{s}_\infty} (t - t_j^\infty)^{n_j^\infty}} \tag{3.3.7}$$

where $N = \sum_i n_i^0 = \sum_j n_j^\infty$ and $f(\xi_0) = 0$,$f(\xi_\infty) = \infty$,$f(\xi_1) = 1$ which correspond to three points on z-sphere fixed by projective transformations.

The function that corresponds to 'loop amplitude' is given similarly by theta functions (appendix A)instead:

$$f(t) = C(t_i^0, t_j^\infty; \xi_0, \xi_\infty, \xi_1) \frac{\prod_{i=1}^{\mathbf{s}_0} \theta(t - t_i^0)^{n_i^0}}{\prod_{j=1}^{\mathbf{s}_\infty} \theta(t - t_j^\infty)^{n_j^\infty}} \tag{3.3.8}$$

this is a map from t-torus to z-sphere. One can calculate possible numbers of the $\mathbb{Z}_2$ points by using Riemann-Hurwitz formula for one incoming and one outgoing string of length N which is conjectured to correspond to vacuum loop amplitude of (bosonic)string theory :

$$B = N - 1 + N - 1 + n_2$$

$$0 = -N + 2N - 2 + n_2$$

so we get $n_2 = 2$ if two locations are distinct. There can possibly be a $\mathbb{Z}_3$ point instead. This is very crucial since $\mathbb{Z}_3$ corresponds to the resolution of the case when two $\mathbb{Z}_2$ coincide on the same Riemann sheet. This will be discussed separately in the last chapter. One can ask why we don't have situations like one twist location surrounding the other? This is not allowed in the same integral since we have a well defined radial ordering on the z-sphere. This can be problematic if one tries to define the Z_2 fields on torus instead. This is left to the discussion of the two loop amplitudes by considering two Z_2 field correlations on a twisted torus[29].

The above function for the one string case

$$z(t) = \Big(\frac{\theta(t-t_0)}{\theta(t_1-t_0)}\Big)^N \Big(\frac{\theta(t_1-t_\infty)}{\theta(t-t_\infty)}\Big)^N \tag{3.3.9}$$

uniformizes the following correlation function in S_N orbifold model:

$$G(z,w)_{i,j} = \frac{\langle \partial X_i(z)\partial X_j(z)\sigma_N(\infty)\sigma_{IJ}(1)\sigma_{KL}(x)\sigma_N(0)\rangle}{\langle \sigma_N(\infty)\sigma_{IJ}(1)\sigma_{KL}(x)\sigma_N(0)\rangle}$$

we can reduce the the sum by using n cyclic symmetry of the each asymptotic twists:

$$G = \langle\langle \partial X_i(z)\partial X_j(z)\sigma_N(\infty)\sigma_{1J}(1)\sigma_{1J}(x)\sigma_N(0)\rangle\rangle \tag{3.3.10}$$

each of N different correlators corresponds to a root of the the map z=$\Big(\frac{\theta(t-t_0)\theta(t_1)}{\theta(t_1-t_0)\theta(t)}\Big)^N$ So by uniformizing this correlation function and using the fact that the correlation function $< \partial X^i(t(z))\partial X(t(w)^j >= \frac{-1}{(t(z)-t(w))^2}$ we obtain the following Green's function on the universal covering of the torus $\mathbb{C}$ (note that we suppress the index M and S which labels the roots t(z))

$$G(z,w) = -\delta^{ij}\frac{t'(z)t'(w)}{(t(z)-t(w))^2} \tag{3.3.11}$$

to find the full Green's function on the torus we use image method mentioned in [30]. Basic idea of the construction is identifying the points $t \sim t+\omega$ on the universal cover $\mathbb{C}$ by summing up the Green's functions:

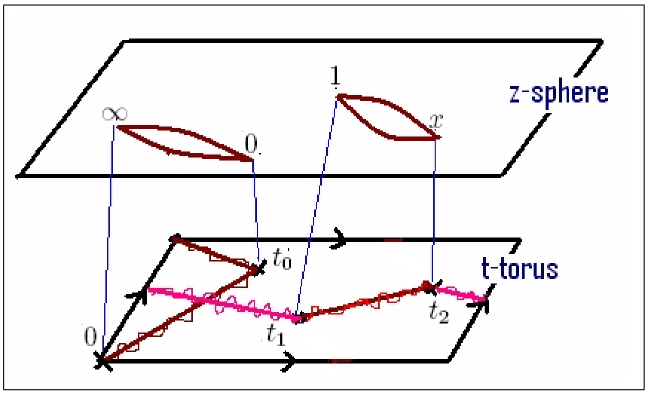
$$G_T(z,w) = -\sum_{\omega\,\epsilon\,\mathbb{L}} \frac{t'(z)t'(w)}{(t(z)-t(w)-\omega)^2} - \frac{t'(z)t'(w)}{(t(z)-t(w))^2} \tag{3.3.12}$$

this Green's function satisfies the required monodromy conditions in the vicinity of the branch points. On the other hand one can add holomorphic differentials without effecting the local analytical properties. The holomorphic differentials are constants on the torus and on the sphere it is $C \times \frac{dt}{dz}dz$ so we obtain the following form of the Green's function

$$G_T(z,w) = -t'(z)t'(w)\wp(t(z)-t(w)) + A(\tau)\times t'(z)t'(w) \tag{3.3.13}$$

for four Z_2 calculation this is very simple. The global monodromy conditions corresponds to integration on the fundamental cycles of the torus. if one considers the condition 4.2.2

Figure 3.3: The pull back of the cycles on the sphere are homotopic to the closed lines on torus in the figure.



$$\oint_{C_1} dzt'(z)\wp(t(z)-t_w) + \oint_{C_1} dz\ t'(z)A + \oint_{C_1} d\bar{z}\bar{t}'(\bar{z})B = 0 \qquad (3.3.14)$$

$$\oint_{C_2} dzt'(z)\wp(t(z)-t_w) + \oint_{C_2} dz\ t'(z)A + \oint_{C_2} d\bar{z}\bar{t}'(\bar{z})B = 0 \qquad (3.3.15)$$

where C_1 corresponds to $[0,\tau]$ and C_2 corresponds to $[0,1]$ cycles so by changing the variables $dz\ t'(z) = dt$ we obtain the same condition as 2.4.31.

$$\int_0^{\tau} dt\wp(t-t_w) + \int_0^{\tau} dtA + \int_0^{\bar{\tau}} d\bar{t}B = 0 \qquad (3.3.16)$$

$$\int_0^{1} dt\wp(t-t_w) + \int_0^{1} dtA + \int_0^{\bar{1}} d\bar{t}B = 0 \qquad (3.3.17)$$

this two equation s give the right holomorphic differentials as in 2.4.36 and by considering the Schwarzian derivative term $\{t,z\} = \frac{1}{12}c\left(\frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2\right)$ with $t' = \left(\frac{e_2-e_1}{\wp'(t)}\right)$ one obtains

$$< T(z) >= \left(\frac{e_2-e_1}{\wp'(t)}\right)^2\left[\wp(2t) - I - \frac{\pi}{Im\tau}\right] \qquad (3.3.18)$$

this simply shows that our method works for the simplest case. However, things are not that smooth for the generic case $N > 2$. First of all the cycles that corresponds to global monodromy on the sphere does not corresponds to the fundamental cycles of the torus (figure 3.3). On the other hand one can show that the integrals give constants A and B as zero order in N for each root t_M. This is good news since in the expansion of the $\{t,z\}$ around x we obtain terms of order N. This means that terms will dominate in the large N and we don't need to find the holomorphic differentials explicitly. Now we extract that terms from

$$\{t,z\} = \frac{c}{12}\Big(\frac{t'''}{t'} - \frac{3}{2}\Big(\frac{t''}{t'}\Big)^2\Big) \tag{3.3.19}$$

.

