Imperial College of Science, Technology and Medicine
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On Classical and Quantum Moduli Spaces of Supersymmetric Gauge Theories

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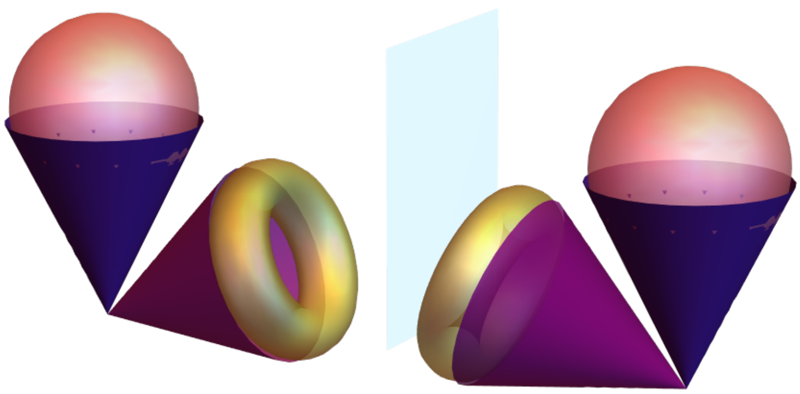

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Abstract

This manuscript presents some results that concern supersymmetric theories with eight supercharges and moduli spaces of instantons. The latter are attacked from two fronts. Firstly we study the quantum-corrected Coulomb branch of three-dimensional $\mathcal{N} = 4$ quiver theories which are encoded in the over-extended Dynkin diagram of a group G. Studying the ring of gauge invariant operators on the Coulomb branch for these theories by means of a generalised monopole formula yields precisely the Hilbert series for the moduli space of G-instantons. We provide results for any G, including non-simply-laced and exceptional groups.

In the second part of the thesis we analyse the Higgs branch of some five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. We provide a description of the quantum-corrected Higgs branch in terms of instanton operators, the glueball superfield and mesons. In particular, a classical nilpotent relation is found to be corrected by bilinears in instanton operators. The analysis depends on a decomposition of the Hilbert series for the moduli space of E_{N_f+1} instantons into $SO(2N_f)$ instantons, which are the known Higgs branch at infinite and finite coupling respectively. The dressing of instanton operators in terms of finite coupling fields is also analysed.

In passing, we also present an interesting phenomenon where the Higgs branch of a given family of gauge theories with eight supercharges and classical gauge and global symmetry groups is not a single hyperKähler cone but rather the union of two such cones with nontrivial intersection.

Declaration

The research presented in this thesis is the author's own work and is the result of two collaborations with Amihay Hanany, Stefano Cremonesi and Noppadol Mekareeya. Chapter 4 and 5 have been published in

- S. Cremonesi, G. Ferlito, A. Hanany, and N. Mekareeya, "Coulomb Branch and the Moduli Space of Instantons," *JHEP* **1412** (2014) 103, arXiv:1408.6835 [hep-th]
- S. Cremonesi, G. Ferlito, A. Hanany, and N. Mekareeya, "Instanton Operators and the Higgs Branch at Infinite Coupling," *JHEP* **04** (2017) 042, arXiv:1505.06302 [hep-th]

The results presented in chapter 3 are at the preprint stage and can be found in

- G. Ferlito and A. Hanany, "A tale of two cones: the Higgs Branch of $Sp(n)$ theories with $2n$ flavours,". arXiv:hep-th/1609.06724[hep-th]

Throughout this manuscript the work of other researchers has been properly cited and acknowledged to the best of the author's knowledge.

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My path through the complex and bizarre world of String Theory has been frustrating and despair inducing at the worst of times. But the glimpse at the beauty of Nature made up for it entirely. I thank my supervisor Amihay Hanany for guiding me through its meanders.

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This thesis is dedicated to Tom, Angela and Miele.

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Chapter 1

Introduction

One of the most fruitful ideas in modern scientific thought is that of gauge theories: not only are they beautiful constructions with far reaching consequences in various abstract Mathematics fields, but, most strikingly, the underlying structure of fundamental particles. Gauge theory has proved itself as the correct formalism describing the interplay of forces and matter and for this reason continues to be a remarkable field of study.

Laboratories have verified the theoretical predictions made through gauge theories, particles have been discovered, constants of nature have been measured to high precision. In a sense, it is as if Nature has been decoded entirely in the description offered by gauge theory. As for any complex idea, issues have sprung up throughout the years and in turn resolutions have been uncovered . A paradigm shift where gauge theories have fallen from grace has not occurred. If anything, new ideas that do not resemble gauge theory in some limit have come under intense scrutiny.

Gauge theories are based on a simple idea: that nature at its fundamental level respects and embodies symmetry. It is no wonder that, once the only other allowed symmetry of spacetime, supersymmetry, was discovered, a new, upgraded, version of gauge theory had to emerge. Quickly enough, supersymmetric gauge theories were given a secure footing and since then they have never left the centre stage of theoretical physics.

Extensive and, in some cases, comprehensive studies of supersymmetric gauge theories have been pursued. Indeed, such theories have properties that make them amenable to more precise computations than their non supersymmetric counterparts, analogously to the difference between real and complex analysis.

There is as of yet no experimental evidence for supersymmetric gauge theories despite intensive searches for their possible phenomenological manifestations.

Nonetheless they offer so much insight that, were their role solely a mathematical simplification of gauge theory without supersymmetry, they would still play a most prominent part.

So much for praising their virtues. Gauge theories, with or without supersymmetry, are difficult mathematical beasts. There are both topological effects and complicated dynamical mechanisms that can only be studied in approximate manners. Chief among the many issues is that perturbation theory is often unable to describe the dominant physics. For example, some gauge theories become strongly coupled at low energies, leading to phenomena like quark confinement. Excluding the often unreachable oasis of lattice field theory, the analytical land looks quite desolate in these regions. Moreover, contrary to some misconceptions, it is not just the strong coupling regime that perturbation theory cannot access. Perturbation theory is also blind to effects analogous to tunnelling in quantum mechanics which are essential to characterise the low energy limit of a gauge theory. Whilst these effects can sometimes be computed in non-supersymmetric theories, it is in the supersymmetric versions of gauge theories that they have yielded the most powerful results. In 1994, Seiberg and Witten [4, 5] succeeded in describing the low energy regime of a supersymmetric gauge theory. Whilst their result was found indirectly using remarkable geometrical arguments, a general understanding of the low energy regime requires knowledge of two key concepts: instanton contributions and the moduli space of vacua.

Instantons are field configurations that minimise the Euclidean action. Such field configurations are the leading non-perturbative contribution to the path integral when the coupling constant is small. They are non-perturbative in the sense that they do not appear in the usual perturbative loop expansion. Even so, as minima of the action, they are valid contributions to the weak coupling expansion. If one wishes for such an expansion, known as a semiclassical approximation, to hold throughout a range of energy scales, the coupling, which is typically non-constant upon quantum corrections, should not be allowed to become too large in the infrared.

In theories that display symmetry breaking, the Higgs field vacuum expectation value acts as an infrared regulator: it freezes the running of the coupling at a given energy scale. For these theories, instanton configurations must be taken into account to fully control the low energy physics. It should be clear then why the physics of instantons occupies such a huge portion of gauge theory analysis. A physics of instantons cannot but foretell a mathematics of instantons. Indeed their formal mathematical characterisation has led to unexpected advances in differential and algebraic geometry. For now, it suffices to mention that instanton field configurations, known as moduli spaces of instantons, form beautiful geometric spaces with highly constrained properties known as hyperKähler varieties (a subclass of the celebrated Calabi Yau spaces of string theory). The interest in moduli spaces of instantons lies in their simplicity and ubiquity: they are to fundamental physics as the regular n-sided polygons are to plane geometry. We will soon explain how they relate to the present work.

The second key idea to introduce is that of moduli spaces of vacua. In any field theory, (scalar) field configurations that minimise the energy play a distinguished role. For supersymmetric field theories such field configurations are numerous and continuously connected: one often talks of a space of vacuum solutions or a moduli space of vacua. Such spaces have rich algebro-geometric properties such as canonical metrics or complicated singularity structures. A given point on the moduli space corresponds to fields taking particular values: physical quantities, like massless and massive particle excitations or topological states, ostensibly depend on which point of the moduli space is being probed. The spectrum is thus dependent on a continuum of vacua – in other words, though they have the same energy, these vacua are inequivalent. Analagously to the effect that spin-orbit coupling has on the hydrogen atom energy levels, we expect quantum corrections to separate field configurations with the same vacuum energy – quantum corrections lift degeneracy. Moduli spaces are no different: an entire space of vacuum solutions should lift up and leave behind a true quantum vacuum point or none at all. For a supersymmetric theory however, such a lifting does not happen generically. The moduli space might be modified but their structure often survives quantum corrections. It thus remains a meaningful object to study both classically and quantum-mechanically [6, 7, 8, 9].

However, it is outside this purely field theoretic context that one must recognise a striking feature of supersymmetric gauge theories: they can often be embedded in string theory. A '70's field theorist with no knowledge of the developments in the '90's would be astonished to learn that field theories can quite simply be drawn on a piece of paper and non-trivial physical information extracted from such sketches. Whilst string theory's claim to fame is as a candidate theory of everything – a statement which might very well be true but that has created more enemies than friends – one of its, only at first sight, less dramatic powers lies in its ability to shed light on the hidden structure of field theory.

Starting from the trend-setting paper of Polchinski, [10], this point of view has returned many new results. In that work – one of a handful marking the beginning of the so called second superstring revolution – a generalisation of Maxwell electrodynamics, the gauge theory par excellence, was identified as underlying the basic objects of string theory, Dp-branes. These are hypersurfaces extended in p+1 spacetime dimensions and are none other than the carriers of the supersymmetric gauge theories that live in the universe. Such an understanding has allowed many gauge theory quantities and phenomena to be translated to effects in brane dynamics, in what has been termed geometric engineering.

One of the first gauge phenomena to be brane engineered is that of instantons. In [11] Witten showed that the moduli space of k instantons on $SO(32)$ could be identified with the moduli space of vacua of k coincident D5-branes in Type I string theory. In particular, the parameters arising from this string embedding precisely match that of the ADHM construction [12] for instantons of classical groups. Soon after, it was realised in [13] that any Dp/D(p-4)-brane system where the lighter brane is entirely lying on the worldvolume of the heavier one gives rise to a supersymmetric gauge theory whose moduli space of vacua corresponds to the moduli space of instantons.

For the sake of readers lacking expertise in this area, let us dwell on this example. Dp-branes can be employed to geometrically engineer supersymmetric gauge theories. The parameters of such string embeddings must reproduce the parameters of the gauge theory: moduli spaces of systems of branes realise moduli spaces of supersymmetric gauge theories. Now comes the crux: the moduli spaces of some given supersymmetric gauge theories are themselves isomorphic to moduli spaces of instantons. So we conclude that moduli spaces of instantons are realised by the brane systems that reproduce those gauge theories. The raison d'être for such flashy stringy realisations of field theory phenomena is that string theory displays complex webs of dualities: exploiting these allows for effortless motion in between seemingly different gauge theories.

A textbook example is the embedding of 3d mirror symmetry [14, 15]. This is a well tested field theory conjecture of the infrared equivalence of pairs of supersymmetric gauge theories in three dimensions which have very different Lagrangian descriptions in the ultraviolet. Whilst such an equivalence could seem at first sight purely academic, it became clear that it was actually a manifestation of a string duality known as S-duality. This interpretation was provided by the work of [16] where a Type IIB string embedding for three dimensional supersymmetric gauge theories involving D3-branes, D5-branes and NS5-branes was introduced. The power of the formalism presented in that work lies in the fact that many more examples can be constructed and generalisations developed. It is now routine to construct mirror pairs thanks to simple rules based on drawing the brane picture for the starting gauge theory and performing a number of moves to obtain its dual.

In three dimensions supersymmetric gauge theories with eight supercharges have moduli spaces that are locally the product of two hyperKähler manifolds [17]. One factor, the Higgs branch, is protected from quantum corrections, whilst the other, the Coulomb branch, is not. For a given theory, mirror symmetry predicts the existence of another theory where the role of the two branches is swapped. Such a conjecture has been tested in a number of ways starting with the work in [18, 19, 20] and culminating with calculations of indices and partition functions [21, 22, 23, 24, 25, 26, 27, 28, 29]: riding on the wave of field-theoretic proofs of mirror symmetry, quantum corrections to the moduli space are better controlled than ever before. In the context of the work we present, this is what is of interest to us. Indeed, in one chapter of this thesis we construct brane systems that engineer Coulomb branches that are isomorphic to moduli spaces of instantons and study them by generalising the results of [30].

Another example of a string embedding is that of five-dimensional minimally supersymmetric gauge theories. These were introduced in [31] and further explored in [32, 33, 34, 35]. The field theoretic properties of $SU(2)$ gauge theories with up to seven N_f flavours are beautifully encoded as the worldvolume theory of one D4-brane probing a background of D8-branes constrained to be living on an interval. Through such a construction the existence of certain 5d ultraviolet fixed points has been identified and the enhancement of global symmetries given a precise description.

A new brane construction involving NS5, D5 and D7 branes was proposed in [36, 37] which, once generalised, offered yet more insight [38, 39] into the dynamics of gauge theories. Though these constructions are tangentially relevant to the computations presented in this thesis, they provide a bridge to the recent developments in exact calculations of field theory quantities in five dimensions [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. For us, the classic results of [31] are already rich enough: in there it was recognised that the moduli space of the low energy gauge theory, which is isomorphic to the moduli space of

$SO(2N_f)$ instantons, grows extra moduli and enhances to the moduli space of E_{N_f+1} instantons. In contrast to 3d where the interesting quantum effects take place in the infrared, in 5d it is the ultraviolet fixed points that have surprises in store.

There are many other instances of gauge theory/string embeddings that have led to dramatic new developments in field theory but we will not touch on them in this thesis. Here we hope to offer some insight on two canonical examples that are connected to moduli spaces of instantons. The author will endeavour to do so in a clear and pedagogical manner.

1.1 Outline of the thesis

This manuscript has been organised in the following way:

- In the interest of pedagogy, a discussion of the essential background for the work in this thesis is provided in chapter 2. Here we take care to explain how moduli spaces arise in supersymmetric gauge theories and why they are algebraic varieties. We specialise in gauge theories which have eight supercharges, which in four dimensions means they have two copies of supersymmetry. Their global symmetry structure is summarised and classic results concerning the moduli spaces of vacua of $SU(n_c)$ theory with $U(N_f)$ flavour symmetry are presented. Much effort has been made to recast the original arguments in the language adopted throughout this thesis. Indeed it is here that the basics of the computational tools we need are introduced, namely the ideas of partition functions that count operators and describe the variety in terms of representation theory.

- The techniques introduced in the background chapter are then straightforwardly applied in chapter 3 to analyse the classical moduli spaces of theories with flavour groups $SU(N_f)$, $SO(N_f)$ and $USp(2N_f)$. Some results have partially appeared in the literature but we reveal an interesting phenomenon that has not been discussed in the necessary detail. Theories with $Sp(n)$ gauge symmetry and $2n$ flavours have a moduli space that is not a simple hyperKähler cone but rather the union of two such cones with a nontrivial intersection. We characterise the two cones and their intersection explicitly in terms of chiral operators satisfying defining equations.

- After the warm up of chapter 3 we switch gears and in chapter 4 we study three-dimensional supersymmetric gauge theories whose quantum corrected Coulomb branch is isomorphic to the moduli space of instantons. The construction presented here offers a representation theoretic description of the moduli space for any instanton number and any gauge group, including exceptional and non-simply laced ones. The analysis follows that of [30] and generalises the results therein. Furthermore, since the work in this chapter provides an independent calculation for the Coulomb branch which does not depend on knowledge of the Higgs branch of the dual theory, our results provide a non-trivial check for the mirror symmetry of [14]. A discussion of the properties of dressed monopole operators as generators of the moduli space of instantons is provided.

- In chapter 5 we switch dimensions and branch. We move to five dimensional minimally supersymmetric gauge theories and analyse the Higgs branch. Theories with $SU(2)$ gauge group and $N_f < 8$ flavours have a modified Higgs branch with global symmetry enhancement at infinite bare coupling. We propose that the mechanism for the enhancement of symmetry on the Higgs branch stems from a correction of the finite coupling relations by means of instanton operators. In order for these corrections to reproduce the classical Higgs branch when the coupling is restored to be finite, the glueball operator must be included in the description of the chiral ring. We elucidate all of these points and perform the computations for the quantum corrected moduli space for this class as well as a few others.

- Chapter 6 will summarise the discussion and provide the concluding remarks. Some calculations are relegated to the appendices.

Chapter 2

Essential Background

2.1 Supersymmetry

Since most of the work carried out in this thesis relates to theories with eight supercharges, it is appropriate to give some background to such supersymmetric theories. Much of our discussion leans on many articles and reviews ([4, 5, 8, 9]) written in the last twenty years (for a recent review on $\mathcal{N} = 2$ see [53]). Here we focus on the essential features needed to fully understand the author's work.

For concreteness let us start with the 4d $\mathcal{N} = 2$ $SU(n_c)$ Lagrangian with N_f flavours[1].The crucial feature of an $\mathcal{N} = 2$ theory is the presence of a global $SU(2)_R$ symmetry which arises as an automorphism of the supersymmetry algebra and under which the various fields transform. The full R-symmetry also has an abelian factor $U(1)_r$, but this is irrelevant in our discussion.

The fields entering the Lagrangian live in vector multiplets and hypermultiplets. These $\mathcal{N} = 2$ multiplets can be decomposed in terms of $\mathcal{N} = 1$ multiplets. The $\mathcal{N} = 2$ vector superfield contains an $\mathcal{N} = 1$ vector multiplet V and an $\mathcal{N} = 1$ chiral multiplet Φ, both in the adjoint of $SU(n_c)$. A hypermultiplet H contains an $\mathcal{N} = 1$ chiral multiplet Q in the fundamental representation of $U(N_f)$ and antifundamental of $SU(n_c)$ and an $\mathcal{N} = 1$ antichiral multiplet $\widetilde{Q}^\dagger$ in the same gauge and flavour representation. The field content is summarised in Table 2.1.

The scalars in the hypermultiplet, q and $\widetilde{q}^\dagger$, transform as a doublet of $SU(2)_R$, whilst the complex scalar in the vector multiplet is a singlet under $SU(2)_R$.

The $\mathcal{N} = 2$ Lagrangian can then be written as a sum of two contributions,

[1] In this thesis, theories with eight supercharges will always be assumed *on-shell* and no attempt will be made to discuss *off-shell* formulations. Working on-shell guarantees that $\mathcal{N} = 2$ superfields can be recast in terms of their $\mathcal{N} = 1$ components without issues.

<table>
<tr><th>$\mathcal{N}=2$</th><th colspan="2">$\mathcal{N}=1$</th><th>$SU(2)_R$</th></tr>
<tr><td rowspan="2">V-plet $\mathbf{\Phi}$</td><td>v-plet V</td><td>A_μ, λ_α</td><td rowspan="2">$\begin{pmatrix} \lambda_\alpha \\ \xi_\alpha \end{pmatrix} \to [1]$</td></tr>
<tr><td>χ-plet, Φ</td><td>$\phi,\ \xi_\alpha$</td></tr>
<tr><td rowspan="2">H-plet H</td><td>χ-plet, Q</td><td>$q,\ \psi_\alpha$</td><td rowspan="2">$\begin{pmatrix} q \\ \widetilde{q}^\dagger \end{pmatrix} \to [1]$</td></tr>
<tr><td>anti χ-plet, $\widetilde{Q}^\dagger$</td><td>$\widetilde{q}^\dagger,\ \widetilde{\psi}^\dagger_{\dot\alpha}$</td></tr>
</table>

Table 2.1: The field content for an $\mathcal{N}=2$ gauge theory and the transformation properties of the nontrivial component fields under the $SU(2)_R$ global symmetry.

one coming from the vector multiplet sector and one from the hypermultiplet sector. The former is given by

$$\mathcal{L}_{\substack{\mathcal{N}=2\\ \text{Vplet}}} = \frac{1}{4\pi}\text{Im}\left[\tau\int \mathrm{d}^4\theta\,\text{Tr}\left(\Phi^\dagger e^V\Phi\right) + \tau\int \mathrm{d}^2\theta\frac{1}{2}\,\text{Tr}\,W^2\right] \tag{2.1.1}$$

where τ is the gauge coupling constant, $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$, $W^2 = W_\alpha W^\alpha$ and W_α is the field strength superfield. This term can be elegantly written in terms of $\mathcal{N}=2$ superspace. Introducing an extra set of Grassmann coordinates $\widetilde{\theta}_\alpha$ and an $\mathcal{N}=2$ superfield $\mathbf{\Phi}$, the vector multiplet contribution becomes an integration over chiral $\mathcal{N}=2$ superspace

$$\mathcal{L}_{\substack{\mathcal{N}=2\\ \text{Vplet}}} = \frac{1}{4\pi}\text{Im}\ \ \text{Tr}\int \mathrm{d}^2\theta\,\mathrm{d}^2\widetilde{\theta}\mathcal{F}\left(\mathbf{\Phi}\right) \tag{2.1.2}$$

$$= \frac{1}{4\pi}\text{Im}\ \ \text{Tr}\int \mathrm{d}^2\theta\,\mathrm{d}^2\widetilde{\theta}\frac{1}{2}\tau\mathbf{\Phi}^2, \tag{2.1.3}$$

where $\mathcal{F}\left(\mathbf{\Phi}\right)$ is the prepotential which, though in this UV description is just a quadratic function, can be any holomorphic function of the $\mathcal{N}=2$ superfield.

The $\mathcal{N}=2$ hypermultiplets contribution to the Lagrangian is given by[2]

$$\mathcal{L}_{\substack{\mathcal{N}=2\\ \text{Hplet}}} = \frac{1}{4\pi}\text{Im}\left[\tau\int \mathrm{d}^4\theta\,\text{tr}\left(Q_i^\dagger e^V Q^i + \widetilde{Q}_i e^V \widetilde{Q}^{i\dagger}\right) + \tau\int \mathrm{d}^2\theta\mathcal{W}\right]\ . \tag{2.1.4}$$

$\mathcal{W}$ is the $\mathcal{N}=2$ superpotential and it is constrained to be

$$\mathcal{W} = \sqrt{2}Q^i\cdot\Phi\cdot\widetilde{Q}_i\ , \tag{2.1.5}$$

[2] We denote the trace in the adjoint representation as $\text{Tr}\,\cdot$ and the trace in fundamental representation as $\text{tr}\,\cdot$.

where $\cdot$ signifies a gauge index contraction to make $\mathcal{W}$ gauge invariant. We do not consider theories with massive hypermultiplets and as such the only UV parameter is the complex gauge coupling τ.

Expanding the full Lagrangian, (2.1.1) + (2.1.4), and keeping terms involving only the scalar fields gives

$$\mathcal{L}_{\text{scalars}} \sim \frac{1}{g^2}\,\text{Tr}[\partial_\mu\phi\partial^\mu\phi^\dagger] + \text{tr}[\partial_\mu q^i\partial^\mu q_i^\dagger] + \text{tr}[\partial^\mu\widetilde{q}^{i\dagger}\partial_\mu\widetilde{q}_i] - V_{UV}(\phi, q^i, \widetilde{q}_i, \phi^\dagger, q_i^\dagger, \widetilde{q}^{i\dagger}) \tag{2.1.6}$$

The potential term for the scalars is given by a sum of the squares of the D-terms and the F-terms

$$V_{UV}(\phi, q^i, \widetilde{q}_i, \phi^\dagger, q_i^\dagger, \widetilde{q}^{i\dagger}) = \frac{1}{2g^2}D^A D_A + \sum_{\{\varphi\}}\left|F_{\mathcal{R}(\varphi)}\right|^2 \tag{2.1.7}$$

where the D-terms and F-terms are, in general, given by

$$D^A = \sum_{\{\varphi\}}\text{Trace}\left(\varphi^\dagger(T^A_{\mathcal{R}})\varphi\right) \tag{2.1.8}$$

$$F_{\mathcal{R}(\varphi)} = \frac{\partial\mathcal{W}}{\partial\varphi}\ , \tag{2.1.9}$$

with φ the various scalars in the theory, $\mathcal{R}$ the representation they carry, Trace$(\cdot)$ the trace in the representation $\mathcal{R}$, and A an adjoint index.

For $SU(n_c)$ with N_f flavours (2.1.8) and (2.1.9) become

$$D^A = -g[\phi,\phi^\dagger]^A - \text{tr}\left(q^i(T^A_{\text{fun}})q_i^\dagger - \widetilde{q}^{\dagger i}(T^A_{\text{fun}})\widetilde{q}_i\right) \tag{2.1.10}$$

$$F_{\mathcal{R}(\phi)} = \frac{\partial\mathcal{W}}{\partial\phi}\quad, F_{\mathcal{R}(q)} = \frac{\partial\mathcal{W}}{\partial q}\quad, F_{\mathcal{R}(\widetilde{q})} = \frac{\partial\mathcal{W}}{\partial\widetilde{q}} \tag{2.1.11}$$

where the adjoint index is $A = 1,\ldots,N_c^2-1$ and we have suppressed both flavour and gauge indices in the F-terms. We will make these explicit shortly.

The existence of solutions to the F-terms and the D-terms determines the existence of a supersymmetric vacuum. In fact, supersymmetry allows for entire spaces worth of inequivalent vacua, as we proceed to explain.

2.2 Moduli spaces

At low energies the theory with ultraviolet (UV) Lagrangian (2.1.1) + (2.1.4) becomes an effective theory and can be written as a supersymmetric non-linear sigma model, the scalar part of which is given by[3]

$$\mathcal{L}_{eff} \sim \mathcal{K}_{I\overline{J}}(\varphi, \varphi^\dagger)\partial_\mu \varphi^I \partial^\mu \varphi^{\dagger \overline{J}} - V(\varphi, \varphi^\dagger) \tag{2.2.1}$$

where $I, J = 1, \ldots$ label the various scalar fields $\{\varphi\}$ that have survived in the infrared (IR), $\overline{I}, \overline{J} = 1, \ldots$ their complex conjugates $\{\varphi^\dagger\}$, whilst V is a nonnegative function of these scalar fields and $\mathcal{K}_{I\overline{J}}$ is the target space metric.

A minimum energy configuration is achieved by setting the scalar fields constant over spacetime and setting the potential to zero. For supersymmetric field theories, the set of constant scalar fields parametrises a Hermitian (hence the bar notation on the indices) manifold $\mathcal{M}^{(0)}$ which satisfies some further geometric conditions dependent on the amount of supersymmetry: for theories with four supercharges $\mathcal{M}^{(0)}$ is Kähler [54]. Extended supersymmetry imposes extra constraints [55, 56, 57]; in particular 4d supersymmetric sigma models where $\mathcal{N} = 2$ is preserved require that $\mathcal{M}^{(0)}$ be hyperKähler.

This space of constant fields must be restricted by imposing that the scalar potential be zero and that field configurations related by gauge transformations be counted only once. Hence the actual scalar manifold is given by

$$\mathcal{M} \equiv \mathcal{M}^{(0)} / \{V(\varphi, \overline{\varphi}) = 0\} / G \tag{2.2.2}$$

The space of constant scalar field configurations subject to the vanishing of the potential and modulo gauge transformations is called the **moduli space of vacua** of the theory.

Supersymmetric field theories typically admit flat directions in the potential. This means that, unlike their non-supersymmetric counterparts where solutions to $V(\varphi, \overline{\varphi})$ usually consist of isolated points, there are continuous solutions (directions) to the above equation. We refer to these solutions as VEVs or moduli.

The definition of $\mathcal{M}$ as given by (2.2.2) is a known as a Kähler quotient construction. The most important consequence of such a construction is that $\mathcal{M} \subset \mathcal{M}^{(0)}$ still retains its Kähler properties (or hyperKähler if $\mathcal{N} = 2$

[3] The effective Lagrangian in the form of (2.2.1) is generic for any group G and for any representation $\mathcal{R}$ of the matter fields.

supersymmetry is preserved by the sigma model) [57].

2.2.1 Higgs branch and Coulomb branch

For supersymmetric theories $\mathcal{K}_{I\overline{J}}$ is a Kähler metric which means that locally it can be written as the second derivative of a function $\mathcal{K}$

$$\mathcal{K}_{I\overline{J}}(\varphi,\overline{\varphi}) = \frac{\partial\mathcal{K}(\varphi,\overline{\varphi})}{\partial\varphi^I\overline{\varphi}^{\overline{J}}} \ . \tag{2.2.3}$$

$\mathcal{K}$ is known as the Kähler potential.

For a theory with $\mathcal{N}=2$ supersymmetry the set of scalar fields of the theory is split into those coming from the vector multiplets, $\{\phi^a\}$, with a ranging over the number of abelian vector multiplets, and those residing in hypermultiplets, $\{q^i,\widetilde{q}^{\dagger i}\}$, with i labelling the various hypermultiplets[4]. In principle the Kähler potential is a generic function of all the scalars and their complex conjugates $\mathcal{K} = \mathcal{K}(\phi^a,q^i,\widetilde{q}^i,\phi^{\dagger\overline{a}},q^{\dagger\overline{i}},\widetilde{q}^{\dagger\overline{i}})$. However, kinetic terms which involve cross-terms of the vector multiplet scalars and the hypermultiplets scalars, such as $\partial_\mu q^i\partial^\mu\phi^{\dagger\overline{a}}$, are not compatible with $\mathcal{N}=2$ supersymmetry[5] so the terms in the metric $\partial_{q^i}\partial_{\phi^{\dagger\overline{a}}}\mathcal{K}$ must vanish. It follows that the Kähler potential can be written as a sum of two separate contributions

$$\mathcal{K} = \mathcal{K}_H(q^i,\widetilde{q}^i,q^{\dagger\overline{i}},\widetilde{q}^{\dagger\overline{i}}) + \mathcal{K}_C(\phi^a,\phi^{\dagger\overline{a}}) \tag{2.2.4}$$

Here comes an important point to stress. The two nonlinear supersymmetric sigma models manifest a profound difference. Indeed the low energy effective theory of scalars coming from hypermultiplets still possesses $\mathcal{N}=2$ supersymmetry and so the target space with potential $\mathcal{K}_H$ is a hyperKähler manifold. On the other hand, scalars in the $\mathcal{N}=2$ vector multiplet originate from only one $\mathcal{N}=1$ chiral multiplet[6]: the nonlinear sigma model for the scalars in the vector multiplet can only have $\mathcal{N}=1$ supersymmetry. The target space with potential $\mathcal{K}_C$ is thus only Kähler[7].

[4] For 4d $\mathcal{N}=2$ $SU(n_c)$ with N_f flavours $i=1,\dots N_f$ and $a=1,\dots,n_c$. However tracelessness of the $\mathfrak{su}(n)$ Lie algebra generators implies there are n_c-1 independent vector multiplet scalars.

[5] Invariance of the action under $SU(2)_R$ and $\mathcal{N}=1$ supersymmetry cannot be simultaneously satisfied.

[6] The second set of supersymmetry is imposed between the chiral multiplet and the $\mathcal{N}=1$ vector mulitplet as we know.

[7] In reality there is still a remnant of the fact that the UV theory had $\mathcal{N}=2$ and the geometry is actually special Kähler. Here it is only relevant that the manifold does not posses a hyperKähler structure unlike $\mathcal{K}_H$.

The fact that the Kähler potential (2.2.4) gives rise to a product target space implies an important property for the space of vacua: it means that the latter can also be locally factorised into

$$\mathcal{M} = \mathcal{M}_{\mathcal{H}} \times \mathcal{M}_{\mathcal{C}} \ , \tag{2.2.5}$$

the first factor obtained by setting to zero the VEVs of the vector multiplet scalars and leaving the hypermultiplets scalars as the only moduli whilst the second factor obtained in the opposite limit. $\mathcal{M}_{\mathcal{H}}$ is known as the **Higgs branch** and $\mathcal{M}_{\mathcal{C}}$ as the **Coulomb branch**. It is helpful to think of the moduli space as the so-called **mixed branch**, where some vector multiplet scalars and some hypermultiplet scalars take nonzero VEVs, and to consider its limiting submanifolds as the Coulomb branch and the Higgs branch. Thanks to the factorisation, the mixed branch remains, at least locally, a product of a Higgs branch submanifold and a Coulomb branch submanifold. In the following discussion mixed branches will not make an appearance and so we do not discuss them any further. Note, furthermore, that the Higgs branch inherits a hyperKähler structure from the target space and the Coulomb branch inherits a Kähler structure.

As a first approach, we can construct Higgs branches and Coulomb branches from the UV Lagrangian description rather than an effective Lagrangian. Indeed (2.1.6) is a sigma model with a canonical metric for both the scalars in the vector multiplet and the ones in the hypermultiplet, since the kinetic terms can be written as

$$\mathcal{L} \sim \frac{1}{g^2}\,\mathrm{Tr}[\partial_\mu \phi \partial^\mu \phi^\dagger] + \delta_{i\bar{j}}\,\mathrm{tr}[\partial_\mu q^i \partial^\mu q^{\bar{j}\dagger}] + \delta_{i\bar{j}}\,\mathrm{tr}[\partial^\mu \widetilde{q}^{i\dagger} \partial_\mu \widetilde{q}^{\bar{j}}] \ . \tag{2.2.6}$$

The target space then locally factorises as

$$\mathcal{M}^{(0)}_{\text{class}} = \mathbb{C}^{(n_c^2-1)} \times \mathbb{C}^{2N_f n_c} \ . \tag{2.2.7}$$

The full moduli space, namely the product space of the Higgs branch and Coulomb branch, appears by taking a quotient of each target space factor with the potential restricted to that branch. Such a restriction simply consists of setting to zero the scalars in the other branch. Once we mod out by gauge equivalence we have

$$\begin{aligned}\mathcal{M}_{\text{class}} &= \mathcal{M}_{\mathcal{H},\text{class}} \times \mathcal{M}_{\mathcal{C},\text{class}} \\ &= \left(\mathbb{C}^{(n_c^2-1)}/\{V=0\}_{\mathcal{C}}/G\right) \times \left(\mathbb{C}^{2N_f n_c}/\{V=0\}_{\mathcal{H}}/G\right) \ , \end{aligned} \tag{2.2.8}$$

where $\{V = 0\}_{\mathcal{C}}$ is the surviving set of constraints after the scalars in the hypermultiplet have been set to zero, and $\{V = 0\}_{\mathcal{H}}$ the set of constraints after the scalars in the vector multiplet have instead been set to zero. These quotient spaces are referred to as the *classical* Coulomb branch and Higgs branch.

Non renormalisation

The classical branches give a coarse indication of the structure of vacua at low energies: quantum mechanical corrections that enter the effective description can in principle modify the moduli spaces. However it is common lore that the Higgs branch is actually not modified by quantum corrections.

The fact that the Higgs branch is unmodified even when taking into account quantum corrections goes by the name of non renormalisation. The argument presented in [9] is very simple. Since the gauge coupling constant τ appears in the prepotential (2.1.3), it can be considered as a background $\mathcal{N} = 2$ vector superfield. After quantum corrections the gauge coupling depends on the dynamically generated scale, $\tau \sim \log \Lambda$. In turn, therefore, Λ is constrained to be a background vector superfield itself. The metric on the Higgs branch does not depend on scalars in vector superfields, hence it cannot receive quantum corrections. The Higgs branch at all scales is thus given by the classical Higgs branch. Later on in this thesis we will argue that this statement does not hold in the context of five-dimensional theories. For the purposes of the present discussion, however, it suffices to recognise that classical computations of the moduli space of hypermultiplet scalars are exact.

Classical moduli space

Since the scalar potential (2.1.7) is a sum of squares, setting it to zero requires that the D-terms and the F-terms vanish independently. With a little rearrangement these conditions become

$$\frac{1}{g^2}[\phi, \phi^\dagger] = 0, \tag{2.2.9}$$

$$q_i^{\dagger b} q^i{}_a - \widetilde{q}_i{}^b \widetilde{q}^{\dagger i}_a = \nu \delta^b{}_a \tag{2.2.10}$$

$$\widetilde{q}_i{}^b q^i{}_a = \eta \delta^b{}_a, \quad \phi^a{}_b \widetilde{q}_i{}^b = 0, \quad q^i{}_b \phi^b{}_a = 0 \ , \tag{2.2.11}$$

where ν and η are a real and a complex number that enforce the tracelessness of the $SU(n_c)$ generators. We can thus identify the conditions dictated by the classical potential restricted on the Coulomb branch,

$$\{V=0\}_{\mathcal{C}} \equiv \{q^i = 0,\ \widetilde{q}_i = 0,\ \phi \neq 0 \mid [\phi, \phi^\dagger] = 0\} \ , \tag{2.2.12}$$

and Higgs branch,

$$\{V=0\}_{\mathcal{H}} \equiv \{\phi = 0,\ q^i \neq 0,\ \widetilde{q}_i \neq 0 \mid q_i^{\dagger b} q^i{}_a - \widetilde{q}_i{}^b \widetilde{q}^{\dagger i}_a = \nu \delta^b{}_a \ , \widetilde{q}_i{}^b q^i{}_a = \eta \delta^b{}_a\} \ . \tag{2.2.13}$$

Coulomb branch

The classical Coulomb branch (2.2.12) is described by a set of complex scalar fields taking values in the adjoint of $\mathfrak{su}(n)$ such that their Lie algebra commutator vanishes. This is precisely the definition of scalars taking values in the Cartan subalgebra of $\mathfrak{su}(n)$, that is one can take

$$\phi = \mathrm{diag}(\phi_1, \ldots, \phi_{n_c}), \qquad \text{with} \quad \sum_{a=1}^{n_c} \phi_a = 0 \ , \tag{2.2.14}$$

leaving $(n_c - 1)$ independent moduli. There is also a residual discrete symmetry acting on the scalars: S_{n_c}, the Weyl group of $SU(n_c)$, acts on the scalars ϕ_a by permutation. This means that the classical Coulomb branch can be identified with $\mathbb{C}^{(n_c-1)}/S_{n_c}$. More generally for a gauge group G we have

$$\mathcal{M}_{\mathcal{C},\text{class}} = \mathbb{C}^r / W \ , \tag{2.2.15}$$

where r is the rank of G and W its Weyl group. A generic point on the classical Coulomb branch is one such that all ϕ_a's take nonzero VEV. This is the Higgs mechanism in full spring: the gauge group is maximally broken to its Cartan subalgebra $G \to U(1)^r$. For G the special unitary group, we have $SU(n_c) \to U(1)^{n_c-1}$, so that only $(n_c - 1)$ photons remain massless.

As we have already mentioned, the Coulomb branch receives quantum corrections and so this classical analysis is not sufficient. In [4], the authors exploited the extra constraints imposed by $\mathcal{N} = 2$ supersymmetry on the geometry of the Coulomb branch, namely that the manifold has to be *rigid special Kähler*, to solve for the effective theory at low energy. This seminal work gave rise to what is now known as Seiberg-Witten theory and led to myriad publications in the field. Despite its outstanding beauty, this aspect of the theory is of no concern to us in this thesis.

Higgs branch

The classical analysis is most useful on the Higgs branch since it provides the exact result at all energy scales. We could study this branch in an analogous fashion to how we proceeded for the Coulomb branch, namely by examining the D-terms and the F-terms as given by (2.2.13). However there is another avenue which turns out to be particularly relevant to us and makes use of a nice result for moduli spaces of supersymmetric gauge theories. In order to understand this description, we must take a pleasant detour. We will return to the Higgs branch in section 2.4.3.

2.3 Moduli spaces as algebraic varieties

The description of the moduli space in terms of vanishing D-terms and F-terms is somewhat redundant. In general it is unnecessary to deal with the constraints set from the vanishing of the D-terms. Despite the simplicity of the result, there are subtleties hidden in its precise mathematical formulation – we will gloss over these issues. It is nonetheless a crucial outcome of supersymmetry so we present a streamlined version of what appears in [58] (but see also [59, 60, 61, 62]).

For the time being, let us concentrate on supersymmetric theories with no superpotential - so that only $\mathcal{N} = 1$ supersymmetry is preserved.

A supersymmetric gauge theory is manifestly invariant under a supergauge transformation that acts on $\mathcal{N} = 1$ chiral and vector superfields as

$$\Phi \to g \cdot \Phi \ , \tag{2.3.1}$$

$$e^V \to g^{-1} \cdot e^V \cdot g \ . \tag{2.3.2}$$

To maintain chirality of Φ and reality of V, the group element, $g \equiv e^{i\Omega}$, is the exponentiation of a chiral superfield parameter Ω. Since the scalar field in a chiral superfield is complex, Ω takes value in the *complexified* Lie algebra $\mathfrak{g}^c$ so that g is an element of the complexified gauge group G^C.

To avoid the complications of non-abelian transformations, let us consider an abelian vector superfield $V(x, \theta, \overline{\theta})$ (in the conventions of [63]). Its expansion in θ and $\overline{\theta}$ is

$$\begin{aligned} V(x,\theta,\overline{\theta}) = {} & C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) \\ & + \frac{i}{2}\theta\theta\,[M(x)+iN(x)] - \frac{i}{2}\overline{\theta\theta}\,[M(x)-iN(x)] \\ & - \theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta}\left[\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\right] \\ & - i\overline{\theta\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right] + \frac{1}{2}\theta\theta\overline{\theta\theta}\left[D(x) + \frac{1}{2}\Box C(x)\right] \end{aligned} \tag{2.3.3}$$

Under a supergauge transformation $V \to V + \Omega + \Omega^\dagger$, where the chiral superfield has components $\Omega = (\omega, \psi, F)$, the vector superfield components transform as

$$C \to C + \omega + \omega* \tag{2.3.4}$$

$$\chi \to \chi - i\sqrt{2}\psi \tag{2.3.5}$$

$$M + iN \to M + iN - 2iF \tag{2.3.6}$$

$$A_\mu \to A_\mu - i\partial_\mu(\omega - \omega*) \tag{2.3.7}$$

$$\lambda \to \lambda \tag{2.3.8}$$

$$D \to D\ . \tag{2.3.9}$$

Notice that both the real and imaginary parts of the scalar field ω appear in the transformations, in C and A_μ respectively, preserving the complexified gauge variation. Typically one chooses the Wess-Zumino (WZ) gauge where C, χ, M, N are all set to zero. In such a gauge the only component transforming nontrivially under the supergauge transformation is A_μ. It is well known that the WZ gauge breaks supersymmetry. In fact, it also breaks the complexified gauge symmetry leaving the theory invariant under only the usual gauge transformation with real parameter $i(\omega - \omega*)$.

Let us choose a different gauge such that $C \neq 0$ and return to the general possibly nonabelian case. For a vector superfield taking values in the adjoint representation of a group G, we have

$$V^A = C^A - \theta\sigma^\mu\overline{\theta}A^A_\mu + i\theta\theta\overline{\theta}\overline{\lambda}^A - i\overline{\theta\theta}\theta\lambda^A + \frac{1}{2}\theta\theta\overline{\theta\theta}D^A \tag{2.3.10}$$

$$= C^A + V^r_{WZ}r \tag{2.3.11}$$

With such a choice, complexified gauge invariance is unbroken. In this gauge, the D-terms for a set of scalar fields $\{\varphi \in \Phi\}$ transforming in some representation of G with generators $T^A_{\mathcal{R}}$ are given by

$$\frac{\partial}{\partial C^A}\left(\varphi^\dagger e^C \varphi\right) = 0 \ . \tag{2.3.12}$$

Consider a field φ that satisfies this D-flatness condition. For such a φ we have

$$\varphi^\dagger e^C T^A \varphi = \ (e^{C/2}\varphi)^\dagger T^A (e^{C/2}\varphi) = \ \widehat{\varphi}^\dagger T^A \widehat{\varphi} = 0 \ , \tag{2.3.13}$$

where we have defined the G^C-equivalent field $\widehat{\varphi} \equiv e^{C/2}\varphi$. The condition (2.3.13) can be rewritten as

$$\frac{\partial}{\partial \widehat{C}^A}\left(\widehat{\varphi}^\dagger e^{\widehat{C}} \widehat{\varphi}\right)\Big|_{\widehat{C}=0} = 0 \ , \tag{2.3.14}$$

that is (2.3.13) is just a D-term in WZ gauge (since for $C = 0$, $V = V_{WZ}$). The last equation, (2.3.14), can be recast in terms of the G^C variation of a gauge invariant quantity. Indeed we can write

$$\frac{\partial}{\partial \widehat{C}^A}\nu\left(e^{\widehat{C}} \widehat{\varphi}\right)\Big|_{\widehat{C}=0} = 0 \ , \tag{2.3.15}$$

where $\nu(a) = a^\dagger a$. Equation (2.3.15) has the following meaning. The G-invariant quantity ν takes a constant value upon G^C gauge variation. Fields $\widehat{\varphi}$ that satisfy this condition lie on specific G-orbits which are named D-orbits. In other words, a D-orbit is defined as the set $\{\widehat{\varphi}\}$ that obey the WZ D-flatness condition. We also define a G^C-orbit as the set of fields that are equivalent under the action of G^C: $\widehat{\varphi}_2 = g \cdot \widehat{\varphi}_1 \implies \widehat{\varphi}_2 \sim \widehat{\varphi}_1$.

Let us fix the superpotential of the theory to be zero for now. The following theorem then holds.

Theorem 2.1. *Every constant field configuration φ_0 is G^C-equivalent to a solution $\widehat{\varphi}$ of the WZ-gauge D-terms.*

We provide a graphical representation of Theorem 2.1 in Figure 2.1. The theorem can be further refined by showing that for every φ_0 the associated G^C-orbit contains exactly one D-orbit with representative $\widehat{\varphi}$. It then holds that the set of scalars obeying the WZ D-terms is equivalent to the set of φ_0 modded out by G^C and thus the classical moduli space of a theory with no superpotential is given by

$$\mathcal{M} \equiv \mathcal{M}^{(0)}/G^C \tag{2.3.16}$$

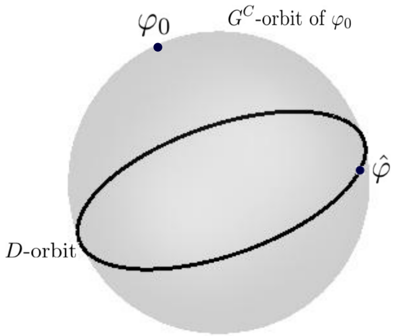

Figure 2.1: A D-orbit, the black circle, contained in a G^C-orbit, the sphere.

where $\mathcal{M}^{(0)}$ is the space of constant scalar fields and the quotient identifies G^C-equivalent scalars. $\mathcal{M}^{(0)}$ has the structure of an affine variety (since it is just given by $\mathbb{C}^n$, where n is the number of scalars).

In the presence of a superpotential $\mathcal{W}$ one has to consider the F-terms in the scalar potential as well as the D-terms. In this case, instead of starting from the space of unconstrained constant scalar fields, it is appropriate to restrict the latter to the subset of scalars that obey

$$F_{(\mathcal{R})} \equiv \frac{\partial \mathcal{W}(\varphi)}{\partial \varphi_{(\mathcal{R})}} = 0 \ , \tag{2.3.17}$$

which is a set of $\dim(\mathcal{R})$ algebraic equations. The set of constant scalar fields $\{\varphi_0\}$ that obey (2.3.17) is referred to as the F-flat variety and denoted $\mathcal{F}$. Then,

$$\mathcal{F} \equiv \mathcal{M}^{(0)} \Big/ \left\{ \frac{\partial \mathcal{W}}{\partial \varphi_{(\mathcal{R})}} = 0 \right\} \ . \tag{2.3.18}$$

The moduli space is then obtained by "applying the D-terms" in the way we have constructed, namely by taking a quotient by the complexified gauge group G^C, so that finally one has

$$\mathcal{M} = \mathcal{F}/G^C \ . \tag{2.3.19}$$

Crucially, in [58], the following theorem is demonstrated.

Theorem 2.2. *For an algebraic variety $\mathcal{A}$ acted upon by a group G^C, the quotient $\mathcal{A}/G^C$ is in bijection with the variety given by the subring R_G of G-invariant elements contained in the ring associated to $\mathcal{A}$.*

In other words the quotient $\mathcal{F}/G^C$ can be identified with the ring of gauge invariant polynomials constructed out of the fields φ with the restriction that (2.3.17) is satisfied. Theorem (2.2) is at the heart of all the calculations in this thesis as it allows tools from algebraic geometry to be used to study moduli spaces of supersymmetric theories. Before proceeding to study the Coulomb and Higgs branches using this result, our discussion can be further refined.

2.3.1 The chiral ring

The reader should at this point put aside the discussion on moduli spaces and ask what any well-trained quantum field theorist would wonder. What about the good old correlation functions of operators? Let the reader be in no doubt: the general answer to such a question is well above the reaches of our humble work. Nonetheless, let us make a few comments.

