Topics in Relativistic Cosmology

Cosmology on the Past Lightcone and in Modified Gravitation

Maye Y.A. Elmardi

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Supervisors: Prof. Chris Clarkson, University of Cape Town
Dr. Julien Larena, University of Cape Town

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Maye Elmardi
Department of Mathematics and Applied Mathematics
University of Cape Town

Abstract

The lightcone gauge is a set of what are called the observational coordinates adapted to our past lightcone. We develop this gauge by producing a perturbed spacetime metric that describes the geometry of our past lightcone where observations are usually obtained. We then connect the produced observational metric to the perturbed Friedmann-Lemaître-Robertson-Walker metric in the standard general gauge or what is the so-called 1+3 gauge. We derive the relations between these perturbations of spacetime in the observational coordinates and those perturbations in the standard metric approach, as well as the dynamical equations for the perturbations in observational coordinates. We also calculate the observables in the lightcone gauge and re-derive them in terms of Bardeen potentials to first order. A verification is made of the observables in the perturbed lightcone gauge with those in the standard gauge. The advantage of the method developed is that the observable relations are simpler than in the standard formalism. We use the perturbed lightcone gauge in galaxy surveys and galaxy number density contrast. The significance of the new gauge is that by considering the null-like light propagations, the calculations are much simpler since angular deviations are not considered.

Standard cosmology based on General Relativity is generally believed to have serious shortcomings, such as the unexplained issues of dark matter and dark energy. As a remedy, many alternative theories of gravitation have been proposed over the years, one of which is $f(R)$ gravity. We explore classes of irrotational-fluid cosmological models in the context of $f(R)$ gravity in an attempt to put some theoretical and mathematical restrictions on the form of the $f(R)$ gravitational Lagrangian. In particular, we investigate the consistency of the linearised dust models for shear-free cases as well as in the limiting cases when either the gravito-magnetic or gravito-electric components of the Weyl tensor vanish. We also discuss the existence and consistency of classes of non-expanding irrotational spacetimes in $f(R)$-gravity.

Furthermore, we explore exact $f(R)$ gravity solutions that mimic Chaplygin-gas inspired ΛCDM cosmology. Starting with the original, generalized and modified Chaplygin gas equations of state, we reconstruct the forms of $f(R)$ Lagrangians. The resulting solutions are generally quadratic in the Ricci scalar, but have appropriate ΛCDM solutions in limiting cases. These solutions, given appropriate initial conditions, can be potential candidates for scalar field-driven early universe expansion (inflation) and dark energy-driven late-time cosmic acceleration.

Keywords: General Relativity, Direct Observational Approach, Observables, Galaxy Surveys, Galaxy Number Count, Density Contrast, Cosmological Perturbations, Past Lightcone Gauge, f(R) Gravity, Modified Gravity, Cosmic Acceleration, Dark Energy, Irrotational Universe, Shear-free Uni-

verse, Chaplygin Gas.

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Conventions and Abbreviations

Throughout this thesis (unless otherwise stated), the natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used, and Greek indices (α , β , μ , $\nu \ldots$) and Latin indices a , b , $c \ldots$ run from 0 to 3 whereas Latin indices (i , $j \ldots$) run from 1 to 3. The symbols ∇ and ; represent the usual covariant derivative whereas ∂ and **,** stand for partial derivatives; $\tilde{\nabla}$ and the overdot $\cdot$ represent the spatial covariant derivative, and differentiation with respect to cosmic time, respectively. We use the $(- + ++)$ spacetime signature and the Riemann tensor is defined by

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\gamma}_{\nu\beta}\Gamma^{\mu}_{\alpha\gamma} - \Gamma^{\sigma}_{\nu\alpha}\Gamma^{\mu}_{\beta\sigma} \ , \tag{1}$$

where the $\Gamma^{\mu}_{\nu\beta}$ are the Christoffel symbols and they are symmetric in the lower indices, defined by

$$\Gamma^{\mu}_{\nu\beta} = \frac{1}{2} g^{\mu\alpha}(g_{\nu\alpha,\beta} + g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha}) \ . \tag{2}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor:

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \ , \tag{3}$$

and the Ricci scalar is given as

$$R = R^{\mu}{}_{\mu} \ . \tag{4}$$

The following are standard notations used in the thesis:

$$g : \det(g_{\mu\nu}), \text{ the determinant of the metric } g_{\mu\nu} \ , \tag{5}$$

$$(\mu\nu) : \text{symmetrization over the indices } \mu \text{ and } \nu \ , \tag{6}$$

$$[\mu\nu] : \text{anti-symmetrization over the indices } \mu \text{ and } \nu \ . \tag{7}$$

The 4-dimensional volume element $\eta_{\mu\nu\gamma\beta}$ is defined such that

$$\eta_{\mu\nu\gamma\beta} = \eta_{[\mu\nu\gamma\beta]}, \quad \eta_{0123} = \sqrt{|\det \ g_{\mu\nu}|} \ . \tag{8}$$

The following are abbreviations frequently used in the thesis:

BBN: Big Bang Nucleosynthesis
CMB: Cosmic Microwave Background
EFEs: Einstein Field Equations
FLRW: Friedmann-Lemaître-Robertson-Walker
GI: Gauge-invariant
GLC: Generalized Lightcone

GR: General Relativity
LTB: Lemaître-Tolman-Bondi
WMAP: Wilkinson Microwave Anisotropy Probe
APM: Automatic Plate Measuring
PSCz: Point Source Catalogue Redshift

Part I

Chapter 1

Introduction

> Nothing exists except atoms and empty space; everything else is opinion.
>
> Democritus

1.1 Curved Spacetime

Cosmology is the study of the origin, evolution, and the eventual fate of the Universe; points of epistemological interest probably as old as human civilisation itself. Modern cosmology in the sense of 'physical cosmology' started with the publication of Einstein's General Relativity (GR) in late 1915 [12, 13]. Einstein wrote down the now so-called *Einstein-Hilbert action*

$$\mathcal{S}_{EH} = \frac{1}{2}\int_{all\ spacetime} d^4x\sqrt{-g}R\ , \tag{1.1}$$

from which we can derive the field equations:

$$G_{\mu\nu} + g_{\mu\nu} = T_{\mu\nu}, \tag{1.2}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \tag{1.3}$$

is the *Einstein tensor*, $g_{\mu\nu}$ is the metric of the spacetime geometry, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and $T_{\mu\nu}$ is the energy momentum tensor sourced by the presence of matter in spacetime [14]. In his formulation of GR, Einstein gave an entirely new and astounding explanation of energy, matter and gravity. He described gravity as the consequence of the bending of spacetime around a massive body, *i.e.*, it replaces Newtonian gravity with curved spacetime. In this view of gravity, a test particle with velocity u^ν follows the geometry of space with a geodesic trajectory given by

$$u^\mu \nabla_\mu u^\nu = 0, \tag{1.4}$$

where

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{1.5}$$

and τ is a proper time along the geodesic [10]. The idea of the geometry of the spacetime that is curved by the existence of matter on it was a new way of understanding space and time, as a spacetime dynamical continuum evolving according to the local content of energy and momentum. Astronomers have applied and tested these new scientific definitions to the extent that they have become the conceptual foundations of modern cosmology.

The model offered as a cosmological solution to Einstein's equation was first suggested by himself in 1917 [15]. In his solution Einstein held the idea of a static universe [16]. His theory was not based on firm observational data, but rather on a mere theoretical simplification. Einstein mistakenly wanted to balance the self attraction of matter on the large scales by adding a new term (Λ) to his equations such that Eq. (1.2) gets modified as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{1.6}$$

in order to keep the Universe static [17,18]. Einstein's approach was then generalised independently by Friedmann in 1922 [19], where he did not try to balance the matter allowing the possibility of an expanding or contracting universe, *i.e.*, evolving cosmos, and by Lemaître in 1927 [20,21] who also predicted the redshift[1] of receding galaxies before its observation by Hubble [22] in 1929. The work of Friedmann and Lemaître was develpoed with geometric properties of spacetime by Robertson in 1929 [23–26], followed by Walker [27].

1.2 The Friedmann-Lemaître-Robertson-Walker Models: the Background Geometry of the Universe

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric (also shortened as FRW or RW) is a solution to the EFEs with a homogeneous and isotropic universe. The cosmological models based on this metric including perturbations are sometimes called the Standard Model of modern cosmology because they describe successfully the major features of the observed universe [28], its expansion from a hot Big Bang leading to the observed galactic redshifts and remnant black body radiation [29]. These models, however, do not describe the real universe well in an essential way, in that the highly idealised degree of symmetry does not correspond to the lumpy real universe [30].

The assumption of large-scale isotropy observed in particular through the Cosmic Microwave Background (CMB), where it shows that our Universe properties are almost identical whatever the direction we look at. The hypothesis of the so-called *Copernican Principle*: the Earth occupies no unique (or central) position in the Universe infer the homogeneity. These two facts are counted as a pillars of the *Cosmological Principle*: the Universe can be modeled as statistically spatially homogeneous[2] and isotropic. Therefore modern cosmology is based on this hypothesis that our universe is to a good approximation homogeneous and isotropic on sufficiently large scales [31].

We know now the real universe is a perturbed one and there are inhomogeneities and anisotropies arising during structure formation, that can be compared in detail with observations, but we can consider it as *almost FLRW* at the same time. In this section we will strictly examine homogeneous and isotropic cosmologies.

1.2.1 The Homogeneous Universe

The Coordinate systems and metric

In order to explain a homogeneous and isotropic space one can admit a slicing of a maximally symmetric space along a fixed time coordinate t, such that the 3-spaces-like hypersurfaces Σ_t are surfaces of intrinsic geometry of homogeneity and isotropy with a spatial metric γ_{ij} of constant time and curvature

$$K = \frac{k}{a^2(t)}\ . \tag{1.7}$$

[1]The redshift of an object is defined to be the fractional Doppler shift of its emitted light (photons) wavelength due to its local peculiar motion. A cosmological redshift originates from the general relativistic effects of space expansion affecting the wavelengths of cosmological events.

[2]From here onwards 'homogeneous' implies spatially homogeneous.

Here k denotes spacetime curvature and takes the values $-1, 0$ or $+1$ depending on whether the Universe is *open, flat* or *closed*, respectively. The normalised metric $d\sigma^2$ characterises a 3-space of normalised constant curvature whose spatial spherical comoving coordinates (χ, θ, ϕ) can be chosen such that

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1.8}$$

The term $\gamma_{ij} dx^i dx^j$ is a function of three spatial coordinates and it can describe an Euclidean space, or hyperbolic space, whereas χ is a radial coordinate, ϕ and θ are two angles in the sky running from 0 to 2π and from $-\pi$ to π respectively. $S(\chi)$ depends on the curvature k, and it can be given as [14]

$$S(\chi) = \begin{cases} \sin(\chi) & \text{for } k = +1, \\ \chi & \text{for } k = 0, \\ \sinh(\chi) & \text{for } k = -1 \ . \end{cases} \tag{1.9}$$

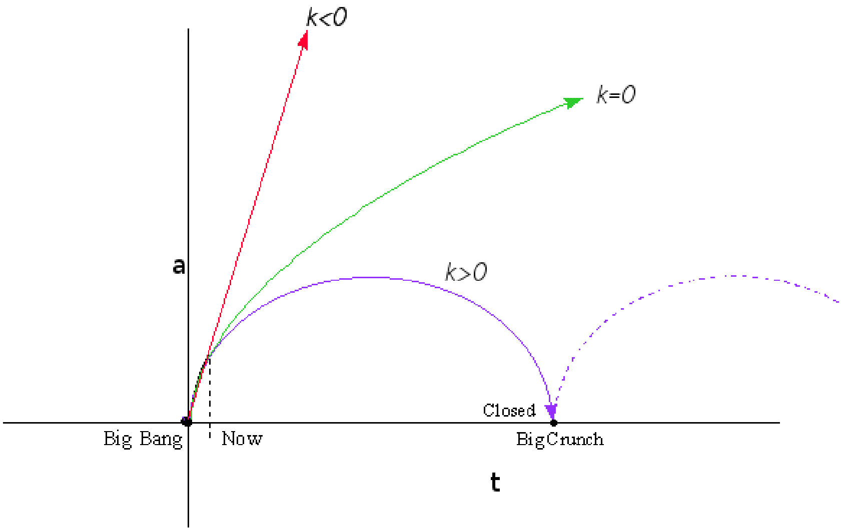


Figure 1.1: The kinematics of the scale factor a in FLRW universe which satisfies the energy condition $\rho + 3p > 0$.

At $k = +1$ the function $S(\chi)$ relates the surface of a comoving sphere to its radius χ, whereas the equation

$$ds^2 = -c^2 dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j \tag{1.10}$$

gives the full spacetime metric widely known today as the FLRW metric. Using the radial coordinate $r = S(\chi)$, the metric can also take the form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right) , \tag{1.11}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$. This construction shows how the Cosmological Principle, and thus the symmetry assumptions, has allowed us to reduce the ten arbitrary functions of the space-time metric into a single function of one variable $a(t)$ and a pure number k.

The cosmological time is related to the conformal time η by

$$dt = a d\eta \ . \tag{1.12}$$

Then we can re-write Eq. (1.10) in the form of

$$ds^2 = -a(\eta)^2 d\eta^2 + a(\eta)^2 \gamma_{ij} dx^i dx^j \ . \tag{1.13}$$

The FLRW solutions to the EFEs are the best fit to the observed universe so far [31].

1.2.2 Kinematics and Dynamics

For matter which respects the assumptions of homogeneity and isotropy, the stress-energy tensor must read

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu + p(t) h_{\mu\nu} \ , \tag{1.14}$$

where ρ and p define the energy density and isotropic pressure (respectively) of all kinds of *standard matter*, *i.e.*, baryonic matter, dark matter, radiation. Each species is characterised by the equation-of-state parameter w, defined as

$$w = \frac{p}{\rho} \ , \tag{1.15}$$

the ratio between its pressure and its energy density. $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projector on spatial sections $t = const.$, hence $h_{\mu 0} = 0$ and $h_{ij} = g_{ij}$.

The EFEs (1.2) reduce to ordinary differential equations, known as *Friedmann's equations*, for the function $a(\eta)$, with ($' = d/d\eta$):

$$\mathcal{H}^2 = \frac{1}{3} a^2 \rho - \frac{k}{a^2} + \frac{1}{3} a^2 \Lambda \ , \tag{1.16}$$

$$\frac{a''}{a} = -\frac{1}{6}(1+3w) a^2 \rho + \frac{1}{3} a^2 \Lambda \ . \tag{1.17}$$

Here $\mathcal{H}$ is the conformal Hubble parameter related to H as follows:

$$H \equiv \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a}, \quad \mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = \frac{a'}{a} = aH \ . \tag{1.18}$$

The energy-momentum conservation equation is given by

$$\rho' = -3\left(\frac{a'}{a}\right)(\rho + p) = -3(1+w)\mathcal{H}\rho \ . \tag{1.19}$$

Here ρ is a combination of cold non-relativistic matter with $p_m = 0$, and radiation with $p_r = 1/3\rho_r$. We can also define the fractional energy densities

$$\Omega_m \equiv \left(\frac{\rho a^2}{3\mathcal{H}^2}\right) \ , \tag{1.20}$$

$$\Omega_k \equiv -\frac{k}{\mathcal{H}^2} \ , \tag{1.21}$$

$$\Omega_\Lambda \equiv \frac{\Lambda a^2}{3\mathcal{H}^2} \ , \tag{1.22}$$

where Ω_m, Ω_k and Ω_Λ are the fractional energy densities for matter, curvature and dark energy density, respectively. The total energy density in the Universe can thus be given in a more compact form by virtue of Eq. (1.16) as

$$1 = \Omega_m + \Omega_\Lambda + \Omega_k \ , \tag{1.23}$$

For $\Omega_m \gg \Omega_k, \Omega_\Lambda$ contributions to Eq. (1.19) the scale factor evolves as

$$a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}} \ , \tag{1.24}$$

when

$$w = \begin{cases} 0 & \text{for nonrelativistic matter ,} \\ 1/3 & \text{for ultrarelativistic component (radiation) .} \end{cases} \tag{1.25}$$

When $\Omega_k \gg \Omega_m, \Omega_\Lambda > 0$ with $w = 0$, we get [32]

$$a(t) \propto \sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}t\right) . \tag{1.26}$$

When $\Lambda = 0$, it coincides with $a \propto t^{2/3}$, but when the cosmological constant is dominant over the contribution of matter then we get $a(t) \propto \exp(t\sqrt{\Lambda/3}) = \exp(tH)$ where the expansion rate will be a constant in this case.

1.3 Some Observations

In this section we are going to discuss some cosmological observations. To measure the observables, we require a specific model, such as the FLRW model at a particular time.

1.3.1 Hubble Diagram

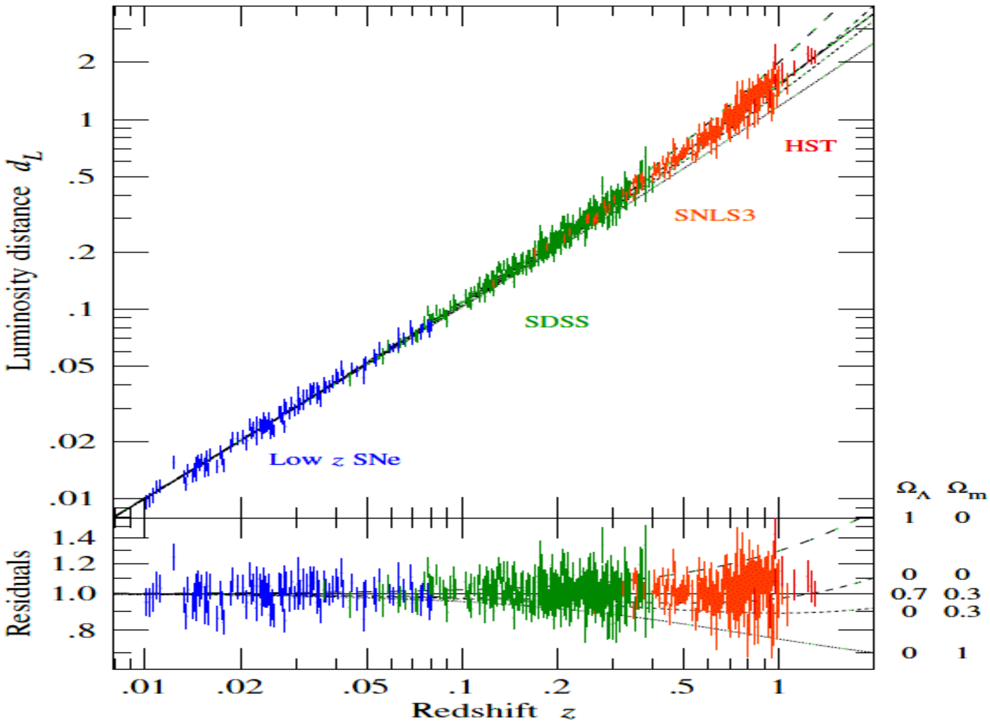


Figure 1.2: The top panel depicts the Hubble diagram (Redshift-Distance Relation) with the ΛCDM best fit; the bottom shows the residuals. This diagram is obtained by the analysis of 740 SNeIa [1].

The diagram represents the plot of luminosity distance d_L of objects with known intrinsic luminosity, *i.e.*, the so-called standard candles where Supernovae type Ia (SNeIa) are the best candidates, with their redshift, showing a redshift increasing with distance as a result of an expanding universe [22]. This recent Hubble diagram obtained by [1], where they plot the joint lightcurves, *i.e.*, the evolution of the luminosity of the event with time, with the redshift (that is determined by spectroscopic measurements), is interpreted assuming that light propagates through a smooth homogeneous and isotropic universe, so that the redshift luminosity distance d_L is measured assuming FLRW model [33].

The Hubble diagram obtained from SNeIa data was a good probe for investigating the existence of dark energy at low redshift. That is how SNe provided the evidences of the accelerated cosmic expansion discovered in the late 1990s.

1.3.2 The Cosmic Microwave Background (CMB)

The CMB is a thermal radiation imprint showing that the Universe was born about 13.81 billion years ago with a Big Bang using the FLRW model. The COsmic Background Explorer (COBE) satellite [34] was the first space-based experiment dedicated to measure the CMB spectrum, then by the Wilkinson Microwave Anisotropy Probe (WMAP) [35]. Recent precision observations of the CMB have been done by the Planck mission in 2013 [36].

The Universe was in its early stages at high enough temperature to be fully ionised; then processes such as Thompson scattering would thermalise the radiation field very efficiently. One would then expect to observe a radiation field which would have retained the black-body spectrum [2], see Fig. [1.3]. When the primordial plasma cooled enough, light thus suddenly stopped being scattered by charged particles, and started propagating freely, following null geodesics. The spatial distribution from the CMB, showing that this happened everywhere at the same cosmic time. These findings were rewarded with the 1978 and 2006 Nobel Prizes in Physics.

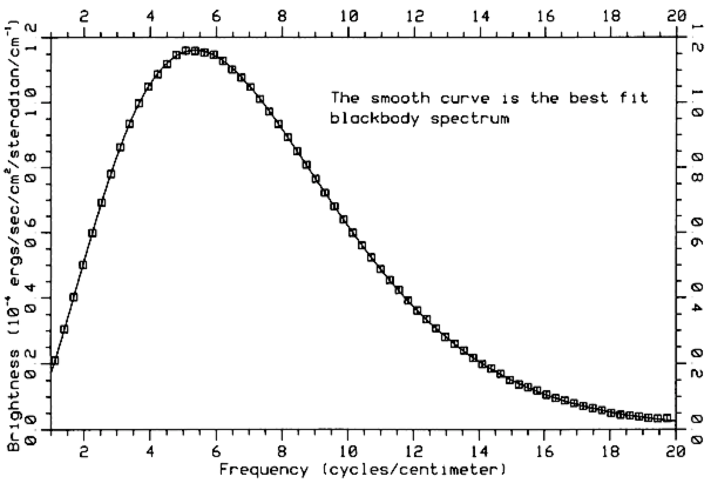


Figure 1.3: The black-body spectrum of the CMB radiation measured with COBE satellite [2].

As observers, we can measure three main things about this radiation: frequency spectrum, temperature and polarisation states. Each of these observables contains information about the creation and evolution of the field and is packed with cosmological information. The observed average temperature is uniform across the sky of $\sim 2.72548 \pm 0.00057$ K [37]. The CMB temperature anisotropy $\delta T/T \sim 10\ \mu$K corresponds to regions of slightly different densities, representing the seeds of all future structure of the Universe; the galaxies of today. We see these temperature fluctuations projected in a 2D spherical sky, see Fig. [1.4]. This implies that, on the CMB temperature map, two points (hot or cold) on the last scattering surface are most likely to be separated by an angle $\theta_* = r_s/r_A$, where r_s is the *sound horizon* and r_A is the area distance measured by using the standard choice of FLRW model. The analysis of the fluctuations of this thermal radiation has given us valuable insights into our universe and its parameters; such as the rate of expansion, the Hubble parameter, the mean matter density of the Universe.

The resulting anisotropies spectrum of the CMB shows a series of acoustic oscillations as shown in Fig. [1.5], which present a snapshot of the CMB sky at the moment when photons decouple from

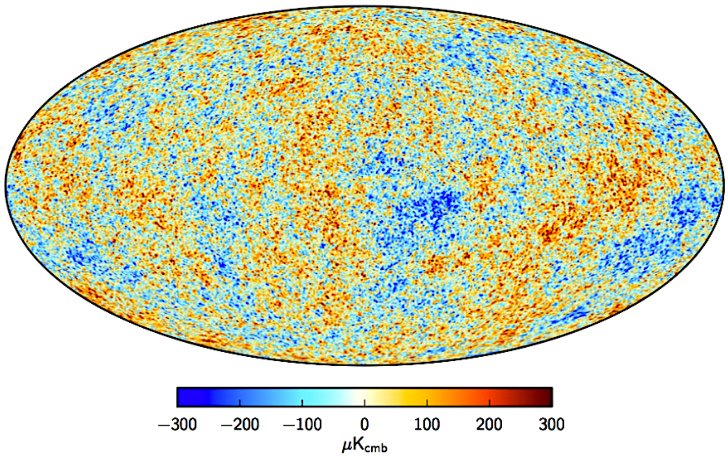


Figure 1.4: The 4 years of data collection by Planck satellite. The colour code corresponds to temperature fluctuations of a few micro-Kelvin [3].

electrons. The details of the physics and the analysis of the CBM spectra can be found in [10, 38]. The CMB physics is well understood now due to the analysis of its fluctuations. It inspires an era

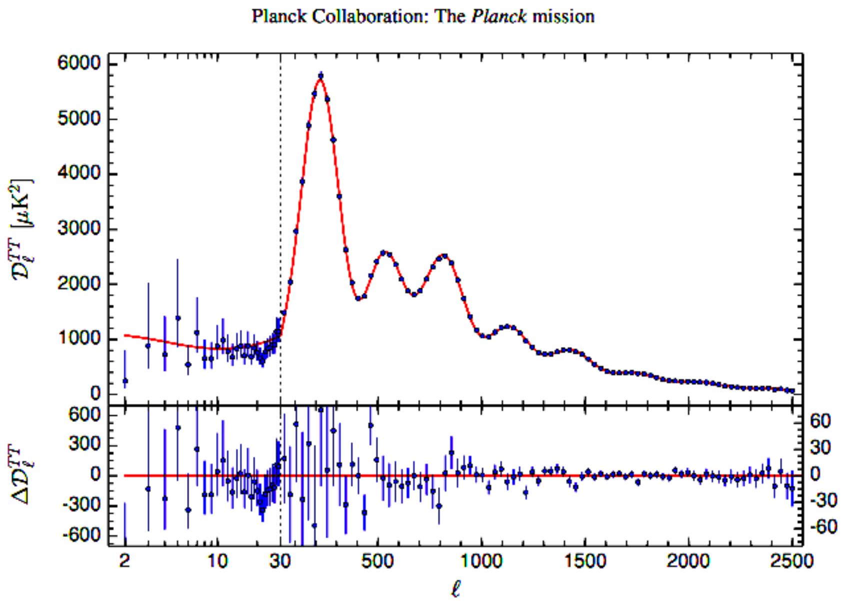


Figure 1.5: The 2013 Planck CMB temperature angular power spectrum [3].

of precision of measurements in cosmology. The main reason why the CMB allows such an accurate determination of cosmological parameters lies in the fact that its anisotropies are small and it can be determined within a first-order perturbation theory.

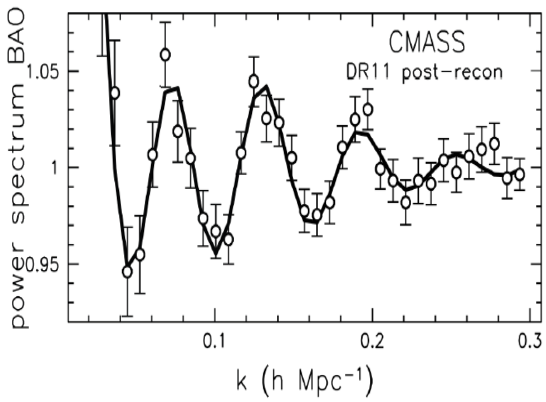
1.3.3 Baryon Acoustic Oscillations (BAO)

The BAO is the observational consequence of the clustering of baryonic matter on certain length scales due to acoustic waves which propagated in the early universe. The primordial plasma would have had very slight overdensities of matter, where they gravitationally attract more matter. But the intense heat of the photon-matter interactions seeks thermal equilibrium, which creates a large amount of outward pressure. Before the decoupling and the emission of the CMB, the pressure results in sound waves of both baryons and photons. These sound waves have imprinted the matter distribution in the late times, leading to characteristic patterns in the galaxy distribution and hence geometry and expansion history. It appears as an acoustic peak of the CMB anisotropy [39].

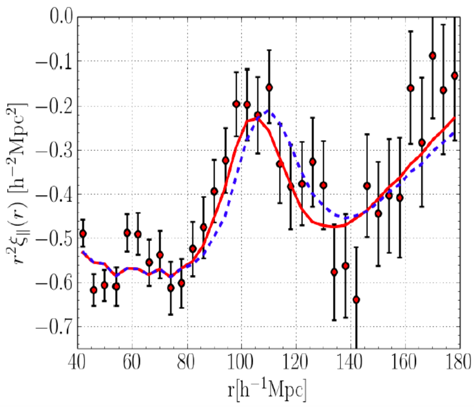
The BAO signal grows with the cosmic expansion, and considered as a cosmic standard ruler, the BAO scale r_d estimated today by

$$r_d = (1 + z_*) r_A(z_*) \theta_* \ , \tag{1.27}$$

where $r_A(z)$ is adapted to the FLRW model. The BAO signal can be observed at some different epochs of the Universe, *e.g*, $z = 1090$ (Fig. [1.5]), $z = 2.34$ (Fig. [1.6b]) and $z = 0.57$ (Fig. [1.6a]).



(a) Power spectrum of galaxy sample, the BAO signal is effectively measured at $z = 0.57$ [40].



(b) Two-point correlation function for objects aligned with the line of sight, the effective redshift is $z = 2.34$ [41].

1.3.4 The Accelerating Universe

A recent study by the Supernova [42,43] Cosmology Group (SCP) and the High-z Supernova Search Team (HZT) for redshift and distance measurements of Ia supernovae was crucial in determining the extension of Hubble's law in large distances $z \lesssim 1$. The results showed that at high redshifts, supernovae are getting dimmer than expected by a factor of about $10 - 15\%$ compared to Hubble law. The unexpected results imply that the Universe is unexpectedly accelerating [44].

Since the accelerated expansion cannot be produced by an ordinary matter and radiation field distribution, then it must be a kind of an exotic fluid generically known as *dark energy*, whose negative pressure counterbalances the gravitational attraction, comprising close to a 70% of the total energy density of the Universe [45].

The non-vanishing cosmological constant Λ is the simplest, most obvious candidate for dark energy acting as a cosmic fluid with equation of state $w = -1$, which can also be thought of as a perfect fluid satisfying the equation of state $p = -\rho$. Since the accelerated expansion in the early Universe needs to end to connect to the radiation dominated Universe, whereas the pure cosmological constant gives rise to an exponential expansion, this model is not responsible for inflation. But it is possible to use the cosmological constant for dark energy since the late acceleration today does not need to end. Due to the cosmological constant the Ω_m reduces and that conclude in reduces the observed angular distance r_A and hence the luminosity distance d_L, therefore observing the SNe with a given z appear dimmer in a Universe whose filled by dark energy than without $z_{\Lambda \neq 0} < z_{\Lambda = 0}$.

There are other candidates for the role of dark energy as well. A plausible alternative is dynamical vacuum energy [46], or a scalar field ϕ with a slowly varying potential [47], represented by the so-called quintessence scalar field. The scalar fields are traditionally used in inflationary models to describe the transition from the quasi-exponential expansion of the early universe to power-law expansion [48]. Moreover it is generally difficult to construct viable quintessence potentials motivated from early-universe particle physics because the field mass responsible for cosmic acceleration today is very small ($m_\phi \sim 10^{-33}$eV) [49]. The question of explaining why the vacuum energy or scalar field dominates the Universe only recently is the *cosmic coincidence problem*. A recently proposed class of simple cosmological models is based on the use of peculiar perfect fluids such as the Chaplygin gas model [50, 51]. In [52] we discussed the possibility of treating Chaplygin gas as a scalar field model in a universe without conventional matter forms in early and late cosmological expansion histories.

Another suggestion proposed is that it could be the energy of the quantum vacuum, but its energy density would be enormously larger than today's dark energy density [53]. Other suggestions include inhomogeneous cosmologies. Such models make use of the fact that the expansion rate is larger in overdense regions, so that if we happen to live inside a particularly large void in the Universe we will measure a lower value of H_0 in our vicinity than the average value in the Universe [54]. There is another approach to explain the acceleration of the Universe: *modified gravity*, *i.e.*, the gravitational theory is modified compared to GR. These are several versions of these models [55–61], but in this thesis we will only consider those gravitational models obtained by the inclusion of higher-order curvature invariants in the Einstein-Hilbert action that result in fourth-order field equations. Although modified theories of gravity may account for the observed dimming of supernovae, it seems the dark energy hypothesis will remain the favorite candidate in astronomy corridors for at least the near foreseeable future. The observations still do not show particular evidence to favour any one of such alternative models.

1.3.5 Visible Matter and Dark Matter

One of the most important observables in cosmology is the amount of matter existing in the Universe. This is because it tells us so much about the evolution of the Universe and consequently its expansion rate. The average density $\langle\rho\rangle$ of the visible luminous matter can be calculated by determining the number of galaxies n_G within the Hubble volume and the average galactic mass M_G [62] as

$$\langle\rho\rangle = n_G \langle M_G \rangle. \tag{1.28}$$

By measuring distances and velocities of stars within the galaxy and using Kepler's Third Law, we can determine the mass of a galaxy as

$$GM(r) = v^2 r \ , \tag{1.29}$$

where r is the radius of the galactic, v the orbital velocity and $M(r)$ the total mass within a spherically symmetric region of radius r. In [63] they measure the gas and the stellar masses of the Coma Cluster M_b and compare it with its total 'dynamical' mass assuming that will equal to the mean cosmological baryonic fraction

$$\frac{M_b}{M_{tot}} = \frac{\Omega_b}{\Omega_m} \ . \tag{1.30}$$

Calculating Ω_b from the analyses of the Big Bang Nucleosynthesis (BBN), they concluded that $\Omega_m \approx 0.28 \pm 0.56$ for $h = 0.5$ which is close to the current admitted value. Nevertheless the total luminous matter in the Universe provides less than 1% of the critical density required for a flat universe [64]

$$\rho_c = \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-29} \mathrm{h}^2 \mathrm{g\ cm}^{-3} \ . \tag{1.31}$$

In [65] they introduce a more subtle method for constraining the cosmological parameters with galaxy clusters were the gas mass fraction extracted from galaxy clusters read

$$f_{gas} = \frac{M_{gas}}{M_{tot}} = A r_A + B(z) r_A^{3/2}(z) \ , \tag{1.32}$$

where A and B are some cosmological parameters-independent values, and $M_{gas} = M_b - M_G$ [65], so that the area distance r_A constrains all the cosmological dependence. Eq. (1.32) is a model-dependent formula because of the $r_A(z)$ term, in this case FLRW. The analysis of Planck data shows that normal matter, making up galaxies and stars, contributes only about 4.9% to the mass and energy density of the Universe.

The near-flat geometry of the Universe that current observations seem to suggest, in particular from the position of the first acoustic peak in the CMB power spectrum, suggest that luminous matter in galaxies cannot be accounted only for the matter density in our Universe. When astronomers measure the rotational velocity curves of our galaxy, and when they extend far from the centre of the galaxy r, they find out that the velocities of some rare stars have constant velocity curves and do not, therefore, follow Keplerian laws of motion [66].

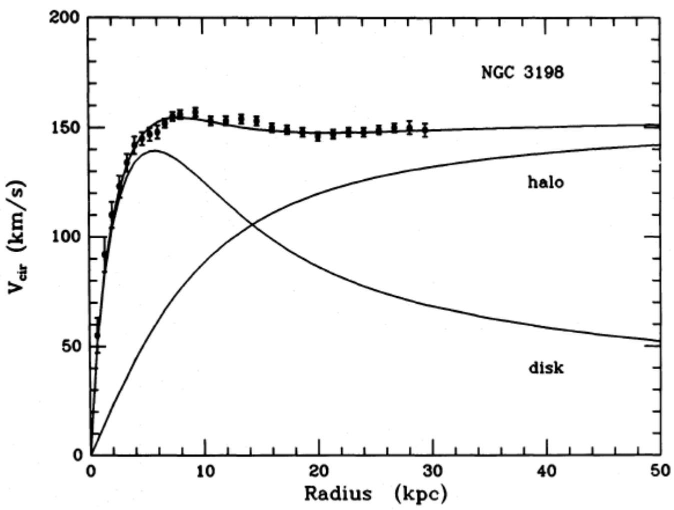


Figure 1.7: Rotation curves of typical spiral galaxy predicted, disk and extended dark matter halo models and observed with error bars [4]. Best fit obtained for an exponential disk model with maximum mass.

This implies that

$$M(r) \propto r \ . \tag{1.33}$$

There seems to be a lot of invisible mass, 'dark matter', extending beyond the visible limits of virtually all spiral galaxies [67]. In an effort to explain the missing mass in the velocity curve of galaxies, Jan Oort postulated the existence of dark matter in 1932 [68]. Thus, measuring average velocity of galaxies and the average inverse distance between galaxies in a cluster can be used to infer its total gravitational mass, shown by Fritz Zwicky in 1933 [69]. He suggested that there must be some form of invisible matter in which it would provide enough mass and gravity to hold the cluster together [5, 70, 71]. The existence and properties of dark matter are inferred from its gravitational effects on visible matter, radiation, and the large-scale structure of the Universe, see Fig. [1.8]. According to the CMB measurements, about 84.5% of the total matter in the Universe exists in dark matter form. On the other hand, dark energy with dark matter constitute 95.1% of the total mass-energy content of the Universe. Needless to say, most of the matter and energy in the Universe is unaccounted for.