Before that we remind the map and its derivative:

$$z(t) = z(t) = \Big(\frac{\theta(t-t_0)}{\theta(t_1-t_0)}\Big)^N\Big(\frac{\theta(t_1)}{\theta(t)}\Big)^N = C\times\Big(\frac{\theta(t-t_0)}{\theta(t)}\Big)^N \tag{3.3.20}$$

$$\frac{dt}{dz} = \frac{1}{NC}\Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^{N-1}\frac{\theta(t)^2}{\theta'(t-t_0)\theta(t)-\theta'(t)\theta(t-t_0)} \tag{3.3.21}$$

for simpicity we chose to write $W = (\theta'(t-t_0)\theta(t) - \theta'(t-t_0)\theta(t))$ and by direct diferentiation one can easily derive that

$$\frac{t''}{t'} = \frac{1}{NC}\Big((1-N)\Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^n + \Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^{N-1}\frac{2\theta(t)\theta'(t)W - W'\theta(t)^2}{W^2}\Big) \tag{3.3.22}$$

what we need is $Res_x t,z = \frac{1}{2\pi i}\oint_{c_x} dz\frac{d}{dz}\Big(\frac{t''}{t'}\Big)$ this is equal to the integral on the torus $\frac{1}{2\pi i}\oint_{c_{t_2}} dt\frac{d}{dt}\Big(\frac{t''}{t'}\Big)$. One has to be carefull here remember we supressed the indices of the functions $t(z)$ the transformation can be done only for a specific root $t_M(z)$ since in fact $z(t)$ is multivalued and $z(t_M))$ is single valued. Onother remark is one does not need to expand all terms in 3.3.22. What we need is the coefficient of the terms of type $(t-t_2)^{-1}$ and the highest order term in N. We remind the reader that $Wt_2 = 0$ which corresponds to the location of the second order branch point or twist field location on sphere $x = z(t_2)$. Using this fact and up on inspection the term is given by

$$lim_{N\to\infty} Res_x\Big(\frac{t''}{t'}\Big) \sim \Big(\frac{1}{NC}(N-1)(N-2)\Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^N \times \frac{1}{\theta(t)\theta(t-t_0)}W'\Big)|_{t_2} \sim \frac{N}{x}\Xi(x) \tag{3.3.23}$$

where $\Xi(x) = \Big(\frac{\theta''(t-t_0)}{\theta(t-t_0)} - \frac{[}{\theta''(t)}\theta(t)\Big)|_{t_2}$. By a very similar inspection of the following term

$$\Big(\frac{t''}{t'}\Big)^2 = \frac{1}{(NC)^2}\Big((n-1)^2\Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^{2N} + \Big(\frac{\theta(t)}{\theta(t-t_0)}\Big)^{2N-2}\frac{(2\theta(t)\theta'(t)W - W'\theta(t)^2)^2}{W^4}\Big) \tag{3.3.24}$$

one can easily derive that we obtain the same term as 3.3.26 from 3.3.24. Eventually the Schwarzian derivative term reduces to the following around x in the large N limit:

$$lim_{N\to\infty} Res_x\{t, z\} \sim \frac{n}{24x}\Xi(x) \tag{3.3.25}$$

by taking into account that we have N terms corresponding to each root $t_M(z)$ so what we obtain is

$$lim_{N\to\infty} Res_x < T(z) > \sim \frac{N^2}{24x}\Xi(x) \tag{3.3.26}$$

3.4 Loop Amplitudes From Other Methods

In this section we will move out of track and talk about another method of showing the correspondence between DVV theory and string theory. It was first proposed by Grignani, Orland, Pniak and Semenoff that in the null compactification of the string theory one can show that the moduli sapce of the string theory is equivalent to the DVV theory defined on a torus. We will just report the approach in a heuristic way for the information of the reader and don't go into technical details.

This approach mainly based on Riemann surface theory proposes a way to represent the string world-sheets of arbitrary genus by a branched coverings of a torus.

$$\Sigma_g \xrightarrow{f} T \tag{3.4.1}$$

they studied the situation where the target space has two compact dimensions. Both world-sheet and space-time has taken to be Euclidean theories. First starting with light-cone compactification

$$(X^0, X^9) \;\sim\; (X^0 + \sqrt{2}\pi i R, X^9 - \sqrt{2}\pi R) \tag{3.4.2}$$

where factor i in the first compact direction is discussed to define the correct partition function for arbitrary genus. Then a second compactification is considered

$$(X^0, X^9) \;\sim\; (X^0 + \beta, X^9) \tag{3.4.3}$$

First compactification was discussed to leave GSO projection invariant. However the second compactification was mentioned to introduce a temperature $T = \frac{1}{k_B \beta}$ by modifying the GSO projection in a way that makes the space-time fermions anti-periodic.

Beginning with a parametrization of the surface of arbitrary genus in terms of Abelian one forms and defining the expansion of the dX in terms of holomorphic and anti-holomorphic abelian differentials one obtain g constraints. Next the authors discusses a way of parametrizing the action in terms of the period matrix of the surface Σ_g, and eventually compactification leads to a constraint in the period matrix such that the number of moduli is reduced from (3g-3) to (2g-3). Whenever the compact dimension are decompactified in the limit to ∞ the full moduli is recovered.

Basically claim of this work is that Riemann surfaces establish that moduli space of infinite momentum-frame superstring world-sheets are identical to those of branched-cover instantons in the matrix-theory conjectured to describe M-theory. In more concrete words the proposed model is a correspondence between string theory considered on $M \times S^1 \times S^1$ target space and DVV theory defined on T^2 embedded to target space $S_N M$. This theory was able to reproduce zero point amplitude (so called thermodynamic partition function) of string theory perfectly.

The main difference here is We we have two compact parameters instead of one. We have to regarg both boundary conditions as twisted (and also the spin structures enter in the picture for each cycle of the toeus for details check [29],[31]and [35])i.e. only considering bosonic theory:

$$X^i(\sigma_1 + 2\pi, \sigma_2) = P X^i(\sigma_1, \sigma_2) P^{-1} \tag{3.4.4}$$

$$X^i(\sigma_1, \sigma_2 + 2\pi) = Q X^i(\sigma_1, \sigma_2) Q^{-1} \tag{3.4.5}$$

where P and Q are two commuting elements in S_N group, $QP = PQ$ definitely required for consistency.

Nevertheless one can reproduce the DLCQ one loop zero amplitude as discussed in [35]. The DLCQ amplitude given by

$$E = -\frac{1}{\sqrt{2}R\beta}\mathcal{H}[e^{-\beta/\sqrt{2}R}] * \Big[\frac{1}{(4\pi^2\alpha'\tau_2)^4}\frac{|\theta_2(0,\tau)|^8}{|\eta|^{24}}\Big]_{\tau=i\nu} \tag{3.4.6}$$

the factor in front is the ration of volumes of R^8 and $R^9 \times S^1$ with compact light-cone. The action of Hecke operator $\mathcal{H}[p]$ on a function $\phi(\tau, \bar{\tau})$ is defined by

$$(H)[p] * \phi(\tau, \bar{\tau}) = \sum_{N=0}^{\infty} \frac{p^N}{N} \sum_{\binom{kr=N}{s \ mod \ k}} \phi\Big(\frac{r\tau + s}{k}, \frac{r\bar{\tau} + s}{k}\Big) \tag{3.4.7}$$

3.5 Symmetric Product on Torus and Two Loop Amplitudes

This section is about a recent result published in June 2007 by Szabo,Kadar and Cove [35] in the lines of the construction mentioned in the previous section. This paper proposes a way of calculating two loop stringtheory amplitudes in the null compactification from DVV theory defined on a torus.

Chapter 4

Appendix

4.1 Appendix-A:Theory of Riemann Surfaces and Elliptic Functions

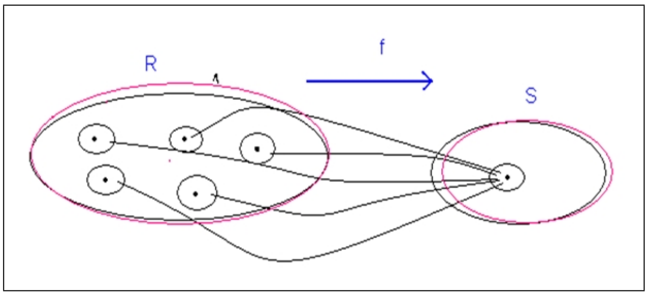
We obtain complex structure by using stereographic map and identifying the two coordinates with complex ones $(x_1, x_2, x_3) \rightarrow \frac{(x_1+ix_2)}{1\pm x_3}$. The holomorphic and anti-holomorphic coordinates on the **Riemann sphere** are defined as follows: The projections of the north pole patch $S-(0,0,-1)$ and south pole patch $S-(0,0,1)$ to R^2 is **holomorphic** and **anti-holomorphic coordinates** respectively. In short $p_+(x) = \frac{x_1+ix_2}{1-x_3}$ and $p_-(x) = \frac{x_1+ix_2}{1-x_3}$ are related by the following transformation

$$[p_-(x)]^1 = \frac{x_1+ix_2}{1+x_3} = \frac{1+x_3}{x_1+ix_2} = \frac{1+x_3}{x_1^2+x_2^2} \times (x_1 - ix_2) = \frac{x_1-ix_2}{1-x_3} = [p_+(x)]^*$$

we denote holomorphic coordinates by z and anti-holomorphic coordinates by $(\bar{z})$. A holomorphic function on the **Riemann sphere** S is defined as $f(z)$ which is function of only z and regular around $z = \infty$.