The scalar fields in the chiral superfields we have encountered so far belong to the set of what are called chiral operators. In general, the lowest component of any chiral superfield is a chiral operator. A chiral operator is an operator $\mathcal{O}$ for which

$$\left[\overline{\mathcal{Q}}_{\dot{\alpha}}, \mathcal{O}\right] = 0 \ , \tag{2.3.20}$$

that is $\mathcal{O}$ is annihilated by one set of the supercharges[8]. It is clear that a product of chiral operators is itself chiral. Importantly the dependence on spacetime position for a chiral operator is

$$\frac{\partial}{\partial x^\mu}\mathcal{O}(x) = [P_\mu, \mathcal{O}(x)] = \left\{\overline{\mathcal{Q}}^{\dot{\alpha}}, [\mathcal{Q}^\alpha, \mathcal{O}]\right\} \ , \tag{2.3.21}$$

where we have used the supersymmetry algebra $\left\{\mathcal{Q}^\alpha, \overline{\mathcal{Q}}^{\dot{\alpha}}\right\} \sim P^\mu$, the super Jacobi identity and the chirality condition (2.3.20). The spacetime dependence for the VEV of $\mathcal{O}(x)$ is then

$$\frac{\partial}{\partial x^\mu}\langle\mathcal{O}(x)\rangle = \langle\frac{\partial}{\partial x^\mu}\mathcal{O}(x)\rangle = \langle 0|\{\overline{\mathcal{Q}}^{\dot{\alpha}}, [\mathcal{Q}^\alpha, \mathcal{O}]\}|0\rangle = 0 \ , \tag{2.3.22}$$

since the vacuum is invariant under supersymmetry transformations, that is $\mathcal{Q}^\alpha|0\rangle = 0$ and $\overline{\mathcal{Q}}^{\dot{\alpha}}|0\rangle = 0$. In general the expectation value of a product of

[8] We remind the reader that in this part of the thesis we are in 4d.

chiral operators is independent of each of their positions

$$\frac{\partial}{\partial x^k}\langle\prod_i \mathcal{O}_i(x)\rangle = 0\ . \tag{2.3.23}$$

As such, it can be factorised with impunity and the spacetime dependence dropped entirely so that

$$\langle\prod_i \mathcal{O}_i(x)\rangle = \langle\mathcal{O}_1\rangle \ldots \langle\mathcal{O}_n\rangle\ . \tag{2.3.24}$$

Thanks to this, correlation functions of chiral operators are greatly simplified and are amenable to detailed study. Firstly it is clear that the quantity $\{\overline{\mathcal{Q}}_{\dot\alpha}, \ldots\}$ plays no role in the VEVs of such operators, hence one can define equivalence classes of operators, $\mathcal{O}_1(x) \sim \mathcal{O}_2(x)$, when

$$\mathcal{O}_1(x) = \mathcal{O}_2(x) + \{\overline{\mathcal{Q}}^{\dot\alpha}, \overline{X}_{\dot\alpha}(x)\} \tag{2.3.25}$$

for some $\overline{X}_{\dot\alpha}$. The set of equivalence classes of chiral operators forms a ring known as the **chiral ring** [64].

If the chiral operators considered are gauge invariant then it becomes clear at once why the chiral ring is relevant to our discussion. It is indeed precisely the ring of gauge invariant holomorphic functions that describe the moduli space of vacua. The equivalence works both ways of course: studying the moduli space of vacua via the rings of gauge invariant polynomials gives access to correlation functions of chiral operators.

Later in this thesis we also talk about the chiral ring in situations where the definitions above do not apply. For example, in 5d minimally supersymmetric theories (8 supercharges) there are no sub-superalgebras with 4 supercharges and as such one cannot define chirality. Nonetheless the ring of holomorphic functions can still be analysed in order to characterise the moduli space of vacua. We will be loose in our language and still refer to this as the chiral ring.

2.4 Counting operators

In light of the correspondence between moduli spaces of vacua and rings of gauge invariant holomorphic functions, powerful yet simple techniques have been developed starting with the work in [65, 66, 67, 68, 69, 70, 71]. These will be in use throughout this thesis and so we summarise the premise and the computational tools there introduced.

The polynomial ring in the scalar fields is denoted $R = \mathbb{C}[\varphi_1, \ldots, \varphi_n]$. The F-terms (2.3.17) furnish an ideal of the polynomial ring

$$I = \langle \{F_{(\mathcal{R}_i)} = 0\}_{i=1}^{n} \rangle \ , \tag{2.4.1}$$

where there is an equation $F_{(\mathcal{R}_i)} = 0$ for each scalar field φ_i generating the polynomial ring. The $\mathcal{F}$-flat variety (2.3.18) is then in bijection with the quotient ring R/I

$$\mathcal{F} \longleftrightarrow \mathbb{C}[\varphi_1, \ldots, \varphi_n] \Big/ \langle \{F_{(\mathcal{R}_i)} = 0\}_{i=1}^{n} \rangle \ . \tag{2.4.2}$$

Application of the D-terms is done by further taking the quotient (2.3.19) which, by Theorem 2.2, is equivalent to considering a subring of gauge invariant polynomials. Let there be a basis set of n' gauge invariant objects,

$$\mathcal{O}_j = \mathcal{O}_j(\{\varphi_i\}_{i=1}^{n}) \ , \tag{2.4.3}$$

constructed from the fields $\{\varphi_i\}_{i=1}^{n}$ which satisfy the F-terms constraints. Let $\mathcal{O}$ be the set of such operators $\mathcal{O} = \{\mathcal{O}_j\}_{j=1}^{n'}$ and regard it as a map from $\mathcal{F} = R/I$ to the ring $\mathbb{C}[\mathcal{O}_1, \ldots, \mathcal{O}_{n'}]$. The image of this map gives rise to an affine variety,

$$\mathcal{M} \simeq \text{Im}\left(\mathbb{C}[\varphi_1, \ldots, \varphi_n] \Big/ \langle \{F_{(\mathcal{R}_i)} = 0\}_{i=1}^{n} \rangle \to \mathbb{C}[\mathcal{O}_1, \ldots, \mathcal{O}_{n'}]\right) \ , \tag{2.4.4}$$

corresponding to the moduli space of vacua.

2.4.1 Hilbert Series

Studying the vacuum variety as defined by Equation (2.4.4) can be done in different ways. Here we exploit the power of a generating function that can be associated to a ring. Such a generating function is known as Hilbert series.

For a ring R over $\mathbb{C}$ with a grading $R = R_0 + R_1 + \ldots$ under addition, the Hilbert function,

$$Hf(R, i) = \dim_{\mathbb{C}} R_i \qquad \text{for } i \in \mathbb{N} \ , \tag{2.4.5}$$

counts the elements in each subspace R_i. In a polynomial ring these are simply the number of homogeneous polynomials at i^{th} degree. By the Hilbert Basis Theorem the ring is finitely generated. Let us take $R = \mathbb{C}[\mathcal{O}_1, \ldots, \mathcal{O}_{n'}]$ so that

$\{\mathcal{O}_j\}_{j=1}^{n'}$ is a generating set. Let the degree of each generator be $\deg(\mathcal{O}_j) = d_j$. The Hilbert series is then defined as

$$H(R,t) = \sum_{i=0}^{\infty} Hf(R,i)t^i \ . \tag{2.4.6}$$

By the Hilbert syzygy theorem there exists a closed form rational function for the power series given by

$$H(R,t) = \frac{P(R,t)}{\prod_{j=1}^{n'}(1-t^{d_j})} \ , \tag{2.4.7}$$

where $P(R,t)$ is a polynomial in t with integer coefficients.

Lemma 2.1. *The dimension d of the affine variety associated to R is given by the limit*

$$\lim_{t\to 1}(1-t)^d H(R,t) \ , \tag{2.4.8}$$

when such a limit is finite and nonzero.

From now on we will drop the redundant argument R and write $H(t)$ instead of $H(R,t)$. We will refer to the parameter t as a fugacity.

Refinement

We can *refine* the Hilbert series by incorporating further gradings. The Hilbert series simply generalises to

$$H(R,t_1,\ldots,t_k) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^k} \dim_{\mathbb{C}} R_{\boldsymbol{\alpha}} \boldsymbol{t}^{\boldsymbol{\alpha}} \ , \tag{2.4.9}$$

where $\boldsymbol{t} = (t_1,\ldots,t_k)$ and $\boldsymbol{\alpha}$ is a multi-index $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_k)$. A ring which admits a multi-grading is associated to a variety which admits as many $\mathbb{C}^*$ actions as there are gradings. For example, when $R = \mathbb{C}[x_1,\ldots x_l]$, the Hilbert series is

$$H(t) = \sum_{i=0}^{\infty} \begin{pmatrix} i+l-1 \\ i \end{pmatrix} t^i = \frac{1}{(1-t)^l} \ . \tag{2.4.10}$$

We can interpret this as counting all possible homogeneous monomials at degree i without distinguishing which x_i's make them up. If instead we make explicit the $(\mathbb{C}^*)^l$ action on $\mathbb{C}^l$, the grading on the ring of polynomials is captured by a set of fugacities $\{t_1, \ldots, t_l\}$ so that the refined Hilbert series is

$$H(t_1, \ldots, t_l) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_l=0}^{\infty} t_1^{i_1} \cdots t_l^{i_l} = \prod_{i=1}^{l} \frac{1}{(1-t_i)} \ , \tag{2.4.11}$$

Conversely, starting from a refined Hilbert series, setting all $t_i = t$, the unrefined Hilbert (2.4.10) series is obtained.

When appropriately manipulated, refined Hilbert series can capture the isometry of the space at hand. In the previous example the map

$$t_1 = x_1 t, \quad t_2 = \frac{x_2}{x_1} t, \quad \cdots, \quad t_{l-1} = \frac{x_{l-1}}{x_{l-2}} t, \quad t_{l-1} = \frac{1}{x_{l-1}} t \tag{2.4.12}$$

allows for the Hilbert series to be rewritten as

$$H(x_1, \ldots, x_{l-1}, t) = \sum_{i_1=0} \chi[i, 0, \ldots, 0]_{SU(l)} t^i \tag{2.4.13}$$

$$= \frac{1}{(1-x_1 t)} \frac{1}{(1-\frac{x_2}{x_1} t)} \cdots \frac{1}{(1-\frac{x_{l-1}}{x_{l-2}} t)} \frac{1}{(1-\frac{1}{x_{l-1}} t)} \ , \tag{2.4.14}$$

where $\chi[i, 0, \ldots, 0]_{SU(l)}$ denotes the character for the representation of $SU(l)$ with Dynkin label $[i, 0, \ldots, 0]$. The isometry group $U(l)$ acting on $\mathbb{C}^l$ has been made explicit by a careful choice of fugacities. By comparing with (2.4.7) we notice that the Hilbert series spells out once again that the generators all have degree 1 in the main grading, as expected.

Computations

Vacuum varieties in supersymmetric gauge theories lend themselves very naturally to calculations of Hilbert series. The procedure for calculating the Hilbert series in such a setting is:

- Identify global and gauge symmetries and their Abelian subalgebras.
- Identify the transformation properties of the (gauge variant) scalar fields under the global and gauge symmetries $\varphi_{(\mathcal{R}_{Gl} \times \mathcal{R}_G)}$.
- For the $SU(2)_R$ symmetry, identify the $U(1)$ subalgebra which selects the highest weight of $SU(2)$ representations. This provides the main grading to the ring.
- Assign the $U(1)$ global, gauge and R charges to the scalar fields.

This procedure leads to a replacement

$$\varphi_{(\mathcal{R}_{Gl} \times \mathcal{R}_G)} \to \chi[\mathcal{R}_{Gl}](\boldsymbol{y}) \times \chi[\mathcal{R}_G](\boldsymbol{x}) \times t^d \ , \tag{2.4.15}$$

where $\chi[\mathcal{R}](\boldsymbol{z})$ signifies the character of the representation $\mathcal{R}$ in the variables $\boldsymbol{z}$ and d is the highest weight of the $SU(2)_R$ representation.

For simplicity, let us restrict for now to a theory without F-terms and consider the affine variety without taking the D-terms into account yet. The ring of gauge *variant* operators, $\mathbb{C}[\varphi_1, \ldots, \varphi_n]$, contains all the unordered k-tuples in the fields $\{\varphi\}_i^n$ - the generators of the ring. Such a ring can be obtained by constructing at each degree all possible symmetric monomials in the fields. The k^{th} graded piece is thus

$$R_k = \{h_k(\varphi_1, \ldots, \varphi_n)\} \ , \tag{2.4.16}$$

where $h_k(\varphi_1, \ldots, \varphi_n)$ is a sum of all distinct products of degree k in a subset of the variables φ_i. Symmetrisation manifestly plays a prominent role in the chiral ring. On the Hilbert series' side, this is captured by a function which counts symmetric products of its argument, known as **Plethystic Exponential** (PE).

Definition 2.1. *For a multivariate function $f(t_1, \ldots, t_n)$ with $f(0, \ldots, 0) = 0$, we define*

$$\text{PE}[f(t_1, \ldots, t_n)] := \exp\left(\sum_{r=1}^{\infty} \frac{f(t_1^r, \ldots, t_n^r)}{r}\right) \ . \tag{2.4.17}$$

Example 2.1. *For $f(t) = t$, the PE is*

$$\text{PE}[t] = \exp\left(\sum_{r=1}^{\infty} \frac{t^r}{r}\right) = \exp\left(-\ln(1-t)\right) = \frac{1}{1-t} \ . \tag{2.4.18}$$

Two properties of the PE are noteworthy. Firstly, $\text{PE}[-\alpha t] = (1-t)^\alpha$ when α is a positive constant. Secondly, $PE[f_1 + f_2] = PE[f_1]PE[f_2]$. Consequently, for a power series in t with positive coefficients, $g(t) = \sum_n a_n t^n$, the PE is

$$\text{PE}\left[\sum_n a_n t^n\right] = \frac{1}{\prod_n (1-t^n)^{a_n}} \ . \tag{2.4.19}$$

Here, the reader should recall the above discussion on the Hilbert series: the PE is intimately related to it. Indeed by considering the ring $R = \mathbb{C}[\varphi_1, \ldots \varphi_n]$, identifying the symmetry group acting on it, $U(n)$, and the charges of the generators $\{\varphi_i\}$ as $\{x_1 t, \frac{x_2}{x_1}t, \cdots, \frac{x_{l-1}}{x_{l-2}}t, \frac{1}{x_{l-1}}t\}$, the Hilbert series associated to the ring is given by

$$\begin{aligned} H(x_1,\ldots,x_{l-1},t) &= \mathrm{PE}\left[\left(x_1 t + \frac{x_2}{x_1}t + \cdots + \frac{x_{l-1}}{x_{l-2}}t + \frac{1}{x_{l-1}}t\right)\right] \\ &= \mathrm{PE}\left[\chi[1,0,\ldots,0]_{SU(n)}t\right] \ , \end{aligned} \tag{2.4.20}$$

which, upon using (2.4.19), recovers precisely (2.4.14).

F-flat Hilbert series

For a quotient ring R/I like (2.4.2), a mere symmetrisation of the generators of R is not sufficient since the ideal provides equations that set to zero some elements in R. In general the Hilbert series for such a quotient can be evaluated using standard computational algebraic geometry packages like Macaulay2 [72] by providing the fundamental fields and the relations between them as determined by (2.4.1). However there is a remarkably useful simplification for supersymmetric gauge theories which can be fully Higgsed. For such theories the F-flat variety $\mathcal{F}$ is a so-called **complete intersection**. A complete intersection (CI) is a variety such that its dimension is given by

$$\dim(CI) = n - m \ , \tag{2.4.21}$$

where n is the degree of the embedding space and m is the number of vanishing polynomials forming the ideal I. Let R be the ring of gauge variant scalar fields $R = \mathbb{C}[\varphi_1,\ldots,\varphi_n]$ and $\{e_i\}_{i=1}^m$ the degrees of the m relations arising from the F-terms. The unrefined Hilbert series for such a CI is

$$H(\mathcal{F},t) = \frac{\prod_{j=1}^m (1-t^{e_j})}{\prod_{i=1}^n (1-t^{d_i})} = \mathrm{PE}\left[\sum_{i=1}^n t^{d_i} - \sum_{j=1}^m t^{e_j}\right] \ . \tag{2.4.22}$$

Upon replacing the gauge variant F-terms by characters,

$$F_{\mathcal{R}_G} \to \chi[\mathcal{R}_G](\boldsymbol{x}) \times t^{e_j} \ , \tag{2.4.23}$$

the refined Hilbert series for $\mathcal{F}$ is given by

$$H(\mathcal{F},\boldsymbol{x},\boldsymbol{y},t) = \mathrm{PE}\left[\sum_i^n \chi[\mathcal{R}_{Gl}(\varphi_i)](\boldsymbol{y})\ \chi[\mathcal{R}_G(\varphi_i)](\boldsymbol{x})\ t^{d_i} - \sum_j^m \chi[\mathcal{R}_G(F_j)](\boldsymbol{x})\ t^{e_j}\right] , \qquad (2.4.24)$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the fugacities for the Cartan subalgebra of the gauge group and flavour group respectively.

The discussion so far has only provided a Hilbert series for $\mathcal{F}$ but this is not sufficient as it does not take into account the D-terms. As has been discussed at length, imposing the D-terms amounts to taking the quotient of $\mathcal{F}$ by the complexified gauge group G^C. The resulting variety is in correspondence with the subring of $\mathcal{F}$ where only gauge invariant elements, that is singlets, are allowed. As far as the Hilbert series is concerned, this means that gauge variant polynomials must be removed from the series. Projecting onto gauge singlets is done by means of the Molien-Weyl integration formula so that the Hilbert series for the ring associated to the moduli space is given by

$$H(\mathcal{M},\boldsymbol{y},t) = \int_G \mathrm{d}\mu_G(\boldsymbol{x}) H(\mathcal{F},\boldsymbol{x},\boldsymbol{y},t) \qquad (2.4.25)$$

where $\mathrm{d}\mu_G$ is the Haar measure for the group G as provided in [73],

$$\int_G \mathrm{d}\mu_G = \oint \cdots \oint \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r} \prod_{\alpha\in\{\Delta^+\}} \left(1 - \prod_{l=1}^r x_l^{\alpha}\right) , \qquad (2.4.26)$$

with $\alpha \in \{\Delta^+\}$ the positive roots in the Lie algebra $\mathfrak{g}$ of rank r. Upon evaluation of the Molien integral, the surjection (2.4.4) has been implemented at the level of the Hilbert series. To summarise, the last set of steps to evaluate the Hilbert series for moduli spaces of vacua is as follows.

- Identify the transformation properties of the F-terms under the gauge symmetry.
- Assign the $U(1)$ global, gauge and R charges to the F-terms and as such replace them by characters.
- Compute the Hilbert series for the F-flat variety in terms of a plethystic exponential.
- Project onto the gauge invariant sector by computing the Molien integral

of the F-flat Hilbert series.

It is important to stress that in general after projecting onto the gauge invariants, the variety is no longer a complete intersection. In particular it does not have the structure of a plethystic exponential.

A useful tool after computation of (2.4.25) is the so-called **Plethystic Logarithm** (PL). The Hilbert series (2.4.25) can be obtained either as a rational function or as a Taylor series in t (depending on how computationally feasible the integral is). In both cases, a surprising amount of information about the variety is contained in the first few terms of the expansion in t. Furthermore, the inverse function of the PE brings to light the generators and the relations of the ring of gauge invariant operators. Such an inverse is the aforementioned plethystic logarithm, which, for a multi-variate function $f(x_1, \ldots, x_n)$ such that $f(0, \ldots, 0) = 1$, is defined as

$$\mathrm{PL}[f(x_1, \ldots, x_n)] = \sum_{k=1}^{\infty} \frac{1}{k} \mu(k) \log f(x_1^k, \ldots, x_n^k) \ .$$

If the space is a complete intersection after projecting onto the gauge invariant sector, the PL is a finite polynomial of terms with positive and negative signs, the former encoding the generators and the latter the relations. If the space is not a complete intersection, the PL does not terminate and it is less helpful: higher syzygies enter the infinite series and make reading off generators and relations a delicate task.

2.4.2 Highest weight generating function

The structure of the moduli space can be captured in a more succinct form through the so-called highest weight generating function (HWG), which was first introduced in [74]. The HWG summarises the full character of a representation using its highest weight and takes on a deep geometrical meaning since, under appropriate and consistent manipulations, it allows for movement in the space of theories. As such it should be considered on a par with the superpotential, partition functions and indices appearing in the literature on spaces of vacua.

A typical Hilbert series which counts holomorphic functions on a given $\mathcal{N} = 2$ vacuum variety has the form

$$\mathrm{HS}(y_1, \ldots, y_s; t) = \sum_k f_k(x_1, \ldots, x_s) t^k \ , \tag{2.4.27}$$

where each $f_k(x_1, .., x_s)$ is a sum of characters for irreducible representations of the global symmetry group coming from evaluation of the Molien integral.

To obtain a HWG, one notices that the character, $\chi_{[n_1,\dots,n_s]}(x_1,\dots,x_s)$, for a given representation can be encoded by the set of coefficients appearing in the corresponding Dynkin label $[n_1,\dots,n_s]$. Choosing a set of fugacities $\{\mu_i\}_{i=1}^{s}$ to keep track of such coefficients, the map

$$\chi_{[n_1,\dots,n_s]}(x_1,\dots,x_s) \mapsto \mu_1^{n_1}\cdots\mu_r^{n_s} \tag{2.4.28}$$

can be applied to (2.4.27) to obtain a generating function in terms of highest weights,

$$\mathrm{HWG}(\mu_1,\dots,\mu_s;t) = \sum_k \left(\mu_1^{n_1}\cdots\mu_r^{n_s}\right)_k t^k \tag{2.4.29}$$

The series can then be resummed as a rational function.

Interestingly, it is often the case that the HWG can be rewritten in the form of a plethystic exponential, even when the Hilbert series cannot. This means that the variety associated to the ring of highest weights is a complete intersection unlike the variety associated to the ring of all weights. The coarse graining implemented by the HWG throws away information on the one hand, but reveals some nice properties that are invisible to the fine graining provided by the Hilbert series.

2.4.3 The Higgs branch as an algebraic variety

In light of the techniques introduced in this section, we can proceed to evaluate the Hilbert series associated to the ring of holomorphic functions which parametrises the Higgs branch of $SU(n_c)$ with N_f flavours.

The first check relates to the dimension of the moduli space. There are $2 \times N_f n_c$ complex scalars $\{q^i_a,\ \widetilde{q}^a_i\}$ subject to $n_c^2 - 1$ real equations coming from the D-terms, (2.2.10), and $n_c^2 - 1$ complex equations coming from the F-terms, (2.2.11). For the latter there are $n_c^2 - 1$ redundant equations due to the gauge invariance of the superpotential. Hence the Higgs branch has complex dimension $2N_f n_c - 2(n_c^2 - 1)$. Reflecting the hyperKähler structure of the Higgs branch, the complex dimension is, as expected, even. The literature usually gives the quaternionic dimension, namely $N_f n_c - (n_c^2 - 1)$. To simplify the discussion we have here assumed the case of $N_f \geq n_c$.

The ring of gauge variant scalar fields is $R = \mathbb{C}[q^i_a, \widetilde{q}^a_i]$ whilst the ideal provided by the F-terms is the last equation in (2.2.13)

$$I = \langle \widetilde{q}_i^a q_b^i - \eta \delta^a{}_b = 0 \rangle \tag{2.4.30}$$

The gauge and global symmetry is $SU(n_c) \times SU(N_f) \times U(1)_B$. Let

$$\begin{aligned} q_a^i &\to [1,0,\dots,0]_{SU(n_c)}(\boldsymbol{x}) \times [0,\dots,0,1]_{SU(N_f)}(\boldsymbol{y}) \times u^{-1} \times t \\ \widetilde{q}_i^a &\to [0,0,\dots,1]_{SU(n_c)}(\boldsymbol{x}) \times [1,0,\dots,0]_{SU(N_f)}(\boldsymbol{y}) \times u \times t \ , \end{aligned}$$

where u is the fugacity for the $U(1)_B$ baryon number and $[r_1, \dots, r_{N_f-1}]$ now stands for the character of the representation, that is we have suppressed the χ in front of the Dynkin label. The ideal transforms in the adjoint of $SU(n_c)$ and has highest weight 2 under $SU(2)_R$. The F-flat variety $\mathcal{F}$ is

$$\begin{aligned} H(\mathcal{F}, \boldsymbol{x}, \boldsymbol{y}, t) = \mathrm{PE}\Big[&[1,0,\dots,0]_{SU(n_c)}(\boldsymbol{x})[0,\dots,0,1]_{SU(N_f)}(\boldsymbol{y})u^{-1}t \\ &+ [0,0,\dots,1]_{SU(n_c)}(\boldsymbol{x})[1,0,\dots,0]_{SU(N_f)}(\boldsymbol{y})ut \\ &-[0,0,\dots,1]_{SU(n_c)}(\boldsymbol{x})t^2 \Big] \end{aligned} \tag{2.4.31}$$

Projecting onto the singlet sector, the Hilbert series can be written as a sum of characters of $SU(N_f)$ in the fugacities $\boldsymbol{y}$, namely $H(\mathcal{M}_H, \boldsymbol{y}, t)$. Converting the series into a HWG with highest weight fugacities $\{\mu_i\}_{i=1}^{N_f-1}$ we obtain

$$\mathrm{HWG}(\mu_1, \dots, \mu_{N_f-1}, t) = \mathrm{PE}\left[t^2 + \sum_{i=1}^{n-1} \mu_i \mu_{N-i} t^{2i} + (\mu_n u + \mu_{N-n} u^{-1}) t^n \right] . \tag{2.4.32}$$

Equation (2.4.32) is valid for $N_f \geq 2n_c$, to which we further restrict from now on for simplicity.

Let us analyse the chiral ring. We look for gauge invariant generators and the relations they satisfy. We construct the gauge invariant generators from first principles but keep the Hilbert series as an aid. In a sense we are reversing our previous rationale: starting from a Hilbert series and its Plethystic Logarithm we extract generators and relations[9]. This procedure is quite subtle as higher syzygies can ruin the delicate balance of plus and minus signs appearing in the PL.

[9] Despite the variety not being a complete intersection, taking an expansion of the HS up to some order and consequently the PL up to that order, the generators and relations can be extracted.

Generators

The HS and the PL both have terms

$$t^2\left([1,0,\dots,0,1]+1\right) \tag{2.4.33}$$

$$t^{n_c}\left(u^{n_c}[0,\dots,0,\underset{n_c^{\text{th}}}{1},0,\dots,0]+u^{-n_c}[0,\dots,0,\underset{(N_f-n_c)^{\text{th}}}{1},0,\dots,0]\right)\ , \tag{2.4.34}$$

where of course these are $SU(N_f)$ characters as the gauge group has been integrated over. The presence of these terms tells us that there are four generators. Let us denote the $SU(N_f)$ fundamental indices $i,j,\dots=1,\dots,N_f$. The appearing representations dictate that the four generators are: $M^i{}_j$, η at t^2 and $B^{[i_1\dots i_n]}$, $\widetilde{B}_{[i_1\dots i_n]}$ at t^{n_c}.

In this case, however, we can rely on the previous knowledge of the ring of gauge *variant* elements $R=\mathbb{C}[q^i_a,\widetilde{q}^a_i]$ and can precisely construct what the gauge invariant operators are. From the Lagrangian fields we can indeed construct two types of basic gauge invariant operators: firstly the mesons, separated into the traceless part and the trace, both quadratic in the squarks and hence $SU(2)_R$ spin-1 operators, and secondly two kinds of baryons, made up of n_c squarks and hence $SU(2)_R$ spin-$\frac{n_c}{2}$. They are constructed as

$$M^i{}_j = q^i{}_a\widetilde{q}^a{}_j - \frac{1}{N_f}q^k{}_a\widetilde{q}^a{}_k\delta^i{}_j \tag{2.4.35}$$

$$\eta = \frac{1}{n_c}\widetilde{q}^a{}_k q^k{}_a \tag{2.4.36}$$

$$B^{i_1\dots i_{n_c}} = \frac{1}{n_c!}q^{i_1}_{a_1}\cdots q^{i_{n_c}}_{a_{n_c}}\epsilon^{a_1\dots a_{n_c}} \tag{2.4.37}$$

$$\widetilde{B}_{i_1\dots i_{n_c}} = \frac{1}{n_c!}\widetilde{q}^{a_1}_{i_1}\cdots\widetilde{q}^{a_{n_c}}_{i_{n_c}}\epsilon_{a_1\dots a_{n_c}}\ . \tag{2.4.38}$$

The meson $M^i{}_j$ is constructed so as to be traceless. The trace of its traceful version, namely the object $q^i{}_a\widetilde{q}^a{}_j$, is proportional to η. This can be seen by taking a contraction of the F-terms (2.4.30), which yields $n_c\eta=\widetilde{q}^a_i q^i{}_a$.

A compact notation, as introduced in [9], is henceforth rather convenient. We define a dot for contraction between one upper and one lower $SU(N_f)$ index and a Hodge star for an epsilon contraction of $SU(N_f)$ indices so that, for example,

$$M\cdot M \equiv M^i{}_j M^j{}_k$$

$$*B_{i_1\cdots i_{N_f-n_c}} \equiv \epsilon_{i_1\cdots i_{N_f-n_c}j_1\cdots j_{n_c}}B^{j_1\cdots j_{n_c}}\ .$$

We will often suppress the indices as well as identity matrices whenever there is no room for confusion.

Relations

The relations between the generators occur at different powers of t: they appear at t^4, t^{n_c+2} and t^{2n_c}. From the HS and the PL, we learn that they transform in some representations of $SU(N_f) \times U(1)_B$ obeying clear patterns of behaviour which we summarise in Appendix D. From the relations in terms of characters of representations, the chiral ring can be reconstructed in terms of defining equations in the basic gauge invariant operators up to numerical factors. As mentioned just above, knowledge of the ring of gauge variant operators – the squarks and the F-terms – can be exploited to write down the precise relations, including the numerical factors. This is however circumstantial; in the spirit of the Hilbert series techniques we will rely only on the representations of the gauge invariant operators as they appear in Appendix D. The chiral ring is then as summarised in Table 2.2 which is carefully constructed from the representations appearing in Table D.1.

$-t^4$	$M \cdot M \propto \mathrm{Tr}\, M^2$
	$\mathrm{Tr}\, M^2 \propto \eta^2$
$-t^{n_c+2}\ u^{n_c}$	$*B \cdot M \propto \eta * B$
	$M \cdot B \propto \eta B$
$-t^{n_c+2}\ u^{-n_c}$	$M \cdot *\widetilde{B} \propto \eta * \widetilde{B}$
	$\widetilde{B} \cdot M \propto \eta \widetilde{B}$
$-t^{2n_c} u^{2n_c}$	$B \cdot *B = 0$
$-t^{2n_c} u^{-2n_c}$	$*\widetilde{B} \cdot \widetilde{B} = 0$
$-t^{2n_c}$	$B^{i_1 i_2 i_3} \widetilde{B}_{j_1 j_2 j_3} \propto M^{[i_1}{}_{j_1} M^{i_2}{}_{j_2} M^{i_3]}{}_{j_3}$

Table 2.2: Chiral ring of 4d $\mathcal{N} = 2$ $SU(n_c)$ with N_f flavours.

2.4.4 The classical Coulomb branch revisited

The Coulomb branch was examined without the help of Theorem (2.2) but it is instructive to apply it on this branch too. The scalar fields on the Coulomb branch are the set $\{\phi^A\}$ where A is an index labelling the adjoint representation of the group G. Thus, for a parametrisation of the Coulomb branch, Theorem (2.2) instructs us to find gauge invariant polynomials in the scalars ϕ^A. In other words the singlet contribution has to be extracted from tensor products of the adjoint representation. In general *invariant tensors* intertwine between singlets and tensor product representations. For tensor products of the adjoint representations the relevant invariant tensors are in one-to-one correspondence with Casimir operators.

Casimir operators of a Lie algebra $\mathfrak{g}$ are of the form

$$C^{(k)} = \sum_{a_1,\dots,a_k=1}^{|\mathfrak{g}|} s_{a_1\dots a_k} T^{a_1} \cdots T^{a_k} \ , \tag{2.4.39}$$

for $s_{a_1\dots a_k}$ a symmetric tensor. In particular $s_{a_1\dots a_k}$ is constructed by taking symmetrised traces

$$s_{a_1\dots a_k} = \frac{1}{k!} \sum_{\text{perm}} \text{Tr}\left(T^{a_1} \cdots T^{a_k}\right) \ . \tag{2.4.40}$$

For a field $\phi = \phi^A T^A$, we can then define monomials

$$\text{Tr}\, \phi^k \equiv s_{a_1\dots a_k} \phi^{a_1} \cdots \phi^{a_k} \ . \tag{2.4.41}$$

For the algebra $\mathfrak{sl}(n)$, there are $n-1$ independent Casimir invariants, $\{\text{Tr}\, \phi^k\}_{k=2}^n$, the first being the quadratic Casimir operator which is given by the Killing form and the others being referred to as the higher Casimirs.

The ring of polynomials in these traces parametrises a variety which is simply $\mathbb{C}^{n-1}$. Notice though that, by construction, the symmetric group S_n has a residual action on the variables. As such, the classical Coulomb branch for a theory with gauge group $SU(n_c)$ is given by

$$\mathcal{M}_{\mathcal{C},\text{class}} = \mathbb{C}^{(n_c-1)}/S_{n_c} \tag{2.4.42}$$

which, once generalised to any (semi)simple Lie group G, recovers (2.2.15).

2.5 Three-dimensional $\mathcal{N} = 4$ theories

Going from four to higher dimensions may seem exotic and fun – fancy superstring theory exists in $D = 10$ after all. But why go down to lower than four dimensions? And why to three? There are many reasons but in the context of this thesis the answer is that the moduli space of vacua becomes particularly rich and holds some unexpected surprises. In preparation for chapter 4 we introduce the basic features of three-dimensional theories with eight supercharges and their moduli spaces.

In three dimensions the spinor is a two-component object; to have eight supercharges requires four copies of such a spinor, hence the notation 3d $\mathcal{N} = 4$. A 3d $\mathcal{N} = 4$ theory is best understood as the compactification of a 6d $\mathcal{N} = 1$ theory [17]: six is the maximum number of dimensions for non-gravitational theories with eight supercharges.

Upon dimensional reduction of a 6d spacetime $\mathbb{R}^{1,5}$ to $\mathbb{R}^{1,2}$, the rotation symmetry in the three compactified directions acts as an R-symmetry for the theory in 3d. The double cover of this is denoted $SU(2)_L$. There is of course also an $SU(2)_R$ symmetry rotating the supercharges as in 6d, so that the full R-symmetry is $SU(2)_L \times SU(2)_R$.

The three real scalar fields arising from the dimensional reduction of the 6d vector multiplet transform as a vector of the rotation group acting on the compactified space, hence they are in a triplet of $SU(2)_L$: we denote them ϕ^i, where $i = 1, 2, 3$.

Again, it makes sense to choose an $\mathcal{N} = 2$ sub-superalgebra (analogously to when we choose $\mathcal{N} = 1$ in 4d) to describe the field content. The vector multiplet of 3d $\mathcal{N} = 4$ can be recognised as a sum of a 3d $\mathcal{N} = 2$ vector multiplet V containing one gauge field, one real scalar, and a Dirac spinor plus a chiral multiplet Φ containing a complex scalar and a Dirac spinor. This is summarised in Table 2.3. When such a description in terms of $\mathcal{N} = 2$ multiplets is adopted, the only visible subgroup of $SU(2)_L$ is the Cartan $U(1)_L$ under which the chiral multiplet Φ has charge 2. We will discuss what happens with the hypermultiplets later on.

3d $\mathcal{N}=4$ Vector Multiplet				
$\mathcal{N}=4$	$\mathcal{N}=2$	Field	Label	$SU(2)_L \times SU(2)_R$
V-plet	V-plet V	gauge	a_μ	$\begin{pmatrix}\lambda_\alpha \\ \xi_\alpha\end{pmatrix} \to [1;1]$
		Dirac spinor	λ_α	
		real scalar	σ	$\begin{pmatrix}\text{Re}\varphi \\ \text{Im}\varphi \\ \sigma\end{pmatrix} \to [2;0]$
	χ-plet Φ	complex scalar	φ	
		Dirac spinor	ξ_α	

Table 2.3: The Lagrangian field content of a 3d $\mathcal{N}=4$ vector multiplet.

2.5.1 The classical Coulomb branch

The action for an $\mathcal{N}=4$ vector multiplet in the adjoint of a gauge group G with $\text{rank}(G)=r$ can be obtained by dimensionally reducing the action for a 6d $\mathcal{N}=1$ vector multiplet. The bosonic sector is then schematically given by

$$\text{Tr}\int_{6d} F\wedge *F \xrightarrow{\text{dim red}}\sim \frac{1}{e^2}\,\text{Tr}\int \mathrm{d}^3x F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi^i|^2 + |[\phi^i,\phi^j]|^2 \ , \quad (2.5.1)$$

where we have suppressed the gauge index $A=1,\dots,|G|$ and $i=1,2,3$. The scalar potential is again the square of a commutator and it vanishes for the scalars taking values in the $U(1)^r$ Cartan subalgebra of G: the choice $\phi^i = \text{diag}(\phi_1^{(i)},\dots,\phi_r^{(i)})$ guarantees a supersymmetric vacuum. For a generic choice of the $3r$ VEVs of the scalars, the adjoint Higgs mechanism ensures that the gauge group is fully broken $G \longrightarrow U(1)^r$ so that there are r massless photons left in the low energy effective theory.

In three dimensions, abelian gauge fields $a_\mu^{(j)}$ can be dualised to scalar fields $\gamma^{(j)}$ via

$$f_{\mu\nu}^{(j)} = \partial_\mu a_\nu^{(j)} - \partial_\nu a_\mu^{(j)} = \epsilon_{\mu\nu\rho}\partial^\rho\gamma^{(j)} \ . \quad (2.5.2)$$

The gauge invariance of $a_\mu^{(j)}$ implies that the scalars $\gamma^{(j)}$ are compact, namely

$$\gamma^{(j)} \sim \gamma^{(j)} + 2\pi \ . \quad (2.5.3)$$

Thus a generic point on the Coulomb branch is parametrised by generic VEVs for *all* the scalar fields including the dual photons, the latter taking values in S^1. Altogether thus, these scalars parametrise a classical Coulomb branch which is

$$\mathcal{M}_{\mathcal{C},\text{class}} = (\mathbb{R}^3 \times S^1)^r / W \ , \tag{2.5.4}$$

where we have again taken into account the residual action of the Weyl group, as explained in the 4d case (2.2.15). The quaternionic dimension of this space is $\dim_{\mathbb{Q}} = r$ and the complex dimension twice that, $\dim_{\mathbb{C}} = 2r$.

The coupling constant e has positive mass dimension in 3d and the theory is free in the ultraviolet. In the infrared it is believed to flow to a superconformal fixed point. Here the effective field theory description is in terms of a supersymmetric non-linear sigma model. Supersymmetry imposes that the target space be hyperKähler since the abelian vector multiplet can be dualised to a chiral multiplet. This should be contrasted with the Coulomb branch in 4d which was previously argued to possess only a (special) Kähler structure. The quaternionic dimension of the hyperKähler Coulomb branch is r, as obtained from the counting of the scalar fields.

Another feature of three dimensions with an abelian gauge group is the presence of a current which is topologically conserved. Indeed, taking

$$j^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} f_{\mu\nu} \ , \tag{2.5.5}$$

one immediately notices that j^μ is conserved by virtue of Bianchi identities. The global symmetry associated to the conserved current is denoted $U(1)_J$. Due to the origin of the $U(1)_J$, its associated charge is called topological. Thus a theory with r abelian vector multiplets possesses a global $U(1)_J^r$ symmetry which is not visible in the Lagrangian: UV fields are $U(1)_J$ neutral.

As previously explained, the Coulomb branch receives quantum corrections and a classical description does not suffice to characterise its chiral ring. In chapter 4 we review a recent breakthrough which allows for a description of the quantum corrected chiral ring. This description bypasses the dualisation of the gauge field since this is not a clear procedure for nonabelian fields occurring at points of symmetry enhancement on the moduli space (for example at the origin where all the VEVs are set to zero and the gauge symmetry is unbroken). We postpone a discussion of the quantum corrected Coulomb branch and the associated references to chapter 4.

2.5.2 The Higgs branch and 3d mirror symmetry

Dimensional reduction of a hypermultiplet is less interesting: scalars go to scalars and no extra constraints arise from the action in the reduced coordinates. Furthermore, the Higgs branch is classically exact, therefore it is independent of whether it is formulated in dimensions from 3 to 6. The upshot is that the Higgs branch of 3d $\mathcal{N} = 4$ theories is precisely the Higgs branch of the 4d $\mathcal{N} = 2$ theories that we studied in section 2.2.

The peculiar phenomena of 3d $\mathcal{N} = 4$ theories are a by-product of the global R-symmetry; recall that in the 4d case, the R-symmetry is $U(1) \times SU(2)_R$ and the nonabelian factor acts non-trivially on the Higgs branch. The appearance of an $SU(2)_L$ in 3d, which acts non-trivially on the Coulomb branch, hints at a symmetry between the Coulomb branch and the Higgs branch. In fact such a symmetry, known as **3d mirror symmetry**, was originally conjectured on the basis of matching symmetries and dimensions of the moduli spaces [14].

Mirror symmetry predicts the infrared equivalence of two theories whose Lagrangian description is very different. Since three-dimensional theories are free in the ultraviolet but flow to interacting superconformal fixed points in the the infrared, it is useful to think about such an equivalence from the opposite point of view: a superconformal fixed point with some manifest global symmetry can be reached as the RG flow of two different Lagrangian theories.

Crucially, the Higgs branch of one theory arises as the Coulomb branch of the other theory and viceversa, namely the duality swaps the $SU(2)_L$ and $SU(2)_R$: it is here that the "mirror effect" is visible. The remarkable property of such a duality should be clear at once: the Coulomb branch, which is quantum corrected, can be obtained classically as the Higgs branch of another theory.

Fixing the desired superconformal fixed point symmetry, the theory whose Coulomb branch enjoys such a symmetry, does not manifestly display it in the ultraviolet. There is a so called enhancement of symmetry from the UV to the IR which is quantum mechanically generated by non-Lagrangian operators known as monopole operators. We will review this phenomenon in chapter 4.

In this thesis, mirror symmetry plays a marginal role. In fact the techniques developed to study the quantum corrected Coulomb branch, which are described and generalised in chapter 4, allow for testing the duality, rather than using it as a working assumption.

2.6 Five-dimensional $\mathcal{N} = 1$ theories

Five-dimensional QFTs are at first sight doomed from the start. They are indeed non-renormalisable. From the free Lagrangian for a gauge field,

$$-\frac{1}{4}\int \mathrm{d}^5x F_{\mu\nu}F^{\mu\nu} \ , \tag{2.6.1}$$

the canonical mass dimension of A_μ in 5d is $3/2$, from which it follows that the gauge coupling constant $[g^2]$ has mass dimension -1. Gauge theories in 5d should then be seen as theories with a cutoff Λ, where the energy scale m at which the theory is effective is set by the inverse coupling constant,

$$m \sim \frac{1}{g^2} \ . \tag{2.6.2}$$

Nonetheless, for certain classes of 5d supersymmetric field theories, arguments have been put forward [31] that there exist UV fixed points from which these gauge theories flow. In particular a UV fixed point means that the coupling constant is taken to infinity,

$$\frac{1}{g^2} \to 0 \ . \tag{2.6.3}$$

At infinite coupling the mass scale is thus lost: the theory is in fact superconformal. The kinetic term for the gauge field, $g^{-2}F \wedge *F$, can thus be considered a relevant deformation of the SCFT. The superconformal fixed point enjoys a larger symmetry than the gauge theory obtained by deforming it: such a field-theoretic effect can be understood via string-theoretic arguments. We will follow [31] to provide the reader with the necessary background.

2.6.1 Field theory

The basic features of the 5d supersymmetry algebra stem from the properties of the four-dimensional spinor representation of $SO(4,1)$. The antisymmetrised tensor product of two such spinors contains a singlet and a vector. The former means that the spinor is pseudoreal and the latter that the anticommutator of two identical supercharges, being a symmetric product, cannot yield the momentum operator, a vector representation. The introduction of an extra index is necessary[10], hence the supersymmetry algebra schematically looks like

$$\{Q^B, \overline{Q}_A\} \sim \delta^B_A P_\mu \gamma^\mu + \dots \ , \tag{2.6.4}$$

where the ... signifies various central charges and $A, B = 1, 2$ are the extended symmetry index. There are thus 8 supercharges and since this is the minimal supersymmetry in 5d it is denoted by $\mathcal{N} = 1$. It is related via dimensional reduction to the other quantum field theories with 8 supercharges as the following sequence shows,

$$6d\ \mathcal{N} = (1, 0) \to 5d\ \mathcal{N} = 1 \to 4d\ \mathcal{N} = 2 \to 3d\ \mathcal{N} = 4 \tag{2.6.5}$$

The extended supersymmetry implies the presence of an R-symmetry $SU(2)_R$ under which the supercharges Q_A transform as a doublet.

As usual for theories with 8 supercharges the massless representations of such an algebra are the vector multiplet and the hypermultiplet. In five dimensions, the field content of such multiplets is

$$\begin{aligned} &\text{V-plet } \Phi : \text{vector } A_\mu \ + \ \text{real scalar } \phi + \text{spinor} \\ &\text{H-plet} : \ 4 \text{ real scalars} + \text{spinor} \end{aligned}$$

Coulomb branch

The Coulomb branch of theories with eight supercharges has been discussed in subsections 2.2.1 and 2.4.4, though in the context of four-dimensional theories. Everything applies analogously here, except for the fact that the scalar in the vector multiplet is real so that the moduli take values over the real rather than the complex numbers.

The VEV of the scalar ϕ in the Cartan subalgebra of a gauge group G gives rise to a moduli space which is isomorphic to

$$\mathcal{C} \cong \frac{\mathbb{R}^r}{W} \ , \tag{2.6.6}$$

where r is the rank of the gauge group G and W is the Weyl group of G as previously mentioned. At a generic point on the Coulomb branch the gauge

[10] Alternatively one should recall that in 4+1d, whilst the usual Majorana condition cannot be self-consistently imposed on one Dirac spinor, a symplectic Majorana spinor can nonetheless be constructed: one introduces a pair of Dirac spinors and imposes a self-consistent reality condition that mixes them by means of an antisymmetric matrix. Doubling the spinors and then “halving" them through the reality condition implies the same counting of eight real degrees of freedom.

symmetry is $U(1)^r$. For $G = U(1)$, no Weyl group is present hence the moduli space is simply $\mathcal{C}_{U(1)} = \mathbb{R}$, whilst for $G = SU(2)$ the moduli are arranged as

$$\begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix} , \tag{2.6.7}$$

with an explicit $\mathbb{Z}_2$ action such that the moduli space is $\mathcal{C}_{SU(2)} = \mathbb{R}/\mathbb{Z}_2$.

Prepotential and BPS spectrum

As in (2.1.3) the action for the vector multiplets can be written in terms of superfields $\mathbf{\Phi}^i$ by means of a holomorphic function $\mathcal{F}\left(\mathbf{\Phi}^i\right)$. In five dimensions this function is constrained to be at most cubic in the superfields [31, 34]

$$\mathcal{F}\left(\mathbf{\Phi}\right) = \frac{1}{2g} \operatorname{Tr} \mathbf{\Phi}^2 + \frac{c}{6} \operatorname{Tr} \mathbf{\Phi}^3 , \tag{2.6.8}$$

where $\frac{1}{2g}$ is the classical gauge coupling and c the classical Chern Simons coefficient. The constant and linear terms are not present since they do not enter the Lagrangian.

For an effective theory of $U(1)^r$ abelian vector multiplets and N_f matter multiplets in representations $\mathbf{r}_f$ with masses m_f, the quantum effects produce extra cubic contributions at one loop and result in an exact prepotential which, in terms of the scalar fields, looks like

$$\mathcal{F} = \frac{1}{2g} h_{ij}\phi^i\phi^j + \frac{c}{6} d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\left(\sum_R |R\cdot\phi|^3 - \sum_f \sum_{W_f} |W_f\cdot\phi + m_f|^3\right) , \tag{2.6.9}$$

where h_{ij} and d_{ijk} are the second and third Casimir operators of $\mathfrak{g}$, R denotes a root of $\mathfrak{g}$, and W_f a weight in the representation r_f. Notice that the terms in brackets have arisen as quantum corrections, so in general, even if the theory has a vanishing Chern Simons term, it will nonetheless be generated at one loop.

For example, for an effective theory with one vector multiplet – which can be obtained from a classical theory with $U(1)$ gauge symmetry or from a Higgsed $SU(2)$ theory – the effective prepotential is

$$\mathcal{F}\left(\phi\right) = \frac{1}{2g}\phi^2 + \frac{c}{6}\phi^3 , \tag{2.6.10}$$

for g and c real. The c term is zero in the classical theory as the groups have no third Casimirs but a cubic contribution is generated as quantum corrections are taken into account. Notice however that for a $U(1)$ theory with no flavours the classical prepotential is exact whilst an $SU(2)$ theory, which has a root system, this is not the case.

The prepotential encodes the key properties of the theory. By taking derivatives, the dual (magnetic) variables, the effective gauge coupling, and the metric on the Coulomb branch can be extracted as

$$\phi_D = \frac{\partial \mathcal{F}}{\partial \phi}(\phi) = \frac{1}{g}\phi + \frac{c}{2}\phi^2 \tag{2.6.11}$$

$$(g_{\text{eff}}^{-2})_{ij} = \partial_i \partial_j \mathcal{F} \tag{2.6.12}$$

$$\mathrm{d}s^2 = (g_{\text{eff}}^{-2})_{ij} \mathrm{d}\phi_i \mathrm{d}\phi_j \tag{2.6.13}$$

The dual scalar is part of the tensor multiplet dual to the vector multiplet. Indeed, in 5d we have that $*F^{(2)} = H^{(3)}$, a 3-form field strength associated to a 2-form gauge field $B^{(2)}$. The objects which are charged under the gauge field and its dual are respectively

$$\mathrm{d} * F^{(2)} = q_e \delta^{(4)} \to \text{electric particle} \tag{2.6.14}$$

$$\mathrm{d}F^{(2)} = q_m \delta^{(3)} \to \text{magnetic monopole string}\ . \tag{2.6.15}$$

These objects belong to the BPS spectrum of the theory. As such they have masses and tensions proportional to the central charges of the $\mathcal{N} = 1$ supersymmetry algebra (for a review see [75]), the electric particle having mass given by the electric central charge and the string having tension given by the magnetic central charge.

In the low energy theory, these central charges are dependent on where in the moduli space the objects are. The electric central charge and hence the mass of the electric BPS particle is simply given by the VEV of the scalar ϕ times the charge q_e, whilst the magnetic central charge is given by the VEV of dual scalar ϕ_D, as given in (2.6.11), times the magnetic charge q_m.

There are other non-perturbative objects that enrich the theory. Their presence stems from a peculiarity of 5d field theories.

Global $U(1)$ symmetry

In five dimensions the one form defined by

$$j^{(1)} = \frac{1}{8\pi^2} \operatorname{Tr} * (F \wedge F) \ , \tag{2.6.16}$$

or in index notation $j^\mu \propto \epsilon^{\mu\nu\rho\sigma\tau} \operatorname{Tr}(F_{\nu\rho}F_{\sigma\tau})$, is topologically conserved by virtue of the Bianchi identity. Hence it is a conserved current. The associated symmetry is denoted $U(1)_I$ and the corresponding charge q_I.