Dark matter is classified as cold, warm and hot dark matter [72]; some combination of them is also possible. The classification is based on the particles that are assumed to make it up, and their typical velocity dispersion of those particles.

The best dark matter candidates are based on the CDM hypothesis, with weakly interacting massive particles (WIMPS) as most favourite particles. There are currently several alternative theories of gravity that try to do away with dark matter, with MOND [73] and TeVeS [74] among the most popular, although neither of these theories can account for the properties of galaxy clusters [75].

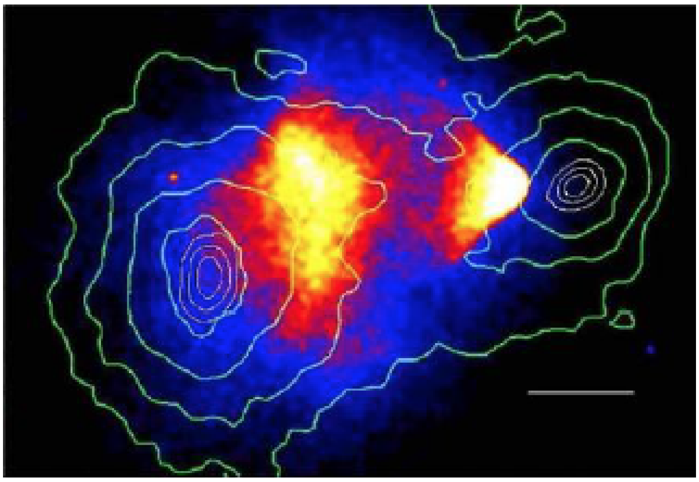


Figure 1.8: Image of the Bullet Cluster obtained by the Chandra X-ray Observatory: The blue color shows the distribution of dark matter, which passed through the collision without slowing down. The red coulour shows the hot X-ray emitting gas. In green are the gravitational mass contours reconstructed from weak-lensing observations [5].

1.3.6 Large-scale Structure

According to the prevailing paradigm, it is sufficient that matter density fluctuations grow from considerable initial fluctuations with amplitudes of order of 10^{-5} to reproduce the cosmic structures observed today [38]. The small deviations from homogeneity and isotropy in the CMB are of uttermost importance since, most probably, they represent the seeds, which, via gravitational instability, have led to the formation of large-scale structure, galaxies and eventually solar systems with planets that support life in the Universe. Gravitational instability for overdense regions is a powerful scenario for structure formation; it is easy to understand and most likely responsible for the structures in our universe [6]. In time, matter accumulates in initially overdense regions that are denser than average. It does not matter how small the initial overdensity was, eventually enough matter will be attracted to the region to form structures (e.g., in typical cosmological scenarios, the overdensity was of order 1 part in 10^5). A mass near an overdense region is attracted to the centre by gravity but at the same time repelled by isothermal pressure. If the region is dense enough, gravity wins and overdensity grows with time, see Fig. [1.9]. In the mean time, the Universe exhibits a uniform isotropic expansion, and because there is slightly more matter in the overdensity, there will be a slightly stronger gravitational attractive force tending to draw that overdense region together and due to that the overdense regions will expand at slower rate than average which can serve to retard the expansion locally. Eventually, the overdensity, or *perturbation* as it is often called, will reach a point where it stops expanding altogether and begins to collapse under its own weight (gravity) [76]. As it collapses it will fragment into thousands or even millions of knots, each of which themselves will collapse, forming a cluster of stars or galaxies, depending on how big the original cloud of gas was to begin with [77]. An overdense region that is larger than the horizon distance cannot be supported by its internal pressure, because any changes in pressure are propagated at a speed that is lower than the speed of light. Nevertheless, the relative density fluctuation within this overdense region does grow with time, although it will grow slowly since the overdense region is expanding at a lower rate than the universe around it. However, the horizon distance increases with time, so eventually, the overdense region will lie within the horizon distance and can respond to internal pressure changes. After this time, an overdense region can be stable against collapse provided that the mass within this region is less than the amount of pressure required for the balance [78]. The fact that the anisotropies measured recently by COBE [79] and then by Planck just coincide with

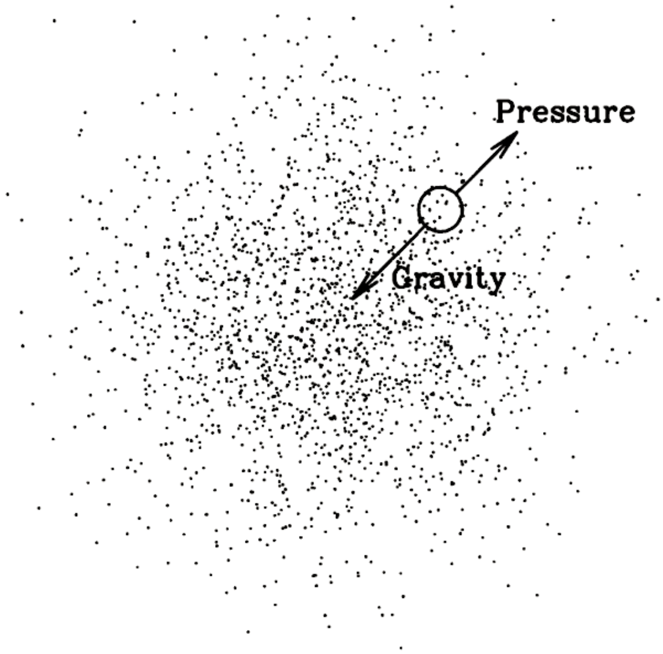


Figure 1.9: Gravitational instability: a nearby mass attracted to the center of an overdense region by gravity and repelled by pressure [6].

the amount of growth necessary to form structures today is taken as a hint that the gravitational instability picture may be correct [80]. Nevertheless, several complications must be addressed to obtain a real picture of the matter distribution in the Universe.

The study of these anisotropies requires a comparison between the observations and theory in order to know the behaviour and distribution of matter and hence the power spectrum, for example, in Newtonian framework, the way to relate the matter overdensity to the gravitational potential at late time in a static universe is determined by the Poisson equation

$$\nabla_i^2 \Phi = \rho \;, \tag{1.34}$$

From the above we can conclude the force of gravitational instability

$$F_i = m a_i \;, \tag{1.35}$$

where a here is the acceleration of the fluid, together with the Euler equation [81]

$$(\partial_t + u_i \nabla_j) u^j = -\frac{\nabla_i p}{\rho} - \nabla_i \Phi \;, \tag{1.36}$$

and the basic equation of mass conservation in fluid dynamics [82]

$$\partial_t \rho = -\nabla_i (\rho u^i) \;. \tag{1.37}$$

These equations describe the evolution of small perturbations around a homogeneous background. By combining the linearised evolution equations (1.37) and (1.36) for the fluctuations, we can get the equation

$$(\partial_t^2 - c_s^2 \nabla^2) \delta\rho = \bar{\rho} \delta\rho \;, \tag{1.38}$$

where c_s is the speed of sound, the total matter $\rho = \bar{\rho} + \delta\rho$, $\bar{\rho}$ describes the homogeneous background

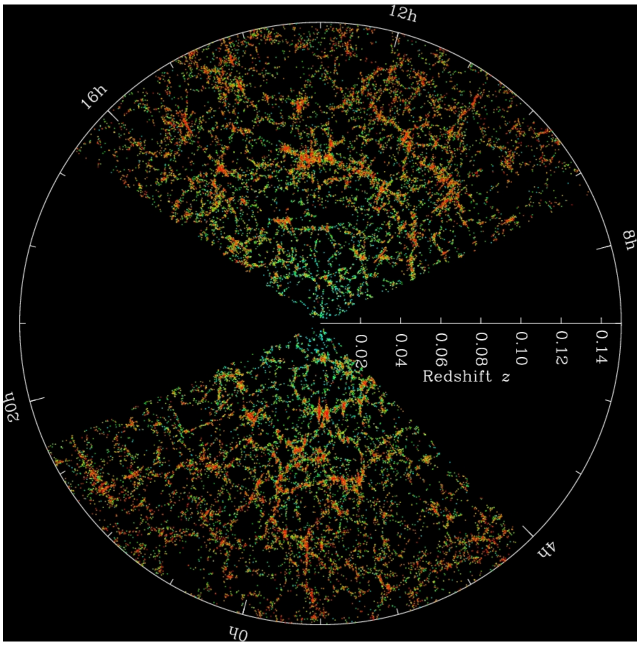


Figure 1.10: Slices through the SDSS 3-dimensional map of the distribution of galaxies and it contains about 100 billion stars. Earth is at the center, and each point represents a galaxy. Galaxies are couloured according to the ages of their stars. The outer circle is at a distance of two billion light years. Both slices contain all galaxies within −1.25 and 1.25 degrees declination [7].

and $\delta\rho$ denotes small inhomogeneous matter fluctuations. Similarly for the potential, pressure and velocity we can decompose them into homogeneous and inhomogeneous parts. The pressure fluctuations are proportional to the density fluctuations as $\delta p = c_s^2 \delta\rho$.

Eq. (1.38) can be solved by a plane wave $\delta\rho = A\exp[i(\omega t - k)]$, where $\omega^2 = c_s^2 k^2 - \bar{\rho}$, and this solution indicates that there is a critical wavenumber, called the *Jeans wavenumber*,

$$k_J = \frac{\sqrt{\bar{\rho}}}{c_s} \,, \tag{1.39}$$

for which the frequency of the matter fluctuations oscillations is zero [81]. On small scales $k > k_J$ the pressure dominates and we get an oscillating solution with a fixed amplitude. On the other hand, on large scales $k < k_J$ gravity dominates and the fluctuations grow exponentially, the crossover happening at the *Jeans length*

$$\lambda_J = \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{\bar{\rho}}} \,. \tag{1.40}$$

In an expanding universe such that $r(t) = a(t)x$, where $r(t)$ and x represent the physical and comoving coordinates, the velocity field can be given by [82]

$$u_i(t) = Hr_i + v_i \,, \tag{1.41}$$

where Hr_i is the Hubble flow, and $v_x = a\dot{x}$. With some algebra we can re-write the *continuity equation* Eq. (1.37) to first order as

$$\left[\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho}\right]\delta + \bar{\rho}\frac{\partial\delta}{\partial t} + \frac{\bar{\rho}}{a}\nabla_i v^i = 0 \,, \tag{1.42}$$

where δ is the fractional density perturbation $\frac{\delta\rho}{\bar{\rho}}$, also known as the *density contrast*. More over, the

Poisson and Euler equations (1.34) and (1.36) reduce to

$$\nabla_i^2 \delta\Phi = a^2 \bar{\rho}\delta \ , \tag{1.43}$$

$$\dot{v}_i + H v_i = -\frac{1}{a\bar{\rho}} \nabla_i \delta p - \frac{1}{a} \nabla_i \delta\Phi \ . \tag{1.44}$$

Using these equations together with Eq. (1.19) results in the *Jeans' instability* equation [3]

$$\ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}\nabla^2\delta = \bar{\rho}\delta \ . \tag{1.45}$$

The second term on the LHS of the above equation of motion plays the role of a damping (friction) force term such that below the Jeans' length the fluctuations oscillate with decreasing amplitude and above the Jeans' length the fluctuations experience power-law growth [81].

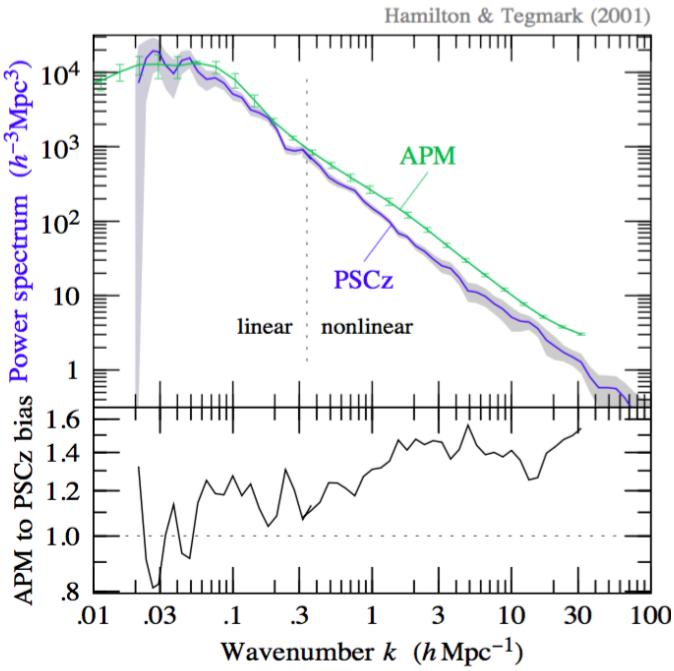


Figure 1.11: A galaxies survey power spectrum from PSCz and APM data survey [4] [8]. The bias is the square root of the ratio of power spectra.

By expanding Eq. (1.38) into its Fourier components

$$\delta\rho(t, r) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}.\mathbf{r}} \delta\rho_{\mathbf{k}}(t) \ , \tag{1.46}$$

we can compute the matter fluctuation spectrum. For each Fourier mode

$$(\partial_t^2 + c_s^2 k^2)\delta\rho_{\mathbf{k}} = 0 \ , \tag{1.47}$$

the solution of which is given by

$$\delta\rho_{\mathbf{k}} = A_{\mathbf{k}} e^{i\omega_k t} + B_{\mathbf{k}} e^{-i\omega_k t} \ , \tag{1.48}$$

[3] More details of this derivation can be found in [76, 78, 81, 82].

[4] The PSCz is a redshift survey of 15,411 galaxies accomplished by [83], and APM is the best power-spectrum determination from the angular correlations survey [84].

where $\omega_k = c_s k$. Fourier modes evolve independently, and the power spectrum is sufficient to completely describe the density field.

1.4 The Cosmological Parameters

Cosmological parameters, including a parameterization of some functions, are simple numbers describing the properties of our Universe whose geometry is well characterised by Friedmann metric. They have been measured within increasing precision over the last decades. The term originally was used to refer to the parameters describing the global dynamics of the Universe, such as its expansion rate $H_0 = 67.74 \pm 0.46$ Km/s/Mpc, often written as $H_0 = h \times 100$ Km/s/Mpc, and the space curvature of the cosmological model $\Omega_k = 0.0008^{+0.0040}_{-0.0039}$, These parameters are from the recent observations made by Planck 2015 based on ΛCDM assumption, more details in [3, 85]. The structure and fate of the universe can be described by the cosmological parameters, such as mean mass density $\Omega_m = 0.3089 \pm 0.0062$ of matter in the universe (baryons $\Omega_b h^2 = 0.02230 \pm 0.00014$. Photons Ω_γ, neutrinos Ω_ν and radiation Ω_r represent much less $\sim 10^{-4}$. The remainder is known of its nonrelativistic manner and it does not interact with the normal matter, it has in particular no electromagnetic signature named cold dark matter Ω_{CDM}, and dark energy $\Omega_\Lambda = 0.6911 \pm 0.0062$ the current value of the cosmological constant, at the present time divided by the critical density [3, 85]. Each of these parameters evolves differently with the redshift, so it is unlikely that two terms will be comparable at any given time [86]. The present photon density is 4.7×10^{-31} kg m^{-3} using the standard models [87].

With these parameters one can test the consistency of the standard relativistic expanding cosmological model that is known as Λ-Cold-Dark-Matter (ΛCDM). However, if $\Omega_m < 1$, then the universe is undergoing from a state of matter domination into a state where either the space curvature or the cosmological constant is dominant. These cosmological parameters can only be believably determined when several independent methods of estimation have been applied and all yield similar values of the parameters. The three main ways of estimation of Ω_m are [86]:

(1) Local low-redshift dynamical tests

(2) Tests that depend on the coordinate distance to high-redshift sources through the angular size distance or the luminosity distance (radio source and the supernova)

(3) Tests using the fluctuations of the microwave background radiation on different angular scales.

The mean matter density today is $\rho_m \sim 3 \times 10^{-27}$ kg/m^3. We need to describe the nature of perturbations in the Universe, through global statistical descriptors such as the matter and radiation power spectra (their study is naturally intertwined with the determination of cosmological parameters). There may also be parameters describing the physical state of the Universe, such as the ionisation as a function of time during and since the recombination era. Typical comparisons of cosmological models with observational data now feature ten parameters [88].

Calculating the observables, *i.e.*, the number count, angular sizes of distant objects, Luminosity distance.. etc., we can track the history of the Universe at least back to where the interaction allows the interchanges between the densities of the different species before the BBN. But the main target is to measure the global cosmological parameters based on model assumption. In this thesis cosmological distances measures are studied in detail, for the homogeneous universe in Sec. 1.5 and for the inhomogeneous universe in Sec. 2.3. In Ch. 3 we will introduce a simpler approach for measuring distances in cosmology where we develop a lightcone gauge using a simple fact that observations are made in our past lightcone.

1.4.1 The History of the Universe

With all the observations that have been made and that we introduced in this chapter we can finally tell the story of our Universe. After the observational confirmation of the expanding Universe, cosmologists concluded that the galaxies will be spread much farther apart in the far future and that the Universe must have been much denser looking back in time. Therefore following the Friedmann dynamics back in time shows that there is a singularity, namely at $a(t) = 0$, we choose the origin of the cosmic time $t = 0$ occurs at the big bang singularity, then present-day-value of $t_0 = 13.81$ Gyr represent *the age of the Universe* [33]. This is generally taken to be the precursor of the current *Big Bang* model [89, 90]. After Planck era $t \sim 10^{-35}$ and within the first $t \sim 10^{-32}$ our Universe is thought to have experienced a first period of accelerated expansion, its called *inflation* introduced by Guth (1981) [91, 92]. This sudden increase in the rate of expansion of the Universe would have increased the size of the Universe by an enormous factor. Prior to inflation the entire universe was small and causally connected; it was during this period that the physical properties evened out. Inflation resolves the horizon problem and the so-called *flatness problem* of the big bang model. It has therefore been accepted as part of the current concordance model of cosmology. Many inflationary models have been proposed and tested with observations [93], but the most popular models of inflation that involves a single scalar field, the inflaton, whose slight inhomogeneities, and due to quantum fluctuations, have been the seeds of the Universe structures that we observe today.

The so called the *reheating* phase [94] started after the end of inflation during which the inflaton decays into particles of the standard model of particle physics. The Universe was about 379000 years old - much before the formation of stars and planets - it was denser, much hotter, and filled with hydrogen plasma. As the Universe expanded further, both the plasma and the radiation filling it grew cooler and at a temperature of about 3000K, it became favourable for protons and electrons to combine into hydrogen neutral atoms, and some stable nuclei (helium and lithium) could be formed from a primordial mixture of protons, neutrons and electrons [95, 96].

This is the *Primordial nucleosynthesis*, or BBN as it is commonly called [97], is the earliest and one of the most stringent tests of Big Bang cosmology. The relevant BBN reactions that played an important role in the history of the Universe took place in the first three minutes, notably (corresponding to temperatures of $T \sim 1$ MeV to $\sim 10^9$ KeV or higher [98]). At the end of this epoch the scale factor reads $a_0/a_{BBN} = 5 \times 10^7$ and dominated by radiation. From Friedmann equations we know that radiation density Ω_r decreases faster than the nonrelativistic matter density, at some point they become equal $\Omega_m\ a_0/a_{equ} = \Omega_m/\Omega_r \sim 3400$. During the *recombination* epoch and at this point atomic nuclei and electrons recombined into atoms, then photons stopped interacting with matter and became allowed to travel freely through space, and so the Universe became transparent [99]. When light started to travel freely through space rather than constantly being scattered by electrons and protons in plasma, a photon *decoupling* of matter and radiation formed. The decoupled photons reach present-day observers as the CMB, and they appear to come from a spherical surface around the observer such that the radius of the shell is the distance each photon has travelled since it was last scattered at the epoch of recombination. Such a surface is referred to as *the last scattering surface* (LSS).

Since light from this epoch has basically remained unaltered to the present day in the CMB imprint, that is what gives it the term *relic radiation*. The Universe remained basically neutral for few hundreds of millions of years *the dark ages*, during this time structure form via gravitational accretion creating the first stars and their light broke some of the neutral atoms into hydrogen ions in the interstellar medium during the epoch of *reionization* $a_0/a_{re} \sim 12$. The next billions of years the formation and evolution of galaxies took place and the large-scale-structure separated by walls and filaments. The cosmological constant starts to dominate over matter density leading to an accelerated cosmic expansion $a_0/a_m \sim 1.3$ [33].

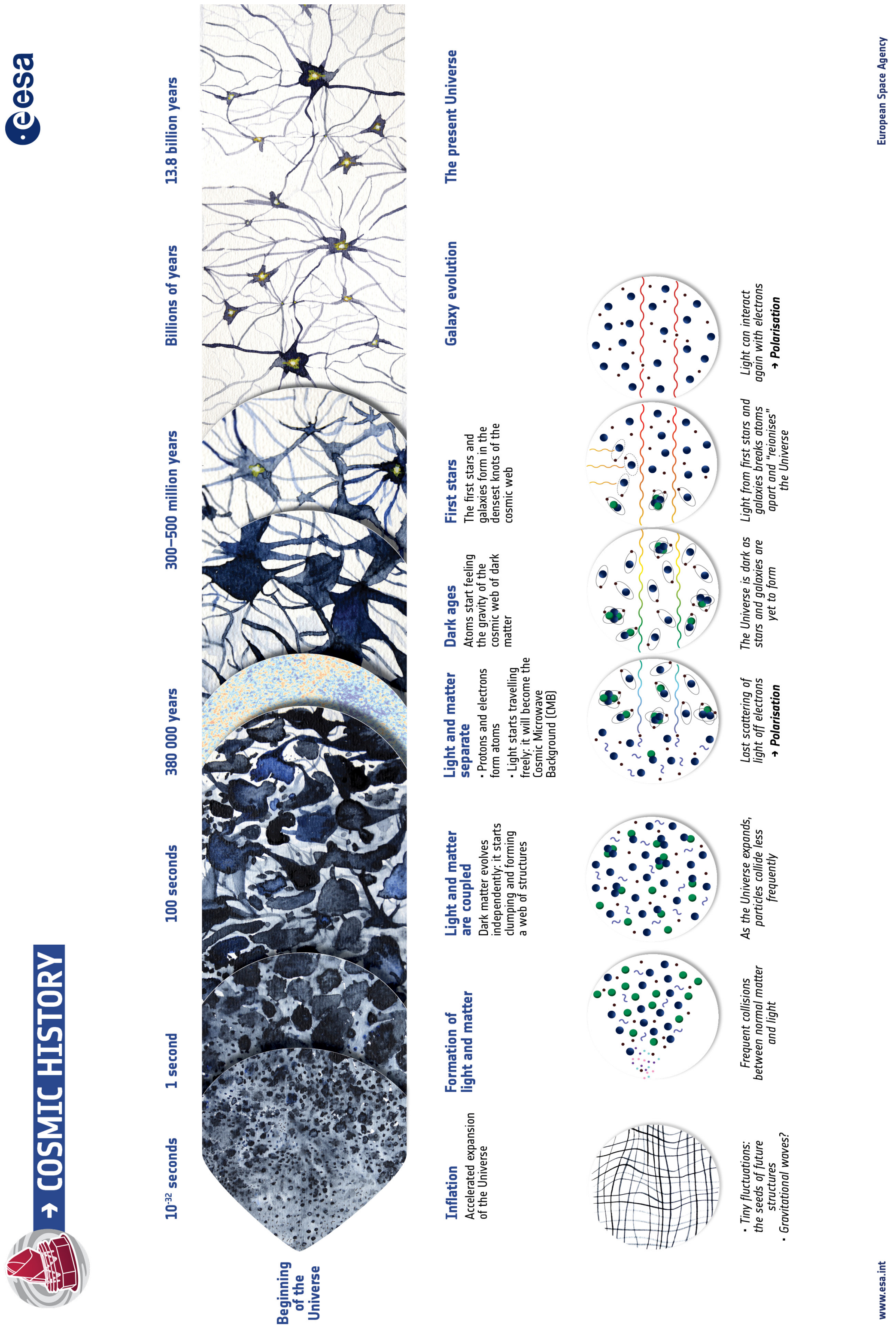


Figure 1.12: The history of the Universe [9].

1.5 Distances in Cosmology

We have shown in the previous sections a brief review of some current status of observational cosmology that fits so well with ΛCDM model. We emphasize that for every observation that involves the relation between angular or luminous distance and redshift area distance, FLRW spacetime is assumed, *i.e*, light propagates through a homogeneous and isotropic universe.

Here we are going to show some theoretical derivations of these distances with standard cosmological model. This will give a framework to interpret cosmological observations and to measure its free parameters. In particular we will show the crucial relation between cosmological distances and redshift for their correct interpretation. Over small distances, the relation between angular diameter distance and redshift, or luminosity distance and redshift, does not depend on whether there is or is not a cosmological constant, or the total value of the matter density Ω_m. These differences only become apparent over much larger distances, which make these expressions sufficient when they are written as functions of redshift z.

In cosmology spatial distance and velocity measurements are important to determine the consistency of relativistic theories and to measure the cosmological parameters. Thus investigations of spatial distance measurement arose from the fact that any specific astronomical measurement of distance carried out in any relativistic model of spacetime must lead to a result which depends upon the particular operations of measurement, and not upon the particular coordinate system used to describe the spacetime. Formulating invariant quantities corresponding to these various astronomical distances depends on the observations of apparent magnitude, apparent size, or apparent luminosity of distant light sources [100, 101]. There are different ways to measure distances in cosmology all of which give the same result in a Minkowski universe but differ in an expanding universe. They are, however, simply related as we shall see [38].

1.5.1 Redshift

A photons trajectory

We can define the four-vector k_μ as the gradient of a wave phase

$$k_\mu = \partial_\mu \phi \ , \tag{1.49}$$

which indicates the local direction of electromagnetic wave propagation through the spacetime. The propagation equation of the waves has to obey in the geometric optics

$$k^\mu k_\mu = 0 \ . \tag{1.50}$$

By taking the gradient of Eq. (1.50) and using Eq. (1.49), we can define the trajectory that followed by the electromagnetic waves

$$k^\mu \nabla_\mu k^\nu = 0 \ . \tag{1.51}$$

The path where k^μ is everywhere tangent are called null geodesics, and it can be interpreted as the world lines of photons or light rays. The geodesic equation (1.51) can also be written.

$$\frac{dk^\mu}{dv} + \Gamma^\mu{}_{\nu\alpha} k^\nu k^\alpha = 0 \ , \tag{1.52}$$

The momentum of the photon is

$$p^\mu = \hbar k^\mu \ , \tag{1.53}$$

where $\hbar = h/(2\pi)$ the reduced Planck constant. The affine parameter v along a given light ray is naturally defined with its tangent vector

$$k^\mu = \frac{dx^\mu}{dv} \ , \tag{1.54}$$

indicating that a small variation in v corresponds to a small displacement $dx^\mu = k^\mu dv$ along the light ray. Then the geodesic equation becomes

$$k^\nu \nabla_\nu k^\mu = \frac{dk^\mu}{dv} + \Gamma^\mu_{\nu\rho} k^\nu k^\rho = \frac{d^2 x^\mu}{dv^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dv} \frac{dx^\rho}{dv} \ , \tag{1.55}$$

where ∇_ν is the covariant derivative with respect to ν. τ denotes as the observer proper time, then the angular frequency is defined as

$$\omega = \left| \frac{d\phi}{d\tau} \right| = |u^\mu \partial_\mu \phi| = u^\mu k_\mu \ . \tag{1.56}$$

In order to detect the wave its propagations have to be in the direction opposite to the direction in which the observer is actually looking. This implies

$$k^\mu = \omega(u^\mu + n^\mu) \ . \tag{1.57}$$

Here n^μ is a unit direction vector of the photons, and u^μ their 4-velocity vector, defined more with the orthonormality relations

$$u^\mu u_\mu = -1, \quad n^\mu n_\mu = 1, \quad u^\mu n_\mu = 0 \ . \tag{1.58}$$

The difference between the frequency emitted by a source ω_s and the actual frequency measured by an observer ω_o is quantified by the redshift z as

$$1 + z = \frac{\omega_s}{\omega_o} = \frac{(u^\mu k_\mu)_s}{(u^\mu k_\mu)_o} \ . \tag{1.59}$$

1.5.2 Proper Distance

If we place the galaxies at equal distance from each other and labelled them by a fixed coordinate x, and if space itself is expanding in time, the scale factor a will depend on time as well, and the relative velocity between two galaxies at distance

$$d = a\Delta x \tag{1.60}$$

is

$$v = \dot{a}\Delta x \ , \tag{1.61}$$

such that

$$v = \frac{da(t)/dt}{a(t)} d = \frac{\dot{a}}{a} d = H_0 d \ , \tag{1.62}$$

where v is the recessional velocity, typically expressed in km/s, and d is the *proper distance* (which can change over time, unlike the *comoving distance*, which is constant) from the galaxy to the observer, measured in megaparsecs (Mpc)[5]. Here H_0 is the *Hubble constant* and corresponds to the value of the *Hubble parameter* $H(t)$ at the time of observation. $H(t)$ is a value that is time dependent and which can be expressed in terms of the scale factor.

It is convenient to normalise the scale factor such that $a_0 = 1$, so that comoving scales become physical scales today. On the other hand, an object at $z \ll 1$ at physical distance d away from us, recedes with speed v, then roughly $d \approx \eta_0 - \eta$ is the time delay between the events (A, B) in the same frame, and the positions of the two events will be changing over time due to the expansion, and therefore, $a_0 \approx a(\eta) + a'(\eta_0 - \eta)$, so that

$$1 + z \approx 1 + \frac{a'}{a}(\eta_0 - \eta) \approx 1 + H_0 d \ . \tag{1.63}$$

[5] 1 parsec = 3.2615638 light years = 3.0856776×10^{16} metres.

Figure 1.13: The proper distance between nearby events (A, B) connected by a unique geodesic $\mathcal{G}$.

Therefore we have $v \approx z$, and $H_0 = v/z$. The proper distance can be written as

$$d = \int_0^z \frac{dz'}{(1+z')H(z')} = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')h(z')} \; , \tag{1.64}$$

where $H(z)$ is the Hubble parameter as a function of redshift, and $h(z) = H(z)/H_0$ is an expansion parameter normalized by the Hubble constant, given by

$$h(z) = \sqrt{\Omega_r(1+z)^4 + \Omega_d(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda} \; . \tag{1.65}$$

1.5.3 Comoving Distance (Line-of-Sight)

A comoving distance between two events in the Universe, at a specific moment of the cosmological time will remain constant, if the two objects are moving with the Hubble flow which will give distance that does not change in time due to the expansion of space [102]. The total line-of-sight comoving distance between two nearby objects along the radial lightray can be given by

$$d_c = \frac{1}{H_0} \int_0^z \frac{dz'}{h(z')} \; . \tag{1.66}$$

The line-of-sight comoving distance is the fundamental distance measure for all other distance measures [102].

1.5.4 Transverse Comoving Distance

The comoving distance between two objects on the sky that are at a constant redshift z, and they are separated by an angle $\delta\theta$ is $\delta\theta d_m$, where d_m is the transverse comoving distance, *i.e.*, the cross-sectional length of an object perpendicular to the light ray. It is related to the line-of-sight comoving distance d_c by [102]

$$d_m = \begin{cases} \frac{d_H}{\sqrt{|\Omega_k|}} \sinh\left[\sqrt{|\Omega_k|} d_c/d_H\right] & \text{if } \Omega_k > 0 \; , \\ d_c & \text{if } \Omega_k = 0 \; , \\ \frac{d_H}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|} d_c/d_H\right] & \text{if } \Omega_k < 0 \; , \end{cases} \tag{1.67}$$

where $d_H = 1/H_0$ is called Hubble distance.

1.5.5 Area Distance

The area distance is also known as the angular diameter distance. This cosmological distance measure relates the proper transverse size of an object to the solid angle in which it is observed, from the directions originating at the object and pointing towards the observer. The area distance r_A at the observer, also called "the corrected luminosity distance" in [103], and "observer area distance" in [104], is defined by,

$$dS_o = r_A^2(O) d\Omega_o \; , \tag{1.68}$$

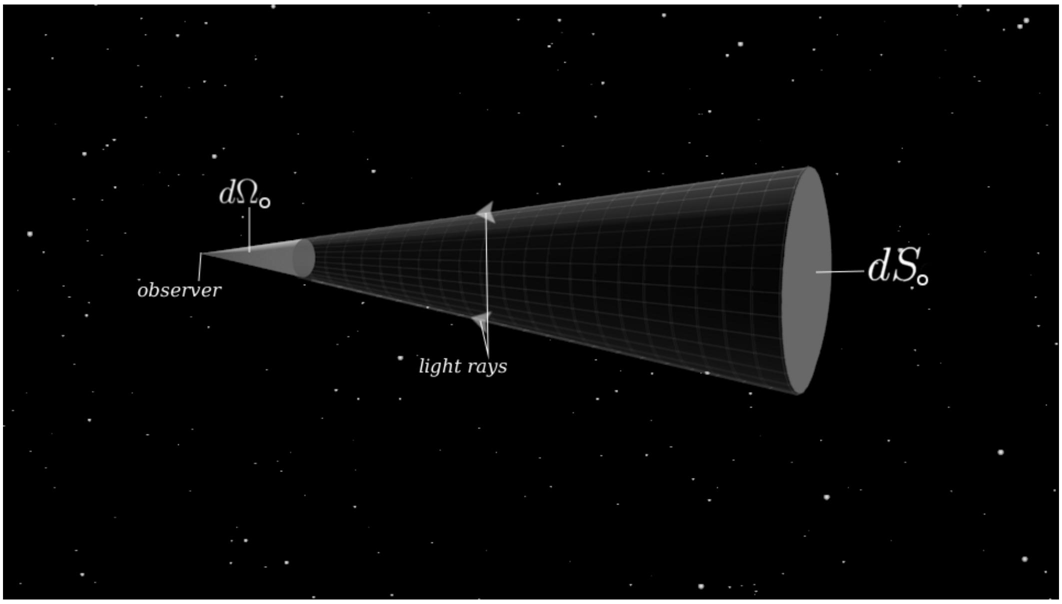


Figure 1.14: In a curved spacetime, light travels on null geodesics, and the area distance r_A of the source from the observer is $r_A^2 = dS_o/d\Omega_o$.

where $d\Omega_o$ is the solid angle subtended by that object at the observer, and dS_o is its cross-sectional area or the transverse size of the object at the observer, see Fig. [1.14]. The area distance measured from the source would be defined using the geodesic bundle diverging from the source and the ratio between the physical transverse size of the source and the solid angle under which it would be observed from the source

$$dS_s = r_A^2(S) d\Omega_s \ . \tag{1.69}$$

From the above definition of the area distance at the source, one can say that it is not directly observable, *i.e.*, an observer can measure dS_s, then the observer cannot without knowing $r_A(S)$ determine the solid angle $d\Omega_s$ into which this radiation was emitted [105]. The relation between $r_A(O)$ and $r_A(S)$ is

$$r_A(S) = (1+z) r_A(O) \ . \tag{1.70}$$

From now on in our calculations of distances we will consider the area distance measured from the observer r_A. The area distance is related also to the transverse comoving distance measured in FLRW by

$$r_A(z) = \frac{d_m}{1+z} \ . \tag{1.71}$$

The difficulty of measuring angular distance in astronomy is that it requires standard rulers or sources of a known size or it can be calibrated by independent experiments. The angular diameter distance is also naturally involved in strong gravitational lensing and time delays experiments.

1.5.6 Luminosity Distance

Distances can be inferred by measuring the flux from an object of known luminosity. In a nonrelativistic picture, the luminosity distance d_L is defined by the relationship between the flux F of the luminosity, *i.e.*, the rate at which radiation crosses a unit area of surface per unit time, and the intrinsic luminosity L of an object, *i.e.*, energy per unit time, arriving at the observer [106]

$$d_L = \sqrt{\frac{L}{4\pi F}} \ . \tag{1.72}$$

A light source not only appears smaller but also fainter as it lies farther from the observer. It is related to the transverse comoving distance and angular diameter distance by

$$d_L(z) = (1+z)d_m = (1+z)^2 r_A \ . \tag{1.73}$$

The factor $(1+z)^2$ can be interpreted as follows: the first $(1+z)$ comes from the shifted energies of the photons at the emission and reception and time dilation between the source and the observer, the second factor $(1+z)$ is due to the exchange of the roles of s and o Eq. (1.70). Because gravitational waves follow null geodesics just like electromagnetic waves, their detections are expected to usher in excellent measurements of luminosity distance to their sources.