A **Riemann surface** is defined as a topological surface with a complex structure and complex structure is given by holomorphic and anti-holomorphic charts on patches. Simply the complex structure says that the surface looks like a disc D in $\mathbb{C}$(or $\bar{\mathbb{C}}$) locally. We say f (comlex valued function) is a **holomorphic function** at $p \epsilon$ R if there exist a chart $\phi : U \rightarrow D$ with $p \epsilon U$, such that the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. it is said to be holomorphic on R if holomorphic at every point p of R. Similarly a meromorphic function is either holomorphic or has singularities of the type $\frac{1}{(z-z_p)^n}$ at a point p on R, where n is called degree of the pole at p. A complex function f is said to be **meromorphic** on R if it is meromorphic at every point of R. We state the following facts without proof, for proofs

Figure 4.1: The holomorphic map f beetween compact connected Riemann surfaces



consult [32]

I- A meromorphic function on sphere $\mathbb{P}^1$ (projective plane) has rational form:

$$f(z) = \frac{P(z)}{Q(z)} \tag{4.1.1}$$

where P and Q are polinomials of certain degree. The total number of zeros and poles of this function is equal ($N_0 = N_\infty$).

II- A meromorphic function on complex torus is doubly periodic function $f(z + 2\omega_1) = f(z + 2\omega_2) = f(z)$ where ω_1 and ω_2 are generators of the two dimensional lattice $\mathbb{L}$. This function cannot have a pole of order 1. The simplest elliptic function is Weierstrass-P function $\wp(t)$ with order 2 pole at t=0, given by

$$\wp(t) = \frac{1}{t^2} + \sum_{\omega \epsilon \mathbb{L}} \left(\frac{1}{(t - 2\omega)^2} - \frac{1}{4\omega^2} \right) \tag{4.1.2}$$

where $\omega = a\omega_1 + b\omega_2$ with $a \neq 0 \neq b$. We will turn to elliptic functions after stating some facts about the topology of Riemann surfaces.

Let $f : R \to S$ be a holomorphic function between compact Riemann surfaces R and S. We say f has a valency number v ($v \ \epsilon \ \mathbb{N}$)at point p if f locally looks like $z \to z^v$ around point p. The degree of the map f is defined then by the total valency number of preimage $f^{-1}(s)$ of point s on S. We emphasize the fact that this does not depend on the chosen point s.

$$deg(f) = \sum_{p \epsilon f^{-1}(s)} v_p(f) \tag{4.1.3}$$

at almost all points f looks like $z \to z$ except some special points where some solutions coincide or crudely missing. The total number of missing solutions is called as total branching index (or remification number)

$$B = \sum_{s\epsilon S} \sum_{r\epsilon f^{-1}(s)} (v_f(r) - 1) = \sum_{s\epsilon S} [deg(f) - (\text{total number of preimages})] \tag{4.1.4}$$

this two numbers are related by famous Riemann-Hurwitz formula, which relates the topologies of the two surfaces R and S:

$$g_R - 1 = deg(f)(g_S - 1) + \frac{1}{2}B \tag{4.1.5}$$

where g denotes the genus of the Riemann surface.

One should remember that oriented compact 2dim surfaces are perfectly classified by their genus, namely all genus zero and genus one surfaces are homeomorphic to sphere and torus and so on. In the case f is a rational function from R to sphere,then R is supposed to be a sphere as well, as one can observe from the formula 4.1.5. This is conluded by unique representation theorem of the maps between speheres. A meromorphic function with certain number of zeros n and poles m (repetations is ignored)is represented globally on a sphere as:

$$z(t) = const \times \frac{\prod_{i=1}^{n}(t - t_i)^{k_i}}{\prod_{j=1}^{m}(t - t_j)^{k_j}} \tag{4.1.6}$$

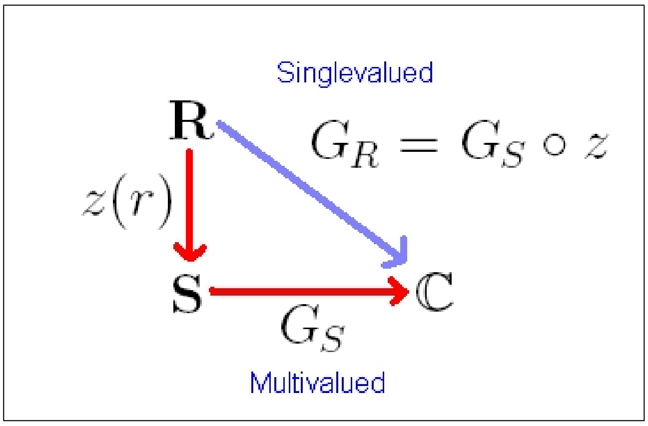
where k_i and k_j degree of the zeros and poles respectively, and total numbers of the zeros and poles are equal ($N_0 = N_\infty$). This can also be considered in a similer manner for the functions on a torus with certain numbers of zeros and poles. For that we have to make use of a function with a simple zero on torus which is called θ function. The definition of the theta function can be related to weierrstrass $\wp$ function but we prefer the following product expansion:

$$\theta(u, \omega) = uexp(-\eta_1 u^2/2\omega_1) \prod_{\omega\epsilon\mathbb{L}} \left[(1 - \frac{u}{\omega}) \exp(\frac{u}{\omega} + \frac{u^2}{2\omega^2})\right] \tag{4.1.7}$$

the ω is defined as half periods as in **??**Pfunc) and $\eta_1 = \frac{1}{2}\int_{-\omega_1}^{\omega_1} \wp du$. The meromorphic function given on the torus defined by the lattice $\mathbb{L}$ has the following form:

$$z(t) = const \times \frac{\prod_{i=1}^{n}(\theta(t - t_i))^{k_i}}{\prod_{j=1}^{m}(\theta(t - t_j))^{k_j}} \tag{4.1.8}$$

Figure 4.2: The uniformization of a function



this form is used to uniformize the multivalued functions on a sphere. Before we mention some crucial properties of the elliptic functions (functions defined on the torus) lets say something about the procedure. Let S be a Riemann sphere and G_S be a multivalued complex function on the sphere. There exist a a surface R and a map $F : R \to S$ such that function $G_R = G_S \circ F$ is single valued (4.2).

The physical procedure the functions of the form 4.1.7 used in the text uis diverse so we give here the most important properties of the theta function. First we note that the function has also two different representations :

$$\theta(v,\tau) = \sum_{-\infty}^{\infty} exp(\pi i n^2 \tau + 2\pi i n v) \tag{4.1.9}$$

$$\theta(v,\tau) = \prod_{m=1}^{\infty} (1-q^m)(1+zq^{m-1/2})(1+z^{-1}q^{m-1/2}) \tag{4.1.10}$$

with $q = exp(2\pi i \tau)$ and $z = exp(2\pi i v)$. It is also useful to define theta function with characteristic

$$\theta[^a_b](v,\tau) = exp(\pi i a^2 \tau + 2\pi i a(v+b))\,\theta(v+a\tau+b,\tau) = \sum_{n=-\infty}^{\infty} exp(\pi i(n+a)^2\tau + 2\pi i(n+a)(v+b)) \tag{4.1.11}$$

We call the theta function we defined θ_1. It has following transformation properties

$$\theta_1(v+1,\tau) = \theta_1(v,\tau)$$
$$\theta_1(v+\tau,\tau) = exp(-\pi i\tau - 2\pi iv)\theta_1(v,\tau) \tag{4.1.12}$$

There are three other θ-functions defined by the translation of the argument by half periods (we use here $2\omega_1 = 1$ and $2\omega_2 = \tau$):

$$\theta_2(v,\tau) = \theta_1(v+\frac{1}{2},\tau)$$
$$\theta_3(v,\tau) = z^{1/2}q^{1/8}\theta_1(v+\frac{1}{2}+\frac{\tau}{2},\tau) \tag{4.1.13}$$
$$\theta_4(v,\tau) = iz^{1/2}q^{1/8}\theta_1(v+\frac{\tau}{2},\tau)$$

The series expansion forms of this functions are:

$$\theta_2(v,\tau) = \sum_{n=-\infty}^{\infty} z^n q^{n^2/2}$$
$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} z^{n+1/2} q^{(n+1/2)^2/2} \tag{4.1.14}$$
$$\theta_4(v,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2/2}$$

these functions have a simple zero in the fundamental parallelogram.