Consider gauging this symmetry by coupling the conserved current to a background vector superfield. The latter has a scalar component which has positive mass dimension; it is nothing but the inverse gauge coupling $\frac{1}{g^2}$, since this is the mass scale of the theory.

There are BPS objects I, charged under the current, which are particle-like and have a mass which is, as usual, proportional to the expectation value of the scalar in the background vector superfield. As we have specified, this is the inverse coupling: $m_I \sim \frac{1}{g^2}$. Hence the BPS spectrum includes the basic objects as summarised in Table 2.4.

electric W-bosons	$m_W \propto \langle\phi\rangle$
magnetic strings	$m_m \propto \langle\phi_D\rangle$
particles I	$m_I \propto \langle\frac{1}{g^2}\rangle$

Table 2.4: Basic objects in the 5d $\mathcal{N} = 1$ BPS spectrum.

To understand what these $U(1)_I$ charged particle-like objects are, it is instructive to look at the conserved charge,

$$\begin{aligned} q_I \equiv \int \mathrm{d}^4x j^0 &\propto \int \mathrm{d}^4x \epsilon^{0\nu\rho\sigma\tau} \operatorname{Tr}(F_{\nu\rho}F_{\sigma\tau}) \\ &= \int \mathrm{d}^4x F_{ij}F^{ij} \ , \end{aligned} \tag{2.6.17}$$

which is precisely the integral that yields the instanton number of 4d Euclidean gauge field configurations. Objects that carry nonzero $U(1)_I$ charge must have nonzero instanton number: they are thus instanton-like solitons. Since instantons are co-dimension 4 objects, in 5d this means they are 1-dimensional objects, that is, particles.

Dynamics

Let us focus on effective theories with one abelian vector multiplet. They arise in the low energy limit of a theory with N_f flavours and $U(1)$ or $SU(2)$ gauge symmetry. Such "high energy" classical theories are of course defined with a cutoff Λ since they are not renormalisable. The Coulomb branch for each theory is summarised in Table 2.5, where we put angle brackets around the scalar fields to emphasise that we are talking about the moduli. The prepotential is given by (2.6.10), though for the $U(1)$ theory there is a discrete global symmetry (acting as $x \to -x$ and $\phi \to -\phi$) which requires that we take absolute values of the modulus. A 1-loop computation in each of the two theories gives the expression for c again tabulated in Table 2.5.

$U(1)$ with N_f electrons	$SU(2)$ with N_f quarks
$\mathcal{C} = \{\langle\phi\rangle \in \mathbb{R}\}$	$\mathcal{C} = \{\langle\phi\rangle \in \mathbb{R}/\mathbb{Z}_2 = \mathbb{R}^+\}$
$\mathcal{F}(\phi) = \frac{1}{2g}\|\phi\|^2 + \frac{c}{6}\|\phi\|^3$	$\mathcal{F}(\phi) = \frac{1}{2g}\phi^2 + \frac{c}{6}\phi^3$
$c = -N_f$	$c = 2(8 - N_f)$

Table 2.5: Coulomb branch of 5d effective theories with one vector multiplet.

The effective couplings for the $U(1)$ and $SU(2)$ theories are then

$$\left(\frac{1}{g^2_{\text{eff}}}\right)_{U(1)} = \frac{1}{g^2} - N_f|\phi| \tag{2.6.18}$$

$$\left(\frac{1}{g^2_{\text{eff}}}\right)_{SU(2)} = \frac{1}{g^2} + 2(8 - N_f)\phi \tag{2.6.19}$$

The effective coupling is dramatically affected by the sign of the quantum correction. In the $U(1)$ case, the correction is negative for any number of flavours. Taking g to be a finite fixed value, there are singularities in the moduli space at the two points $\langle\phi\rangle = \mp\frac{1}{N_f g^2}$. Similar singularities emerge also in the $SU(2)$ theory when $N_f > 8$. On the other hand, the $SU(2)$ theory with $N_f < 8$ has no such singularities in the moduli space. Moreover nothing dangerous happens when the bare coupling is taken to infinity. Indeed, as $g \to \infty$, the effective coupling stays finite everywhere (except at the origin): the field theory still makes sense despite the loss of a gauge description.

2.6.2 String Embedding

Engineering these supersymmetric gauge theories in string theory provides a deeper understanding of what is at work here. Consider first Type I string theory. It has 16 supercharges, giving $\mathcal{N} = 1$ supersymmetry in D=10 spacetime dimensions. At low energy it consists of a supergravity theory coupled to a Yang-Mills theory with gauge group $SO(32)$. The RR-field content of the theory allows for the presence of D9, D5 and D1-branes besides of course the NS-NS sector with the string F1 and the NS5-brane. Most commonly Type I string theory is obtained by a projection of Type IIB: the worldsheet orientation is gauged so that in perturbation theory the worldsheet is unoriented. One should think of this as an orientifold projection enacted by an $O9^-$ plane. Such an orientifold plane is charged under the D9-brane potential and has charge -16 in units where the D9-brane has charge +1. In the presence of an $O9^-$ plane, 16 D9-branes must be present to ensure invariance of the action under gauge transformation. The orientation reversal induced by the $O9^-$ plane can be accounted for by placing 16 D9 images. Strings stretching between the branes and their images give rise to light modes which form an $SO(32)$ gauge field.

Now consider compactification of Type I on a S^1. The resulting theory is known as Type I′. A subsequent T-duality on the circle is the last step to engineer the required background, namely Type I′ on the interval $S^1_{1/R}/\mathbb{Z}_2$. Figure 2.2 is a duality diagram relating Type IIA, Type IIB, Type I, Type I′ and Heterotic SO(32).

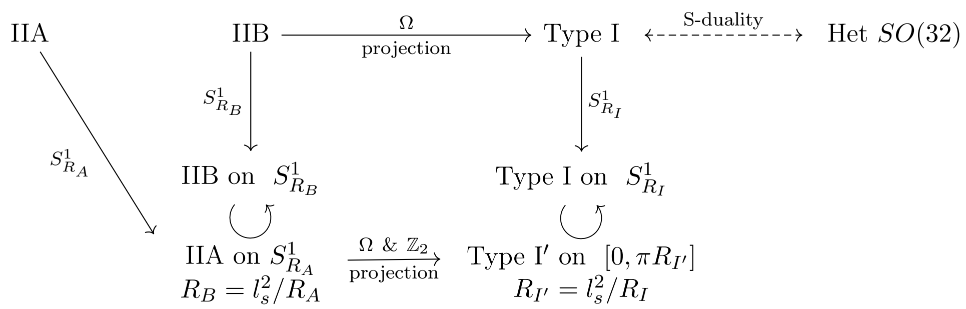

Figure 2.2: A duality diagram showing the Type I/Type I′/Heterotic SO(32). The circular loops denote T-duality. The dashed line represents S-duality.

The effect on the Type I branes D1, D5, D9 wrapping the S^1 is as follows.

A D1-brane becomes a point particle, that is a D0-brane whilst a D5-brane becomes a D4-brane. The D9-branes become D8-branes on the interval $S^1_{1/R}/\mathbb{Z}_2$. The wrapped O9$^-$ plane appears as one O8$^-$ plane at the two ends of the interval $[0, \pi R']$, with $R' = \frac{1}{R}$. The background in 8+1 dimensions is as sketched in Figure 2.3

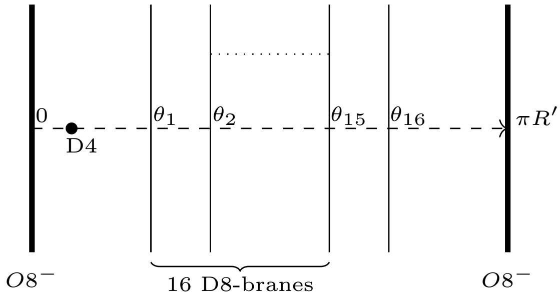

Figure 2.3: Type I$'$ background: the compactified direction is x^9 and the $\mathbb{Z}_2$ projection results in an interval $[0, \pi R']$. The position of the 16 D8-branes along the interval are labelled θ_i. A D4-brane probes the background.

The spacetime moduli of the Type I$'$ background are provided by the size of the $S^1/\mathbb{Z}_2$ and the locations of the D8-branes or equivalently the 16 Wilson lines and the size of the S^1 in the Type I picture.

The D4-brane acts as a probe for this background. Its $U(1)$ gauge field is enhanced in the neighbourhood of the orientifold plane: here the light stringy states form an $SU(2)$ gauge field. The gauge symmetry provided by the heavy D8-branes becomes a flavour symmetry for an observer on the D4-brane. Hence the string embedding engineers an $SU(2)$ gauge theory coupled to $N_f = 16$ hypermultiplets. There are two very special points probed by the D4-brane.

- When N_f of the D8-branes are located at $\theta = 0$, that is they coincide with the orientifold, the theory for the D4-brane is $SU(2)$ with $SO(2N_f)$ flavour symmetry.
- If all the D8-branes are located at some $\theta \neq 0$, away from the orientifold, the theory for the D4-brane is $U(1)$ with $SU(N_f)$ flavour symmetry.

Starting from a configuration where N_f of the D8-branes coincide with the O8$^-$ plane, one physical D8-brane can be pulled away, leaving a $SO(2N_f - 2)$ flavour symmetry. The procedure can be repeated until there are no more branes and a pure $SU(2)$ gauge theory is left on the worldvolume of the D4-brane.

The Yang-Mills coupling constant for a gauge theory on a Dp-brane is given in terms of the brane tension and the string length l_s: $g_{YM}^{-2} \sim l_s^4 T_{Dp}$. The tension for a Dp-brane is given by

$$T_{Dp} \sim (g_s l_s^{p+1})^{-1} \ , \tag{2.6.20}$$

where g_s is the string coupling. Hence for the D4-brane in the Type I′ with string coupling constant $g_{I'}$

$$\left(\frac{1}{g_{YM}}\right)^2 \sim \frac{1}{l_s g_{I'}} \ . \tag{2.6.21}$$

Since the string coupling is determined dynamically via the VEV for the dilaton, $g_{I'} \sim e^{\langle \Phi_D \rangle}$, information about the gauge coupling for the theory on the D4 probe can be gained through a study of the behaviour of the dilaton in the Type I′ background.

Heuristically, the D8-branes fill the $\mathbb{R}^{1,8}$ and they act as a spacetime boundary with a discontinuity in their "electric" field. It results in the dilaton not being constant but having to satisfy

$$\left(\frac{1}{g_{I'}}\right)'' = -\sum q_i \delta(\theta - \theta_i) \ , \tag{2.6.22}$$

where the double dash signifies differentiation with respect to θ. The solution is a piecewise linear function as sketched in Figure 2.4 and explicitly given by

$$\frac{1}{g_{I'}}(\theta) = \frac{1}{g_0} + 4|\theta| - \frac{1}{2}\sum_{i=1}^{16} |\theta - \theta_i| + 4|\theta - \pi R'| - 4\pi R' + \frac{1}{2}\sum_{i=1}^{16} \theta_i \tag{2.6.23}$$

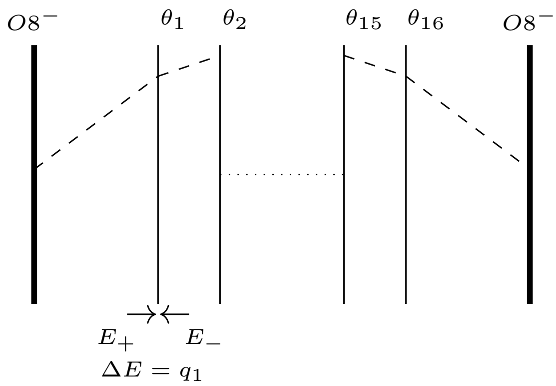

Figure 2.4: Type I′ dilaton profile. The discontinuity in the derivative is due to the presence of the D8-branes which act as domain walls along the θ direction.

Consider now the behaviour of the dilaton near the left orientifold when $n_L < 8$ D8-branes are on top of it whilst $n_R > 8$ D8-branes are away from it, say for simplicity at $\theta = \pi R$. A D4-brane probing the background near the left orientifold will be far away from the n_R D8-branes and will not feel their effect. On the other hand, it will be sensitive to the n_L branes: they will provide an $SO(2n_L)$ flavour symmetry to its $SU(2)$ gauge theory. Let $\phi \equiv \theta$ near $\theta = 0$. The dilaton behaviour is then

$$\frac{1}{g_{I'}}(\phi) = \frac{1}{g_0} + (8 - n_L)\phi \ , \tag{2.6.24}$$

which reproduces (2.6.19) up to a factor of 2. It is now evident that the field theoretic behaviour of the 5d super-Yang-Mills effective coupling has a spacetime interpretation in terms of the dilaton profile in Type I′ string theory.

The string embedding provides an answer as to what type of field theory one obtains when $\frac{1}{g_0}$ diverges. The reasoning, again provided in [31], heavily relies on string theory dualities.

Recall that the Type I′ construction also includes D0-branes, that is spacetime particles. They act as instantons in the gauge theory on the D4-brane [13]. The mass of these particles is, using (2.6.20),

$$m \sim \frac{1}{g_{I'} l_s} \ , \tag{2.6.25}$$

which is equivalent to the inverse coupling constant squared for the Yang-Mills theory on the D4-brane, (2.6.21). The D0-branes are thus precisely the instantonic particles described in the field theory! In the field theory these particles carry non zero integer charge under a $U(1)$ global symmetry. The next question is: what is the spacetime description of such a $U(1)$?

$U(1)$ global symmetry in the string embedding

The Yang-Mills gauge coupling (2.6.21) was given in terms of the Type I′ coupling constant $g_{I'}$. Since Type I and Type I′ are T-duality related[11]

$$\left(\frac{1}{g_{YM}}\right)^2 \sim \frac{1}{l_s g_{I'}} \sim \frac{R_I}{l_s^2 g_I} , \tag{2.6.26}$$

where g_I is the Type I string coupling. Returning to the Type I description is essential to exploit another duality: Type I is in fact S-dual to the Heterotic string theory with gauge group SO(32) [76, 77, 78, 79]. In particular the D1-brane in Type I is S-dual to the F1 string in the Heterotic theory. The D0-brane in Type I′ is T-dual to a D1-brane wrapping $S^1_{R_I}$. Consequently we can regard the D0-brane as an heterotic string wrapping $S^1_{R_I}$. Indeed, looking at the righthand side of (2.6.26) we recognise the mass of a winding state. The winding number of the wrapped heterotic string gives an integer which we identify as the $U(1)$ global charge.

Gauge enhancement

When compactifying the Type I $SO(32)$ theory, Wilson lines on the S^1 can be turned on, giving rise to gauge symmetry breaking. For example when, in the Type I′ language, eight D8-branes are at one orientifold plane and eight are at the other, the original $SO(32)$ gauge group is broken to $SO(16) \times SO(16)$ as we have mentioned. For other choices of the Wilson lines, different patterns of symmetry breaking emerge. In particular when $n_l < 8$ branes are on one orientifold and $16 - n_l$ are on the other, the corresponding Type I/Heterotic $SO(32)$ symmetry breaking is to $SO(2n_l) \times SO(32-2n_l)$ [80]. However there is also a $U(1)\times U(1)$ symmetry associated to the KK momentum and the winding number of the compactified heterotic string. Hence the unbroken symmetry is really $SO(2n_l) \times SO(32-2n_l) \times U(1)^2$. Importantly when the heterotic radius of compactification R_H is at a critical value there is a further enhancement of $SO(2n_l)\times U(1))$ to E_{n_l+1} . This happens because at that radius heterotic string

[11] Recall the classic T-duality between Type IIA and IIB which works as follows. The string length is the same in both theories $l_s = l_A = l_B$ whilst the radii of compactification are related by $R_A = l_s^{-2} R_B$. It leads to the string couplings obeying $g_A^2 R_A^{-1} = g_B^2 R_B^{-1}$ and thus for a D4-brane $g_{YM} = (l_s g_A)^{-1} = R_B (l_s^2 g_A)^{-1}$.

winding states with winding number ± 1[12] become massless and conspire to enlarge the symmetry. Since string winding states are the heterotic description of D0-branes in the Type I′, when such objects become massless we expect, through the chain of dualities, the same enhancement pattern.

The upshot of the string embedding should by now be clear: $SU(2)$ gauge theories with $N_f < 8$ flavours have a UV completion at infinite bare coupling consisting of a SCFT with global symmetry E_{N_f+1} where $E_5 = SO(10)$, $E_4 = SU(5)$, $E_3 = SU(3) \times SU(2)$, $E_2 = SU(2) \times U(1)$ and $E_1 = SU(2)$.[13]

We can regard the 5d fixed point with global symmetry E_8 as the starting theory and trigger various RG flows by either turning on a mass for one of the quarks (pulling away a D9-brane) or by turning on a mass for the D0-brane. In the former case we land on the E_7 theory, whilst in the latter we engineer an effective $SO(14) \times U(1)_I$ gauge theory. Continuing the procedure we reach the various strongly coupled fixed points or the gauge theories with fewer flavours.

[12] For $n_l = 6, 7$ winding states with ± 2 are also massless.

[13] In chapter 5, two extra theories are discussed. They are limiting cases of this sequence.

Chapter 3

Classical Moduli Spaces

This chapter is a warm up to the more advanced techniques introduced in the two subsequent chapters. Here we make use of the tools discussed in the previous chapter to study classical Higgs branches. The theories we study yield some interesting results and highlight the interplay between supersymmetric theories, representation theory and geometry.

3.1 Motivation

In this section, we aim to draw attention on a phenomenon which concerns $Sp(N)$ gauge theories with $2N$ flavours. Amusingly, the Higgs branch of such theories is not a single hyperKähler cone but rather the union of two such cones with a non trivial intersection. Examples of such a phenomenon are known in theories with less supersymmetry, for example in the XYZ model, but are very rare in $\mathcal{N}=2$ theories. It was actually first observed in the context of Seiberg Witten theory with matter [5] for the case of $SU(2)$ with $N_f=2$ flavours and its generalisation mentioned in [81] and briefly discussed in [82].

Here we aim to give an explicit description of the two cones and their intersection. In order to perform such an analysis we rely on the machinery of the Hilbert series and its associated highest weight generating function.

The outline of this chapter as follows. In section 3.2 we recall the description of [5] for the case of $SU(2)$ with 2 flavours and recast their calculations in the language we will use to check for higher rank cases. In section 3.3 we provide the chiral ring partition functions for $\mathcal{N}=2$ theories with classical gauge groups and matter in the bifundamental representations. These expressions are straightforward applications of the usual hyperKähler quotient which gives rise to the Higgs branch. Their form is very suggestive from the point of view of representation theory, in the sense that one can deduce special cases without much effort. In section 3.4 we specialise to the case of $Sp(n)$ with $2n$ flavours and we provide evidence for the statement that the Higgs branch of such theories splits into two cones. In Appendix C we provide the $3d$ mirror dual.

3.2 $SU(2)$ with 2 flavours

In this section we will briefly review the description of the Higgs branch of an $\mathcal{N} = 2$ theory with gauge group $SU(2)$ and 2 hypermultiplets in the fundamental representation. The vector multiplet contains a gauge field, one Dirac fermion and a complex scalar all in the adjoint representation of $SU(2)$. The fields are arranged into an $\mathcal{N} = 1$ vector multiplet and an $\mathcal{N} = 1$ chiral multiplet Φ. Each one of the two hypermultiplets contains two $\mathcal{N} = 1$ chiral superfields Q^i_a and $\widetilde{Q}_{ia}$ where $i = 1, 2$ is the flavour index and $a = 1, 2$ is the gauge index. The flavour symmetry is locally $SO(4) \times SU(2)_R \times U(1)_{\mathcal{R}}$[1].

Let us analyse the chiral ring on the Higgs branch of this theory as follows. We consider the polynomial ring generated by all the fields Q^r_a, with $a = 1, 2$, $r = 1, .., 4$, where we now choose to make explicit the $SO(4)$ symmetry acting on the hypermultiplets when they are massless. The ideal of this ring is generated by taking the F-terms on the Higgs branch, namely by writing the superpotential

$$\mathcal{W} = Q^r_a \epsilon^{ab} \Phi_{bc} \epsilon^{cd} Q^s_d \delta_{rs} \; , \tag{3.2.1}$$

minimising it with respect to the fields and choosing the branch where the quarks expectation value doesn't vanish. The procedure yields three equations:

$$\begin{aligned} I &= \langle \{ F_{ab} \equiv \frac{\partial \mathcal{W}}{\partial \Phi^{ab}} = 0 \} \rangle \\ &= \langle \{ Q^r_a Q^r_b = 0 \} \rangle \end{aligned} \tag{3.2.2}$$

From this ideal, one can evaluate the Hilbert series associated to the quotient $\mathbb{C}[Q^r_a]/I$, using standard mathematical packages[2]. The rational function

[1] The global symmetry is actually $O(4)$, but this subtlety is not important in our discussion.

[2] It is worth to stress that in this instance, Hilbert series techniques are not necessary: the vacuum variety can be analysed simply by studying the basic chiral operators as done in

obtained $\mathcal{F}_b(t, z, x_1, x_2)$, where t and z are fugacities for the $SU(2)_R$ spin and the $SU(2)$ gauge group spin respectively[3] and x_1 , x_2 are the fugacities for the $SO(4)$ flavour symmetry, is then integrated over the gauge group $SU(2)$ to project onto the singlet sector and thus yield only gauge invariant contributions.

$$\mathrm{HS}(t; x_1, x_2) = \int \mathrm{d}\mu_{SU(2)} \mathcal{F}_b(t, z, x_1, x_2) \tag{3.2.3}$$

The resulting rational function we obtain is:

$$\mathrm{HS}(t; x_1, x_2) = \frac{1-t^4}{(1-t^2)(1-x_1^2t^2)(1-x_1^{-2}t^2)} + \frac{1-t^4}{(1-t^2)(1-x_2^2t^2)(1-x_2^{-2}t^2)} - 1 \tag{3.2.4}$$

$$= \mathrm{HS}\left(\mathbb{C}^2/\mathbb{Z}_2; t, x_1\right) + \mathrm{HS}\left(\mathbb{C}^2/\mathbb{Z}_2; t, x_2\right) - 1 \tag{3.2.5}$$

The last equality shows explicitly that the Higgs branch of $SU(2)$ with 2 flavours is the union of two hyperKähler cones $\mathbb{C}^2/\mathbb{Z}_2$, which intersect at the origin. From the Hilbert series the plethystic logarithm, as introduced in the previous chapter, can be evaluated straightforwardly as an expansion in t. The first few terms in such an expansion encode the generators and the relations between them, a set of equations which define the chiral ring on the moduli space. In the plethystic logarithm the first terms with positive sign are generators, whilst the subsequent negative contributions are relations. Evaluating the PL of $\mathrm{HS}(t; x_1, x_2)$, gives the expansion:

$$\mathrm{PL}(t; x_1, x_2) = ([2; 0] + [0; 2])t^2 - ([2; 2] + 2[0; 0])t^4 + ... \tag{3.2.6}$$

where $[m; n]$ are characters of the corresponding representation of $SO(4)$. At t^2 we notice the reducible adjoint representation $([2; 0] + [0; 2])$ which corresponds to the operator $V^{rs} = Q^r_a Q^s_b \epsilon^{ab}$, which is antisymmetric in r, s and has highest weight 2 under $SU(2)_R$. At t^4 there is a reducible relation transforming in the $[2, 2] + [0, 0]$: it is quadratic in the generators since it has highest weight 4 under $SU(2)_R$. Such an operator can be constructed by squaring the matrix V^{rs}; the relation sets it to zero

$$V^{rt} V^{ts} = 0 \tag{3.2.7}$$

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[5]. However we proceed with this technique as it is most suitably generalised to higher rank cases.

[3] More appropriately, t is a fugacity that keeps track of the highest weight for a $SU(2)_R$ representation whilst z is a fugacity for the weights of the $SU(2)$ gauge group representations.

where the singlet relation corresponds to the trace of the full relation. However there is another singlet relation at t^4; using the epsilon tensor, it is construct as:

$$\epsilon_{rstu} V^{rs} V^{tu} = 0 \tag{3.2.8}$$

The two singlet relations correspond to the vanishing of the two quadratic Casimir operators of $SO(4)$. Using (3.2.7) and (3.2.8), the Higgs branch can be concisely written as a variety:

$\mathcal{H}($ 

$) = \left\{ V \in \mathbb{C}^{4\times 4} \mid V = -V^T, \ V^2 = 0, \ \mathrm{rank}(V) \leq 2 \right\}$ (3.2.9)

Crucially for our discussion, the quadratic generator V^{rs} is in a reducible representation. In particular it can be decomposed into a self-dual and anti-self-dual part. Let's write these as

$$V^L_{\alpha\beta} = \gamma^{rs}_{\alpha\beta} V^{rs} \tag{3.2.10}$$

$$V^R_{\dot\alpha\dot\beta} = \gamma^{rs}_{\dot\alpha\dot\beta} V^{rs} \tag{3.2.11}$$

where we have introduced $SO(4)$ gamma matrices $(\gamma^r)_{\alpha\dot\alpha}$ and their antisymmetric product $\gamma^{rs}_{\alpha\beta} = \gamma^{[r}_{\alpha\dot\alpha}\gamma^{s]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$ and $\gamma^{rs}_{\dot\alpha\dot\beta} = \gamma^{[r}_{\alpha\dot\alpha}\gamma^{s]}_{\beta\dot\beta}\epsilon^{\alpha\beta}$. Since $\gamma^{rs}_{\alpha\beta}$ is $(\alpha\ ,\beta)$ symmetric, $V_{\alpha\beta}$ transforms precisely as the $[2,0]$ and similarly $V_{\dot\alpha\dot\beta}$ as the $[0,2]$.

The relations can now be identified as follows. The $[2,2]$ component of (3.2.7) is quadratic in the V's and mixes the self-dual and antiself-dual parts thus, when rewritten, it implies that:

$$V^L_{\alpha\beta} V^R_{\dot\alpha\dot\beta} = 0 \tag{3.2.12}$$

which means that the varieties generated by the two operators are "orthogonal", namely they intersect only at the origin of the Higgs branch.

The two singlet relations at t^4 can now be interpreted as the vanishing of the trace of these two operators:

$$V^L_{\alpha\beta} V^L_{\rho\gamma} \epsilon^{\beta\rho}\epsilon^{\alpha\gamma} = 0 \tag{3.2.13}$$

$$V^R_{\dot\alpha\dot\beta} V^R_{\dot\rho\dot\gamma} \epsilon^{\dot\beta\dot\rho}\epsilon^{\dot\alpha\dot\gamma} = 0 \tag{3.2.14}$$

which correspond to $V^L_{11}V^L_{22} = (V^L_{12})^2$ and $V^R_{11}V^R_{22} = (V^R_{12})^2$, namely the defining equations for two $\mathbb{C}^2/\mathbb{Z}_2$ as already discussed in [5]. Hence the Higgs branch is realised as a union of two cones meeting at the origin.

In the case of (3.2.4), the rational functions can each be expanded in a series, the characters of the two $SU(2)$ replaced by fugacities keeping track of the highest weight associated to the representations and the new series finally resummed. In so doing we precisely construct the highest weight generating function (HWG) [74] for this theory.

After simple manipulations, the resulting HWG is:

$$\mathrm{HWG}(t;\mu_1,\mu_2) = \mathrm{PE}\left[(\mu_1^2+\mu_2^2)t^2 - \mu_1^2\mu_2^2 t^4\right] \tag{3.2.15}$$

where μ_1, μ_2 are the fugacities for the highest weight of $SU(2)\times SU(2) \cong SO(4)$, so that, e.g., μ_1^2 represents the $[2,0]$, μ_2^2 represents the $[0,2]$ and $\mu_1^2\mu_2^2$ the $[2,2]$.

When proceeding to higher rank cases, it is precisely the form of (3.2.15) that turns out to be the most useful for generalised statements about the Higgs branch of the theories at hand.

3.3 $\mathcal{N}=2$ theories with classical gauge groups and fundamental flavours

Using the standard techniques in computations of the HS we can obtain the highest weight generating function of $U(k)$, $Sp(k)$ and $O(k)$ gauge theories with fundamental flavours. The flavour symmetry is $SU(N)$, $SO(N)$ and $Sp(N)$ respectively.

The quivers, HWG functions and the condition between the rank of the group and the number of flavours are given in Table 3.1.

The restriction on the ranks of the gauge group in Table 3.1 is determined just by considering when the representations "degenerate" as follows.

For the theories with $SU(N)$ flavour group, the addends in the plethystic exponential are the highest weights corresponding to the following pattern of $SU(N)$ representations: $[1,0,...,0,1]$, $[0,1,0,...,0,1,0]$, $[0,0,1,0,...,0,1,0,0]$, etc. The sequence terminates when the numbers of representations equals the rank of the gauge group k. In order for such a sequence to exist it is necessary that the number of flavours be at least twice the rank of the gauge group. This is precisely the rank condition appearing in the third column of the first row.

For the theories with $Sp(N)$ flavour group, the summation in the plethystic exponential starts with the highest weight corresponding to the adjoint repre-

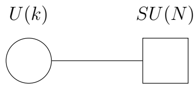
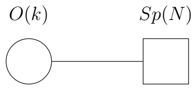
Quiver	HWG $(t;\ \mu_1, ..., \mu_N)$	Rank condition	Variety
	$\mathrm{PE}\Big[\sum_{i=1}^{k} \mu_i \mu_{N-i} t^{2i}\Big]$	$N \geq 2k$	$\{M_{N\times N} \mid \mathrm{Tr}\, M = 0,$ $M^2 = 0,\ \mathrm{rk}(M) \leq k\}$
	$\mathrm{PE}\Big[\sum_{i=1}^{k} \mu_i^2 t^{2i}\Big]$	$N \geq k$	$\{M_{2N\times 2N} \mid M = M^T,$ $M^2 = 0,\ \mathrm{rk}(M) \leq k\}$
	$\mathrm{PE}\Big[\sum_{i=1}^{k} \mu_{2i} t^{2i}\Big]$	$N \geq 4k+3$	$\{M_{N\times N} \mid M = -M^T,$ $M^2 = 0,\ \mathrm{rk}(M) \leq 2k\}$

Table 3.1: HWG for rank k classical gauge groups with fundamental flavours. A fugacity μ_i labels the i^{th} fundamental weight of the flavour group, whilst t is a fugacity that tracks the $SU(2)_R$ highest weight.

sentation $[2,0,..,0]$. Subsequent representations are obtained by pushing the 2 onto the next Dynkin label, k times. The rank condition is straightforward: the pattern is exhausted with the last Dynkin label of $Sp(N)$.

For $SO(N)$ flavour group, again the addends follow a pattern that starts with the highest weight for the adjoint representation $[0,1,0,...,0]$, which is also the 2nd-rank antisymmetric representation. Subsequent terms in the plethystic are even-rank antisymmetric representations. The condition here is more subtle than in previous cases. One needs to take into account that the last, or last two, Dynkin labels (depending on whether N is odd or even) are spinorial labels. For $N = 2n+1$ the nth Dynkin label is spinorial, thus $2k \leq n-1$; for $N = 2n$ the nth and $(n-1)$th labels are spinorial, hence $2k \leq n-2$. Combining these two inequalities for general N, the rank condition in the last row of Table 3.1 is obtained. For example, for $Sp(2)$ with $N = 10$, the condition is not satisfied because the 4^{th} rank antisymmetric representation of $SO(10)$ is the [0,0,0,1,1]. The corresponding highest weight generating function gets modified to $\mathrm{HWG}_{Sp(2),SO(10)}(t;\ \mu_1, ..., \mu_5) = \mathrm{PE}\big[\mu_2 t^2 + \mu_4\mu_5 t^4\big]$.

3.3.1 Low rank exceptions

For theories with $SU(N)$ and $Sp(N)$ flavour group the rank condition in Table 3.1 is exhaustive: representation theory for such groups does not allow for exceptions. On the contrary, for the case of theories with $SO(N)$ flavour symmetry, the rank condition does not exhaust all the cases. There are three exceptions that, whilst violating the rank condition as stated in Table 3.1, possess nonetheless a simple expression for the associated Hilbert series.

Rank Condition	HWG $(t;\ \mu_1, ..., \mu_N)$
$N \geq 4k+3$	$\mathrm{PE}\left[\sum_{i=1}^{k} \mu_{2i} t^{2i}\right]$
$N = 4k+2$	$\mathrm{PE}\left[\sum_{i=1}^{k-1} \mu_{2i} t^{2i} + \mu_{2k}\mu_{2k+1} t^{2k}\right]$
$N = 4k+1$	$\mathrm{PE}\left[\sum_{i=1}^{k-1} \mu_{2i} t^{2i} + \mu_{2k}^2 t^{2k}\right]$
$N = 4k$	$\mathrm{PE}\left[\sum_{i=1}^{k-1} \mu_{2i} t^{2i} + (\mu_{2k-1}^2 + \mu_{2k}^2) t^{2k} - \mu_{2k-1}^2 \mu_{2k}^2 t^{4k}\right]$

Table 3.2: Exhaustive list of rank condition for theories with orthogonal group as flavour symmetry and associated highest weight generating function. The HWG appearing in the fourth row is discussed extensively in section 3.4.

3.4 A special family

3.4.1 Preamble

Here we look in more detail into the case of $Sp(k)$ theories with $2n$ flavours, i.e the one associated to the quiver

By setting $N = 4n$ in the first column of Table 3.2 one can notice that, for fixed k, the theory can fall in two classes only: $n \geq k+1$, which has the HWG as given in the first line of the table, or $n = k$, which has the HWG as in the last line of the table.

In both cases, the Higgs branch variety can explicitly be written as the space generated by a $4n \times 4n$ antisymmetric matrix M^{ab}, with $a, b = 1, ..., 4n$, with spin-1 under $SU(2)_R$, subject to:

$$M^{a_1 a_2} M^{a_2 a_3} = 0 \tag{3.4.1}$$

$$\epsilon_{a_1 \cdots a_{4n}} M^{a_1 a_2} \cdots M^{a_{2k-1} a_{2k}} M^{a_{2k+1} a_{2k+2}} = 0 \ , \tag{3.4.2}$$

the first equation expressing a nilpotency of degree 2 for the matrix M whilst the second equation simply restricting the rank of the matrix: $\text{rank}(M) \leq 2k$. (3.4.1) and (3.4.2) are direct consequences of the F-terms.

For the case $n \geq k+1$, the space has dimension $k(4n-2k-1)$ and is a single hyperKähler cone. This ceases to be the case when one flavour is removed: for theories where $n = k$, an interesting phenomenon occurs which we discuss below.

3.4.2 $Sp(n)$ with $2n$ flavours

This subfamily of theories is very special. Ignoring the violation of the bound and following the prescription that the terms in the HWG summation for orthogonal flavour group - last row in Table 3.1 - are the highest weights for even-rank antisymmetric representations, we expect the $(2n)$th rank antisymmetric of $SO(4n)$ to appear. This one, however, is a reducible representation:

$$\wedge^{2n}[1,0,...,0,0]_{SO(4n)} = [0,...,0,2,0] + [0,...,0,0,2] \tag{3.4.3}$$

Remarkably, it is this splitting of the $(2n)$th rank antisymmetric representation that lies at the heart of the geometric splitting of the Higgs branch into two hyperKähler cones, as anticipated in the introduction.

Thus, at the very least, the last summand appearing in the HWG should be modified and account for this splitting. In fact, after a hyperKähler quotient calculation we obtain that:

$$\text{HWG}_{Sp(n),SO(4n)} = \text{PE}\left[\sum_{i=1}^{n-1} \mu_{2i} t^{2i} + (\mu_{2n-1}^2 + \mu_{2n}^2) t^{2n} - \mu_{2n-1}^2 \mu_{2n}^2 t^{4n}\right] \tag{3.4.4}$$

The term inside the round brackets corresponds indeed to the reducible $(2n)$th antisymmetric representation of $SO(4n)$ but there is also an extra negative contribution.

The unrefined Hilbert series that can be extracted from the HWG generating function in (3.4.4) has the general form:

$$\mathrm{HS}_{Sp(n),SO(4n)}(t)=\frac{N_{2n(2n-1)+2}(t)}{(1-t^2)^{2n(2n-1)}} \tag{3.4.5}$$

where $N_{2n(2n-1)+2}(t)$ is a polynomial in t of degree $2n(2n-1)+2$ whose coefficients are *not* all positive integers. We will return to the form of this HS shortly and comment on this observation.

The algebraic variety associated to this theory is given by (3.4.1) and (3.4.2), with $k=n$, i.e. the matrix of generators, M, is degree 2 nilpotent and has rank less than or equal to $2n$.

After manipulation, (3.4.4) can be written as a sum of plethystic exponentials:

$$\begin{aligned}\mathrm{HWG}_{Sp(n),SO(4n)} =& \mathrm{PE}\left[\sum_{i=1}^{n-1}\mu_{2i}t^{2i}+\mu_{2n-1}^2t^{2n}\right]+\mathrm{PE}\left[\sum_{i=1}^{n-1}\mu_{2i}t^{2i}+\mu_{2n}^2t^{2n}\right]\\ &-\mathrm{PE}\left[\sum_{i=1}^{n-1}\mu_{2i}t^{2i}\right]\end{aligned} \tag{3.4.6}$$

Such a simplified form is of crucial importance: it allows to identify the Higgs branch of these theories as a union of two hyperKähler cones (the two positive terms) with a non trivial intersection (the negative term). This is a remarkable and rare phenomenon on which we aim to draw attention.

The intersection variety is straightforwardly recognisable as the Higgs branch of $Sp(n-1)$ with $SO(4n)$ flavour symmetry as can be evinced by comparing the negative term of (3.4.6) and the last row of Table 3.1. The variety is defined by the equations in (3.4.1) and (3.4.2), where $k=n-1$.

The structure of the two intersecting cones is also straightforward to extract. Indeed, when $n=k$, (3.4.2) sets the $(2n+2)$th-rank of M to zero. The $4n\times 4n$ antisymmetric matrix has thus rank at most $2n$ and in particular the tensor $\epsilon_{a_1\cdots a_{4n}}M^{a_1a_2}\cdots M^{a_{2n-1}a_{2n}}$, which transforms in the $(2n)$th-rank representation, is non-vanishing. Being a reducible representation, its two components can be written:

$$M^{a_1a_2}\cdots M^{a_{2n-1}a_{2n}}(\gamma^{a_1\cdots a_{2n}})_{\alpha\beta} \tag{3.4.7}$$

$$M^{a_1a_2}\cdots M^{a_{2n-1}a_{2n}}(\gamma^{a_1\cdots a_{2n}})_{\dot\alpha\dot\beta} \tag{3.4.8}$$

where we have defined the multi-index gamma matrices as

$$\gamma^{a_1\cdots a_{2n}}{}_{\alpha_1\alpha_{2n}} \equiv \gamma^{[a_1}{}_{\alpha_1\dot{\alpha}_1}\cdots\gamma^{a_{2n}]}{}_{\alpha_{2n}\dot{\alpha}_{2n}}\eta^{\dot{\alpha}_1\dot{\alpha}_2}\cdots\eta^{\dot{\alpha}_{2n-1}\dot{\alpha}_{2n}}\eta^{\alpha_2\alpha_3}\cdots\eta^{\alpha_{2n-2}\alpha_{2n-1}}, \tag{3.4.9}$$

with $\eta^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}$ and $\eta^{\alpha\beta} = \epsilon^{\alpha\beta}$ if $n = 1 \bmod 2$ whilst $\eta^{\dot{\alpha}\dot{\beta}} = \delta^{\dot{\alpha}\dot{\beta}}$ and $\eta^{\alpha\beta} = \delta^{\alpha\beta}$ if $n = 0 \bmod 2$, due to the fact that in the former case the spinor representation is symplectic and in the latter it is orthogonal. The matrix $\gamma^{a_1\cdots a_{2n}}{}_{\dot{\alpha}\dot{\beta}}$ is defined analogously to the undotted case.

The two cones can be constructed by setting one of these two components to zero, whilst keeping the other non-vanishing and vice versa. Then the first cone is generated by the same $4n\times 4n$ matrix M^{ab} as before, subject to:

$$M^{a_1a_2}M^{a_2a_3} = 0 \tag{3.4.10}$$

$$M^{a_1a_2}\cdots M^{a_{2n-1}a_{2n}}(\gamma^{a_1\cdots a_{2n}})_{\dot{\alpha}\dot{\beta}} = 0\ , \tag{3.4.11}$$

whilst the second cone is again generated by M^{ab} and the variety is defined by:

$$M^{a_1a_2}M^{a_2a_3} = 0 \tag{3.4.12}$$

$$M^{a_1a_2}\cdots M^{a_{2n-1}a_{2n}}(\gamma^{a_1\cdots a_{2n}})_{\alpha\beta} = 0\ . \tag{3.4.13}$$

3.4.3 Discussion

At first sight the most puzzling element of the discussion so far is the fact that the Hilbert series in (3.4.5) has a numerator with negative coefficients. In particular this means that in this instance the ring of holomorphic functions defined by the F-terms ideal is not Cohen-Macaulay. Indeed the following theorem [83] holds.

Theorem 3.1 (Macaulay)**.** *The Hilbert series of a Cohen-Macaulay graded ring R, where all generators have degree 1, has the form*

$$HS(R,t) = \frac{P(R,t)}{(1-t)^d} \tag{3.4.14}$$

where $P(R,t)$ is a polynomial in t with $P(R,1)\neq 0$ and such that $P(R,t)$ has positive integer coefficients.

Why then is the Higgs branch of $Sp(n)$ with $2n$ flavours not a Cohen-Macaulay ring?

To clarify the situation, it is helpful to look at the (identical) contribution to the unrefined Hilbert series from each cone. It is a rational function of t in the form:

$$HS_{1-\text{cone}}(t) = \frac{N_{2n(2n-1)}}{(1-t^2)^{2n(2n-1)}} \tag{3.4.15}$$

with the numerator having positive coefficients. This subvariety is thus Cohen-Macaulay[4]. This implies that each cone is a normal variety, by Serre's criterion [84]. However singular (HK) cones whose generators have all degree one are classified by the (closure of) nilpotent orbits of a semisimple Lie algebra [85]. With this statement at hand and comparing with theorems in [86] it is easy to recognise that the Higgs branch of $Sp(n)$ with $2n$ flavours is in fact isomorphic to the nilpotent cone associated to the very even partition $\rho = \{2^{2n}\}$ of $SO(4n)$. The non-normality of the variety is thus expected, as the theory just falls in the class of the very even nilpotent orbits of special orthogonal groups.

[4]Moreover the singular locus has *codim* ≥ 2

Chapter 4

Coulomb Branch and the Moduli Space of Instantons

4.1 Introduction

Instantons were first introduced as Euclidean finite action solutions of the self-dual pure Yang-Mills equations [87, 88]. The space of such solutions, graded by an integer number k, the Pontryagin number (or charge) of the instanton, is known as the moduli space of instantons. An algebraic prescription to construct instanton solutions for classical gauge groups $SU(N)$, $SO(N)$, $USp(2N)$ on $\mathbb{R}^4$ was developed by Atiyah, Drinfeld, Hitchin and Manin in [12]. With the advent of D-branes as dynamical objects, the ADHM construction was given geometric light by means of a brane realization [11, 13]: Dp-branes inside $D(p+4)$-branes are codimension 4 objects, which dissolve into instantons for the worldvolume gauge fields of the $D(p+4)$-branes. For the gauge theory living on the Dp-brane, which has 8 supercharges, the Higgs branch of the moduli space therefore corresponds to the moduli space of instantons of the $D(p+4)$ gauge group.

In order to compute moduli spaces of instantons for classical gauge groups, one avenue is thus analyzing the Higgs branch of the ADHM quiver gauge theory. This is done by considering the constraints given by the F and D terms in the supersymmetric gauge theory and modding out by the gauge group. The Higgs branch for theories with 8 supercharges is classically exact [9] and therefore identical when formulated in dimensions between 3 and 6. Another avenue for computing moduli spaces of instantons, where no such simplification is available, is through the *Coulomb branch* of certain 3d gauge theories with eight supercharges and gauge group $\mathcal{G}$ whose details we specify below. These two routes, via the Higgs branch and the Coulomb branch, are independent of each other, though calculating exact quantities on both sides can furnish a test of mirror symmetry and relate one to the other [14]. In this section we will exclusively study theories whose Coulomb branch is the moduli space of instantons, without resorting to mirror symmetry.

The stringy realization of moduli spaces of instantons through brane constructions has led to new insights. Indeed the ADHM construction exists only for classical gauge groups and, until recently, the instanton partition functions for exceptional gauge groups were only possible by means of superconformal indices [21, 22, 29] of theories obtained by wrapping $M5$-branes on punctured Riemann surfaces [89] as in [90] for $E_{6,7,8}$ instantons or by extrapolating the blow-up equations of [91, 92] as in [93]. Here we focus our attention on supersymmetric gauge theories with 8 supercharges whose moduli spaces include moduli spaces of instantons and study their associated Hilbert series [69, 70, 71].

As it has been explained in chapter 2, the HS provides useful exact information about the moduli space: we can extract the group theoretic properties of the generators of the moduli space and of the relations between them. Salient features of the theories, such as the enhancement of global symmetries, are also neatly exposed by this treatment. For moduli spaces of k pure Yang-Mills instantons, the Hilbert series is also the five-dimensional (or K-theoretic) k instanton partition function of [94], as discussed also in [70, 91, 92, 95].

In [71] the Hilbert series for instantons of charge $k = 2$ were approached from the Higgs branch point of view, the calculations being a generalization of [69] with an increased level of difficulty. Here we attack the problem from the Coulomb branch perspective in the wake of the new developments of [30], where a simple formula for the Hilbert series of the Coulomb branch of $d = 3$ $\mathcal{N} = 4$ *good* or *ugly* [81] superconformal field theories was introduced. The methods introduced in [30] have already given fruitful results [96, 97]. Here we continue to exploit the techniques to analyze the moduli spaces of higher k G-instantons, where G is any simple Lie group. Our results include instantons for gauge groups whose Dynkin diagrams are non-simply laced, which have escaped the Coulomb construction so far.

The Coulomb branch of three-dimensional theories with 8 supercharges receives quantum corrections and it is precisely this which begets the non-trivial structure of the space. As we will review in section 4.3, the chiral operators which parametrize the Coulomb branch are gauge invariant combinations of supersymmetric 't Hooft monopole operators V_m [98] labeled by a magnetic charge m, which break the gauge group $\mathcal{G}$ to a subgroup H_m by the adjoint Higgs mechanism, and of the classical complex scalar fields ϕ_m in the adjoint representation of the residual gauge group H_m. The HS of the Coulomb branch counts gauge invariant either bare (*i.e.* built out of V_m only) or dressed (*i.e.* built out of V_m and ϕ_m) supersymmetric monopole operators according to their quantum numbers, namely the topological charges J and the R-charge under the $U(1)_C$ Cartan subgroup of the $SU(2)_C$ R-symmetry which acts on the Coulomb branch.

Since we want to study moduli spaces of instantons we must make precise which theories, whose Coulomb branch we will investigate, are of interest to us. We extend the correspondence between the Coulomb branch of ADE quivers [99, 100] and the moduli space of ADE instantons, first pointed out for one instanton in [14] and then generalized to higher instanton number in [15, 101]. We claim that the moduli spaces of instantons for any simple gauge group can be obtained as the Coulomb branch of quivers constructed using the over-extension of the Dynkin diagrams for the associated finite Lie algebras. Whilst this has already been expounded using Hilbert series in [30, 97] for ADE quivers, here we complete the treatment by generalizing the previous formula to non-simply laced quivers. The crucial formula that prescribes how to deal with multiple laces is (4.3.3).

Some remarks are in order on the relation of Coulomb branch Hilbert series to superconformal indices. It was recently realized [102] that the Coulomb branch Hilbert series of $d = 3$ $N = 4$ “good" or “ugly" theories with a Lagrangian description [30] is also captured by a limit of the superconformal index of the theory. (The derivation of this limit in [102] is for a $U(N)$ gauge group, but it can be easily generalized to any gauge group and matter content.)

Some of the theories of our interest – those associated to the ADE quivers – have a Lagrangian description, and the standard formula [23, 28, 103, 104] for the superconformal index can be written down. Involving an infinite sum over magnetic charges of integrals over the gauge group, this formula is not of simple evaluation and to our knowledge it has not been computed for the over-extended ADE quivers. Our focus here is on the moduli spaces of instantons, which arise as Coulomb branches of such quiver gauge theories. So we are interested in computing the Hilbert series of the Coulomb branch rather than the whole superconformal index, which also receives contributions from chiral operators in the hypermultiplet moduli space as well as non-chiral operators. In contrast to the standard formula for the superconformal index, the monopole formula of [30] for the Coulomb branch Hilbert series does not involve integrals and can be more easily evaluated.

On the other hand, the theories whose Coulomb branches are moduli spaces of instantons of non-simply laced gauge groups have no known Lagrangian description and the standard formula for the superconformal index is not even available. Our proposal for dealing with multiple laces in the monopole formula is a natural generalization of the formula obtained in [30] for Lagrangian theories. It allows us to study in a uniform way the moduli spaces of instantons of all simple Lie groups. It is at present unclear how to extend this to the full superconformal index.

The plan for the rest of this chapter is as follows. Section 4.2 is a brief summary of a particular type of brane construction that realizes instanton moduli spaces in string theory both from the Higgs branch and the Coulomb branch point of view. From the brane picture we are able to motivate the quiver theories that we use to compute the Hilbert series of moduli spaces of instantons. In section 4.3 we review the monopole formula for the Hilbert series of Coulomb branches and we show how to modify the expression to account for generalized quivers built from non-simply laced Dynkin diagrams. In section 4.4 we provide a step-by-step calculation for the moduli space of k G_2 instantons and give the explicit result for the Hilbert series associated to the moduli space of 3 G_2 instantons. In sections 4.5, 4.6, 4.7 we display formulae for the Hilbert series of $SO(2N+1)$, $USp(2N)$ and F_4 instantons. In section 4.8 we sketch some of the group theoretic features of the moduli space of instantons as an algebraic variety, providing the transformation laws of the generators and the first relations.