1.6 Observations on the Past Lightcone

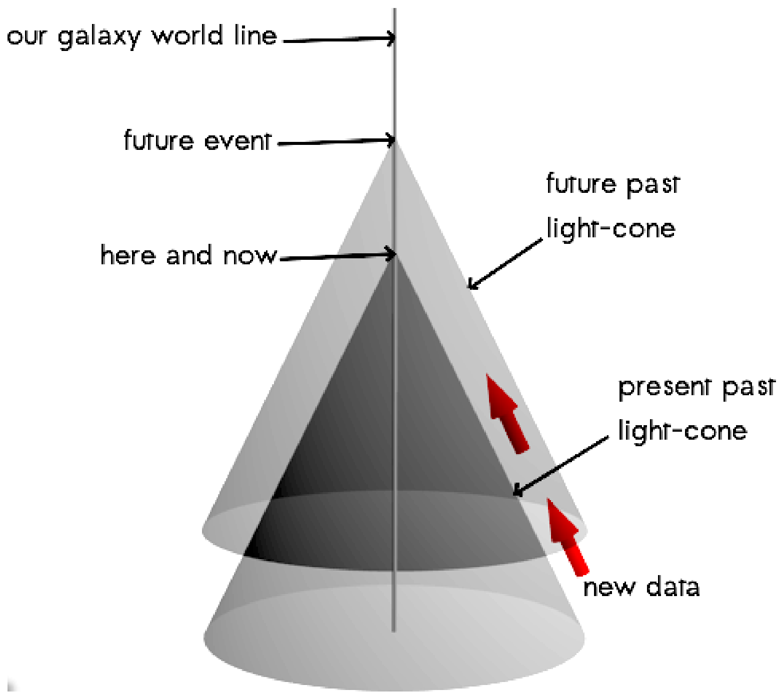


Figure 1.15: Observations from the vertices of our past lightcone and future past lightcone.

One of the fundamental features of cosmology is that there is only one Universe, on which we cannot experiment; we can only observe it [105]. The Universe is so large that the spatial distances and time scales involved are also very large. Thus, we must distinguish between the observable Universe for which we have data, and the Universe which includes regions we cannot directly influence or observe. In GR light travels on the null cone which is the surface of the lightcone. Therefore an observer on the apex can only observe another event crossing the null cone.

On cosmological scales we are it is not possible for us to move away from our local galactic worldline C. Therefore observations give direct access only to our past lightcone C^-, at one cosmological time q: here and now. This is a fundamental constraint on what is empirically decidable in cosmology [107]. Therefore an observer on the central worldline C can only observe distant galaxies and quasars and a radiation background. All the astrophysical data obtained from actual observations are mainly localised on our past lightcone in a 3-dimensional hypersurface around our past worldline. Since all these observations are made at different times and in different directions, they are assumed to have been obtained by observing from a single point on $C(q)$ [108]. The observation of a cosmological quantity (massive stars, black holes, etc.) in the sky requires a set of lightcones along the observation time interval, where data needs to be located only in one past lightcone [108].

However, the time variation of the same object is expected to be extremely long compared to the time interval in which the observations are made, hence we can only consider one lightcone $C^{-}(q)$ to specify our data.

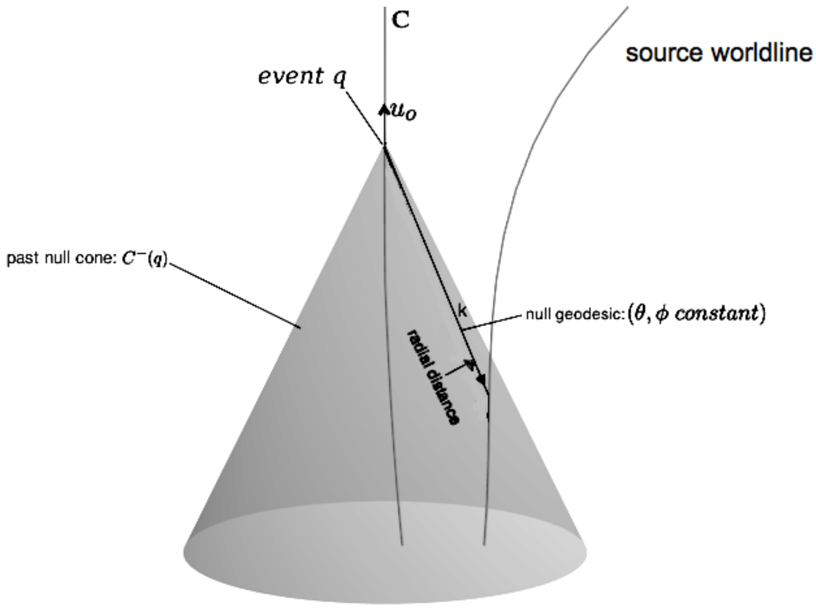


Figure 1.16: Observations of an event on the null cone.

There are particular restrictions down our past lightcone to the limiting redshift beyond which either we cannot perform astronomical observations, or due to the fact the actual structure of the past lightcones is not ideal, but rather very complicated due to the null caustics-points, that occur near to these giant cosmological objects. One needs to consider the existence of such caustics when observations are interpreted, a challenge that can be overcome assuming the Universe is filled with a perfect fluid in which cosmological quantities could be considered as particles [108]. Now with these smoothing approximation techniques we may have the ideal past lightcone ready to set up for observation.

1.7 The Scope of this Thesis

The lightcone gauge is a set of observational coordinates adapted to our past lightcone. And it is made at one point in spacetime as initial data for the field equations without any a *priori* assumptions about the spacetime geometry, such as assuming that spacetime is isotropic and homogeneous. The attempt is to deduce the large-scale structure of the Universe by using idealized astronomical observations, and then confront these interpretations of the astronomical observations with cosmological theory.

In the work we will present here we intend to use and examine cosmology in our past lightcone with cosmological observations. Assuming that our galactic world line is a regular geodesic, where observations are made, then metric variables and curvature components as one approaches the central world line can be achieved. We will add perturbations to the lightcone to obtain a perturbed metric to first-order calculations. We will derive the observables quantities within our perturbed lightcone, and compare them with what has been obtained from the standard approach. The point of this new gauge is to convince you as a reader that the calculations of the cosmological observables are much easier since we consider signals moving in straight lines and hence no light deflection and space distortion need to be worried about. We will also use the new gauge to calculate the density fluctuations, then we will make gauge transformations to our result in the perturbed lightcone gauge to the general gauge and see if our result is compatible with the results obtained in the standard gauge. We prove that our perturbed metric is genuine and it fulfills the EFE degrees of freedom.

The second topic is to address the recent observations of the accelerated cosmic expansion rate by using modified theories of gravity. We analyze models of $f(R)$ gravity that allow non-rotating fluid solutions, *i.e.*, when the vorticity is zero. We consider several sub-cases, such as shear-free cases, non-expanding cosmologies, etc...

In the scramble for the understanding of the nature of dark matter and dark energy, it has recently been suggested that the change of behavior of the missing energy density might be regulated by the changes in the equation of state of the background fluid. Chaplygin gas models are based on the existence of some kind of exotic fluids out there in the Universe. Our approach attempts to replace such exotic fluids with modification of gravity (geometry rather than extra fluid). We will attempt to produce $f(R)$ models that replace Chaplygin gas solutions. This work aims to bring to light a geometric interpretation of the model by re-writing the different toy Chaplygin gas models in terms of exact $f(R)$ gravity solutions that are generally quadratic in the Ricci scalar with appropriate ΛCDM limiting solutions.

Chapter 2

Perturbation Theory in Cosmology

Nothing happens until something moves.

Albert Einstein

2.1 The Inhomogeneous Universe: Gauge-invariant Cosmological Perturbation Theory

The measurements of the anisotropies in the CMB of the Universe show only small initial fluctuations from FLRW background. This spatial variation of the density growth in the actual universe is not quite uniform in all directions. The evolution of these fluctuations until the time when they become of order unity can be studied within linear perturbation theory [38].

2.1.1 The Metric of the Perturbed Spacetime

We shall now slightly perturb the FLRW model with the basic perturbation equations [10]

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + a^2 \delta g_{\mu\nu} \ , \tag{2.1}$$

where $\overline{g}_{\mu\nu}$ represents the unperturbed Friedmann metric, and $\delta g_{\mu\nu}$ can be parametrised in terms of its components respectively as

$$\delta g_{\mu\nu} dx^\mu dx^\nu = -2\phi d\eta^2 + 2B_i d\eta dx^i + 2C_{ij} dx^i dx^j \ . \tag{2.2}$$

Here B_i is a rank-1 tensor and C_{ij} is a rank-2 tensor [10]. For linear perturbation theory, the spacetime metric is thus

$$ds^2 = a^2(\eta) \left[-(1+2\phi) d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + 2C_{ij}) dx^i dx^j \right] \ . \tag{2.3}$$

The inverse components and the Christoffel symbols associated with this metric are given in Appendix A.1.

2.1.2 Decomposition of the Perturbation Variables

The perturbation variables in Eq. (2.2) can be decomposed by splitting the vector field B_i as a sum of a gradient of a scalar and a divergenceless vector as [10]:

$$B_i \equiv \nabla_i B + \overline{B}_i \quad \text{where} \quad \nabla^i \overline{B}_i = 0\,. \tag{2.4}$$

For a velocity field, B will be the potential, and $\overline{B}_i$ the vorticity. Here we can see that the 3-components of the vector have been split into 1 scalar (B) and 2 vector ($\overline{B}_i$) components. Analogously, the symmetric rank-2 tensor C_{ij} can be decomposed as

$$C_{ij} \equiv -\psi\gamma_{ij} + \nabla_i\nabla_j E + \nabla_i F_j + \frac{1}{2}h_{ij} \quad \text{with} \quad \nabla^i F_i = \nabla^i h_{ij} = h^i{}_i = 0\,. \tag{2.5}$$

The 6 components of C_{ij} have thus been split into 2 scalar (ψ and E), 2 vector (F_i) and 2 tensor (h_{ij}) components. Therefore, $\delta g_{\mu\nu}$ is decomposed into 10 components with 10 degrees of freedom of the metric. The advantages of this Scalar-Vector-Tensor (SVT) decomposition lies in the fact that the three types of perturbations are decoupled and can thus be studied separately.

2.1.3 The Gauge Problem

A problem in perturbation theory can be summarised in the 'choice of gauge'. If coordinates are chosen in the background manifold $\overline{s}$ then the correspondence Ω introduces a coordinate system on the physical manifold s. With tensor fields Q (*e.g.*, the Ricci scalar, the energy density ρ, *etc.*) on s, and the corresponding physical quantity $\overline{Q}$ on $\overline{s}$, then we define the perturbation δQ of Q at the point $x \in s$ by [10]

$$\delta Q(x) = Q(x) - \overline{Q}(x,t). \tag{2.6}$$

It is usually understood that the perturbation δQ is small. However, δQ can be assigned at any point on the physical manifold s by simply altering the correspondence Ω.

In order to map between this two manifolds, we need to compare two different spacetimes. We will consider for the physical observable universe s with a spacetime manifold $\mathcal{M}$, metric g and energy momentum tensor T, which in some sense must be close to the FLRW universe, and a fictitious background spacetime $\overline{s}$ with manifold $\overline{\mathcal{M}}$ and metric $\overline{g}$ and energy momentum tensor $\overline{T}$. This implies that there exist some unphysical degrees of freedom related to the choice of the coordinate systems on the two manifolds [10].

Similarly, $\rho(x)$ describes the actual value of the energy density at point x, while $\overline{\rho}(x)$ describes the fictitious background value there. The quantity $\delta\rho = \rho - \overline{\rho}$ is the variation in density of the background model and the realistic lumpy model [109]. In general, the mapping is given as

$$\Omega : \overline{g}_{\mu\nu} \to g_{\mu\nu}. \tag{2.7}$$

An interesting and yet nontrivial problem is how to identify points in the background $\overline{s}$ with corresponding points in the realistic spacetime s, in accord with

$$\overline{g}_{\mu\nu} \to g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{2.8}$$

This is related to the difficulties of the background choice. We will assume the existence of backgrounds that lead to the FLRW universe. There are may be many different ways to these different backgrounds leading to slightly different FLRW backgrounds. But since $|g - \overline{g}|$ is small, of order ϵ, the differences of FLRW backgrounds must also be small, of order ϵ, and can be regarded as part of the perturbations [38].

The only possibilities for gauge-invariant quantities are: a scalar or a tensor, which are constant in $\overline{\mathcal{M}}$. These quantities can be given any value we desire by altering that map; we can, *e.g.*, set it to zero by choosing the real surfaces of constant energy density to be the background surfaces

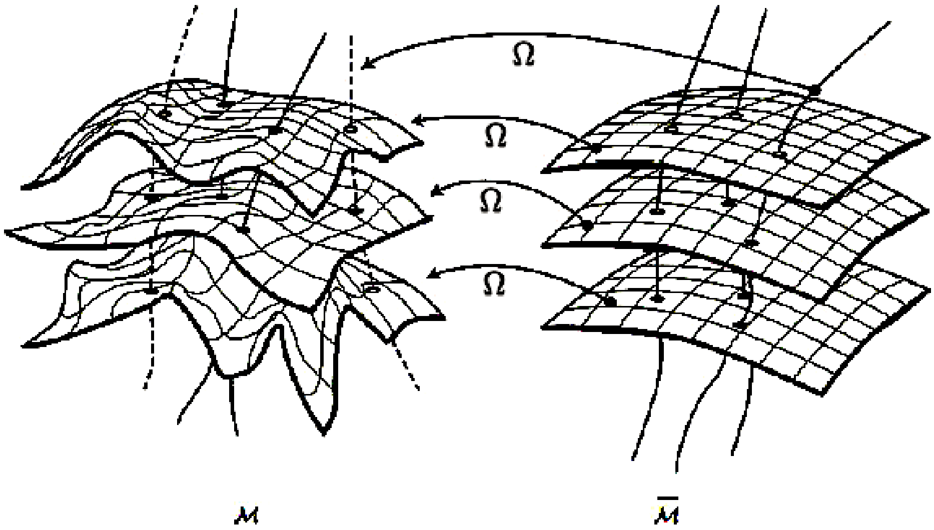


Figure 2.1: Any perturbed quantity is defined by the mapping between the background universe FLRW, $\overline{\mathcal{M}}$, and the actual perturbed universe $\mathcal{M}$ [10].

of constant time and, hence, of constant energy density. Consequently, perturbation equations written in terms of this variable have as solution both physical modes and gauge modes, the latter corresponding to the variation of gauge choice rather than to physical variation, because they depend intrinsically on the manifold choice, and this is what is called the *gauge problem*. One way to solve this is by very carefully keeping track of the gauge choice used and the resulting gauge freedom. The perturbations are not unique but Eq. (2.6) is invariant under coordinate transformations. And as long as it does not matter how we map $\Omega : s \to \bar{s}$ when it leaves the background manifold $\bar{s}$ unchanged, the mapping is called *a gauge transformation* [110].

2.1.4 Gauge Transformation

The gauge transformation reflects the freedom of choosing different diffeomorphism $\{x^\mu\}$ in the physical manifold $\mathcal{M}$

$$x^\mu \leftarrow \overline{x}^\mu = x^\mu + \zeta^\mu \ , \tag{2.9}$$

where ζ^μ is a vector field. Thus, if we alter (push forward) the initial correspondence $\overline{Q}(x)$ maybe only slightly to $\overline{Q}(x+\zeta)$ to obtain the new identification map Ω^*, this describes the same geometry as g [38]. Since we have chosen the background metric $\overline{g}$ we only allow diffeomorphisms which leave $\overline{g}$ invariant, but they can deviate at first order. The definition of the perturbation is now

$$\delta Q^*(x) = Q(x) - \overline{Q}(\Omega^*(x)) \ . \tag{2.10}$$

It is evident that the difference is

$$\Delta Q(x) = \delta Q^*(x) - \delta Q(x) = \overline{Q}(\Omega(x)) - \overline{Q}(\Omega^*(x)), \tag{2.11}$$

and constitutes the fundamental gauge-invariance [111]. The only gauge-invariant quantities are the ones for which

$$\mathcal{L}_\zeta \overline{Q} = 0 \ , \quad \forall \zeta \ , \tag{2.12}$$

with vanishing or constant contribution to the background [38,111]. The perturbation of an arbitrary tensor field $Q = \overline{Q} + \epsilon\delta Q$ obeys the gauge transformation to first order as

$$\delta Q \longrightarrow \delta Q + \mathcal{L}_\zeta \overline{Q} \,, \tag{2.13}$$

and the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\zeta \overline{g}_{\mu\nu} \,. \tag{2.14}$$

Since all relativistic equations are covariant by definition, they can always be written in the form $Q = 0$, where Q is a tensor field [10,111]. It is thus always possible to write all the relevant equations in perturbations up to first order in terms of gauge-invariant quantities only. Such a treatment would ensure that no gauge, *i.e.*., unphysical, degree of freedom, is misleadingly used.

2.1.5 Gauge-invariant Variables

In order to extract the physical degrees of freedom, we will consider an active transformation of the coordinate system defined by the displacement vector ζ. Therefore the coordinates of any point will change according to Eq. (2.9). The vector field ζ^μ is decomposed into 2 scalar degrees of freedom (T and L), and 2 vector degrees of freedom ($\overline{L}^i$), which is divergenceless, *i.e.*, $\nabla^i \overline{L}_i = 0$, as

$$\zeta^0 = T, \quad \zeta_i = L_i = \nabla_i L + \overline{L}_i \,. \tag{2.15}$$

The perturbation variables under gauge transformations look like [10]

$$\phi \to \phi + T' + \mathcal{H}T \,, \tag{2.16}$$
$$B_i \to B_i - \nabla_i T + {L'}_i \,, \tag{2.17}$$
$$C_{ij} \to C_{ij} + \nabla_i L_j + \nabla_j L_i + 2\mathcal{H}T\gamma_{ij} \,. \tag{2.18}$$

Then the metric g transforms as

$$\mathcal{L}_\zeta \overline{g} = a^2[-2(\mathcal{H}T + T')d\eta^2 + 2(L'^i - \nabla^i T)d\eta dx_i + (2\mathcal{H}T\gamma^{ij} + \nabla^j L^i + \nabla^i L^j)dx_i dx_j] \,. \tag{2.19}$$

Now using the above decomposed perturbation variables, and under gauge transformations they become [10]

$$\phi \to \phi + T' + \mathcal{H}T \,, \tag{2.20}$$
$$\overline{B}_i \to \overline{B}_i + L^i \,, \tag{2.21}$$
$$B \to B - T + L' \,, \tag{2.22}$$
$$\psi \to \psi + \mathcal{H}T \,, \tag{2.23}$$
$$E \to E + L \,, \tag{2.24}$$
$$F^i \to F^i + L^i \,, \tag{2.25}$$
$$h_{ij} \to h_{ij} \,. \tag{2.26}$$

Since one of the gauge-invariant quantities corresponds exactly to the Newtonian potential energy from Newtonian gravity, the Newtonian limit can be performed easily. By putting the scalar perturbations $E = -L$ and $T = L'$, $E = B = 0$ one gets the so-called *longitudinal gauge*:

$$\delta g = -2\Phi d\eta^2 + 2\Psi\gamma_{ij}dx^i dx^j \,. \tag{2.27}$$

For vector perturbation it is convenient to set $F^i = -L^i$ so that F^i will vanish and we will have

$$\delta g_{0i} = \overline{\Phi}_i = (F'_i - \overline{B}_i) \,. \tag{2.28}$$

This is a 'vector gauge' and it is gauge invariant. Clearly there are no tensorial gauge transformations and hence h_{ij} is also gauge invariant [38].

The traditionally favoured choice for cosmological perturbations is the metric perturbation theory. In the so-called *Bardeen's approach* [112] for metric perturbations, we can construct a set of gauge-invariant independent quantities related to density perturbations, but these quantities are not perturbations themselves in terms of the background metric, *i.e.*, they do not depend on L^i and T. For instance

$$\phi = \Phi - \mathcal{H}(B - E') - (B - E')' \ , \tag{2.29}$$

$$\psi = \Psi + \mathcal{H}(B - E') \ , \tag{2.30}$$

where Φ and Ψ are called *Bardeen potentials*. The Bardeen approach to cosmological perturbation theory is widely used and fundamentally important.

A new method developed by Ellis and Bruni (1989) [109] as an alternative description of spacetime in terms of scalars, 3-vectors and projected symmetric trace-free (PSTF) tensors is the so-called *1+3 covariant approach* [113], and the later is what we are going to use and call as standard gauge. It has the freedom of choosing a set of coordinates and transforming to another if wished. This work is built on the work of Ehlers [114], Hawking (1966) [115], Olson [116] and Stewart and Walker [111], and does not use the metric directly but describes the spacetime instead by means of covariant and gauge-invariant quantities obtained by a local foliation of spacetime into 'space' and 'time'.

2.1.6 The Four-velocity

A comoving observer's trajectory along a worldline orthogonal to the hypersurfaces Σ_t is given by, u^μ the four-velocity of the observer

$$u^\mu = \overline{u}^\mu + \delta u^\mu \ , \tag{2.31}$$

satisfying the condition given in Eqs. (1.58). With a time-like geodesic, the normalisation condition of $\overline{u}^\mu$ to zeroth order provides the solution

$$\overline{u}^\mu = a^{-1}\delta^\mu_0, \quad \overline{u}_\mu = -a\delta^0_\mu \ , \tag{2.32}$$

and hence $u^i = 0$ at zeroth order. From Eqs. (1.58) we see that

$$g_{\mu\nu}u^\mu u^\nu = -1 \ , \tag{2.33}$$

and therefore

$$-a^2(1+2\phi){u^0}^2 = -1 \ . \tag{2.34}$$

This leads to

$$u^0 = \frac{1}{a}\left(1 - \phi\right) \ , \tag{2.35}$$

and

$$\delta u^0 = -a^{-1}\phi \ . \tag{2.36}$$

We then write

$$\delta u^i = a^{-1}v^i \ , \tag{2.37}$$

where v^i is the peculiar velocity of the object. It is also easy to show that

$$\delta u_0 = -a\phi, \quad \text{and} \quad \delta u_i = a(v_i + B_i) \ . \tag{2.38}$$

We can decompose v_i into a scalar and a tensor part according to the form of Eq. (2.4):

$$v_i = \nabla_i v + \overline{v}_i. \tag{2.39}$$

And upon performing a gauge transformation on δu^μ,

$$\delta u^\mu \to \delta u^\mu + \mathcal{L}_\varsigma \overline{u}^\mu, \tag{2.40}$$

where

$$\mathcal{L}_\zeta \overline{u}^\mu = \zeta^\alpha \nabla_\alpha \overline{u}^\mu - u^\alpha \nabla_\alpha \zeta^\mu \ . \tag{2.41}$$

After SVT decompositions on v and v_i, they will transform as

$$v \to v - L' \ , \tag{2.42}$$

$$\overline{v}_i \to \overline{v}_i - \overline{L}_i' \ . \tag{2.43}$$

Like the case of metric perturbative gauge-invariant quantities, we can define the gauge-invariant quantities associated with quantities in Eqs. (5.54), (2.43):

$$V \equiv v - E' \ , \tag{2.44}$$

$$\overline{V}_i \equiv \overline{v}_i - \overline{B}_i \ , \tag{2.45}$$

$$\overline{W}_i \equiv \overline{v}_i - F_i' \ . \tag{2.46}$$

Different choices are possible [10, 106], and it is also worth noting that we can relate different gauge-invariant variables like

$$\overline{W}_i = \overline{V}_i - \overline{\Phi}_i \ . \tag{2.47}$$

2.2 Perturbations of Null Geodesics

We would like now to study the effects of perturbation on the light rays coming from a source to the observer.

2.2.1 Effects on Source Position

On astronomical and cosmological scales, light gets deflected by any inhomogeneous gravitational field and affects the source's apparent position, therefore when measuring distances of a distorted image or the brightness of images one has to study the geodesic deviation of null geodesics. Consequently, such measurements rely on our good understanding of light propagation through the cosmos, in particular the way that light beams are focused by matter lying between us as the main observer and the sources.

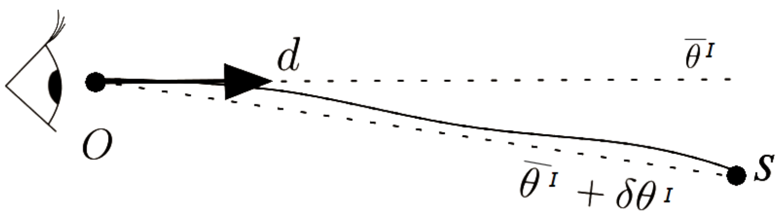


Figure 2.2: The deviation of the angular coordinates $\bar{\theta}^I$ of the source given an observational direction d.

For example, if we want to study the position of a far distant source, on the observer's celestial sphere, we could consider an event S, and a straight line of sight d corresponding to the direction $\bar{\theta}^I|_{I=2,3} = (\theta, \phi)$ towards which the observer is looking, where (θ, ϕ) are two angles in the sky. Now due to a perturbed spacetime, between the source and the observer, the deflection of light implies that the light ray deviates from its radial straight line $\theta^I = \bar{\theta}^I$, to angular coordinates corresponding to

$$\theta^I = \bar{\theta}^I + \delta\bar{\theta}^I \ . \tag{2.48}$$

This new direction corresponds to the direction in which the observer would see the image if light did follow a radial straight line [33]. The above equation relates the real position of the source to its observable position.

2.2.2 Effects on Null-vector

The geodesic of photons is a worldline $x^\mu(\lambda)$ where λ is an affine parameter. Applying perturbation to the geodesic equation (1.52) to first order, we get

$$k^\mu = (\bar{k}^\mu + \delta k^\mu)\ , \Gamma^\mu{}_{\nu\alpha} = (\bar{\Gamma}^\mu{}_{\nu\alpha} + \delta\Gamma^\mu{}_{\nu\alpha})\ , \tag{2.49}$$

$$\frac{d}{d\lambda}\delta k^\mu + \bar{k}^\nu \bar{k}^\alpha \delta\Gamma^\mu{}_{\nu\alpha} + 2\bar{k}^\nu \bar{\Gamma}^\mu{}_{\nu\alpha}\delta k^\alpha = 0\ . \tag{2.50}$$

The tangent vector of the null geodesic satisfies the condition

$$g_{\mu\nu}k^\mu k^\nu = 0\ . \tag{2.51}$$

Perturbing the above Eq. (2.51) also to first order, we will get

$$\bar{k}^\mu \bar{k}^\nu \delta g_{\mu\nu} + \bar{k}^\mu \bar{g}_{\mu\nu}\delta k^\nu + \bar{k}^\nu \bar{g}_{\mu\nu}\delta k^\mu = 0\ . \tag{2.52}$$

2.2.2.1 Conformal Trick

We can use the perturbed FLRW metric as in Eq. (2.2) and by connecting two metrics using Sachs' and Wolfe's conformal techniques [117] such that

$$ds^2 = a^2 \widetilde{ds}^2\ , \tag{2.53}$$

where

$$\widetilde{ds}^2 = \left[-(1+2\phi)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2C_{ij})dx^i dx^j\right]\ . \tag{2.54}$$

$\widetilde{ds}^2$ is a conformal space metric, and it is called a *perturbed Minkowski metric*. It has all the geometrical properties of the FLRW metric but its space is not expanding. The lightlike geodesics of ds^2 coincide with those of $\widetilde{ds}^2$. If $\tilde{k}$ is a null geodesic for the metric $\widetilde{ds}^2$ with affine parameter $\tilde{\lambda}$, then

$$\widetilde{k}^\mu = a^2 k^\mu \tag{2.55}$$

is a null geodesic for ds^2 with affine parameter λ. However the affine parameters do not coincide and are related by

$$d\tilde{\lambda} = a^{-2} d\lambda\ . \tag{2.56}$$

In the following, we will first calculate $\widetilde{k}^\mu$, and then we will use Eq. (2.55) to determine k^μ. Lightlike geodesics are invariant under conformal transformations; we are doing this to make our calculations easier. We will drop the tilde ($\tilde{}$) and keep in mind the calculations are made on the conformal space unless otherwise stated. The 0-components of the geodesic equation (2.50), with the use of Eq. (2.52), can be written as

$$\begin{aligned}&\frac{d}{d\lambda}\delta k^0 + \bar{k}^0\bar{k}^0\delta\Gamma^0{}_{00} + 2\bar{k}^0\Gamma^0{}_{00}\delta k^0 + 2\bar{k}^0\bar{k}^i\delta\Gamma^0{}_{0i} + 2\bar{k}^i\Gamma^0{}_{0i}\\ &+\delta k^0 + 2\bar{k}^0\Gamma^0{}_{0i}\delta k^i + \bar{k}^i\bar{k}^j\delta\Gamma^0{}_{ij} + \bar{k}^j\Gamma^0{}_{ij}\delta k^i + \bar{k}^i\Gamma^0{}_{ij}\delta k^j = 0\ ,\end{aligned} \tag{2.57}$$

and with a conventional choice of $\bar{k}^\mu = (1, n^i)$ at the background, where n^i is the unit vector, we have

$$\frac{d}{d\lambda}\delta k^0 + \phi' + 2n^i\nabla_i\phi + n^i n^j\left(C'_{ij} - \nabla_i B_j\right) = 0\ . \tag{2.58}$$

We also define the backward affine parameter along the past lightcone, such that

$$\phi' = \frac{d\phi}{d\lambda} - n^i\nabla_i\phi\ . \tag{2.59}$$

Then we can rewrite Eq. (2.58) as

$$\frac{d}{d\lambda}\delta k^0 = \phi' - 2\frac{d}{d\lambda}\phi - n^i n^j \left(C'_{ij} - \nabla_i B_j\right) . \tag{2.60}$$

Integrating both sides of Eq. (2.60) yields

$$\delta k^0_o - \delta k^0_s = \int_s^o \left[\phi' - 2\frac{d}{d\lambda}\phi - n^i n^j \left(C'_{ij} - \nabla_i B_j\right)\right] d\lambda . \tag{2.61}$$

k^0 is a component along the light trajectory between the observer o and the source s is given by

$$k^0_o - k^0_s = \overline{k}^0_o + \delta k^0_o - \overline{k}^0_s - \delta k^0_s . \tag{2.62}$$

And since the background is Minkowski, the background term is the same both at the source and observer points. Therefore

$$k^0_o - k^0_s = \int_s^o \left[\phi' - 2\frac{d}{d\lambda}\phi - n^i n^j \left(C'_{ij} - \nabla_i B_j\right)\right] d\lambda . \tag{2.63}$$

Now calculating the i-component of the null vector,

$$\begin{aligned}&\frac{d}{d\lambda}\delta k^i + \overline{k}^0\overline{k}^0\delta\Gamma^i{}_{00} + 2\overline{k}^0\Gamma^i{}_{00}\delta k^0 + 2\overline{k}^0\overline{k}^j\delta\Gamma^i{}_{0j} + 2\overline{k}^j\Gamma^i{}_{0j}\delta k^0 \\ &+2\overline{k}^0\Gamma^i{}_{0j}\delta k^j + \overline{k}^j\overline{k}^k\delta\Gamma^i{}_{jk} + \overline{k}^j\Gamma^i{}_{jk}\delta k^k + \overline{k}^k\Gamma^i{}_{jk}\delta k^j = 0 ,\end{aligned} \tag{2.64}$$

and hence

$$\frac{d}{d\lambda}\delta k^i + B^{i\prime} + \nabla^i\phi + 2n^j\left(C^{i\prime}{}_j + \frac{1}{2}(\nabla_j B^i - \nabla^i B_j)\right) + n^j n^k\left(\nabla_k C^i{}_j + \nabla_j C^i{}_k - \nabla^i C_{jk}\right) = 0 , \tag{2.65}$$

which we can rewrite after integrating both sides, as

$$\begin{aligned}\delta k^i_o - \delta k^i_s \;=\; &\int_s^o \Bigg[-B^{i\prime} - \nabla^i\phi - 2n^j\left(C^{i\prime}{}_j + \frac{1}{2}(\nabla_j B^i - \nabla^i B_j)\right) \\ &-n^j n^k\left(\nabla_k C^i{}_j + \nabla_j C^i{}_k - \nabla^i C_{jk}\right)\Bigg] d\lambda .\end{aligned} \tag{2.66}$$

k^i is a vector along the path between the observer o and the source s

$$k^i_o - k^i_s = \overline{k}^i_o + \delta k^i_o - \overline{k}^i_s - \delta k^i_s . \tag{2.67}$$

In the Minkowski background, this yields

$$\begin{aligned}k^i_o - k^i_s \;=\; &\int_s^o \Bigg[-B^{i\prime} - \nabla^i\phi - 2n^j\left(C^{i\prime}{}_j + \frac{1}{2}(\nabla_j B^i - \nabla^i B_j)\right) \\ &-n^j n^k\left(\nabla_k C^i{}_j + \nabla_j C^i{}_k - \nabla^i C_{jk}\right)\Bigg] d\lambda ,\end{aligned} \tag{2.68}$$

and since we are dealing with a conformal space the background terms are constant.