We note that in $< Z_2Z_2Z_2Z_2 >$ calculation following facts are useful.

$$\wp(v) = \wp(1/2) + \Big[\frac{\theta'(0)}{\theta_1(v)}\frac{\theta_2(v)}{\theta_2(0)}\Big] = \wp(1/2+\tau/2) + \Big[\frac{\theta'(0)}{\theta_1(v)}\frac{\theta_3(v)}{\theta_3(0)}\Big] = \wp(\tau/2) + \Big[\frac{\theta'(0)}{\theta_1(v)}\frac{\theta_4(v)}{\theta_4(0)}\Big] \tag{4.1.15}$$

where we use the convention $\wp(1/2) = e_1$, $\wp(1/2+\tau/2) = e_2$ and $\wp(\tau/2) = e_3$. We also note the relation $e_1 + e_2 + e_3 = 0$.[33]

Another useful elliptic function is Dedekind η-function defined in the following form:

$$\eta(\tau) = q^{1/4}\prod_{m=1}^{\infty}(1-q^m) = \Big[\frac{\partial_v\theta_1(0,\tau)}{2\pi}\Big]^{1/3} \tag{4.1.16}$$

4.2 Appendix-B:Geometry of Orbifolds and CFT

-Global Monodromy and Holomorphicity

The green's function g(z,w), required to satisfy local monodromy conditions, is unique up to addition of the bi-linear combination of the n_{cl} holomorphic fields $\partial X^{(n)}(z;z_i)$,satisfying classical equation of motion and local monodromy conditions. In other words one can add n_{cl}^2 terms to $g(z,w;z_i)$

$$A_{mn}\partial X^{(n)}(z;z_i)\partial\bar{X}^{(m)}(w;z_i) \tag{4.2.1}$$

to determine the constant we need to impose global monodromy conditions.i.e we need to specify what happens when $X(z,\bar{z})$ transported around the closed loop $\mathcal{C}$ which encircles two or more twist fields. The vertex can be thought as puncture,so the circles enclosing twist fields are topologically nontrivial. Twist fields provide local boundary conditions, moreover they also provide global information to fix g(z,w). To get this global information we need to know something about the background geometry of orbifolds.

A typical orbifold is $\Omega = R^d/S$ under the discrete group action $X \to \theta X + v$ on an embedding X^μ : World-sheet → Space-time, with θ: rotations and v:translations.

The most useful fact for our purposes is, when tracing contour enclosing all the twist fields the scalar field stays single valued. So the following integral vanishes

$$0 = \Delta_{C_I} X_{qu} = \oint_{C_I} dz\partial X + \oint_{C_I} d\bar{z}\bar{\partial} X \tag{4.2.2}$$

this is simply because the quantum piece of the correlation function transforms homogenously under the discreet group action

$$X_{qu} \to \theta^j X_{qu} \tag{4.2.3}$$

for our purposes what happens to classical piece is irrelevant since we omit the disconnected piece in the correlators and more important than that we work in the large N sector, namely the radius of fundamental string goes to infinity.

Each twist field corresponds to a conjugacy class of the orbifold space group.

4.3 Appendix-C: A Note on Bosonization

The simplest example of the cocycle appears in the compact scalar field theory. The separation of the chiral theories gives nontrivial commutation relations and OPEs

$$[x_L, p_L] = [x_R, p_R] = i \tag{4.3.1}$$

the field decomposed into $X = X_L(z) + X_R(\bar{z})$ and has following OPEs

$$X_L(z_1)X_L(z_2) \sim \frac{-\alpha}{2} ln(z_{12}) \tag{4.3.2}$$

$$X_R(\bar{z_1})X_R(\bar{z_2}) \sim \frac{-\alpha}{2} ln(\bar{z}_{12}) \tag{4.3.3}$$

$$X_L X_R \sim 0 \tag{4.3.4}$$

the vertex operator corresponding to state $|0; k_L, k_R >$ is

$$V_{k_L k_R}(z_1, \bar{z}^1) =: exp(ik_L X_L + ik_R X_R) : \tag{4.3.5}$$

with the following OPE

$$V_{k_L k_R}(z_1, \bar{z}_1) V_{k'_L k'_R}(z_2, \bar{z}_2) \sim z_{12}^{\alpha' k_L k'_L/2} \bar{z}_{12}^{\alpha' k_R k'_R/2} V_{(k+k)'_L (k+k')_R}(z_2, \bar{z}_2) \tag{4.3.6}$$

one tour of z_1 around z_2 brings the phase $exp(\pi i \alpha'(k_L k'_L - k_R k'_R))$ by considering the allowed momentum states of the compact scalar($k_L - k_R = \frac{2wR}{\alpha'}$ and $k_L + k_R = \frac{2n}{R}$) we conclude that the phase $exp(2\pi i(wn' + nw')) = 1$. On the other hand, exchanging $z_1 \leftrightarrow z_2$ and $k \leftrightarrow k'$ we obtain the phase $exp(i\pi \frac{\alpha'}{4}(k_L - k_R)(k'_L + k'_R) - (k'_L - k'_R)(k_L + k_R)) = exp(i\pi(nw' - wn'))$ in the RHS of the eq.2.36, although LHS symmetric. The vertex defined in the following way satisfies our inquiry for the symmetric LHS:

$$V_{k_L k_R}(z_1, \bar{z}_1) = exp(i\pi \frac{\alpha'}{4}(k'_L - k'_R)(p_L + p_R)) : exp(ik_L X_L + ik_R X_R) : \tag{4.3.7}$$

Now the representation is Bosonic for even and Fermionic for odd values of the constant $(nw' - wn)$. The extra term in front of the vertex operator is called cocyle. More generally, we define the proper vertex operator of bosonized fermionic theory as

$$V_{k_L k_R}(z_1,\bar{z}_1) = C_k(\alpha)_0 : exp(ik_L.H_L + ik_R.H_R) : \tag{4.3.8}$$

for now we assume that k_α is basis of a lattice $\Gamma = n_\alpha k_\alpha$ where we expand zero mode operators in terms of the basis of the lattice vectors: $\alpha_0 = \alpha_{0\beta} k_\alpha$. Soon we will clarify the fact that Γ just corresponds to the weight lattice of the algebra in context. The cocyle is merely

$$C_k(\alpha_0) = exp(i\pi \sum_{\alpha>\beta} n_\alpha \alpha_{0\beta} k_\beta \diamond k_\beta) \tag{4.3.9}$$

when $k \diamond k$ is even the vertex 2.38 is commuting with any vertices and if it is odd the vertex is anti-commuting with other vertices having odd k products. An example, bosonization of Spin(8) algebra, will be given in the 3rd chapter while calculating the three amplitudes of DVV matrix string theory.

$SU(4) \times U(1)$ **formalism and bosonization of SO(8) algebra**

4.4 Appendix-D: Symmetric Group

We will list some crucial properties of the Symmetric group and give some concrete examples of cycle decomposition, breaking or combining of cycles up on transposition of two elements. A cycle of length n of group element $g \, \epsilon \, S_N$ is defined by the action on the set of N elements.

$$x \xrightarrow{g} gx \xrightarrow{g} \ldots \xrightarrow{g} g^{n-1}x \xrightarrow{g} x \tag{4.4.1}$$

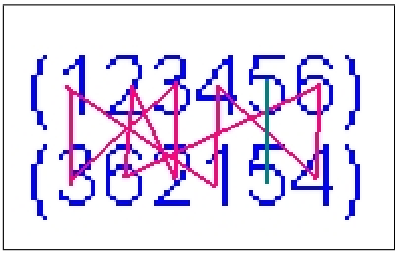
we demonstrate the fact that the conjugacy class of the elment g defined by the $[g] =: \{hgh^{-1} : for \; \forall \; h \; \epsilon \; S_N\}$ has a unique cycle decomposition. We can demonstrate this by considering the cycles of the hgh^{-1} one finds the same structure as 4.4.1:

$$hx \xrightarrow{g} ghx \xrightarrow{g} \ldots \xrightarrow{g} g^{n-1}hx \xrightarrow{g} x \tag{4.4.2}$$

which means the decomposition

$$[g] = (1)^{N_1}(2)^{N_1} \ldots (s)^{N_s} \tag{4.4.3}$$

Figure 4.3: Following the lines one can find the cycles in the diagram



defines an equivalence relation so we have the same decomposition for any element in the conjugacy class. This is simply because N_i different orbits of the group g ϵ S_N as in 4.4.1 is also has the same number of different orbits in the hgh^{-1}.

Moreover this cycles are broken when two elements in a cycle is transposed and two cycle is combined when two elements in different cycles are transposed. We can prove this fact by looking at the diagram 4.4.1. Assume we transpose the elements x and $g^j x$ in the cycle. First start with g^j applying g we get gx and so on and eventually $g(g^{j-1})x = g^j x$, which defines a cycle of length $(j-1)$. This is the same for x, starting from x and one gets $g(x) = g^{j+1}x$ and eventually $g(g^{n-1}) = x$ and this defines a cycle of length (n-j+1).