4.2 Brane realization of instantons

In this section we summarize various brane constructions for moduli spaces of instantons of classical gauge groups [13, 11, 105, 106, 107, 108]. String dualities which realize mirror symmetry relate the Higgs branch and the Coulomb branch brane picture. However we stress that the Coulomb branch construction that will be used later on does not require mirror symmetry.

An instanton is a solitonic object of codimension 4. Dp-branes inside $D(p+4)$-branes, with or without $O(p+4)$-planes, provide a realization of instantons for classical gauge groups. To realize the kind of three-dimensional theory that we are interested in, we consider $D2$-branes in the background of $D6$-branes. The $D6$-branes provide the gauge group whilst k $D2$-branes, when lying on top of the $D6$-branes, give rise to instanton configurations of charge k on $\mathbb{C}^2$. The classical gauge group on the worldvolume of the $D6$-branes depends on which type of orientifold $O6$-plane is added to the construction. In particular N parallel $D6$-branes provide a $U(N)$ low energy effective theory, as sketched in Table 4.1. With the addition of k $D2$-branes, the system living on the latter becomes that of a quiver theory with gauge group $U(k)$ and $SU(N)$ flavor symmetry, since the $U(1)$ factor inside $U(N)$ is gauged.

In order to realize $SO(2N+1)$ instantons we construct a background with N parallel $D6$-branes on top of an orientifold plane $\widetilde{O}6^-$. The orientifold allows for strings to end on it, thus reproducing the B_N root system. The quiver for such a construction is given by a gauge group $USp(2k)$ with matter in the antisymmetric representation and $2N+1$ fundamental half-hypermultiplets with flavor symmetry $SO(2N+1)$.[1]

For $USp(2N)$-instantons, the brane construction involves N $D6$-branes on top of an $O6^+$ or $\widetilde{O}6^+$ plane. k half $D2$-branes in such a background give rise to a quiver gauge theory with $O(k)$ gauge group, matter in the symmetric representation and $2N$ fundamental half-hypermultiplets with flavor symmetry $USp(2N)$.

Lastly, in presence of k $D2$-branes, N $D6$-branes and an orientifold $O6^-$, the D_N root system is realized, allowing for a quiver with $USp(2k)$ gauge symmetry, matter in the antisymmetric representation and $2N$ fundamental half-hypermultiplets with flavor symmetry $SO(2N)$.

The Higgs branch of these theories is achieved when the $D2$-branes are inside the $D6$-branes; the Coulomb branch is realized when the $D2$-branes are away from the $D6$-branes. Thus it is the Higgs branch of these quiver gauge theories that reproduces the moduli space of G-instantons, where G is the flavor symmetry group of the quiver. We show the brane constructions and the corresponding quivers in Table 4.1.

For exceptional groups we do not have a perturbative open string description on the Higgs branch. However progress can be made appealing to mirror

[1] We have glossed over a subtlety: the $\widetilde{O}6^-$ plane requires the presence of a Romans mass. This $D8$-brane charge translates into a Chern-Simons coupling in the parity anomalous gauge theory on $D2$-branes, which reduces supersymmetry and lifts the Coulomb branch. The moduli space of B_N instantons is the subvariety of the total moduli space of vacua of the supersymmetric Chern-Simons theory with vanishing expectation values for monopole operators.

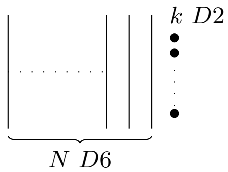
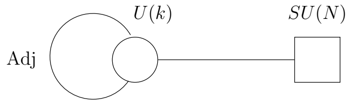
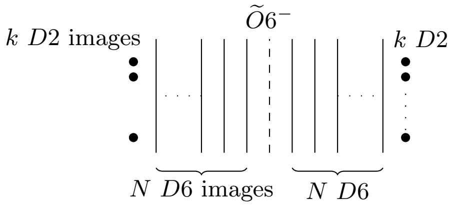
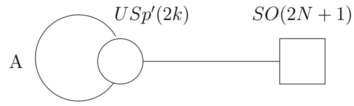
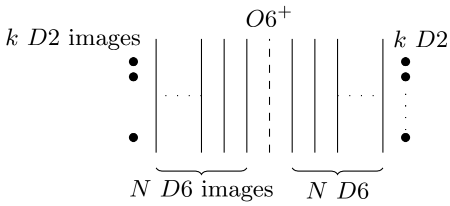
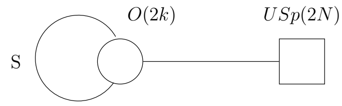
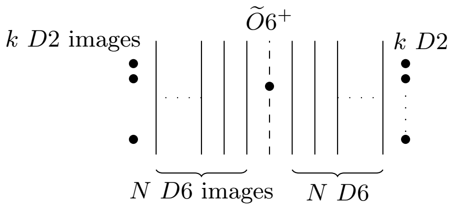
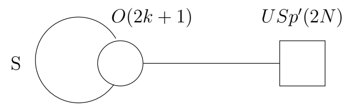
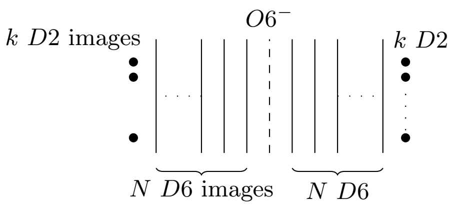
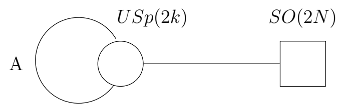
G	Brane configurations from which Higgs branch can be realized	ADHM quiver
A_{N-1}		
B_N		
C_N		
C_N		
D_N		

Table 4.1: Brane constructions and quiver diagrams whose Higgs branch correspond to k G-instantons on $\mathbb{C}^2$. To describe the moduli space of instantons, all D2 branes are dissolved on coincident D6 branes and orientifold planes. In the pictures the $D6$ branes are separated from each other and the orientifold for clarity. Note that there exist constructions of the moduli space of E-instantons in terms of M5-branes on a sphere with punctures. However it is unknown how to realize such moduli spaces as perturbative open string backgrounds.

symmetry and generalizing the lessons learned for classical groups. We can implement this symmetry on the previous constructions by performing T-duality to Type IIB, and then S-duality to realize mirror symmetry. Under T-duality along a direction transverse to the $D2$-branes and parallel to the $D6$-branes, the $D2$-brane becomes a $D3$-brane on S^1 and the $D6$-brane becomes a $D5$-brane.[2] After S-duality, the $D3$-brane is unchanged whilst the $D5$-brane turns into a $NS5$-brane. In the absence of orientifolds, i.e. for the case of $G = A_{N-1}$ in Table 4.3, the application of these dualities results in a necklace quiver gauge theory with N $U(k)$ gauge nodes.

Moreover, and crucially, since mirror symmetry exchanges Higgs branches with Coulomb branches, it is now the Coulomb branch of this new dual theory which corresponds to the moduli space of instantons.

The action of mirror symmetry on the four orientifold planes we considered is illustrated in Table 4.2. Note in particular that T-duality results in a restriction to an interval defined by two separated $O5$ planes and that S-duality turns an $O5$ into an ON orientifold.

Orientifold	T-duality	S-duality
$\widetilde{O6}^-$	$O5^-$ & $\widetilde{O}5^-$	ON^- & $\widetilde{ON}^-$
$O6^+$	$O5^+$ & $O5^+$	ON^+ & ON^+
$O6^-$	$O5^-$ & $O5^-$	ON^- & ON^-
$\widetilde{O}6^+$	$\widetilde{O}5^+$ & $O5^+$	$\widetilde{O}N^+$ & ON^+

Table 4.2: The effect of T- and S-dualities on orientifold planes.

The effect of mirror symmetry, through action on branes and orientifolds, on the brane constructions in Table 4.1 is summarized in Table 4.3. For example consider the brane realization on the Higgs branch of one C_N instanton (i.e with $k = 1$ $D2$-branes). The $O6^+$ background is turned into an interval bounded by ON^+ on the left and an ON^+ on the right. The N parallel $NS5$-branes lie within this interval.

As befits a magnetically charged object, the $D3$-brane is to be viewed as a root of the Langlands dual algebra, here B_N. When stretching onto the ON^+, the $D3$-brane reproduces a short root: it ends on the ON^+. Finally, one

[2] More precisely, we view $\mathbb{C}^2 = \mathbb{R}^4$ as an "A_0" hyperKähler space, namely a circle fibration over $\mathbb{R}^3$ with a fixed point, and perform T-duality along the fiber. The fixed point of the circle action is dualized to an $NS5$-brane. We will return to this point in the following.

balances the number of $D3$-branes stretching between neighboring $NS5$, in this case one. The result is sketched in Figure 4.1.

After engineering the dual brane construction, we can associate to it a quiver. The rank of each node in the quiver is read off from the number of D-branes: since we have one $D3$-brane between each neighboring $NS5$, the gauge groups are all $U(1)$.

To account for the different length of the last root on the left and on the right, we use the double lace notation of Dynkin diagrams. In the next section we will specify how to deal with multiple laces. The quiver we end up with is the *Dynkin diagram of the untwisted affine algebra* $C_N^{(1)}$, with the dual Coxeter labels (or Kac labels/comarks) $a_i^\vee$, $i = 0, \ldots, r = \mathrm{rk}(G)$, providing the rank of the gauge groups. For instanton number k the ranks of the unitary gauge groups are given by $ka_i^\vee$.

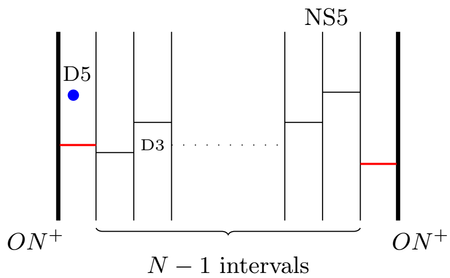


Figure 4.1: Brane construction for the C_N affine Dynkin diagram with the attached $U(1)$ node. Each type of brane is indicated in the diagram. Here there is one $D3$-brane per interval. The red and black segments indicate D3-branes in correspondence with the simple roots of the B-type algebra, which is dual to the C-type algebra associated with ON^+. The blue dot in the leftmost interval indicates the D5-brane corresponding to the over-extended $U(1)$ node.

In a completely analogous fashion to this example, the quivers that we analyze for the moduli space of G-instantons are precisely the Dynkin diagrams for the untwisted affine algebras of G type, with the crucial addition of an extra node, the nature of which we explain below.[3]

[3]We have chosen to use the untwisted affine Dynkin diagrams associated to *electric* objects, rather than the Langlands dual Dynkin diagrams associated to *magnetic objects*, which are obtained by reversing the arrows. The prescription that we will provide for the HS of instanton moduli spaces from the Coulomb branch can be phrased equally well in terms of dual diagrams.

4.2.1 Over-extended node

The quiver gauge theories constructed from the affine Dynkin diagrams are not sufficient to obtain the moduli spaces of instantons. In particular, for $k > 1$ instanton number, the parametrization of the instanton solution on $\mathbb{C}^2$ mixes nontrivially with the parametrization of the instanton in the gauge group G.

For $k = 1$, i.e. a single $D3$ brane stretching on a circle, the fugacity associated with $\mathbb{C}^2$ factorizes:

$$g_{1,G}(t, x, \boldsymbol{u}) = \frac{1}{(1 - tx)(1 - tx^{-1})} \widetilde{g}_{1,G}(t, \boldsymbol{u}) \ . \tag{4.2.1}$$

Here $\boldsymbol{u}$ are the fugacities associated to G,[4] x is the fugacity associated to $SU(2)$ rotations of $\mathbb{C}^2$, and t the fugacity for the highest weight of the $SU(2)$ R-symmetry. After factoring out the center of mass degree of freedom, we are left with the Hilbert series $\widetilde{g}_{1,G}$ of the reduced moduli space of 1 G-instanton, which does not depend on x.

For $k > 1$ one can similarly extract the center of mass mode,

$$g_{k,G}(t, x, \boldsymbol{u}) = \frac{1}{(1 - tx)(1 - tx^{-1})} \widetilde{g}_{k,G}(t, x, \boldsymbol{u}) \ , \tag{4.2.2}$$

but the Hilbert series $\widetilde{g}_{k,G}$ of the reduced moduli space of k G-instantons depends on the $SU(2)$ fugacity x for $k > 1$. In fact, as we will explain in section 4.8.6, for $k > 1$ there are two different global $SU(2)$ symmetries, one acting on the center of mass and the other on the reduced moduli space of instantons.

In order to see the centre of mass of the instantons and the $SU(2)_x$ rotation symmetry of $\mathbb{C}^2$ in the Type IIB brane construction, we need to follow the chain of dualities more carefully (see footnote 2).

The T-duality from Type IIA to Type IIB is done along a circle direction with a fixed point: this results in an extra $NS5$ brane in Type IIB, in addition to the $D5$-branes and $O5$-planes discussed above. The $NS5$-brane ensures that the matter fields in the 2-index tensor representation of the ADHM quiver gauge groups transform as denoted in Table 4.1 rather than the adjoint representation. S-duality maps this $NS5$-brane into a $D5$-brane, which fixes the origin of $\mathbb{C}^2$. The $D5$-brane $U(1)$ symmetry acts as a flavor group for the worldvolume theory on the $D3$-branes: it attaches a square node to the extended node of the affine Dynkin diagram, as in [15, 101, 105].

Even though this $U(1)$ node appears naturally as a flavor node in the brane

[4] We use simple roots fugacities $\boldsymbol{u}$ instead of highest weight fugacities $\boldsymbol{y}$ for convenience.

construction, it is useful to treat it on the same footing as the other gauge nodes, and then ungauge an overall diagonal $U(1)$ gauge symmetry under which no matter fields are charged. The relevant quivers for the moduli space of instantons on $\mathbb{C}^2$ are then the so-called *over-extended* Dynkin diagrams [109], with a rank 1 over-extended node connected to the extended (or affine) node. The gauge fixing of the decoupled $U(1)$ gauge symmetry can be done at any node of the quiver: fixing the $U(1)$ of the over-extended node reduces it to a flavor node as is natural in the brane construction; fixing a $U(1)$ inside a $U(N)$ gauge factor leaves an $SU(N)/\mathbb{Z}_N$ gauge group. In section 4.3 we explain how to implement this gauge fixing and how to identify the global symmetries acting on the instanton moduli space in the Coulomb branch Hilbert series.

G	$\mathcal{L}$	Coulomb branch quivers
E_6	Y	$\circ_1 - \bullet_k - \circ_{2k} - \circ_{3k} - \circ_{2k} - \circ_k$, with $\circ_{3k} - \circ_{2k} - \circ_k$
E_7	Y	$\circ_1 - \bullet_k - \circ_{2k} - \circ_{3k} - \circ_{4k} - \circ_{3k} - \circ_{2k} - \circ_k$, with $\circ_{4k} - \circ_{2k}$
E_8	Y	$\circ_1 - \bullet_k - \circ_{2k} - \circ_{3k} - \circ_{4k} - \circ_{5k} - \circ_{6k} - \circ_{4k} - \circ_{2k}$, with $\circ_{6k} - \circ_{3k}$
F_4	N	$\circ_1 - \bullet_k - \circ_{2k} - \circ_{3k} \Rightarrow \circ_{2k} - \circ_k$
G_2	N	$\circ_1 - \bullet_k - \circ_{2k} \Rrightarrow \circ_k$

Table 4.4: Quiver diagrams from which the Hilbert series of the moduli space of k instantons in exceptional gauge groups can be computed using the monopole formula for the Coulomb branch. For these cases there is no known brane construction analogous to Table 4.3.

4.3 The Hilbert series for the moduli space of k G-instantons

The purpose of this section is to review the essential tools for the computation of the Hilbert series for the quantum corrected Coulomb branch of $3d$ $\mathcal{N} = 4$ quiver gauge theories where the gauge group is a product of $U(N)$ factors. As we have detailed in the previous section, for suitable generalized quivers, possibly including non-simple laces, this method provides the Hilbert series of the moduli space of instantons.

Three-dimensional $\mathcal{N} = 4$ theories are described by vector multiplets in the ad-

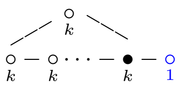
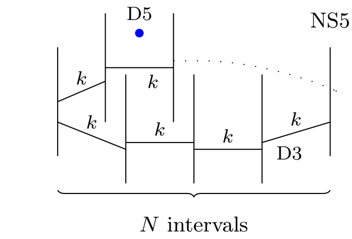
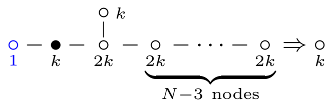
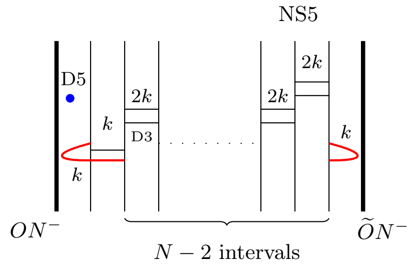
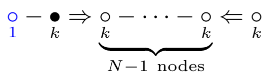
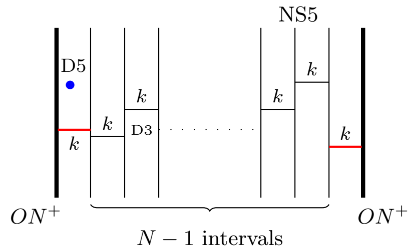
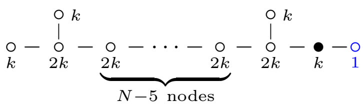
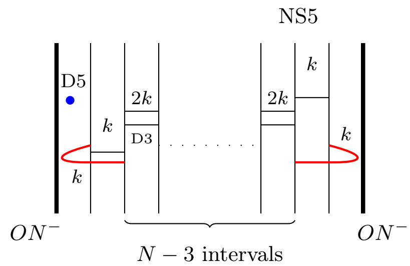
G	$\mathcal{L}$	Coulomb branch quivers	Brane set-up
A_N	Y		
B_N	N		
C_N	N		
D_N	Y		

Table 4.3: Quiver diagrams from which the Hilbert series of the moduli space of k instanton in classical gauge groups can be computed using the monopole formula for the Coulomb branch. The corresponding brane configuration is depicted next to each quiver. Note that the configurations associated with the left boundary condition for B_N and the left and right boundary conditions for D_N involve an ON^- plane and an NS5 brane, whose combination is usually called ON^0 [107]; this type of configuration was pointed out in [110, 106]. The second column indicates whether a Lagrangian is available or not.

joint representation and matter fields (hypermultiplets or half-hypermultiplets) transforming in some representation of the gauge group. At a generic point on the Coulomb branch the scalars in the vector multiplet acquire non-zero VEV, breaking the gauge group $\mathcal{G}$ of rank r to $U(1)^r$, its maximal torus; matter fields and W-bosons acquire mass and are integrated out, while the r massless gauge fields, the photons, can be dualized to scalars. So at low energies on the Coulomb branch, what is left is an effective theory of r abelian vector multiplets which, by virtue of the gauge field dualization to a scalar, can be themselves dualized to twisted hypermultiplets.

The previous description breaks down at subvarieties of the Coulomb branch where the residual gauge group is non-abelian. In particular it fails to describe the origin of the Coulomb branch, which flows to a SCFT in the IR. The dualization of a non-abelian vector multiplet is not understood. Instead, a more fruitful exposition takes advantage of special disorder operators, which can be defined directly at the infrared fixed point [98] and which are not polynomial in the microscopic degrees of freedom: they are called 't Hooft monopole operators and are defined by prescribing a Dirac monopole singularity at an insertion point in the Euclidean path integral [111]. Monopole operators are classified by embedding $U(1) \hookrightarrow \mathcal{G}$, and are labeled by magnetic charges which, by a generalized Dirac quantization [112], take value in the weight lattice $\Gamma_{\mathcal{G}^\vee}$ of the GNO or Langlands dual group $\mathcal{G}^\vee$ [113, 114]. The monopole flux breaks the gauge group $\mathcal{G}$ to a residual gauge group H_m by the adjoint Higgs mechanism. Restricting to gauge invariant monopole operators is achieved by modding out by the Weyl symmetry group, thus restricting $m \in \Gamma_{\mathcal{G}^\vee}/\mathcal{W}_{\mathcal{G}}$.

In a three-dimensional $\mathcal{N} = 2$ theory one can define half-BPS monopole operators which sit in chiral multiplets. Crucially, there exists a *unique* BPS monopole operator V_m for each choice of magnetic charge m [19]. If the theory has $\mathcal{N} = 4$ supersymmetry, the $\mathcal{N} = 4$ vector multiplet decomposes into an $\mathcal{N} = 2$ vector multiplet V and a chiral multiplet Φ in the adjoint representation. To describe the Coulomb branch, V is replaced by monopole operators V_m, which now can be dressed by the classical complex scalar ϕ inside Φ. This dressing preserves the same supersymmetry of a chiral multiplet [20] if and only if ϕ is restricted to ϕ_m, a constant element of the Lie algebra of the residual gauge group H_m [30]. The monopole operators which parametrise the Coulomb branch of an $\mathcal{N} = 4$ field theory are thus polynomials of V_m and ϕ_m, which are made gauge invariant by averaging over the action of the Weyl group [30].

For us the gauge group $\mathcal{G}$ will mostly be be a product of $U(N_i)$ unitary groups, which are self-dual. For $U(N)$ monopole operators $V_{\boldsymbol{m}}$, with magnetic charge $\boldsymbol{m} = \mathrm{diag}(m_1, ..., m_N)$, the weight lattice of the dual group is given by $\Gamma_{U(N)} = \mathbb{Z}^N = \{m_i \in \mathbb{Z}, i = 1, .., N\}$. Modding out by the Weyl group S_N restricts the lattice to the Weyl chamber $\Gamma_{U(N)}/S_N = \left\{\boldsymbol{m} \in \mathbb{Z}^N | m_1 \geq m_2 \geq ... \geq m_N\right\}$.

For $U(N)$ gauge groups, which are not simply connected, the center $\mathcal{Z}(\mathcal{G}^\vee) = U(1)$ engenders a topological $U(1)_J$ symmetry group. Classically, monopole operators are only charged under this symmetry. To each such $U(N_i)$ gauge group, we associate a fugacity z_i for the topological $U(1)_{J_i}$ symmetry with conserved current $* \operatorname{Tr} F_i$, where F_i is the field strength of the i-th gauge group. Other charges are acquired quantum-mechanically: in particular, monopole operators become charged under the Cartan $U(1)_C$ of the $SU(2)_C$ R-symmetry acting on the Coulomb branch. For a Lagrangian $\mathcal{N} = 4$ gauge theory, this charge is given by the formula

$$\Delta(\boldsymbol{m}) = - \sum_{\boldsymbol{\alpha} \in \Delta_+} |\boldsymbol{\alpha}(\boldsymbol{m})| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\boldsymbol{\rho}_i \in \mathcal{R}_i} |\boldsymbol{\rho}_i(\boldsymbol{m})| \ , \tag{4.3.1}$$

where the first contribution, arising from vector multiplets, is a sum over the positive roots of the gauge group, while the second contribution is a sum over the weights of the gauge group representations of the hypermultiplets. The fugacity for this R-symmetry is called t^2 in the following. The dimension formula (4.3.1) was conjectured in [81] based on a weak coupling computation in [19], and later proven exactly in [115, 116]. For the theories that we will be studying, which are good or ugly in the sense of [81], (4.3.1) is believed to equal the scaling dimension in the IR CFT.

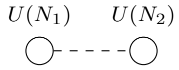
For gauge theories described by (possibly non-simply laced) Dynkin diagrams, we propose the following prescription for computing the R-charge of a monopole operator, generalizing the Lagrangian formula (4.3.1). Each diagram is constructed from two basic building blocks: a node and a line. A $U(N)$ node, with magnetic charge $\boldsymbol{m}$, contributes to the Coulomb branch Hilbert series as follows:

$U(N)$
◯

$$\Delta_{\mathrm{vec}}(\boldsymbol{m}) = - \sum_{1 \leq i < j \leq N} |m_i - m_j| \ . \tag{4.3.2}$$

A line connecting the nodes $U(N_1)$ and $U(N_2)$ can be either a single bond ($-$), a double bond ($\Rightarrow$) or a triple bond ($\Rrightarrow$), which we take to be oriented from node 1 to node 2. Let us assign magnetic charges $\boldsymbol{m}^{(1)}$ and $\boldsymbol{m}^{(2)}$ to $U(N_1)$ and

$U(N_2)$ respectively. We propose that the contribution from a line is:



$$\Delta_{\text{hyp}}(\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}) = \frac{1}{2} \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \left| \lambda m_j^{(1)} - m_k^{(2)} \right| \tag{4.3.3}$$

where $\lambda = 1$ for a single bond, $\lambda = 2$ for a double bond and $\lambda = 3$ for a triple bond. If $\lambda > 1$, (4.3.3) does not arise from matter fields transforming in a genuine representation of $U(N_1) \times U(N_2)$.[5]

We stress that formula (4.3.3) is the crucial ingredient that will allow us to compute the Hilbert series of instanton moduli spaces for any simple Lie group. We will successfully test our proposal by comparing with known results and by studying general properties of the Hilbert series that can be extracted from the Coulomb branch formula.

The dimension formula, given by the sum of the two contributions, (4.3.2) for each node and (4.3.3) for each line, makes the quivers associated to the affine Dynkin diagrams (*i.e.* before adding the over-extended node) balanced in the sense of [81]: each unitary gauge group has an effective number of flavors equal to twice the number of colors.[6]

Once we have classified gauge invariant chiral operators (classical operators, bare and dressed monopole operators) on the Coulomb branch of non-simply laced quivers by their quantum number J_i and Δ, we enumerate them by means of a generating function that grades them by their charges. The Hilbert series of the Coulomb branch of a $d = 3$ $\mathcal{N} = 4$ good or ugly superconformal field theory is then given by [30]

$$HS(t, \boldsymbol{z}) = \sum_{\boldsymbol{m} \in \Gamma_{\mathcal{G}^\vee}/\mathcal{W}_{\mathcal{G}}} \boldsymbol{z}^{\boldsymbol{J}(\boldsymbol{m})}\, t^{2\Delta(\boldsymbol{m})} P_{\mathcal{G}}(t; \boldsymbol{m}) \ , \tag{4.3.4}$$

where $\boldsymbol{z}^{\boldsymbol{J}(\boldsymbol{m})} = \prod_i z_i^{J_i(\boldsymbol{m})}$. The sum is over GNO magnetic sectors [113], restricted to a Weyl chamber to impose invariance under the gauge group $\mathcal{G}$. There is one bare monopole operator per magnetic charge sector [19]. The factors $\boldsymbol{z}^{\boldsymbol{J}(\boldsymbol{m})}\, t^{2\Delta(\boldsymbol{m})}$ account for the topological charges and conformal

[5] Conceivably, this prescription could be derived from a Lagrangian quiver gauge theory associated to an unfolded simply laced quiver, further orbifolded by an outer automorphism group of the quiver. We will not pursue this possibility here. We thank Jan Troost for discussions on this point.

[6] The effective number of flavors for a gauge group is obtained by adding up the ranks of all the gauge groups connected to it by an edge, appropriately weighted by λ. For instance, for F_4 node 2 has $3k$ colors and $2k + 2(2k) = 6k$ effective flavors, while node 3 has $2k$ colors and $3k + k = 4k$ flavors.

dimension of bare monopole operators of magnetic charge $\boldsymbol{m}$. Finally, the factor $P_{\mathcal{G}}(t;\boldsymbol{m})$ reflects the dressing of a bare monopole operator $V_{\boldsymbol{m}}$ by polynomials of the classical adjoint scalar $\phi_m \in \mathfrak{h}_{\boldsymbol{m}}$ which are gauge invariant under the residual gauge group $H_{\boldsymbol{m}}$ left unbroken by the monopole flux. The contribution of this dressing factor to the Hilbert series is given by the generating function of $H_{\boldsymbol{m}}$ Casimir invariants

$$P_{\mathcal{G}}(t;\boldsymbol{m}) = \prod_{i=1}^{\mathrm{rk}(\mathcal{G})} \frac{1}{1-t^{2d_i(\boldsymbol{m})}} \tag{4.3.5}$$

where $d_i(\boldsymbol{m})$ are the degrees of the Casimir invariants of $H_{\boldsymbol{m}}$.[7] We refer the readers to Appendix A of [30] for more details on these classical dressing factors.

In the next sections we will apply formula (4.3.4) to the non-simply laced quivers discussed in section 4.2 and compute exactly the Hilbert series of the corresponding three instanton moduli spaces. To make contact with moduli spaces of G-instantons, we first need to specify how the fugacities $\boldsymbol{z}$ of the topological symmetry are related to the fugacities x and $\boldsymbol{u}$ of the global $SU(2)_x \times G_{\boldsymbol{u}}$ symmetry acting on G-instantons.

4.3.1 Refinement

Consider a generalized quiver gauge theory corresponding to an over-extended affine Dynkin diagram from Tables 4.3 and 4.4. We label the nodes as follows: $i = 1, \ldots, r = \mathrm{rk}(G)$ for the nodes of the Dynkin diagram of the finite Lie algebra Lie(G), $i = 0$ for the affine node corresponding to the null root, and $i = -1$ for the over-extended node attached to the $i = 0$ node. The ranks N_i of the associated unitary groups are given by $N_{-1} = 1$ for the over-extended node and by $N_i = k a_i^\vee$, $i = 0, \ldots, r$, for the nodes of the affine Dynkin diagram. Each unitary gauge group has a topological symmetry $U(1)_{J_i}$ with fugacity z_i.

When all the nodes are treated as gauge groups, an overall diagonal $U(1)$ is decoupled and needs to be factored out. This decoupled $U(1)$ corresponds to the shift symmetry

$$m_{-1} \to m_{-1} + c\ , \qquad m_i \to m_i + c\, \frac{a_i}{a_i^\vee}\, \mathbb{1}_{k a_i^\vee} \qquad (i = 0, \ldots r)\ , \qquad c \in \mathbb{Z} \tag{4.3.6}$$

[7](4.3.5) assumes that the ring of Casimir invariants is freely generated, as is the case for semisimple Lie groups. The assumption could fail if the gauge group contains extra discrete factors, in which case (4.3.5) is to be replaced by the appropriate Molien formula. We will not encounter this subtlety in this thesis.

in the dimension formula, where $\mathbb{1}_n$ denotes the $n \times n$ unit matrix, a_i are the Coxeter labels and $a_i^\vee$ are the dual Coxeter labels of the untwisted affine algebra (in particular $a_0 = a_0^\vee = 1$). Note that for the untwisted affine algebras the ratio $a_i/a_i^\vee$ is an integer. The decoupled $U(1)$ is factored out by fixing the shift symmetry (4.3.6), multiplying the Coulomb branch Hilbert series by its inverse classical factor $(1-t^2)$, and setting to 1 the fugacity of the associated topological symmetry:

$$z_{-1}\left(\prod_{i=0}^{r} z_i^{a_i}\right)^k = 1 \ . \tag{4.3.7}$$

The constraint (4.3.7) on the fugacities ensures that the shift (4.3.6) does not affect the Hilbert series and determines z_{-1} in terms of the remaining $r+1$ fugacities z_i, $i = 0, \ldots, r$, associated to the nodes of the untwisted affine Dynkin diagram. The fugacities z_i, $i = 1, \ldots, r$, associated to the nodes of the Dynkin diagram of Lie(G) are simple root fugacities for the global symmetry G, therefore in (4.2.2) we can identify

$$u_i = z_i \ , \qquad i = 1, \ldots, r \ . \tag{4.3.8}$$

The fugacity x for the $SU(2)$ rotational symmetry is determined by identifying the two unique monopole operators of dimension $\Delta = \frac{1}{2}$, which generate the $\mathbb{C}^2$ moduli space of the center of mass of the instantons. The tower of monopole operators obtained by rescaling these magnetic fluxes by an integer then reconstructs the prefactor in (4.2.2). Let us focus on a monopole operator which generates a $\mathbb{C}$ subspace of the $\mathbb{C}^2$ moduli space of the center of mass, and assign to it weight tx in the Hilbert series for definiteness.[8] Up to the shift (4.3.6), the magnetic charge of this monopole operator (written in matrix notation) can be taken to be[9]

$$m_{-1} = 0 \ , \qquad m_i = \mathrm{diag}(1, 0^{k-1}) \otimes \frac{a_i}{a_i^\vee} \ \mathbb{1}_{a_i^\vee} \ , \quad i = 0, \ldots, r \ . \tag{4.3.9}$$

It is straightforward to see that the monopole operator with magnetic charge (4.3.9) has dimension $\Delta = \frac{1}{2}$: because the contributions to Δ coming from the untwisted affine Dynkin diagram cancel out, while the contribution of the edge connecting the extended node to the over-extended node is $\frac{1}{2}$. From the

[8]The monopole operator with weight weight tx^{-1} is obtained by flipping sign to the magnetic flux and acting with the Weyl group to bring the resulting flux to the positive Weyl chamber.

[9]We use the shorthand notation $(r^s) = (\underbrace{r, \cdots, r}_{s \text{ times}})$.

topological charge of the monopole operator of magnetic charge (4.3.9) we read off the fugacity for the $SU(2)_x$ rotational symmetry,

$$x = \prod_{i=0}^{r} z_i^{a_i} = z_0 \prod_{i=1}^{r} u_i^{a_i} \ . \tag{4.3.10}$$

In the last equality we have used $a_0 = 1$ and the identification (4.3.8). (4.3.10) can be used to express z_0 in terms of x and $\boldsymbol{u}$. The constraint (4.3.7) from the removal of the decoupled $U(1)$ then determines z_{-1} as

$$z_{-1} = x^{-k} \ . \tag{4.3.11}$$

4.4 k G_2 instantons

The theory whose Coulomb branch is the moduli space of k G_2 instantons on $\mathbb{C}^2$ is described by the quiver diagram

$$\underset{1}{\circ} - \underset{k}{\bullet} - \underset{2k}{\circ} \Rrightarrow \underset{k}{\circ} \tag{4.4.1}$$

where each number denotes the rank of each unitary gauge group and an overall $U(1)$ symmetry is factored out.

The dimension formula for k G_2 instantons can be extracted from this quiver using the prescription of Section 4.3:

$$\begin{aligned} \Delta_{k,G_2}(\boldsymbol{m},\boldsymbol{n},\boldsymbol{s}) = &\sum_{i=1}^{k} |m_i| + \sum_{i=1}^{k}\sum_{j=1}^{2k} |m_i - n_j| + \sum_{j=1}^{2k}\sum_{\ell=1}^{k} |3n_j - s_\ell| \\ &- 2\left(\sum_{1\le i<i'\le k} |m_i - m_{i'}| + \sum_{1\le j<j'\le 2k} |n_j - n_{j'}| + \sum_{1\le \ell<\ell'\le k} |s_\ell - s_{\ell'}| \right) , \end{aligned} \tag{4.4.2}$$

where $\boldsymbol{m} = (m_1, ..., m_k)$, $\boldsymbol{n} = (n_1, ..., n_{2k})$ and $\boldsymbol{s} = (s_1, ..., s_k)$. Note the factor of 3 in front of n_j for the triply laced bifundamental contribution. Here we have gauge fixed the decoupled $U(1)$ by setting the monopole flux of the over-extended node (indicated in blue) to zero.

The Hilbert series for the moduli space of k G_2 instantons can thus be computed as follows:

$$g_{k,G_2}(t;\boldsymbol{z}) = \sum_{m_1\geq\cdots\geq m_k>-\infty} \sum_{n_1\geq\cdots\geq n_{2k}>-\infty} \sum_{s_1\geq\cdots\geq s_k>-\infty} t^{\Delta_{k,G_2}(\boldsymbol{m},\boldsymbol{n},\boldsymbol{s})} P_{U(k)}(t;\boldsymbol{m})P_{U(2k)}(t;\boldsymbol{n})P_{U(k)}(t;\boldsymbol{s}) \times z_0^{\sum_{i=1}^k m_i} z_1^{\sum_{j=1}^{2k} n_j} z_2^{\sum_{\ell=1}^{k} s_\ell} \ , \tag{4.4.3}$$

where the fugacities $\boldsymbol{z}$ are associated to the topological symmetry.

For $k=1$, the result of (4.4.3) can be written as

$$g_{1,G_2}(t;\boldsymbol{z}) = \frac{1}{(1-tx)(1-tx^{-1})} \sum_{p=0}^{\infty} \chi^{G_2}_{[p,0]}(u_1,u_2)t^{2p} \ , \tag{4.4.4}$$

where $[1,0]$ is the adjoint representation of G_2 and

$$x = z_0 z_1^2 z_2^3 \ , \qquad u_1 = z_1, \qquad u_2 = z_2 \ . \tag{4.4.5}$$

This agrees with (5.46) of [69].

It is worth mentioning that, for $k \geq 2$, the Hilbert series (4.4.3) can alternatively be computed using the Hall-Littlewood formula and the gluing technique discussed in [96, 97]. Indeed quiver (4.4.1) can be constructed by gluing the following two basic building blocks

$$T_{(k,k-1,1)}(SU(2k)):\ (1)-(k)-[2k]\ , \qquad T_{(k,k)}(SU(2k)):\ [2k]-(k)\ , \tag{4.4.6}$$

once the edge $[2k]-(k)$ in the second building block is converted to $[2k] \Rrightarrow (k)$ by tripling the value of the background magnetic charges in the Coulomb branch Hilbert series of $T_{(k,k)}(SU(2k))$. The two building blocks are glued by gauging the common flavor symmetry $U(2k)/U(1)$. The final expression of the Hilbert series in question is given by

$$\begin{aligned} g_{k,G_2}(t;\boldsymbol{a},\boldsymbol{b}) = \sum_{n_1\geq n_2\geq\ldots\geq n_{2k-1}\geq n_{2k}=0} & t^{-2\delta_{U(2k)}(\boldsymbol{n})}(1-t^2)P_{U(2k)}(t;n_1,\ldots,n_{2k})\times \\ & H[T_{(k,k-1,1)}(SU(2k))](t;a_1,a_2,a_3;n_1,\ldots,n_{2k})\times \\ & H[T_{(k,k)}(SU(2k))](t;b_1,b_2,b_3;3n_1,\ldots,3n_{2k}) \ . \end{aligned} \tag{4.4.7}$$

The Hall-Littlewood formulae for the Coulomb branch HS of (4.4.6) are given by

$$\begin{aligned} &H[T_{(k,k-1,1)}(SU(2k))](t;a_1,a_2,a_3;\boldsymbol{n}) \\ &\quad = t^{\delta_{U(2k)}(\boldsymbol{n})}(1-t^2)^{2k}K_{(k,k-1,1)}(t;a_1,a_2,a_3)\Psi^{\boldsymbol{n}}_{U(2k)}(\boldsymbol{v}_{(k,k-1,1)};t) \ , \end{aligned} \tag{4.4.8}$$

$$\begin{aligned} &H[T_{(k,k)}(SU(2k))](t;b_1,b_2;\boldsymbol{n}) \\ &\quad = t^{\delta_{U(2k)}(\boldsymbol{n})}(1-t^2)^{2k}K_{(k,k)}(t;b_1,b_2)\Psi^{\boldsymbol{n}}_{U(2k)}(\boldsymbol{v}_{(k,k)};t) \ , \end{aligned} \tag{4.4.9}$$

where the Hall-Littlewood polynomial is defined as

$$\Psi^{\boldsymbol{n}}_{U(N)}(x_1,\dots,x_N;t) = \sum_{\sigma\in S_N} x^{n_1}_{\sigma(1)}\dots x^{n_N}_{\sigma(N)} \prod_{1\le i<j\le N} \frac{1-tx^{-1}_{\sigma(i)}x_{\sigma(j)}}{1-x^{-1}_{\sigma(i)}x_{\sigma(j)}} \ , \tag{4.4.10}$$

and the parameters and prefactors are given by

$$\delta_{U(2k)}(\boldsymbol{n}) = \sum_{1\le i<j\le 2k}(n_i-n_j) \ , \tag{4.4.11}$$

$$\begin{aligned} \boldsymbol{v}_{(k,k-1,1)} = \Big(&t^{k-1}a_1, t^{k-3}a_1,\dots,t^{-(k-3)}a_1,t^{-(k-1)}a_1, \\ &t^{k-3}a_2,t^{k-5}a_2,\dots,t^{-(k-5)}a_2,t^{-(k-3)}a_2,a_3\Big) \ , \end{aligned} \tag{4.4.12}$$

$$\begin{aligned} \boldsymbol{v}_{(k,k)} = \Big(&t^{k-1}b_1,t^{k-3}b_1,\dots,t^{-(k-3)}b_1,t^{-(k-1)}b_1, \\ &t^{k-1}b_2,t^{k-3}b_2,\dots,t^{-(k-3)}b_2,t^{-(k-1)}b_2\Big) \ , \end{aligned} \tag{4.4.13}$$

$$\begin{aligned} K_{(k,k-1,1)}(t;\boldsymbol{a}) = \mathrm{PE}\Bigg[&(t^2+t^{2k}) + 2\sum_{m=1}^{k-1} t^{2m} \\ &+ (a_2a_3^{-1}+a_2^{-1}a_3)t^k + (a_1a_3^{-1}+a_1^{-1}a_3)t^{k+1} \\ &+ (2+a_1a_2^{-1}+a_2a_1^{-1})\sum_{m=1}^{k} t^{2m-1}\Bigg] \ , \end{aligned}$$

$$K_{(k,k)}(t;\boldsymbol{b}) = \mathrm{PE}\left[\left(2+b_1b_2^{-1}+b_1^{-1}b_2\right)\sum_{m=1}^{k} t^{2m}\right] \ .$$

The fugacities can be set as follows:

$$a_1^k a_2^{k-1} a_3 = 1 \ , \qquad b_1^k b_2^k = 1 \ . \tag{4.4.14}$$

The relations between the fugacities $\boldsymbol{a}$ and $\boldsymbol{b}$ to the topological fugacity of each node in quiver (4.4.1) are given by (see (3.13) of [96])

$$z_{-1} = a_3a_2^{-1}, \quad z_0 = a_2a_1^{-1}, \quad z_1 = a_1b_1^3, \quad z_2 = b_2b_1^{-1} \ , \tag{4.4.15}$$

and by factoring out the overall $U(1)$ we have the following condition (cf. (3.3) of [97]):

$$z_{-1}(z_0 z_1^2 z_2^3)^k = 1 \; . \tag{4.4.16}$$

From (4.3.8) and (4.3.11), we find that the relations between $\boldsymbol{a}, \boldsymbol{b}$ and the fugacities x associated with $SU(2)$ and u_1, u_2 associated with G_2 are

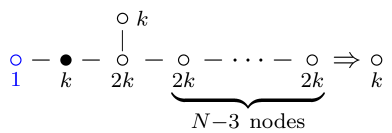
$$\begin{aligned} x &= z_0 z_1^2 z_2^3 = a_1 a_2 (b_1 b_2)^3 \; , \\ z_1 &= a_1 b_1^3 \; , \quad z_2 = b_2 b_1^{-1} \; . \end{aligned} \tag{4.4.17}$$

For $k = 2$ we recover the Hilbert series (9.3) and (9.5)[10] of [71]. For $k = 3$ let us report only the result with z_i being set to unity; the unrefined Hilbert series of the reduced three G_2 instanton moduli space is

$$\begin{aligned} \widetilde{g}_{3,G_2}(t) = \frac{1-t}{(1-t^2)^7(1-t^3)^9(1-t^4)^7} \Big(& 1 + t + 11t^2 + 34t^3 + 124t^4 + 352t^5 \\ & + 1055t^6 + 2657t^7 + 6584t^8 + 14635t^9 + 31194t^{10} + 61229t^{11} \\ & + 114367t^{12} + 198932t^{13} + 329172t^{14} + 511194t^{15} + 755093t^{16} \\ & + 1051845t^{17} + 1394817t^{18} + 1749632t^{19} + 2091341t^{20} + 2368619t^{21} \\ & + 2557449t^{22} + 2619060t^{23} + 2557449t^{24} + \text{palindrome up to } t^{46} \Big) \; . \end{aligned} \tag{4.4.18}$$

4.5 k B_N instantons

The theory whose Coulomb branch is the moduli space of k $SO(2N+1)$ instantons on $\mathbb{C}^2$ is described by the quiver diagram



(4.5.1)

where each number denotes the rank of a unitary gauge group and the decoupled overall $U(1)$ symmetry is removed. For $k = 2$ we recover the results given in Section 5 of [71].

[10] There is a typo in Eq. (9.5) of [71]: the power of $(1 + t + t^2)$ in the denominator should be 7.

The unrefined Hilbert series of the reduced 3 $SO(7)$ instanton moduli space is

$$\begin{aligned}\widetilde{g}_{3,SO(7)}(t) = {} & \frac{(1-t)^2}{(1-t^2)^9\,(1-t^3)^{12}\,(1-t^4)^9}\Big(1+2t+18t^2+68t^3+292t^4+1024t^5 \\ & +3565t^6+11012t^7+32587t^8+88764t^9+229405t^{10}+554642t^{11} \\ & +1271439t^{12}+2749154t^{13}+5648717t^{14}+11006976t^{15}+20431264t^{16} \\ & +36104898t^{17}+60918929t^{18}+98135686t^{19}+151245678t^{20} \\ & +10417596422t^{40}+315153966t^{22}+426792414t^{23}+554536028t^{24} \\ & +691345362t^{25}+827700194t^{26}+951603050t^{27}+1051256831t^{28} \\ & +1115766454t^{29}+1138239548t^{30}+1115766454t^{31} \\ & +\text{palindrome up to } t^{60}\Big)\ . \end{aligned} \tag{4.5.2}$$

4.6 k C_N instantons

The theory whose Coulomb branch is the moduli space of k $USp(2N)$ instantons on $\mathbb{C}^2$ is described by the quiver diagram

$$\underset{1}{\circ} - \underset{k}{\bullet} \Rightarrow \underbrace{\underset{k}{\circ} - \cdots - \underset{k}{\circ}}_{N-1\ \text{nodes}} \Leftarrow \underset{k}{\circ} \tag{4.6.1}$$

where each number denotes the rank of a unitary gauge group and an overall $U(1)$ symmetry decouples. For $k=2$ we recover the results given in Section 4.2 of [71]. Below we present the unrefined Hilbert series for 3 instantons and small values of N.

The unrefined Hilbert series of the reduced 3 $USp(4)$ instanton moduli space is

$$\begin{aligned}\widetilde{g}_{3,USp(4)}(t) = {} & \frac{1}{(1-t^2)^5(1-t^3)^6(1-t^4)^5}\Big(1+8t^2+18t^3+61t^4+142t^5 \\ & +388t^6+792t^7+1691t^8+2996t^9+5255t^{10}+7994t^{11}+11713t^{12} \\ & +15134t^{13}+18773t^{14}+20796t^{15}+21980t^{16}+20796t^{17}+18773t^{18} \\ & +\text{palindrome up to } t^{32}\Big)\ . \end{aligned} \tag{4.6.2}$$

The unrefined Hilbert series of the reduced 3 $USp(6)$ instanton moduli space is

$$\begin{aligned}\widetilde{g}_{3,USp(6)}(t) = &\frac{1}{(1-t^2)^7(1-t^3)^8(1-t^4)^7}\Big(1 + 17t^2 + 38t^3 + 209t^4 + 644t^5 \\ &+ 2260t^6 + 6382t^7 + 17808t^8 + 43106t^9 + 99660t^{10} + 206484t^{11} \\ &+ 404244t^{12} + 724452t^{13} + 1224332t^{14} + 1917162t^{15} + 2834175t^{16} \\ &+ 3909874t^{17} + 5102043t^{18} + 6239722t^{19} + 7227435t^{20} + 7864776t^{21} \\ &+ 8110736t^{22} + 7864776t^{23} + \text{palindrome up to } t^{44}\Big) \ .\end{aligned} \tag{4.6.3}$$

The unrefined Hilbert series of the reduced 3 $USp(8)$ instanton moduli space is

$$\begin{aligned}\widetilde{g}_{3,USp(8)}(t) = &\frac{1}{(1-t^2)^9(1-t^3)^{10}(1-t^4)^9}\Big(1 + 30t^2 + 66t^3 + 564t^4 + 1978t^5 \\ &+ 8986t^6 + 31320t^7 + 108588t^8 + 327552t^9 + 938028t^{10} + 2428438t^{11} \\ &+ 5923950t^{12} + 13333518t^{13} + 28288029t^{14} + 56057448t^{15} \\ &+ 105000098t^{16} + 185111036t^{17} + 309423948t^{18} + 489269266t^{19} \\ &+ 735494922t^{20} + 1049537386t^{21} + 1426754090t^{22} + 1845578580t^{23} \\ &+ 2277688217t^{24} + 2678999920t^{25} + 3009187465t^{26} + 3224258916t^{27} \\ &+ 3300770520t^{28} + 3224258916t^{29} + 3009187465t^{30} \\ &+ \text{palindrome up to } t^{56}\Big) \ .\end{aligned} \tag{4.6.4}$$

The unrefined Hilbert series of the reduced 3 $USp(10)$ instanton moduli space is

$$\begin{aligned}\widetilde{g}_{3,USp(10)}(t) = &\frac{(1-t)^2}{(1-t^2)^{13}(1-t^3)^{12}(1-t^4)^{11}}\Big(1 + 2t + 48t^2 + 196t^3 + 1533t^4 \\ &+ 7458t^5 + 39083t^6 + 173746t^7 + 729193t^8 + 2753342t^9 + 9659061t^{10} \\ &+ 31142740t^{11} + 93620178t^{12} + 262065600t^{13} + 688287079t^{14} \\ &+ 1698315214t^{15} + 3955023058t^{16} + 8708306700t^{17} + 18185341012t^{18} \\ &+ 36076921166t^{19} + 68144856266t^{20} + 122727426896t^{21} \\ &+ 211098608616t^{22} + 347187234006t^{23} + 546680541199t^{24} \\ &+ 824886510488t^{25} + 1193911094540t^{26} + 1658736457996t^{27} \\ &+ 2213773962229t^{28} + 2839692757258t^{29} + 3502903178369t^{30} \\ &+ 4156849878890t^{31} + 4747242880506t^{32} + 5218604879584t^{33}\end{aligned}$$

$$+ 5523278387053t^{34} + 5628609146268t^{35} + 5523278387053t^{36} + \text{palindrome up to } t^{70}\Big) . \quad (4.6.5)$$

For higher number of instantons, the Hilbert series can be computed more easily from the Higgs branch of the ADHM quiver. We demonstrate this computation in Appendix A.1. Let us report here the unrefined Hilbert series (*i.e.* $x = 1$ and $z_i = 1$ for all i) for $k = 5$ and small values of N:

$$\widetilde{g}_{5,USp(2)}(t) = \frac{1}{(1-t^2)^4(1-t^3)^4(1-t^4)^3(1-t^5)^4(1-t^6)^3} \times \Big(1 + 2t^2 + 6t^3 + 14t^4 + 26t^5 + 59t^6 + 108t^7 + 216t^8 + 382t^9 + 669t^{10} + 1090t^{11} + 1788t^{12} + 2718t^{13} + 4080t^{14} + 5844t^{15} + 8166t^{16} + 10902t^{17} + 14271t^{18} + 17886t^{19} + 21899t^{20} + 25824t^{21} + 29701t^{22} + 32898t^{23} + 35621t^{24} + 37152t^{25} + 37792t^{26} + 37152t^{27} + \text{palindrome up to } t^{52}\Big) . \quad (4.6.6)$$

$$\widetilde{g}_{5,USp(4)}(t) = \frac{1}{(1-t^2)^5(1-t^3)^6(1-t^4)^6(1-t^5)^6(1-t^6)^5} \times \Big(1 + 8t^2 + 18t^3 + 65t^4 + 184t^5 + 568t^6 + 1486t^7 + 4068t^8 + 10202t^9 + 25294t^{10} + 59530t^{11} + 136840t^{12} + 301276t^{13} + 645420t^{14} + 1332274t^{15} + 2669897t^{16} + 5173382t^{17} + 9731196t^{18} + 17732334t^{19} + 31384129t^{20} + 53895904t^{21} + 89958111t^{22} + 145882550t^{23} + 230128561t^{24} + 353099760t^{25} + 527468664t^{26} + 767161840t^{27} + 1087152304t^{28} + 1501274126t^{29} + 2021417792t^{30} + 2654217372t^{31} + 3400290035t^{32} + 4250584996t^{33} + 5186895160t^{34} + 6179265798t^{35} + 7189118462t^{36} + 8168673774t^{37} + 9067212695t^{38} + 9832235886t^{39} + 10417596422t^{40} + 10784743772t^{41} + 10910252456t^{42} + 10784743772t^{43} + \text{palindrome up to } t^{84}\Big) . \quad (4.6.7)$$

4.7 k F_4 instantons

The theory whose Coulomb branch is the moduli space of k F_4 instantons on $\mathbb{C}^2$ is described by the quiver diagram

$$\underset{1}{\circ} - \underset{k}{\bullet} - \underset{2k}{\circ} - \underset{3k}{\circ} \Rightarrow \underset{2k}{\circ} - \underset{k}{\circ} \tag{4.7.1}$$

where each number denotes the rank of a unitary gauge group and an overall $U(1)$ symmetry is factored out.