Now applying Eq. (2.55) and Eq. (2.56) to our null vector results Eqs. (2.63) and (2.68) to move back from conformal space to FLRW space, then for the 0-components will get

$$k^0_s = \frac{a_o^2}{a_s^2}k^0_o - \frac{1}{a_s^2}\int_s^o a^{-2}\left[\phi' - 2a^2\frac{d}{d\lambda}\phi - n^i n^j \left(C'_{ij} - \nabla_i B_j\right)\right] d\lambda . \tag{2.69}$$

In particular the observer sends a light ray of a past-pointed null cone vectors. Then this implies

$$-g_{ab}k^a u^b < 0\,. \tag{2.70}$$

We can normalize the null vector $k^0_o = -1$ at the observer position

$$k^0_s = -\frac{1}{a_s^2} - \frac{1}{a_s^2}\int_s^o a^{-2}\left[\phi' - 2a^2\frac{d}{d\lambda}\phi - n^i n^j\left(C'_{ij} - \nabla_i B_j\right)\right]d\lambda\,. \tag{2.71}$$

Applying the same conformal transformation to the i-components, we will get

$$\begin{aligned} k^i_s \;&=\; \frac{a_o^2}{a_s^2}k^i_o - \frac{1}{a_s^2}\int_s^o a^{-2}\left[-B^{i\prime} - \nabla^i\phi - 2n^j\left(C^{i\prime}{}_j + \frac{1}{2}(\nabla_j B^i - \nabla^i B_j)\right)\right. \\ &\left. -n^j n^k\left(\nabla_k C^i{}_j + \nabla_j C^i{}_k - \nabla^i C_{jk}\right)\right]d\lambda\,, \end{aligned} \tag{2.72}$$

where $k^i_o = n^i_o$ at the observer, then we get slightly simpler expression in the FLRW space

$$\begin{aligned} k^i_s \;&=\; \frac{n^i_o}{a_s^2} - \frac{1}{a_s^2}\int_s^o a^{-2}\left[-B^{i\prime} - \nabla^i\phi - 2n^j\left(C^{i\prime}{}_j + \frac{1}{2}(\nabla_j B^i - \nabla^i B_j)\right)\right. \\ &\left. -n^j n^k\left(\nabla_k C^i{}_j + \nabla_j C^i{}_k - \nabla^i C_{jk}\right)\right]d\lambda\,. \end{aligned} \tag{2.73}$$

By lowering the indices, we obtain

$$k^s_\eta \;=\; 1 + \int_s^o a^{-2}\phi' d\lambda - \int_s^o a^{-2}[n^i n^j\left(C'_{ij} - \nabla_{(j}B_{i)}\right)]d\lambda - B_i n^i\,, \tag{2.74}$$

and

$$\begin{aligned} k^s_i \;&=\; n^o_i + \int_s^o a^{-2}\nabla_i\phi d\lambda + \int_s^o a^{-2}B'_i d\lambda + 2\int_s^o a^{-2}\left[n^j\left(C'_{ij} + \frac{1}{2}(\nabla_j B_i - \nabla_i B_j)\right)\right]d\lambda \\ &+ \int_s^o a^{-2}\left[n^l n^j\left(2\nabla_j C_{il} - \nabla_i C_{lj}\right)\right]d\lambda - B_i|^o_s + 2n^j C_{ij}|^o_s\,. \end{aligned} \tag{2.75}$$

2.2.3 Effect on Frequency

Now we can calculate the effect of perturbations on the observed frequency of the light signal by using Eq. (1.59) to first-order perturbation along with the 4-velocity we define in Sec. 2.1.6 and Eqs. (2.71), (2.73) we get

$$\begin{aligned} 1+z \;&=\; \frac{\overline{\omega}_s + \delta\omega_s}{\overline{\omega}_o + \delta\omega_o} = \frac{\left[(\overline{g}_{\mu\nu} + \delta g_{\mu\nu})(\overline{k}^\mu + \delta k^\mu)(\overline{u}^\nu + \delta u^\nu)\right]_s}{\left[(\overline{g}_{\gamma\sigma} + \delta g_{\gamma\sigma})(\overline{k}^\gamma + \delta k^\gamma)(\overline{u}^\sigma + \delta u^\sigma)\right]_o}\,, \\ 1+z \;&=\; \frac{1}{a(\eta_s)}\left[[1 - \phi + n^i v_i - n^i\nabla_i B - n^i\overline{B}_i]^o_s + \int_s^o(\phi+\psi)' d\eta\right. \\ &\left. - \int_s^o n^i n^j\left(\nabla_i\nabla_j E' + \nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \frac{1}{2}(\nabla_i\nabla_j B + \nabla_j\overline{B}_i + \nabla_i\nabla_j B + \nabla_i\overline{B}_j)\right)d\eta\right]\,, \end{aligned} \tag{2.76}$$

where we have used $a^{-2}d\lambda = d\eta$. Substituting for the *Bardeen potentials* quantities using Eqs. (2.29, 2.30) and further simplify the term $\nabla_i\nabla_j(B - E')$ as follows:

$$
\begin{aligned}
n^i n^j \nabla_i \nabla_j (B - E') &= n^i \nabla_i \left(\frac{d}{d\eta} - \frac{\partial}{\partial \eta} \right) (B - E') \, , && (2.77) \\
&= -\left(\frac{d}{d\eta} - \frac{\partial}{\partial \eta} \right) (B - E')' + n^i \nabla_i \frac{d}{d\eta} (B - E') \, , && (2.78) \\
&= (B - E')'' - \frac{d}{d\eta} (B - E')' + n^i \nabla_i \frac{d}{d\eta} (B - E') \, . && (2.79)
\end{aligned}
$$

Thus, integrating both sides leads to

$$
\int_s^o n^i n^j \nabla_i \nabla_j (B - E') d\eta = \int_s^o (B - E')'' d\eta - (B - E')'|_s^o + n^i \nabla_i (B - E')|_s^o \, . \quad (2.80)
$$

Moreover, let us decompose v_i using Eq. (2.39) into scalar and tensor parts:

$$
\begin{aligned}
v_i &= [\nabla_i v + \overline{v}_i] \, , && (2.81) \\
&= [\nabla_i (V + E') + (\overline{V}_i + \overline{B}_i)] \, , && (2.82) \\
&= V_i + [\nabla_i E' + \overline{B}_i] \, , && (2.83)
\end{aligned}
$$

where V_i is the gauge-invariant velocity perturbation of the baryon fluid, and now we can obtain

$$
1 + z = \frac{1}{a(\eta)_s} \left[[1 - \Phi + \mathcal{H}(B - E') + V_i n^i]_s^o + \int_s^o (\Phi + \Psi)' d\eta - \int_s^o n^i n^j \left(\nabla_i F_j' + \frac{1}{2} h_{ij}' - \nabla_i \overline{B}_j \right) d\eta \right] . \quad (2.84)
$$

Using (2.28) in the above equation results

$$
1 + z = \frac{1}{a(\eta)_s} \left[[1 - \Phi + \mathcal{H}(B - E') + V_i n^i]_s^o + \int_s^o (\Phi + \Psi)' d\eta - \int_s^o n^i n^j \left(\nabla_i \overline{\Phi}_j + \frac{1}{2} h_{ij}' \right) d\eta \right] . \quad (2.85)
$$

This redshift relation is in a general gauge presentation, which we can reduce to any sort of gauge we want. The redshifts calculated are gauge-invariant if and only if they relate to the observed redshift of emission points on an invariantly specified surface in our past lightcone $C^-(q)$. This equation is known as the *Sachs-Wolfe* (SW) *equation* as well and it relates photons' present potential and energy to their potential and energy at the emission. The interpretation given the perturbations considered is based on a splitting of perturbations into scalar, vector, and tensor parts [10]. The *ordinary Sachs-Wolfe term*,

$$
\Theta_{SW} \equiv [-\Phi + \mathcal{H}(B - E')]_s^o \quad (2.86)
$$

is the scalar only contribution whereas the *Doppler term*

$$
\Theta_{dop} \equiv [V_i n^i - \overline{\Phi}_i n^i]_s^o \quad (2.87)
$$

is the scalar-and- vector contribution and indicates that the emitter source and the observer do not have the same velocity. Finally, the term

$$
\Theta_{ISW} \equiv \int_s^o \left((\Phi + \Psi)' - \overline{\Phi'}_i n^i + \frac{1}{2} h_{ij}' n^i n^j \right) d\eta \quad (2.88)
$$

contains all three kinds of perturbations and is called the *integrated Sachs-Wolfe term*. The physical meaning of this splitting is non-local, and its implications depend on the details of the gauge choice made. All terms in the above equation are gauge invariant. One may think the term $\mathcal{H}(B - E')$ is not gauge invariant under gauge transformation, but can proven to be so using the transformations;

$$\eta \longrightarrow \eta - T \ , \tag{2.89}$$
$$x^i \longrightarrow x^i - L^i \ , \tag{2.90}$$

and thus

$$B - E' \longrightarrow B - E' - T, \tag{2.91}$$
$$a(\eta) \longrightarrow a(\eta)[1 - \mathcal{H}T]. \tag{2.92}$$

Therefore, $1 + z$ is gauge invariant.

2.3 The Perturbative Cosmological Distances

We are going to calculate the cosmological distances in a perturbed FLRW space. The perturbed distance is a gauge-invariant quantity. In any arbitrary geometry of spacetime it is easier to measure the distance in a gauge of our choice.

A distance measure from a moving source with 4-velocity u_s and an observer moving with 4-velocity u_o, can be obtained as a solution of the Sachs focusing equation [118, 119], which rules the relative acceleration between neighbouring geodesic motions, it represents the deviation of the geodesic equation

$$\frac{d^2 D}{d\lambda^2} = -\left(\mathcal{R} + |\Sigma|^2\right) D \ , \tag{2.93}$$

Σ is the complex shear of the light ray, this equation surely provides the best geometrical interpretation of curvature, where

$$\mathcal{R} = \frac{1}{2} R_{\mu\nu} k^\mu k^\nu, \quad |\Sigma|^2 = \Sigma_{\mu\nu}\Sigma^{\mu\nu} \ . \tag{2.94}$$

$\Sigma_{\mu\nu}$ describes the shear

$$\Sigma_{\mu\nu} = \Sigma_{\langle\mu\nu\rangle} \equiv N_{(\mu}^{\ \alpha} N_{\nu)}^{\ \sigma} \nabla_\alpha k_\sigma - \frac{1}{2}\theta N_{\mu\nu} \ , \tag{2.95}$$

where $N_{\mu\nu}$ is the screen space tensor given by Eq. (4.1), θ describes the expansion of a bundle of light rays [120]. The shear related to the photon ray vector by

$$N_\mu^\alpha N_\nu^\beta \nabla_\alpha k_\beta = \frac{1}{2}\theta N_{\mu\nu} + \Sigma_{\mu\nu} \ . \tag{2.96}$$

In perturbed FLRW metric to first order, the Sachs focusing equation reduces to [119]

$$\frac{d^2 D}{d\lambda^2} = -\mathcal{R} D \tag{2.97}$$

since the complex scalar shear vanishes for a conformally flat spacetime and Σ^2 contributes only at second order [119]. We consider a light bundle with vertex at the source; this leads to the following initial conditions:

$$D(\lambda_s) = 0, \quad \frac{dD(\lambda_s)}{d\lambda} = \omega_s = -(g_{\mu\nu} k^\mu u^\nu)_s \ . \tag{2.98}$$

These are general initial conditions for an arbitrary affine parameter λ for distance measure. We can normalise λ such that $\omega_s = (1 + z_s)$ then one could obtain the redshift luminosity distance, while $\omega_s = (1 + z_s)^{-1}$ gives us the redshift angular diameter distance [119].

2.3.1 Area Distance

One considers a past-oriented lightlike geodesic, so we can normalise $\lambda = \lambda_s$ such that $\omega_s = 1$. That will ease the calculations for us for now. We will get the area distance in terms of the direction $\mathbf{n}$

and the time of emission η_s. If we would like to obtain the real observed area distance simply we need to measure the source redshift in the same direction $\mathbf{n}$.
The area distance will be given by

$$r_A(\lambda_o) \quad = \quad (\lambda_o - \lambda_s) - \int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} \mathcal{R}(\lambda - \lambda_s) d\lambda' d\lambda \ . \tag{2.99}$$

Note that as we showed in Sec. 2.2.2.1, a perturbed FLRW universe is conformally related to a perturbed Minkowski spacetime by the scale factor a. The null geodesics are invariant under conformal transformations, therefore the distance D will not be affected by the conformal transformation that we are going to apply. Then the effect of the expansion on the distance simply leads to a rescaling [121]. The two conformally related area distances will be

$$r_A = a_s \ \tilde{r}_A \ . \tag{2.100}$$

We will drop the *tilde* and bear in mind that the following calculations have been done in conformal space. From Eq. (2.99) we can calculate the area distance using the right initial conditions of the Sachs focusing equation for an arbitrary affine parameter λ. At the source s it is useful to express the affine parameter in terms of the conformal time

$$\int_s^o k^\eta d\lambda \quad = \quad \int_s^o d\eta \ , \tag{2.101}$$

and from Eq. (2.63) we get

$$\begin{aligned}\lambda_o - \lambda_s \quad = \quad & (\eta_o - \eta_s) + \int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} \Big[-\frac{d}{d\lambda}(\Phi + \Psi) + n^i \nabla_i (\Phi - \Psi) \\ & -2n^i \nabla_i \Phi - n^i n^j \left[\nabla_i \nabla_j E' + \nabla_{(i} F'_{j)} + \frac{1}{2} h'_{ij} - \nabla_{(i} B_{j)} \right] \Big] d\lambda' d\lambda \ . \end{aligned} \tag{2.102}$$

This leads to

$$\begin{aligned}\lambda_o - \lambda_s \quad = \quad & (\eta_o - \eta_s) \Big[1 - (\Phi + \Psi) + (B - E)' \Big] \\ & + \int_{\eta_s}^{\eta_o} (\eta_o - \eta) \Big[-n^i \nabla_i (\Phi + \Psi) - n^i \nabla_i (B - E')' \\ & + n^i n^j \nabla_i \nabla_j (B - E') - n^i n^j \left(\nabla_{(i} F'_{j)} + \frac{1}{2} h'_{ij} + \nabla_{(i} \overline{B}_{j)} \right) \Big] d\eta \ , \end{aligned} \tag{2.103}$$

where we have used the identity B.60. We need to calculate

$$\mathcal{R} = \frac{1}{2}(R_{00} + 2R_{0i} n^i + R_{ij} n^i n^j) \ . \tag{2.104}$$

Using a Maple code, we have calculated the Ricci tensor components, and therefore

$$\begin{aligned} - \int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} (\lambda - \lambda_s) \mathcal{R} d\lambda' d\lambda \quad = \quad & -\frac{1}{2} \int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} (\lambda - \lambda_s) \Big[-2 \frac{d^2}{d\lambda^2} [\Phi - \mathcal{H}(B - E') - (B - E)'] \\ & + (\nabla^2 - n^i n^j \nabla_i \nabla_j)(\Phi + \Psi) + (\nabla^2 - n^i n^j \nabla_i \nabla_j)(B - E')' \\ & - \nabla^2 [n^i (\overline{B}_i - F'_i)] - n^i n^j [\nabla_{(i} \overline{B}'_{j)} - \nabla_{(i} F''_{j)}] - n^i n^j \nabla^2 h_{ij} \\ & - 2n^i n^j \nabla_i \nabla_j (B - E') \Big] d\lambda' d\lambda \ , \end{aligned} \tag{2.105}$$

where $\nabla^2 = \nabla^i \nabla_i$. By joining Eqs. (2.103) and (2.105), we can give the area distance in FLRW space by

$$\begin{aligned}
r_A(\mathbf{n},\eta_s) &= a(\eta_s)(\eta_o-\eta_s)\Bigg[1-\Psi_s-\mathcal{H}(B-E') \\
&+\frac{1}{2}\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)(\eta_o-\eta)\Big((\nabla^2-n^in^j\nabla_i\nabla_j-\frac{2}{\eta_o-\eta}n^i\nabla_i)(\Phi+\Psi) \\
&-n^i\nabla^2(\overline{B}_i-F_i')-n^in^j(\nabla_{(i}\overline{B}_{j)}'-\nabla_{(i}F_{j)}'')-n^in^j\nabla^2h_{ij}-\frac{2}{\eta_o-\eta}(\nabla_{(i}F_{j)}' \\
&+\frac{1}{2}h_{ij}'+\nabla_{(i}\overline{B}_{j)})n^in^j\Big)\,d\eta\Bigg]\,, \qquad (2.106)
\end{aligned}$$

where we have used the conformal transformation as mentioned in Eq. (2.56).

2.3.2 Luminosity Distance

For the perturbed luminosity distance we need

$$\begin{aligned}
(1+z)^2 &= \frac{1}{a^2(\eta_s)}\Bigg[1-2\Phi_o+2\Phi_s+2\mathcal{H}(B-E')+2V_in^i \\
&+2\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta-2n^in^j\int_{\eta_s}^{\eta_o}\Big(\nabla_iF_j'+\frac{1}{2}h_{ij}'-\nabla_i\overline{B}_j\Big)\,d\eta\Bigg]\,. \qquad (2.107)
\end{aligned}$$

Using Eq. (2.106), the luminosity distance is given by

$$\begin{aligned}
d_L(\mathbf{n},\eta_s) &= a(\eta_s)^{-1}(\eta_o-\eta_s)\Bigg[1-2\Phi_o+2\Phi_s-\Psi_s+\mathcal{H}(B-E')+2V_in^i+2\int_s^o(\Phi+\Psi)'d\eta \\
&-2n^in^j\int_s^o\Big(\nabla_iF_j'+\frac{1}{2}h_{ij}'-\nabla_i\overline{B}_j\Big)\,d\eta \\
&+\frac{1}{2}\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)(\eta_o-\eta)\Big([\nabla^2-n^in^j\nabla_i\nabla_j-\frac{2}{\eta_o-\eta}n^i\nabla_i](\Phi+\Psi) \\
&-\nabla^2n^i(\overline{B}_i-F_i')-n^in^j[\nabla_{(i}\overline{B}_{j)}'-\nabla_{(i}F_{j)}'']-n^in^j\nabla^2h_{ij}-\frac{2}{\eta_o-\eta}[\nabla_{(i}F_{j)}'+\frac{1}{2}h_{ij}' \\
&+\nabla_{(i}\overline{B}_{j)}]n^in^j\Big)d\eta\Bigg]\,. \qquad (2.108)
\end{aligned}$$

We can re-write the term $\int(\Phi+\Psi)'d\eta$ as follows, with the use of Eq. (2.59) and Eq. (B.55):

$$\begin{aligned}
\int(\Phi+\Psi)'d\eta &= \int_{\eta_s}^{\eta_o}\Phi'd\eta+\int_{\eta_s}^{\eta_o}\Psi'd\eta\,, \qquad (2.109) \\
&= \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)\Phi'd\eta+\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}\Phi'd\eta'd\eta+\int_{\eta_s}^{\eta_o}\Psi'd\eta\,, \qquad (2.110)
\end{aligned}$$

or

$$\begin{aligned}
\int(\Phi+\Psi)'d\eta &= \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)\frac{d}{d\eta}\Phi d\eta-\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta \\
&+\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}\frac{d}{d\eta'}\Phi d\eta'd\eta-\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Phi d\eta'd\eta \\
&+\int_{\eta_s}^{\eta_o}\frac{d}{d\eta}\Psi d\eta-\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta\,, \qquad (2.111)
\end{aligned}$$

$$\begin{aligned}
\int(\Phi+\Psi)'d\eta &= \frac{1}{\eta_o-\eta_s}[(\eta-\eta_s)\Phi]_{\eta_s}^{\eta_o} - \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta + \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\Phi d\eta \\
&\quad -\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta} n^i\nabla_i\Phi d\eta' d\eta + \Psi - \int_{\eta_s}^{\eta_o} n^i\nabla_i\Psi d\eta\ , \qquad (2.112)\\
&= (\Phi_o+\Psi) - \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta + \frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\Phi d\eta \\
&\quad -\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta} n^i\nabla_i\Phi d\eta' d\eta - \int_{\eta_s}^{\eta_o} n^i\nabla_i\Psi d\eta\ . \qquad (2.113)
\end{aligned}$$

Substituting the above decomposed term back into the $d_L(\mathbf{n},\eta_s)$, we will get

$$\begin{aligned}
d_L(\mathbf{n},\eta_s) &= \frac{(\eta_o-\eta_s)}{a_s(\eta)}\Bigg[1+2\Psi_o+2\Phi_s-3\Psi_s+\mathcal{H}(B-E')+2V_in^i+\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\Phi d\eta \\
&\quad -\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}\frac{(\eta'-\eta_s)}{(\eta_s-\eta)}n^i\nabla_i(\Phi+\Psi)d\eta'd\eta - \frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta \\
&\quad -\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Phi d\eta'd\eta - \int_{\eta_s}^{\eta_o}n^in^j\left(\nabla_{(i}F'_{j)}+\frac{1}{2}h'_{ij}-\nabla_{(i}\overline{B}_{j)}\right)d\eta \\
&\quad -2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta + \frac{1}{2}\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Big([\nabla^2-n^in^j\nabla_i\nabla_j](\Phi+\Psi) \\
&\quad -\nabla^2\overline{B}_in^i-\nabla_{(i}\overline{B}'_{j)}n^in^j+\frac{2}{(\eta_s-\eta)}\nabla_{(i}\overline{B}_{j)}n^in^j+\nabla^2n_iF^{i'}+\nabla_{(i}F''_{j)}n^in^j \\
&\quad -\frac{2}{(\eta_s-\eta)}\nabla_{(i}F'_{j)}n^in^j+\frac{2}{(\eta_s-\eta)}n^in^jh'_{ij}-\nabla^2h_{ij}n^in^j\Big)d\eta'd\eta\Bigg]\ .
\end{aligned} \qquad (2.114)$$

Converting the double integrals into a single integral, this yields

$$\begin{aligned}
d_L(\mathbf{n},\eta_s) &= \frac{(\eta_o-\eta_s)}{a_s(\eta)}\Bigg[1+2\Psi_o+2\Phi_s-3\Psi_s+\mathcal{H}(B-E')+2V_in^i-\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta \\
&\quad +\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i(\Phi+\Psi)d\eta - 2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta - \frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Phi d\eta'd\eta \\
&\quad +\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\Phi d\eta - \int_{\eta_s}^{\eta_o}n^in^j\left(\nabla_{(i}F'_{j)}+\frac{1}{2}h'_{ij}-\nabla_{(i}\overline{B}_{j)}\right)d\eta \\
&\quad +\frac{1}{2}\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Big([\nabla^2-n^in^j\nabla_i\nabla_j](\Phi+\Psi)-\nabla^2\overline{B}_in^i-\nabla_{(i}\overline{B}'_{j)}n^in^j \\
&\quad -\frac{2}{(\eta-\eta_s)}\nabla_{(i}\overline{B}_{j)}n^in^j+\nabla^2n_iF^{i'}+\nabla_{(i}F''_{j)}n^in^j+\frac{2}{(\eta-\eta_s)}\nabla_{(i}F'_{j)}n^in^j-\frac{2}{(\eta-\eta_s)}n^in^jh'_{ij} \\
&\quad -\nabla^2h_{ij}n^in^j\Big)d\eta\Bigg]\ .
\end{aligned} \qquad (2.115)$$

2.3.2.1 The Observed Luminosity Distance

The conformal time and the background redshift are not observable quantities. A luminosity distance as a function of these is not a directly measurable quantity; we need to relate it to a gauge-dependent quantity. What we do measure (observe) instead is the redshift of the source

$$z_s = \overline{z}_s + \delta z_s\ , \qquad (2.116)$$

where

$$\begin{aligned}\delta z_s &= (1+\overline{z}_s)\Big[\Psi_o - \Psi_s + \mathcal{H}(B-E') + V_i n^i - \int_{\eta_s}^{\eta_o} n^i \nabla_i(\Phi+\Psi) d\eta \\ &\quad - \int_{\eta_s}^{\eta_o} n^i n^j \left(\nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \nabla_{(i}\overline{B}_{j)}\right) d\eta\Big] . \end{aligned} \tag{2.117}$$

Writing

$$d_L(\mathbf{n},\eta_s) = d_L(\mathbf{n},\eta(\overline{z}_s)) \equiv d_L(\mathbf{n},\overline{z}_s) \ , \tag{2.118}$$

by taking the Taylor expansion of $d_L(\mathbf{n},\overline{z}_s)$ around z_s we get [119, 121]

$$d_L(\mathbf{n},\overline{z}_s) = d_L(\mathbf{n},z_s) - \frac{d}{dz_s} d_L(\mathbf{n},z_s)|_{z=\overline{z}} \quad \delta z_s \ , \tag{2.119}$$

with

$$\frac{d}{dz_s} d_L(\mathbf{n},z_s)|_{z=\overline{z}} = \frac{d}{d(\overline{z}_s+\delta z_s)} d_L(\mathbf{n},(\overline{z}_s+\delta z_s))|_{z=\overline{z}} \ , \tag{2.120}$$

$$= \frac{d}{d\overline{z}_s} d_L(\mathbf{n},\overline{z}_s)\left(1-\left(\frac{\delta z_s}{\overline{z}_s}\right)^2\right)|_{z=\overline{z}} \ , \tag{2.121}$$

$$= \frac{d}{d\overline{z}_s} d_L(\mathbf{n},\overline{z}_s) + O(1) \ , \tag{2.122}$$

$$= (1+z_s)^{-1} d_L + \mathcal{H}_s^{-1} + O(1) \ , \tag{2.123}$$

where we have used the fact that $\overline{z}_s + 1 = 1/a(\eta_s)$. This leads to

$$\frac{d}{dz_s} d_L(\mathbf{n},z_s)|_{z=\overline{z}} = -\left((1+z_s)^{-1} d_L + \mathcal{H}_s^{-1}\right)\delta z_s \ , \tag{2.124}$$

$$\begin{aligned} &= (1+z_s)\Big[-(\eta_o - \eta_s + \mathcal{H}_s^{-1})\Big]\Big[\Psi_o - \Psi_s + \mathcal{H}(B-E') + V_i n^i \\ &\quad - \int_{\eta_s}^{\eta_o} n^i \nabla_i(\Phi+\Psi) d\eta - \int_{\eta_s}^{\eta_o} n^i n^j \left(\nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \nabla_{(i}\overline{B}_{j)}\right) d\eta\Big] . \end{aligned} \tag{2.125}$$

Then the redshift luminosity distance will be given as

$$\begin{aligned} d_L(\mathbf{n},z_s) &= (1+z_s)\Big[(\eta_o-\eta_s) + [(\eta_o-\eta_s) - \mathcal{H}_s^{-1}]\Psi_o - [2(\eta_o-\eta_s) - \mathcal{H}_s^{-1}]\Psi_s \\ &\quad + 2(\eta_o-\eta_s)\Phi_s - \mathcal{H}_s^{-1}\mathcal{H}(B-E') + [(\eta_o-\eta_s) - \mathcal{H}_s^{-1}]V_i n^i + 2\int_{\eta_s}^{\eta_o} \Phi d\eta \\ &\quad + \int_{\eta_s}^{\eta_o} (\eta_s - \eta) n^i \nabla_i(-3\Phi+\Psi) d\eta + [(\eta_o-\eta_s) - \mathcal{H}_s^{-1}] \int_{\eta_s}^{\eta_o} n^i \nabla_i(-\Psi+\Phi) d\eta \\ &\quad + n^i n^j \mathcal{H}_s^{-1} \int_{\eta_s}^{\eta_o} \left(\nabla_i F'_j + \frac{1}{2}h'_{ij} - \nabla_i \overline{B}_j\right) d\eta \\ &\quad + \frac{1}{2}\int_{\eta_s}^{\eta_o} (\eta-\eta_s)(\eta_o-\eta)\Big([\nabla^2 - n^i n^j \nabla_i \nabla_j](\Phi+\Psi) - \nabla^2 n^i(\overline{B}_i - F'_i) \\ &\quad - n^i n^j[\nabla_{(i}\,\overline{B}'_{j)} - \nabla_{(i}F''_{j)}] - n^i n^j \nabla^2 h_{ij} - \frac{2}{\eta_o - \eta}[\nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} + \nabla_{(i}\overline{B}_{j)}]n^i n^j\Big) d\eta\Big] . \end{aligned} \tag{2.126}$$

This equation is the final expression we have got for the luminosity distance [122,123] in a perturbed Friedmann universe in general gauge, as a function of the measured source redshift z_s and its direction $\mathbf{n}$. This equation is complicated and contains the angular and redshift fluctuations of the luminosity distance, which may also contain important information about our universe. Nevertheless the matter distribution and the geometry have fluctuations too, but to first order in perturbation theory these fluctuations can average out in the mean and are therefore expected to be small.

Part II

Chapter 3

Ideal Observational Cosmology

Time is the true driving force of the Universe.

Khalid Masood

Cosmological observable quantities, henceforth simply referred to as *observables*, encode information about the state of the Universe at a particular cosmological redshift. In our past lightcone we can obtain these observables which can give us our connection to the rest of the Universe. Hence a precise measure of cosmological observables can directly determine the geometry of the observable part of the spacetime, in the so-called observational approach [124]. Furthermore we can assume a dynamical theory for the spacetime curvature of the past lightcone, *i.e*, GR.

The observations are taken so that we can discover what these observations imply about the large-scale structure of the Universe. The idea was first discussed in [103], and Refs. [108, 125–127] discussed the construction of the spacetime metric and ways to determine local matter density in the Universe directly from astronomical observations on our past lightcone as initial data for the field equations, and later to establish what is called now the lightcone gauge based on an observational coordinates set. The main aim was to a great extent that cosmology rather be a directly observational subject [108]. Therefore, they bring cosmologically interpretable astronomical observations into a confrontation with the cosmological theories, to reveal the structure of distant regions in the Universe.

3.1 Observational Coordinates

A lightcone gauge has been constructed and adapted to observations made on the null cone using observational coordinates as following. A spacetime consists of a manifold $\mathcal{M}$ with metric g. We shall assume the spacetime filled with a perfect fluid of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) \ , \tag{3.1}$$

where $T_{\mu\nu}$ is the stress energy tensor. The first step in constructing a set of observational coordinates is to identify fundamental observers. The integral curves of the velocity vector u^μ and their normalised 4-velocity, represent the worldlines of these fundamental observers, *i.e*, they are comoving with the galaxies. If τ is the proper time along these worldlines, then

$$u^\mu = \frac{dx^\mu}{d\tau} \ , \quad u^\mu u_\mu = -1 \ . \tag{3.2}$$

Let us now single out our worldline C, where C is a set of timelike geodesics generated by u^μ at the event attached to us, on Earth.

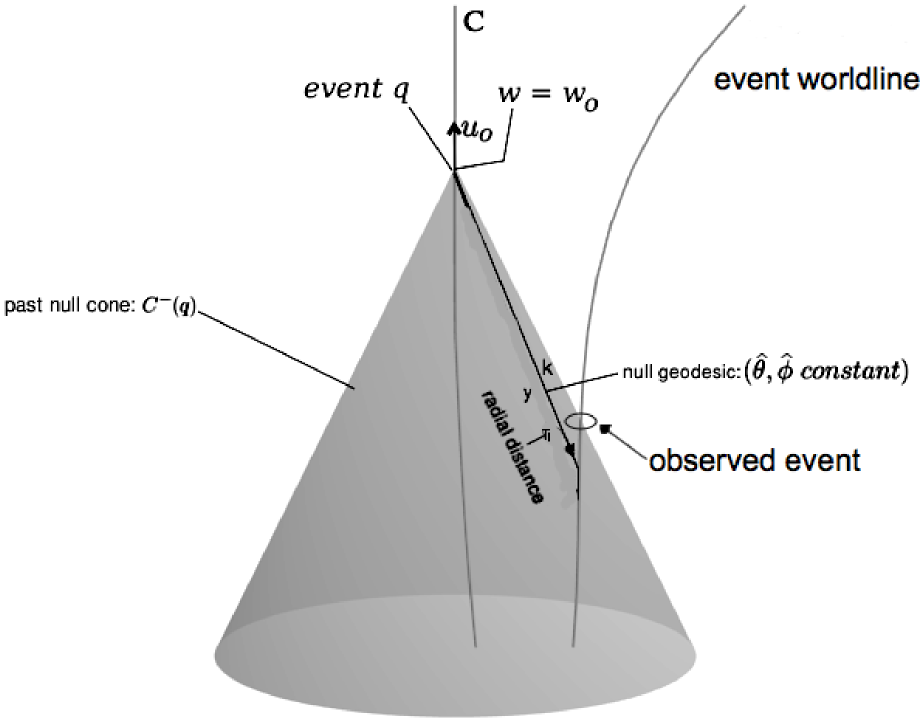


Figure 3.1: Observational coordinates $\{w, y, \hat{\theta}, \hat{\phi}\}$ based on the event q on the worldline C. w is the time of observation; $\hat{\theta}, \hat{\phi}$ represent the direction of observation; and y is a measure of distance to the object observed.

We will introduce the set of observational coordinates $x^\mu =: \{w, y, \hat{\theta}, \hat{\phi}\}$, constructed as follows: the coordinate w is the past lightcones of the events on C, generated along our worldline. It can be normalized by measuring the proper time along the central worldline C (in other words, $w|_C = \tau|_C$). By choosing $w = w_0$ to correspond to the event q *here and now*, the null cone generated then will represents the surfaces of events that happened on our past lightcone at constant w_0. Then generically q will be at the vertices of the lightcones where we receive information and signals from the Universe. Then w is completely determined when w_0 has been chosen. The null geodesic vector field k^μ and ν is the affine parameter along this null geodesic, generating the geodesics of these lightcones, will be written as

$$k^\mu = dx^\mu/d\nu \ , \tag{3.3}$$

where

$$k_\mu \equiv w_{,\mu} \Rightarrow k^\mu k_\mu = 0 \ . \tag{3.4}$$

This definition necessarily implies that k is hypersurface-orthogonal [128],

$$k_{\mu;\nu} = k_{\nu;\mu} \ . \tag{3.5}$$

Null geodesic vector fields are orthogonal to the null surfaces and generate the past-directed null geodesics along the past lightcone, on which w is constant:

$$k^\mu{}_{;\nu} k^\nu = 0 \Rightarrow w_{,\mu} k^\mu = 0 \ . \tag{3.6}$$

Once the null geodesic vector condition is satisfied at the central worldline, this implies

$$k_\mu u^\mu = w_{,\mu} u^\mu \Leftrightarrow k_\mu u^\mu|_c = 1 \ , \tag{3.7}$$

and this shows that the affine parameter ν is uniquely defined geometrically on the null geodesics, and this defines the *central condition*. If we specify that $\nu = 0$ on the worldline C, so the event "q" is given by

$$w = w_0 \ , \nu = 0 \ . \tag{3.8}$$

The coordinate y measures distances down the null geodesics, and so represents spatial distance from "q" [108]. There are various choices of y that might be suitable for different purposes, for example [108]:

(1) $y = \nu$, the unique affine parameter down the null geodesics through C determined by the central conditions on C ($\nu|_c = 0, u^\mu k_\mu|_C = 1$). The spacetime metric will be simplified, but one loses the beautiful physical interpretation of observational coordinates;

(2) $y = r_A$, the area distance down the null cones from C;

(3) $y = z$, galactic redshift observed from C, imposing y=cst along matter worldlines;

(4) y chosen as in one of (1)-(3) on the initial null cone $w = w_0$, and then specified thereafter to be comoving with the fluid; $y_{,\mu} u^\mu = 0$.

When one of these specific choices has been made, y is uniquely defined on all the null cones. We will use such a choice of coordinate y as a coordinate comoving with the fluid, and determined by a unique specification on the initial null cone $w = w_0$. From equations (3.3) and (3.4), and because $w_{,\mu} k^\mu = 0$ we will have [108]

$$k_\mu = \delta^0_\mu \ , \quad k^\mu = dx^\mu / d\nu = (1/\beta)\delta^\mu_1 \Rightarrow (1/\beta) = dy/d\nu \quad \text{for} \quad \beta > 0 \ , \tag{3.9}$$

where β is some function that determines the relation of the affine parameter ν to the coordinate y. Eq. (3.9) shows the rate of change of the coordinate y down the null geodesics relative to the affine parameter ν. $\beta = cst$ when y is affine parameter; and $\beta \to 1$ when we choose $y = r_A$ as $y \to 0$ [108]. Different values of y with constant values of ν represent an event at the same distance from q down the null cone in different directions. The coordinates $(\hat{\theta}, \hat{\phi})$ are angles on the "physical" sky. The observer sees the sky as the superposition of 2-spheres embedded in the lightcone, then we can redefine $\hat{\theta}, \hat{\phi}$ as spherical coordinates on the celestial sphere with respect to the (physically non-rotating) reference frame $\mathbf{e}_\mu$[1]. They label the geodesics generating the past lightcone (they are constant along such geodesics) [108]. At a constant surface w, the null-cone geodesics are generated by constant $\hat{\theta}$ and $\hat{\phi}$, so

$$k^\mu \hat{\theta}_{,\mu} = k^\mu \hat{\phi}_{,\mu} = 0 \ . \tag{3.10}$$

They are based on a parallelly propagated orthonormal tetrad $\mathbf{e}_\mu$ [108] along C. Then at a constant w and ν,

$$\lim_{\nu \to 0} \left\{ \frac{ds^2}{\nu^2} \bigg|_{\substack{w=cst \\ \nu=cst}} \right\} = d\Omega^2 = d\hat{\theta}^2 + \sin^2\hat{\theta} d\hat{\phi}^2 \ . \tag{3.11}$$

These coordinates do not necessarily cover all the spacetime, but they do cover that part which is observable from the worldline C.

3.2 The Observational Metric

The metric components can be obtained from the previous discussions. From Eqs. (3.4) and (3.9) we see that

$$k^\mu k_\mu = 0 \Rightarrow w_{,\mu} g^{\mu\nu} w_{,\nu} \implies g^{00} = 0 \ , \tag{3.12}$$

$$k^\mu = g^{\mu\nu} k_\nu \Rightarrow g^{\mu 0} = (1/\beta)\delta^\mu_1 \ , \tag{3.13}$$

[1] These tetrad vectors e_μ are defined through the conditions: $(\mathbf{u} = \mathbf{e_0}, \mathbf{u} \cdot \mathbf{e_i} = \mathbf{0}, \mathbf{e_i e_j} = \delta_{\mathbf{ij}})$, and thus satisfy the parallel propagation along C: $\nabla_u \mathbf{e}_\mu|_\mathbf{c} = \mathbf{0}$, with $\mathbf{u}$ the velocity of the comoving geodesic observer.

and thus

$$g^{\mu\nu}g_{\nu\gamma} = \delta^{\mu}_{\gamma} \Rightarrow g^{0\nu}g_{\nu\gamma} = \delta^{0}_{\gamma} \Rightarrow g_{1\gamma} = \beta\delta^{0}_{\gamma} \ . \tag{3.14}$$

We can get the general expression for $g^{\mu\nu}$ and compute its inverse by introducing new functions for the non-constrained components. We thus have [108],

$$g_{\mu\nu} = \begin{pmatrix} \alpha & \beta & v_2 & v_3 \\ \beta & 0 & 0 & 0 \\ v_2 & 0 & h_{22} & h_{23} \\ v_3 & 0 & h_{23} & h_{33} \end{pmatrix} , \quad g^{\nu\gamma} = \begin{pmatrix} 0 & 1/\beta & 0 & 0 \\ 1/\beta & \delta & \sigma_2 & \sigma_3 \\ 0 & \sigma_2 & h_{33}/h & -h_{23}/h \\ 0 & \sigma_3 & -h_{23}/h & h_{22}/h \end{pmatrix} , \tag{3.15}$$

where

$$h = \det(h_{IJ}) = h_{22}h_{33} - (h_{23})^2 \ , \tag{3.16}$$
$$\delta = -(\alpha + \beta(v_2\sigma_2 + v_3\sigma_3))/\beta^2 \ . \tag{3.17}$$

Here we have defined

$$\sigma_2 = -(v_2h_{33} - v_3h_{23})/\beta h \ , \tag{3.18}$$
$$\sigma_3 = -\left(v_3h_{22} - v_2h_{23}\right)/\beta h \ , \tag{3.19}$$

where $(I, J) \in \{2, 3\}^2$. The angular distance r_A can be defined in terms of the proposed functions above,

$$r_A^4 \sin^2\hat{\theta} = h = \det(h_{IJ}) \ , \quad f_{IJ} = h_{IJ}/r_A^2 \Rightarrow \det(f_{IJ}) = \sin^2\hat{\theta} \ , \tag{3.20}$$

where f_{IJ} give an alternative representation of the quantities h_{IJ}. The metric form above implies that the surface $w|_{cst}$ are null surfaces. But it does not, as it stands, guarantee that these null surfaces are the past lightcones of the geodesic worldline C. To set this feature, one has to impose some limits on the behaviour of the metric tensor components near the worldline C [108]. When the coordinate y is taken to be the affine parameter or the area distance, these essential limits are [108]:

$$\lim_{y\to 0}\alpha = -1, \quad \lim_{y\to 0}\beta = 1, \quad \lim_{y\to 0}(v_I/y^2) = 0, \quad \lim_{y\to 0} h_{IJ}dx^I dx^J/y^2 = d\Omega^2 \ . \tag{3.21}$$

When this coordinate y is taken to be the last case (4), from what we obtained above, and by making a coordinate transformation $y' = y'(w, y, \hat{\theta}, \hat{\phi}), w' = w, \hat{\theta}' = \hat{\theta}, \hat{\phi}' = \hat{\phi}$, as $y \to 0$, one finds the limits are found to be [108]

$$\lim_{y\to 0}\alpha = -1 \ , \quad \lim_{y\to 0}\beta = \beta_0(w, x^I) \ , \quad \lim_{y\to 0} v_I = 0 \ , \quad \lim_{y\to 0} h_{IJ}dx^I dx^J/y^2 = \beta_0^2 d\Omega^2 \ . \tag{3.22}$$

These limits we just introduced guarantee the necessary conditions to make the null hypersurfaces to be the past lightcones of observer of the worldline C. Finally we can say that we have observational coordinates *if and only if* the metric tensor components obey Eqs. (3.15) and (3.21) [108].