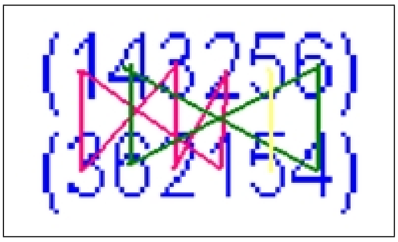
$$x \xrightarrow{g} g^{j+1}x \xrightarrow{g} \ldots \xrightarrow{g} g^{n-1}x \xrightarrow{g} x$$

$$g^j x \xrightarrow{g} gx \xrightarrow{g} \ldots \xrightarrow{g} g^{j-1}x \xrightarrow{g} g^j x$$

this can be illustrated by the diagram 4.3 and 4.4 for a simple permutation of the 6 elements.

Following the lines that closes we obtain the cycles of length (1) and (5), namely (5)(12346) .By transposition of two elements 2 and 4 the cycle of length (5)is broken into two cycles of length (3) and (2). This can also be thought vice verse.

Figure 4.4: The big cycle is broken up on transposition of the elements



4.5 Spin(8) Clifford Algebra

The description of Dirac matrices for spin(8) requires a Clifford algebra with eight anticommuting matrices. This is important in understanding the structure of LC type II A theory and $S_N R^8$ model so we present it here explicitly.

The Dirac algebra of SO(8) requires 16-dimensional matrices corresponding to reducible 8_s+8_c representation of spin(8). These matrices can be written in block form

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{a\dot{a}} \\ \gamma^i_{\dot{b}b} & 0 \end{pmatrix} \tag{4.5.1}$$

where $\gamma^i_{\dot{a}a}$ is transpose of $\gamma^i_{a\dot{a}}$. T he equation $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ is satisfied if

$$\gamma^i_{a\dot{a}}\gamma^j_{\dot{a}b} + \gamma^j_{a\dot{a}}\gamma^i_{\dot{a}b} = 2\delta^{ij}\delta_{ab} \; i, j = 1, ..., 8 \tag{4.5.2}$$

and similarly with dotted and undotted indices interchanged. A specific set of matrices that satisfies the equation is

$$\begin{aligned} \gamma^1 &= \epsilon \otimes \epsilon \otimes \epsilon & \gamma^2 &= 1 \otimes \tau_1 \otimes \epsilon \\ \gamma^3 &= 1 \otimes \tau_3 \otimes \epsilon & \gamma^4 &= \tau_1 \otimes \epsilon \otimes 1 \\ \gamma^5 &= \tau_3 \otimes \epsilon \otimes 1 & \gamma^6 &= \epsilon \otimes 1 \otimes \tau_1 \\ \gamma^7 &= \epsilon \otimes 1 \otimes \tau_3 & \gamma^8 &= 1 \otimes 1 \otimes 1 \end{aligned}$$

where $\epsilon = i\tau_2$ and τ_i are Pauli matrices. We define

$$\gamma^{ij}_{ab} = \frac{1}{2}(\gamma^i_{a\dot{a}}\gamma^j_{\dot{a}b} - \gamma^i_{a\dot{a}}\gamma^j_{\dot{a}b}) \tag{4.5.3}$$

and similarly for the $\Gamma^{ij}_{\dot{a}\dot{b}}$.

The ten dimensional Majorana has 32 real components but the Weyl $\Gamma^{11}\lambda = \lambda$ condition eliminates halve of them. In terms of the transverse subgroup SO(8) of SO(9,1), the surviving 16 components are given by $8_s + 8_c$.Dirac matrices can be expressed in this basis in terms of eight matrices γ_i defined above. The $\gamma^9 = \gamma^1...\gamma^8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Also note that Dirac equation Majorana-Weyl spinors $(\lambda^a_s, \lambda^{\dot{a}}_c)$ decomposes into left and right going components:

$$\partial_+ \lambda^a_s + \gamma^i_{a\dot{a}} \partial_i \lambda^{\dot{a}}_c = 0 \tag{4.5.4}$$

$$\partial_- \lambda^a_c + \gamma^i_{\dot{a}a} \partial_i \lambda^a_s = 0 \tag{4.5.5}$$

4.6 Appendix-F:Four graviton scattering amplitude

This is a calculation done by Arutyunov and Frolov we just review the main lines of the calculation here. The details can be

As stated in the section 2.6 S_N orbifold sigma model corrected by the DVV vertex is a well defined perturbation theory. A general amplitude of order g^n_s with straightforward definitions of asymptotic states (or highest weight states in CFT jargon)

$$< f\,|S|\,i >\sim \int \prod_i d^2 z_i < f|V(z_1)....V(z_n)|i > \tag{4.6.1}$$

the four point amplitude we wish to calculate is then

$$< f\,|S|\,i >= \frac{-1}{2}\Big(\frac{\lambda N}{2\pi}\Big)^2 < f| \int d^2 z_1 d^2 z_2 |z_1||z_1| T(V_{int}(z_1,\bar{z}_1) V_{int}(z_2,\bar{z}_2))|i > \tag{4.6.2}$$

where T means time ordering: $|z_1| > |z_2|$. Where DVV interaction vertex V_{int} defined in **??**

$$V_{int}(z,\bar{z}) = \sum_{I<J} V_{IJ}(z,\bar{z}) \tag{4.6.3}$$

The asymptotic states are created by the S_N invariant vertex where k_i and ξ_i stands for the momentum and polarization vectors respectively.

$$|i>= C_0 V_{[g_0]}[k_1,\zeta_1;k_2,\zeta_2](0,0)|0> \tag{4.6.4}$$

$$< f| = C_\infty \lim_{z_\infty\to\infty} |z_\infty|^{4\Delta_\infty} < 0|V_{[g_\infty]}[k_2,\zeta_2;k_3,\zeta_3](z_\infty,\bar{z}_\infty) \tag{4.6.5}$$

The decomposition of the group elements g_{infty} and g_0 is

$$g_0 = n_0(N-n_0), \quad g_\infty = n_\infty(N-n_\infty), \tag{4.6.6}$$

and the normalization constants of the asymptotic states are

$$C_0 = \sqrt{\frac{N!}{n_0(N-n_0)}}, \quad C_\infty = \sqrt{\frac{N!}{n_\infty(N-n_\infty)}} \tag{4.6.7}$$

We introduce the light-cone momenta of the initial and final states as:

$$k_1^+ = \frac{n_0}{N}, \quad k_2^+ = \frac{N-n_0}{N}, \quad k_3^+ = -\frac{n_\infty}{N}, \quad k_4^+ = -\frac{N-n_\infty}{N} \tag{4.6.8}$$

which satisfies the mass shell condition $k_a^+k_a^- - \mathbf{k}_a\mathbf{k}_a = 0$ for each a=1,..,4. According to [23] the S-matrix elements can be expressed as:

$$< f|S|i >= -i2\lambda^2 N^3 \delta(k_1^- + k_2^- + k_3^- + k_4^-)M \tag{4.6.9}$$

where delta function indicates the conservation of the total length of strings and the matrix element M is given by

$$M = \int d^2u|u|F(u,\bar{u}) \tag{4.6.10}$$

here we introduce the notation

$$F(u,\bar{u}) =< f|T(V_{int}(1,1)V_{int}(u,\bar{u}))|i >=$$

$$C_0C_\infty \sum_{I<J;K<L} < 0|V_{[g_\infty]}[k_3,\zeta_3;k_4,\zeta_4](\infty)T(V_{IJ}(1,1)V_{KL}(u,\bar{u}))V_{[g_0]}[k_1,\zeta_1;k_2,\zeta_2](0,0)|0 >$$

In what follows it is assumed that $|u| < 1$. The vertices includes sum over the conjugacy classes and we can express the function F as following form by using the global properties of the S_N orbifold.

$$F(u,\bar{u}) = \frac{C_0 C_\infty}{N!} \sum_{h_\infty\ \epsilon S_N} \sum_{I<J;K<L} < V_{h_\infty^{-1} g_\infty h_\infty}(\infty) V_{IJ}(1,1) V_{KL}(u,\bar{u}) V_{g_0}(0,0) > . \tag{4.6.11}$$

by using the symmetries of the action i.e. world-sheet parity symmetry and space reflection symmetry one can simplify this expression.

$$h_\infty^{-1} g_\infty h_\infty g_{IJ} g_{KL} g_0 = 1 \Rightarrow h_\infty^{-1} g_\infty h_\infty = g_0^{-1} g_{KL} g_{IJ} \tag{4.6.12}$$

$$< V_{h_\infty^{-1} g_\infty h_\infty} V_{IJ} V_{KL} V_{g_0} >=< \tilde{V}_{g_{IJ} g_{KL} g_0^{-1}} V_{IJ} V_{KL} \tilde{V}_{g_0^{-1}} > \tag{4.6.13}$$

$$< V_{h_\infty^{-1} g_\infty h_\infty} V_{IJ} V_{KL} V_{g_0} >=< \tilde{V}_{g_{I'J'} g_{K'L'} g_0^{-1}} V_{I'J'} V_{K'L'} \tilde{V}_{g_0^{-1}} > \tag{4.6.14}$$

$$< V_{h_\infty^{-1} g_\infty h_\infty} V_{IJ} V_{KL} V_{g_0} >=< V_{g_{I'J'} g_{K'L'} g_0^{-1}} V_{I'J'} V_{K'L'} V_{g_0^{-1}} > \tag{4.6.15}$$

one can also show that correlation function $F(u,\bar{u})$ is real by considering the complex conjugate of the following correlator.