The Hilbert series of k F_4 instantons can be computed using the monopole formula given by (4.3.4). For $k \geq 2$, (4.3.4) is more easily calculated using the gluing technique discussed in [97]. Indeed quiver (4.7.1) can be constructed from the building blocks

$$\begin{aligned} T_{(k,k,k-1,1)}(SU(3k)): &\quad (1) - (k) - (2k) - [3k] \ , \\ T_{(k,k,k)}(SU(3k)): &\quad [3k] - (2k) - (k) \ , \end{aligned} \tag{4.7.2}$$

once the edge $[3k]-(2k)$ in the second building block is converted to $[3k] \Rightarrow (2k)$ by doubling the value of the background magnetic charges in the Coulomb branch Hilbert series of $T_{(k,k,k)}(SU(3k))$. The two building blocks are glued by gauging the common flavor symmetry $U(3k)/U(1)$.

The final expression of the Hilbert series in question is given by

$$\begin{aligned} g_{k,F_4}(t;\boldsymbol{a},\boldsymbol{b}) = \sum_{m_1 \geq m_2 \geq \ldots \geq m_{3k}=0} & t^{-2\delta_{U(3k)}(\boldsymbol{m})}(1-t^2)P_{U(3k)}(t;m_1,\ldots,m_{3k})\times \\ & H[T_{(k,k,k-1,1)}(SU(3k))](t;a_1,a_2,a_3,a_4;m_1,\ldots,m_{3k})\times \\ & H[T_{(k,k,k)}(SU(3k))](t;b_1,b_2,b_3;2m_1,\ldots,2m_{3k}) \ . \end{aligned} \tag{4.7.3}$$

The Coulomb branch Hilbert series of $T_{(k,k,k-1,1)}(SU(3k))$ is given by

$$\begin{aligned} & H[T_{(k,k,k-1,1)}(SU(2k))](t;a_1,a_2,a_3,a_4;\boldsymbol{n}) \\ & \quad = t^{\delta_{U(3k)}(\boldsymbol{n})}(1-t^2)^{3k}K_{(k,k,k-1,1)}(t;a_1,a_2,a_3,a_4)\Psi^{\boldsymbol{n}}_{U(3k)}(\boldsymbol{v}_{(k,k,k-1,1)};t) \ , \end{aligned} \tag{4.7.4}$$

with

$$\delta_{U(3k)}(\boldsymbol{n}) = \sum_{1 \leq i < j \leq 3k} (n_i - n_j) \ , \tag{4.7.5}$$

$$\begin{aligned}\boldsymbol{v}_{(k,k,k-1,1)} = \Big(&t^{k-1}a_1, t^{k-3}a_1, \ldots, t^{-(k-3)}a_1, t^{-(k-1)}a_1, \\ &t^{k-1}a_2, t^{k-3}a_2, \ldots, t^{-(k-3)}a_2, t^{-(k-1)}a_2\ , \\ &t^{k-2}a_3, t^{k-4}a_3, \ldots, t^{-(k-4)}a_3, t^{-(k-2)}a_3, a_4\Big)\ ,\end{aligned} \tag{4.7.6}$$

$$\begin{aligned}&K_{(k,k,k-1,1)}(t;\boldsymbol{a}) \\ &= \mathrm{PE}\left[(t^2+t^{2k}) + \sum_{m=1}^{k-1} t^{2m} + (a_3a_4^{-1}+a_4^{-1}a_3)t^k \right. \\ &\qquad + (a_1a_4^{-1}+a_1^{-1}a_4+a_2a_4^{-1}+a_2^{-1}a_4)t^{k+1} \\ &\qquad + (a_1a_3^{-1}+a_1^{-1}a_3+a_2a_3^{-1}+a_2^{-1}a_3)\sum_{m=1}^{k} t^{2m-1} \\ &\qquad \left. + (2+a_1a_2^{-1}+a_2a_1^{-1})\sum_{m=1}^{k} t^{2m}\right]\ .\end{aligned} \tag{4.7.7}$$

On the other hand, the Coulomb branch Hilbert series of $T_{(k,k,k)}(SU(3k))$ is

$$\begin{aligned}&H[T_{(k,k,k)}(SU(2k))](t;b_1,b_2,b_3;\boldsymbol{n}) \\ &\quad = t^{\delta_{U(3k)}(\boldsymbol{n})}(1-t^2)^{3k}K_{(k,k,k)}(t;b_1,b_2,b_3)\Psi^{\boldsymbol{n}}_{U(3k)}(\boldsymbol{v}_{(k,k,k)};t)\ ,\end{aligned} \tag{4.7.8}$$

with

$$\begin{aligned}\boldsymbol{v}_{(k,k,k)} = \Big(&t^{k-1}b_1, t^{k-3}b_1, \ldots, t^{-(k-3)}b_1, t^{-(k-1)}b_1, \\ &t^{k-1}b_2, t^{k-3}b_2, \ldots, t^{-(k-3)}b_2, t^{-(k-1)}b_2\ , \\ &t^{k-1}b_3, t^{k-3}b_3, \ldots, t^{-(k-3)}b_3, t^{-(k-1)}b_3\Big)\ ,\end{aligned} \tag{4.7.9}$$

$$K_{(k,k,k)}(t;\boldsymbol{b}) = \mathrm{PE}\left[\left(\sum_{1\le i,j\le 3} b_i b_j^{-1}\right)\sum_{m=1}^{k} t^{2m}\right]\ . \tag{4.7.10}$$

The fugacities can be set as follows:

$$a_1^k a_2^k a_3^{k-1} a_4 = 1\ , \qquad b_1^k b_2^k b_3^k = 1\ . \tag{4.7.11}$$

The relations between the fugacities $\boldsymbol{a}$ and $\boldsymbol{b}$ to the topological fugacity of each node in quiver (4.7.1) are given by (see (3.13) of [96])

$$\begin{aligned}&z_{-1} = a_4a_3^{-1}, \quad z_0 = a_3a_2^{-1}, \quad z_1 = a_2a_1^{-1}, \\ &z_2 = a_1b_1^2, \quad z_3 = b_2b_1^{-1}, \quad z_4 = b_3b_2^{-1}\ ,\end{aligned} \tag{4.7.12}$$

and by factoring out the overall $U(1)$ we have the following condition (cf. (3.3) of [97]):

$$z_{-1}(z_0 z_1^2 z_2^3 z_3^4 z_4^2)^k = 1 \ . \tag{4.7.13}$$

From (4.3.8) and (4.3.11), we find that the relations between $\boldsymbol{a}, \boldsymbol{b}$ and the fugacities x associated with $SU(2)$ and u_1, u_2, u_3, u_4 associated with F_4 are

$$\begin{aligned} x &= z_0 z_1^2 z_2^3 z_3^4 z_4^2 = a_1 a_2 a_3 (b_1 b_2 b_3)^2 \ , \\ u_1 &= a_2 a_1^{-1} \ , \quad u_2 = a_1 b_1^2 \ , \quad u_3 = b_2 b_1^{-1} \ , \quad u_4 = b_3 b_2^{-1} \ . \end{aligned} \tag{4.7.14}$$

For $k = 2$ we recover the results given in (10.2) and (10.4) of [71].

4.8 The moduli space of instantons as an algebraic variety

4.8.1 One instanton

The reduced moduli space of one G instanton is the orbit of the highest root vector in the complexification of the Lie algebra of G [117, 118, 119], also known as minimal nilpotent orbit. The space of holomorphic functions on such a reduced moduli space was studied in [69].[11] The Hilbert series can be obtained as

$$H(t, \boldsymbol{u}) = \sum_{p=0}^{\infty} \chi^G_{p \cdot \mathbf{Adj}}(\boldsymbol{u}) t^{2p} \ , \tag{4.8.1}$$

where $p \cdot \mathbf{Adj}$ denotes the irreducible representation of G whose highest weight is p times that of the adjoint representation. The plethystic logarithm of this Hilbert series reads

$$\mathrm{PL}\left[H(t, \boldsymbol{u})\right] = \chi^G_{\mathbf{Adj}}(\boldsymbol{u}) t^2 - \left(\chi^G_{\mathrm{Sym}^2 \mathbf{Adj}}(\boldsymbol{u}) - \chi^G_{2 \cdot \mathbf{Adj}}(\boldsymbol{u})\right) t^4 + \dots \ . \tag{4.8.2}$$

The meaning of the plethystic logarithm is as follows.

The generator M of the reduced moduli space is of order 2 and transforms in the adjoint representation of G. There are relations at order 4 transforming

[11] See [120, 121] for a mathematical perspective on this type of varieties, independent of instantons.

in the representation $\mathrm{Sym}^2\mathbf{Adj} - 2\cdot\mathbf{Adj}$, where the minus sign means that the irreducible representation $2\cdot\mathbf{Adj}$ is removed from the decomposition of $\mathrm{Sym}^2\mathbf{Adj}$. These are known as the *Joseph relations* [122] (see also [123]).

For the case of $G = SU(N)$, $\mathrm{Sym}^2\mathbf{Adj}$ decomposes as

$$\begin{aligned}\mathrm{Sym}^2\mathbf{Adj} &= \mathrm{Sym}^2[1,0,\dots,0,1] \\ &= [2,0,\dots,0,2] + [1,0,\dots,0,1] + [0,\dots,0] + [0,1,0,\dots,0,1,0]\ .\end{aligned} \tag{4.8.3}$$

Thus,

$$\mathrm{Sym}^2\mathbf{Adj} - 2\cdot\mathbf{Adj} = [1,0,\dots,0,1] + [0,\dots,0] + [0,1,0,\dots,0,1,0]\ . \tag{4.8.4}$$

In this case, the generator M of the reduced moduli space is an $N\times N$ traceless matrix, and the Joseph relations can be explicitly written as

$$M^{a_1}_{\ a_2}M^{a_2}_{\ a_3} = (M^2)^{a_1}_{\ a_3} = 0\ , \qquad \epsilon^{b_1\dots b_N}\epsilon_{a_1\dots a_N}M^{a_1}_{\ b_1}M^{a_2}_{\ b_2} = 0\ , \tag{4.8.5}$$

where the indices $a_1, a_2, \dots, a_N, b_1, \dots, b_N = 1, \dots, N$ are the fundamental indices of $SU(N)$. Note that the first relations, which indicate that M is a nilpotent matrix, transform in the representation $[1,0,\dots,0,1]+[0,\dots,0]$ of $SU(N)$. The second relations transform in the representation $[0,1,0,\dots,0,1,0]$ of $SU(N)$.

4.8.2 Two instantons

The generators of the reduced moduli space of two G instantons on $\mathbb{C}^2$ transform under the global symmetry $SU(2)\times G$ as stated in Table 4.5.

Order	Representation of $SU(2)\times G$
2	$[2;\ \mathbf{0}]+[0;\ \mathbf{Adj}]$
3	$[1;\ \mathbf{Adj}]$

Table 4.5: Generators of the reduced moduli space of two G instantons on $\mathbb{C}^2$ and how they transform under the global symmetry $SU(2)\times G$.

There is one relation at order 4 in the representation $[0;\mathbf{0}]$ of $SU(2)\times G$.

Explicitly, this relation can be written as

$$\det X + c\text{Tr}(M^2) = 0 \ , \tag{4.8.6}$$

where X and M are the generators at order 2 in the representation $[2; 0]$ and $[0; \mathbf{Adj}]$ of $SU(2) \times G$ respectively; the determinant corresponds to the $SU(2)$ group and Tr denotes the trace in the adjoint representation of G; the constant c depends on the group G.

There are also relations at order 5 in the representation $[1; \mathbf{Adj}]+[1; \text{Sym}^2\mathbf{Adj} - 2 \cdot \mathbf{Adj}]$, where the notation $\text{Sym}^2\mathbf{Adj} - 2 \cdot \mathbf{Adj}$ is as before. This result agrees with the plethystic logarithm of the expression (3.11) in [93].

4.8.3 Three instantons

The generators of the reduced moduli space of three G instantons on $\mathbb{C}^2$ transform under the global symmetry $SU(2) \times G$ as stated in Table 4.6.

Order	Representation of $SU(2) \times G$
2	$[2; \mathbf{0}]+ [0; \mathbf{Adj}]$
3	$[3; \mathbf{0}]+ [1; \mathbf{Adj}]$
4	$[2; \mathbf{Adj}]$

Table 4.6: Generators of the reduced moduli space of three G instantons on $\mathbb{C}^2$ and how they transform under the global symmetry $SU(2) \times G$.

There is a set of relations at order 5 in the representation $[1; \mathbf{0}]$ of $SU(2) \times G$. Explicitly, this relation can be written as

$$M_a G_a^\alpha = 0 \ , \tag{4.8.7}$$

where M_a are the generators of the moduli space at order 2 in the representation $[0; \mathbf{Adj}]$ and G_a^α are the generators at order 3 in the representation $[1; \mathbf{Adj}]$. Here $a = 1, \ldots, \dim\, G$ is an adjoint index of G and $\alpha = 1, 2$ is an $SU(2)$ fundamental index.

Analytical properties of Hilbert series for three instantons

As discussed around (2.4) of [71], the Hilbert series of three G instantons on $\mathbb{C}^2$ shares certain analytical properties with the third symmetric power of the

Hilbert series of one G instanton on $\mathbb{C}^2$, namely

$$\begin{aligned}&\lim_{x\to a}(1-t^2x^{-2})(1-t^3x^{-3})\widetilde{g}_{\mathrm{Sym}^3\mathcal{M}_{1,G}}(t;x;\boldsymbol{u})\\&=\lim_{x\to a}(1-t^2x^{-2})(1-t^3x^{-3})\widetilde{g}_{3,G}(t;x;\boldsymbol{u})\ ,\quad \text{with } a=\pm t,\ e^{\pm 2\pi i/3}t\ ,\end{aligned} \tag{4.8.8}$$

where a tilde denotes the Hilbert series of a *reduced* instanton moduli space, x is the fugacity of $SU(2)$, and $\boldsymbol{u}$ denote the fugacities of the group G, and the third symmetric power is given by

$$\begin{aligned}\widetilde{g}_{\mathrm{Sym}^3\mathcal{M}_{1,G}}(t,x,\boldsymbol{u})=\frac{1}{6}\Bigg[&\frac{1}{(1-tx^{\pm1})^2}\widetilde{g}_{1,G}(t,\boldsymbol{u})^3+3\frac{1}{1-t^2x^{\pm2}}\widetilde{g}_{1,G}(t,\boldsymbol{u})\widetilde{g}_{1,G}(t^2,\boldsymbol{u}^2)\\&+2\frac{1-tx^{\pm1}}{1-t^3x^{\pm3}}\widetilde{g}_{1,G}(t^3,\boldsymbol{u}^3)\Bigg]\ .\end{aligned} \tag{4.8.9}$$

Explicitly, (4.8.8) can be rewritten as follows:

$$\begin{aligned}\lim_{x\to t}(1-t^2x^{-2})(1-t^3x^{-3})\widetilde{g}_{3,G}(t;x;\boldsymbol{u})&=\frac{\widetilde{g}_{1,G}(t,\boldsymbol{u})^3}{(1-t^2)^2}\ ,\\ \lim_{x\to -t}(1-t^2x^{-2})(1-t^3x^{-3})\widetilde{g}_{3,G}(t;x;\boldsymbol{u})&=\frac{\widetilde{g}_{1,G}(t,\boldsymbol{u})\widetilde{g}_{1,G}(t^2,\boldsymbol{u}^2)}{1-t^4}\ ,\\ \lim_{x\to \omega t}(1-t^2x^{-2})(1-t^3x^{-3})\widetilde{g}_{3,G}(t;x;\boldsymbol{u})&=\frac{1-\omega t^2}{1-t^6}\widetilde{g}_{1,G}(t^3,\boldsymbol{u}^3)\ ,\quad \omega=e^{\pm\frac{2\pi i}{3}}\ .\end{aligned} \tag{4.8.10}$$

The properties (4.8.10) together with the fact that the numerator of the unrefined Hilbert series $\widetilde{g}_{3,G}(t;x=1;\boldsymbol{u}=\boldsymbol{1})$ is palindromic can be used to check our results on the Hilbert series of three instantons.

Let us demonstrate this for the case of 3 G_2 instantons. The numerator of the unrefined Hilbert series (4.4.18) is palindromic. In order to make use of (4.8.10), one needs to compute a refined Hilbert series at least with respect to x. To keep the presentation brief, let us report the result for 3 G_2 instantons up to order t^9:

$$\begin{aligned}&\widetilde{g}_{3,G_2}(t;x;\boldsymbol{u}=\boldsymbol{1})\\&=1+t^2\left(x^2+\frac{1}{x^2}+15\right)+t^3\left(x^3+\frac{1}{x^3}+15x+\frac{15}{x}\right)\\&\quad+t^4\left(x^4+\frac{1}{x^4}+29x^2+\frac{29}{x^2}+135\right)+t^5\left(x^5+\frac{1}{x^5}+30x^3+\frac{30}{x^3}+240x+\frac{240}{x}\right)\end{aligned}$$

$$
\begin{aligned}
&+ t^6\Big(2x^6 + \frac{2}{x^6} + 44x^4 + \frac{44}{x^4} + 437x^2 + \frac{437}{x^2} + 1102\Big) \\
&+ t^7\left(x^7 + \frac{1}{x^7} + 44x^5 + \frac{44}{x^5} + 542x^3 + \frac{542}{x^3} + 2292x + \frac{2292}{x}\right) \\
&+ t^8\left(2x^8 + \frac{2}{x^8} + 59x^6 + \frac{59}{x^6} + 739x^4 + \frac{739}{x^4} + 4232x^2 + \frac{4232}{x^2} + 7964\right) \\
&+ t^9\left(2x^9 + \frac{2}{x^9} + 59x^7 + \frac{59}{x^7} + 844x^5 + \frac{844}{x^5} + 5962x^3 + \frac{5962}{x^3} + 17057x + \frac{17057}{x}\right) \\
&+ \ldots \, , \qquad (4.8.11)
\end{aligned}
$$

and for 1 G_2 instanton we have

$$
\begin{aligned}
\widetilde{g}_{1,G_2}(t;\boldsymbol{u}=\boldsymbol{1}) &= \sum_{p=0}^{\infty} \mathrm{dim}_{G_2}[p,0] t^{2p} \\
&= 1 + 14t^2 + 77t^4 + 273t^6 + 748t^8 + 1729t^{10} + \ldots \, . \qquad (4.8.12)
\end{aligned}
$$

These can be substituted in (4.8.10) and the agreement on each equality can be obtained perturbatively up to order t^4.

4.8.4 Higher instanton numbers

Explicit computations reveal that the generators of the reduced moduli space of five G instantons on $\mathbb{C}^2$ transform under the global symmetry $SU(2) \times G$ as stated in Table 4.7.

Order	Representation of $SU(2) \times G$
2	[2; **0**]+ [0; **Adj**]
3	[3; **0**]+[1; **Adj**]
4	[4; **0**]+[2; **Adj**]
5	[5; **0**]+[3; **Adj**]
6	[4; **Adj**]

Table 4.7: Generators of the reduced moduli space of five G instantons on $\mathbb{C}^2$ and how they transform under the global symmetry $SU(2) \times G$.

4.8.5 Generators of the reduced instanton moduli spaces

The data gathered in the previous subsection leads us to conjecture that the reduced moduli space of k G instantons on $\mathbb{C}^2$ is generated by two sets of holomorphic functions transforming in:

1. representations $[p;\mathbf{0}]$ of $SU(2)\times G$ at order p, for all $2\le p\le k$;
2. representations $[p;\mathbf{Adj}]$ of $SU(2)\times G$ at order $p+2$, for all $0\le p\le k-1$.

These two sets of generators can be systematically understood from the Coulomb branch viewpoint, as we now explain.

The generators transforming in the representation $[p;\mathbf{0}]$ are all monopole operators. To describe them, it is useful to introduce a class of monopole operators that are obtained by embedding $U(k)$ monopoles into the $\prod_{i=0}^{r} U(ka_i^\vee)$ gauge group of the quiver. Let $M=\mathrm{diag}(m_1,m_2,\ldots,m_k)$ be a $U(k)$ magnetic charge and consider the monopole operators of magnetic charge

$$m_{-1}=0\ ,\qquad m_i=M\otimes\frac{a_i}{a_i^\vee}\,\mathbb{1}_{a_i^\vee}\ ,\quad i=0,\ldots,r\ ,\tag{4.8.13}$$

generalizing (4.3.9). The dimension of these monopole operators can be easily computed: the contributions of nodes and edges of the affine Dynkin diagram cancel out because the quiver is balanced, while the edge attached to the over-extended node yields $\Delta=\frac{1}{2}\sum_{i=1}^{k}|m_i|$.[12] Taking into account the charge under the topological symmetry group, the monopole operators (4.8.13) appear in the HS with weight $x^{\sum_i m_i}t^{\sum_i |m_i|}$.

Next, let

$$\sigma_{p,\ell}\equiv\mathrm{diag}(1^{p-\ell},(-1)^{\ell})\ ,\qquad \ell=0,1,\ldots,p\tag{4.8.14}$$

be a $p\times p$ diagonal matrix with entries equal to ±1, which may be thought of as a collection of spins $\pm\frac{1}{2}$ for an abstract $SU(2)$. This abstract $SU(2)$ is identified with the $SU(2)_x$ global symmetry of the instanton moduli space by specializing the matrix M in (4.8.13) to

$$M=\mathrm{diag}(\sigma_{p,\ell},0^{k-p})\tag{4.8.15}$$

up to Weyl reflections, where $p=1,2,\ldots,k$ so that the $p\times p$ matrix $\sigma_{p,\ell}$ fits in the $k\times k$ matrix M. The case $p=1$ gives the generators of the center of

[12] For instance, for F_4 we compute

$$\Delta=\tfrac{1}{2}\sum_i|m_i|-\tfrac{1}{2}\sum_{i,j}|m_i-m_j|(2+6+12+4)-\sum_{i<j}|m_i-m_j|(1+4+9+8+2)=\tfrac{1}{2}\sum_i|m_i|\ .$$

the instanton, that was discussed in (4.3.9). The cases $p = 2, \dots, k$ yield the generators of the reduced instanton moduli space in the representations $[p; \mathbf{0}]$ of $SU(2) \times G$. Indeed the monopole operators of magnetic charge (4.8.13), (4.8.15) appear in the HS with weights $x^{p-2\ell} t^p$. As $\ell = 0, 1, \dots, p$ at fixed p, they span the representation $[p; \mathbf{0}]$ of $SU(2) \times G$.

One can similarly identify the generators at order $p + 2$ transforming in the representation $[p; \mathbf{Adj}]$, where $p = 0, 1, \dots, k-1$. Let us first restrict to the positive roots $\boldsymbol{\alpha}$ of G, keeping all weights of $SU(2)$ representations. The generators are monopole operators of magnetic charges

$$m_{-1} = 0 \ , \qquad m_i = \mathrm{diag}\left(R_i^{(\boldsymbol{\alpha})}, \sigma_{p,\ell} \otimes \frac{a_i}{a_i^\vee} \mathbb{1}_{a_i^\vee}, 0^{\frac{a_i}{a_i^\vee}(k-1-p)} \right) \ , \quad i = 0, \dots, r \ , \tag{4.8.16}$$

where $R_i^{(\boldsymbol{\alpha})}$ is an $a_i^\vee \times a_i^\vee$ diagonal matrix whose elements are tabulated in Appendix A.2 for non-simply laced groups and can be found in [124] for simply-laced groups. $R_0^{(\boldsymbol{\alpha})}$ is always zero. Note that p necessarily runs from 0 to $k-1$. The contribution of $R_i^{(\boldsymbol{\alpha})}$ to the topological charge of the monopole operator reproduces the positive root $\boldsymbol{\alpha}$ of G, whereas $\sigma_{p,\ell}$ is responsible for the $SU(2)$ weight $p - 2\ell$ as above. For negative roots of G, $R_i^{(\boldsymbol{\alpha})}$ is replaced by its negative. For the Cartan elements of G, $R_i^{(\boldsymbol{\alpha})}$ are set to zero and the monopole operators are dressed by the classical field at the i-th node of the Dynkin diagram of G.

4.8.6 Monopole operators and global symmetries

The global symmetry group acting on the Coulomb branch of a $3d$ $\mathcal{N} = 4$ superconformal field theory takes the form $SU(2)_C \times G_J$. $SU(2)_C$ is an R-symmetry which rotates the triplet of complex structures of the hyperKähler manifold. The holomorphic functions with respect to a fixed complex structure that are counted by the HS are highest weights of $SU(2)_C$ representations. The associated fugacity is t. On the other hand, G_J commutes with the supercharges. A subgroup of G_J is manifest in the UV Lagrangian of the gauge theory: it consists of the topological symmetry group which is generated by the topologically conserved currents $J_i = * \operatorname{Tr} F_i$, where F_i are the field strength 2-forms of the i-th $U(N_i)$ gauge group. More generally, the topological symmetry is the center $\mathcal{Z}(\mathcal{G}^\vee)$ of the dual of the gauge group. The topological symmetry group, which is $U(1)^{r+1}$ for the theories here considered, acts on monopole operators. The associated fugacities are z_i.

At the IR fixed point of a three-dimensional gauge theory, the manifest topological symmetry group can enhance to a non-abelian symmetry group G_J. The conserved currents of the hidden symmetry are monopole operators. In a $3d$ $\mathcal{N}=4$ superconformal field theory, conserved currents sit in the same multiplet as dimension $\Delta = 1$ chiral operators [81] (see also [116, 124]). Thus the non-R global symmetry can be deduced from the Hilbert series: the order t^2 term gives the adjoint representation of G_J.

Applying this strategy to the quivers whose Coulomb branches are the moduli spaces of instantons, one can see that the global non-R symmetry enhances from $U(1)^{r+1}$ to $G_J = SU(2)\times G$ for $k=1$ instanton and to $G_J = SU(2)\times SU(2)\times G$ for $k>1$ instantons, as we now explain.

The maximal torus $U(1)^r$ of G is the manifest topological symmetry associated to the nodes of the Dynkin diagram of G in the quiver. The $\Delta = 1$ states counted by the Hilbert series are $\operatorname{Tr}\Phi_i$, $i=1,\dots,r$, where Φ_i is the adjoint chiral multiplet in the $\mathcal{N}=4$ vector multiplet of the i-th gauge group. The global symmetry enhancement is due to dimension 1 monopole operators in one-to-one correspondence with the roots of G. For positive roots $\boldsymbol{\alpha}$, these dimension 1 monopole operators take the form

$$m_{-1} = 0\ , \quad m_i = \operatorname{diag}\left(R_i^{(\boldsymbol{\alpha})}, 0^{\frac{a_i}{a_i^\vee}(k-1)}\right)\ , \quad i=0,\dots,r\ , \tag{4.8.17}$$

where $R_i^{(\boldsymbol{\alpha})}$ is an $a_i^\vee \times a_i^\vee$ diagonal matrix whose elements are tabulated in Appendix A.2 for non-simply laced groups and can be found in [124] for simply-laced groups. Note that $R_0^{(\boldsymbol{\alpha})}$ is always zero. The topological charge $\operatorname{Tr} R_i^{(\boldsymbol{\alpha})}$ of the monopole operator is the component of the positive root $\boldsymbol{\alpha}$ of G along the i-th simple root of G. For instance, for $G = SU(N+1)$, the positive roots are $\boldsymbol{\alpha}_{ij} = \sum_{p=i}^{j-1} \boldsymbol{\gamma}_p$, with $\boldsymbol{\gamma}_p$ the simple roots and $1\le i<j\le N$. Then $R_p^{(\boldsymbol{\alpha}_{ij})} = (1)$ if $i \le p < j$ and $R_p^{(\boldsymbol{\alpha}_{ij})} = (0)$ otherwise. The negative roots of G are obtained by flipping sign to the magnetic charges (4.8.17).

Next we explain the $SU(2)$ groups. The $SU(2)$ symmetry that is present for any instanton number k acts on the two complex variables parametrizing the center of the instanton configuration, namely the monopole operators of magnetic charges ± 1 times (4.3.9). The squares of those monopole generators, corresponding to magnetic charges ± 2 times (4.3.9), provide the roots of $SU(2)$; the classical field $\sum_{i=0}^r \frac{a_i}{a_i^\vee} \operatorname{Tr}\Phi_i$ associated to the remaining $U(1)$ topological symmetry provides the Cartan element of $SU(2)$.

For instanton number $k>1$ there is an additional $SU(2)$ which acts on the reduced moduli space of instantons. The adjoint representation of this additional

$SU(2)$ is spanned by monopole operators of the form (4.8.13), (4.8.15), where $p = 2$ in (4.8.14).

Note that the characters of the adjoint representations of the two $SU(2)$ factors that appear in the HS at order t^2 involve the same fugacity x for the diagonal $SU(2)$ defined in (4.3.10). Since the symmetry is $SU(2) \times SU(2)$, it should be possible to further refine the Hilbert series of the instanton moduli space and distinguish the two $SU(2)$ factors. However, for one of the $SU(2)$ groups, not even the Cartan subalgebra is manifest, but rather it is generated by a monopole operator. This difficulty can be circumvented because the center of the instanton is factored in the Hilbert series and is represented by a free twisted hypermultiplet. One can always *a posteriori* assign different fugacities to the two $SU(2)$ factors (cf. (3.3) of [90]), modifying (4.2.2) as follows:

$$g_{k,G}(t, x_1, x_2, \boldsymbol{u}) = \frac{1}{(1 - t x_1)(1 - t x_1^{-1})} \widetilde{g}_{k,G}(t, x_2, \boldsymbol{u}) \ . \qquad (4.8.18)$$

Chapter 5

Instanton Operators and the Higgs Branch at Infinite Coupling

5.1 Introduction

In Chapter 2, the interesting features of minimally supersymmetric five dimensional supersymmetric gauge theories were presented. Let us recall that, despite looking nonrenormalisable from the Lagrangian perspective, a number of such field theories can be considered as flowing from certain non-trivial superconformal field theories in the ultraviolet (UV) [31, 32, 33, 34]. The UV fixed points at infinite gauge coupling may furthermore exhibit an enhancement of the global symmetry. In particular, in the seminal work [31], it was pointed out that the UV fixed point of 5d $\mathcal{N} = 1$ $SU(2)$ gauge theory with $N_f \leq 7$ flavours exhibits E_{N_f+1} flavour symmetry, which enhances from the global symmetry $SO(2N_f) \times U(1)$ apparent in the Lagrangian at finite coupling.

Since then a large class of five-dimensional supersymmetric field theories have been constructed using webs of five-branes [36, 37, 38] and the enhancement of the global symmetry of these theories has been studied using various approaches, including superconformal indices [42, 125, 44, 126, 127, 46, 47, 48, 128, 50, 129, 130], Nekrasov partition functions and (refined) topological string partition functions [41, 43, 131, 45, 132, 133, 134, 135, 136].

As previously explained, the source of enhancement can be ascribed to instanton-like particles that are charged under the the $U(1)_I$ global symmetry associated with the topological conserved current $J = \frac{1}{8\pi^2} \operatorname{Tr} *(F \wedge F)$. In the

UV superconformal field theory, the instanton particles are created by local operators known as *instanton operators*, that insert a topological defect at a spacetime point and impose certain singular boundary conditions on the fields [51, 52, 49]. These operators play an important role in enhancing the global symmetry of the theory. For 5d $\mathcal{N} = 1$ field theories at infinite coupling, it was argued that instanton operators with charge $I = \pm 1$, form a multiplet under the supersymmetry and flavour symmetry [49]. In 5d $\mathcal{N} = 2$ Yang-Mills theory with simply laced gauge group, it is believed that the instanton operators constitute the Kaluza-Klein tower that enhances the Poincaré symmetry and provides the UV completion by uplifting this five-dimensional theory to the $6d$ $\mathcal{N} = (2, 0)$ CFT [137, 138, 51].

Standard lore says that the Higgs branch of theories with 8 supercharges in dimensions 3 to 6 are classically exact, and do not receive quantum corrections. In 5 dimensions, this statement turns out to be imprecise, and should be corrected. In fact, one of the main points in this part of the thesis, is that there are three different regimes, given by 0, finite, and infinite gauge coupling. The hypermultiplet moduli space, which we always refer to as the Higgs branch, turns out to be different in each of these regimes, and hence our analysis corrects and sharpens the standard lore. The main goal of this chapter is to understand how, at infinite coupling, instanton operators correct the chiral ring relations satisfied by the classical fields at finite coupling.

In order to perform such an analysis we start from the known Higgs branch at infinite coupling and write the Hilbert series of such a moduli space for various 5d $\mathcal{N} = 1$ theories. We mostly focus on the $SU(2)$ gauge theories with N_f flavours, for which string theory arguments show that the Higgs branch at infinite coupling is the reduced moduli space of one E_{N_f+1} instanton on $\mathbb{C}^2$ [31, 37]. The Hilbert series counts the holomorphic functions that parametrise the Higgs branch, graded with respect to the Cartan subalgebra of the (enhanced) flavour symmetry and the highest weight of the $SU(2)$ R-symmetry of the theory:

$$H(t, y) = \mathrm{Tr}_{\mathcal{H}}\left(t^{2R} y_A^{H_A}\right) , \tag{5.1.1}$$

where $\mathcal{H}$ is the Hilbert space of chiral operators of the SCFT, R the $SU(2)_R$ isospin and H_A the Cartan generators of the enhanced global symmetry.

Such a Hilbert series can then be expressed in terms of the global symmetry of the theory at finite coupling — the latter is a subgroup of the enhanced symmetry at infinite coupling:

$$H(t, y(x, q)) = \mathrm{Tr}_{\mathcal{H}}\left(t^{2R} q^I x_a^{H_a}\right) , \tag{5.1.2}$$

where I is the topological charge and H_a the Cartan generators of the $SO(2N_f)$ flavour symmetry. This decomposition allows us to extract the contributions of the classical fields and the instanton operators to the Higgs branch chiral ring and explicitly write down the relations they satisfy.

This chapter is organised as follows. In section 5.2 we study the Higgs branch of $SU(2)$ gauge theories with $N_f \leq 7$ flavours, spell out the relations in the chiral ring in terms of mesons, glueball and instanton operators, and discuss the dressing of instanton operators. We generalise the analysis to pure $USp(2k)$ Yang-Mills theories with an antisymmetric hypermultiplet in sections 5.3 and 5.4, and to pure $SU(N)$ Yang-Mills in section 5.5. Several technical results are relegated to Appendix B

5.2 $SU(2)$ with N_f flavours: one E_{N_f+1} instanton on $\mathbb{C}^2$

The dynamics of 5d $\mathcal{N} = 1$ $SU(2)$ gauge theory with $N_f \leq 7$ flavours was studied in detail in [31]. In there it was argued that, despite being power counting non-renormalisable, these theories possess strongly interacting UV fixed points. Moreover a classification was proposed where the global symmetry, which at finite coupling is $SO(2N_f) \times U(1)_I$, with $U(1)_I$ the global symmetry associated with a topologically conserved current, enhances to E_{N_f+1}, where $\widetilde{E}_1 = U(1), E_1 = SU(2)$, $E_2 = SU(2) \times U(1)$, $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = SO(10)$ and E_6, E_7, E_8 are the usual exceptional symmetries.

The analysis presented in this chapter focuses on how the Higgs branch of these 5d theories changes along the RG flow. In particular we take care in distinguishing three different regimes for these theories, the operators that contribute to the chiral ring on the Higgs branch[1] and the defining equations that these operators satisfy:

[1]Even though we discuss theories with minimal $\mathcal{N} = 1$ supersymmetry (that is 8 Poincaré supercharges) in 5 dimensions, we are interested in the chiral ring as defined in terms of a subsuperalgebra with 4 supercharges, and the Higgs branch as a complex algebraic variety. We therefore use 4d $\mathcal{N} = 1$ notation and terminology throughout this chapter. Even though this formalism is not consistent with Poincaré supersymmetry in five dimensions, it is necessary to discuss chiral operators and holomorphic functions on the Higgs branch.

- In the classical regime, where fermions are neglected, these 5d theories have the usual Higgs branch which is just given by $\widetilde{\mathcal{M}}_{1,SO(2N_f)}$, the centred (or *reduced*) moduli space of one $SO(2N_f)$ instanton. The gauge invariant operators that generate this space are mesons M^{ab}, constructed out of chiral matter superfields in the bifundamental of the $SU(2)$ gauge group and $SO(2N_f)$ flavour group. The relations that these generators satisfy on the moduli space can be extrapolated from its description as the minimal nilpotent orbit of $SO(2N_f)$ [117]. They are the usual Joseph relations [122] and their transformation properties can be read off from the decomposition of the second symmetric product of the adjoint, the representation in which the generator transforms. Let $V(\theta)$ denote the adjoint representation. The decomposition

$$\mathrm{Sym}^2 V(\theta) = V(2\theta) + \mathfrak{I}_2 \tag{5.2.1}$$

prescribes that the relations transform in the representation $\mathfrak{I}_2$.

For $SO(2N_f)$

$$\mathfrak{I}_2 = \mathrm{Sym}^2[1,0,...] + \wedge^4[1,0,...] \ . \tag{5.2.2}$$

We can construct these representations from the adjoint mesons M^{ab} as follows. Take M to be an antisymmetric $2N_f \times 2N_f$ matrix, $M^{ab} = -M^{ba}$, $a, b = 1, .., 2N_f$. Then the two terms of (5.2.2) correspond respectively to:

$$M^2 = 0 \tag{5.2.3}$$
$$M^{[ab}M^{cd]} = 0 \ . \tag{5.2.4}$$

We call the last equation the rank 2 condition.

- When the coupling is finite, one needs to take into account the contribution from the gaugino sector. In particular, the glueball superfield S, which is a chiral superfield bilinear in the gaugino superfield $\mathcal{W}$, is now no longer suppressed and will *de jure* appear in the chiral ring. This operator satisfies a classical relation in the chiral ring as in four dimensions [64], namely

$$S^2 = 0 \ , \tag{5.2.5}$$

hence S is the only extra operator that one needs to consider at finite coupling. At first sight it might seem counterintuitive that S contributes to the Higgs branch as it is a bilinear in the vector multiplet. In fact in 5d the Higgs branch is the *only* complex branch of the full moduli space. As such, any chiral operator, and in particular the glueball superfield S, belongs to the class of Higgs operators. This will become even clearer later, when we recover the finite coupling Higgs branch from the one at infinite coupling.

Geometrically we interpret the operator S as generating a 2-point space, which by a slight abuse of notation we denote by $\mathbb{Z}_2$. Algebraically the Hilbert series for this space is simply written as

$$HS(\mathbb{Z}_2; t) = 1 + t^2 \tag{5.2.6}$$

where 1 signifies the identity operator and the t^2 term is associated to the quadratic operator S. The fugacity t grades operators by their $SU(2)_R$ representation and the normalisation is chosen so that the power is twice the isospin. The meson M^{ab} and the glueball superfield S obey the chiral ring relation [64, 139]

$$SM^{ab} = 0 \ . \tag{5.2.7}$$

This signifies that the spaces $\widetilde{\mathcal{M}}_{1,SO(2N_f)}$ and $\mathbb{Z}_2$ intersect only at the origin.

From an algebraic perspective, when two moduli spaces X and Y intersect, the Hilbert series of their union is given by the surgery formula

$$H_{X\cup Y} = H_X + H_Y - H_{X\cap Y} \ , \tag{5.2.8}$$

where the subtraction is done to avoid double counting [140]. Thus, when $\mathbb{Z}_2$ is glued to $\widetilde{\mathcal{M}}_{1,SO(2N_f)}$, the net effect on the Hilbert series is simply that of adding a t^2 to the Hilbert series of $\widetilde{\mathcal{M}}_{1,SO(2N_f)}$.

The plethystic logarithm of this newly obtained expression is interesting: it shows that at order t^4 there are two extra relations compared to the classical regime, one transforming in the singlet and one transforming in the adjoint of $SO(2N_f)$. The singlet relation is (5.2.5). For the adjoint relation the only possible extra operator that one can construct in such a representation is SM^{ab}. The adjoint relation is then precisely (5.2.7).

- At infinite coupling, the moduli space is a different space altogether. Instanton operators, carrying charge under $U(1)_I$, contribute to the chiral ring and are responsible for prompting symmetry enhancement: the Higgs branch in this regime becomes isomorphic to the reduced moduli space $\widetilde{\mathcal{M}}_{1,E_{N_f+1}}$ of one E_{N_f+1} instanton on $\mathbb{C}^2$ [31]. In order for this to happen a crucial event on the chiral ring takes place: instanton and anti-instanton operators I and $\widetilde{I}$ of $U(1)_I$ charge ± 1 correct the relation (5.2.5).[2]

  This is the most dramatic dynamical mechanism happening at infinite coupling: the operator S is no longer a nilpotent bilinear in the vector multiplet and it becomes, for all intents and purposes, a chiral bosonic operator on the Higgs branch. The contribution of S to the chiral ring will no longer amount to (5.2.6), but instead an infinite tower of operators will appear generating a factor $(1-t^2)^{-1}$ in the Hilbert series.

The purpose of the work presented in this chapter is to explore these statements quantitatively for known cases of UV-IR pairs of theories. We do this as follows. We start from the UV theory at infinite gauge coupling, which has E_{N_f+1} symmetry acting on the hypermultiplet moduli space. As soon as the dimensionful gauge coupling becomes finite, a term is added to the scalar potential which is proportional to the norm squared of the moment maps of the broken symmetries in the breaking $E_{N_f+1} \to SO(2N_f) \times U(1)_I$. Consequently, the broken moment maps must vanish on the Higgs branch of the theory at finite coupling. In terms of the chiral ring, this sets to zero the instanton operators I and $\widetilde{I}$.[3]

Computationally, one starts with the Hilbert series of the reduced one E_{N_f+1} instanton moduli space written in terms of representations of E_{N_f+1} [69] and decomposes them into representations of $SO(2N_f) \times U(1)_I$. For all theories of our interest, the Hilbert series after this decomposition admits a very simple expression in terms of the highest weight generating function [74]. This allows us to analyse the generators of the moduli space in terms of instanton operators and classical fields, and in many cases the relations between such generators are sufficiently simple to be written down explicitly.

[2] We call the instanton operator $\widetilde{I}$ of topological charge -1 "anti-instanton operator", even though it is mutually BPS with the positively charged instanton operator I.

[3] Although this argument applies to most of the theories we study here, it is in general not useful for theories where instanton operators have $SU(2)_R$ spin higher than 1, *e.g.* as in section 5.5.

5.2.1 E_0

The E_0 theory is the trivial case. There is no hypermultiplet moduli space. Consequently the Hilbert series for this theory is just given by 1, corresponding to the identity operator. The theory has no RG flow. Its interest lies in it being the limiting case of all the theories we consider in this section since none of the operators $(M,S,I,\widetilde{I})$ makes an appearance.

5.2.2 $N_f = 0$

A pure $SU(2)$ SYM theory with $\mathcal{N} = 1$ supersymmetry in 5d can be obtained by flowing from two UV fixed points which have different global symmetry. The existence of these two theories is dictated by a discrete θ parameter taking value in $\pi_4(Sp(1)) = \mathbb{Z}_2$ [33]. For the non-trivial element the global symmetry at infinite coupling is $\widetilde{E}_1 = U(1)$ whilst for the identity element the global symmetry is $E_1 = SU(2)$.

The $\widetilde{E}_1$ theory

For the theory with $\theta = \pi$ no enhancement of the global symmetry occurs: the global symmetry at finite and infinite coupling is the instanton charge symmetry $U(1)_I$. Here instanton operators are absent and the generator of the moduli space is just S obeying $S^2 = 0$, both at infinite and finite coupling. The moduli space generated by this operator is simply $\mathbb{Z}_2$. Classically the moduli space is trivial.

The E_1 theory

For the theory associated to the trivial element of the $\mathbb{Z}_2$ valued θ parameter the $U(1)_I$ topological symmetry is enhanced to $SU(2)$ by instanton operators at infinite coupling. In this regime the Higgs branch of the theory is isomorphic to the reduced moduli space of one-$SU(2)$ instanton $\widetilde{\mathcal{M}}_{1,SU(2)}$, which is the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. This theory is the prototypical example of the class we study. Since there is no flavour symmetry, we can understand the three regimes by means of simple physical arguments.

As we flow away from the UV fixed point, the Higgs branch is lifted and its only remnant is a discrete $\mathbb{Z}_2$ space generated by S. Classically, even this contribution can be neglected and the Higgs branch is completely absent. This is a remarkable effect whereby from no Higgs branch in the classical regime a full Higgs branch opens up at infinite coupling.

Algebraically we start from the Hilbert series for $\mathbb{C}^2/\mathbb{Z}_2$ and decompose it in representations of $U(1)_I$ so that we can identify the contribution from instanton operators, as well as the finite coupling chiral operators, and their relations.

The Hilbert series for $\mathbb{C}^2/\mathbb{Z}_2$ can be written as

$$H[\widetilde{\mathcal{M}}_{1,SU(2)}](t;x) = \sum_{n=0}^{\infty}[2n]_x t^{2n} = \frac{1-t^4}{(1-t^2x^2)(1-t^2)(1-t^2x^{-2})} \ , \qquad (5.2.9)$$

where t is the fugacity for the $SU(2)_R$ symmetry, x is the fugacity for the $SU(2)$ global symmetry acting on $\mathbb{C}^2/\mathbb{Z}_2$, and $[2n]_x$ stands for the character, as a function of x, of the representation of $SU(2)$ with such a Dynkin label. Identifying the Cartan subalgebra of the $SU(2)$ symmetry with $U(1)_I$, we obtain

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{1,SU(2)}](t;q^{1/2}) &= \frac{1-t^4}{(1-t^2q)(1-t^2)(1-t^2q^{-1})} \\ &= \frac{1}{1-t^2}\sum_{j=-\infty}^{\infty} t^{2|j|}q^j \ . \end{aligned} \qquad (5.2.10)$$

The generators and their relations

Eq. (5.2.10) has a natural interpretation in terms of operators at infinite coupling:

- Each term in the sum $t^{2|j|}q^j$ corresponds to an instanton operator $I_{+|j|}$ for $j>0$ and an anti-instanton operator $I_{-|j|}$ for $j<0$ that is the highest weight state of the $SU(2)_R$ representation with highest weight $2|j|$.[4] q is the fugacity for the instanton number $U(1)_I$. The plethystic logarithm of the Hilbert series shows that the instanton operator $I_{+|j|}$ is generated by the charge 1 operator $I_{+1} \equiv I$ through the relation $I_{+|j|} = (I)^j$. Similarly $I_{-|j|} = (\widetilde{I})^j$ where $\widetilde{I} \equiv I_{-1}$.

- The tower of operators generated by S can be identified with the factor $(1-t^2)^{-1}$. This enhancement in the number of operators constructed from powers of S is crucial: at infinite coupling S is a full-on operator on the Higgs branch and, together with the instanton and anti-instanton operators I, $\widetilde{I}$, forms a triplet of the $SU(2)$ that generates $\mathbb{C}^2/\mathbb{Z}_2$.

[4]Notice how the $SU(2)_R$ spin of an instanton operator of charge $\pm j$ is $|j|$. Whilst we can easily extract the $SU(2)_R$ spin as a function of instanton number, it is not clear how to do so for the representation under the global symmetry, as will be seen for the cases with higher number of flavours.