Now we could characterise the cosmological quantities on the null cone at $w = w_0$ down to some distance y^*, and with the knowledge of the following: the metric tensor components $g_{\mu\nu} = \{\alpha, \beta, v_I, h_{IJ}\}$, the matter 4-velocity components u^μ, the total number density n of the sources, and the radiation density ρ_{rad} at each point of the null cone. Assuming the equation of state of the matter is known from local astronomical observations, knowledge of n will determine the local rest mass density of matter ρ_m at each point and the isotropic pressure p_m will be determined, and so the matter contribution to the total stress tensor will be known [108].

3.3 The Perturbed Lightcone Gauge

The Perturbed Lightcone Gauge (PLG) is a gauge built on our understanding of the observational coordinates that we introduced in Sec. 3.1. The PLG links the perturbed variables $(\alpha, \beta, v_I, h_{IJ})$

introduced in the lightcone metric, into their equivalent variables in the perturbed FLRW. The PLG uses the same coordinates $(w, y, \hat{\theta}, \hat{\phi})$ [2] as the observational coordinates with the same definitions. The produced expressions we obtained are in a general gauge, and we could reduce them into any gauge we desire.

3.3.1 The PLG Coordinates Transformations

The coordinate transformation will depend on all coordinates. We can think of this coordinate transformation like projecting a lightcone of a fundamental observer expressed with observational coordinates, into a spatial hypersurface presented by 1+3 general coordinates system, see Fig. 3.2. Considering the perturbed FLRW metric with coordinates $x^\mu = (\eta, \chi, \theta, \phi)$ as given in Eq. (2.3)

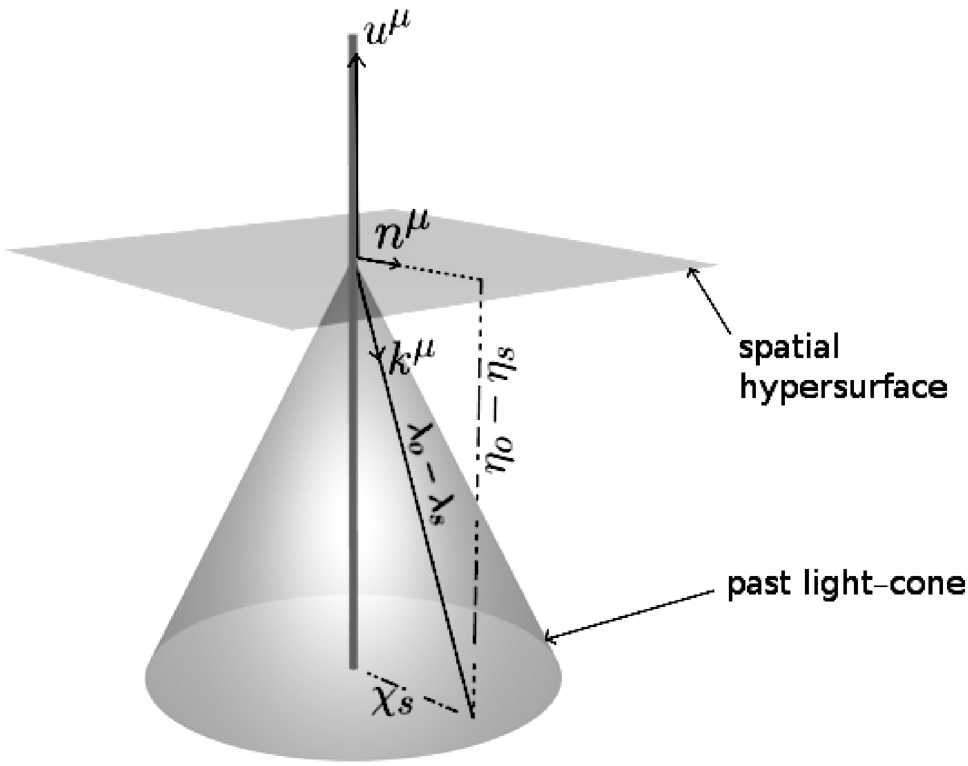


Figure 3.2: Spacetime diagram showing the transformations between the observational coordinates into a spatial hypersurface in general coordinates.

with

$$\gamma_{ij} = \delta_i^1 \delta_j^1 + S^2(\chi) \left(\delta_i^2 \delta_j^2 + \sin^2(\theta) \delta_i^3 \delta_j^3 \right) , \tag{3.23}$$

the '$_{0i}$' and '$_{ij}$' perturbed components are decomposed as given in (2.4) and (2.5). We will define the four-vectors ξ^μ and the tensor $\hat{\delta}^\mu$ associated with the coordinates transformation as

$$x^{\hat{\mu}} = x^\mu + \xi^\mu + \epsilon \hat{\delta}^\mu , \tag{3.24}$$

where ϵ is a small parameter that keeps the equation as it is needed for possible higher-order terms, $\xi^\mu = (\chi, 0, 0, 0)$, and the $\hat{\delta}^\mu$ expresses the coordinate change of the perturbed part of the observational metric. We can therefore define the new coordinates in terms of the FLRW coordinates as follows:

$$x^{\hat{0}} = w = \eta + \chi + \epsilon \hat{\delta} w(\eta, \theta, \phi) , \tag{3.25}$$

$$x^{\hat{1}} = y = \chi + \epsilon \hat{\delta} y(\eta, \chi, \theta, \phi) , \tag{3.26}$$

[2]We will use the overhat symbol ˆ to indicate the use of the observational coordinates.

$$x^{\hat{2}} = \hat{\theta} = \theta + \epsilon\hat{\delta}\hat{\theta}(\eta,\chi,\theta,\phi) \ , \tag{3.27}$$

$$x^{\hat{3}} = \hat{\phi} = \phi + \epsilon\hat{\delta}\hat{\phi}(\eta,\chi,\theta,\phi) \ . \tag{3.28}$$

The associated Jacobian matrix is

$$\frac{\partial x^{\hat{\mu}}}{\partial x^{\nu}} = \begin{pmatrix} 1+\epsilon\frac{\partial\hat{\delta}w}{\partial\eta} & 1+\epsilon\frac{\partial\hat{\delta}w}{\partial\chi} & \epsilon\frac{\partial\hat{\delta}w}{\partial\theta} & \epsilon\frac{\partial\hat{\delta}w}{\partial\phi} \\ \epsilon\frac{\partial\hat{\delta}y}{\partial\eta} & 1+\epsilon\frac{\partial\hat{\delta}y}{\partial\chi} & \epsilon\frac{\partial\hat{\delta}y}{\partial\theta} & \epsilon\frac{\partial\hat{\delta}y}{\partial\phi} \\ \epsilon\frac{\partial\hat{\delta}\theta}{\partial\eta} & \epsilon\frac{\partial\hat{\delta}\theta}{\partial\chi} & 1+\epsilon\frac{\partial\hat{\delta}\theta}{\partial\theta} & \epsilon\frac{\partial\hat{\delta}\theta}{\partial\phi} \\ \epsilon\frac{\partial\hat{\delta}\phi}{\partial\eta} & \epsilon\frac{\partial\hat{\delta}\phi}{\partial\chi} & \epsilon\frac{\partial\hat{\delta}\phi}{\partial\theta} & 1+\epsilon\frac{\partial\hat{\delta}\phi}{\partial\phi} \end{pmatrix} = \delta^{\hat{\mu}}_{\nu} + \delta^{\hat{\mu}}_{0}\delta^{1}_{\nu} + \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\nu}} \ . \tag{3.29}$$

The matrix can be inverted according to the relation

$$(A+\epsilon H)^{-1} = A^{-1} - \epsilon A^{-1}HA^{-1} \ , \tag{3.30}$$

where A is the background metric, and H is the perturbation. Thus we get

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\nu}}} = \delta^{\mu}_{\nu} - \delta^{\mu}_{0}\delta^{1}_{\nu} - \epsilon\partial_{\nu}\hat{\delta}^{\hat{\mu}} + \epsilon\delta^{1}_{\nu}\partial_{w}\hat{\delta}^{\hat{\mu}} + \epsilon\delta^{\mu}_{0}\partial_{\nu}\hat{\delta}y - \epsilon\delta^{\mu}_{0}\delta^{1}_{\nu}\partial_{w}\hat{\delta}y \ . \tag{3.31}$$

In general, the last term in the RHS of (3.29) is given by

$$\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\nu}} = \frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{\gamma}}}\frac{\partial x^{\hat{\gamma}}}{\partial x^{\nu}}, \tag{3.32}$$

$$= \frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{\nu}}} + \frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{w}}\delta^{1}_{\nu} \ . \tag{3.33}$$

Re-writing Eq. (3.31) using Eq. (3.33) we will get

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\nu}}} = \delta^{\mu}_{\nu} - \delta^{\mu}_{0}\delta^{1}_{\nu} - \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{\nu}}} - \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{w}}\delta^{1}_{\nu} + \epsilon\delta^{1}_{\nu}\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{w}} + \epsilon\delta^{\mu}_{0}\frac{\partial\hat{\delta}y}{\partial x^{\hat{\nu}}} + \epsilon\frac{\partial\hat{\delta}y}{\partial x^{w}}\delta^{1}_{\nu}\delta^{\mu}_{0} - \epsilon\delta^{\mu}_{0}\delta^{1}_{\nu}\frac{\partial\hat{\delta}y}{\partial x^{w}} \ , \tag{3.34}$$

or

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\nu}}} = \delta^{\mu}_{\nu} - \delta^{\mu}_{0}\delta^{1}_{\nu} - \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{\nu}}} + \epsilon\delta^{\mu}_{0}\frac{\partial\hat{\delta}y}{\partial x^{\hat{\nu}}} \ . \tag{3.35}$$

And again we can rewrite Eq. (3.29) as

$$\frac{\partial x^{\hat{\mu}}}{\partial x^{\nu}} = \delta^{\hat{\mu}}_{\nu} + \delta^{\hat{\mu}}_{0}\delta^{1}_{\nu} + \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{\nu}}} + \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{w}}\delta^{1}_{\nu} \ . \tag{3.36}$$

Eqs. (3.35) and (3.36) are our key equations for the transformations. Now γ_{ij} will transform as follows:

$$\gamma_{ij} = \frac{\partial x^{\hat{\mu}}}{\partial x^{i}}\frac{\partial x^{\hat{\nu}}}{\partial x^{j}}\hat{\gamma}_{\hat{\mu}\hat{\nu}} \ , \tag{3.37}$$

$$= \left(\delta^{\hat{\mu}}_{i} + \delta^{\hat{\mu}}_{0}\delta^{1}_{i} + \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{i}}\right)\left(\delta^{\hat{\nu}}_{j} + \delta^{\hat{\nu}}_{0}\delta^{1}_{j} + \epsilon\frac{\partial\hat{\delta}^{\hat{\nu}}}{\partial x^{j}}\right)\hat{\gamma}_{\hat{\mu}\hat{\nu}} \ , \tag{3.38}$$

$$= \hat{\gamma}_{\hat{i}\hat{j}} + \epsilon\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{i}}\hat{\gamma}_{\hat{\mu}\hat{j}} + \epsilon\frac{\partial\hat{\delta}^{\hat{\nu}}}{\partial x^{j}}\hat{\gamma}_{\hat{i}\hat{\nu}} \ . \tag{3.39}$$

Applying Eq. (3.33), one can rewrite this as

$$\gamma_{ij} = \hat{\gamma}_{\hat{i}\hat{j}} + \epsilon\left(\frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{\hat{i}}} + \frac{\partial\hat{\delta}^{\hat{\mu}}}{\partial x^{w}}\delta^1_i\right)\hat{\gamma}_{\hat{\mu}\hat{j}} + \epsilon\left(\frac{\partial\hat{\delta}^{\hat{\nu}}}{\partial x^{\hat{j}}} + \frac{\partial\hat{\delta}^{\hat{\nu}}}{\partial x^{w}}\delta^1_j\right)\hat{\gamma}_{\hat{i}\hat{\nu}} \ . \tag{3.40}$$

At the background we can write

$$\xi^{\mu} = x^{\hat{\mu}} - x^{\mu}. \tag{3.41}$$

Hence we have the following:

$$\frac{\partial\xi^{\mu}}{\partial x^{\hat{\nu}}} = \frac{\partial\xi^{\mu}}{\partial x^{\gamma}}\frac{\partial x^{\gamma}}{\partial x^{\hat{\nu}}} \ , \tag{3.42}$$

$$= \delta^{\mu}_0\delta^1_{\gamma}\left(\delta^{\gamma}_{\nu} - \delta^{\gamma}_0\delta^1_{\nu} - \epsilon\frac{\partial\hat{\delta}^{\hat{\gamma}}}{\partial x^{\hat{\nu}}} + \epsilon\delta^{\gamma}_0\frac{\partial\hat{\delta}y}{\partial x^{\hat{\nu}}}\right) , \tag{3.43}$$

$$= \delta^{\mu}_0\delta^1_{\nu} - \delta^{\mu}_0\epsilon\frac{\partial\hat{\delta}y}{\partial x^{\hat{\nu}}} \ . \tag{3.44}$$

And finally, the metric tensor $g_{\mu\nu}$ transforms as

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\gamma}}{\partial x^{\hat{\mu}}}\frac{\partial x^{\sigma}}{\partial x^{\hat{\nu}}}g_{\gamma\sigma} \ . \tag{3.45}$$

These transformations are valid up to first order. Note that $\gamma_{\hat{i}\hat{j}}$ is a zeroth-order term. The new metric elements are then: (we will stop writing ϵ for simplicity)

$$g_{ww} = a^2(w-y)\left[-1 - 2\left(\phi - \partial_w\hat{\delta}w + \partial_w\hat{\delta}y\right)\right] , \tag{3.46}$$

$$g_{w\hat{i}} = a^2(w-y)\left[\delta^1_i + 2\phi\delta^1_i + B_i + \partial_{\hat{i}}\hat{\delta}w - \partial_{\hat{i}}\hat{\delta}y - \delta^1_i\partial_w\hat{\delta}w + \delta^1_i\partial_w\hat{\delta}y - \gamma_{\hat{i}\hat{j}}\partial_w\hat{\delta}^{\hat{j}}\right] , \tag{3.47}$$

$$g_{\hat{i}\hat{j}} = a^2(w-y)\left[\gamma_{\hat{i}\hat{j}} - \delta^1_i\delta^1_j - 2\phi\delta^1_i\delta^1_j - 2\delta^1_{(i}\partial_{\hat{j})}\hat{\delta}w + 2\delta^1_{(j}\partial_{\hat{i})}\hat{\delta}y - 2B_{(i}\delta^1_{j)} + 2C_{\hat{i}\hat{j}} - 2\gamma_{\hat{k}(\hat{i}}\partial_{\hat{j})}\hat{\delta}^{\hat{k}}\right] . \tag{3.48}$$

Let us now introduce the definitions for the perturbed metric elements:

$$\delta g_{00} = \delta\alpha = -2\left(\phi - \partial_w\hat{\delta}w + \partial_w\hat{\delta}y\right) \ , \tag{3.49}$$

$$\delta g_{01} = \delta\beta = 2\phi + B_{\chi} + \partial_y\hat{\delta}w - \partial_y\hat{\delta}y - \partial_w\hat{\delta}w \ , \tag{3.50}$$

$$\delta g_{0I} = v_I = B_I + \partial_{\hat{I}}\hat{\delta}w - \partial_{\hat{I}}\hat{\delta}y - \gamma_{\hat{I}\hat{J}}\partial_w\hat{\delta}^{\hat{J}} \ , \tag{3.51}$$

$$\delta g_{\hat{1}\hat{1}} = 0 = -2\phi - 2\partial_y\hat{\delta}w - 2B_{\chi} + 2C_{\chi\chi} \ , \tag{3.52}$$

$$\delta g_{\hat{1}\hat{I}} = 0 = -\partial_{\hat{I}}\hat{\delta}w + \partial_{\hat{I}}\hat{\delta}y - B_{\hat{I}} + 2C_{\hat{1}\hat{I}} - 2\gamma_{\hat{k}(\hat{1}}\partial_{I)}\hat{\delta}^{\hat{k}} \ , \tag{3.53}$$

$$\delta g_{IJ} = H_{\hat{I}\hat{J}} = 2C_{\hat{I}\hat{J}} - 2\gamma_{\hat{K}(\hat{I}}\partial_{\hat{J})}\hat{\delta}^{\hat{K}} \ , \tag{3.54}$$

with $(I,J,K) \in \{2,3\}^2$. The perturbed metric describes the past lightcone in the observational coordinates $x^{\hat{a}} = (w, y, \hat{\theta}, \hat{\phi})$ given by

$$ds^2 = a^2(w-y)\left[(-1+\delta\alpha)dw^2 + 2(1+\delta\beta)dwdy + 2v_{\hat{I}}dx^{\hat{I}}dw + h_{\hat{I}\hat{J}}dx^{\hat{I}}dx^{\hat{J}}\right] \ . \tag{3.55}$$

The tensor $h_{\hat{I}\hat{J}} = \left(\Omega_{\hat{I}\hat{J}} + H_{\hat{I}\hat{J}}\right)$, where $\Omega_{\hat{I}\hat{J}} = S^2(y)\left(\delta^2_{\hat{I}}\delta^2_{\hat{J}} + \sin^2(\hat{\theta})\delta^3_{\hat{I}}\delta^3_{\hat{J}}\right)$, is a spatial tensor given by

$$h_{\hat{I}\hat{J}} = \begin{pmatrix} \Omega_{\theta\theta} + H_{\theta\theta} & H_{\theta\phi} \\ H_{\theta\phi} & \Omega_{\phi\phi} + H_{\phi\phi} \end{pmatrix} \ . \tag{3.56}$$

We can further split the tensor $h_{\hat{I}\hat{J}}$ into trace H^T and traceless parts H, where

$$H^T = H_{\theta\theta} + \frac{1}{\sin^2\hat{\theta}} H_{\phi\phi} \ , \tag{3.57}$$

is the trace of the perturbed $h_{\hat{I}\hat{J}}$ on the 2-sphere. Eqs. (3.52) and (3.53) are constrains that define the observational metric.

3.3.2 The PLG Four-velocity

We will use the $u^{\hat{\mu}}$ as the perturbed 4-velocity of a fundamental observer in the observational coordinates, which is defined by

$$u^{\hat{\mu}} = \left(u^w, u^{\hat{i}}\right) \ , \tag{3.58}$$

where

$$u^{\hat{i}} = \frac{v^{\hat{i}}}{a(w-y)} \ . \tag{3.59}$$

The full perturbed 4-velocity is given by

$$u^{\hat{\mu}} = \frac{1}{a(w-y)} \left((1+\delta u^w), \delta u^y, \delta u^{\hat{\theta}}, \delta u^{\hat{\phi}}\right) \ . \tag{3.60}$$

Expanding terms out using the Einstein summation rule with Eq. (3.2), yields

$$g_{00}u^0u^0 + 2g_{01}u^0u^1 + 2g_{0I}u^0u^I + g_{IJ}u^Iu^J = -1 \ , \tag{3.61}$$

and therefore, using the expressions for the components of the metric tensor from the preceding section, we get

$$\begin{aligned}&(-1+\delta\alpha)(1+\delta u^0)^2 + 2(1+\delta\beta)(1+\delta u^0)\delta u^1 + \\ &2v_I(1+\delta u^0)\delta u^I + (\Omega_{IJ} + H_{IJ})\,\delta u^I \delta u^J = -1 \ .\end{aligned} \tag{3.62}$$

Further simplifying gives

$$-2\delta u^0 + \delta\alpha + 2\delta u^1 = 0 \ , \tag{3.63}$$

whence we can conclude

$$\delta u^0 = \delta u^1 + \frac{\delta\alpha}{2} \ . \tag{3.64}$$

It is now easy to see that using lowering operation on the indices $u_{\hat{\mu}} = g_{\hat{\mu}\hat{\nu}}u^{\hat{\nu}}$, the zeroth component of the covariant 4-velocity is given by

$$\begin{aligned} u_0 &= g_{00}u^0 + g_{01}u^1 + g_{0I}u^I \ , && (3.65)\\ &= a(-1+\delta\alpha - \delta u^0 + \delta u^0 - \frac{\delta\alpha}{2}) \ , && (3.66)\\ &= a(-1+\frac{\delta\alpha}{2}) \ , && (3.67)\end{aligned}$$

where we have used Eq. (3.64) in the last two steps. Thus,

$$u^0 = \frac{1}{a}(1+\delta\alpha/2) \ . \tag{3.68}$$

Moreover,

$$\begin{aligned} u_1 &= g_{10}u^0 + g_{11}u^1 + g_{1I}u^I \ , && (3.69)\\ &= a(1+\delta\beta+\delta u^0) \ , && (3.70)\end{aligned}$$

and finally,

$$u_I = g_{I0}u^0 + g_{I1}u^1 + g_{IJ}u^J \ , \tag{3.71}$$
$$= a(v_I + \Omega_{IJ}\delta u^J) \ . \tag{3.72}$$

3.3.3 Connecting the Perturbations $\hat{\delta}^{\mu}$ with the Perturbed FLRW

To determine the local structure of the Universe, we wish to know the values of $\hat{\delta}\mu$, where it is always labeled with the observation coordinates $\hat{\mu} = \{w, y, \hat{\theta}, \hat{\phi}\}$, and we can do that directly via the definition of the k^{μ} in observational coordinates given by Eq. (3.9). The most important thing here is that in order to calculate the null vector k^{μ} in the observational coordinates, we do not have to follow the long procedure we did in calculating the null vector k^{μ} in 1+3 general coordinates system in Sec. 2.2.2. However we can reach the same result if we do so. From Eq. (B.5) at the background and Eq. (3.9), where we defined $\beta = (1+\delta\beta)$,we can write

$$k^{\hat{\mu}} = a^{-2}(1-\delta\beta)\delta^{\hat{\mu}}_y \ . \tag{3.73}$$

It is to be recalled that $k^{\mu} = dx^{\mu}/d\lambda$ where λ is the affine parameter with $d\lambda = a^{-2}d\eta$ and $a^2\frac{d}{d\lambda} = \frac{\partial}{\partial\eta} + \frac{\partial}{\partial\chi}$. Thus we can write

$$k^w = 0 = \frac{dw}{d\lambda} \ , \tag{3.74}$$
$$= \frac{d\eta}{d\lambda} + \frac{d\chi}{d\lambda} + \frac{d\hat{\delta}w}{d\lambda} \ , \tag{3.75}$$
$$= k^{\eta} + k^{\chi} + \frac{d\hat{\delta}w}{d\lambda} \ , \tag{3.76}$$

from which

$$\hat{\delta}w = -\int (k^{\eta} + k^{\chi})d\lambda \ . \tag{3.77}$$

Similarly,

$$k^y = a^{-2}(1-\delta\beta) = \frac{dy}{d\lambda} \ , \tag{3.78}$$
$$a^{-2}(1-\delta\beta) = \frac{d\chi}{d\lambda} + \frac{d\hat{\delta}y}{d\lambda} \ , \tag{3.79}$$

and hence

$$-\frac{\delta\beta}{a^2} = \frac{d\hat{\delta}y}{d\lambda} \ . \tag{3.80}$$

This leads to

$$\hat{\delta}y = -\int \left(\frac{\delta\beta}{a^2}\right) d\lambda = -\int \delta\beta d\eta \ . \tag{3.81}$$

And for $\hat{\delta}\hat{I} \equiv (\hat{\delta}\hat{\theta}, \hat{\delta}\hat{\phi})$, where $\hat{I} \in \{2,3\}^2$,

$$k^{\hat{I}} = 0 = \frac{d\hat{I}}{d\lambda} \ , \tag{3.82}$$
$$= \frac{dI}{d\lambda} + \frac{d\hat{\delta}\hat{I}}{d\lambda} \ , \tag{3.83}$$

and therefore

$$\hat{\delta}\hat{I} = -\int k^I \delta^{\hat{I}}_I d\lambda \ . \tag{3.84}$$

Here we connected the perturbations of the observational metric with the perturbations of the perturbed FLRW metric. Applying all the above into the constrain Eq. (3.52) we get

$$0 = -\phi - B_\chi + C_{\chi\chi} - \partial_y \hat{\delta} w \, , \tag{3.85}$$

$$0 = -\phi - B_\chi + C_{\chi\chi} + \partial_y \int (k^\eta + k^\chi) d\lambda \, . \tag{3.86}$$

We can use the facts that

$$\frac{\partial}{\partial y}|_{w=cst} = \frac{\partial}{\partial \eta}|_{\chi=cst} + \frac{\partial}{\partial \chi}|_{y=cst} = \frac{d}{d\eta} \, , \tag{3.87}$$

$$\frac{\partial}{\partial w}|_{y=cst} = \frac{\partial}{\partial \eta}|_{\chi=cst} \, , \tag{3.88}$$

which means that y mimics the behaviour of the affine parameter along the light ray. And we can define also the identity

$$k^\eta + k^\chi = \frac{1}{a^2}\left(n^\chi(\phi+\psi) + B^\chi + n^\chi \partial_\chi \partial^\chi E + n^\chi \partial_\chi F^\chi + n^\chi \frac{1}{2} h_\chi{}^\chi\right). \tag{3.89}$$

This result makes use of the null geodesic equations (2.71) and (2.73). Therefore, we see that the very definition of the observable coordinates via the relation $k^{\hat{a}} = \beta^{-1}\delta^{\hat{a}}_y$ ensures that the metric has the correct form again.

3.4 Einstein Field Equations in the PLG

Using our observational metric (3.55) to calculate the 10 partial differential equations of the EFEs, we are going to consider a cosmological fluid with

$$\partial_w p \equiv \omega \dot{\rho} \Rightarrow \dot{\rho} \equiv -3\mathcal{H}(1+\omega)\rho \equiv -\partial_y p \, , \tag{3.90}$$

where $\mathcal{H} = \partial_w a/a$ and $\mathcal{H} = -\partial_y a/a$. We also note the following useful relations:

$$\frac{\partial_w^2 a}{a} = -\partial_w \mathcal{H} + \mathcal{H}^2 \, , \tag{3.91}$$

$$\frac{\partial_y^2 a}{a} = \partial_y \mathcal{H} + \mathcal{H}^2. \tag{3.92}$$

Then at the background the EFEs read:

$$\begin{aligned}
E^{(0)}_{ww} &= 3\mathcal{H}^2 + 3k - 8\pi G a^2 \rho - a^2 \Lambda = 0 \, , && (3.93)\\
E^{(0)}_{wy} &= -3\mathcal{H}^2 - 3k + 8\pi G a^2 \rho + a^2 \Lambda = 0 \, , && (3.94)\\
E^{(0)}_{yy} &= -2\mathcal{H}p + 2\mathcal{H}^2 - 8\pi G a^2 \rho + 2k - 8\pi G a^2 p = 0 \, , && (3.95)\\
E^{(0)}_{\theta\theta} &= S^2\mathcal{H}^2 - S^2 k + a^2\Lambda S^2 - 2S^2(\mathcal{H}p + \mathcal{H}^2) - 8\pi G a^2 S^2 p = 0 \, , && (3.96)\\
E^{(0)}_{\phi\phi} &= S^2 \sin^2(\theta)\mathcal{H}^2 - S^2\sin^2(\theta) k - 8\pi G a^2 S^2 \sin^2(\theta) p + a^2 \Lambda S^2 \sin^2(\theta) \\
&\quad -2S^2\sin^2(\theta)(\mathcal{H}p + \mathcal{H}^2) = 0 \, , && (3.97)
\end{aligned}$$

and

$$E^{(0)}_{w\theta} = E^{(0)}_{w\phi} = E^{(0)}_{\theta\phi} = E^{(0)}_{y\theta} = E^{(0)}_{y\phi} = 0 \, . \tag{3.98}$$

We have calculated the 10 equations using the observational metric (3.55). As we can see, at the background the EFEs satisfy the observational metric justification.

We are going to derive the dynamical equations of the perturbations in observational coordinates.

In order to simplify the system we need to use the spherical decomposition dynamics, as follows:

$$\delta\rho(w,y) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left[\delta\rho^{lm}(w,y)Y^{lm}(I)\right] , \tag{3.99}$$

$$\delta u_0(w,y) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left[\delta u_0^{lm}(w,y)Y^{lm}(I)\right] , \tag{3.100}$$

$$\delta\alpha(w,y) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left[\delta\alpha^{lm}(w,y)Y^{lm}(I)\right] , \tag{3.101}$$

$$\delta\beta(w,y) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left[\delta\beta^{lm}(w,y)Y^{lm}(I)\right] , \tag{3.102}$$

$$v_I(w,y,I) = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\left[v^{lm}(w,y)Y_I^{lm}(I) + \overline{v}^{lm}(w,y)\overline{Y}_I^{lm}(I)\right] , \tag{3.103}$$

$$\delta u_I(w,y,I) = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\left[\delta u^{lm}(w,y)Y_I^{lm}(I) + \overline{\delta u}^{lm}(w,y)\overline{Y}_I^{lm}(I)\right] , \tag{3.104}$$

$$\begin{aligned} H_{IJ}(w,y,I) = \frac{1}{2}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} H_{lm}^{T}(w,y)\gamma_{IJ}Y^{lm}(I) + \sum_{l=2}^{+\infty}\sum_{m=-l}^{l}\Big[H^{lm}(w,y)Y_{IJ}^{lm}(I) \\ + \overline{H}^{lm}(w,y)\overline{Y}_{IJ}^{lm}(I)\Big] , \end{aligned} \tag{3.105}$$

to enable us to decompose them harmonically and produce functions suitable for the extraction of observational quantities, see Sec. A.2 for more details. Using the above definitions and a Maple code we calculated the 10 equations using the observation metric (3.55). The conservation equations $\nabla_\mu T^\mu_x$, where $\{x : w, y, \hat{I}\}$ are given by

$$\begin{aligned} (1) \quad \nabla_\mu T^\mu_w \;=\; & \Big[-(1+\omega)\rho\left(\frac{1}{2}\partial_w H^T + \partial_w\delta u^0 + \partial_w\beta - l(l+1)\delta u\right) - \partial_w\delta\rho \\ & -3H(1+\omega)\delta\rho\Big]Y = 0 , \end{aligned} \tag{3.106}$$

$$\begin{aligned} (2) \quad \nabla_\mu T^\mu_y \;=\; & \Big[-\frac{2}{3}a(4-3\omega)\dot{\rho}\delta u^0 - (1+\omega)\rho\Big(\partial_y\delta u^0 - 2\partial_w\delta u^0\Big) + a(-\frac{1}{3}+\omega)\dot{\rho}\beta \\ & +2(1+\omega)\rho\partial_w\beta + \frac{1}{2}(1+\omega)\rho\partial_w H^T + (1+\omega)\partial_w\delta\rho + \omega\partial_y\delta\rho \\ & +3\mathcal{H}(1+\omega)\delta\rho - \rho(1+\omega)l(l+1)\delta u\Big]Y = 0 , \end{aligned} \tag{3.107}$$

$$\begin{aligned} (3) \quad \nabla_\mu T^\mu_I \;=\; & \Big[S^2(1+\omega)\rho\partial_w\overline{\delta u} + S^2\Big(\omega\dot{\rho} - 16\pi G a^2(1+\frac{1}{3})\omega\rho + 4\mathcal{H}\rho\Big)\overline{\delta u} \\ & +(1+\omega)\rho\partial_w\overline{v} + \Big(\omega\dot{\rho} - 16\pi G a^2(1+\frac{1}{3})\omega\rho + 4\mathcal{H}\rho\Big)\overline{v}\Big]\overline{Y}_I \\ & + \Big[-(1+\omega)\rho\delta u^0 + S^2(1+\omega)\rho\partial_w\delta u + S^2\Big(\omega\dot{\rho} - 16\pi G a^2(1+\frac{1}{3})\omega\rho\Big) \\ & +4\mathcal{H}\rho\delta u + (1+\omega)\rho\partial_w v + \Big(\omega\dot{\rho} - 16\pi G a^2(1+\frac{1}{3})\omega\rho + 4\mathcal{H}\rho\Big)v \end{aligned}$$

$$+\omega\delta\rho\Big]Y_I = 0 \,. \tag{3.108}$$

We also substituted for the scalar $\delta\alpha$ by the vector δu^w using Eq. (3.68). Since our observer is comoving with the matter, we can consider $u^1 = \frac{dy}{d\tau} = 0$, where τ is the matter proper time. And the field equations are:

$$\begin{aligned}
(4)\quad E_{IJ} \;=\; & \Big[\partial_y\overline{v} - 2\mathcal{H}\overline{v} - 2S^2\partial^2_{wy}\overline{H} - S^2\partial^2_y\overline{H} + 2(S^2\mathcal{H} - S(1-S^2k)^{\frac{1}{2}})\partial_w\overline{H} \\
& -2S(1-S^2k)^{\frac{1}{2}}\partial_y\overline{H}\Big]\overline{Y}_{IJ} + \Big[-2\beta + 2\partial_y v - 4\mathcal{H}v - 2S^2\partial^2_{wy}H - S^2\partial^2_y H \\
& +2(S^2\mathcal{H} - S(1-S^2k)^{\frac{1}{2}})\partial_w H - 2S(1-S^2k)^{\frac{1}{2}}\partial_y H\Big]Y_{IJ} = 0 \,,
\end{aligned} \tag{3.109}$$

$$\begin{aligned}
(5)\quad E_I{}^I \;=\; & \Big[8S^2\Big(\frac{(1-S^2k)^{\frac{1}{2}}}{S}\mathcal{H} + 4\pi Ga^2\omega\rho - \frac{1}{2}\Lambda a^2\Big)\delta u^0 + (8S^2\mathcal{H} - 4S(1-S^2k)^{\frac{1}{2}})\partial_y\delta u^0 \\
& -2S^2\partial^2_y\delta u^0 + (4S\mathcal{H}(1-S^2k)^{\frac{1}{2}} + 4S^2k)\beta + (4S^2\mathcal{H} - 2S(1-S^2k)^{\frac{1}{2}})\partial_y\beta \\
& +2S^2\partial^2_{wy}\beta + 2S^2\partial^2_{wy}H^T + \frac{1}{2}S^2\partial^2_y H^T + 2(-S^2\mathcal{H} + S(1-S^2k)^{\frac{1}{2}})\partial_w H^T \\
& +2S(1-S^2k)^{\frac{1}{2}}\partial_y H^T - 16\pi Ga^2S^2\omega\delta\rho - l(l+1)\beta + l(l+1)\partial_y v \\
& -2\mathcal{H}l(l+1)v\Big]Y = 0 \,,
\end{aligned} \tag{3.110}$$

$$\begin{aligned}
(6)\quad E_{wI} \;=\; & \Big[16\pi GS^4a^2(1+\omega)\rho\delta\overline{u} + l(l+1)\overline{v} - S^2\partial^2_{wy}\overline{v} - S^2\partial^2_y\overline{v} + 2S(1-S^2k)^{\frac{1}{2}}\partial_w\overline{v} \\
& +(16\pi GS^2(1+\omega)\rho a^2 - 4S^2k)\overline{v} + (2-l(l+1))S^2\partial_w\overline{H}\Big]\overline{Y}_I \\
& +\Big[2S^2\partial_y\delta u^0 - 4\mathcal{H}S^2\delta u^0 + 16\pi GS^4a^2(1+\omega)\rho\delta u - S^2\partial_w\beta + S^2\partial_y\beta \\
& -S^2\partial^2_{wy}v - S^2\partial^2_y v + 2S(1-S^2k)^{\frac{1}{2}}\partial_w v + \Big(16\pi GS^2(1+\omega)\rho a^2 - 4S^2k\Big)v \\
& -\frac{1}{2}S^2\partial_w H^T + \frac{1}{2}S^2\partial_w H(2-l(l+1))\Big]Y_I = 0 \,,
\end{aligned} \tag{3.111}$$

$$\begin{aligned}
(7)\quad E_{yI} \;=\; & \Big[-2\mathcal{H}S^2\partial_y\overline{v} + \Big(-2 + 4S\mathcal{H}(1-S^2k)^{\frac{1}{2}} - 16\pi GS^2(1+\omega)\rho a^2\Big)\overline{v} \\
& +S^2\partial^2_y v - 16\pi GS^4a^2(1+\omega)\rho\delta\overline{u} + (2-l(l+1))S^2\partial_y\overline{H}\Big]\overline{Y}_I \\
& +\Big[-2\mathcal{H}S^2\partial_y v + \Big(-2 + 4S\mathcal{H}(1-S^2k)^{\frac{1}{2}} - 16\pi GS^2(1+\omega)\rho a^2\Big)\overline{v} \\
& +S^2\partial^2_y v - 16\pi GS^4a^2(1+\omega)\rho\delta u - S^2\partial_y\beta - 2S(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\beta \\
& -\frac{1}{2}S^2\partial_y H^T + \frac{1}{2}(2-l(l+1))S^2\partial_y H\Big]Y_I = 0 \,,
\end{aligned} \tag{3.112}$$

$$
\begin{aligned}
(8) \quad E_{yy} \quad &= \quad \Big[-32\pi G S a^2(1+\omega)\rho\delta u^0 - 32\pi G S a^2(1+\omega)\rho\beta - 16\pi G a^2 S(1+\omega)\delta\rho \\
&\quad -4(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\partial_y\beta - S\partial_y^2 H^T - 2(1-S^2k)^{\frac{1}{2}}\partial_y H^T \Big] Y = 0 \,, \qquad (3.113)
\end{aligned}
$$

$$
\begin{aligned}
(9) \quad E_{wy} \quad &= \quad \Big[16\pi G S^3 a^2(1+\omega)\rho\delta u^0 - 4S(1 - 4S\mathcal{H}(1-S^2k)^{\frac{1}{2}} + S^2\Lambda a^2 \\
&\quad -4S^2k + 4\pi G S^2(1+\omega)\rho a^2)\delta u^0 + 4S^2(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\partial_y\delta u^0 \\
&\quad -4S(1 - 2S\mathcal{H}(1-S^2k)^{\frac{1}{2}} - 2S^2k)\beta - Sl(l+1)\beta + 16\pi G S^3 a^2(1+\omega)\rho\beta \\
&\quad +4S^2(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\partial_y\beta + 16\pi G S^3 a^2\delta\rho - Sl(l+1)\partial_y v \\
&\quad -4(S\mathcal{H} - \frac{1}{2}(1-S^2k)^{\frac{1}{2}})l(l+1)v + S^3\partial_y^2 H^T + S^3\partial_{wy}^2 H^T \\
&\quad +SH^T + \frac{9}{2}(1-S^2k)^{\frac{1}{2}} S^2\partial_y H^T - 2S^2(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\partial_w H^T \\
&\quad -\frac{1}{2}Sl(l+1)H^T - S\frac{l^2(l+1)^2}{2}H + Sl(l+1)H \Big] Y = 0 \,, \qquad (3.114)
\end{aligned}
$$

$$
\begin{aligned}
(10) \quad E_{ww} \quad &= \quad \Bigg[4S\Bigg(1 - 4S\mathcal{H}(1-S^2k)^{\frac{1}{2}} + (\Lambda + 8\pi G\rho)S^2a^2 - 6S^2k\Bigg)\delta u^0 \\
&\quad +2Sl(l+1)\delta u^0 - 4S^2\Bigg(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}}\Bigg)\partial_w\delta u^0 \\
&\quad -4S^2\Bigg(2S\mathcal{H} - (1-S^2k)^{\frac{1}{2}}\Bigg)\partial_y\delta u^0 - 16\pi G a^3 S^2\delta\rho + 2Sl(l+1)\beta \\
&\quad +4S^2(1-S^2k)^{\frac{1}{2}}\partial_w\beta - 4S^2\Bigg(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}}\Bigg)\partial_y\beta \\
&\quad -4S\Bigg(3S^2k + 2S\mathcal{H}(1-S^2k)^{\frac{1}{2}} - 1\Bigg)\beta - 2Sl(l+1)\partial_y v - 2Sl(l+1)\partial_w v \\
&\quad +2(2S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})l(l+1)v - S^3\partial_w^2 H^T - S^3\partial_y^2 H^T - 2S^3\partial_{wy}^2 H^T \\
&\quad -SH^T - \frac{9}{2}S^2(1-S^2k)^{\frac{1}{2}}\partial_y H^T + 2S^2(S\mathcal{H} - (1-S^2k)^{\frac{1}{2}})\partial_w H^T \\
&\quad +\frac{1}{2}Sl(l+1)H^T + S\frac{l^2(l+1)^2}{2}H - Sl(l+1)H \Bigg] Y = 0 \,. \qquad (3.115)
\end{aligned}
$$