$$< V_{g_\infty}[k_2,\zeta_2;k_3,\zeta_3](\infty) T(V_{IJ}(1,1) V_{KL}(u,\bar{u})) V_{g_0}[k_1,\zeta_1;k_2,\zeta_2](0,0) >^* = \tag{4.6.16}$$

$$\lim_{z_\infty \to \infty} \lim_{z_0 \to 0} |z_\infty|^{-4\Delta_\infty[\{k_3,k_4\}]} |z_0|^{-4\Delta_{g_0}[\{k_1,k_2\}]} |u^{-6}| \times \tag{4.6.17}$$

:

$$< 0| V_{g_0^{-1}}[-k_1,\zeta_1;-k_2,\zeta_2](\frac{1}{z_\infty},\frac{1}{\bar{z}_{infty}}) T(V_{KL}(\frac{1}{u},\frac{1}{\bar{u}}) V_{IJ}(1,1)) V_{g_\infty^{-1}}[-k_3,\zeta_1;-k_4,\zeta_2](\frac{1}{z_0},\frac{1}{\bar{z}_0})|0>$$

Due to the SO(8) invariance we can make the replacement -$\tilde{k}_a \to k_a$ and after performing the transformation $z \to \frac{1}{z}$ we obtain

$$(V_g[\{k_\alpha\}](z))^\dagger = z^{-2\Delta_g[\{k_\alpha\}]}V_{g^{-1}}[\{-k_\alpha\}](\frac{1}{z}) \tag{4.6.18}$$

$$< V_{g_\infty}[k_2,\zeta_2;k_3,\zeta_3](\infty)T(V_{IJ}(1,1)V_{KL}(u,\bar{u}))V_{g_0}[k_1,\zeta_1;k_2,\zeta_2](0,0) >^*=$$
$$< V_{g_\infty^{-1}}[k_2,\zeta_2;k_3,\zeta_3](\infty)T(V_{IJ}(1,1)V_{KL}(u,\bar{u}))V_{g_0^{-1}}[k_1,\zeta_1;k_2,\zeta_2](0,0) >=$$
$$< V_{g'_\infty}[k_2,\zeta_2;k_3,\zeta_3](\infty)T(V_{I'J'}(1,1)V_{K'L'}(u,\bar{u}))V_{g_0}[k_1,\zeta_1;k_2,\zeta_2](0,0) >=$$

where $h \,\epsilon\, S_N$ is the solution of $h^{-1}g_0^{-1}h = g_0$ and

$$h^{-1}g_\infty^{-1}h = g'_\infty, \quad h^{-1}g_{IJ}h = g_{I'J'}, \quad h^{-1}g_{KL}h = g_{K'L'} \tag{4.6.19}$$

now we apply this result to find the complex conjugate of $F(u,\bar(u))$

$$F(u,\bar{u})^* = \frac{C_0C_\infty}{N!}\sum_{h_\infty\,\epsilon S_N}\sum_{I<J;K<L} < V_{h_\infty^{-1}g_\infty h_\infty}(\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})V_{g_0} >^*$$
$$= \frac{C_0C_\infty}{N!}\sum_{h_\infty\,\epsilon S_N}\sum_{I<J;K<L} < V_{h_\infty^{'-1}g_\infty h'_\infty}(\infty)V_{I'J'}(1,1)V_{K'L'}(u,\bar{u})V_{g_0} >$$
$$\frac{C_0C_\infty}{N!}\sum_{h'_\infty\,\epsilon S_N}\sum_{I'<J';K'<L'} < V_{h_\infty^{'-1}g_\infty h'_\infty}(\infty)V_{I'J'}(1,1)V_{K'L'}(u,\bar{u})V_{g_0} >$$
$$= F(u,\bar{u}))$$

so $F(u,\bar{u})$ is real. On the other hand by using S_N invariance of the model one can express $F(u,\bar{u})$ in the following form

$$\begin{aligned} F(u,\bar{u}) = 2N^2\sqrt{k_1^+k_2^+k_3^+k_4^+}\Big(&\sum_{I=1}^{n_\infty} < V_{g_\infty(I)}(\infty)V_{I,I+N-n_\infty}(1,1)V_{n_0N}(u,\bar{u})V_{g_0}(0,0) > \\ &+ \sum_{I=1}^{N-n_\infty} < V_{g_\infty(I)}(\infty)V_{I,I+n_\infty}(1,1)V_{n_0N}(u,\bar{u})V_{g_0}(0,0) > \\ &+ \sum_{J=n_0+1}^{n_\infty} < V_{g_\infty(J)}(\infty)V_{n_0,J}(1,1)V_{n_\infty N}(u,\bar{u})V_{g_0}(0,0) > \\ &+ \sum_{J=n_0+n_\infty+1}^{N} < V_{g_\infty(J)}(\infty)V_{n_0,J}(1,1)V_{n_0+n_\infty,N}(u,\bar{u})V_{g_0}(0,0) > \Big) \end{aligned} \tag{4.6.20}$$

As a result we are left with finding the following type of correlation functions

$$G_{IJKL}(u,\bar{u}) \equiv < V_{g_\infty}(\infty)V_{IJ}(1,1)V_{KL}(u,\bar{u})V_{g_0} > \tag{4.6.21}$$

Note that one can obtain the same result for $|u| > 1$ by exchanging $(u, \bar{u} \leftrightarrow (1,1))$

CORRELATION FUNCTION

$$G_{IJKL}(u,\bar{u}) = G^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}_{IJKL}\zeta_1^{\dot{\mu}_1}\zeta_2^{\dot{\mu}_2}\zeta_3^{\dot{\mu}_3}\zeta_4^{\dot{\mu}_4} \tag{4.6.22}$$

where

$$G^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}_{IJKL} =< \sigma_{g_\infty}[k_3/2,k_4/2](\infty)\tau^i_{IJ}(1)\tau^j_{KL}(u)\sigma_{g_0}[k_1/2,k_2/2](0) >< \Sigma^{\dot{\mu}_3\dot{\mu}_4}_{g_\infty}(\infty)\Sigma^i_{IJ}(1)\Sigma^j_{Kl}(u)\Sigma^{\dot{\mu}_1\dot{\mu}_2}_{g_0}(0) > \tag{4.6.23}$$

to calculate this correlator we use a map from t-sphere to z-sphere to uniformize the correlator on the t-sphere so that the twist fields are identity operators and we can safely use Wick's theorem. The map is given by

$$z = \left(\frac{t}{t_1}^{n_0}\right)\left(\frac{t-t_0}{t_1-t_0}\right)^{N-n_0}\left(\frac{t_1-t_\infty}{t-t_\infty}\right)^{N-n_\infty} \equiv u(t) \tag{4.6.24}$$

with an implicit parametrization of the map with respect to branch point x on the sphere

$$\begin{aligned} t_0 &= x-1 \\ t_\infty &= x - \frac{(N-n_\infty)x}{(N-n_0)x+n_0}, \\ t_1 &= \frac{N-n_\infty-n_0}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)} \end{aligned}$$

this reparametrization can be seen as a $2(N-n_0)$-fold covering of the u-sphere by the x sphere. Since the number of the nontrivial correlation function are also equal to $2(N-n_0)$, t-sphere can be represented as union of $2(N-n_0)$ domain and each domain is denoted by V_{IJKL}.