From this form of the Hilbert series we can also give another interpretation to the Higgs branch at infinite coupling. Instanton operators on the Higgs branch in 5d $\mathcal{N}=1$ theories play a similar role to monopole operators in 3d $\mathcal{N}=4$ [30] and $\mathcal{N}=2$ theories [141, 142]: in this sense (5.2.10) can be interpreted as the space of *dressed* instanton operators, where the factor $\frac{1}{1-t^2}$ is the dressing from the operator S and it is freely generated.

The numerator in the rational function of (5.2.10) signifies a relation quadratic in the operators which can only be given by

$$S^2 = I\widetilde{I}\ , \tag{5.2.11}$$

the defining equation for $\mathbb{C}^2/\mathbb{Z}_2$.

At finite coupling, where $I, \widetilde{I} = 0$, we recover the known chiral ring relation (5.2.5), i.e. the nilpotency of the operator S. As we have explained, the only remnant of $\mathbb{C}^2/\mathbb{Z}_2$ is a residual $\mathbb{Z}_2$ generated precisely by S.

Classically, we can set $S=0$ and lift the Higgs branch entirely.

5.2.3 $N_f = 1$

For $N_f=1$ and $N_f=2$ the infinite coupling Higgs branch is the moduli space of one instanton for a product gauge group. In such cases the moduli space is given by the union of the one instanton moduli space for each factor. For the case of $N_f=1$, i.e $E_2 = SU(2)\times U(1)$, the Higgs branch at infinite coupling is thus the union of the one $SU(2)$ and the one $U(1)$ instanton moduli spaces.

For the $U(1)$ instanton moduli space, there are two possible ADHM constructions that one may consider: (1) $USp(2)$ gauge theory with one flavour, and (2) $U(1)$ gauge theory with one flavour. As analysed below, the Higgs branch of the former is $\mathbb{Z}_2$ whereas the Higgs branch of the latter is a point. A priori it might not be apparent which option is the correct one but consistency with the finite coupling regime points out that the right choice is the former. We provide an independent argument below.

Let us begin with the first option. The Higgs branch of the ADHM gauge theory given by $USp(2)$ with one flavour describes the moduli space of one $SO(2)$ instanton.[5] There is only one operator in the chiral ring, P, subject to a quadratic nilpotency relation, $P^2=0$. The moduli space of one $SO(2)$

[5] To be precise, the flavour symmetry of the quiver gauge theory is $O(2)$, not $SO(2)$. (We thank the referee for raising this point.) However the moduli space of instantons in question is insensitive to the difference between the two groups.

instanton is thus $\mathbb{Z}_2$.[6]

On the other hand, one may consider a $U(1)$ gauge theory with one flavour, whose Higgs branch is often referred to as "the moduli space of one $U(1)$ instanton". The gauge invariant quantity is $Q\widetilde{Q}$ but is set to zero by the F-terms. The moduli space is thus trivial: it consists of one point only rather than two.

The reduced moduli space $\widetilde{\mathcal{M}}_{1,E_2}$ of one E_2 instanton is thus either isomorphic to the space $\mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{Z}_2$ or to $\mathbb{C}^2/\mathbb{Z}_2 \cup \{1\}$, depending on which of the above options is correct.

With the first option, the Hilbert series of $\widetilde{\mathcal{M}}_{1,E_2}$ can be written using (5.2.8) as:

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{1,E_2}](t;x) &= H[\widetilde{\mathcal{M}}_{1,SU(2)}] + H[\mathbb{Z}_2] - 1 \\ &= \frac{1-t^4}{(1-x^2t^2)(1-t^2)(1-x^{-2}t^2)} + t^2 \end{aligned} \tag{5.2.12}$$

where $H[\mathbb{Z}_2] = 1 + t^2$ is generated by P.

With the second option, the Hilbert series of $\widetilde{\mathcal{M}}_{1,E_2}$ is

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{1,E_2}](t;x) &= H[\widetilde{\mathcal{M}}_{1,SU(2)}] \\ &= \frac{1-t^4}{(1-x^2t^2)(1-t^2)(1-x^{-2}t^2)} \ . \end{aligned} \tag{5.2.13}$$

The generator of the $\mathbb{C}^2/\mathbb{Z}_2$ factor is Φ^{ij}, $i = 1, 2$, with $\Phi^{ij} = \Phi^{ji}$ and it obeys the quadratic nilpotency:

$$\Phi^{ij}\epsilon_{jk}\Phi^{kl} = 0 \tag{5.2.14}$$

where ϵ_{ij} is defined by its antisymmetry property and $\epsilon_{12} = 1$.

The extra generator, P, is there only in the case of a union of $\mathbb{C}^2/\mathbb{Z}_2$ with a two point moduli space. In its presence, beside (5.2.14), two further relations hold:

$$\begin{aligned} P^2 &= 0 \\ P\Phi^{ij} &= 0 \end{aligned} \tag{5.2.15}$$

(5.2.14) is the usual Joseph relation for the $SU(2)$ minimal nilpotent orbit $\mathbb{C}^2/\mathbb{Z}_2$. The last equation encodes the fact that the two spaces, $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{Z}_2$,

[6] Note that as rings $\mathbb{C}[P]/\langle P^2\rangle \neq \mathbb{C}[P]/\langle P\rangle$.

only intersect at one point, the origin of the moduli space.

Let us proceed without making any assumption on whether $\widetilde{\mathcal{M}}_{1,E_2}$ is given by $\mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{Z}_2$ or $\mathbb{C}^2/\mathbb{Z}_2 \cup \{1\}$. In the next subsection, we show that consistency with the finite coupling result tells us that the correct choice is the former.

The generators and their relations

The theory at finite coupling has a Higgs branch which is isomorphic to the union of $\widetilde{\mathcal{M}}_{1,SO(2)}$ with $\mathbb{Z}_2$, the former generated by a meson, M, subject to a quadratic nilpotency and the latter by the glueball superfield S, itself quadratically nilpotent. The finite coupling chiral ring is thus defined by:

$$M^2 = S^2 = SM = 0 \tag{5.2.16}$$

where the last equation signifies that the two spaces, $\widetilde{\mathcal{M}}_{1,SO(2)}$ generated by M and $\mathbb{Z}_2$ generated by S, are orthogonal to each other and intersect only at the origin. Moreover since $\widetilde{\mathcal{M}}_{1,SO(2)} \cong \mathbb{Z}_2$, the Higgs branch at finite coupling is given by $\mathbb{Z}_2 \cup \mathbb{Z}_2$.

The goal is to reproduce the set of equations (5.2.16) from the ones at infinite coupling by setting the instanton operators appearing there to zero. This can be achieved as follows. Decompose the generators Φ^{ij} of $\widetilde{\mathcal{M}}_{1,E_2}$ by letting

$$\Phi^{11} = I \tag{5.2.17}$$

$$\Phi^{12} = M \tag{5.2.18}$$

$$\Phi^{22} = -\widetilde{I} \tag{5.2.19}$$

where M is the $SO(2)$ mesonic operator and $I, \widetilde{I}$ are the instanton and anti-instanton operators respectively. The relation (5.2.14) can be rewritten as

$$M^2 = I\widetilde{I} \; . \tag{5.2.20}$$

It is clear that, by setting the instanton operators to zero, only one of the three equations in (5.2.16) can be recovered for the finite coupling limit. However, if the extra operator P and the extra relations in (5.2.15) are also taken into account, the classical regime can be precisely recovered. To this avail, let P be decomposed as:

$$P = S - M \; , \tag{5.2.21}$$

i.e. a linear combination of the meson M and the glueball S. Then (5.2.14) and (5.2.15) together can be rewritten as:

$$M^2 = \widetilde{I}I \tag{5.2.22}$$
$$S^2 = \widetilde{I}I \tag{5.2.23}$$
$$SM = \widetilde{I}I \tag{5.2.24}$$
$$MI = SI \tag{5.2.25}$$
$$\widetilde{I}M = \widetilde{I}S\ . \tag{5.2.26}$$

This time, setting $I, \widetilde{I} = 0$, the finite coupling relations (5.2.16) are finally recovered.

In the classical regime, where we neglect the contribution from S, we recover the space $\mathbb{Z}_2$, the reduced moduli space of one $SO(2)$ instanton generated by M, such that $M^2 = 0$.

This is the required consistency that we mentioned above: $\widetilde{\mathcal{M}}_{1,E_2}$ is indeed $\mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{Z}_2$, the latter being given by the ADHM construction of $USp(2)$ with 1 flavour.

Let us provide a complementary argument based on symmetries that supports the identification of $\widetilde{\mathcal{M}}_{1,E_2}$ with $\mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{Z}_2$. The ADHM construction for $U(1)$ with N_f flavours provides the moduli space of $U(N_f)/U(1)$ instantons, which for $N_f = 1$ corresponds to an empty symmetry group and thus a trivial moduli space. Furthermore, in the presence of a flavour symmetry, an $SU(2)_R$ spin-1 operator is a necessary requirement for the existence of a linear hypermultiplet containing the conserved current. For a $U(1)$ gauge theory with 1 flavour, there is no flavour symmetry and hence no associated generator. Identifying $\widetilde{\mathcal{M}}_{1,E_2}$ with $\mathbb{C}^2/\mathbb{Z}_2 \cup \{1\}$, there would be only three generators transforming in the adjoint representation of $SU(2)$ associated with $\mathbb{C}^2/\mathbb{Z}_2$ but no extra generator associated with the aforementioned $U(1)$ symmetry, as in (5.2.13). On the other hand, for a $USp(2)$ gauge theory with 1 flavour, there is an $SO(2) \cong U(1)$ flavour symmetry; hence there is a generator at order t^2 associated with this symmetry. We see that only when we identify $\widetilde{\mathcal{M}}_{1,E_2}$ with $\mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{Z}_2$ there are four generators transforming in the adjoint representation of the global symmetry $SU(2) \times U(1) \cong E_2$ as one can see explicitly in (5.2.12).

Expansion in the instanton fugacity

It is instructive to rewrite (5.2.12) as an expansion in q, the $U(1)_I$ fugacity. Replacing x, the fugacity for $SU(2)$, by $q^{1/2}$ we have that:

$$H[\widetilde{\mathcal{M}}_{1,E_2}](t; y, q^{1/2}) = \frac{1}{(1-t^2)} \sum_{n=-\infty}^{\infty} q^n t^{|2n|} + t^2 \ . \tag{5.2.27}$$

Hence a bare instanton operator with $U(1)_I$ charge n is the highest weight state of the spin $|n|$ representation of the $SU(2)_R$ symmetry. For $n \neq 0$, the tower of states originating from the glueball $(1-t^2)^{-1}$, i.e the space $\mathbb{C}$, acts as a dressing for the instanton operators. For $n = 0$, the dressing is a different space, due to the presence of an extra piece of the moduli space unaffected by instantons. It is in fact the space generated by S and M, subject to the relations $SM = 0$ and $M^2 = 0$, i.e $\mathbb{C} \cup \mathbb{Z}_2$.

5.2.4 $N_f = 2$

The reduced moduli space of one $E_3 = SU(3) \times SU(2)_A$ instanton[7] is isomorphic to the union of two hyperKähler cones, the reduced moduli space of one $SU(3)$ instanton, $\widetilde{\mathcal{M}}_{1,SU(3)}$, and the reduced moduli space of one $SU(2)_A$ instanton $\widetilde{\mathcal{M}}_{1,SU(2)_A}$, meeting at a point. As an algebraic variety it is generated by operators transforming in the reducible adjoint representation subject to the Joseph relations, which can be extracted from (5.2.1). The Hilbert series can again be written using the surgery formula (5.2.8) as

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{1,E_3}](t; \boldsymbol{x}, y) &= H[\widetilde{\mathcal{M}}_{1,SU(3)}](t; \boldsymbol{x}) + H[\widetilde{\mathcal{M}}_{1,SU(2)_A}](t; y) - 1 \\ &= \sum_{m_1=0}^{\infty} [m_1, m_1]_{\boldsymbol{x}}^{SU(3)} t^{2m_1} + \sum_{m_2=0}^{\infty} [2m_2]_y^{SU(2)_A} t^{2m_2} - 1 \ , \end{aligned} \tag{5.2.28}$$

where $\boldsymbol{x} = (x_1, x_2)$ are the fugacities for $SU(3)$ and y is the fugacity for $SU(2)_A$.

The $SU(3)$ factor of the enhanced global symmetry E_3 is broken to $SU(2)_B \times U(1)_I$ when one flows away from the fixed point. The $U(1)$ factor is identified with the topological symmetry $U(1)_I$, up to a normalisation of charges that is explained below. The $SU(2)_B$ factor instead combines with the $SU(2)_A$ factor in E_3, which acts as a spectator for the breaking, and together they form a

[7] The subscript A is used to differentiate from $SU(2)_B$ which is defined in the next paragraph.

global symmetry $SO(4)$. Hence, we decompose the representations of $SU(3)$ in (5.2.28), whilst keeping the representations of $SU(2)_A$, i.e we break:

$$SU(3) \times SU(2)_A \supset SU(2)_B \times SU(2)_A \times U(1)_I \cong SO(4) \times U(1)_I \quad (5.2.29)$$

A possible projection matrix that maps the weights of $SU(3)$ to $SU(2)_B \times U(1)$ is given by

$$P_{SU(3)\to SU(2)_B\times U(1)} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} , \quad (5.2.30)$$

Let $\boldsymbol{x} = (x_1, x_2)$ be the fugacities of $SU(3)$; z and w be those of $SU(2)_B$ and $U(1)$ respectively (the fugacity w for the $U(1)$ factor will be related to the fugacity q for $U(1)_I$ shortly). Under the action of this matrix, the weights of the fundamental representation of $SU(3)$ are mapped as follows:

$$(1,0) \to (0,2) , \qquad (-1,1) \to (1,-1) . \quad (5.2.31)$$

In other words, we have

$$x_1 = w^2 , \quad x_2 x_1^{-1} = z w^{-1} \qquad \Leftrightarrow \qquad x_1 = w^2 , \quad x_2 = z w . \quad (5.2.32)$$

The character of the fundamental representation of $SU(3)$ is mapped to that of $SU(2)_B \times U(1)$ as

$$[1,0] = x_1 + x_2 x_1^{-1} + x_2^{-1} = w^2 + z w^{-1} + z^{-1} w^{-1} = [0_2] + [1_{-1}] , \quad (5.2.33)$$

while the adjoint representation decomposes as

$$[1,1] \to [0_0] + [2_0] + [1_3] + [1_{-3}] . \quad (5.2.34)$$

The $U(1)$ charge is a multiple of 3 for states in the root lattice. To obtain integer instanton numbers $I \in \mathbb{Z}$, we set $w^3 = q$, where q is the fugacity for $U(1)_I$.

Under this map, the Hilbert series of the reduced moduli space of one $SU(3)$

instanton becomes

$$H[\widetilde{\mathcal{M}}_{1,SU(3)}](t;z,q) = \sum_{m=0}^{\infty}\sum_{n_1=0}^{m}\sum_{n_2=0}^{m}[n_1+n_2]_z q^{n_1-n_2}t^{2m} \ , \tag{5.2.35}$$

where z is the $SU(2)_B$ fugacity and q is the $U(1)_I$ fugacity.

The highest weight generating function[8] [74] associated to this Hilbert series is

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,SU(3)}](t;\mu,q) = \text{PE}\left[(1+\mu q+\mu q^{-1}+\mu^2)t^2-\mu^2t^4\right] \ , \tag{5.2.37}$$

where μ is the fugacity for the highest weight of $SU(2)_B$.

Thus, the highest weight generating function for (5.2.28) becomes

$$\begin{aligned}\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_3}](t;\mu,\nu,q) = &\ \text{PE}\left[(1+\mu q+\mu q^{-1}+\mu^2)t^2-\mu^2t^4\right] \\ &+\text{PE}[\nu^2t^2]-1 \ ,\end{aligned} \tag{5.2.38}$$

where μ and ν are the fugacities corresponding to the highest weights of $SO(4)\cong SU(2)_A\times SU(2)_B$.

The highest weight generating function (5.2.38) provides five dominant representations that generate the highest weight lattice in a simple way. The information can be read as follows. Inside the first PE we can identify the $SU(2)_R$ spin-1 generators: the singlet S, the instanton operator μq which we denote by $I\equiv I_1$, the anti-instanton operator μq^{-1} which we denote by $\widetilde{I}\equiv I_{-1}$, and the meson transforming in the adjoint of $SU(2)_B$, μ^2, which we denote by $T^{\alpha\beta}$ and is subject to the traceless condition $T^{\alpha\beta}\epsilon_{\alpha\beta}=0$. We also identify a relation quadratic in the generators and transforming in the adjoint representation of $SU(2)_B$, the term $-\mu^2t^4$. The second PE is the contribution from the spectator $SU(2)_A$, with the only representation ν^2, the inert meson that we denote by $\widetilde{T}^{\dot\alpha\dot\beta}$.

Eq. (5.2.38) is an expression that carries information about the representation theory more concisely than the Hilbert series and furthermore the lattice it encodes is a complete intersection. However in order to write the relations between the operators on the chiral ring explicitly, we consider what the Joseph

[8] The highest weight generating function for group of rank r is defined as follows:

$$\mathcal{G}(t;\mu_i) = \sum_{n_i,k} b_{n_1,\dots,n_r,k}\ \mu_1^{n_1}\dots\mu_r^{n_r}\ t^k \tag{5.2.36}$$

where $\{\mu_i\}_{i=1}^r$ are highest weight fugacities s.t. $[n_1,\dots,n_r]\mapsto\mu_1^{n_1}\dots\mu_r^{n_r}$ and $\{b_{n_1,\dots,n_r,k}\}$ are the series coefficients.

relations for $\widetilde{\mathcal{M}}_{1,E_3}$ imply.

The generators and their relations

For the $\widetilde{\mathcal{M}}_{1,E_3}$ case, the generators are $\Phi^i{}_j$, with $i = 1,2,3$ and $\Phi^i{}_i = 0$, transforming in the $[1,1;0]$ of $SU(3) \times SU(2)_A$, and $\widetilde{T}^{\dot\alpha\dot\beta}$ with $\widetilde{T}^{\dot\alpha\dot\beta}\epsilon_{\dot\alpha\dot\beta} = 0$, transforming in the $[0,0;2]$ of $SU(3) \times SU(2)_A$. The relations can be read off from (5.2.1):

$$\begin{aligned} \mathrm{Sym}^2([1,1;0]+[0,0;2]) = &\ \mathrm{Sym}^2[1,1;0] + \mathrm{Sym}^2[0,0;2] + [1,1;2] \quad \text{where} \\ \mathrm{Sym}^2([1,1;0]) = &\ [2,2;0]+[1,1;0]+[0,0;0] \\ \mathrm{Sym}^2([0,0;2]) = &\ [0,0;4]+[0,0;0] \end{aligned} \tag{5.2.39}$$

Hence the generator $\Phi^i{}_j$ obeys a quadratic relation transforming in the reducible representation $[1,1;0]+[0,0;0]$ whilst $\widetilde{T}^{\dot\alpha\dot\beta}$ obeys a singlet relation. This is to be expected, since the minimal nilpotent orbit of traceless 2×2 matrix is the subset of matrices with zero determinant. There is also a quadratic relation mixing $\Phi^i{}_j$ and $\widetilde{T}^{\dot\alpha\dot\beta}$ transforming in the $[1,1;2]$. We can write these relations as follows:[9]

$$\begin{aligned} &[1,1;0]+[0,0;0] \quad &:& \quad \Phi^i{}_j \Phi^j{}_k = 0 \\ &[0,0;0] &:& \quad \mathrm{Tr}(\widetilde{T}^2) \equiv \widetilde{T}^{\dot\alpha\dot\beta}\epsilon_{\dot\alpha\dot\sigma}\epsilon_{\dot\beta\dot\rho}\widetilde{T}^{\dot\rho\dot\sigma} = 0 \\ &[1,1;2] &:& \quad \Phi^i{}_j \widetilde{T}^{\dot\alpha\dot\beta} = 0 \ , \end{aligned} \tag{5.2.40}$$

where the indices of $\widetilde{T}$ are contracted by the epsilon tensor, *e.g.* $(\widetilde{T}^2)^{\dot\alpha\dot\sigma} = \widetilde{T}^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\rho}\widetilde{T}^{\dot\rho\dot\sigma}$.

The glueball operator, the instanton and anti-instanton operators and the meson are embedded into the generator $\Phi^i{}_j$ since this is the one transforming nontrivially under the $SU(3)$ factor that breaks into $SU(2)_B \times U(1)$. We choose

[9] For $\widetilde{T}$ a symmetric 2×2 matrix, *i.e.* $\widetilde{T}^{\dot\alpha\dot\beta}\epsilon_{\dot\alpha\dot\beta} = 0$, the following statements are equivalent: $\widetilde{T}^2 = 0$, $\det \widetilde{T} = 0$ and $\mathrm{Tr}\,\widetilde{T}^2 = 0$.

the following embedding:

$$\begin{aligned} &\Phi^\alpha{}_\beta = T^{\alpha\gamma}\epsilon_{\gamma\beta} - \tfrac{1}{2}S\delta^\alpha{}_\beta \qquad \alpha,\beta = 1,2 \\ &\Phi^\alpha{}_3 = I^\alpha \\ &\Phi^3{}_\alpha = \epsilon_{\alpha\beta}\widetilde{I}^\beta \\ &\Phi^3{}_3 = S \end{aligned} \tag{5.2.41}$$

where $T^{\alpha\beta}$ is a traceless 2×2 matrix, $T^{\alpha\beta}\epsilon_{\alpha\beta} = 0$. Notice that the choice of $\Phi^\alpha{}_\beta$ ensures that $\Phi^i{}_j$ is traceless since $\Phi^i{}_i = \Phi^\alpha{}_\alpha + \Phi^3{}_3 = 0$.

The aim is to decompose the relations in the first and third equations of (5.2.40). Under $SU(3)\times SU(2)_A \supset SU(2)_B \times U(1)_I \times SU(2)_A$ the representations decompose as

$$\begin{aligned} [1,1;0]+[0,0;0] \quad &\to \quad [2_0;0]+[1_1,0]+[1_{-1},0]+2[0_0,0] \\ [1,1;2] \quad &\to \quad [2_0;2]+[1_1;2]+[1_{-1};2]+[0_0;2]\ . \end{aligned} \tag{5.2.42}$$

Thus the relations in the first equation of (5.2.40) decompose into the five relations

$$\begin{aligned} [2_0;0]: \qquad & ST^{\alpha\beta} = -I^\alpha\widetilde{I}^\beta + \tfrac{1}{2}(I^\rho\epsilon_{\rho\sigma}\widetilde{I}^\sigma)\epsilon^{\alpha\beta} \\ [1_1,0]: \qquad & I^\beta\epsilon_{\beta\gamma}T^{\gamma\alpha} = \tfrac{1}{2}I^\alpha S \\ [1_{-1},0]: \qquad & \widetilde{I}^\beta\epsilon_{\beta\gamma}T^{\gamma\alpha} = -\tfrac{1}{2}\widetilde{I}^\alpha S \\ 2[0_0,0]: \qquad & S^2 = \widetilde{I}^\alpha\epsilon_{\alpha\beta}I^\beta = 2\,\mathrm{Tr}(T^2)\ . \end{aligned} \tag{5.2.43}$$

The relations in the second line of (5.2.42) can be explicitly written as:

$$\begin{aligned} [2_0;2]: \qquad & T^{\alpha\beta}\widetilde{T}^{\dot\alpha\dot\beta} = 0 \\ [1_1;2]: \qquad & I^\alpha\widetilde{T}^{\dot\alpha\dot\beta} = 0 \\ [1_{-1};2]: \qquad & \widetilde{I}^\alpha\widetilde{T}^{\dot\alpha\dot\beta} = 0 \\ [0_0;2]: \qquad & S\widetilde{T}^{\dot\alpha\dot\beta} = 0\ . \end{aligned} \tag{5.2.44}$$

Recall also from (5.2.40) that

$$[0_0;0]: \quad \mathrm{Tr}(\widetilde{T}^2) = 0 \ . \tag{5.2.45}$$

In total there are thus 10 equations, namely (5.2.43), (5.2.44) and (5.2.45).[10]

The finite coupling result that S be nilpotent is obtained by virtue of the last equation of (5.2.43) when we set $I, \widetilde{I} = 0$. Consequently we also restore the condition $\mathrm{Tr}(T^2) = 0$, which, for a traceless 2×2 matrix, is equivalent to $T^2 = 0$, the classical relation. Moreover (5.2.7) is also recovered.

Another approach to see these 10 relations between the operators at infinite coupling is to rewrite (5.2.28) in terms of characters of representations of $SO(4) \times U(1)$ and compute its plethystic logarithm. For reference, we present such a Hilbert series up to order t^4 as follows:

$$\begin{aligned} H[E_3](t; x_1, x_2, q) = 1 + \Big(1 + [2,0] + [0,2] + (q + q^{-1})[1,0]\Big)t^2 + \\ + \Big(1 + [2,0] + [4,0] + [0,4] + (q+q^{-1})([1,0] + [3,0]) + (q^2 + q^{-2})[2,0]\Big)t^4 + \dots \ . \end{aligned} \tag{5.2.46}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} \mathrm{PL}\left[H[E_3](t; x_1, x_2, q)\right] = \Big(1 + [2,0] + [0,2] + (q + q^{-1})[1,0]\Big)t^2 + \\ - \Big(3 + [2,0] + [0,2] + [2,2] + (q+q^{-1})([1,2] + [1,0])\Big)t^4 + \dots \ . \end{aligned} \tag{5.2.47}$$

Indeed, the 10 relations listed in (5.2.43), (5.2.44) and (5.2.45) are in correspondence with the terms at order t^4 in (5.2.47). We emphasise here that the computation of the plethystic logarithm provides an efficient way to write down the relations that are crucial to describe the moduli space. This method is applied for the cases of higher N_f in subsequent sections.

We can rewrite these relations in terms of a 4×4 adjoint matrix M^{ab}, with

[10] Notice that the meson $\widetilde{T}^{\dot\alpha\dot\beta}$, the generator for the spectator $SU(2)_A$, is made up of the same fundamental fields (quarks) as the meson $T^{\alpha\beta}$. Before considering gauge invariant combinations, the quarks $Q^{\alpha\dot\alpha}{}_{\mathfrak{a}}$, with $\alpha, \dot\alpha = 1, 2$ and $\mathfrak{a} = 1, 2$, transform in the vector representation of the global symmetry $SO(4) \cong SU(2)_A \times SU(2)_B$ and in the fundamental representation of the *gauge* group $SU(2)$. Out of these quarks the following gauge invariant mesons can be constructed: $T^{\alpha\beta} = Q^{\alpha\dot\alpha}{}_{\mathfrak{a}} Q^{\beta\dot\beta}{}_{\mathfrak{b}} \epsilon^{\mathfrak{a}\mathfrak{b}} \epsilon_{\dot\alpha\dot\beta}$ and $\widetilde{T}^{\dot\alpha\dot\beta} = Q^{\alpha\dot\alpha}{}_{\mathfrak{a}} Q^{\beta\dot\beta}{}_{\mathfrak{b}} \epsilon^{\mathfrak{a}\mathfrak{b}} \epsilon^{\alpha\beta}$. The difference between these two mesons is in the relations they satisfy at infinite coupling, one being quantum corrected whilst the other being unaffected: $\mathrm{Tr}(\widetilde{T}^2) = 0$ vs $2\,\mathrm{Tr}(T^2) = S^2 = I \cdot \widetilde{I}$.

$a, b, c, d = 1, \ldots, 4$ vector indices of $SO(4)$, such that

$$M^{ab} = -M^{ba} \ , \tag{5.2.48}$$

as follows:

$$[2,2]+[0,0]: \quad M^{ab}M^{bc} = (\epsilon_{\alpha\beta}I^{\alpha}\widetilde{I}^{\beta})\delta^{ac} \tag{5.2.49}$$
$$[0,0]: \quad \epsilon_{abcd}M^{ab}M^{cd} = \epsilon_{\alpha\beta}I^{\alpha}\widetilde{I}^{\beta} \tag{5.2.50}$$
$$[0,0]: \quad S^2 = \epsilon_{\alpha\beta}I^{\alpha}\widetilde{I}^{\beta} \tag{5.2.51}$$
$$[2,0]: \quad SM^{ab}(\gamma^{ab})^{\alpha\beta} = \widetilde{I}^{(\alpha}I^{\beta)} \tag{5.2.52}$$
$$[0,2]: \quad SM^{ab}(\gamma^{ab})_{\dot{\alpha}\dot{\beta}} = 0 \tag{5.2.53}$$
$$q([1,2]+[1,0]): \quad M^{ab}I^{\beta}(\gamma^{b})_{\beta\dot{\alpha}} = SI^{\beta}(\gamma^{a})_{\beta\dot{\alpha}} \tag{5.2.54}$$
$$q^{-1}([1,2]+[1,0]): \quad M^{ab}\widetilde{I}^{\beta}(\gamma^{b})_{\beta\dot{\alpha}} = S\widetilde{I}^{\beta}(\gamma^{a})_{\beta\dot{\alpha}} \ . \tag{5.2.55}$$

The gamma matrices γ^a for $SO(4)$ take the following index form:

$$(\gamma^a)_{\alpha\dot{\alpha}} \tag{5.2.56}$$

and the product of two gamma matrices is defined as:

$$(\gamma^{ab})_{\alpha\beta} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b]})_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \quad \text{and} \quad (\gamma^{ab})_{\dot{\alpha}\dot{\beta}} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b]})_{\beta\dot{\beta}}\epsilon^{\alpha\beta} \ ; \tag{5.2.57}$$

where the spinor indices are raised and lowered with the epsilon tensor.

Expansion in the instanton fugacity

It is useful to rewrite (5.2.38) in terms of an expansion in q:

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_3}](t;\mu,\nu,q) = \frac{1}{(1-t^2)(1-t^2\mu^2)}\sum_{n=-\infty}^{\infty} q^n t^{2|n|}\mu^{|n|} + \frac{1}{1-\nu^2t^2} - 1 \tag{5.2.58}$$

From here, we can extract the transformation properties of instanton operators of charge n under the $U(1)_I$. They transform as spin $|n|$ highest weight states for $SU(2)_R$ and as spin $|n|/2$ representations of $SU(2)_B$.

The classical dressing for each q^n instanton operator, the factor outside the sum, is, for $n \neq 0$, a space generated by the $SU(2)_B$ adjoint meson

$T^{\alpha\beta} = M^{ab}(\gamma^{ab})^{\alpha\beta}$ and the glueball operator S obeying the relation:

$$\mathrm{Tr}(T^2) = S^2 \tag{5.2.59}$$

For $n = 0$ there is a contribution coming from the $SU(2)_A$, the second term in (5.2.58), which modifies the classical dressing entirely. The latter is in fact, for this charge zero sector, generated by M^{ab} and S subject to the following relations:

$$[2,2]+[0,0]: \quad M^{ab}M^{bc} = S^2\delta^{ac} \tag{5.2.60}$$

$$[0,0]: \quad \epsilon_{abcd}M^{ab}M^{cd} = S^2 \tag{5.2.61}$$

$$[0,2]: \quad SM^{ab}(\gamma^{ab})_{\dot\alpha\dot\beta} = 0 \tag{5.2.62}$$

These relations are a subset of (5.2.49) - (5.2.55) constructed in the following way: we take the first two equations and we substitute the instanton bilinear on the right hand side with the glueball operator by means of (5.2.51). Moreover we keep (5.2.53) as it is a relation not corrected by instanton operators.

5.2.5 $N_f = 3$

The moduli space of the reduced one $E_4 = SU(5)$ instanton, $\widetilde{\mathcal{M}}_{1,E_4=SU(5)}$, is the nilpotent orbit generated by the adjoint representation of $SU(5)$. Its associated Hilbert series can thus be written as

$$H[\widetilde{\mathcal{M}}_{1,SU(5)}](t;\boldsymbol{x}) = \sum_{n=0}^{\infty}[n,0,0,n]_{\boldsymbol{x}}t^{2n} \ , \tag{5.2.63}$$

where $[1,0,0,1]_{\boldsymbol{x}}$ is the character of the adjoint representation of $SU(5)$ with fugacities $\boldsymbol{x} = (x_1,x_2,x_3,x_4)$. In order to proceed with a decomposition from weights of $SU(5)$ representations to those of $SO(6)\times U(1)$, we choose the projection matrix

$$P_{A_4\to D_3\times U(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} , \tag{5.2.64}$$

which gives the fugacity map

$$\begin{aligned} & x_1 = w^4 \ , \quad x_2 x_1^{-1} = y_3 w^{-1} \ , \quad x_3 x_2^{-1} = y_1 y_3^{-1} w^{-1} \ , \quad x_4 x_3^{-1} = y_1^{-1} y_2 w^{-1} \ , \\ \Leftrightarrow \quad & x_1 = w^4 \ , \qquad x_2 = y_3 w^3 \ , \qquad x_3 = y_1 w^2 \ , \qquad x_4 = y_2 w \ . \end{aligned} \tag{5.2.65}$$

States in the root lattice carry a charge multiple of 5 for the $U(1)$ associated to the fugacity w, hence we set $w^5 = q$ in the following, where q is the fugacity for the integer quantized instanton number $U(1)_I$. Then (5.2.63) can be written in terms of the character expansion of $SO(6) \times U(1) \supset SU(5)$ as

$$H[\widetilde{\mathcal{M}}_{1,SU(5)}](t;\boldsymbol{y},q) = \sum_{n=0}^{\infty} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} [0,n_1,n_2]_{\boldsymbol{y}} q^{n_1-n_2} t^{2n} \ , \tag{5.2.66}$$

where $[p_1,p_2,p_3]_{\boldsymbol{y}}$ is the character of a representation of $SO(6)$ as a function of fugacities $\boldsymbol{y} = (y_1,y_2,y_3)$. The information contained in this equation can be carried compactly by means of the associated highest weight generating function

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,SU(5)}](t;\mu_2,\mu_3;q) = \mathrm{PE}\left[t^2(1+\mu_2 q + \mu_3 q^{-1} + \mu_2\mu_3) - t^4 \mu_2\mu_3\right] \tag{5.2.67}$$

where at t^2 we can again recognise the contribution of S, a singlet of $SO(6)$, the instanton and the anti-instanton operators in the spinor $[0,1,0]$ and cospinor $[0,0,1]$ representations, and the meson in the adjoint representation $[0,1,1]$, while at order t^4 is the basic relation between the operators. Notice that (5.2.67) is a generating function for a lattice with conifold structure.

The generators and their relations

The generators and the relations can be extracted from the plethystic logarithm of the Hilbert series. The Hilbert series of the reduced moduli space of 1 E_4 instanton can be written in terms of characters of $SO(6) \times U(1)$ up to

$O(t^4)$ as:

$$\begin{aligned} H[E_4](t;\boldsymbol{x},q) = & 1 + (1 + [0,1,1] + q^{-1}[0,0,1] + q[0,1,0])t^2 + \\ & + \Big(1 + [0,1,1] + [0,2,2] + q^{-1}([0,0,1] + [0,1,2]) + \\ & + q([0,1,0] + [0,2,1]) + q^{-2}[0,0,2] + q^2[0,2,0]\Big)t^4 + \dots \ . \end{aligned} \tag{5.2.68}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} & \mathrm{PL}\left[H[E_4](t;\boldsymbol{x},q)\right] \\ & = (1 + [0,1,1] + q^{-1}[0,0,1] + q[0,1,0])t^2 - \Big(2 + 2[0,1,1] + [2,0,0] + \\ & + q([1,0,1] + [0,1,0]) + q^{-1}([1,1,0] + [0,0,1])\Big)t^4 + \dots \ . \end{aligned} \tag{5.2.69}$$

Below we write down the generators corresponding to the terms at t^2 and the explicit relations corresponding to the terms at order t^4 of (5.2.69).

For $SO(6)$, we use $a, b, c, d = 1, \dots, 6$ to denote vector indices and use $\alpha, \beta, \rho, \sigma = 1, \dots, 4$ to denote spinor indices. Note that the spinor representation of $SO(6)$ is complex. The delta symbol carries has one upper and one lower index:

$$\delta^{\alpha}_{\beta} \ . \tag{5.2.70}$$

The gamma matrices γ^a can take the following forms:

$$(\gamma^a)_{\alpha\beta} \qquad \text{and} \qquad (\gamma^b)^{\alpha\beta} \ , \tag{5.2.71}$$

where the α, β indices are antisymmetric. The product of two gamma matrices has one lower spinor index and one upper spinor index:

$$(\gamma^{ab})^{\alpha}_{\ \rho} \equiv (\gamma^{[a})^{\alpha\beta}(\gamma^{b]})_{\beta\rho} \ . \tag{5.2.72}$$

From (5.2.69) the generators of the moduli space are M^{ab}, a 6×6 antisymmetric matrix, the instanton operators I^{α} and $\widetilde{I}_{\alpha}$ and the gaugino bilinear S. The relations corresponding to the terms at order t^4 of (5.2.69) can be written as follows:

$$\begin{aligned}
[2,0,0]+[0,0,0]: &\qquad M^{ab}M^{bc} = (I^{\alpha}\widetilde{I}_{\alpha})\delta^{ac} && (5.2.73)\\
[0,1,1]: &\qquad \epsilon^{abcdef}M^{cd}M^{ef} = \widetilde{I}_{\beta}(\gamma^{ab})^{\beta}{}_{\alpha}I^{\alpha} && (5.2.74)\\
[0,0,0]: &\qquad S^2 = I^{\alpha}\widetilde{I}_{\alpha} && (5.2.75)\\
[0,1,1]: &\qquad SM^{ab} = \widetilde{I}_{\beta}(\gamma^{ab})^{\beta}{}_{\alpha}I^{\alpha} && (5.2.76)\\
q([1,0,1]+[0,1,0]): &\qquad M^{ab}I^{\alpha}(\gamma^{b})_{\alpha\beta} = SI^{\alpha}(\gamma^{a})_{\alpha\beta} && (5.2.77)\\
q^{-1}([1,1,0]+[0,0,1]): &\qquad M^{ab}\widetilde{I}_{\alpha}(\gamma^{b})^{\alpha\beta} = S\widetilde{I}_{\alpha}(\gamma^{a})^{\alpha\beta}\ . && (5.2.78)
\end{aligned}$$

As can be seen, the classical relations are corrected by instanton bilinears and this is a recurrent feature for all number of flavours. These relations can also be rewritten in terms of an $SU(4)$ matrix $M^{\alpha}_{\ \beta}$ using the following relation

$$M^{ab} = M^{\alpha}_{\ \beta}(\gamma^{ab})^{\beta}{}_{\alpha}\ . \qquad (5.2.79)$$

Expansion in the instanton fugacity

We rewrite (5.2.67) as an expansion in q as follows:

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,SU(5)}](t;\mu_2,\mu_3,q) = \frac{1}{(1-t^2)(1-t^2\mu_2\mu_3)}\left(\sum_{n\geq 0} q^n(t^2\mu_2)^n + \sum_{n<0} q^n(t^2\mu_3)^{-n}\right)\ . \qquad (5.2.80)$$

Two very interesting features emerge from the q expansion. Firstly, an instanton operator of charge n has $SU(2)_R$ spin $|n|$ and it transforms as an $|n|$-spinor — a representation with $|n|$ on a spinor Dynkin label — of the global flavour group $SO(6)$. Whilst in [49] it was found that this result holds for $n=1$, here we find a prediction for all n.

Secondly the instanton operators are dressed by a factor, the one in front of the sum, which is generated by S and M^{ab}, subject to the following relations:

$$\begin{aligned}
[2,0,0]+[0,0,0]: &\qquad M^{ab}M^{bc} = S^2\delta^{ac} && (5.2.81)\\
[0,1,1]: &\qquad \epsilon^{abcdef}M^{cd}M^{ef} = SM^{ab}\ . && (5.2.82)
\end{aligned}$$

Interestingly, such relations can be extracted directly from (5.2.73) - (5.2.78) by keeping only those relations that are not corrected by the instanton operators. This feature is a recurrent theme for higher number of flavours.

5.2.6 $N_f = 4$

The Higgs branch at infinite coupling for an $SU(2)$ theory with $N_f = 4$ flavours is isomorphic to the reduced moduli space of one $E_5 = SO(10)$ instanton $\widetilde{\mathcal{M}}_{1,E_5=SO(10)}$, which is given by the minimal nilpotent orbit of $SO(10)$. Its Hilbert series is

$$H[\widetilde{\mathcal{M}}_{1,SO(10)}](t;\boldsymbol{x}) = \sum_{n=0}^{\infty}[0,n,0,0,0]_{\boldsymbol{x}} t^{2n} \ , \tag{5.2.83}$$

where $[0,1,0,0,0]_{\boldsymbol{x}}$ is the character of the adjoint representation of $SO(10)$.

At finite coupling the theory has a global symmetry $SO(8) \times U(1)$. Hence we rewrite this Hilbert series in terms of an $SO(8) \times U(1)$ character expansion as

$$H[\widetilde{\mathcal{M}}_{1,SO(10)}](t;\boldsymbol{y},q) = \frac{1}{1-t^2} \sum_{n_1,n_2,n_3 \geq 0} [0,n_1,0,n_2+n_3]_{\boldsymbol{y}} q^{n_2-n_3} t^{2n_1+2n_2+2n_3} \ , \tag{5.2.84}$$

where we decompose representations of $SO(8) \times U(1) \subset SO(10)$ using a projection matrix that maps the weights of $SO(10)$ representations to those of $SO(8) \times U(1)$ as follows

$$P_{D_5 \to D_4 \times U(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & -2 & -2 & -1 & -1 \end{pmatrix} \ . \tag{5.2.85}$$

Under the action of this matrix, the fugacities $\boldsymbol{x}$ of $SO(10)$ are mapped to the fugacities $\boldsymbol{y}$ of $SO(8)$ and w of $U(1)$ as follows:

$$\begin{aligned} (x_1, x_2 x_1^{-1}, x_3 x_2^{-1}, x_4 x_5 x_3^{-1}, x_5 x_4^{-1}) &= \left(w^{-2}, y_4, y_2 y_4^{-1}, y_1 y_2^{-1} y_3, y_1 y_3^{-1}\right) \\ \Leftrightarrow \qquad (x_1, x_2, x_3, x_4, x_5) &= \left(\frac{1}{w^2}, \frac{y_4}{w^2}, \frac{y_2}{w^2}, \frac{y_3}{w}, \frac{y_1}{w}\right) \ . \end{aligned} \tag{5.2.86}$$

In (5.2.84) we set $w^2 = q$ to have integer instanton numbers, rather than even.

The corresponding highest weight generating function is

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,SO(10)}](t;\mu_2,\mu_4;q) = \mathrm{PE}\left[t^2(1+\mu_2+\mu_4 q+\mu_4 q^{-1})\right] \tag{5.2.87}$$

where we recognise the usual $SU(2)_R$ spin-2 generators: the glueball superfield S, a singlet of $SO(8)$, the instanton operators I_α and $\widetilde{I}_\alpha$ associated to $\mu_4 q$ and $\mu_4 q^{-1}$, both transforming in the same spinor representation of $SO(8)$ with opposite $U(1)$ charge, as well as the meson M^{ab}, associated to μ_2. The highest weight lattice is freely, generated as we see from the lack of relations at order t^4.

The generators and their relations

The expansion of (5.2.84) up to order t^4 is given by

$$\begin{aligned} H[E_5](t;\boldsymbol{x},q) = 1 &+ \Big(1+[0,1,0,0]+(q+q^{-1})[0,0,0,1]\Big)t^2+ \\ &+\Big(1+[0,1,0,0]+[0,0,0,2]+[0,2,0,0]+ \\ &+(q+q^{-1})([0,0,0,1]+[0,1,0,1])+(q^2+q^{-2})[0,0,0,2]\Big)t^4+\dots\ . \end{aligned} \tag{5.2.88}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} \mathrm{PL}\left[H[E_5](t;\boldsymbol{x},q)\right] = &\Big(1+[0,1,0,0]+(q+q^{-1})[0,0,0,1]\Big)t^2+ \\ &-\Big(2+[2,0,0,0]+[0,1,0,0]+[0,0,2,0]+[0,0,0,2]+ \\ &+(q+q^{-1})([1,0,1,0]+[0,0,0,1])+(q^2+q^{-2})\Big)t^4+\dots\ . \end{aligned} \tag{5.2.89}$$

From this collection of representations we can write the defining equations for the Higgs branch at infinite coupling by constructing the relevant operators. For $SO(8)$, we use $a,b,c,d=1,\dots,8$ to denote the vector indices, $\alpha,\beta,\rho,\sigma = 1,\dots,8$ to denote those in the spinor representation $[0,0,0,1]$ and $\dot{\alpha},\dot{\beta},\dot{\rho},\dot{\sigma} = 1,\dots,8$ to denote those in the conjugate spinor representation $[0,0,1,0]$. The delta symbol has the following forms:

$$\delta^{\alpha\beta} \quad \text{or} \quad \delta_{\alpha\beta} \quad \text{or} \quad \delta^{\dot{\alpha}\dot{\beta}} \quad \text{or} \quad \delta_{\dot{\alpha}\dot{\beta}}\ . \tag{5.2.90}$$

The gamma matrices γ^a can take the following forms:

$$(\gamma^a)_{\alpha\dot{\alpha}} \qquad \text{or} \qquad (\gamma^a)^{\alpha\dot{\alpha}} \; . \tag{5.2.91}$$

The product of two gamma matrices has the following forms:

$$(\gamma^{ab})_{\alpha\beta} \equiv (\gamma^{[a})_{\alpha\dot{\beta}}(\gamma^{b]})_{\beta\dot{\beta}} \quad \text{and} \quad (\gamma^{ab})_{\dot{\alpha}\dot{\beta}} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b]})_{\alpha\dot{\beta}} \tag{5.2.92}$$

and similarly for both upper indices; the indices α, β and $\dot{\alpha}, \dot{\beta}$ are antisymmetric. The product of four gamma matrices has the following forms:

$$\begin{aligned}(\gamma^{abcd})_{\alpha\beta} &\equiv (\gamma^{[a})_{\alpha\dot{\beta}}(\gamma^{b})_{\rho\dot{\beta}}(\gamma^{c})_{\rho\dot{\sigma}}(\gamma^{d]})_{\beta\dot{\sigma}} \\ (\gamma^{abcd})_{\dot{\alpha}\dot{\beta}} &\equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b})_{\alpha\dot{\rho}}(\gamma^{c})_{\rho\dot{\rho}}(\gamma^{d]})_{\rho\dot{\beta}}\end{aligned} \tag{5.2.93}$$

and similarly for both upper indices; the indices α, β and $\dot{\alpha}, \dot{\beta}$ are symmetric.

The generators of the moduli space are M^{ab}, which is a 8×8 antisymmetric matrix; the instanton operators I_α and $\widetilde{I}_\alpha$; and the glueball superfield S.

The relations corresponding to terms at order t^4 of (5.2.89) can be written as

$$[2,0,0,0]+[0,0,0,0]: \qquad M^{ab}M^{bc} = (I_\alpha \widetilde{I}_\alpha)\delta^{ac} \tag{5.2.94}$$

$$[0,0,2,0]: \qquad M^{ab}M^{cd}(\gamma^{abcd})_{\dot{\alpha}\dot{\beta}} = 0 \tag{5.2.95}$$

$$[0,0,0,2]: \qquad M^{ab}M^{cd}(\gamma^{abcd})_{\alpha\beta} = I_{(\alpha}\widetilde{I}_{\beta)} - \frac{1}{8}(I_\rho \widetilde{I}_\rho)\delta_{\alpha\beta} \tag{5.2.96}$$

$$[0,0,0,0]: \qquad S^2 = I_\alpha \widetilde{I}_\beta \delta^{\alpha\beta} \tag{5.2.97}$$

$$[0,1,0,0]: \qquad SM^{ab} = I_\alpha \widetilde{I}_\beta (\gamma^{ab})_{\alpha\beta} \tag{5.2.98}$$

$$q([1,0,1,0]+[0,0,0,1]): \qquad M^{ab}I_\beta(\gamma^b)_{\beta\dot{\alpha}} = SI_\beta(\gamma^a)_{\beta\dot{\alpha}} \tag{5.2.99}$$

$$q^{-1}([1,0,1,0]+[0,0,0,1]): \qquad M^{ab}\widetilde{I}_\beta(\gamma^b)_{\beta\dot{\alpha}} = S\widetilde{I}_\beta(\gamma^a)_{\beta\dot{\alpha}} \tag{5.2.100}$$

$$(q^2+q^{-2})[0,0,0,0]: \qquad I_\alpha I_\beta \delta_{\alpha\beta} = \widetilde{I}_\alpha \widetilde{I}_\beta \delta_{\alpha\beta} = 0 \; . \tag{5.2.101}$$

Expansion in the instanton fugacity

In terms of an expansion in q, (5.2.87) can be written as

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,SO(10)}](t;\mu_2,\mu_4;q) = \frac{1}{(1-t^2)(1-\mu_2 t^2)(1-\mu_4^2 t^4)} \sum_{n=-\infty}^{\infty} q^n \mu_4^{|n|} t^{2|n|} \; . \tag{5.2.102}$$

Here again we find that instanton operators of charge n are spin $|n|$ of $SU(2)_R$ and transform in $|n|$-spinor representations of $SO(8)$.

However the interpretation of the classical dressing is more subtle than in previous cases. The prefactor in the q expansion signifies a space which is algebraically determined by some of the conditions that define the moduli space of one $SO(8)$ instanton; in particular it is a space generated by two operators, M^{ab}, in the adjoint representation $[0,1,0,0]$ of $SO(8)$, and S, in the singlet $[0,0,0,0]$, subject to relations that transform in the representations $[2,0,0,0]$, $[0,0,0,0]$ and $[0,0,2,0]$. Explicitly these relations are:

$$[2,0,0,0]+[0,0,0,0]: \qquad M^{ab}M^{bc} = S^2\delta^{ac} \tag{5.2.103}$$

$$[0,0,2,0]: \qquad M^{ab}M^{cd}(\gamma^{abcd})_{\dot{\alpha}\dot{\beta}} = 0 \; . \tag{5.2.104}$$

The following features can be observed. Whilst the classical moduli space of one $SO(8)$ instanton is generated by (5.2.3) and (5.2.4), here the anti-self-dual 4th rank antisymmetric representation is missing[11]. Such a space has complex dimension 13 and, by adding the dimension originating from the sum over the instanton number, the correct 14 dimensional moduli space of one $SO(10)$ instanton is recovered. Again, the classical dressing can be guessed from the set of equations in (5.2.94)-(5.2.101) by keeping only the relations that are not corrected by the instanton operators.