Here we have used $\partial_y^2 S(y) = -S(y)k$, and

$$
\frac{(\partial_y S(y))^2}{S(y)^2} = \frac{1}{S(y)^2} - k \,. \qquad (3.116)
$$

The perturbed observational metric has the 10 degrees of freedom: 3 from the scalar $(\delta\beta, \delta\rho, \delta u^0)$, 2 from the vector v_I, 2 from the vector δu^I and 3 from the tensor H_{IJ}. More work in the future will be added to the calculated first-order EFE in the PLG metric.

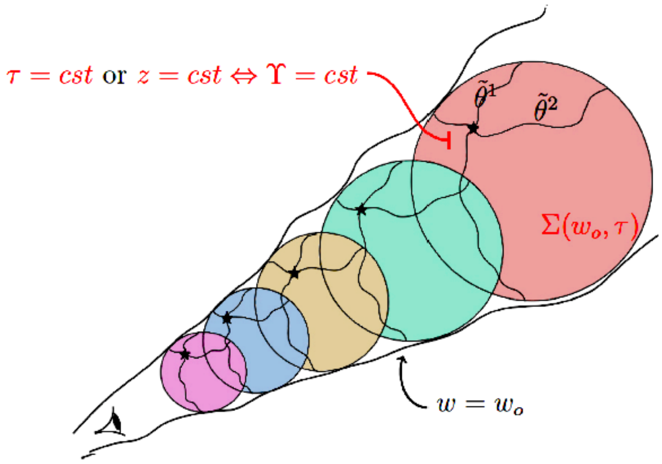


Figure 3.3: Illustration of the GLC coordinates of an inhomogeneous lightcone parametrized by GLC coordinates: The observer sees the sky as the superposition of 2-spheres [11].

3.5 The Geodesic Lightcone Gauge

The Geodesic Lightcone Gauge (GLC) has been introduced and discussed in detail in [129–134], where they introduced a lightcone metric close to the observational coordinates and it has its differences from the observational coordinates as well. Their metric consists of 6 arbitrary functions $(\Upsilon, U^a, \gamma_{ab})$ [11]. It is totally gauge fixed, with a specially adapted coordinate system corresponding to constant-time hypersurfaces, $x^\mu = (w, \tau, \theta^a)$ where $a = 1$,2, and τ is a coordinate to describe the proper time measured by an observer in geodesic motion. Its line element is given by

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw) \ , \tag{3.117}$$

w representing a null coordinate defining past light cones, where $(\partial_\mu w \partial^\mu w = 0)$, with an observer moving with a proper time τ along her geodesic. Υ is an inhomogeneous scale factor, U^a is a shift-vector, and the symmetric 2×2 matrix γ_{ab} is the metric inside $\Sigma(w, \tau)$. The angles $\tilde{\theta}^a$ are the angles were the photons keep their path orthogonal to a 2-sphere $\Sigma(w, \tau)$ of constant time in our past lightcone. The $\partial_\mu \tau$ defines a geodesic flow [129]

$$(\partial^\nu \tau)\nabla_\nu(\partial_\mu \tau) = 0 \ . \tag{3.118}$$

Then we can obtain $g^{\tau\tau}_{GLC}$=-1. The metric and its inverse will take the form

$$g^{GLC}_{\mu\nu} = \begin{pmatrix} 0 & -\Upsilon & \vec{0} \\ -\Upsilon & \Upsilon^2 + U^2 & -U_b \\ \vec{0}^T & -U^T_a & \gamma_{ab} \end{pmatrix} , \quad g^{\nu\gamma}_{GLC} = \begin{pmatrix} -1 & -\Upsilon^{-1} & -U^b/\Upsilon \\ -\Upsilon^{-1} & 0 & 0 \\ -(U^a)^T/\Upsilon & \vec{0}^T & \gamma^{ab} \end{pmatrix} , \tag{3.119}$$

where $\vec{0} = (0, 0)$ and $U_b = (U_1, U_2)$. With $g \equiv \det g^{GLC}_{\mu\nu}$, and $\gamma \equiv \det\gamma_{ab}$, we get

$$\sqrt{-g} = \Upsilon\sqrt{|\gamma|} \ . \tag{3.120}$$

This GLC metric has the 6 degrees of freedom to describe the geometry of spacetime.

For flat FLRW metric, the transformation into GLC coordinates and the meaning of the metric components are easily obtained from Eq. (3.119):

$$\tau = t \quad , \quad w = r + \eta \quad , \quad \theta^1 = \theta \quad , \quad \theta^2 = \phi \, , \tag{3.121}$$
$$\Upsilon = a(t) \quad , \quad U^a = 0 \quad , \quad \gamma_{ab} d\theta^a d\theta^b = a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2) \, , \tag{3.122}$$

where r is defined from (1.11). The constant τ and constant w hypersurfaces intersect each other on 2-dimensional surface (w_0, τ_s), see Fig. [3.3]. Notice that this 2-sphere can be expressed in a constant redshift hypersurface [129], where it corresponds to the constant Υ hypersurface on the past lightcone.

To measure the distance from the observer down to the object in the GLC we need to calculate the differences between the observer proper time and the object proper time.

$$\tau = \int_{\tau_s}^{\tau_o} d\tau' \, . \tag{3.123}$$

To move from GLC gauge to the PLG gauge simply by applying the coordinates transformations using Eqs. (3.35) and (3.36) that we have mentioned earlier.

3.6 Differences Between the PLG and the GLC

We can summarise the differences between the PLG and the GLC as follows:

- The PLG uses the observational coordinates $(w, y, \hat{\theta}, \hat{\phi})$, whereas the GLC metric uses $(\tau, w, \tilde{\theta}^a)$, the first two coordinates has created the main difference between the two gauges.
- The y coordinate in the PLG represents a spatial radial distance or a null distance down the lightcone. But in GLC the y coordinate has been replaced by the proper time coordinate τ and the differences on the proper time of the event crossing the null cone and the fundamental observer moving along its worldline represent the distance down the null cone. And by this distance replacement definitions they will enhance the significance of the GLC gauge to have a coordinate system both adapted to the observations of the source and the proper time of the observer [135].
- An important difference is that the GLC gauge corresponds to a complete 'gauge fixing' of the observational coordinates. We can describe the GLC gauge as a 'gauge fixing' with 6 degrees of freedom, unlike the PLG gauge which is expressed in a general gauge with 10 degrees of freedom.
- The PLG gauge of the observational coordinates is set by the condition $w = \tau|_C$, where τ is the proper time of the observer. On the other hand, in the GLC gauge the lightcone set by $w = \int d\tau/a(\tau) \equiv \eta$, where $a(\tau)$ is defined as the homogeneous limit of the Υ function.
- Both gauges break down when caustics appear down on the past lightcone [108, 135].
- The GLC is well adapted to computations of quantities related to light signals, to simplify the so-called averaging on the lightcone [135].

Chapter 4

Observables in the Past Lightcone Gauge

To go wrong in one's own way is better than to go right in someone else's.

Fyodor Dostoyevsky

We are going to present here the observables that we can measure on our past lightcone. By using the observational coordinates $(w, y, \hat{\theta}, \hat{\phi})$ introduced in Ch 3, then we are going to get a set of observables defined by the PLG parameters, where they are more simple but different definitions from what we introduced in Sec. 2.3. To justify our observables that can cover the perturbed FLRW limits, we need to transfer them from the PLG gauge on the lightcone into 1+3 general gauge on a hypersurface that is orthogonal to the observer 4-velocity and described by a perturbed FLRW metric. In order to do so we need to understand the screen space lying on the hypersurface.

4.1 The Screen Space

We are going to show how the covariant description of light beams, are naturally defined and they can be turned to a more observation-oriented description. This requires to introduce a notion of screen, on which observers can project the beam and characterise its shape and extension. The screen space is orthogonal to the observer 4-velocity, and the tensor [120]

$$N_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu \tag{4.1}$$

projects quantities into the screen space and satisfies the following conditions [120]:

$$N^\mu_{\ \mu} = 2, \quad N_{\mu\alpha} N^\alpha_{\ \nu} = N_{\mu\nu}, \quad N_{\mu\nu} k^\mu = N_{\mu\nu} u^\mu = N_{\mu\nu} n^\mu = 0\ . \tag{4.2}$$

4.1.1 Derivatives and Integrals in the Screen Space

We can isolate the components into parallel and orthogonal components to the light ray. We denote the parts lying in the screen space by $_\perp$ and the parts parallel to it by $_\parallel$. Furthermore, let us define the covariant angular derivative $\nabla_{\perp i}$ on the screen space and the derivative $\nabla_\parallel$ along the direction of observations, respectively, as [120, 136]

$$\nabla_{\perp i} X_j \equiv (\gamma_i^{\ k} - n_i n^k)(\gamma_j^{\ l} - n_j n^l) \nabla_{\perp k} X_l\ , \tag{4.3}$$

$$\nabla_{\parallel} \equiv n^i \nabla_i X \,. \tag{4.4}$$

The radial derivative can be interchanged with the derivative along a null geodesic using the relation

$$\nabla_{\parallel} X = \partial_\eta X - \frac{d}{d\lambda} X \,. \tag{4.5}$$

At the background, one can switch the affine parameter λ with the conformal time η, and both are related to the radial distance along the past lightcone by $\chi = \lambda_o - \lambda = \eta_o - \eta = \eta - \eta_s$. The spatial derivative is in general decomposed into parts along the null geodesic and in the screen space as

$$\begin{aligned} \nabla_i &= n_i \nabla_{\parallel} + \nabla_{\perp i} \,, & (4.6)\\ &= n_i(\partial\eta - \frac{d}{d\eta}) + \nabla_{\perp i} \,. & (4.7) \end{aligned}$$

Then the 3-D Laplacian on the Minkowski background becomes

$$\begin{aligned} \nabla^2 &= \gamma^{ij}[n_j \nabla_{\parallel} + \nabla_{\perp_j}][n_i \nabla_{\parallel} + \nabla_{\perp_i}] \,, & (4.8)\\ &= \gamma^{ij}[n_i n_j \nabla_{\parallel}^2 + n_j \nabla_{\parallel} \nabla_{\perp_i} + \nabla_{\perp_j}[n_i \nabla_{\parallel}] + \nabla_{\perp_i}\nabla_{\perp_j}] \,, & (4.9)\\ &= \gamma^{ij}[n_i n_j \nabla_{\parallel}^2 + n_j \nabla_{\perp_i} \nabla_{\parallel} + \nabla_{\perp_j} n_i \nabla_{\parallel} + n_i \nabla_{\perp_j} \nabla_{\parallel} + \nabla_{\perp_i}\nabla_{\perp_j}], & (4.10)\\ &= \gamma^{ij}[n_i n_j \nabla_{\parallel}^2 + n_{(i} \nabla_{\perp_j)} \nabla_{\parallel} + \nabla_{\perp_j} n_i \nabla_{\parallel} + \nabla_{\perp_i}\nabla_{\perp_j}] \,, & (4.11)\\ &= \nabla_{\parallel}^2 + \gamma^{ij} \nabla_{\perp_j} n_i \nabla_{\parallel} + \nabla_{\perp}^2 \,, & (4.12) \end{aligned}$$

where [120]

$$\nabla_{\perp i} n_j = \frac{1}{\lambda_o - \lambda}(\gamma_{ij} - n_i n_j) = \frac{1}{\chi}(\gamma_{ij} - n_i n_j) \,. \tag{4.13}$$

This leads to the relation

$$\nabla^2 = \nabla_{\parallel}^2 + \frac{2}{\chi}\nabla_{\parallel} + \nabla_{\perp}^2 \,, \tag{4.14}$$

which can also be rewritten as

$$\nabla_{\perp}^2 = -n^i n^j \nabla_i \nabla_j - \frac{2}{\eta_o - \eta} n^i \nabla_i + \nabla^2 \,. \tag{4.15}$$

Since the radial derivative can be interchanged with the derivative along a null geodesic [120], one can write

$$\begin{aligned} n^i \nabla_i n^j \nabla_j X = \nabla_{\parallel}^2 &= \nabla_{\parallel}(\nabla_{\parallel} X) \,, & (4.16)\\ &= (X'' - d\eta X') - d\eta(X' - d\eta X) \,, & (4.17)\\ &= X'' - 2d\eta X' - d^2\eta X \,, & (4.18) \end{aligned}$$

and we can conclude that

$$\nabla_i \nabla_j = n_i n_j \nabla_{\parallel}^2 + 2 n_{(i} \nabla_{\perp_j)} \nabla_{\parallel} + \frac{1}{\chi}(\gamma_{ij} - n_i n_j)\nabla_{\parallel} + \nabla_{\perp_i}\nabla_{\perp_j} \,. \tag{4.19}$$

4.1.2 Screen Space and SVT Decompositions

We can decompose a spatial vector v^i that is orthogonal to the 4-velocity u^μ on a perturbed spacetime into [120]

$$v^i = n^i v_{\parallel} + v_{\perp}^i \,, \tag{4.20}$$

where

$$v_{\parallel} = n_j v^j, \quad v_{\perp}^i = N^i{}_j v^j \,. \tag{4.21}$$

And for a symmetric trace-free spatial tensor h_{ij}, the screen space decomposition will look like

$$h_{ij} = h_{\|}(n_i n_j - \frac{1}{2}N_{ij}) + 2h_{\perp|(i}n_{j)} + h_{\perp ij} , \tag{4.22}$$

where

$$h_{\perp ij} = N_i^k N_j^l h_{kl}, \quad h_{\perp|(i}n_{j)} = N^k{}_{(i}n_{j)}n^l h_{kl} , \tag{4.23}$$

and

$$h^i_{\perp|}n_i = 0 = n_i h^{ij}_{\perp} = N_{ij}h^{ij}_{\perp} . \tag{4.24}$$

4.2 The Redshift of Distant Galaxies in the PLG

The distance-redshift relations for far-away cosmological objects play an important role in cosmology. It has led to the discovery of the expansion of the Universe [137] and later on to the discovery of its accelerated expansion [138].

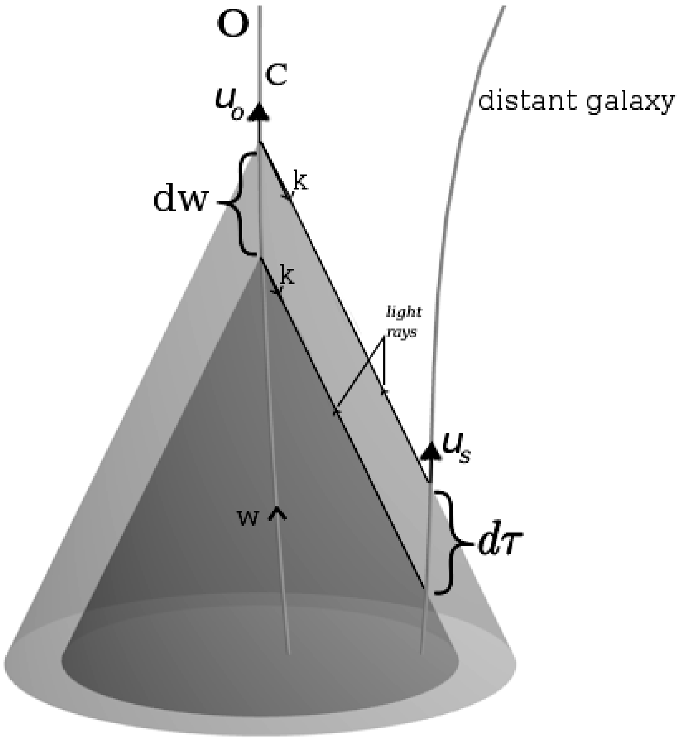


Figure 4.1: A time interval $d\tau$ at the observed galaxy is measured as a time interval dw by the observer.

The redshift of a source crossing the lightcone is the time dilation observed from $C(w, y, \hat{\theta}, \hat{\phi})$ of a source of a proper time τ along its worldlines, crossing our past lightcone is determined by the ratio $dw/d\tau$ along our worldline. Using the relations (3.4) and (4.25), then the observed redshift z of its emitted light is determined by

$$1 + z = \frac{\lambda_o}{\lambda_s} = \frac{a_c(w)}{a(w, y)} = \frac{dw}{d\tau} = u^w , \tag{4.25}$$

where $a_c(w)$ is the scale factor along the central worldline C at singular point $w_0|_C$ (it can be taken equal to 1 today). Or we can use the expression

$$1+z=\frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o}, \tag{4.26}$$

where we can normalise $(k^\mu u_\mu)_o = 1$, and we can re-write (4.26) as

$$1+z=(k_\mu u^\mu)_s, \tag{4.27}$$

using (3.4) again, we will get

$$\begin{aligned} 1+z &= (\delta^0_\mu u^\mu)_s \ , & (4.28)\\ &= (u^w)_s \ . & (4.29)\end{aligned}$$

This shows that in the lightcone gauge the redshift of the source is its 4-velocity. Where the 4-velocity of the source is directly observable because the redshift is directly measurable from the observed source spectrum. Furthermore

$$u^i=\frac{dx^i}{d\tau}=(1+z)\frac{dx^i}{dw} \ . \tag{4.30}$$

And we can obtain the velocity components $u_\mu = g_{\mu\nu}u^\nu$ using (3.15) [108].

4.2.1 Gauge Transformations of the Redshift

To show that Eq. (4.29) is fulfilling the right form of redshift in the standard model we need to transform our result using the PLG to first-order perturbation. Using the coordinates transformations mentioned in the previous chapter, we can transform the u^η of Eq. (2.35) into u^w using the coordinate transformations given by Eq. (3.36), getting

$$\begin{aligned} 1+z=u^w &= \frac{dx^w}{dx^a}u^a \ , & (4.31)\\ &= \frac{dx^w}{dx^\eta}u^\eta+\frac{dx^w}{dx^i}u^i \ , & (4.32)\\ &= \frac{1}{a(w-y)}\left(\delta^0_0+\frac{\partial\hat{\delta}w}{\partial x^w}\right)(1-\phi)\\ &\quad +\frac{1}{a(w-y)}\left(\delta^0_0\delta^1_i+\frac{\partial\hat{\delta}w}{\partial x^{\hat{i}}}+\frac{\partial\hat{\delta}w}{\partial x^w}\delta^1_i\right)v^i \ ,\\ &= \frac{1}{a(w-y)}\left[1+\partial_w\hat{\delta}w-\phi+v^i\delta^1_i\right] \ . & (4.33)\end{aligned}$$

We can use the definition of the 4-velocity of Eq. (3.68) and substitute for $\delta\alpha$ by Eq. (3.49)

$$u^w=\frac{1}{a}(1+\delta\alpha/2)=\frac{1}{a}(1+(\partial_w\hat{\delta}w-\phi-\partial_w\hat{\delta}y)) \ . \tag{4.34}$$

Comparing the above with Eq. (4.33), we can conclude that

$$\partial_w\hat{\delta}y=-v^i\delta^1_i \ . \tag{4.35}$$

We could decompose v_i into scalar and tensor parts according to Eqs. (2.81, 2.83)

$$v^i = V^i+\gamma^{ij}[\nabla_j E'+\overline{B}_j] \ . \tag{4.36}$$

We need now to calculate $\partial_w \hat{\delta} w$, by using the fact that

$$k_\mu = \partial_\mu w \ . \tag{4.37}$$

Thus we get

$$k_\eta = \partial_\eta w = \partial_\eta(\eta + \chi + \hat{\delta} w) = 1 + \partial_\eta \hat{\delta} w \ . \tag{4.38}$$

Applying Eq. (3.88) leads to

$$\partial_w \hat{\delta} w = k_\eta - 1 \ . \tag{4.39}$$

We know that

$$k_\eta = 1 + \int_s^o a^{-2}\phi' d\lambda - \int_s^o a^{-2}\left[n^i n^j \left(C'_{ij} - \nabla_{(i}B_{j)}\right)\right] d\lambda - B_i n^i \ , \tag{4.40}$$

which, upon substituting for B_i and C_{ij} by Eqs. (2.4, 2.5), becomes

$$\begin{aligned} k_\eta \ &= \ 1 + \int_s^o a^{-2}\phi' d\lambda \\ &- \int_s^o a^{-2}\left[n^i n^j \left(-\psi'\delta_{ij} + \nabla_i\nabla_j E' + \nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \ \frac{1}{2}(\nabla_i\nabla_j B - \nabla_j \overline{B}_i + \nabla_i \nabla_j B - \nabla_i \overline{B}_j)\right)\right] d\lambda \\ &- \nabla_i B n^i - \overline{B}_i n^i \ . \end{aligned} \tag{4.41}$$

Further, we substitute for the gauge-invariant quantities and re-write Eq. (4.41) as

$$\begin{aligned} k_\eta \ &= \ 1 + \int_s^o a^{-2}\Phi' d\lambda \\ &- \int_s^o a^{-2}(B - E')'' d\lambda - \int_s^o a^{-2}\left[n^i n^j \left(-\Psi'\delta_{ij} + \nabla_i\nabla_j E' \ + \nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \nabla_i\nabla_j B - \nabla_{(i}\overline{B}_{j)}\right)\right] d\lambda \\ &- \nabla_i B n^i - \overline{B}_i n^i \ . \end{aligned} \tag{4.42}$$

Substituting Eq. (2.80) in Eq. (4.42), we then will get

$$\begin{aligned} k_\eta \ &= \ 1 + \int_s^o \Phi' d\eta - \int_s^o (B - E')'' d\eta + \int_s^o (B - E')'' d\eta - (B - E')'|_s^o + n^i \nabla_i (B - E')|_s^o \\ &- \int_s^o n^i n^j \left(-\Psi'\delta_{ij} + \nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \nabla_{(i}\overline{B}_{j)}\right) d\eta - n^i \nabla_i B - \overline{B}_i n^i \ , \end{aligned} \tag{4.43}$$

$$\begin{aligned} &= \ 1 + \int_s^o \Phi' d\eta - (B - E')' - n^i \nabla_i E' - \int_s^o n^i n^j \left(-\Psi'\delta_{ij} + \nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} \ - \nabla_{(i}\overline{B}_{j)}\right) d\eta \\ &- \overline{B}_i n^i \ . \end{aligned} \tag{4.44}$$

We can therefore write

$$\partial_w \hat{\delta} w = \int_s^o (\Phi + \Psi)' d\eta - (B - E')' - n^i \nabla_i E' - \int_s^o n^i n^j \left(\nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} - \nabla_{(i}\overline{B}_{j)}\right) d\eta - \overline{B}_i n^i \ . \tag{4.45}$$

Now substituting the above equation and Eq. (4.36) in Eq. (4.33) yields

$$\begin{aligned} 1 + z \ &= \ \frac{1}{a(\eta)}\left[1 + \int_s^o (\Phi + \Psi)' d\eta - (B - E')' - n^i \nabla_i E' - n^i \overline{B}_i \right. \\ &\left. - \int_s^o n^i n^j \left(\nabla_{(i}F'_{j)} + \frac{1}{2}h'_{ij} \ - \nabla_{(i}\overline{B}_{j)}\right) d\eta - \Phi + \mathcal{H}(B - E') + (B - E')' + V_i n^i + \gamma^{ij}[\nabla_j E' + \overline{B}_j]\delta_i^1 \right] . \end{aligned} \tag{4.46}$$

We can write the above as

$$1+z = \frac{1}{a(\eta)}\left[1-\Phi+\mathcal{H}(B-E')+V_\chi+\int_s^o(\Phi+\Psi)'d\eta-\int_s^o\left(\nabla_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\nabla_\chi\overline{B}_\chi\right)d\eta\right]. \tag{4.47}$$

Using the gauge-invariant variable of Eq. (2.28), the above result can be summarized as

$$1+z=\frac{1}{a(\eta)}\left\{[1-\Phi+\mathcal{H}(B-E')+V_\chi]_s^o+\int_s^o(\Phi+\Psi)'d\eta-\int_s^o\nabla_\chi\overline{\Phi}_\chi d\eta-\frac{1}{2}\int_s^o h'_{\chi\chi}d\eta\right\}. \tag{4.48}$$

And this is the exact answer of the redshift in general coordinates system that we got earlier in Eq. (2.84), which makes the statement of our redshift in observational coordinates a correct statement, and the observational approach more trusted.

4.3 The Area Distance in the PLG

The shape and size of the image of the source depends on the path taken by the light rays from the source to the observer through the spacetime by the null geodesics; *i.e*, it depends on the spacetime curvature. In fact they are both represented by the metric components h_{IJ}, which are *in principle*, directly measurable. For an object of known size and shape observed at time w_0 and lying at distance y in the direction $\hat{\theta},\hat{\phi}$ one has [108]

$$dl^2=h_{IJ}(w_0,y,\hat{\theta},\hat{\phi})dx^Idx^J\ , \tag{4.49}$$

where dl represents distance of the object perpendicular to the line of sight, which are known if the size, shape and orientation of the object are known. The term dx^I represents the corresponding angular displacements at the image, which are directly measurable [108]. Comparing the angular measurements with the known dimensions, one can deduce h_{IJ}. And directly from (4.49) and (3.21), we get the area distance r_A given by (3.20) [108]

$$r_A=\left[\frac{\det[h_{IJ}]}{\sin^2\hat{\theta}}\right]^{\frac{1}{4}}, \tag{4.50}$$

where

$$h_{IJ}=\begin{pmatrix} a^2S^2(1+H_{\hat{\theta}\hat{\theta}}) & a^2S^2H_{\hat{\theta}\hat{\phi}} \\ a^2S^2H_{\hat{\theta}\hat{\phi}} & a^2S^2(\sin^2\hat{\theta}+H_{\hat{\phi}\hat{\phi}})\end{pmatrix}. \tag{4.51}$$

For simplicity we express the embedded $S(y)$ in H_{IJ}, thus defining the determinant as

$$\det[h]=a^4S^4(\sin^2\hat{\theta}+H_{\hat{\phi}\hat{\phi}}+\sin^2\theta H_{\hat{\theta}\hat{\theta}})-a^4S^4H^2_{\hat{\theta}\hat{\phi}}\ , \tag{4.52}$$

and therefore

$$r_A=\left(a^4S^4[1+H_{\hat{\theta}\hat{\theta}}+\frac{1}{\sin^2\hat{\theta}}H_{\hat{\phi}\hat{\phi}}]\right)^{\frac{1}{4}}. \tag{4.53}$$

This means the area distance using the PLG can be written in a very simple way as

$$r_A=a((w-y),\eta)S(y)\left[1+\frac{1}{4}H^T\right]. \tag{4.54}$$

Nice and simple expression of area distance in the lightcone gauge, but we need to verify this expression with the one in the general gauge, therefore we need to do the gauge transformation.