The overall phase of the correlation function $G_{IJKL}(u)$ cannot be determined and depend on I,J,K,L. By using the symmetries of the theory one can show correlation function of te holomorphic theory is complex conjugate of the anti-holomorphic correlator such that the phases cancel. We have to take into account the symmetry of twist fields:

$$\sigma_g[k/2] \leftrightarrow \bar{\sigma}_{g^{-1}}[\bar{k}/2] \quad and \quad \Sigma_g^{\dot{\mu}} \leftrightarrow \bar{\Sigma}_{g^{-1}}^{\mu} \tag{4.6.25}$$

We obtain the equality

$$\begin{aligned}
&< 0|V_{[g_\infty]}[k_3,\zeta_3;k_4,\zeta_4](\infty)V_{IJ}(1)V_{KL}(u)V_{[g_0]}[k_1,\zeta_1;k_2,\zeta_2](0)|0> \\
=&< 0|\bar{V}_{[g_\infty^{-1}]}[\tilde{k}_3,\tilde{\zeta}_3;\tilde{k}_4,\tilde{\zeta}_4](\infty)\bar{V}_{IJ}(1)\bar{V}_{KL}(u)\bar{V}_{[g_0^{-1}]}[\tilde{k}_1,\tilde{\zeta}_1;\tilde{k}_2,\tilde{\zeta}_2](0)|0>
\end{aligned}$$

than complex conjugating one obtains the equation

$$\begin{aligned}
&< 0|\bar{V}_{[g_\infty^{-1}]}[\tilde{k}_3,\tilde{\zeta}_3;\tilde{k}_4,\tilde{\zeta}_4](\infty)\bar{V}_{IJ}(1)\bar{V}_{KL}(u)\bar{V}_{[g_0^{-1}]}[\tilde{k}_1,\tilde{\zeta}1;\tilde{k}_2,\tilde{\zeta}_2](0)|0>^* \\
&= \lim_{z_\infty\to\infty}\lim_{z_0\to 0} z_\infty^{-2\Delta_{\infty[\{k_3,k_4\}]}} z_0^{-2\Delta_{g_0}[\{k_1,k_2\}]}|u^{-3}| \times \\
&< 0|\bar{V}_{g_0^{-1}}[-\tilde{k}_1,\tilde{\zeta}_1;-\tilde{k}_2,\tilde{\zeta}_2](\frac{1}{z_\infty},\frac{1}{\bar{z}_{infty}})\bar{V}_{KL}(\frac{1}{u},\frac{1}{\bar{u}})\bar{V}_{IJ}(1,1)\bar{V}_{g_\infty^{-1}}[-\tilde{k}_3,\tilde{\zeta}_1;-\tilde{k}_4,\tilde{\zeta}_2](\frac{1}{z_0},\frac{1}{\bar{z}_0})|0>
\end{aligned}$$

By SO(8) invariance of the theory we can make the transformation $-\tilde{k}\to k$, $\tilde{\zeta}\to\zeta$ and after using conformal transformation $z\to\frac{1}{z}$ we obtain

$$\begin{aligned}
&< 0|V_{[g_\infty]}[k_3,\zeta_3;k_4,\zeta_4](\infty)V_{IJ}(1)V_{KL}(u)V_{[g_0]}[k_1,\zeta_1;k_2,\zeta_2](0)|0>^* \\
=&< 0|\bar{V}_{[g_\infty]}[k_3,\zeta_3;k_4,\zeta_4](\infty)\bar{V}_{IJ}(1)\bar{V}_{KL}(u)\bar{V}_{[g_0]}[k_1,\zeta_1;k_2,\zeta_2](0)|0>
\end{aligned}$$

Now we represent the solution for the correlation function, for the details of the calculation one can consult original papers [22-24] G_{IJKL}.

$$< \tau_i\tau_j > (u) = -\delta^{ij}\frac{4x(x-1)(x+\frac{n_0}{N-n_0})(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-N})}{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)} + < \tau_i\tau_j >_k \tag{4.6.26}$$

$$\begin{aligned}
< \tau_i\tau_j >_k = -\Big(&\frac{(x+\frac{n_0}{N-n_0})}{n_0}k_1^i + \frac{(x-\frac{N-n_0-n_\infty}{N-n_0})}{n_\infty}k_3^i + \frac{1}{N-n_0}k_4^i\Big) \times \\
&\Big((x-1)k_1^j + xk_3^j + \frac{n_\infty-n_0}{N-n^0}(x-\frac{n_0}{n_0-n_\infty})k_4^j\Big)
\end{aligned}$$

where the corelation function $G^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}$ is equal to

$$G^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)=\kappa^{1/2}\frac{iR^4}{2^6(n_\infty-n_0)(N-n_0)}\Big(\frac{n_\infty n_0(N-n_0)}{N-n_0}\Big)^{1/2}\Big(\frac{n_\infty-n_0}{N-n_0}\Big)^{\frac{1}{4}k_1k_3}\frac{(x-\frac{n_0}{n_0-n_\infty})^3}{u^{3/2}(x-\alpha^1)^2(x-\alpha^2)^2}$$
$$\times\Big(\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{(x-\frac{n_0}{n_0-N})}\Big)^{1+\frac{1}{4}k_1k_4}\Big(\frac{(x-1)(x+\frac{n_0}{N-n_0})}{(x-\frac{n_0}{n_0-N})}\Big)^{1+\frac{1}{4}k_3k_4}T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)$$

where $T^{\dot\mu_1\dot\mu_2\dot\mu_3\dot\mu_4}_{IJKL}(u)$ is defined in the $SU(4)\times U(1)$ basis according to

$$T^{A_1A_2A_3A_4A_5A_6}_{IJKL}(u)=C(g_0,g_\infty)\times\frac{x^{d_0}(x-1)^{d_1}(x+\frac{n_0}{N-n_0})^{d_2}(x-\frac{N-n_0-n_\infty}{N-n_0})^{d_3}(x-\frac{n_0}{n_0-n_\infty})^{d_4}}{((x-\alpha^1)(x-\alpha^2))^{d_5}} \tag{4.6.27}$$

the coefficients are given by

$$d_0=p_1p_4+p_6p_1+p_6p4,\quad d_1=p_6p_3+p_6p_4+p_3p4,$$
$$d_2=p_1p_2+p_6p_1+p_6p2,\quad d_3=p_6p_2+p_6p_3+p_2p3,$$
$$d_4=p_1p_6+p_6p_3+p_1p3,\quad d_5=p_5p_6,$$
$$d_6=p_1p_5+p_3p_5+p_1p3-p_2p6,$$

and the structure constants are

$$|C(g_0,g_\infty)|=\frac{n_0^{p_1p_5}n_\infty^{p_3p_5}(N-n_\infty)^{p_4p_6}(n_\infty-n_0)^{d_4-d_5}}{(N-n_0)^{d_6}} \tag{4.6.28}$$

Computation of the fermionic correlators $<\Sigma^{\dot\mu_3\dot\mu_4}_{g_\infty}(\infty)\Sigma^i_{IJ}(1)\Sigma^j_{Kl}(u)\Sigma^{\dot\mu_1\dot\mu_2}_{g_0}(0)>$ was done by bosonization of the fermions [23] in the fracmework of $SU(4)\times U(1)$ formalism which is concisely presented in Appendix D.

Before we revies the calculation of the scattering amplitude it is useful to observe the following fact.

$$<\tau_+\tau_\mu>_k=0=<\tau_\mu\tau_+>_k \tag{4.6.29}$$

The scattering amplitude was calculated by using the relations mentioned above.

THE SCATTERING AMPLITUDE

Until now we considered the correlation function as defined on the disc $|u| < 1$. The correlation function for $|u| > 1$ is also given in the same form 4.6.22, so we can ignore the time ordering. Consequently from 4.6.20, 4.6.21 and 4.6.22 we obtain

$$M = 2N^2\sqrt{k_1^+k_2^+k_3^+k_4^+}\sum_{IJKL}\int d^2u|u|G_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}\bar{G}_{IJKL}^{\nu_1\nu_2\nu_3\nu_4}(\bar{u})\zeta_1^{\dot{\mu}_1\nu_1}\zeta_2^{\dot{\mu}_2\nu_2}\zeta_3^{\dot{\mu}_3\nu_3}\zeta_4^{\dot{\mu}_4\nu_4} \tag{4.6.30}$$

Substituting the holomorphic correlator $G_{IJKL}(u, \bar{u})$ and its complex conjugate we get rid of the phase ambiguity and end up with the expression

$$\begin{aligned} M =& \frac{R^8}{2^8\sqrt{k_1^+k_2^+k_3^+k_4^+}}\Big(\frac{n_0 n_\infty(N-n_\infty)}{N(N-n_0)}\Big)^2\Big(\frac{n_\infty - n_0}{N-n_0}\Big)^{\frac{1}{2}k_1k_3} \\ &\times\int d^2\Big|\frac{du}{dx}\Big|^2\Big|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\Big|^{\frac{1}{2}k_1k_4}\Big|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{(x-\frac{n_0}{n_0-n_\infty})}\Big|^{\frac{1}{2}k_3k_4} \\ &\times\sum_{IJKL}T_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}(u)T_{IJKL}^{\nu_1\nu_2\nu_3\nu_4}(u)\zeta_1^{\dot{\mu}_1\nu_1}\zeta_2^{\dot{\mu}_2\nu_2}\zeta_3^{\dot{\mu}_3\nu_3}\zeta_4^{\dot{\mu}_4\nu_4} \end{aligned}$$

where we introduced the notation

$$T_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}(u) =< \tau_i\tau_j > T_{IJKL}^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4 ij}(u) \tag{4.6.31}$$

recall that under the transformation $u \to x$ the u sphere is mapped on the domain V_{IJKL}. Taking this into account and performing the transformations

$$z = \frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{\frac{n_\infty - n_0}{N-n_0(x-\frac{n_0}{n_0-n_{infty}})}} \quad\Rightarrow\quad dz = \frac{(x-\alpha_1)(x-\alpha_2)}{\frac{n_\infty-n_0}{N-n_0}(x-\frac{n_0}{n_0-n_{infty}})^2} \tag{4.6.32}$$

we express M in the form

$$\begin{aligned} M =& \frac{R^8}{2^8\sqrt{k_1^+k_2^+k_3^+k_4^+}}\Big(\frac{n_0 n_\infty(N-n_\infty)}{N(N-n_0)}\Big)^2 \\ &\times\int d^2z\Big|\frac{dx}{dz}\Big|^2|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k4}T^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}(z)T^{\nu_1\nu_2\nu_3\nu_4}(\bar{z})\zeta_1^{\dot{\mu}_1\nu_1}\zeta_2^{\dot{\mu}_2\nu_2}\zeta_3^{\dot{\mu}_3\nu_3}\zeta_4^{\dot{\mu}_4\nu_4} \end{aligned}$$