5.2.7 $N_f = 5$

The Hilbert series of $\widetilde{\mathcal{M}}_{1,E_6}$ can be written as

$$H[\widetilde{\mathcal{M}}_{1,E_6}](t;\boldsymbol{x}) = \sum_{n=0}^{\infty} \begin{bmatrix} & & n & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\boldsymbol{x}} t^{2n} \; . \tag{5.2.105}$$

[11] Recall that for $SO(8)$, $\wedge^4[1,0,0,0] = [0,0,2,0]+[0,0,0,2]$ is a reducible representation.

A projection matrix that maps the weights of E_6 to those of $D_5 \times U(1)$ is given by

$$P_{E_6 \to D_5 \times U(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & -3 & -5 & -6 & -4 & -2 \end{pmatrix} . \tag{5.2.106}$$

Under the action of this matrix, the fugacities of $\boldsymbol{x}$ of E_6 are mapped to the fugacities $\boldsymbol{y}$ of $SO(10)$ and w of $U(1)$ as follows:

$$\begin{aligned} (x_1, x_1 x_2^{-1}, x_1 x_3^{-1}, x_2 x_6^{-1}, x_3 x_5^{-1}, x_3 x_4^{-1}) &= \left(\frac{1}{w^4}, \frac{1}{w y_5}, \frac{w}{y_4}, \frac{y_5}{w y_1}, \frac{y_4}{w y_2}, \frac{w y_4}{y_3} \right) \\ \Leftrightarrow \qquad (x_1, x_2, x_3, x_4, x_5, x_6) &= \left(\frac{1}{w^4}, \frac{y_5}{w^3}, \frac{y_4}{w^5}, \frac{y_3}{w^6}, \frac{y_2}{w^4}, \frac{y_1}{w^2} \right) . \end{aligned} \tag{5.2.107}$$

The fugacity of $U(1)_I$ is $q = w^3$. Thus, the Hilbert series $H[\widetilde{\mathcal{M}}_{1,E_6}]$ can be written in terms of characters of representations of $SO(10) \times U(1)_I$ as

$$H[\widetilde{\mathcal{M}}_{1,E_6}](t; \boldsymbol{y}, q) = \frac{1}{1-t^2} \sum_{n_1, n_2, n_3 \geq 0} [0, n_1, 0, n_2, n_3]_{\boldsymbol{y}} q^{n_2 - n_3} t^{2n_1 + 2n_2 + 2n_3} , \tag{5.2.108}$$

The corresponding highest weight generating function is

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_6}](t; \mu_2, \mu_4; q) = \mathrm{PE}\left[t^2 (1 + \mu_2 + \mu_4 q + \mu_5 q^{-1}) \right] . \tag{5.2.109}$$

The generators and their relations

The expansion of (5.2.108) up to order t^4 is given by

$$\begin{aligned} H[E_6](t;\boldsymbol{x},q) = 1 &+ (1+[0,1,0,0,0]+q^{-1}[0,0,0,0,1]+q[0,0,0,1,0])t^2 + \\ &+ \Big(1+[0,1,0,0,0]+[0,2,0,0,0]+[0,0,0,1,1]+ \\ &\quad + q^{-1}([0,0,0,0,1]+[0,1,0,0,1])+q([0,0,0,1,0]+[0,1,0,1,0])+ \\ &\quad + q^{-2}[0,0,0,0,2]+q^2[0,0,0,2,0]\Big)t^4+\ldots \ . \end{aligned} \tag{5.2.110}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} \mathrm{PL}\left[H[E_6](t;\boldsymbol{x},q)\right] &= (1+[0,1,0,0,0]+q^{-1}[0,0,0,0,1]+q[0,0,0,1,0])t^2 + \\ &- \Big(2+[0,1,0,0,0]+[2,0,0,0,0]+[0,0,0,1,1]+ \\ &\quad + q([1,0,0,0,1]+[0,0,0,1,0])+q^{-1}([1,0,0,1,0]+[0,0,0,0,1])+ \\ &\quad + (q^2+q^{-2})[1,0,0,0,0]\Big)t^4+\ldots \ . \end{aligned} \tag{5.2.111}$$

For $SO(10)$, we use $a,b,c,d = 1,\ldots,10$ to denote vector indices and $\alpha,\beta,\rho,\sigma = 1,\ldots,16$ to denote spinor indices. Note that the spinor representation of $SO(10)$ is complex. The delta symbol has the following form:

$$\delta^{\alpha}_{\beta} \ . \tag{5.2.112}$$

The gamma matrices γ^a can take the following forms:

$$(\gamma^a)_{\alpha\beta} \qquad \text{and} \qquad (\gamma^a)^{\alpha\beta} \ , \tag{5.2.113}$$

where the α,β indices are symmetric. The product of two gamma matrices has the following form:

$$(\gamma^{ab})^{\alpha}_{\ \rho} \equiv (\gamma^{[a})^{\alpha\beta}(\gamma^{b]})_{\beta\rho} \ . \tag{5.2.114}$$

The product of four gamma matrices has the following form:

$$(\gamma^{abcd})^{\alpha}_{\ \beta} \equiv (\gamma^{[a})^{\alpha\sigma_1}(\gamma^{b})_{\sigma_1\sigma_2}(\gamma^{c})^{\sigma_2\sigma_3}(\gamma^{d]})_{\sigma_3\beta} \ . \tag{5.2.115}$$

The generators of the moduli space are M^{ab}, which is a 10×10 antisymmetric matrix; the instanton operators I^α and $\widetilde{I}_\alpha$; and the gaugino superfield S.

The relations appearing in the plethystic logarithm (5.2.111) are as follows:

$$\begin{aligned}
[2,0,0,0,0]+[0,0,0,0,0]: &\qquad M^{ab}M^{bc}=(I^\alpha\widetilde{I}_\alpha)\delta^{ac}\ , && (5.2.116)\\
[0,0,0,1,1]: &\qquad M^{[a_1a_2}M^{a_3a_4]}=\widetilde{I}_\beta(\gamma^{a_1\cdots a_4})^\beta{}_\alpha I^\alpha\ , && (5.2.117)\\
[0,0,0,0,0]: &\qquad S^2=I^\alpha\widetilde{I}_\alpha\ , && (5.2.118)\\
[0,1,0,0,0]: &\qquad SM^{ab}=\widetilde{I}_\beta(\gamma^{ab})^\beta{}_\alpha I^\alpha\ , && (5.2.119)\\
q([1,0,0,0,1]+[0,0,0,1,0]): &\qquad M^{ab}I^\alpha(\gamma^b)^\beta{}_\alpha=SI^\alpha(\gamma^a)^\beta{}_\alpha\ , && (5.2.120)\\
q^{-1}([1,0,0,0,1]+[0,0,0,1,0]): &\qquad M^{ab}\widetilde{I}_\beta(\gamma^b)^\beta{}_\alpha=S\widetilde{I}_\beta(\gamma^a)^\beta{}_\alpha\ , && (5.2.121)\\
(q^2+q^{-2})[1,0,0,0,0]: &\qquad I^\alpha I^\beta(\gamma^a)_{\alpha\beta}=\widetilde{I}_\alpha\widetilde{I}_\beta(\gamma^a)^{\alpha\beta}=0\ . && (5.2.122)
\end{aligned}$$

Expansion in the instanton fugacity

The highest weight generating function (5.2.109) can be expanded in the instanton number fugacity q as

$$\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_6}](t;\mu_2,\mu_4,,\mu_5;q)=\frac{1}{(1-t^2)(1-t^2\mu_2)(1-t^4\mu_4\mu_5)}\times \left(\sum_{n\geq 0}q^n(t^2\mu_4)^n+\sum_{n<0}q^n(t^2\mu_5)^{-n}\right)\ . \tag{5.2.123}$$

From this formula we see that the instanton operators of charge n are spin $|n|$ highest weight states under $SU(2)_R$ and transform in the n-spinor representation $[0,0,0,n,0]$ of $SO(10)$ for $n>0$ and the conjugate $|n|$-spinor representation $[0,0,0,0,|n|]$ for $n<0$.

The dressing factor has the features previously encountered in that is generated by the classical operators M^{ab} and S, subject to the relations

$$[2,0,0,0,0]+[0,0,0,0,0]:\qquad M^{ab}M^{bc}=S^2\delta^{ac}\ . \tag{5.2.124}$$

Comparing this space to the moduli space of one $SO(10)$ instanton given by (5.2.3) and (5.2.4), it is clear that here the rank-2 condition (5.2.4) is missing altogether. As we have explained in the previous case, this can be at once read off from the relations (5.2.116)-(5.2.122), by keeping only the ones which are not corrected by instanton bilinears. The classical dressing is a space of dimension 21 and again, by adding the contribution from the sum over instantons, we recover the correct 22-dimensional moduli space of one E_6 instanton.

5.2.8 $N_f = 6$

The Hilbert series of $\widetilde{\mathcal{M}}_{1,E_7}$ can be written as

$$H[\widetilde{\mathcal{M}}_{1,E_7}](t;\boldsymbol{x}) = \sum_{n=0}^{\infty} \begin{bmatrix} & 0 & \\ n & 0\ 0\ 0\ 0 & 0 \end{bmatrix}_{\boldsymbol{x}} t^{2n} \ . \qquad (5.2.125)$$

The E_7 representations can be decomposed into those of $SO(12) \times U(1)$ using the projection matrix:

$$P_{E_7 \to D_6 \times U(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -3 & -4 & -3 & -2 & -1 \end{pmatrix} \ . \qquad (5.2.126)$$

Under the action of this matrix, the fugacities $\boldsymbol{x}$ of E_7 are mapped to the fugacities $\boldsymbol{y}$ of $SO(12)$ and the fugacity q of $U(1)$ as

$$\boldsymbol{x} = \left(\frac{1}{q^2}, \frac{y_6}{q^2}, \frac{y_5}{q^3}, \frac{y_4}{q^4}, \frac{y_3}{q^3}, \frac{y_2}{q^2}, \frac{y_1}{q} \right) \ . \qquad (5.2.127)$$

We then have the following highest weight generating function:

$$\begin{aligned} &\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_7}](t;\mu_2,\mu_4,\mu_5;q) \\ &= \mathrm{PE}\left[\Big(1+\mu_2+\mu_5(q+q^{-1})+(q^2+q^{-2})\Big)t^2+\mu_4 t^4 \right] , \end{aligned} \qquad (5.2.128)$$

where at order t^2 we recognise the contributions of: S, which is a singlet of $SO(12)$; the instanton and the anti-instanton operators with $U(1)_I$ charge ± 1 in the spinor representation $[0,0,0,0,1,0]$; the instanton and the anti-instanton operators with $U(1)_I$ charge ± 2 which are singlets of $SO(12)$; the meson in the adjoint representation $[0,1,0,0,0,0]$. In addition there is a fourth-rank antisymmetric tensor of $SO(12)$ at order t^4.

The generators and their relations

The expansion up to order t^4 of (5.2.128) is given by

$$\begin{aligned} &H[E_7](t;\boldsymbol{x},q) \\ &= 1 + \Big(1 + [0,1,0,0,0,0] + (q+q^{-1})[0,0,0,0,1,0] + (q^2+q^{-2})\Big)t^2 \\ &\quad + \Big(2 + [0,2,0,0,0,0] + [0,0,0,0,2,0] + [0,0,0,1,0,0] + [0,1,0,0,0,0] \\ &\qquad + (q+q^{-1})(2[0,0,0,0,1,0] + [0,1,0,0,1,0]) \\ &\qquad + (q^2+q^{-2})(1 + [0,0,0,0,2,0] + [0,1,0,0,0,0]) \\ &\qquad + (q^3+q^{-3})[0,0,0,0,1,0] + (q^4+q^{-4})\Big)t^4 + \ldots \; . \end{aligned} \tag{5.2.129}$$

The plethystic logarithm of this Hilbert series is given by

$$\begin{aligned} &\mathrm{PL}\left[H[E_7](t;\boldsymbol{x},q)\right] \\ &= \Big(1 + [0,1,0,0,0,0] + (q+q^{-1})[0,0,0,0,1,0] + (q^2+q^{-2})\Big)t^2 - \\ &\quad - \Big(2 + [0,0,0,1,0,0] + [0,1,0,0,0,0] + [2,0,0,0,0,0] \\ &\qquad + (q+q^{-1})([0,0,0,0,1,0] + [1,0,0,0,0,1]) + (q^2+q^{-2})[0,1,0,0,0,0]\Big)t^4 + \ldots \; . \end{aligned} \tag{5.2.130}$$

For $SO(12)$, we use $a,b,c,d = 1,\ldots,12$ to denote vector indices, $\alpha,\beta,\rho,\sigma = 1,\ldots,32$ to denote indices of the spinor representation $[0,0,0,0,1,0]$, and $\dot\alpha,\dot\beta,\dot\rho,\dot\sigma = 1,\ldots,32$ to denote indices of the conjugate spinor representation $[0,0,0,0,0,1]$. The spinor representation of $SO(12)$ is pseudoreal, hence all contractions of the spinor indices are made with the epsilon tensor, which takes the forms

$$\epsilon_{\alpha\beta} \quad \text{or} \quad \epsilon^{\alpha\beta} \quad \text{or} \quad \epsilon_{\dot\alpha\dot\beta} \quad \text{or} \quad \epsilon^{\dot\alpha\dot\beta} \; . \tag{5.2.131}$$

Gamma matrices γ^a take the forms

$$(\gamma^a)_{\alpha\dot\beta} \; . \tag{5.2.132}$$

The product of two gamma matrices has the following forms:

$$(\gamma^{ab})_{\alpha\beta} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b]})_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \quad \text{and} \quad (\gamma^{ab})_{\dot{\alpha}\dot{\beta}} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b]})_{\beta\dot{\beta}}\epsilon^{\alpha\beta} \ , \tag{5.2.133}$$

where the spinor indices are symmetric. The product of four gamma matrices has the following forms:

$$(\gamma^{abcd})_{\alpha\sigma} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b})_{\beta\dot{\beta}}(\gamma^{c})_{\rho\dot{\rho}}(\gamma^{d]})_{\sigma\dot{\sigma}}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\beta\rho}\epsilon^{\dot{\rho}\dot{\sigma}} \tag{5.2.134}$$

$$(\gamma^{abcd})_{\dot{\alpha}\dot{\sigma}} \equiv (\gamma^{[a})_{\alpha\dot{\alpha}}(\gamma^{b})_{\beta\dot{\beta}}(\gamma^{c})_{\rho\dot{\rho}}(\gamma^{d]})_{\sigma\dot{\sigma}}\epsilon^{\alpha\beta}\epsilon^{\dot{\beta}\dot{\rho}}\epsilon^{\rho\sigma} \ , \tag{5.2.135}$$

where the spinor indices are antisymmetric.

The generators of the moduli space are M^{ab}, which is a 12×12 antisymmetric matrix, the instanton operators $I^{\alpha}_{1+}, I^{\alpha}_{1-}$ and I_{2+}, I_{2-}, and the glueball superfield S.

From (5.2.130), we have the following sets of relations:

$$[2,0,0,0,0,0]+[0,0,0,0,0,0]: \quad M^{ab}M^{bc} = (I^{\alpha}_{1+}\epsilon_{\alpha\beta}I^{\beta}_{1-})\delta^{ac} \tag{5.2.136}$$

$$[0,0,0,1,0,0]: \quad M^{[a_1a_2}M^{a_3a_4]} = I^{\alpha}_{1+}I^{\beta}_{1-}(\gamma^{a_1\cdots a_4})_{\alpha\beta} \tag{5.2.137}$$

$$[0,0,0,0,0,0]: \quad S^2 + I_{2+}I_{2-} = I^{\alpha}_{1+}I^{\beta}_{1-}\epsilon_{\alpha\beta} \tag{5.2.138}$$

$$[0,1,0,0,0,0]: \quad SM^{ab} = I^{\alpha}_{1+}I^{\beta}_{1-}(\gamma^{ab})_{\alpha\beta} \tag{5.2.139}$$

$$(q^2+q^{-2})[0,1,0,0,0,0]: \quad I_{2\pm}M^{ab} = I^{\alpha}_{1\pm}I^{\beta}_{1\pm}(\gamma^{ab})_{\alpha\beta} \tag{5.2.140}$$

$$(q+q^{-1})([1,0,0,0,0,1]+[0,0,0,0,1,0]): \quad M^{ab}I^{\alpha}_{1\pm}(\gamma^{b})_{\alpha\dot{\beta}} = (SI^{\alpha}_{1\pm} + I_{2\pm}I^{\alpha}_{1\mp})(\gamma^{a})_{\alpha\dot{\beta}} \ . \tag{5.2.141}$$

To aid computations it is useful to rewrite (5.2.129) and (5.2.130) in terms of characters of $SO(12)\times SU(2)$. The reader can find the relevant formulae in Appendix B.2.

Expansion in the instanton fugacity

The highest weight generating function (5.2.128) can be expanded in powers of the instanton number fugacity q as

$$\begin{aligned}
&\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_7}](t;\mu_2,\mu_4,\mu_5;q) \\
&= \frac{1}{(1-t^2)(1-\mu_2 t^2)(1-\mu_4 t^4)(1-\mu_5^2 t^4)(1-t^4)} \sum_{m\in\mathbb{Z}} (t^2\mu_5)^{|m|} q^m \sum_{n\in\mathbb{Z}} t^{2|n|} q^{2n} \\
&= \quad \mathrm{PE}[(\mu_5^2+1+\mu_2)t^2+\mu_4 t^4+\mu_5^2 t^6] \\
&\quad \times \Big(\frac{1+\mu_5^2 t^4}{1-t^4} \sum_{m \text{ even}} t^{|m|} q^m - \frac{(t\mu_5)^2}{1-\mu_5^2 t^4} \sum_{m \text{ even}} \mu_5^{|m|} t^{2|m|} q^m \\
&\quad + \frac{(1+t^2)\mu_5 t}{1-t^4} \sum_{m \text{ odd}} t^{|m|} q^m - \frac{(t\mu_5)^2}{1-\mu_5^2 t^4} \sum_{m \text{ odd}} \mu_5^{|m|} t^{2|m|} q^m \Big) \ .
\end{aligned} \tag{5.2.142}$$

The first equality is a q expansion in terms of a double sum. This separates the classical dressing from the one and two instanton contributions. It is precisely the presence of both types of instantons as quadratic generators that, for $N_f > 5$, complicates the features of the q expansion in terms of a one sum only. We still write such an expansion in the second equality, splitting it into odd and even terms.

5.2.9 $N_f = 7$

The Hilbert series of $\widetilde{\mathcal{M}}_{1,E_8}$ can be written as

$$H[\widetilde{\mathcal{M}}_{1,E_8}](t;\boldsymbol{x}) = \sum_{n=0}^{\infty} \begin{bmatrix} & & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}_{\boldsymbol{x}} t^{2n} \ . \tag{5.2.143}$$

The E_8 representations can be decomposed into those of $SO(14) \times U(1)$ using the projection matrix

$$P_{E_8 \to D_7 \times U(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -5 & -7 & -10 & -8 & -6 & -4 & -2 \end{pmatrix} . \quad (5.2.144)$$

Under the action of this matrix, the fugacities $\boldsymbol{x}$ of E_8 are mapped to the fugacities $\boldsymbol{y}$ of $SO(14)$ and the fugacity q of $U(1)$ as

$$\boldsymbol{x} = \left(\frac{1}{q^4}, \frac{y_7}{q^5}, \frac{y_6}{q^7}, \frac{y_5}{q^{10}}, \frac{y_4}{q^8}, \frac{y_3}{q^6}, \frac{y_2}{q^4}, \frac{y_1}{q^2} \right) . \quad (5.2.145)$$

We then have the following highest weight generating function:

$$\begin{aligned} \mathcal{G}[\widetilde{\mathcal{M}}_{1,E_8}](t;\boldsymbol{\mu};q) = \mathrm{PE} \Big[& t^2 \left(1 + \mu_2 + \mu_6 q + \mu_7 q^{-1} + \mu_1(q^2 + q^{-2})\right) \\ & + t^4 \left(1 + \mu_2 + \mu_4 + \mu_6 q + \mu_7 q^{-1} + \mu_3(q^2 + q^{-2})\right) \\ & + t^6 \left(\mu_4 + \mu_5(q^2 + q^{-2})\right) \Big] . \end{aligned} \quad (5.2.146)$$

The generators and their relations

The Hilbert series of the reduced moduli space of 1 E_8 instanton can be written in terms of characters of $SO(14) \times U(1)$ as

$$\begin{aligned} & H[E_8](t;\boldsymbol{x},q) \\ & = 1 + \Big((1 + [0,1,0,0,0,0,0]) + [0,0,0,0,0,1,0]q + [0,0,0,0,0,0,1]q^{-1} \end{aligned}$$

$$
\begin{aligned}
&+[1,0,0,0,0,0,0](q^2+q^{-2})\Big)t^2+\Big(2+[0,0,0,0,0,1,1]+[0,0,0,1,0,0,0]\\
&+2[0,1,0,0,0,0,0]+[0,2,0,0,0,0,0]+[2,0,0,0,0,0,0]\\
&+(2[0,0,0,0,0,1,0]+[0,1,0,0,0,1,0]+[1,0,0,0,0,0,1])q\\
&+(2[0,0,0,0,0,0,1]+[0,1,0,0,0,0,1]+[1,0,0,0,0,1,0])q^{-1}\\
&+([0,0,0,0,0,2,0]+[0,0,1,0,0,0,0]+[1,0,0,0,0,0,0]+[1,1,0,0,0,0,0])q^2\\
&+([0,0,0,0,0,0,2]+[0,0,1,0,0,0,0]+[1,0,0,0,0,0,0]+[1,1,0,0,0,0,0])q^{-2}\\
&+[1,0,0,0,0,1,0](q^3+q^{-3})+[2,0,0,0,0,0,0](q^4+q^{-4})\Big)t^4+\dots\ .
\end{aligned}
\tag{5.2.147}
$$

The plethystic logarithm of this Hilbert series is given by

$$
\begin{aligned}
&\mathrm{PL}\left[H[E_8](t;\boldsymbol{x},q)\right]\\
&=\Big((1+[0,1,0,0,0,0,0])+[0,0,0,0,0,1,0]q+[0,0,0,0,0,0,1]q^{-1}\\
&\quad+[1,0,0,0,0,0,0](q^2+q^{-2})\Big)t^2-\Big(2+[2,0,0,0,0,0,0]+[0,0,0,1,0,0,0]+[0,1,0,0,0,0,0]\\
&\quad+([0,0,0,0,0,1,0]+[1,0,0,0,0,0,1])q+([0,0,0,0,0,0,1]+[1,0,0,0,0,1,0])q^{-1}\\
&\quad+([0,0,1,0,0,0,0]+[1,0,0,0,0,0,0])(q^2+q^{-2})\\
&\quad+[0,0,0,0,0,0,1]q^3+[0,0,0,0,0,1,0]q^{-3}+(q^4+q^{-4})\Big)t^4+\dots\ .
\end{aligned}
\tag{5.2.148}
$$

It is also useful to write the Hilbert series written in terms of characters of representations of $SO(16)$:

$$
\begin{aligned}
&H[E_8](t;\boldsymbol{z})\\
&=1+([0,0,0,0,0,0,0,1]+[0,1,0,0,0,0,0,0])t^2+\\
&\quad(1+[0,0,0,0,0,0,0,1]+[0,0,0,0,0,0,0,2]\\
&\quad+[0,0,0,1,0,0,0,0]+[0,1,0,0,0,0,0,1]+[0,2,0,0,0,0,0,0])t^4\dots\ .
\end{aligned}
\tag{5.2.149}
$$

The plethystic logarithm of this Hilbert series is

$$
\begin{aligned}
&\mathrm{PL}[H[E_8](t;\boldsymbol{z})]\\
&=([0,0,0,0,0,0,0,1]+[0,1,0,0,0,0,0,0])t^2-\Big(1+[0,0,0,1,0,0,0,0]\\
&\quad+[1,0,0,0,0,0,1,0]+[2,0,0,0,0,0,0,0]\Big)t^4+\dots\ .
\end{aligned}
\tag{5.2.150}
$$

Note that the spinor representation $[0,0,0,0,0,0,0,1]$ of $SO(16)$ branches to those of $SO(14)\times U(1)$ as

$$[0,0,0,0,0,0,0,1] \longrightarrow [0,0,0,0,0,0,1]_{-1} + [0,0,0,0,0,1,0]_{+1} \ , \qquad (5.2.151)$$

corresponding to the charge ± 1 instanton operators I_{1-} and I_{1+}, whereas the field X in the adjoint representation $[0,1,0,0,0,0,0,0]$ of $SO(16)$ contains the charge ± 2 instanton operators I_{2+}, I_{2-}, the glueball superfields S and the meson M.

Thus, one independent singlet at order t^4 of (5.2.149) implies that $I_{1+}I_{1-}$ is proportional to the singlet formed by I_{2+}, I_{2-}, S and M in X. The adjoint field X of $SO(16)$ satisfies the matrix relation

$$X^2 = 0 \ , \qquad (5.2.152)$$

transforming in the rank two symmetric representation $[2,0,0,0,0,0,0,0] + [0,0,0,0,0,0,0,0]$ of $SO(16)$. This representation branches into those of $SO(14)\times U(1)$ as

$$\begin{aligned}[2,0,0,0,0,0,0,0] \longrightarrow\ & 1 + [0,0,0,0,0,0,0]_{-4} + [0,0,0,0,0,0,0]_{+4} \\ & + [1,0,0,0,0,0,0]_{-2} + [1,0,0,0,0,0,0]_{+2} + [2,0,0,0,0,0,0]_0 .\end{aligned} \qquad (5.2.153)$$

Upon expanding (5.2.152) in components, we see that the vanishing components $(X^2)_{15,15}$, $(X^2)_{16,16}$ and $(X^2)_{15,16}$ imply that

$$I^a_{2+}I^a_{2+} = 0 \ , \qquad I^a_{2-}I^a_{2-} = 0 \ , \qquad S^2 + I^a_{2+}I^a_{2-} = 0 \ . \qquad (5.2.154)$$

These relations are collected in (5.2.164) and (5.2.174).

For future reference, the branching rule of the representation $[1,0,0,0,0,0,1,0]$ of $SO(16)$ to those of $SO(14)\times U(1)$ is

$$\begin{aligned}[1,0,0,0,0,0,1,0] \longrightarrow\ & [0,0,0,0,0,0,1]_{-3} + [0,0,0,0,0,0,1]_{+1} + [0,0,0,0,0,1,0]_{-1} \\ & + [0,0,0,0,0,1,0]_{+3} + [1,0,0,0,0,0,1]_{-1} + [1,0,0,0,0,1,0]_{+1} \ ,\end{aligned} \qquad (5.2.155)$$

and the branching rule of the representation $[0,0,0,1,0,0,0,0]$ of $SO(16)$ is

$$\begin{aligned}[0,0,0,1,0,0,0,0] \longrightarrow & [0,0,0,1,0,0,0]_0 + [0,0,1,0,0,0,0]_{-2} + [0,0,1,0,0,0,0]_{+2} \\ & + [0,1,0,0,0,0,0]_0 \ . \end{aligned} \tag{5.2.156}$$

For $SO(14)$, we use $a, b, c, d = 1, \dots, 14$ to denote vector indices and $\alpha, \beta, \rho, \sigma = 1, \dots, 64$ to denote the spinor indices. Note that the spinor representation of $SO(14)$ is complex. The delta symbol has the form

$$\delta^{\alpha}_{\beta} \ . \tag{5.2.157}$$

The gamma matrices γ^a can take the following forms:

$$(\gamma^a)_{\alpha\beta} \qquad \text{or} \qquad (\gamma^a)^{\alpha\beta} \ , \tag{5.2.158}$$

where the α, β indices are antisymmetric. The product of two gamma matrices is

$$(\gamma^{ab})^{\alpha}{}_{\rho} \equiv (\gamma^{[a})^{\alpha\beta}(\gamma^{b]})_{\beta\rho} \ . \tag{5.2.159}$$

The product of three gamma matrices has the forms

$$(\gamma^{abc})_{\alpha\rho} \equiv (\gamma^{[a})_{\alpha\beta}(\gamma^{b})^{\beta\sigma}(\gamma^{c]})_{\sigma\rho} \qquad \text{and} \qquad (\gamma^{abc})^{\alpha\rho} \equiv (\gamma^{[a})^{\alpha\beta}(\gamma^{b})_{\beta\sigma}(\gamma^{c]})^{\sigma\rho} \ , \tag{5.2.160}$$

symmetric in the spinor indices. The product of four gamma matrices is

$$(\gamma^{abcd})^{\alpha}{}_{\beta} \equiv (\gamma^{[a})^{\alpha\sigma_1}(\gamma^{b})_{\sigma_1\sigma_2}(\gamma^{c})^{\sigma_2\sigma_3}(\gamma^{d]})_{\sigma_3\beta} \ . \tag{5.2.161}$$

The generators of the moduli space are M^{ab}, which is a 14×14 antisymmetric matrix; the instanton operators I^{α} and $\widetilde{I}_{\alpha}$; and the gaugino superfield S.

The relations corresponding to order t^4 of (5.2.148) are as follows:

$$[2,0,0,0,0,0,0] + [0,0,0,0,0,0,0] : M^{ab}M^{bc} + I^{(a}_{2+}I^{c)}_{2-} = I^{\alpha}_{1+}(I_{1-})_{\alpha}\delta^{ac} \tag{5.2.162}$$

$$[0,0,0,1,0,0,0] : M^{[a_1a_2}M^{a_3a_4]} = (I_{1-})_{\beta}(\gamma^{a_1\cdots a_4})^{\beta}{}_{\alpha}I^{\alpha}_{1+} \tag{5.2.163}$$

$$[0,0,0,0,0,0,0] : S^2 + I^a_{2+}I^a_{2-} = 0 \tag{5.2.164}$$

$$[0,1,0,0,0,0,0] : SM^{ab} + I_{2+}^{[a} I_{2-}^{b]} = I_{1+}^{\alpha} (I_{1-})_{\beta} (\gamma^{ab})^{\beta}{}_{\alpha} \tag{5.2.165}$$

$$q([0,0,0,0,0,1,0] + [1,0,0,0,0,0,1]) : M^{ab} I_{1+}^{\alpha} (\gamma^{b})_{\alpha\beta} = S I_{1+}^{\alpha} (\gamma^{a})_{\alpha\beta} + I_{2+}^{a} (I_{1-})_{\beta} \tag{5.2.166}$$

$$q^{-1}([0,0,0,0,0,0,1] + [1,0,0,0,0,1,0]) : M^{ab} (I_{1-})_{\alpha} (\gamma^{b})^{\alpha\beta} = S (I_{1-})_{\alpha} (\gamma^{a})^{\alpha\beta} + I_{2-}^{a} I_{1+}^{\beta} \tag{5.2.167}$$

$$q^{2}[0,0,1,0,0,0,0] : M^{[ab} I_{2+}^{c]} = I_{1+}^{\alpha} (\gamma^{abc})_{\alpha\beta} I_{1+}^{\beta} \tag{5.2.168}$$

$$q^{-2}[0,0,1,0,0,0,0] : M^{[ab} I_{2-}^{c]} = (I_{1-})_{\alpha} (\gamma^{abc})^{\alpha\beta} (I_{1-})_{\beta} \tag{5.2.169}$$

$$q^{2}[1,0,0,0,0,0,0] : M^{ab} I_{2+}^{b} = S I_{2+}^{a} \tag{5.2.170}$$

$$q^{-2}[1,0,0,0,0,0,0] : M^{ab} I_{2-}^{b} = S I_{2-}^{a} \tag{5.2.171}$$

$$q^{3}[0,0,0,0,0,0,1] : I_{2+}^{a} I_{1+}^{\alpha} (\gamma^{a})_{\alpha\beta} = 0 \tag{5.2.172}$$

$$q^{-3}[0,0,0,0,0,0,1] : I_{2-}^{a} (I_{1-})_{\alpha} (\gamma^{a})^{\alpha\beta} = 0 \tag{5.2.173}$$

$$(q^{4} + q^{-4})[0,0,0,0,0,0,0] : I_{2+}^{a} I_{2+}^{a} = I_{2-}^{a} I_{2-}^{a} = 0\ . \tag{5.2.174}$$

Expansion in the instanton fugacity

The highest weight generating function (5.2.146) can be rewritten in terms of an implicit expansion in q involving 5 sums:

$$\begin{aligned}
\mathcal{G}[\widetilde{\mathcal{M}}_{1,E_8}](t;\boldsymbol{\mu};q) &= \mathrm{PE}\left[(1+\mu_2)\,t^2 + (1+\mu_2+\mu_4)\,t^4 + \mu_4 t^6\right] \\
&\times \mathrm{PE}\left[(\mu_6\mu_7+\mu_1^2)t^4 + (\mu_6\mu_7+\mu_3^2)t^8 + \mu_5^2 t^{12}\right] \\
&\times \Big(\sum_{n_1\geq 0} (\mu_6 t^2 q)^{n_1} + \sum_{n_1<0} (\mu_7 t^2)^{-n_1} q^{n_1}\Big) \sum_{n_2\in\mathbb{Z}} (\mu_1 t^2)^{|n_2|} q^{2n_2} \\
&\times \Big(\sum_{n_3\geq 0} (\mu_6 t^4 q)^{n_3} + \sum_{n_3<0} (\mu_7 t^4)^{-n_3} q^{n_3}\Big) \sum_{n_4\in\mathbb{Z}} (\mu_3 t^4)^{|n_4|} q^{2n_4} \sum_{n_5\in\mathbb{Z}} (\mu_5 t^6)^{|n_5|} q^{2n_5}\ .
\end{aligned} \tag{5.2.175}$$

5.3 $USp(4)$ with one antisymmetric hypermultiplet

In this theory, we pick the trivial value of the discrete theta angle for the $USp(4)$ gauge group. The Higgs branch at infinite coupling of this theory is identified with the reduced moduli space of 2 $SU(2)$ instantons on $\mathbb{C}^2$ [34], whose global symmetry is $SU(2) \times SU(2)$. The Hilbert series is given by (3.14) of [71]. For reference, we provide here the explicit expression of the Hilbert series up to order t^6:

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{2,SU(2)}](t;y,x) = 1 &+ ([0;2]+[2;0])t^2 + [1;2]t^3 + (1+[0;4]+[2;2]+[4;0])t^4 \\ &+ ([1;2]+[1;4]+[3;2])t^5 + ([0;2]+[0;6]+[2;0] \\ &+ 2[2;4]+[4;2]+[6;0])t^6 + \dots \ . \end{aligned} \tag{5.3.1}$$

The plethystic logarithm of this expression is

$$\begin{aligned} \mathrm{PL}\left[H[\widetilde{\mathcal{M}}_{2,SU(2)}](t;y,x)\right] &= ([0;2]+[2;0])t^2 + [1;2]t^3 - t^4 - ([1;2]+[1;0])t^5 \\ &\quad - ([2;0]+[0;2])t^6 + \dots \ . \end{aligned} \tag{5.3.2}$$

The corresponding highest weight generating function is (see (4.25) of [74])

$$\mathcal{G}[\widetilde{\mathcal{M}}_{2,SU(2)}](t;\mu_1,\mu_2) = \mathrm{PE}\left[(\mu_1^2+\mu_2^2)t^2 + \mu_1\mu_2^2 t^3 + t^4 + \mu_1\mu_2^2 t^5 - \mu_1^2\mu_2^4 t^{10}\right] \ , \tag{5.3.3}$$

where μ_1 and μ_2 are respectively the fugacities for the highest weights of the $SU(2)$ acting on the centre of instantons and the $SU(2)$ associated with the internal degrees of freedom.

Let us use the indices $a,b,c,d = 1,2$ for the first $SU(2)$ and $i,j,k,l = 1,2$ for the second $SU(2)$. The generators of the moduli space are as follows.

- **Order t^2:** The rank two symmetric tensors P_{ab} and M_{ij} in the representation $[2;0]$ and $[0;2]$ of $SU(2)\times SU(2)$:

$$P_{ab} = P_{ba} \ , \qquad M_{ij} = M_{ji} \ . \tag{5.3.4}$$

- **Order t^3:** A doublet of rank two symmetric tensors $(A_a)_{ij}$, with

$$(A_a)_{ij} = (A_a)_{ji} \ , \tag{5.3.5}$$

  in the representation $[1;2]$ of $SU(2)\times SU(2)$.

The singlet relation at order t^4 can be written as

$$[0;0]t^4 : \qquad \mathrm{Tr}(P^2) = \mathrm{Tr}(M^2) \ . \tag{5.3.6}$$

The relations at order t^5 are

$$[1;0]t^5: \qquad \epsilon^{ii'}\epsilon^{jj'}(A_a)_{ij}M_{i'j'} = 0 \ , \tag{5.3.7}$$

$$[1;2]t^5: \qquad \epsilon^{bb'}P_{ab}(A_{b'})_{ij} = \epsilon^{kk'}M_{ik}(A_a)_{k'j} + (i \leftrightarrow j) \ . \tag{5.3.8}$$

The relations at order t^6 are

$$[2;0]t^6: \qquad \mathrm{Tr}(P^2)P_{ab} = \epsilon^{ii'}\epsilon^{jj'}(A_a)_{ij}(A_b)_{i'j'} \ , \tag{5.3.9}$$

$$[0;2]t^6: \qquad \mathrm{Tr}(M^2)M_{ij} = \epsilon^{ab}\epsilon^{kk'}(A_a)_{ik}(A_b)_{k'j} \ . \tag{5.3.10}$$

Let us now rewrite the above statements in $SU(2) \times U(1)$ language. Up to charge normalisation, we identify the Cartan subalgebra of the latter $SU(2)$ associated with μ_2 with the $U(1)_I$ symmetry. More precisely, if w is the fugacity associated to the Cartan generator of the latter $SU(2)$, then $q = w^2$ is the fugacity for the topological symmetry. The highest weight generating function can then be written as

$$\mathcal{G}[\widetilde{\mathcal{M}}_{2,SU(2)}](t;\mu_1;q) = \mathrm{PE}\Big[\Big(1+\mu_1^2+(q+q^{-1})\Big)t^2 + \Big(\mu_1+\mu_1(q+q^{-1})\Big)t^3 - \mu_1 t^5 - \mu_1^2 t^6\Big] \ . \tag{5.3.11}$$

This can be written as a power series in q as

$$\mathcal{G}[\widetilde{\mathcal{M}}_{2,SU(2)}](t;\mu_1;q) = \frac{1}{(1-t^2)(1-t^4)(1-\mu_1 t)\left(1-\mu_1^2 t^2\right)(1-\mu_1 t^3)} \times \left((1-\mu_1^2 t^6)\sum_{j=-\infty}^{\infty} q^j t^{2|j|} - \left(1-t^4\right)\sum_{j=-\infty}^{\infty} q^j t^{2|j|}(\mu_1 t)^{|j|+1}\right) \ . \tag{5.3.12}$$

The Hilbert series up to order t^6 can be written explicitly as follows:

$$\begin{aligned}H[\widetilde{\mathcal{M}}_{2,SU(2)}](t;y,q) = 1 &+ \Big(1+[2]+(q+q^{-1})\Big)t^2 + \Big([1]+[1](q+q^{-1})\Big)t^3 \\ &+ \Big(2+[2]+[4]+(1+[2])(q+q^{-1})+(q^2+q^{-2})\Big)t^4 \\ &+ \Big(2[1]+[3]+(2[1]+[3])(q+q^{-1})+[1](q^2+q^{-2})\Big)t^5 \\ &+ \Big(2+3[2]+[4]+[6]+(2+2[2]+[4])(q+q^{-1}) \\ &\quad + (1+2[2])(q^2+q^{-2})+(q^3+q^{-3})\Big)t^6 + \dots \; .\end{aligned} \tag{5.3.13}$$

The plethystic logarithm of this Hilbert series is given by

$$\begin{aligned}\mathrm{PL}\left[H[\widetilde{\mathcal{M}}_{2,SU(2)}](t;y,q)\right] &= \Big(1+[2]+(q+q^{-1})\Big)t^2 + \Big([1]+[1](q+q^{-1})\Big)t^3 - t^4 \\ &\quad - \Big(2[1]+[1](q+q^{-1})\Big)t^5 - \Big(1+[2]+(q+q^{-1})\Big)t^6 \\ &\quad + \dots \; .\end{aligned} \tag{5.3.14}$$

The generators. At order t^2, the generators are

$$[2]: \qquad P_{ab} \quad \text{with } P_{ab} = P_{ba} \; , \tag{5.3.15}$$

$$q,\ q^{-1},\ 1: \qquad I,\ \widetilde{I},\ S \; . \tag{5.3.16}$$

The generators P_{ab} are identified as a product of two antisymmetric tensors:

$$P_{ab} = \mathrm{Tr}(X_a X_b) \; . \tag{5.3.17}$$

At order 3, the generators are denoted by

$$q[1],\ q^{-1}[1],\ [1]: \qquad J_a\ , \quad \widetilde{J}_a\ , \quad T_a\ . \tag{5.3.18}$$

where the generators T_a are identified as a product of two gauginos and one antisymmetric tensor

$$T_a = \mathrm{Tr}\left(X_a \mathcal{W}\mathcal{W}\right) \; . \tag{5.3.19}$$

The relations. The relation at order t^4 can be written as

$$[0]t^4: \qquad \mathrm{Tr}(P^2) + S^2 = I\widetilde{I} \; . \tag{5.3.20}$$

The relations at order t^5 can be written as

$$
\begin{aligned}
[1]t^5 : &\qquad ST_a = \widetilde{I} J_a + I \widetilde{J}_a \ , &(5.3.21)\\
q[1]t^5 : &\qquad P_{ab} J_{b'} \epsilon^{bb'} + I T_a + S J_a = 0 \ , &(5.3.22)\\
[1]t^5 : &\qquad P_{ab} T_{b'} \epsilon^{bb'} + 2 S T_a = 0 \ , &(5.3.23)\\
q^{-1}[1]t^5 : &\qquad P_{ab} \widetilde{J}_{b'} \epsilon^{bb'} + \widetilde{I} T_a + S \widetilde{J_a} = 0 \ . &(5.3.24)
\end{aligned}
$$

The relations at order t^6 can be written as

$$
\begin{aligned}
[2]t^6 : &\qquad S^2 P_{ab} + T_a T_b = J_{(a} \widetilde{J}_{b)} + I \widetilde{I} P_{ab} \ , &(5.3.25)\\
q t^6 : &\qquad S^2 I = \epsilon^{ab} J_a T_b + I^2 \widetilde{I} \ , &(5.3.26)\\
t^6 : &\qquad S^3 = \epsilon^{ab} J_a \widetilde{J}_b + S I \widetilde{I} \ , &(5.3.27)\\
q^{-1} t^6 : &\qquad S^2 \widetilde{I} = \epsilon^{ab} \widetilde{J}_a T_b + \widetilde{I}^2 I \ . &(5.3.28)
\end{aligned}
$$

5.4 $USp(2k)$ with one antisymmetric hypermultiplet

As in the previous sections, we pick the trivial value of the discrete theta angle for $USp(2k)$ gauge group. The Higgs branch of the conformal field theory at infinite coupling is identified with the moduli space of k $SU(2)$ instantons on $\mathbb{C}^2$ [34]. Below we consider the moduli space of the theory at finite coupling.

For $k = 1$, the Higgs branch at finite coupling is

$$\mathbb{C}^2 \times \mathbb{Z}_2 \ , \qquad (5.4.1)$$

where $\mathbb{C}^2$ is the classical moduli space of a $USp(2)$ gauge theory with 1 antisymmetric hypermultiplet and $\mathbb{Z}_2$ is the moduli space generated by the glueball superfield S such that $S^2 = 0$. The Hilbert series is then given by

$$
\begin{aligned}
H_{k=1}(t; x, w) &= H[\mathbb{Z}_2](t; w) H[\mathbb{C}^2](t; x) \\
&= (1 + w^2 t^2)\, \mathrm{PE}\left[t(x + x^{-1})\right] = \frac{1 + w^2 t^2}{(1 - tx)(1 - tx^{-1})} \ ,
\end{aligned} \qquad (5.4.2)
$$

where the fugacity w corresponds to the number of gaugino superfields.

For higher k, the theory in question can be realised as the worldvolume theory of k coincident D4-branes on an $O8^-$ plane. Hence, the moduli space is expected to be the k-th symmetric power of $\mathbb{C}^2 \times \mathbb{Z}_2$, whose Hilbert series is given by

$$\begin{aligned} H_k(t,x,w) &= \oint_{|\nu|=1} \frac{\mathrm{d}\nu}{2\pi i \nu^{k+1}} \exp\left(\sum_{m=1}^{\infty} \frac{\nu^m}{m} H_{k=1}(t^m; x^m, w^m) \right) \\ &= \sum_{j=0}^{k} (wt)^{2j} H[\mathrm{Sym}^j \mathbb{C}^2](t,x) H[\mathrm{Sym}^{k-j}\mathbb{C}^2](t,x) \ , \end{aligned} \tag{5.4.3}$$

where $H[\mathrm{Sym}^n \mathbb{C}^2](t,x)$ is the Hilbert series for the n-th symmetric power of $\mathbb{C}^2$:

$$H[\mathrm{Sym}^n \mathbb{C}^2](t,x) = \oint_{|\nu|=1} \frac{\mathrm{d}\nu}{2\pi i \nu^{n+1}} \exp\left(\sum_{m=1}^{\infty} \frac{\nu^m}{m} \frac{1}{(1-t^m x^m)(1-t^m x^{-m})} \right) \ . \tag{5.4.4}$$

We tested the result for $k=2$ directly from the field theory side using `Macaulay2`; the details are presented in Appendix B.1.

Note that this result also holds for $USp(2k)$ gauge theory with 1 antisymmetric hypermultiplet and 1 fundamental hypermultiplet. This is because the classical moduli space of this theory is the moduli space of k $SO(2)$ instantons on $\mathbb{C}^2$ — this space is in fact the k-symmetric power of the moduli space of 1 $SO(2)$ instanton on $\mathbb{C}^2$, which is identical to $\mathbb{C}^2$.

Since the symmetric product $\mathrm{Sym}^k(\mathbb{C}^2 \times \mathbb{Z}_2)$ has a $\mathbb{C}^2$ component that can be factored out, it is natural to define the Hilbert series $\widetilde{H}_k(t;x,w)$ of the reduced moduli space as follows:

$$H_k(t;x,w) = H[\mathbb{C}^2](t;x) \widetilde{H}_k(t;x,w) = \frac{1}{(1-tx)(1-tx^{-1})} \widetilde{H}_k(t;x,w) \ . \tag{5.4.5}$$

Examples

For $k=2$, we have

$$\begin{aligned} \widetilde{H}_{k=2}(t,x,w) &= (1+w^4t^4)(1-t^4)\,\mathrm{PE}[(x^2+1+x^{-2})t^2] + (wt)^2\,\mathrm{PE}[(x+x^{-1})t] \\ &= 1 + ([2]+w^2)t^2 + [1]w^2t^3 + ([4]+[2]w^2+w^4)t^4 + ([3]w^2)t^5 \\ &\quad + ([6]+[4]w^2+[2]w^4)t^6 + \dots \ . \end{aligned} \tag{5.4.6}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} &\mathrm{PL}[\widetilde{H}_{k=2}(t,x,w)] \\ &= ([2]+w^2)t^2 + [1]w^2t^3 - t^4 - [1](w^2+w^4)t^5 - ([2]w^4+w^6)t^6 + \ldots \ . \end{aligned} \tag{5.4.7}$$

For $k=3$, we have

$$\begin{aligned} \widetilde{H}_{k=3}(t,x,w) = 1 &+ ([2]+w^2)t^2 + ([3]+[1]w^2)t^3 + (1+[4]+2[2]w^2+w^4)t^4 \\ &+ ([3]+[5]+([1]+2[3])w^2+[1]w^4)t^5 + \Big([2]+2[6] \\ &+ (1+[2]+3[4])w^2 + 2[2]w^4 + w^6\Big)t^6 + \ldots \ . \end{aligned} \tag{5.4.8}$$

The plethystic logarithm of this Hilbert series is

$$\begin{aligned} \mathrm{PL}[\widetilde{H}_{k=3}(t,x,w)] = ([2]+w^2)t^2 + ([3]+[1]w^2)t^3 + [2]w^2t^4 - [1]t^5 \\ - \Big([2]+(1+[2])w^2+[2]w^4\Big)t^6 + \ldots \ . \end{aligned} \tag{5.4.9}$$

General k.

For general k, we have two sets of generators transforming in:

1. representation $[p]$ at order t^p, for all $2 \leq p \leq k$;
2. representation $[p]w^2$ at order t^{p+2}, for all $0 \leq p \leq k-1$;

these follow from the generators of the moduli space of two instantons, given by section 8.5 of [1]. Explicitly, these generators are

$$\begin{aligned} &\mathrm{Tr}(X_{a_1}X_{a_2}), \mathrm{Tr}(X_{a_1}X_{a_2}X_{a_3}), \ldots \ , \mathrm{Tr}(X_{a_1}X_{a_2}\cdots X_{a_k}), \\ &\mathrm{Tr}(\mathcal{W}\mathcal{W}),\ \mathrm{Tr}(X_{a_1}\mathcal{W}\mathcal{W}),\ \mathrm{Tr}(X_{(a_1}X_{a_2)}\mathcal{W}\mathcal{W})\ , \ldots,\ \mathrm{Tr}(X_{(a_1}\cdots X_{a_{k-1})}\mathcal{W}\mathcal{W}) \end{aligned} \tag{5.4.10}$$

where $a_1, a_2, \ldots, a_k = 1, 2$. The set of relations with the lowest dimension transform in the representation $[k-2]$ at order t^{k+2}.

In the limit $k \to \infty$, the moduli space is thus freely generated by (5.4.10).[12] A similar situation was considered in [66], where it was pointed out that the generating function of multi-trace operators for one brane is equal to that of single trace operators for infinitely many branes.

[12] We would like to express our thanks to Nick Dorey for his nice presentation at the Swansea workshop and especially for discussing this point.