4.3.1 The Gauge Transformations of the Area Distance

In the subsequent calculations, we need to justify whether our results are consistent with the ones in general $1+3$ coordinates. Note that $w|_C$ θ, ϕ are constants at the surface. We recall that the area distance for Eq. (4.54), and from Eq. (3.54) we have

$$H_{IJ} = 2C_{IJ} - 2\gamma_{\hat{K}(I}\nabla_{J)}\hat{\delta}^{\hat{K}} \ ,$$

in observational coordinates, and since

$$\gamma^{IJ}H_{IJ} \equiv H^{I}{}_{I} = H^{T} \ , \tag{4.55}$$

one can write

$$\begin{aligned} H^{T} &= 2\gamma^{IJ}C_{IJ} - \gamma^{IJ}\gamma_{KI}\nabla_{J}\hat{\delta}^{\hat{K}} - \gamma^{IJ}\gamma_{\hat{K}J}\nabla_{I}\hat{\delta}^{\hat{K}} \ , \qquad (4.56)\\ &= 2C^{I}{}_{I} - 2\nabla^{I}\hat{\delta}I \ , \qquad (4.57) \end{aligned}$$

which, when substituting

$$\hat{\delta}I = -\int_{\lambda_s}^{\lambda} k^{I} d\lambda' \tag{4.58}$$

yields

$$H^{T} = 2C^{I}{}_{I} + 2\nabla_{I}\int_{\lambda_s}^{\lambda} k^{I} d\lambda' \ , \tag{4.59}$$

and according to Eq. (B.49), we have

$$H^{T} = 2C^{I}{}_{I} + 2\int_{\lambda_s}^{\lambda} \nabla_{I}k^{I} d\lambda' \ . \tag{4.60}$$

We will write the above equation in terms of the perpendicular and parallel components to the screen space's spatial tensor N_{ij}, where

$$X^{\perp i} = N^{i}{}_{j}X^{j} \quad \text{and} \quad n^{\perp i} = 0 \ . \tag{4.61}$$

We can think of it as a 2-sphere metric embedded in one higher dimension of Euclidean space with constant radius, therefore

$$H^{T} = 2C^{\perp i}_{\perp i} + 2\int_{\lambda_s}^{\lambda} \nabla_{\perp i}k^{\perp i} d\lambda' \ . \tag{4.62}$$

Then the null vector from the source position at x_s and along the light trajectory to the observer position at x_o as a function of the affine parameter λ is given by

$$\begin{aligned} k^{\perp i} &= -\frac{1}{a^2}\int_{\lambda_s}^{\lambda} a^{-2}\left[-B^{\perp i\prime} - \nabla^{\perp i}\phi - 2n^{j}\left(C^{\perp i\prime}{}_{j} + \frac{1}{2}(\nabla_{j}B^{\perp i} - \nabla^{\perp i}B_{j})\right)\right.\\ &\qquad \left. -n^{j}n^{k}\left(\nabla_{k}C^{\perp i}_{j} + \nabla_{j}C^{\perp i}_{k} - \nabla^{\perp i}C_{jk}\right)\right]d\lambda' \ , \qquad (4.63)\\ &= -\frac{1}{a^2}\frac{1}{(\lambda_o - \lambda_s)}\int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} a^{-2}\left[-B^{\perp i\prime} - \nabla^{\perp i}\phi - 2n^{j}\left(C^{\perp i\prime}{}_{j} + \frac{1}{2}(\nabla_{j}B^{\perp i} - \nabla^{\perp i}B_{j})\right)\right.\\ &\qquad \left. - n^{j}n^{k}\left(\nabla_{k}C^{\perp i}_{j} + \nabla_{j}C^{\perp i}_{k} - \nabla^{\perp i}C_{jk}\right)\right]d\lambda' d\lambda \ . \qquad (4.64) \end{aligned}$$

One can expand further Eq. (4.62) using (4.63) to obtain

$$
\begin{aligned}
H^T &= 2C^{\perp i}_{\perp i} \\
&+2\int_{\lambda_s}^{\lambda} a^{-2}\nabla_{\perp i}\Bigg\{\frac{n^{\perp i}}{a^2}-\frac{1}{a^2}\frac{1}{(\lambda_o-\lambda_s)}\int_{\lambda_s}^{\lambda_o}\int_{\lambda_s}^{\lambda} a^{-2}\Bigg(-B^{\perp i\prime}-\nabla^{\perp i}\phi-2n^j\left(C^{\perp i\prime}{}_j+\frac{1}{2}(\nabla_j B^{\perp i}-\nabla^{\perp i}B_j)\right) \\
&-n^jn^k\left(\nabla_k C^{\perp i}_{\ j}+\nabla_j C^{\perp i}_{\ k}-\nabla^{\perp i}C_{jk}\right)\Bigg)d\lambda' d\lambda\Bigg\}d\lambda' ,
\end{aligned}
\tag{4.65}
$$

where $a^{-2}d\lambda \to d\eta$ in the perturbed spacetime, and (θ,ϕ) are constant angles along the light ray between the source at η_s and the observer at η_o. We can therefore consider the unit radial vector $n^{\perp i}=0$, reducing Eq. (4.65) to

$$
\begin{aligned}
H^T \quad &= 2C^{\perp i}_{\perp i}+2\int_{\eta_s}^{\eta}\nabla_{\perp i}\Bigg\{-\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta'-\eta_s)\Bigg[-B^{\perp i\prime}-\nabla^{\perp i}\phi \\
&-2n^j\left(C^{\perp i\prime}{}_j+\frac{1}{2}(\nabla_j B^{\perp i}-\nabla^{\perp i}B_j)\right)-n^jn^k\left(\nabla_k C^{\perp i}_{\ j}+\nabla_j C^{\perp i}_{\ k}-\nabla^{\perp i}C_{jk}\right)\Bigg]d\eta\Bigg\}d\eta' ,
\end{aligned}
\tag{4.66}
$$

$$
\begin{aligned}
&= 2C^{\perp i}_{\perp i}+\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Bigg[\nabla_{\perp i}B^{\perp i\prime}+\nabla_{\perp i}\nabla^{\perp i}\phi-2\nabla_{\perp i}\,n^jC^{\perp i\prime}{}_j \\
&+\nabla_{\perp i}n^j\nabla_j B^{\perp i}-\nabla_{\perp i}\nabla^{\perp i}B_jn^j+\nabla_{\perp i}n^jn^k\nabla_k C^{\perp i}_{\ j}+\nabla_{\perp i}n^jn^k\nabla_j C^{\perp i}_{\ k} \\
&-\nabla_{\perp i}\nabla^{\perp i}C_{jk}n^jn^k\Bigg]d\eta' d\eta .
\end{aligned}
\tag{4.67}
$$

Expanding more by substituting for B_i and C_{ij} by Eqs. (2.4), (2.5), and applying the Bardeen gauge-invariant potentials of Ch. 2 into Eq. (4.67), we get

$$
\begin{aligned}
H^T \quad &= -4\Psi_s-4\mathcal{H}(B-E')+2\nabla_{\perp i}\nabla^{\perp i}E+2\nabla_{\perp i}F^{\perp i}+h^{\perp i}_{\perp i} \\
&+\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Bigg[\nabla_{\perp i}\nabla^{\perp i}\Phi+\nabla_{\perp i}\nabla^{\perp i}\Psi-\nabla_{\perp i}\nabla^{\perp i}\nabla_k\nabla_jEn^jn^k \\
&+\nabla_{\perp i}\nabla^{\perp i}E''+\nabla_{\perp i}n^jn^k\nabla_j\nabla^{\perp i}\nabla_kE+2\nabla_{\perp i}\nabla^{\perp i}\nabla_jE'n^j+\nabla_{\perp i}n^jn^k\nabla_k\nabla^{\perp i}\nabla_jE \\
&+\nabla_{\perp i}n^j\nabla_j\overline{B}^{\perp i}-\nabla_{\perp i}\nabla^{\perp i}\overline{B}_jn^j+\nabla_{\perp i}\overline{B}^{\perp i\prime}+2\nabla_{\perp i}n^j\nabla_{(j}F^{\perp i)\prime}+\nabla_{\perp i}n^jn^k\nabla_k\nabla_{(j}F^{\perp i)} \\
&-\nabla_{\perp i}\nabla^{\perp i}\nabla_{(k}F_{j)}n^jn^k+\nabla_{\perp i}n^jn^k\nabla_j\nabla_{(k}F^{\perp i)}+\frac{1}{2}\nabla_{\perp i}n^jn^k\nabla_jh_k{}^{\perp i}+\frac{1}{2}\nabla_{\perp i}n^jn^k\nabla_kh_j{}^{\perp i} \\
&+\nabla_{\perp i}n^jh^{\perp i\prime}{}_j-\frac{1}{2}\nabla_{\perp i}\nabla^{\perp i}h_{jk}n^jn^k\Bigg]d\eta' d\eta .
\end{aligned}
\tag{4.68}
$$

Now we will apply the relations from Sec. (4.1) on the terms above, and we will write each term derivations separately for simplicity, see B.4. Putting all the pieces together we will get

$$
\begin{aligned}
H^T \quad = \quad &-4\Psi_s-4\mathcal{H}(B-E')+\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Bigg[(\nabla^2-\nabla_\parallel^2-\frac{2}{\chi}\nabla_\parallel)(\Phi+\Psi) \\
&-\nabla^2\overline{B}_in^i-\nabla_{(i}\overline{B}'_{j)}n^in^j+\frac{2}{\chi}\nabla_{(i}\overline{B}_{j)}n^in^j+\nabla^2n_jF^{j\prime}+\nabla_{(i}F''_{j)}n^jn^i-\frac{2}{\chi}\nabla_{(i}F'_{j)}n^jn^i \\
&+\frac{2}{\chi}n^in^j\frac{1}{2}h'_{ij}-\nabla^2h_{ij}n^in^j\Bigg]d\eta' d\eta .
\end{aligned}
\tag{4.69}
$$

Now substituting back in (4.54), the area distance is given by

$$
\begin{aligned}
r_A(\mathbf{n},\eta) \quad = \quad & a(\eta_s)(\eta_o-\eta_s)\Big[1-\Psi_s-\mathcal{H}(B-E')+\frac{1}{2}\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)(\eta_o-\eta) \\
& \times\Big((\nabla^2-n^in^j\nabla_i\nabla_j-\frac{2}{(\eta_o-\eta)}n^i\nabla_i)(\Phi+\Psi)-n^i\nabla^2\overline{B}_i+n^i\nabla^2F_i'-\nabla_{(i}\overline{B}_{j)}'n^in^j \\
& +\nabla_{(i}F_{j)}''n^jn^i-\nabla^2h_{ij}n^in^j-\frac{2}{(\eta_o-\eta)}(\nabla_{(i}F_{j)}'n^jn^i+n^in^j\frac{1}{2}h_{ij}'+\nabla_{(i}\overline{B}_{j)}n^in^j)\Big)d\eta\Big]\,,
\end{aligned}
\tag{4.70}
$$

where $S(\chi)=(\eta_o-\eta_s)$. This equation represents the area distance of an object in a general gauge including the vector and tensor modes contributions in the Friedmann universe, see Eq. (2.106), and it is equivalent to the expression of the area distance in our PLG, see Eq. (4.54). The double integrals term in Eq. (4.70) represents the integrated effects proportional to line-of-sight integrals of the scalar, vector and tensor modes and their time derivatives.

4.4 The Luminosity Distance in the PLG

The luminosity distance received from a source of area distance r_A, observed at redshift z, is defined in the past-lightcone as

$$d_L=\big(1+z\big)^2r_A\,, \tag{4.71}$$

and is independent of other spacetime properties. This quantity is directly measurable. The luminosity distance between a source at η_s along the way of sight of an observer at η_o, is given by using the PLG as

$$d_L=a(w-y)S(y)\big(1+z\big)^2\Big[1+\frac{1}{4}H^T\Big]\,. \tag{4.72}$$

4.4.1 The Gauge Transformations of the Luminosity Distance

We are going now to transform the above-mentioned expression of the luminosity distance in PLG to the general gauge, and justify our Eq. (4.72). From Eq (4.70), we can write

$$
\begin{aligned}
d_L(\mathbf{n},\eta_s) \quad = \quad & \frac{S(\chi)}{a_s(\eta)}\Big[1-2\Phi_o+2\Phi_s-\Psi_s+\mathcal{H}(B-E')+2n^iV_i \\
& +2\int(\Phi+\Psi)'d\eta-\int n^in^j\left(\nabla_{(i}F_{j)}'\;+\frac{1}{2}h_{ij}'-\nabla_{(i}\overline{B}_{j)}\right)d\eta \\
& +\frac{1}{2}\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Big([\nabla^2-n^in^j\nabla_i\nabla_j-\frac{2}{(\eta_s-\eta)}n^i\nabla_i](\Phi+\Psi) \\
& -\nabla^2\overline{B}_in^i-\nabla_{(i}\overline{B}_{j)}'n^in^j+\frac{2}{(\eta_s-\eta)}\nabla_{(i}\overline{B}_{j)}n^in^j+\nabla^2n_jF^{j\prime}+n^jn^i\nabla_{(i}F_{j)}'' \\
& -\frac{2}{(\eta_s-\eta)}\nabla_{(i}F_{j)}'n^jn^i+\frac{2}{(\eta_s-\eta)}n^in^jh_{ij}'-\nabla^2h_{ij}n^in^j\Big)d\eta'd\eta\Big]\,.
\end{aligned}
\tag{4.73}
$$

Converting the double integrals into single integral with $(\eta_o-\eta)=(\eta_s-\eta)$ results in

$$
\begin{aligned}
d_L(\mathbf{n},\eta_s) \;=\;& \frac{(\eta_o-\eta_s)}{a_s(\eta)}\Bigg[1+2\Psi_o+2\Phi_s-3\Psi_s+\mathcal{H}(B-E')+2n^iV_i-\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i\Phi d\eta \\
&+\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta-\eta_s)n^i\nabla_i(\Phi+\Psi)d\eta-2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta-\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Phi d\eta' d\eta \\
&+\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\Phi d\eta-\int n^in^j\left(\nabla_{(i}F'_{j)}+\frac{1}{2}h'_{ij}-\nabla_{(i}\overline{B}_{j)}\right)d\eta \\
&+\frac{1}{2}\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Bigg((\nabla^2-n^in^j\nabla_i\nabla_j)(\Phi+\Psi)-\nabla^2\overline{B}_in^i-\nabla_{(i}\overline{B}'_{j)}n^in^j \\
&-\frac{2}{(\eta-\eta_s)}\nabla_{(i}\overline{B}_{j)}n^in^j+\nabla^2n_jF^{j\prime}+\nabla_{(i}F''_{j)}n^jn^i+\frac{2}{(\eta-\eta_s)}\nabla_{(i}F'_{j)}n^jn^i-\frac{2}{(\eta-\eta_s)}n^in^jh'_{ij} \\
&-\nabla^2h_{ij}n^in^j\Bigg)d\eta\Bigg]\,, \qquad (4.74)
\end{aligned}
$$

where we have used Eq (2.113) as well. We can relate the peculiar velocities of the source to the Bardeen potential via the first-oder perturbations of Einstein's equations using $u^\mu = a^{-1}(1-\Phi, v^i)$ where we can get

$$
v^i(\eta) = -\frac{1}{4\pi G a^2(\rho+p)}\left(\frac{a'}{a}\nabla_i\Phi+\nabla_i\Phi'\right)\,. \qquad (4.75)
$$

Eq. (4.74) is the luminosity distance of a source in direction $\mathbf{n}$ at conformal time η_s. It is presented in a general gauge, with the scalar, vector and tensor modes together. Then the real observed luminosity distance will be given as

$$
\begin{aligned}
d_L(\mathbf{n},z_s) \;=\;& (1+z_s)\Bigg[(\eta_o-\eta_s)+(\eta_o-\eta_s-\mathcal{H}_s^{-1})\Psi_o-(2(\eta_o-\eta_s)-\mathcal{H}_s^{-1})\Psi_s \\
&+2(\eta_o-\eta_s)\Phi_s-\mathcal{H}_s^{-1}\mathcal{H}(B-E')+(\eta_o-\eta_s-\mathcal{H}_s^{-1})n^iV_i+2\int_{\eta_s}^{\eta_o}\Phi d\eta \\
&+\int_{\eta_s}^{\eta_o}(\eta_s-\eta)n^i\nabla_i(-3\Phi+\Psi)d\eta+(\eta_o-\eta_s-\mathcal{H}_s^{-1})\int_{\eta_s}^{\eta_o}n^i\nabla_i(-\Psi+\Phi)d\eta \\
&+\mathcal{H}_s^{-1}\int_{\eta_s}^{\eta_o}n^in^j\left(\nabla_{(i}F'_{j)}+\frac{1}{2}h'_{ij}-\nabla_{(i}\overline{B}_{j)}\right)d\eta \\
&+\frac{1}{2}\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Bigg([\nabla^2-n^in^j\nabla_i\nabla_j](\Phi+\Psi)-\nabla^2n^i(\overline{B}_i+F'_j)-n^in^j(\nabla_{(i}\overline{B}'_{j)}+\nabla_{(i}F''_{j)}) \\
&-\nabla^2h_{ij}n^in^j-\frac{2}{(\eta-\eta_s)}[\nabla_{(i}\overline{B}_{j)}+\nabla_{(i}F'_{j)}+\frac{1}{2}h'_{ij}]n^in^j\Bigg)d\eta\Bigg]\,, \qquad (4.76)
\end{aligned}
$$

where we have use Eq (2.125). It is equivalent to our expression of luminosity distance in PLG Eq. (4.72) which we expressed in one single line.

4.4.2 The Luminosity Distance in the Longitudinal Gauge

We are now going to rewrite an expression for the luminosity distance from Eq. (4.74) in longitudinal gauge, which it has been done in [121]. To do so, we will consider a perfect fluid where $\Phi = \Psi$. This gives

$$
\begin{aligned}
d_L(\mathbf{n},\eta_s) &= (1+z_s)(\eta_o-\eta_s)\Bigg[1+2\Psi_o-\Psi_s+2n^iV_i^o+2n^i\left(\frac{1}{4\pi Ga_s^2(\rho+p)(\eta_s)}\left(\mathcal{H}\nabla_i\Psi+\nabla_i\Psi'\right)(\eta_s)\right)\\
&+2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta-\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\Psi d\eta+\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Psi d\eta' d\eta\\
&+\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Big((\nabla^2-n^in^j\nabla_i\nabla_j)\Psi\Big)d\eta\Bigg]\,. \qquad (4.77)
\end{aligned}
$$

Using this and the expression for the redshift fluctuations

$$
\delta z_s=(1+z_s)\left[\Psi_o-\Psi_s+n^i(V_i^o-V_i^s)+2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta\right], \qquad (4.78)
$$

where we have used Eq (2.59), then

$$
\begin{aligned}
\left((1+z_s)^{-1}d_L+\mathcal{H}_s^{-1}\right)\delta z_s &= (1+z_s)[-(\eta_o-\eta_s)-\mathcal{H}_s^{-1}]\Big[\Psi_o\\
&-\Psi_s+n^i(V_i^o-V_i^s)+2\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta\Big]\,. \qquad (4.79)
\end{aligned}
$$

Then the redshift luminosity distance in the longitudinal gauge is

$$
\begin{aligned}
d_L(\mathbf{n},z_s) &= (1+z_s)\Big[(\eta_o-\eta_s)+[(\eta_o-\eta_s)-\mathcal{H}_s^{-1}]\Psi_o+\mathcal{H}_s^{-1}\Psi_s\\
&+[(\eta_o-\eta_s)-\mathcal{H}_s^{-1}]n^iV_i^o+n^i\left(\frac{(\eta_o-\eta_s-\mathcal{H}_s^{-1})}{4\pi Ga_s^2(\rho+p)(\eta_s)}\left(\mathcal{H}\nabla_i\Psi+\nabla_i\Psi'\right)\right)(\eta_s)\\
&-2\int_{\eta_s}^{\eta_o}\Psi d\eta+2\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}n^i\nabla_i\Psi d\eta' d\eta-2\mathcal{H}_s^{-1}\int_{\eta_s}^{\eta_o}n^i\nabla_i\Psi d\eta\\
&+\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Big((\nabla^2-n^in^j\nabla_i\nabla_j)\Psi\Big)d\eta\Big]\,. \qquad (4.80)
\end{aligned}
$$

We can re-write the above equation by substituting for each $n^i\nabla_i$ the terms $(\frac{d}{d\eta}-\partial_\eta)$ and with using of Eq. (4.75) again to obtain

$$
\begin{aligned}
d_L(\mathbf{n},z_s) &= (1+z_s)(\eta_o-\eta_s)\Big[1+(1-\frac{1}{(\eta_o-\eta_s)\mathcal{H}_s})\Psi_o+\frac{1}{(\eta_o-\eta_s)\mathcal{H}_s}\Psi_s\\
&+(1-\frac{1}{(\eta_o-\eta_s)\mathcal{H}_s})n^iV_i^o-(1-\frac{1}{(\eta_o-\eta_s)\mathcal{H}_s})n^iV_i^s-\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\Psi d\eta\\
&+\frac{2}{(\eta_o-\eta_s)\mathcal{H}_s}\int_{\eta_s}^{\eta_o}\Psi' d\eta-\frac{2}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}\Psi' d\eta' d\eta\\
&-\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}(\eta'-\eta_s)\Psi'' d\eta' d\eta+\int_{\eta_s}^{\eta_o}\frac{(\eta_o-\eta)(\eta-\eta_s)}{(\eta_o-\eta_s)}\nabla^2\Psi d\eta\Big]\,. \qquad (4.81)
\end{aligned}
$$

In this equation the first line, apart from the background contribution, contains what can be identified as "gravitational redshift". This is, however, not entirely correct since this term does not vanish even if $\Psi_s=\Psi_o$ [121]. The second line can be the terms due to peculiar motion of the observer and emitter (Doppler terms). The third and fourth lines collect integrated effects proportional to line-of-sight integrals of Ψ and its time derivative, and the fifth and last line represents the lensing term with $\nabla^2\Psi\propto\delta\rho$.

4.4.3 Vector Contributions to the Luminosity Distance

Vector perturbations they may serve as a source of cosmic strings or primordial magnetic fields [139, 140]. The vector contribution to the luminosity distance at η_s and from Eq. (4.74) we can be written as

$$\begin{aligned} d_L(\mathbf{n},\eta_s) &= \frac{(\eta_o-\eta_s)}{a_s(\eta)}\Bigg[1-\int n^i n^j(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)})d\eta \\ &+\frac{1}{(\eta_o-\eta_s)}\int_{\eta_s}^{\eta_o} n^i n^j(\eta_o-\eta)(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)})d\eta \\ &+\frac{1}{2}\int_{\eta_s}^{\eta_o}\frac{(\eta_o-\eta)(\eta-\eta_s)}{(\eta_o-\eta_s)}\Big(\nabla^2(F'_i-\overline{B}_i)n^i+\nabla_{(i}F''_{j)}n^i n^j-\nabla_{(i}\overline{B}'_{j)}n^i n^j\Big)d\eta\Bigg]\,. \end{aligned} \tag{4.82}$$

Using

$$\delta z_s = (1+\overline{z}_s)\bigg[-\int n^i n^j\Big(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)}\Big)d\eta\bigg]\,, \tag{4.83}$$

and

$$-\left((1+z_s)^{-1}d_L+\mathcal{H}_s^{-1}\right)\delta z_s = (1+z_s)\Big[(\eta_o-\eta_s+\mathcal{H}_s^{-1})\Big]\bigg[\int n^i n^j\Big(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)}\Big)d\eta\bigg]\,, \tag{4.84}$$

we can rewrite this contribution as

$$\begin{aligned} d_L(\mathbf{n},z_s) &= (1+z_s)\bigg[(\eta_o-\eta_s)+\mathcal{H}_s^{-1}\int n^i n^j\Big(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)}\Big)d\eta \\ &+\int_{\eta_s}^{\eta_o} n^i n^j(\eta_o-\eta)(\nabla_{(i}F'_{j)}-\nabla_{(i}\overline{B}_{j)})d\eta \\ &+\frac{1}{2}\int_{\eta_s}^{\eta_o}(\eta_o-\eta)(\eta-\eta_s)\Big((\nabla^2(F'_i-\overline{B}_i)n^i+\nabla_{(i}F''_{j)}n^i n^j-\nabla_{(i}\overline{B}'_{j)}n^i n^j\Big)d\eta\bigg]\,. \end{aligned} \tag{4.85}$$

This expression depends only on the gauge-invariant quantities F_i and B_i, and as expected it is a completely geometrical equation.

4.4.4 Tensor Contributions to the Luminosity Distance

Tensor perturbations are generically produced during inflation, and hence their contribution has to be added for completeness. A gravitational wave from a far away passing arbitrary object could generate a tensor perturbation added to the luminosity distance, and in principle could be detected [119]. The tensor contributions for an object at η_s and from Eq. (4.74) are given by

$$\begin{aligned} d_L(\mathbf{n},\eta_s) &= \frac{(\eta_o-\eta_s)}{a_s(\eta)}\bigg[1-\frac{1}{2}\int_{\eta_s}^{\eta_o} n^i n^j h'_{ij}d\eta-\int_{\eta_s}^{\eta_o} n^i n^j\frac{(\eta_o-\eta)}{(\eta_o-\eta_s)}h'_{ij}d\eta \\ &-\frac{1}{2}\int_{\eta_s}^{\eta_o} n^i n^j\frac{(\eta_o-\eta)(\eta-\eta_s)}{(\eta_o-\eta_s)}\nabla^2 h_{ij}d\eta\bigg]\,. \end{aligned} \tag{4.86}$$

And

$$\delta z_s = (1+\overline{z}_s)\bigg[-\frac{1}{2}\int_{\eta_s}^{\eta_o} n^i n^j h'_{ij}d\eta\bigg]\,, \tag{4.87}$$

for a measurable relation

$$-\left((1+z_s)^{-1}d_L+\mathcal{H}_s^{-1}\right)\delta z_s = (1+z_s)\Big[(\eta_o-\eta_s+\mathcal{H}_s^{-1})\Big]\bigg[\frac{1}{2}\int_{\eta_s}^{\eta_o} n^i n^j h'_{ij}d\eta\bigg]\,, \tag{4.88}$$

and hence

$$
\begin{aligned}
d_L(\mathbf{n}, z_s) \quad &= \quad (1+z_s)\Bigg[(\eta_o - \eta_s) - \frac{1}{2\mathcal{H}_s}\int_{\eta_s}^{\eta_o} n^i n^j h'_{ij} d\eta - \int_{\eta_s}^{\eta_o} n^i n^j (\eta_o - \eta) h'_{ij} d\eta \\
&\quad - \frac{1}{2}\int_{\eta_s}^{\eta_o} n^i n^j (\eta_o - \eta)(\eta - \eta_s)\nabla^2 h_{ij} d\eta \Bigg] . \qquad (4.89)
\end{aligned}
$$

The origin of the different terms in this relation is as follows: the first term is the unperturbed expression for the luminosity distance in a Friedmann-Lemaître universe at the observed redshift z_s, the second term derives from the redshift correction, the third term from the relation between the conformal time η and the affine parameter λ, and the last one we can interpret as a lensing effect. The first two terms come from the perturbation of the redshift.

4.4.5 The Shear on the Lightcone

The shear σ_{IJ} of the null geodesics generating the null cone is a measure of the rate of distortion down these geodesics. It is usually defined by projecting the covariant derivative of k_μ down the null geodesics into the "screen space" orthogonal to k^μ and the 4-velocity u^μ of the fundamental observers, and then subtracting off the trace [108]. The trace-free part is then the shear of the light rays:

$$
\hat{\sigma}_{IJ} = \left(\frac{r_A^2}{2\beta}\right)\delta_i^I \delta_j^J \frac{\partial f_{IJ}}{\partial y} . \qquad (4.90)
$$

The magnitude of the shear is $\hat{\sigma}$, determined by

$$
\hat{\sigma}^2 = \frac{1}{2}\sigma_{IJ} g^{IK} \sigma_{KL} g^{JL} . \qquad (4.91)
$$

Both r_A and f_{IJ} can, *in principle*, be directly observed, whereas the shear can only be determined from the knowledge of h_{IJ} [108]. From Eq. (4.90), where

$$
f_{IJ} = h_{IJ}/r_A^2 \quad \Rightarrow \quad det(f_{IJ}) = \sin^2\theta, \qquad (4.92)
$$

give an alternative representation of the quantities h_{IJ} as its conformal two-structure. So

$$
\begin{aligned}
\hat{\sigma}_{\theta\theta} \quad &= \quad \frac{S^2}{2}\left(\partial y H_{\theta\theta} - \frac{1}{2}\partial y H^T\right), \qquad (4.93) \\
\hat{\sigma}_{\phi\phi} \quad &= \quad \frac{S^2}{2}\left(\partial y H_{\phi\phi} - \frac{1}{2}\partial y H^T\right), \qquad (4.94) \\
\hat{\sigma}_{\theta\phi} \quad &= \quad \frac{S^2}{2}\left(\partial y H_{\theta\phi}\right) . \qquad (4.95)
\end{aligned}
$$

As the size and shape of the image are independent of the motion of the observer, one may carry out this calculation for the particular observer moving with zero proper motion and zero redshift; the corresponding 4-velocity will then be $u^w = \frac{1}{a(\eta)}\Big[1 + \partial_w \hat{\delta} w - \phi + v^\chi\Big]$, and the projection into the screen space, defined by its components $h_{\hat{I}\hat{J}} = \delta_i^{\hat{I}}\delta_j^{\hat{J}} g_{\hat{I}\hat{J}}$. From these results it follows that while $h_{\hat{I}\hat{J}}$, and so both r_A and $f_{\hat{I}\hat{J}}$, can in principle be directly observed, the null geodesic expansion $\hat{\theta}$ and shear $\hat{\sigma}_{ab}$ can only be determined from this knowledge of $h_{\hat{I}\hat{J}}$ if the metric component β is known, this implies that $\hat{\theta}$ and shear $\hat{\sigma}_{ab}$ are not directly observable.

Chapter 5

The Galaxy Number Count

If what appears little be universally despised, nothing greater can be attained; for all that is great was at first little, and rose to its present bulk by gradual accessions and accumulated labours.

Samuel Johnson

The large-scale cosmic structure contains lots of information about the global properties of our Universe, and by analysing maps of galaxies we can probe the initial conditions of the Big Bang and its physical processes that have operated subsequently [141, 142]. Galaxy clusters are the largest bound structures in the Universe. Clusters vary from groups of tens of galaxies to more than 1000 galaxies. They represent nonlinear overdensities. Statistical measurements of galaxy motions and clustering with the weak gravitational lensing, provide some of the strongest evidence to date that Einstein's GR is an accurate description of gravity on cosmological scales. However, what are not yet exactly known are the content and the mechanisms responsible for the Universe's acceleration that put GR in an increasingly more awkward position.

5.1 The Galaxy Surveys

Galaxies are the building blocks which define the large-scale distribution of visible matter in the Universe and it can be used to trace the underlying dark matter distribution. Without dark matter, galaxy formation would occur substantially later in the Universe than it is observed. After this all dark matter ripples could grow freely, forming seeds into which the baryons could later fall. Such information requires a combination of the galaxies' location in three dimensions and distance information from its redshift [143].

From the analysis of the SDSS data, it has been concluded that the irregularities in the galaxy density are still on the level of a few percent on scales of $100h^{-1}$Mpc. Also, about 10% of the observed galaxies are found in gravitationally-bound clusters, such as the nearby Virgo and Coma clusters [144–146]. It also seems to exist in larger structure forms, such as superclusters with densities about twice the average density of the Universe. Recent large galaxy surveys have been performed with SDSS, and the 2dF Galaxy Redshift Survey that measured the spectra of hundreds of thousands of objects and obtained precise three-dimensional maps of the deep sky about the distribution of matter in the Universe [147, 148]. The latest generation of the SDSS (SDSS-IV, 2014-2020) will extend its precision by expanding its revolutionary infrared spectroscopic survey of the Galaxy in the northern and southern hemispheres, with the collaboration of the extended Baryon Oscillation Spectroscopic Survey (eBOSS) and APO Galaxy Evolution Experiment 2 (APOGEE-2). Spatially resolved maps of individual galaxies Mapping Nearby Galaxies at APO (MaNGA) will be made using the Sloan spectrographs [149].

The galaxy survey map could be a good test for models beyond ΛCDM models like modified gravity models and dark energy models.

5.1.1 The Real Measure in Galaxy Surveys

What we measure from galaxy surveys is an incomplete picture because what we actually measure are the galaxy redshifts (z) and the sky positions (θ, ϕ), we cannot provide the true positions of the galaxies [150]. If the Universe were completely homogeneous and isotropic, where light propagates on straight lines, we could know the true position of galaxies directly. But we know that the Universe is not homogeneous and isotropic, and there are small but significant inhomogeneities between us and other galaxies that distort our coordinate systems, as well as change the redshift and the photons' propagation slightly. They also change the shape and the size of the image and therefore we see a distorted image of the galaxies and hence we see a distorted version of the dark matter distribution. That effects the galaxy number count and the convergence κ (or the magnification). To interpret the observed data properly, we need to understand these distortions. Two well-known examples are the redshift-space distortions due to the peculiar velocity of the galaxies and the gravitational lensing which affects the size and the shape of the object's image.

5.1.2 The Galaxy Bias

We cannot always assume in advance that any given class of galaxy traces cosmological density in an unbiased fashion, and we should therefore look at the properties of a variety of tracers. The galaxy bias is the relationship between the spatial distribution of galaxies and the underlying dark matter density field (matter density contrast)

$$\delta_g(z,k) = b(z,k)\delta_m(z,k) \;, \tag{5.1}$$

known as local bias. Here we define the linear galaxy bias b as the ratio of the mean galaxy number density contrast to the mean density contrast of dark matter. And the scale-independent local bias is defined as

$$\delta_g(z,k) = b(z)\delta_m(z,k) \;. \tag{5.2}$$

A scale independent galaxy bias factor b assumed in one gauge appears as a scale dependent galaxy bias factor $b(k)$ in another gauge. This translates into a perfect degeneracy in the power spectrum:

$$P_g(z,k) = b^2(z)P_m(z,k) \;. \tag{5.3}$$

The empirical degree of bias today is probably small, but naturally larger at earlier epochs of galaxy formation. At high redshifts, the first galaxies to form will be the first structures to collapse, which will be biased tracers of the mass. The galaxy bias is expected to be a strong function of redshift, initially $b > 1$ at high redshift and approaching unity over time. This is consistent with the idea that galaxies formed early on in the most overdense regions of space, which are biased. For $\Omega \leq 0.3$ would imply that optically-selected galaxies are anti-biased, where $b < 1$, indicating that galaxies are less clustered than the dark matter distribution, whereas substantial bias must exist for $\Omega = 1$ models, for which there is a strongly non-monotonic function of scale. The real-space power spectra show only a weak scale-dependent relative bias of $b \simeq 1.15$ on large scales (scales larger than 60 Mpc) and tend towards a constant value, increasing to $b \simeq 1.5$ on the smallest scales (scales smaller than 30 Mpc).

The galaxy bias of a given observational sample is often inferred by comparing the observed clustering of galaxies with the clustering of dark matter measured in a cosmological simulation.

Therefore, the bias depends on the cosmological model used in the simulation. A very important cosmological parameter is σ_8, defined as the standard deviation of galaxy count fluctuations in a sphere of radius $8h^{-1}$ Mpc, and the absolute bias value inferred can be simply scaled with the assumed value of σ_8. The relative bias as a function of scale can be defined as the square root of the ratio of the power spectra [151–156].

5.2 Galaxy Number Count

The number count is what we can measure when we divide the map of galaxy surveys with beams at fixed redshift and solid angle, see Fig. [5.1]. By counting the galaxies in each pixel separately we can study the fluctuation of the galaxy number and the distribution of dark matter, *i.e.*, if there is no galaxy in the region, that means no dark matter there. The geometric properties of spacetime

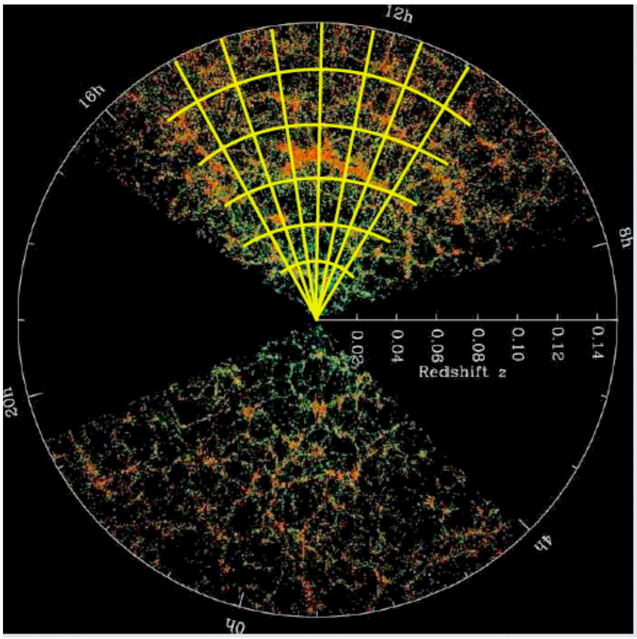


Figure 5.1: The number count of galaxies per pixel at fixed redshift and solid angle.

play a role in the determination of the distribution of galaxies. The *number count* is the number of these galaxies observed in a given solid angle $d\Omega$ in the distance range $(r, r + dr)$, the corresponding volume of which is given by

$$dV = a^3(\eta)r^2 dr d\Omega \ . \tag{5.4}$$

If the source number density is $n(\eta_s)$ and the probability of detection is P, the number of sources observed is

$$dN = P\left[\frac{n(\eta)}{(1+z)^3}\right]a^3(\eta)f^2(r)drd\Omega \ , \tag{5.5}$$

where the quantity in brackets is constant if source numbers are conserved in a FLRW model, and dr can be expressed in terms of observable quantities such as dz. f is the *selection function* representing the fraction of galaxies in dV that are actually detected and included in the number count. In general f depends on η, r, θ and ϕ. We can estimate f from the knowledge of the galactic brightness distribution and spectrum and redshift z, and the detection limit, there are many undetectable objects in the sky, including entire galaxies, because they lie below the detection line.

5.2.1 Distortions in Galaxy Number Counts

The observed redshift and position of galaxies are affected by the matter fluctuations and the gravity waves between the source galaxies and the observer. The volume element constructed using the observed redshift and observed angle is different from the real physical volume occupied by the observed galaxies. The observed flux and redshift of the source galaxies are also different from their intrinsic properties. Therefore, the observed galaxy fluctuation field contains additional contributions arising from the distortion in observable quantities and these include tensor contributions as well as numerous scalar contributions [157].

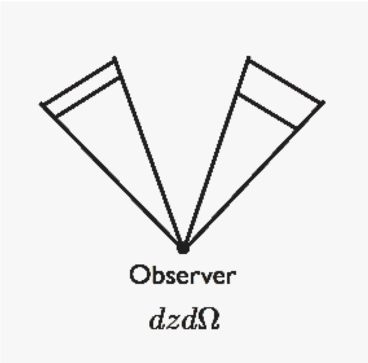


Figure 5.2: Same redshift bin different physical volume: Where in one of the pixel all the galaxies moving toward the centre of the pixel.

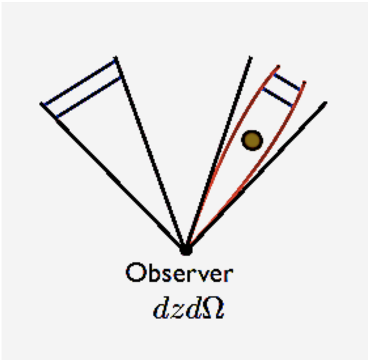


Figure 5.3: Same solid angle different physical volume: An overdensity cluster bending the light beams emitted by the galaxies.

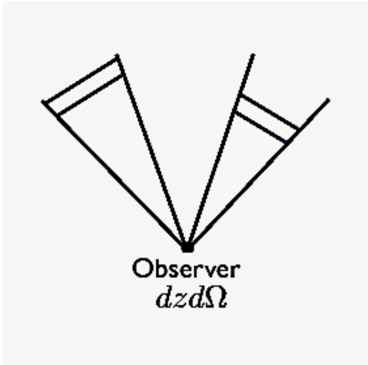


Figure 5.4: Same radial bin different distance: The whole galaxies in the pixel is moving toward the observer.

Credit: C. Bonvin.

In a galaxy redshift survey, at redshift z the observer looks at different part of the sky and selects two pixels in two different directions with the same solid angle and redshift bin to measure the number of galaxies $N(\mathbf{n}, z)d\Omega_{\mathbf{n}}dz$ in each of them. By averaging over the solid angle he can obtain the redshift distribution $\langle N\rangle(z)dz$ of these galaxies. This observer can experience different cases of distortions in the galaxy number count:

- If one of these pixels is inside an overdensity region, that will make all the galaxies move toward the centre of the pixel and the number of galaxies in the pixel will squeeze in size and therefore the observer will count two different galaxy number in two different physical volume with the same solid angle and redshift bin, see Fig. [5.2].
- In Fig. [5.3], a case where the same observer have the same redshift bin and solid angle of the two pixels, but an overdensity cluster is observed in the way between one of the pixels and the observer. The overdensity cluster will bend the light beams coming from the galaxies changing the size of the solid angle, and the observer will see a different physical volume between the two pixels.
- Fig. [5.4] shows a case where in one of the pixels all galaxies are moving in the same direction towards the observer. Due to the expansion, the galaxy number will be diluted and the observer will see fewer galaxies in that pixel.

Therefore, the observed galaxy number density is affected by perturbations given the total number of observed galaxies, and it contains additional contributions from the distortions in the observable quantities, compared to the standard description that galaxies simply trace the underlying matter distribution.