The limit $R \to \infty$ expression of S-matrix is

$$< f|S|i >= -i\lambda^2 2^{-8}2N\delta_{m_1+m_2+m_3+m_4,0}\delta(\sum_i k_i^-)\delta^D(\sum_i k_i)\sqrt{\frac{\prod_{i=1}^4 (k_i)^{\epsilon(\dot{\mu}_i)}(k_i)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}}I(\zeta;k) \tag{4.6.33}$$

where we define

$$I(\zeta;k) =\Big(\frac{n_0n_\infty(N-n_\infty)}{N(N-n_0)}\Big)^2\sqrt{\prod_{i=1}^{4}(k_i)^{-\epsilon(\dot{\mu}_i)}(k_i)^{-\epsilon(\nu_i)}} \tag{4.6.34}$$

$$\times\int d^2z\Big|\frac{dx}{dz}\Big|^2|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}T^{\dot{\mu}_1\dot{\mu}_2\dot{\mu}_3\dot{\mu}_4}(z)T^{\nu_1\nu_2\nu_3\nu_4}(\bar{z})\zeta_1^{\dot{\mu}_1\nu_1}\zeta_2^{\dot{\mu}_2\nu_2}\zeta_3^{\dot{\mu}_3\nu_3}\zeta_4^{\dot{\mu}_4\nu_4} \tag{4.6.35}$$

$$< f|S|i >= -i\lambda^2\delta^{D+2}(\sum_i k_i^\mu)\sqrt{\frac{\prod_{i=1}^4(k_i)^{\epsilon(\dot{\mu}_i)}(k_i)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}}2^{-8}I(\zeta;k)$$

By using the reduction formula to extract the scattering amplitude from the S-matrix

$$< f|S|i >= -i\delta^{D+2}(\sum_i k_i^\mu)\sqrt{\frac{\prod_{i=1}^4(k_i)^{\epsilon(\dot{\mu}_i)}(k_i)^{\epsilon(\nu_i)}}{\prod_{i=1}^4 k_i^+}}A(1,2,3,4) \tag{4.6.36}$$

where

$$A(1,2,3,4) = 2^{-8}\lambda^2 I(\zeta;k) \tag{4.6.37}$$

in the superstring theory

$$I(\zeta;k) = K(\zeta;k)K(\zeta;k)C(s,t,u), \tag{4.6.38}$$

where

$$C(s,t,u) = -\pi\frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} \tag{4.6.39}$$

the problem is reduced to calculations of factor $I(\zeta;k)$.Reader can find the necessary information about so called kinematical factors in original paper[24]. Where the calculation of the kinematical factors of all physical particles are performed which simply means the calculation above wa extended to all physical particles. As conclusion this constitutes an evidence for the DVV conjecture in three level.

References

[1] M.B. Green,J. Schwarz,E. Witten "Superstring Theory Vol I-II," *Cambridge University Press*, 1987.

[2] J. Polchinski "String Theory Vol I-II" *Cambridge University Press*, 1998.

[3] S. Theisen,D. Lust "Lectures on String Theory" *Springer Verlag*,Berlin-Heidelberg, 1989.

[4] E. Kritsis "Introduction to Superstring Theory",hep-th/9709062.

[5] G. Arutyunov, "Lectures on String Theory",Utrecht University *unpublished.*

[6] E. D'Hocker, D. H. Pong "Geometry of String Perturbation Theory" *Reviews of Modern Physics,Vol. 60,No.4,October 1988.*

[7] S. B. Giddings,"Conformal techniques in String theory and String field theory", *Phys. Rep.* 170 (1988) 167.

[8] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, "M theory as a Matrix Model: a Conjecture", Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[9] L. Motl, "Proposals on nonperturbative superstring interactions", hep-th/9701025

[10] T. Banks and N. Seiberg, "Strings from Matrices", Nucl. Phys. B497 (1997) 41, hep-th/9702187.

[11] L. Susskind, Another Conjecture about M(atrix) Theory, hep-th/9704080.

[12] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Matrix String Theory", Nucl. Phys. B500 (1997) 43, hep-th/9703030.

[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, "Notes on Matrix and Micro Strings", Nucl. Phys. B (Proc. Suppl.) 62 (1998) 348, hep-th/9709107.

[14] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, "Elliptic Genera of Symmetric Products and Second Quantized Strings", Comm. Math. Phys. 185 (1997) 197, hep-th/9608096.

[15] W. Taylor, M(atrix) theory: Matrix quantum mechanics as a fundamental theory, Rev. Mod. Phys. 73, 419 (2001) [arXiv:hep-th/0101126].

[16] R. Dijkgraaf, "Fields, Strings matrices and Symmetric Products", hep-th/9912104.

[17] F.Hacquebord, "Symmetries and Interactions in Matrix String Theory", hep-th/9909227.

[18] T. Wynter, Gauge fields and interactions in matrix string theory, Phys. Lett. B 415, 349 (1997) [arXiv:hep-th/9709029].

[19] I.K. Kostov and P. Vanhove "Matrix String Partition Functions", Phys. Lett. B 444 (1998) 196 [hep-th/9809130].

[20] G. Bonelli, L. Bonora, F. Nesti and A. Tomasiello, Matrix string theory and its moduli space, Nucl. Phys. B 554, 103 (1999) [arXiv:hep-th/9901093].

[21] L.Dixon, D.Friedan, E.Martinec and S.Shenker,"Orbifold Conformal Field Theory", Nucl.Phys. B282 (1987) 13.

[22] G. E. Arutyunov and S. A. Frolov, Virasoro amplitude from the S(N) R**24 orbifold sigma model, Theor. Math. Phys. 114, 43 (1998) [arXiv:hep-th/9708129].

[23] G. E. Arutyunov and S. A. Frolov, Four graviton scattering amplitude from S(N) R**8 supersymmetric orbifold sigma model, Nucl. Phys. B 524, 159 (1998) [arXiv:hep-th/9712061].

[24] G. Arutyunov, S. Frolov and A. Polishchuk, On Lorentz invariance and supersymmetry of four particle scattering amplitudes in S(N) R**8 orbifold sigma model, Phys. Rev. D 60, 066003 (1999) [arXiv:hep-th/9812119].

[25] S. J. Rey, Heterotic M(atrix) strings and their interactions, Nucl. Phys. B 502, 170 (1997) [arXiv:hep-th/9704158].

[26] Lowe,D. A., "Heterotic Matrix String Theory", 1997b, Phys. Lett. B403, 243; hep-th/9704041.

[27] H McKean,V. Moll, "Elliptic Curves: Function Theory, Geometry, Arithmetics",*Cambridge University Press*,1997 .

[28] W. Taylor ;"Lectures on D-branes, Gauge Theory and M(atrices)*" ;hep-th/9801182

[29] R. Szabo,Z ,Kadar,H.Cove ;"DLCQ Strings, Twist Fields and One-Loop Correlators on a Permutation Orbifold."; hep-th:0706.3382v2

[30] Friedel T.J. Epple; "Propagators on One-Loop Worldsheets for Supersymmetric Orientifolds and Intersecting Branes";hep-th/0410177

[31] G. Grignani and G. W. Semenoff ; "The thermodynamic partition function of matrix string theory"; hep-th/9903246.

[32] R.Miranda;"Algebraic Curves and Riemann Surfaces";Graduate Studies in Math., vol. 5, Amer. Math. Soc., Providence, RI, 1995.

[33] C. Teleman "Riemann Surfaces";*Lecture notes*,St. John's College, Cambridge, UK, 2001.

[34] F. D'Francesco,P. Mathieu,D. Sénéchal "Conformal Field theory" *Springer-Verlag*, 1997.

[35] G. W. Semenoff ; "DLCQ strings and branched coverings of Torii.";hep-th/0112043