5.5 Pure super Yang-Mills theories

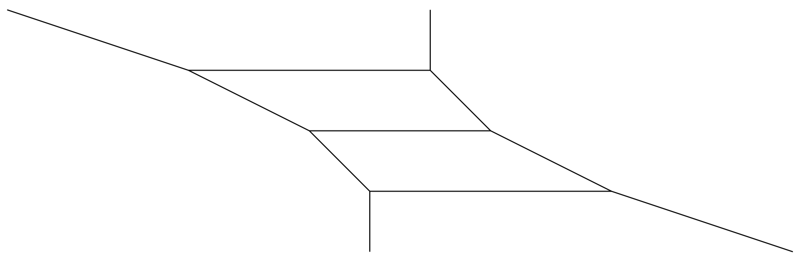
For 5d $\mathcal{N} = 1$ pure Yang-Mills theory, the Higgs branch at infinite coupling takes a simple orbifold structure. Field theoretic and stringy arguments can be provided for this statement.

In [49] it was argued by counting zero modes that for an $SU(N)$ gauge group the instanton operators transform in the spin-$\frac{N}{2}$ representation of $SU(2)_R$. In [50] the result was generalised to arbitrary gauge groups. Using the observation of [125] that the instanton contribution to the superconformal index is given by an "$SU(2)$-covariantized" version of the Hilbert series, the $SU(2)_R$ spin of instanton operators in pure Yang-Mills theories is given by $\frac{1}{2}h^\vee_G$, where $h^\vee_G$ is the dual Coxeter number of the group G. It is then straightforward to construct the relation between the instanton operators and the glueball operator:

$$S^{h^\vee_G} = I\widetilde{I} \ . \tag{5.5.1}$$

which reduces to the standard nilpotency for S [64] at finite coupling where the instanton operators are set to zero. The Higgs branch at infinite coupling is thus the orbifold $\mathbb{C}^2/\mathbb{Z}_{h^\vee_G}$.

For $SU(N)$ pure Yang-Mills a stringy construction provides a complementary viewpoint. For this therory, an $SL(2,\mathbb{Z})$ transformation on the 5-brane web can be exploited to set the charges of the external 5-brane legs to $(p_1, q_1) = (N, -1)$ and $(p_2, q_2) = (0, 1)$. In this basis, the web can be depicted as follows (this example is for $N = 3$):



(5.5.2)

At infinite coupling, the two 5-branes intersect and move apart, giving a one quaternionic dimensional Higgs branch, which has a cone structure. Using the classification of hyperKähler cones of dimension 1, the space has to be an ADE singularity. The existence in the chiral ring of the operator S, which has spin-1 under $SU(2)_R$, rules out the D and E cases, implying that the Higgs branch has to be $\mathbb{C}^2/\mathbb{Z}_m$, for some m. The value of m can be deduced by considering the intersection number, which is given by:

$$p_1 q_2 - p_2 q_1 = N \ . \tag{5.5.3}$$

The Higgs branch at infinite coupling is therefore $\mathbb{C}^2/\mathbb{Z}_N$.[13]

The generators of the Higgs branch at infinite coupling are I, S, $\widetilde{I}$, singlets under $SU(N)$, and with $U(1)_I$ charge $+1$, 0 and -1 respectively. For $N > 2$, the isometry group of $\mathbb{C}^2/\mathbb{Z}_N$ is identified with $U(1)_I$. For $N = 2$, the isometry of the Higgs branch is enhanced to $SU(2)$ and the operators form a triplet $(I, S, \widetilde{I})$.

The construction can be generalised by means of orientifold planes [37] to give analogous results for the case of classical gauge groups.

[13] We thank Cumrun Vafa for discussions about this point.

Chapter 6

Conclusion

The purpose of this thesis was to present techniques that allow for exploration of vacuum varieties in supersymmetric gauge theories and the role that some local operators that are not visible in the Lagrangian play in the structure of the vacuum.

Firstly we explicitly showed how the structure of vacuum varieties displays highly non-trivial behaviour even in its classical limit: a nice class of theories was presented whose classical moduli space is the union of two cones. The phenomenon is an interesting playground to show the links between the geometry of the vacuum variety and representation theory. The following two chapters were instead dedicated to quantum corrected moduli spaces.

In particular, chapter 4 was dedicated to proposing and testing a simple formula for the Hilbert series of moduli spaces of pure Yang-Mills instantons, which arise as Coulomb branches of three-dimensional $\mathcal{N} = 4$ generalised quiver gauge theories whose quiver diagrams are given by over-extended Dynkin diagrams. A natural modification of the monopole formula for the Coulomb branch Hilbert series introduced in [30] allowed for a comprehensive study of instantons in all simple Lie groups, including the non-simply laced ones. The proposal has been successfully tested against previous works for one and two instantons and produced new results for higher instanton numbers. General features of the moduli spaces of instantons can be systematically deduced from the formalism here presented. It would be interesting to derive the explicit ring structure of the moduli spaces by a careful analysis of monopole operators through the results presented here and using the recent proposal of [143].

Some natural questions remain open. Firstly, it would be nice to derive the formula from a path integral by folding the appropriate simply laced quiver via an outer automorphism group. This would help to understand the Higgs branch of such quivers and compute superconformal indices [23, 28, 103, 104, 102] . Secondly, the Coulomb branch formalism should also allow for the computation of the hyperKähler metric on the moduli spaces of instantons [14, 15]. Indeed, formulae (4.2)–(4.4) in [15] could be generalised to non-simply laced quivers by inserting the multiplicity λ in the matter contribution to the metric in analogy with (4.3.3). For classical groups, this suggestion can be tested against the metric obtained from the hyperKähler quotient in the Higgs branch of the corresponding ADHM quiver.

As far as the Higgs branch of 5d $\mathcal{N} = 1$ theories are concerned, a coherent picture for all values of the gauge coupling emerges in the work of chapter 5. In particular, we have presented explicit relations that define the chiral ring at infinite coupling and are consistent with those at finite coupling. A crucial result of this work is the correction to the glueball superfield, S, which at finite coupling is a nilpotent bilinear in the gaugino superfield and at infinite coupling becomes an ordinary chiral operator on the Higgs branch.

For pure $SU(2)$ theories with $N_f \leq 7$ flavours a nice pattern was established. The finite coupling relations involving mesons and the glueball operator are corrected at infinite coupling by bilinears in the instanton operators, in the obvious way dictated by representation theory. New relations also arise which exist uniquely at infinite coupling.

By expanding the highest weight generating function of the Hilbert series at infinite coupling in powers of q, we have analysed the dressing of instanton operators by mesons and gauginos. For $N_f \leq 5$ the defining equations for the space associated to the dressing can be obtained by keeping the relations at infinite coupling which are not corrected by the instanton operators. For $N_f = 6, 7$, the presence of charge ± 2 instanton operators as generators independent from the charge ± 1 ones complicates the picture and leaves the interpretation of the classical dressing in a preliminary and unsatisfactory stage.

The techniques developed here could also be applied to other 5d $\mathcal{N} = 1$ theories with known Higgs branch at infinite coupling. We leave this to future work. The long term goal is to better understand supersymmetric instanton operators and their dressing from first principles and use such knowledge to derive a general formula for the Hilbert series associated to the Higgs branch at infinite coupling.

It is furthermore the hope of the author that the pedagogical presentation strived for in this thesis can somewhat serve present and future researchers in the quest for further developments in the field.

Appendices

Appendix A

A.1 The Hilbert series for k $USp(2N)$ instantons for odd k via Higgs branch

For higher number of instantons, the Hilbert series can be computed more easily from the Higgs branch of the ADHM quiver. In particular for k odd the Hilbert series is given by

$$\begin{aligned} & g_{k,USp(2N)}(t;x;\boldsymbol{u}) \\ & = \frac{1}{2}\sum_{\omega=\pm 1}\int \mathrm{d}\mu_{SO(k)}(\boldsymbol{z})\,\mathrm{PE}\left[\omega\chi^{USp(2N)}_{[1,0,\ldots,0]}(\boldsymbol{u})\chi^{SO(k)}_{[1,0,\ldots,0]}(\boldsymbol{z})t \right. \\ & \quad \left. + (x+x^{-1})(\chi^{SO(k)}_{[2,0,\ldots,0]}(\boldsymbol{z})+1)t - t^2\chi^{SO(k)}_{[0,1,0,\ldots,0]}(\boldsymbol{z})\right]\,, \qquad (k \text{ odd}) \end{aligned} \tag{A.1.1}$$

where for $SO(k)$, the Dynkin labels $[1,0,\ldots,0]$, $[2,0,\ldots,0]$, $[0,1,0,\ldots,0]$ denotes the vector, the symmetric traceless, and the adjoint representations respectively. Here ω corresponds to the parity action ± 1 of $O(k) = SO(k)\times\{\pm 1\}$ for odd k. The Haar measure of $SO(2k+1)$ is given by

$$\begin{aligned} & \int \mathrm{d}\mu_{SO(2k+1)}(\boldsymbol{z}) \\ & = \oint_{|z_1|=1}\frac{\mathrm{d}z_1}{2\pi i z_1}\cdots\oint_{|z_k|=1}\frac{\mathrm{d}z_k}{2\pi i z_k}\prod_{1\le i<j\le k}(1-z_iz_j)\left(1-z_iz_j^{-1}\right)\prod_{m=1}^{k}(1-z_m)\,, \end{aligned} \tag{A.1.2}$$

where the adjoint representation is taken as

$$\chi^{SO(2k+1)}_{[0,1,0,\ldots,0]}(\boldsymbol{z}) = \sum_{1\le i<j\le k}\left(z_iz_j + z_iz_j^{-1}\right) + \sum_{m=1}^{k} z_m \tag{A.1.3}$$

The Hilbert series for the reduced moduli space of instantons is then given by

$$\widetilde{g}_{k,USp(2N)}(t;x;\boldsymbol{u}) = (1-tx)(1-tx^{-1})g_{k,USp(2N)}(t;x;\boldsymbol{u})\,. \tag{A.1.4}$$

A.2 Monopole operators and symmetry enhancement

A.2.1 G_2

The relevant diagram for k G_2 instantons is depicted below.

$$\underset{1}{\circ} - \overset{\boldsymbol{\alpha}_0}{\underset{k}{\bullet}} - \overset{\boldsymbol{\alpha}_1}{\underset{2k}{\circ}} \Rrightarrow \overset{\boldsymbol{\alpha}_2}{\underset{k}{\circ}} \tag{A.2.1}$$

where the simple roots $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ are indicated above the nodes. The positive roots of G_2 are of the form $c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2$, with (c_1, c_2) listed in Table A.1. For each positive root, we tabulate the monopole operators associated with it.

Positive root	$R^{(\boldsymbol{\alpha}_0)}$	$R^{(\boldsymbol{\alpha}_1)}$	$R^{(\boldsymbol{\alpha}_2)}$
$(1,0)$	(0)	$(1,0)$	(0)
$(0,1)$	(0)	$(0,0)$	(1)
$(1,1)$	(0)	$(1,0)$	(1)
$(1,2)$	(0)	$(1,0)$	(2)
$(1,3)$	(0)	$(1,0)$	(3)
$(2,3)$	(0)	$(1,1)$	(3)

Table A.1: Magnetic charges $R_i^{(\boldsymbol{\alpha})}$ of the monopole operators that contribute to each positive root $\boldsymbol{\alpha}$ of G_2 for $k = 1$ instanton.

A.2.2 F_4

The relevant diagram for k F_4 instantons is depicted below.

$$\underset{1}{\circ} - \overset{\boldsymbol{\alpha}_0}{\underset{k}{\bullet}} - \overset{\boldsymbol{\alpha}_1}{\underset{2k}{\circ}} - \overset{\boldsymbol{\alpha}_2}{\underset{3k}{\circ}} \Rightarrow \overset{\boldsymbol{\alpha}_3}{\underset{2k}{\circ}} - \overset{\boldsymbol{\alpha}_4}{\underset{k}{\circ}} \tag{A.2.2}$$

The 24 positive roots of F_4 are of the form $\sum_{i=1}^{4} c_i\boldsymbol{\alpha}_i$, with $(c_1, \ldots, c_4)$ listed in Table A.2. For each positive root, we tabulate the monopole operators associated with it.

Positive root	$R^{(\boldsymbol{\alpha}_0)}$	$R^{(\boldsymbol{\alpha}_1)}$	$R^{(\boldsymbol{\alpha}_2)}$	$R^{(\boldsymbol{\alpha}_3)}$	$R^{(\boldsymbol{\alpha}_4)}$
$(1,0,0,0)$	(0)	$(1,0)$	$(0,0,0)$	$(0,0)$	(0)
$(0,1,0,0)$	(0)	$(0,0)$	$(1,0,0)$	$(0,0)$	(0)
$(0,0,1,0)$	(0)	$(0,0)$	$(0,0,0)$	$(1,0)$	(0)
$(0,0,0,1)$	(0)	$(0,0)$	$(0,0,0)$	$(0,0)$	(1)
$(1,1,0,0)$	(0)	$(1,0)$	$(1,0,0)$	$(0,0)$	(0)
$(0,1,1,0)$	(0)	$(0,0)$	$(1,0,0)$	$(1,0)$	(0)
$(0,0,1,1)$	(0)	$(0,0)$	$(0,0,0)$	$(1,0)$	(1)
$(0,1,1,1)$	(0)	$(0,0)$	$(1,0,0)$	$(1,0)$	(1)
$(1,1,1,0)$	(0)	$(1,0)$	$(1,0,0)$	$(1,0)$	(0)
$(1,1,1,1)$	(0)	$(1,0)$	$(1,0,0)$	$(1,0)$	(1)
$(0,1,2,0)$	(0)	$(0,0)$	$(1,0,0)$	$(2,0)$	(0)
$(1,1,2,0)$	(0)	$(1,0)$	$(1,0,0)$	$(2,0)$	(0)
$(0,1,2,1)$	(0)	$(0,0)$	$(1,0,0)$	$(2,0)$	(1)
$(1,2,2,0)$	(0)	$(1,0)$	$(1,1,0)$	$(2,0)$	(0)
$(1,1,2,1)$	(0)	$(1,0)$	$(1,0,0)$	$(2,0)$	(1)
$(0,1,2,2)$	(0)	$(0,0)$	$(1,0,0)$	$(2,0)$	(2)
$(1,2,2,1)$	(0)	$(1,0)$	$(1,1,0)$	$(2,0)$	(1)
$(1,1,2,2)$	(0)	$(1,0)$	$(1,0,0)$	$(2,0)$	(2)
$(1,2,3,1)$	(0)	$(1,0)$	$(1,1,0)$	$(2,1)$	(1)
$(1,2,2,2)$	(0)	$(1,0)$	$(1,1,0)$	$(2,0)$	(2)
$(1,2,3,2)$	(0)	$(1,0)$	$(1,1,0)$	$(2,1)$	(2)
$(1,2,4,2)$	(0)	$(1,0)$	$(1,1,0)$	$(2,2)$	(2)
$(1,3,4,2)$	(0)	$(1,0)$	$(1,1,1)$	$(2,2)$	(2)
$(2,3,4,2)$	(0)	$(1,1)$	$(1,1,1)$	$(2,2)$	(2)

Table A.2: Magnetic charges $R_i^{(\boldsymbol{\alpha})}$ of the monopole operators that contribute to each positive root $\boldsymbol{\alpha}$ of F_4 for $k=1$ instanton.

A.2.3 C_N

The relevant diagram for k $USp(2N)$ instantons is depicted below.

$$\underset{1}{\circ} - \overset{\boldsymbol{\alpha}_0}{\underset{k}{\bullet}} \Rightarrow \underbrace{\overset{\boldsymbol{\alpha}_1}{\underset{k}{\circ}} - \cdots - \overset{\boldsymbol{\alpha}_{N-1}}{\underset{k}{\circ}}}_{N-1 \text{ nodes}} \Leftarrow \overset{\boldsymbol{\alpha}_N}{\underset{k}{\circ}} \tag{A.2.3}$$

where the simple roots are indicated above each node. The positive roots of $USp(2N)$ are

$$\Delta_+ = \{\boldsymbol{e}_i + \boldsymbol{e}_j\}_{i<j} \cup \{\boldsymbol{e}_i - \boldsymbol{e}_j\}_{i<j} \cup \{2\boldsymbol{e}_i\}_{i=1}^{N} \ , \tag{A.2.4}$$

where $\{\boldsymbol{e}_i\}$ is the standard basis. The simple roots of $USp(2N)$ can be written as

$$\begin{aligned} \boldsymbol{\alpha}_\ell &= \boldsymbol{e}_\ell - \boldsymbol{e}_{\ell+1} \ , \qquad 1 \le \ell \le N-1 \ , \\ \boldsymbol{\alpha}_N &= 2\boldsymbol{e}_N \ . \end{aligned} \tag{A.2.5}$$

The positive roots can be written in terms of the simple roots as

$$\begin{aligned} \boldsymbol{e}_i - \boldsymbol{e}_j &= \sum_{\ell=i}^{j-1} \alpha_\ell \ , \\ 2\boldsymbol{e}_i &= 2\sum_{\ell=i}^{N-1} \boldsymbol{\alpha}_\ell + \boldsymbol{\alpha}_N \ , \\ \boldsymbol{e}_i + \boldsymbol{e}_j &= \sum_{\ell=i}^{j-1} \boldsymbol{\alpha}_\ell + 2\sum_{\ell=j}^{N-1} \boldsymbol{\alpha}_\ell + \boldsymbol{\alpha}_N \ . \end{aligned} \tag{A.2.6}$$

The magnetic charges $R_i^{(\boldsymbol{\alpha})}$ of the monopole operators that contribute to each positive root $\boldsymbol{\alpha}$ of C_N for $k = 1$ instanton are as follows:

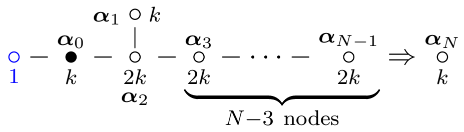
- $\boldsymbol{e}_i - \boldsymbol{e}_j$: (1) from nodes $\boldsymbol{\alpha}_p$ with $1 \le p \le j-1$, and (0) from other nodes.

- $2\boldsymbol{e}_i$: (0) from node $\boldsymbol{\alpha}_0$, (2) from nodes $\boldsymbol{\alpha}_p$ with $i \le p \le N-1$, and (1) from node $\boldsymbol{\alpha}_N$.

- $\boldsymbol{e}_i + \boldsymbol{e}_j$: (0) from node $\boldsymbol{\alpha}_0$, (1) from node $\boldsymbol{\alpha}_p$ with $1 \le p \le j-1$, (2) from node $\boldsymbol{\alpha}_q$ with $j \le q \le N-1$, and (1) from node $\boldsymbol{\alpha}_N$.

A.2.4 B_N

The relevant diagram for k $SO(2N+1)$ instantons is depicted below.



(A.2.7)

where the simple roots are indicated at each node. The positive roots of $SO(2N+1)$ are

$$\Delta_+ = \{\boldsymbol{e}_i + \boldsymbol{e}_j\}_{i<j} \cup \{\boldsymbol{e}_i - \boldsymbol{e}_j\}_{i<j} \cup \{\boldsymbol{e}_i\}_{i=1}^N \ , \tag{A.2.8}$$

where $\{\boldsymbol{e}_i\}$ is the standard basis. The simple roots of $SO(2N+1)$ can be written as

$$\begin{aligned} \boldsymbol{\alpha}_\ell &= \boldsymbol{e}_\ell - \boldsymbol{e}_{\ell+1} \ , \qquad 1 \le \ell \le N-1 \ , \\ \boldsymbol{\alpha}_N &= \boldsymbol{e}_N \ . \end{aligned} \tag{A.2.9}$$

The positive roots can be written in terms of the simple roots as

$$\begin{aligned} \boldsymbol{e}_i - \boldsymbol{e}_j &= \sum_{\ell=i}^{j-1} \alpha_\ell \ , \\ \boldsymbol{e}_i &= \sum_{\ell=i}^{N} \boldsymbol{\alpha}_\ell \ , \\ \boldsymbol{e}_i + \boldsymbol{e}_j &= \sum_{\ell=i}^{j-1} \boldsymbol{\alpha}_\ell + 2\sum_{\ell=j}^{N} \boldsymbol{\alpha}_\ell \ . \end{aligned} \tag{A.2.10}$$

The magnetic charges of the monopole operators that contribute to each positive root $\boldsymbol{\alpha}$ of B_N for any instanton number are as follows:

- $\boldsymbol{e}_i$: $(1, \boldsymbol{0})$ from the nodes $\boldsymbol{\alpha}_p$ with $i \le p \le N$, and $(\boldsymbol{0})$ from other nodes.
- $\boldsymbol{e}_i - \boldsymbol{e}_j$: $(1, \boldsymbol{0})$ from the nodes $\boldsymbol{\alpha}_p$ with $i \le p \le j-1$, and $(\boldsymbol{0})$ from other nodes.
- $\boldsymbol{e}_i + \boldsymbol{e}_j$: $(1, \boldsymbol{0})$ from the nodes $\boldsymbol{\alpha}_p$ with $i \le p \le j-1$, $(1^2, \boldsymbol{0})$ from the nodes $\boldsymbol{\alpha}_q$ with $j \le q \le N-1$, (2) from the node $\boldsymbol{\alpha}_N$.

Appendix B

B.1 Hilbert series of chiral rings with gaugino superfields

In this appendix we present a method to compute the Hilbert series of the Higgs branch at finite coupling. In this computation we include the classical chiral operators as well as the gaugino superfield $\mathcal{W}$.

In five dimensions, the gaugino λ_I^A carries the $USp(4)$ spin index $A = 1, \ldots, 4$ and the $SU(2)_R$ index $I = 1, 2$. Since we focus on holomorphic functions, which are highest weights of $SU(2)_R$ representations, we restrict ourselves to $I = 1$. In 4d $\mathcal{N} = 1$ language the fundamental representation of $USp(4)$ decomposes to $[1;0] + [0;1]$ of $SU(2) \times SU(2)$. These are usually denoted by undotted and dotted indices, respectively. Since the latter correspond to non-chiral operators in the 4d $\mathcal{N} = 1$ holomorphic approach, we adhere to the undotted $SU(2)$ spinor index. The gaugino superfield is henceforth denoted as $\mathcal{W}_\alpha$.

We will see that the 4d $\mathcal{N} = 1$ formalism adopted in this appendix yields results for the Hilbert series that are consistent with the chiral ring obtained by setting instanton and anti-instanton operators to zero in the five-dimensional UV fixed point, which is discussed in the Chapter 5.

B.1.1 $SU(2)$ gauge theory with N_f flavours

Let us denote the chiral matter fields appearing in the Lagrangian by Q^i_a, with $i = 1, \ldots, 2N_f$ and $a = 1, 2$. The F-terms relevant to the classical Higgs branch are[1]

$$\epsilon^{ab}\epsilon^{cd}Q^i_a Q^i_d = 0 \ . \tag{B.1.1}$$

These relations are symmetric under the interchange of the indices b and c.

Now let us discuss the inclusion of the gaugino superfield $(\mathcal{W}_\alpha)_{ab}$. $\mathcal{W}_\alpha$ is

[1] Here and in Chapter 5 , our relations are valid in the chiral ring. As operator relations, they hold up to a superderivative.

adjoint valued and is chosen to be a traceless symmetric 2-index tensor:

$$\epsilon^{ab}(\mathcal{W}_\alpha)_{ab} = 0 \ . \tag{B.1.2}$$

Moreover, we impose the following conditions (see section 2 of [64]) :

$$\text{Each component of } (\mathcal{W}_\alpha)_{ab} \text{ is an anti-commuting variable ,} \tag{B.1.3}$$
$$\epsilon^{bc}(\mathcal{W}_\alpha)_{ab}(\mathcal{W}_\beta)_{cd} + (\beta \leftrightarrow \alpha) = 0 \quad \forall\ \alpha, \beta = 1, 2,\ a, d = 1, 2, \tag{B.1.4}$$
$$\epsilon^{bc}(\mathcal{W}_\alpha)_{ab}Q^i_c = 0 \qquad \forall\ \alpha = 1, 2,\ a = 1, 2,\ i = 1, \dots, N_f \ . \tag{B.1.5}$$

The condition (B.1.3) follows from the fact that the lowest component of the gaugino superfield is fermionic. The relation (B.1.4) follows from gauge invariance and supersymmetry. The relation (B.1.5) indicates how the gaugino superfield acts on fundamental fields.

The Hilbert series of the ring of variables Q^i_a, $(\mathcal{W}_\alpha)_{ab}$ subject to the conditions (B.1.1), (B.1.2), (B.1.3), (B.1.4) and (B.1.5) can be computed using `Macaulay2`. For reference, we provide the Macaulay2 code for the case of $N_f = 3$ in source code (SC) 1.

After integrating over the $SU(2)$ gauge group and restricting to the scalar sector under the Lorentz group, we obtain the Hilbert series of the space

$$\widetilde{\mathcal{M}}_{1,SO(2N_f)} \cup \mathbb{Z}_2 \ , \tag{B.1.6}$$

where $\widetilde{\mathcal{M}}_{1,SO(2N_f)}$ is the reduced moduli space of one $SO(2N_f)$ instanton on $\mathbb{C}^2$ and $\mathbb{Z}_2$ is the moduli space generated by the glueball superfield S such that $S^2 = 0$

$$\begin{aligned} H[\widetilde{\mathcal{M}}_{1,SO(2N_f)} \cup \mathbb{Z}_2](t; \boldsymbol{x}, w) &= H[\mathbb{Z}_2](t; w) + H[\widetilde{\mathcal{M}}_{1,SO(2N_f)}](t; \boldsymbol{x}) - 1 \\ &= w^2t^2 + \sum_{p=0}^{\infty}[0, p, 0, \dots, 0]t^{2p} \ , \end{aligned} \tag{B.1.7}$$

where the fugacity w counts the number of gaugino superfields $\mathcal{W}$ and $\boldsymbol{x}$ are the fugacities of $SO(2N_f)$. The plethystic logarithm up to order t^4 of this is

$$\begin{aligned} \mathrm{PL}\left[H[\widetilde{\mathcal{M}}_{1,SO(2N_f)} \cup \mathbb{Z}_2](t; x, w)\right] &= ([0, 1, 0, \dots, 0] + w^2)t^2 - \Big(1 + [2, 0, \dots, 0] \\ &\quad + [0, 0, 0, 1, 0, \dots, 0] + w^2[0, 1, 0, \dots, 0] + w^4\Big)t^4 + \dots \ . \end{aligned} \tag{B.1.8}$$

This shows that the generators are the meson $M^{ij} = -M^{ji}$, in the adjoint representation of $SO(2N_f)$, and the glueball $S = -\frac{1}{32\pi^2} \operatorname{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha$, subject to the relations

$$M^{ij} M^{jk} = 0 \ , \quad M^{[ij} M^{kl]} = 0 \ , \quad S M^{ij} = 0 \ , \quad S^2 = 0 \ . \tag{B.1.9}$$

```
R=QQ[Q11,Q12,Q13,Q14,Q15,Q16,Q21,Q22,Q23,Q24,Q25,Q26,w111,w121,w211,
  w221,w112,w122,w212,w222,SkewCommutative=>{w111,w121,w211,w221,w112,
  w122,w212,w222},Degrees=>{{1,0,1,0},{1,0,1,0},{1,0,1,0},{1,0,1,0},{1
  ,0,1,0},{1,0,1,0},{1,0,-1,0},{1,0,-1,0},{1,0,-1,0},{1,0,-1,0},{1,0,-
  1,0},{1,0,-1,0},{1,1,2,1},{1,1,0,1},{1,1,0,1},{1,1,-2,1},{1,1,2,-1},
  {1,1,0,-1},{1,1,0,-1},{1,1,-2,-1}}]
I=ideal(w121 - w211, w122 - w212,-2*Q21*Q24 - 2*Q22*Q25 - 2*Q23*Q26,
  2*Q14*Q21 + 2*Q15*Q22 + 2*Q16*Q23 + 2*Q11*Q24 + 2*Q12*Q25 +
  2*Q13*Q26, 0, -2*Q11*Q14 - 2*Q12*Q15 - 2*Q13*Q16,2*w111*w211
  - 2*w121*w111, 2*w111*w221 - 2*w121*w121, 2*w211*w211 -
  2*w221*w111, 2*w211*w221 - 2*w221*w121, w111*w212 + w112*w211
  - w121*w112 - w122*w111, w111*w222 + w112*w221 - w121*w122
  - w122*w121, w211*w212 + w212*w211 - w221*w112 - w222*w111,
  w211*w222 + w212*w221 - w221*w122 - w222*w121, w111*w212 +
  w112*w211 - w121*w112 - w122*w111, w111*w222 + w112*w221 -
  w121*w122 - w122*w121, w211*w212 + w212*w211 - w221*w112 -
  w222*w111, w211*w222 + w212*w221 - w221*w122 - w222*w121,
  2*w112*w212 - 2*w122*w112, 2*w112*w222 - 2*w122*w122,
  2*w212*w212 - 2*w222*w112, 2*w212*w222 - 2*w222*w122,Q21*w111
  - Q11*w121, Q21*w211 - Q11*w221, Q21*w112 - Q11*w122, Q21*w212
  - Q11*w222, Q22*w111 - Q12*w121, Q22*w211 - Q12*w221, Q22*w112
  - Q12*w122, Q22*w212 - Q12*w222, Q23*w111 - Q13*w121, Q23*w211
  - Q13*w221, Q23*w112 - Q13*w122, Q23*w212 - Q13*w222, Q24*w111
  - Q14*w121, Q24*w211 - Q14*w221, Q24*w112 - Q14*w122, Q24*w212
  - Q14*w222, Q25*w111 - Q15*w121, Q25*w211 - Q15*w221, Q25*w112
  - Q15*w122, Q25*w212 - Q15*w222, Q26*w111 - Q16*w121,
  Q26*w211 - Q16*w221, Q26*w112 - Q16*w122, Q26*w212 - Q16*w222)
toString hilbertSeries(R/I, Reduce=>true)
```

SC 1: A `Macaulay2` code to compute the Hilbert series of the ring of variables Q^i_a, $(\mathcal{W}_\alpha)_{ab}$, with $N_f = 3$, subject to the conditions (B.1.1), (B.1.2), (B.1.3), (B.1.4) and (B.1.5). Here we write Q^i_a as `Qai` and $(\mathcal{W}_\alpha)_{ab}$ as `wab`α. The ring `R` is multi-graded with respect to the following charges (in order): 1. the R-charge associated with the fugacity t, 2. the number of gaugino superfields associated with the fugacity w, 3. the weights of the $SU(2)$ gauge group, and 4. the weights of the $SU(2)$ symmetry associated with the index α.

B.1.2 $USp(2k)$ gauge theory with one antisymmetric hypermultiplet

The analysis is similar to the previous subsection. Let us denote the antisymmetric fields by X_a^{ij}, where $a = 1, 2$ and $i, j = 1, \ldots, 2k$ are the $USp(2k)$ gauge indices. The F-terms associated to the classical Higgs branch is

$$J_{ii'} J_{jj'} J_{kk'} \epsilon^{ab} X_a^{ij} X_b^{k'i'} = 0 \ , \tag{B.1.10}$$

where J_{ij} is the symplectic matrix associated with $USp(2k)$.

For the gaugino superfield $\mathcal{W}_\alpha^{ij}$ (with $\alpha = 1, 2$), we impose the conditions [64]

$$\mathcal{W}_\alpha^{ij} = \mathcal{W}_\alpha^{ji} \ , \tag{B.1.11}$$

$$\text{each component of } \mathcal{W}_\alpha^{ij} \text{ is an anti-commuting variable} \ , \tag{B.1.12}$$

$$J_{jk} \mathcal{W}_{(\alpha}^{ij} \mathcal{W}_{\beta)}^{kl} = 0, \tag{B.1.13}$$

$$J_{jk} (\mathcal{W}_\alpha^{ij} X_a^{kl} - X_a^{ij} \mathcal{W}_\alpha^{kl}) = 0 \ . \tag{B.1.14}$$

After integrating over the $USp(2k)$ gauge group and restricting to the scalar sector under the Lorentz group, we obtain the Hilbert series of the space

$$\text{Sym}^k \left(\mathbb{C}^2 \times \mathbb{Z}_2 \right) \ , \tag{B.1.15}$$

In particular, for $k = 2$, we recover the Hilbert series (5.4.6).

B.2 $N_f = 6$ in representations of $SO(12) \times SU(2)$

Here we rewrite (5.2.129) and (5.2.130) in terms of characters of representations of $SO(12) \times SU(2)$:

$$\begin{aligned} &H[E_7](t; \boldsymbol{x}, y) \\ &= 1 + ([0,0,0,0,0,0;2] + [0,0,0,0,1,0;1] + [0,1,0,0,0,0;0])t^2 \\ &\quad + (1 + [0,0,0,0,0,0;4] + [0,0,0,0,1,0;1] + [0,0,0,0,1,0;3] + [0,0,0,0,2,0;2] \\ &\quad + [0,0,0,1,0,0;0] + [0,1,0,0,0,0;2] + [0,1,0,0,1,0;1] + [0,2,0,0,0,0;0])t^4 \\ &\quad + \ldots \ . \end{aligned} \tag{B.2.1}$$

The plethystic logarithm of (B.2.1) is

$$\begin{aligned}
&\mathrm{PL}\left[H[E_7](t;\boldsymbol{x},y)\right] \\
&= ([0,0,0,0,0,0;2]+[0,0,0,0,1,0;1]+[0,1,0,0,0,0;0])t^2 - \Big(2+[0,0,0,1,0,0;0] \\
&\quad +[2,0,0,0,0,0;0]+[0,0,0,0,1,0;1]+[1,0,0,0,0,1;1]+[0,1,0,0,0,0;2]\Big)t^4 \\
&\quad + \dots \ .
\end{aligned} \tag{B.2.2}$$

The representation $[0,0,0,0,0,0;2]$ corresponds to I_{2+}, I_{2-} and S, $[0,0,0,0,1,0;1]$ to $I_{1\pm}$ and $[0,1,0,0,0,0;0]$ to M. In the Hilbert series (B.2.1) there is only one independent singlet at order t^4: this means that the singlets coming from these three sets of operators must be proportional to each other. These indeed correspond to the trace part of (5.2.136) and the relation (5.2.138).

Appendix C

C.1 The dual theory of $Sp(n)$ with $2n$ flavours under 3d mirror symmetry

Since the Higgs branch of theories with 8 supercharges receives no quantum corrections [9] and independent of the spacetime dimension, we can consider the theories at hand to be defined in $3d$ with $\mathcal{N} = 4$ supersymmetry. This perspective is useful because 3d mirror symmetry [14] can then be exploited. In so doing a dual theory, whose Coulomb branch is identical to the Higgs branch we have studied, can be identified. This can be accomplished using the brane engineering introduced in [16], through the generalisation by means of orientifold planes in [106, 107] that allows for brane constructions for theories with $SO(2N)$ flavour symmetry.

The original gauge theory with $SO(2N)$ flavour symmetry is best engineered in its Coulomb branch, as here the brane picture is clear. A sequence of brane moves allows for the Higgs branch to be reached. Subsequently S-duality on the branes can be implemented, which corresponds to effecting 3d mirror symmetry for the gauge theory on the world-volume of the branes. At this stage the brane construction is depicting the 3d dual theory in its Coulomb branch. Thus, the specifics of this dual gauge theory can now be read off from the branes.

The brane construction for $Sp(k)$ with $SO(2N)$ flavour symmetry can be implemented in the following way. We take Type IIB and orientifold it by means of an $O5^-$ along the 012789 directions, i.e we take the quotient $\mathbb{R}^{1,9}/\Omega\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reversing each of the 3456 coordinates and Ω is the worldsheet parity operator. We place a NS5 brane that stretches through the 012345 directions at some distance away on the positive $s = 6$ direction (w.r.t. the orientifold position which we set as the origin). Moreover we add N D5 branes that stretch through the same directions as the orientifold but again at some distance away on the positive $s = 6$ direction. This configuration preserves eight supercharges. k half-D3 branes can be added at any point along the 345 directions and stretching along the 0126 directions, without further breaking supersymmetry. The orientifold induces brane images to its left along the $s = 6$ direction, i.e one NS5 brane image and N D5 brane images, as well as opposite images along the 345 directions, i.e k half-D3 brane images. Its action on the field theory living on the world volume of the branes is to project out some string states, leaving an $SO(2N)$ gauge symmetry on the stack of $D5$-branes and an $Sp(k)$ gauge symmetry on the D3 branes. For an observer on the latter, the result is an $Sp(k)$ gauge theory with $SO(2N)$ flavour symmetry. The brane construction is sketched in Figure C.1.

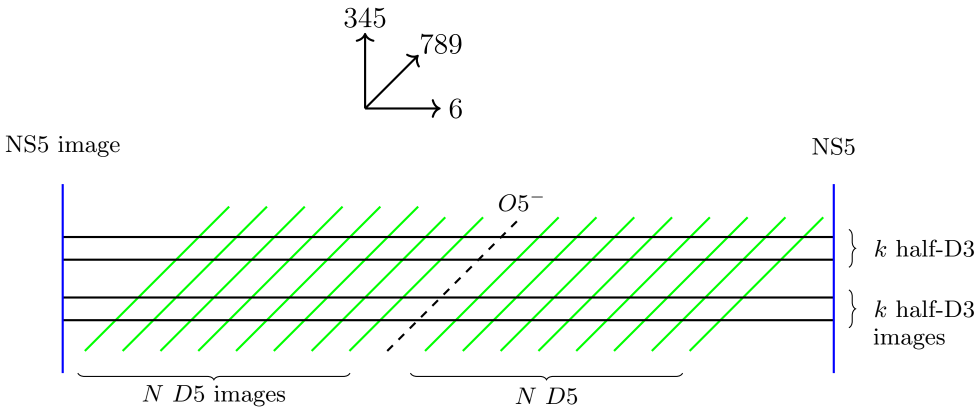


Figure C.1: Coulomb branch of $Sp(k)$ with N flavours. Each black line corresponds to a half-D3 brane. Here $k = 2$ and $N = 8$. The one presented here is the double cover of the orientifold $O5^-$ theory.

Ensuring the $D5$ branes are positioned at the origin of the 345 directions, as shown in Figure C.1, sets the masses to zero. In order to go to the origin of the Coulomb branch the k D3 branes are shifted along the 345 directions so that they touch the N D5 branes. We sketch this in Figure C.2.

When the D5 branes and the D3 branes sit at the same point on the 345 directions, the latter can maximally split, turning on all the moduli that parametrise the Higgs branch. The splitting must take into account the fact that the $O5^-$ projection doesn't allow a D3 brane to stretch between a $D5$ brane and its mirror image .

Moreover, the maximally splitting of the $D3$ branes has to be achieved supersymmetrically: the non supersymmetric s-configuration, namely more than one D3 brane stretching between an NS5 and a D5 brane, is not allowed.

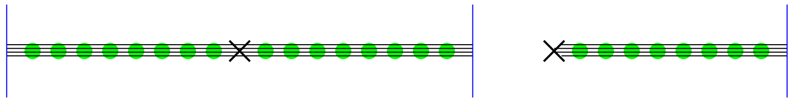
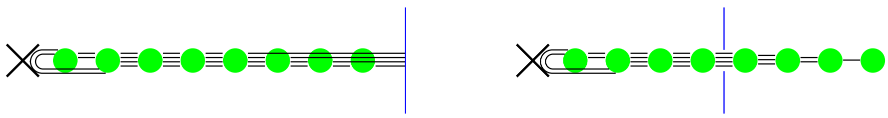


Figure C.2: The origin of the moduli space for $Sp(k)$ with N flavours: the $2k$ half-D3 branes are at the same position as the N D5 branes and the $O5^{-}$ on the 345 direction. On the left is the double cover of the origin of the moduli space and on the right the physical space. The picture has been simplified: the green dots represent D5 branes (and their images), the cross is the orientifold plane and the blue line the NS5 brane (and its image).

Thus, if a D3 brane is already stretched between a D5 and an NS5 brane, the “next" D3 brane can't split there and has to stretch all the way to the next available D5 brane. The resulting configuration is sketched in (a) of Figure C.3
.

The last step is executed for convenience: the NS5 brane can be moved across the D5 branes intervals $2k$ times: each such time a D3 brane is destroyed. The result is sketched in (b) of Figure C.3.



(a) (b)

Figure C.3: The Higgs branch is achieved by maximally breaking the D3 branes between the D5 branes. Near the orientifold plane, the right projection must be adopted, i.e. D3 branes cannot stretch between a D5 brane and its image. At the NS5 end of the system, caution must also be used: a supersymmetric configuration is achieved when at most one D3 brane stretches between a D5 and an NS5 brane. A D3 brane that stretches leftward from the NS5 brane towards a D5 brane can end on the latter provided it is the first to do so: otherwise it must continue onwards to the next left D5 brane. This is how the configuration sketched in (a) is achieved. There is still freedom to move the NS5 brane across the D5 branes, as this does not affect the moduli space. Each motion of the NS5 across a D5 brane results in the annihilation of a $D3$ brane. Moving the NS5 brane across $2k$ intervals results in the set-up of (b)

Now that the Higgs branch of $Sp(k)$ with N flavours has been engineered via branes, mirror symmetry in the form of S-duality can be performed: it acts by converting NS5 branes into D5 branes and vice versa, D3 branes into themselves and the $O5^-$ into an ON^-. The resulting brane construction is sketched in Figure C.4 (a). After mirror symmetry the Higgs branch of the original theory is exchanged with the Coulomb branch of the new dual theory: so the configuration of branes in Figure C.4 (a) depicts the Coulomb branch of the mirror theory.

But from branes engineering Coulomb branches it is easy to read off the associated quiver gauge theory: in so doing we obtain the quiver in Figure C.4 (b), which corresponds to the dual theory of $Sp(k)$ with N flavours and reproduces the mirror quiver appearing in [107]. The $SO(2N)$ symmetry in this dual theory is manifest: the quiver is the flavoured, balanced (in the sense of [81][1]), D_N Dynkin diagram, with ranks as in the figure. It is precisely the set of balanced nodes that forms the Dynkin diagram of the global symmetry on the Coulomb branch.

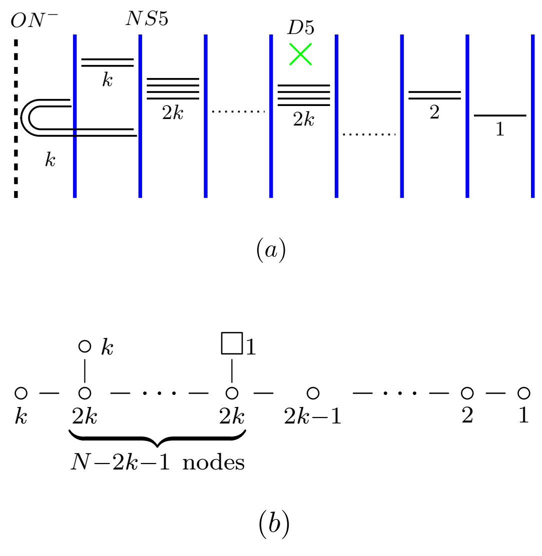


Figure C.4: (a) The brane set up for the Coulomb branch of the mirror dual of $Sp(k)$ with $2N$ flavours. (b) The resulting quiver gauge theory can be read off directly from the branes.

The most relevant part in our discussion is to examine the limit cases, namely the instances where the number of flavours N approaches the number of colours $2k$. The most general quiver occurs when there is at least one node with rank $2k$. The limiting cases can be thus obtained by setting $N - 2k - 1 = 1, 0, -1$

[1] for each node $N_f = 2N_c$

respectively. We tabulate these in Table C.1.

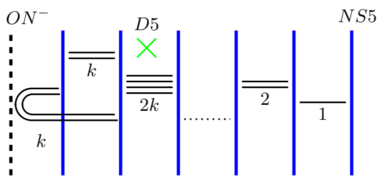
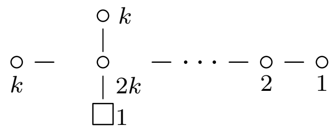
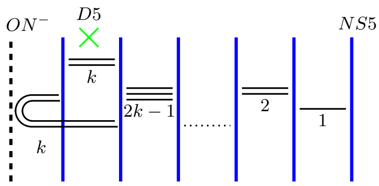
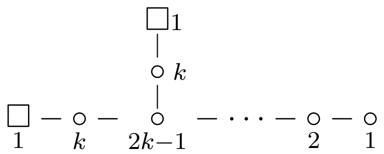
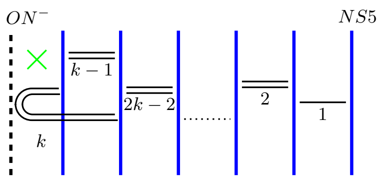
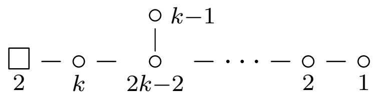
Flavours	Branes	Quivers
$N = 2k + 2$		
$N = 2k + 1$		
$N = 2k$		

Table C.1: The quivers associated to theories where N approaches $2k$. The quiver in the first line falls into the general class since there is one node with rank $2k$: it is precisely this one that gets flavoured. When the last node in the linear chain is $2k - 1$, the $D5$ brane gives a $U(1)$ flavour symmetry to both the spinor nodes. Lastly, for $N = 2k$, the flavour node has moved all the way to frame one of the two spinor nodes.

The last row of the table is the case we are interested in. To make contact with the previous section it is useful to let $N = 2n$. Then the quiver theory in the last row corresponds then to the case $n = k$ and is precisely the mirror dual of $Sp(n)$ with $2n$ flavours. The Coulomb branch of the former should be isomorphic to the Higgs branch of the latter. The Hilbert series for the ring of invariants on the Coulomb branch can be studied using the techniques introduced in [30]. Let's take the simplest example of $n = 1$. This degenerate case corresponds to the quiver:

$$\underset{2}{\square} - \underset{1}{\circ}$$

i.e. $U(1)$ with 2 flavours. The Coulomb branch of this theory is $\mathbb{C}^2/\mathbb{Z}_2$, which means we recover only one of the two (identical) cones that contribute to the Higgs branch of $SU(2)$ with 2 flavours.

Computing the Hilbert series of the Coulomb branch for higher values of n, the same conclusion is reached: not the union, but only a single hyperKähler cone is obtained. This can be understood by recognising that the flavour node in the flavoured D_{2n} quiver reaches one of the spinor roots of the Dynkin diagram. But flavouring the cospinor node is an equally allowed choice and the Coulomb branch associated to this quiver corresponds to the second cone that makes up the variety. The brane construction reflects the ambiguity: the two spinor representations are equivalent and physically undistinguishable.

This class of theories is quite special. Three-dimensional mirror symmetry has here a very awkward realisation: on the one side a single quiver, whilst on the other side two different quivers, equivalent by relabelling. It is nonetheless a legitimate pair, if for no other reason than the fact that it is the natural limit of a standard family of mirror pairs.

Field-theoretically the waters are still murky: what is the precise Lagrangian for the mirror theory of $Sp(n)$ with $2n$ flavours?

Appendix D

D.1 Chiral ring of $SU(n_c)$ with N_f flavours

The representations for the relations of $SU(n_c)$ with N_f with $N_f \geq 2n_c$ are tabulated below.

$-t^4$	$([1,0,\ldots,0,1]+[0,\ldots,0])$
$-t^{n_c+2}\ u^{n_c}$	$\Big([1,0,\ldots,0,\underset{(n_c-1)^{\rm th}}{1},0,\ldots,0]+[0,\ldots,0,\underset{(n_c+1)^{\rm th}}{1},0,\ldots,0,1]$ $+[0,\ldots,0,\underset{(n_c)^{\rm th}}{1},0,\ldots,0]\Big)$
$-t^{n_c+2}\ u^{-n_c}$	$\Big([0,\ldots,0,\underset{(N_f-n_c+1)^{\rm th}}{1},0,\ldots,0,1]+[1,0,\ldots,0,\underset{(N_f-n_c-1)^{\rm th}}{1},0,\ldots,0]$ $+[0,\ldots,0,\underset{(N_f-n_c)^{\rm th}}{1},0,\ldots,0]\Big)$
$-t^{2n_c}u^{2n_c}$	$\Big([0,\ldots,0,\underset{(n_c-2)^{\rm th}}{1},0,\ldots,0,\underset{(n_c+2)^{\rm th}}{1},0,\ldots,0]$ $+[0,\ldots,0,\underset{(n_c-4)^{\rm th}}{1},0,\ldots,0,\underset{(n_c+4)^{\rm th}}{1},0,\ldots,0]+\cdots\Big)$
$-t^{2n_c}u^{-2n_c}$	$\Big([0,\ldots,0,\underset{(N_f-n_c-2)^{\rm th}}{1},0,\ldots,0,\underset{(N_f-n_c+2)^{\rm th}}{1},0,\ldots,0]$ $+[0,\ldots,0,\underset{(N_f-n_c-4)^{\rm th}}{1},0,\ldots,0,\underset{(N_f-n_c+4)^{\rm th}}{1},0,\ldots,0]+\cdots\Big)$
$-t^{2n_c}$	$\Big([0,\ldots,0,\underset{(n_c)^{\rm th}}{1},0,\ldots,0,\underset{(N_f-n_c)^{\rm th}}{1},0,\ldots,0]$ $+[0,\ldots,0,\underset{(n_c-1)^{\rm th}}{1},0,\ldots,0,\underset{(N_f-n_c+1)^{\rm th}}{1},0,\ldots,0]+\cdots$ $+[1,0,\ldots,0,1]+[0,\ldots,0]\Big)^*$

Table D.1: Relations for the Higgs branch of 4d $\mathcal{N}=2$ $SU(n_c)$ with N_f flavours.

*This is not the full story. Because of higher syzygies the representations uncharged under $U(1)_B$ at t^{2n_c} are slightly modified according to whether n_c is odd or even. For n_c odd the representation $-t^{2n_c}[1,0,\ldots,0,1]$, i.e. the adjoint in the last line of the table, does **not** appear in the PL. It is screened by an adjoint coming with a plus sign (a higher syzygy). On the other hand for even n_c the representations $[1,0,\ldots,0,1]+[0,\ldots,0]$ on the last line appear twice rather than once as indicated on the table! Again, a higher syzygy adjoint and singlet show up, both with a minus sign, thus adding two (spurious) contributions to the set of relations.

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