5.2.2 The Galaxy Number Count on the Lightcone Gauge

Suppose one counts the galaxies seen in a solid angle $d\Omega_0$ around the direction of observation $(\hat{\theta}, \hat{\phi})$, down to a distance y. An increment from $y \longrightarrow y + dy$ will result in including dN new galaxies in the count, where dN is the number of galaxies detected in a volume of size as $(dy, d\hat{\theta}, d\hat{\phi})$ around a point on our past lightcone

$$dV = (r_A^2 d\Omega)(u^\mu k_\mu d\nu) \tag{5.6}$$

corresponding to the range $(y \longrightarrow y + dy)$ in the coordinate y. If the number density of galaxies at the position y is n, then $(n\ dV)$ is the number of galaxies that will be contained in this volume. However there is a chance that not all of the galaxies will be counted in dV, in general, for some will be too faint to be detected, while others may not be selected for inclusion in the galactic number count because of various selection effects (e.g. they may be confused with stars if their images are very small) [108]. We will write dN in the form

$$dN = f_m dV \ , \tag{5.7}$$

where f_m is the selection function representing the fraction of galaxies in dV that are actually detected and included in the number count; one can estimate f_m from knowledge of the galactic brightness distribution and spectrum, the area distance r_A and redshift z. In general, f_m will depend on $w, y, \hat{\theta}$ and $\hat{\phi}$. The number count of galaxies in a box of size $(dy, d\hat{\theta}, d\hat{\phi})$ around a point on our past lightcone can also be calculated as

$$dN = f_m r_A^2 (1+z) d_{\Omega_0} \beta dy \ , \tag{5.8}$$

where $d\nu = \beta dy$, and $\beta = a^2(1 + \delta\beta)$. If y has been chosen to be an observable quantity, then dN is directly measurable. As z and r_A are known, one can estimate the selection function f_m, which depends on r_A, z, the galaxy properties and the observational limits and selection effects. Therefore, *in principle*, one could determine the quantity β in terms of known quantities.

5.3 The Perturbation of Galaxy Number Counts Δ

The galaxy number density contrast in one pixel can be given by

$$\Delta = \frac{N - \bar{N}}{\bar{N}} = b \cdot \frac{\delta\rho}{\rho} = b \cdot \delta \ . \tag{5.9}$$

The number count Δ is an observable quantity; it relates the number of the galaxies in each pixel to the average numbers of the galaxies $\bar{N}$, and the distribution of dark matter δ and its bias b. Since the density of the Universe is changing with time, it is useful to express the magnitude of density variation using the relative density fluctuation, which is defined as follows:

$$\frac{\delta\rho}{\rho} = \frac{\text{density within a fluctuation} - \text{mean density of the Universe}}{\text{mean density of the Universe}} \ . \tag{5.10}$$

5.3.1 The Δ Calculations in Redshift Space

Galaxy formation is a local process and its relation to the underlying matter density should be well defined and gauge invariant. The observable quantities such as observed galaxy counting should be independent of a choice of the gauge condition. The large-scale distribution of galaxies, the density fluctuation $\delta(\mathbf{x}, t)$ which we calculate in a given Friedmann background, is not a gauge invariant, this is "the cosmological gauge problem" [38]. Since it depends on the background Friedmann universe we compare the observed $\rho(x,t)$ with [150].

$$\delta(\mathbf{x},t) \equiv \frac{\rho(\mathbf{x},t)-\bar{\rho}(t)}{\bar{\rho}(t)} \; . \tag{5.11}$$

In order to fix this problem, one has to consider individual observational effects like the redshift space distortions [158, 159], the Alcock-Pacinski [160] or lensing [150, 161].

For *unbiased* distribution as we will consider for this section, $\bar{\rho}$, is the mean galaxy density, i.e.,

$$\bar{\rho} = \langle \rho \rangle \; . \tag{5.12}$$

The physical survey volume density per redshift bin per solid angle can be written as a background part in a homogeneous world and a fluctuated quantity, since the solid angle and the redshift bin are distorted between the source and the observer:

$$V(\mathbf{n},z) = V(z) + \delta V(\mathbf{n},z) \; . \tag{5.13}$$

Then following the discussion in [150], the redshift density perturbation can be written as

$$\begin{aligned}
\delta_z(\mathbf{n},z) &= \frac{\rho(\mathbf{n},z)-\langle\rho\rangle(z)}{\langle\rho\rangle(z)} \, , && (5.14)\\
&= \frac{\frac{N(\mathbf{n},z)}{V(\mathbf{n},z)}-\frac{\langle N\rangle(z)}{V(z)}}{\frac{\langle N\rangle(z)}{V(z)}} \, , && (5.15)\\
&= \left(\frac{N(\mathbf{n},z)V(z)}{V(z)+\delta V(\mathbf{n},z)} - \langle N\rangle(z)\right)\left(\frac{1}{\langle N\rangle(z)}\right) \, , && (5.16)\\
&= \left(N(\mathbf{n},z) - N(\mathbf{n},z)\frac{\delta V(\mathbf{n},z)}{V(z)} - \langle N\rangle(z)\right)\left(\frac{1}{\langle N\rangle(z)}\right) \, , && (5.17)\\
&= \frac{N(\mathbf{n},z)-\langle N\rangle(z)}{\langle N\rangle(z)} - \frac{N(\mathbf{n},z)\delta V(\mathbf{n},z)}{V(z)\langle N\rangle(z)} \, , && (5.18)\\
&= \frac{N(\mathbf{n},z)-\langle N\rangle(z)}{\langle N\rangle(z)} - \frac{\Big(\langle N\rangle(z)+\delta N(\mathbf{n},z)\Big)\delta V(\mathbf{n},z)}{V(z)\langle N\rangle(z)} \, , && (5.19)\\
&= \frac{N(\mathbf{n},z)-\langle N\rangle(z)}{\langle N\rangle(z)} - \frac{\delta V(\mathbf{n},z)}{V(z)} \, . && (5.20)
\end{aligned}$$

The perturbation in the number density of galaxies is an observed quantity, and the volume perturbation also can be measured with other tracers than galaxies, and it is therefore measurable by itself and hence gauge invariant [150]. Therefore they are gauge-invariant quantities. And hence $\delta_z(\mathbf{n},z)$ is a gauge invariant. We can re-write the above expression as

$$\Delta(\mathbf{n},z) = \frac{N(\mathbf{n},z)-\langle N\rangle(z)}{\langle N\rangle(z)} = \delta_z(\mathbf{n},z) + \frac{\delta V(\mathbf{n},z)}{V(z)} \, , \tag{5.21}$$

which is a gauge-invariant expression.

5.3.2 The Computation of $\delta_z(\mathbf{n},z)$

We need to relate $\delta_z(\mathbf{n},z) \to \delta(\mathbf{x},t)$ to a gauge-independent quantity. So to first order

$$\begin{aligned}
\delta_z(\mathbf{n},z) &= \frac{\rho(\mathbf{n},z)-\overline{\rho}(z)}{\overline{\rho}(z)} = \frac{(\overline{\rho}(\overline{z})+\delta\rho(\mathbf{n},z))-\overline{\rho}(z)}{\overline{\rho}(z)} \, ,\\
&= \frac{(\overline{\rho}(z-\delta z)+\delta\rho(\mathbf{n},z))-\overline{\rho}(z)}{\overline{\rho}(z)} \, . && (5.22)
\end{aligned}$$

Taylor expansion of $\overline{\rho}(z)$ around $\overline{z}$ gives

$$\overline{\rho}(z) = \overline{\rho}(\overline{z}+\delta z)\ , \tag{5.23}$$

$$= \overline{\rho}(\overline{z}) + \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))\ . \tag{5.24}$$

Using the fact that $z = \overline{z} + \delta z$ and Eq. (5.23) in Eq. (5.22) yields

$$\delta_z(\mathbf{n}, z) = \frac{\overline{\rho}(\overline{z}) + \delta\rho(\mathbf{n}, z) - \overline{\rho}(z)}{\overline{\rho}(\overline{z}) + \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))}\ , \tag{5.25}$$

$$= \frac{\delta\rho(\mathbf{n}, z) - \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z}) + \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))}\ , \tag{5.26}$$

$$= \frac{\delta\rho(\mathbf{n}, z)}{\overline{\rho}(\overline{z}) + \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))} - \frac{d\overline{\rho}}{d\overline{z}}\frac{\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z}) + \frac{d\overline{\rho}}{d\overline{z}}\delta z(\mathbf{n}, z(\eta))}\ , \tag{5.27}$$

$$= \frac{\delta\rho(\mathbf{n}, z)}{\overline{\rho}(\overline{z})}\left(1 - \frac{d\overline{\rho}}{d\overline{z}}\frac{\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z})}\right) - \frac{d\overline{\rho}}{d\overline{z}}\frac{\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z})}\left(1 - \frac{d\overline{\rho}}{d\overline{z}}\frac{\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z})}\right)\ . \tag{5.28}$$

Thus, to first order

$$\delta_z(\mathbf{n}, z) = \delta(\mathbf{n}, z) - \frac{d\overline{\rho}}{d\overline{z}}\frac{\delta z(\mathbf{n}, z(\eta))}{\overline{\rho}(\overline{z})}\ , \tag{5.29}$$

where $\frac{\delta\rho(\mathbf{n},z)}{\overline{\rho}(\overline{z})} = \delta(\mathbf{n}, z)$, and by using

$$\frac{d\overline{\rho}}{d\overline{z}} = -3\frac{\overline{\rho}}{1+\overline{z}}\ , \tag{5.30}$$

the matter fluctuation (at the observed redshift) is given by

$$\delta_z(\mathbf{n}, z) = \delta(\mathbf{n}, z) + 3\frac{\delta z(\mathbf{n}, z(\eta))}{(1+\overline{z})}\ . \tag{5.31}$$

Here we relate the perturbation variables in direction $\mathbf{n}$ at redshift z to their unperturbed position and time η. $\overline{z} = \overline{z}(\eta)$ is the redshift of the background universe that we measure on and δz is the redshift perturbation to this universe, $\delta_z(\mathbf{n}, z) = \delta(\mathbf{n}, z)$ in a uniform-redshift frame $\delta z = 0$. It is gauge invariant since is defined by observable quantities, where the time slicing is set by the observed redshift z, rather than by an arbitrary choice of coordinate systems or gauge conditions as for $\delta(\mathbf{n}, z)$ and differs in its value contingent upon the gauge choice [162].

Moreover, by solving the background relation $\overline{z} = \overline{z}(\eta)$, we can write

$$\rho(\mathbf{n}, \overline{z}(\eta)) = \overline{\rho}(\eta) + \delta\rho(\mathbf{n}, \eta)\ . \tag{5.32}$$

Note that $\overline{\rho}(z) = \overline{\rho}(\overline{z} + \delta z)$ deviates to first order from $\overline{\rho}(\overline{z})$. Both δz and $\delta\rho$ depend on the chosen background and are, hence, gauge dependent; however their combination in Eq. (5.29) must turn out to be gauge invariant as it is *in principle* observable.
Gauge-invariant density fluctuations and velocity perturbations can be found by combining δ, v and v_i with metric perturbations. We shall use the Bardeen relations [112]

$$V \equiv v - E' = v^{long}\ , \tag{5.33}$$

$$D_s \equiv \delta + 3(1+w)\mathcal{H}(E' - B) \equiv \delta^{long}\ , \tag{5.34}$$

$$\begin{aligned} D &\equiv \delta^{long} + 3(1+w)\mathcal{H}V = \delta + 3(1+w)\mathcal{H}(v - B) \\ &= D_s + 3(1+w)\mathcal{H}V\ , \end{aligned} \tag{5.35}$$

$$\begin{aligned} D_g &\equiv \delta + 3(1+w)(\psi) = \delta^{long} - 3(1+w)\Psi \\ &= D_s + 3(1+w)\Psi\ . \end{aligned} \tag{5.36}$$

Here v^{long} and δ^{long} are the velocity and density perturbations in the longitudinal gauge and D_g is density fluctuation on the uniform curvature hypersurface. On sub-horizon scales the differences between $\delta, \delta^{Long}, D_g$ and D are negligible as are the differences between v and V. Since measurements of density and velocity perturbations can only be made on sub-horizon scales, we may therefore use any of the gauge-invariant perturbation variables to compare with measurements [38].

If we want to introduce a bias between the matter density and the galaxy density, it would probably be most physical to assume that both galaxies and dark matter follow the same velocity field, as they experience the same gravitational acceleration. We then expect that biasing should be applied to the density fluctuation in comoving gauge D and not to D_g. On small scales, such differences are irrelevant, but on large scales they do become relevant, as becomes clear when considering the (linear) power spectra for the different density fluctuation variables [150].

5.3.3 The Volume Distortion

The volume perturbation $\frac{\delta V}{V}$ should be gauge invariant, because it is, *in principle*, a measurable quantity given unbiased volume tracers. The differential volume element (seen by a source with 4-velocity u^μ) is given by

$$dV = \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}u^\mu dx^\nu dx^\alpha dx^\beta \ . \tag{5.37}$$

In terms of the observed redshift z and sky position determined by the polar angles (θ, ϕ) at the observation time,

$$\begin{aligned} dV &= \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}u^\mu \frac{\partial x^\nu}{\partial z}\frac{\partial x^\alpha}{\partial \theta_s}\frac{\partial x^\beta}{\partial \varphi_s}\left|\frac{\partial(\theta_s,\varphi_s)}{\partial(\theta_o,\varphi_o)}\right| dzd\theta_o d\varphi_o \ , \\ &= v(z,\theta_o\varphi_o)dzd\theta_o d\varphi_o \ , \end{aligned} \tag{5.38}$$

where v is a *volume density*, which determines the volume perturbation

$$\frac{\delta V}{V} = \frac{v(z) - \overline{v}(z)}{\overline{v}(\overline{z})} = \frac{\delta v}{\overline{v}} \ . \tag{5.39}$$

The determinant of the Jacobian matrix is

$$\left|\frac{\partial(\theta_s,\varphi_s)}{\partial(\theta_o,\varphi_o)}\right| = \det \mathbf{J} = \begin{bmatrix} \frac{\partial z}{\partial z} & \frac{\partial z}{\partial \theta_o} & \frac{\partial z}{\partial \varphi_o} \\ \frac{\partial \theta_s}{\partial z} & \frac{\partial \theta_s}{\partial \theta_0} & \frac{\partial \theta_s}{\partial \varphi_o} \\ \frac{\partial \varphi_s}{\partial z} & \frac{\partial \varphi_s}{\partial \theta_o} & \frac{\partial \varphi_s}{\partial \varphi_0} \end{bmatrix} \ , \tag{5.40}$$

and gives the transformation matrix from the angles at the source to the angles at the observer. In homogeneous and isotropic backgrounds, the geodesics are straight lines, that is,

$$\theta_s = \theta_o \ , \varphi_s = \varphi_o \ . \tag{5.41}$$

But in a perturbed universe, angles are perturbed with respect to

$$\theta_s = \theta_0 + \delta\theta \ , \quad \varphi_s = \varphi_o + \delta\varphi \ . \tag{5.42}$$

Thus using (5.42) in the Jacobian expression given in (5.40), to first order of Taylor expansion, one gets

$$\left|\frac{\partial(\theta_s,\varphi_s)}{\partial(\theta_o,\varphi_o)}\right| = 1 + \frac{\partial\delta\theta}{\partial\theta} + \frac{\partial\delta\varphi}{\partial\varphi} \ . \tag{5.43}$$

We discussed in the last two subsections a general simplification steps for computing the two terms of Eq. (5.21). We can apply them to any spacetime metric. In [150] they calculated the perturbed galaxy number counts using the perturbed FLRW metric. In the up coming section we are going to compute the perturbed galaxy number counts using our perturbed PLG metric and use the simplified steps we just introduced above. Then we will do a coordinate transformation to our final answer to the perturbed FLRW space and compare it with what [150] obtained.

5.4 Galaxy Number Counts Fluctuations with the PLG

Let us now consider the geodesic of a photon emitted from a galaxy in our lightcone background which arrives to us at the vertex of our lightcone, moving in straight null-like vectors pointed in direction $\mathbf{n}$ (hence, to lowest order, it is seen under the direction $-\mathbf{n}$ from the observer). The observer receives the photon redshifted by

$$1 + z = u^w = \frac{1}{a}(1 + \frac{\delta\alpha}{2}) = (1 + \bar{z})(1 + \frac{\delta\alpha}{2}) \ . \tag{5.44}$$

Since

$$\delta z = z - \overline{z} \tag{5.45}$$

and

$$\delta z = \frac{1}{2}(1 + \bar{z})\delta\alpha \ , \tag{5.46}$$

we have

$$\delta_z(\mathbf{n}, z) = \delta_m(\mathbf{n}, z) + \frac{3}{2}\delta\alpha \ . \tag{5.47}$$

The volume perturbation in terms of redshift and sky position is determined by our observation coordinates $(w, y, \hat{\theta}, \hat{\phi})$, and therefore

$$v \quad = \quad \sqrt{-g}\epsilon_{0123}u^w \frac{\partial y}{\partial z}\frac{\partial \hat{\theta}}{\partial \hat{\theta}_s}\frac{\partial \hat{\phi}}{\partial \hat{\phi}_s} + \sqrt{-g}\epsilon_{1230}u^y \frac{\partial w}{\partial z}\frac{\partial \hat{\theta}}{\partial \hat{\theta}_s}\frac{\partial \hat{\phi}}{\partial \hat{\phi}_s} \ , \tag{5.48}$$

where the components of ϵ satisfy

$$\epsilon_{0321} = \epsilon_{0231} = \epsilon_{0312} = \epsilon_{0132} = \epsilon_{0213} = \epsilon_{0123} = 1, \tag{5.49}$$

$$\epsilon_{1320} = \epsilon_{1230} = \epsilon_{1302} = \epsilon_{1203} = \epsilon_{1023} = \epsilon_{1032} = -1 \ , \tag{5.50}$$

and the angles between the source and the observer are fixed, it can be shown that there is no angular displacement

$$\delta\hat{\theta} = \delta\hat{\phi} = 0 \ . \tag{5.51}$$

That will lead to the volume perturbation being given as

$$v \quad = \quad \sqrt{-g}u^w \frac{\partial y}{\partial z} - \sqrt{-g}u^y \frac{\partial w}{\partial z} \ . \tag{5.52}$$

$$\tag{5.53}$$

With constant light cone $w|_{cst}$, we get

$$v = \sqrt{-g}u^w \frac{\partial y}{\partial z} \ . \tag{5.54}$$

Furthermore

$$\sqrt{-g} = a^4 S^2 \sin(\theta)\left(1 + \frac{H^T}{2} + \delta\beta\right) \ , \tag{5.55}$$

and the 4-velocity of the source according to the observational coordinates

$$u = \{(1 + z), n_i v^i\} \ , \tag{5.56}$$

where $v^i \equiv v^y, v^{\hat{\theta}}$ and $v^{\hat{\phi}}$. Since $\frac{dy}{dz}$ is the change in comoving distance y with redshift along the photon geodesic, we can re-write it as

$$
\begin{aligned}
\frac{dy}{dz} &= \frac{d(\overline{y}+\delta y)}{d(\overline{z}+\delta z)} = \frac{d[\overline{y}(1+\frac{\delta y}{\overline{y}})]}{d[\overline{z}(1+\frac{\delta z}{\overline{z}})]} \, , \\
&= \frac{d\overline{y}(1+\frac{\delta y}{\overline{y}}) + \overline{y}d(1+\frac{\delta y}{\overline{y}})}{d\overline{z}(1+\frac{\delta z}{\overline{z}}) + \overline{z}d(1+\frac{\delta z}{\overline{z}})} \, , \\
&= \frac{d\overline{y}}{d\overline{z}+d\delta z} + \frac{d\delta y}{d\overline{z}+d\delta z} \, , \\
&= \frac{d\overline{y}}{d\overline{z}}(1-\frac{d\delta z}{d\overline{z}}) + \frac{d\delta y}{d\overline{z}}(1-\frac{d\delta z}{d\overline{z}}) \, , \\
&= \frac{d\overline{y}}{d\overline{z}} - \frac{d\overline{y}}{d\overline{z}}\frac{d\delta z}{d\overline{z}} + \frac{d\delta y}{d\overline{z}} \, .
\end{aligned}
\tag{5.57}
$$

Using the fact that $\overline{y} = \chi$, we can rewrite the above result as

$$
\frac{dy}{dz} = \frac{d\chi}{d\overline{z}} - \frac{d\chi}{d\overline{z}}\frac{d\delta z}{d\overline{z}} + \frac{d\delta y}{d\overline{z}} = \left(\frac{d\chi}{d\eta} - \frac{d\chi}{d\overline{z}}\frac{d\delta z}{d\eta} + \frac{d\delta y}{d\eta}\right)\frac{d\eta}{d\overline{z}} \, . \tag{5.58}
$$

Here dy/dz is to be understood as the change in comoving distance y with respect to the redshift along the photon geodesic. The distinction between z and $\overline{z}$ is only relevant for background quantities.

Appling our reslut above to Eq. (5.54), then it will look like

$$
v(z) = a^4 S^2 \sin(\theta)\left(1+\frac{H^T}{2}+\delta\beta\right)(1+z)\left(\frac{d\chi}{d\eta} - \frac{d\chi}{d\overline{z}}\frac{d\delta z}{d\eta} + \frac{d\delta y}{d\eta}\right)\frac{d\eta}{d\overline{z}} \, . \tag{5.59}
$$

In Eq. (5.58)), the last term contains the redshift-space distortion, which will turn out to be the biggest correction to the power spectrum [150]. To lowest order along the photon geodesic, with $1+z = \frac{a_0}{a} = \frac{1}{a}$, we have

$$
\begin{aligned}
\frac{d\eta}{d\overline{z}} &= \frac{d\eta}{d\overline{z}}\frac{da}{da} \, , && (5.60)\\
&= \frac{d\eta}{da}\frac{da}{d\overline{z}} = \frac{d\eta}{da}\frac{d}{d\overline{z}}\left(\frac{1}{1+\overline{z}}\right) \, , && (5.61)\\
&= -\frac{1}{a'}(1+\overline{z})^{-2} = -\frac{a}{a'}(1+\overline{z})^{-1} \, , && (5.62)\\
&= -a\mathcal{H}^{-1} = -H^{-1} \, . && (5.63)
\end{aligned}
$$

With all the above taken into account, the volume element becomes

$$
v(z) = -a^4 S(y)^2 \sin(\theta)\left(1+\frac{H^T}{2}+\delta\beta\right)\left(1+\frac{1}{2}\delta\alpha\right)\left(\frac{d\chi}{d\eta} - \frac{1}{\mathcal{H}(1+\overline{z})}\frac{d\delta z}{d\eta} + \frac{d\delta y}{d\eta}\right)\mathcal{H}^{-1} \, , \tag{5.64}
$$

or

$$
v(z) = \frac{a^4 S(y)^2 \sin(\theta)}{\mathcal{H}}\left(-1 + \frac{1}{\mathcal{H}(1+\overline{z})}\frac{\partial\delta z}{\partial y} - \frac{\partial \delta y}{\partial y} - \frac{H^T}{2} - \delta\beta - \frac{1}{2}\delta\alpha\right) \, , \tag{5.65}
$$

where we have used the relations (3.87). Furthermore, if we introduce the volume density as

$$
\frac{\delta v}{\overline{v}} = \frac{v(z) - \overline{v}(z)}{\overline{v}(\overline{z})} \, , \tag{5.66}
$$

and $\overline{v}(z)$ is given by the Taylor expansion of $\overline{v}(z)$ around $\overline{z}$, we get

$$\overline{v}(z) = \overline{v}(\overline{z} + \delta z) \ , \tag{5.67}$$

$$= \overline{v}(\overline{z}) + \frac{d\overline{v}}{d\overline{z}}\delta z \ , \tag{5.68}$$

$$\overline{v}(z) = \overline{v}(\overline{z}) + \frac{d\overline{v}(\overline{z})}{d\overline{z}}\delta z(\mathbf{n}, z) \ . \tag{5.69}$$

To obtain the fluctuation of v just subtract the unperturbed part $\overline{v}(z)$ from v of Eq.(5.65) (and additional $1/a$ factor coming from the background part of [1+z] term), using $a = 1/(\overline{z}+1)$

$$\overline{v}(\overline{z}) = \frac{S(y)^2 \sin(\theta)}{(1+\overline{z})^4 \mathcal{H}} \ , \tag{5.70}$$

obtaining

$$\begin{aligned}
\frac{d\overline{v}(\overline{z})}{d\overline{z}} &= \sin\theta \frac{d}{d\overline{z}}\left[\frac{S(y)^2}{(1+\overline{z})^4\mathcal{H}}\right] , \\
&= \sin\theta\left[\frac{2S(y)dS(y)/d\overline{z}(1+\overline{z})^4\mathcal{H} - S(y)^2[4(1+\overline{z})^3\mathcal{H} + (1+\overline{z})^4\frac{d\mathcal{H}}{d\overline{z}}}{(1+\overline{z})^8\mathcal{H}^2}\right] , \\
&= \sin\theta\left[\frac{2S(y)(dS(y)/d\chi).(d\chi/d\overline{z})(1+\overline{z})^4\mathcal{H} - S(y)^2[4(1+\overline{z})^3\mathcal{H} + (1+\overline{z})^4\frac{d\mathcal{H}}{d\eta}\frac{d\eta}{d\overline{z}}]}{(1+\overline{z})^8\mathcal{H}^2}\right] , \\
&= \sin\theta\left[\frac{2S(y)\frac{a}{\mathcal{H}}(1+\overline{z})^4\mathcal{H} - S(y)^2[4(1+\overline{z})^3\mathcal{H} + (1+\overline{z})^4\mathcal{H}'(-a\mathcal{H}^{-1})]}{(1+\overline{z})^8\mathcal{H}^2}\right] , \\
&= \sin\theta\left[2S(y)a(1+\overline{z})^{-4}\mathcal{H}^{-2} - S(y)^2[4(1+\overline{z})^{-5}\mathcal{H}^{-1} - (1+\overline{z})^{-4}a\mathcal{H}'\mathcal{H}^{-3}]\right] , \qquad (5.71) \\
&= \overline{v}(\overline{z})\left[\frac{2}{S(y)\mathcal{H}} - 4 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right](1+\overline{z})^{-1} \ . \qquad (5.72)
\end{aligned}$$

Thus,

$$\overline{v}(z) = \overline{v}(\overline{z})\left(1 + \left[\frac{2}{S(y)\mathcal{H}} - 4 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right]\frac{\delta z}{1+\overline{z}}\right) , \tag{5.73}$$

and

$$\frac{\delta v}{\overline{v}} = \left(\frac{1}{\mathcal{H}(1+\overline{z})}\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial y} - \frac{H^T}{2} - \delta\beta + \left[\frac{2}{S(y)\mathcal{H}} + \frac{\mathcal{H}'}{\mathcal{H}^2} - 1\right]\frac{\delta\alpha}{2}\right) . \tag{5.74}$$

And since from Eq. (3.81) we can conclude

$$\frac{\partial \delta y}{\partial y} = \frac{d\delta y}{d\eta} = -\delta\beta \ , \tag{5.75}$$

then

$$\frac{\delta v}{\overline{v}} = \left(\frac{1}{\mathcal{H}(1+\overline{z})}\frac{\partial \delta z}{\partial y} - \frac{H^T}{2} + \left[\frac{2}{S(y)\mathcal{H}} + \frac{\mathcal{H}'}{\mathcal{H}^2} - 1\right]\frac{\delta\alpha}{2}\right) . \tag{5.76}$$

With $S(y) = \overline{y} = (\eta_o - \eta_s)$ in flat space, and $\delta z = \frac{1}{2}(1+\overline{z})\delta\alpha$, one can then write

$$\Delta(\mathbf{n}, z) = \delta(\mathbf{n}, z) + \frac{1}{2\mathcal{H}}\frac{\partial \delta\alpha}{\partial y} - \frac{H^T}{2} + \left[\frac{2}{\overline{y}\mathcal{H}} + \frac{\mathcal{H}'}{\mathcal{H}^2} - 1\right]\frac{\delta\alpha}{2} \ . \tag{5.77}$$

This is the expression for the density redshift perturbation in observational coordinates using the observational metric, and as indicated does not include unmeasurable monopole terms or a dipole term $(n^i V_i)_o$ that usually arises by the perturbation at the observer position. Moreover it does not depend on the peculiar velocity of observer and emitter, and we do not need to compute the deviation vectors that relate the perturbed geodesic to the unperturbed one. It only depends on quantities in terms of the perturbed metric and in principle all measurable.

Eq. (5.77) is gauge-invariant expression, the first term we have discussed it earlier, the second term contains the Doppler term, the integrated sachs-wolfe, the gravitational redshift and the redshift-space distortion. The third term contains the lensing distortion and time delay, and the last term contains the redshift perturbation of the volume.

5.4.1 Gauge Transformations of the Density Fluctuations

We need now to convert our result in Eq. (5.77) from the PLG into its equivalent expression in the general coordinates and then we will contrast the outcomes. The upcoming result is already obtained in [150, 157, 162]. Now we will convert each term one by one:

$$
\begin{aligned}
\frac{1}{2\mathcal{H}}\frac{\partial\delta\alpha}{\partial y} &= \frac{1}{\mathcal{H}(1+\bar{z})}\frac{\partial\delta z}{\partial y} = \frac{1}{\mathcal{H}(1+\bar{z})}\frac{d\delta z}{d\eta}\,, \qquad (5.78)\\
&= -\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta-\int_{\eta_s}^{\eta_o}\left(\partial_{(\chi}F'_{\chi)}+\frac{1}{2}h'_{\chi\chi}-\partial_{(\chi}\overline{B}_{\chi)}\right)d\eta\\
&+\frac{1}{\mathcal{H}}\frac{d}{d\eta}\Bigg[-\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta-\int_{\eta_s}^{\eta_o}\left(\partial_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\partial_\chi\overline{B}_\chi\right)d\eta\Bigg]\,, \qquad (5.79)\\
&= -\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta-\int_{\eta_s}^{\eta_o}\left(\partial_{(\chi}F'_{\chi)}+\frac{1}{2}h'_{\chi\chi}-\partial_{(\chi}\overline{B}_{\chi)}\right)d\eta\\
&+\frac{1}{\mathcal{H}}\left[\partial_\chi V_\chi+\Psi'-\left(\partial_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\partial_\chi\overline{B}_\chi\right)\right]\,, \qquad (5.80)
\end{aligned}
$$

where for a pressureless matter moving along the geodesics, we know that

$$
\dot{V}_\chi-\mathcal{H}V_\chi-\partial_\chi\Psi=0\,. \qquad (5.81)
$$

and for

$$
\begin{aligned}
\frac{\delta\alpha}{2} &= \frac{\delta z}{(1+\bar{z})}\,, \qquad (5.82)\\
&= \Bigg[-\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta\\
&-\int_{\eta_s}^{\eta_o}\left(\partial_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\partial_{(\chi}\overline{B}_{\chi)}\right)d\eta\Bigg]\,. \qquad (5.83)
\end{aligned}
$$

Finally, the H^T term, you may see B.5 for more detailed derivation. Using the definition of Δ_Ω as the angular part of the Laplacian, which is denoted as

$$
\Delta_\Omega=(\partial_\theta^2+\cot\theta\partial_\theta+\frac{1}{\sin^2\theta}\partial_\phi^2)\equiv\nabla_{\perp i}\nabla^{\perp i}\,, \qquad (5.84)
$$

we can show that

$$
\begin{aligned}
H^T &= \Bigg[-2\Psi-2\mathcal{H}(B-E')\\
&+\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}\int_{\eta_s}^{\eta}\left(\Delta_\Omega\left[(\Phi+\Psi)-\overline{B}_\chi+F'_\chi-\frac{1}{2}h_{\chi\chi}\right]\right)d\eta' d\eta\Bigg]\,. \qquad (5.85)
\end{aligned}
$$

Now using Eqs. (5.34) and (5.36) together with the above expressions, we get

$$\begin{aligned}\Delta(\mathbf{n},z) &= D_g-\Psi-\mathcal{H}(B-E')+\frac{1}{\mathcal{H}}\left[\Psi'+\partial_\chi V_\chi-\left(\partial_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\partial_\chi\overline{B}_\chi\right)\right]\\ &+\left[\frac{2}{\overline{y}\mathcal{H}}+\frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left[-\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta-\int_{\eta_s}^{\eta_o}\left(\partial_\chi F'_\chi+\frac{1}{2}h'_{\chi\chi}-\partial_{(\chi}\overline{B}_{\chi)}\right)d\eta\right]\\ &-\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta'-\eta_s)\Delta_\Omega\left((\Phi+\Psi)-\overline{B}_\chi+F'_\chi-\frac{1}{2}h_{\chi\chi}\right)d\eta'\ .\end{aligned} \tag{5.86}$$

For only scalars we have

$$\begin{aligned}\Delta(\mathbf{n},z) &= D_g-\Psi-\mathcal{H}(B-E')+\frac{1}{\mathcal{H}}\left[\Psi'+\partial_\chi V_\chi\right]\\ &+\left[\frac{2}{\overline{y}\mathcal{H}}+\frac{\mathcal{H}'}{\mathcal{H}^2}\right]\left[-\Phi+\mathcal{H}(B-E')+V_\chi+\int_{\eta_s}^{\eta_o}(\Phi+\Psi)'d\eta\right]\\ &-\frac{1}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o}(\eta'-\eta_s)\Delta_\Omega(\Phi+\Psi)d\eta'\ .\end{aligned} \tag{5.87}$$

Eq. (5.87) represents the gauge-invariant redshift density fluctuation using a FLRW metric. The $\mathcal{H}^{-1}\partial_\chi\Psi$ term is the gravitational redshift. The light emitted from a galaxy has to pass via that potential field and reach the observer. In so doing, the photon has to lose some of its own energy and hence become redshifted. That will result in changing the redshift of the beam. The term $\mathcal{H}^{-1}\partial_\chi(V_\chi)$ is the redshift space distortion due to the galaxies' peculiar velocity relative to the observer line of sight, and this is considered the largest signal correction on the intermediate scales [150]. The middle line comes from the redshift perturbation of the volume, and it contains a Doppler term; it also contains the ordinary and integrated Sachs-Wolfe terms. The third line (the integral) is the lensing distortion which corresponds to the change in the solid angle causing radial and angular volume distortions and time delay [163–165]; it is relevant especially on large scales. The rest of the terms have very small relativistic effects.

The standard Newtonian description of the galaxy power spectrum breaks down and the general relativistic description is therefore essential for understanding the observed galaxy power spectrum and deriving correct constraints from these measurements. The relativistic effects progressively become significant at low angular multipoles at high redshifts $z \geq 2$, where the relativistic effects are dominant and significant on the horizon scale but they break the symmetry of the correlation function [166]. Due to these effects Δ contains additional information δ, V, Φ, Ψ, this can help testing gravity by probing the relation between density, velocity and gravitational potentials. They do affect our observables and by measuring these effects we can use them to test the relations between the density, velocity and gravitational potentials. Using the PLG gauge was an attempt to make the measuring of these relativistic effects achievable in most simple way. Our results will be most significant for future galaxy survey catalogs like BOSS, DES, Euclid, and of significance to SLOAN-7 data analysis.

Part III

Chapter 6

$f(R)$ Theories of Gravity

> The great tragedy of science - the slaying of a beautiful hypothesis by an ugly fact.
>
> Thomas Huxley

Relativistic cosmology based on Einstein's Field Equations (1.2) describes a universe with its geometric properties. It is derived from a density variational principle Lagrangian (L) of the EH action with scalar curvature R.

If L is chosen to be a function of R, we can consider that as an Extended Theory of Gravity [58]. This approach is aimed to address problems and shortcomings from the standard cosmological model, such as the recently discovered, arguably not yet explained, phenomenon of cosmic acceleration and the lack of a final theory of quantum gravity. It is also important to point out that any extended theory also has to address where GR's positive results had been obtained, *i.e.*, such as Big Bang cosmology, inflation, dark energy, local gravity constraints, cosmological perturbations, and spherically symmetric solutions in weak and strong gravitational backgrounds.

The $f(R)$ model of gravity is a family of theories, each one defined by a different function of the Ricci scalar. Via the arbitrary function introduced in L, we have the freedom to explain the accelerated expansion and the structure formation of the Universe without adding unknown forms of energy or exotic matter. However, not all functional forms of these models can be accepted as viable cosmological models; a wide range of them can be ruled out based on observations, cosmological and astrophysical principles, while others can be rejected because of theoretical pathologies.

$f(R)$ theories were first speculated on by Buchdahl [167] in 1970, but it gained more popularity in cosmology after further developments by Starobinsky [168]. The recent interest has centred on their potential candidacy as possible infrared (IR) and ultraviolet (UV) completions of GR [59, 60, 169–172]. More recently, these gravitational alternative theories have found cosmological applications in, *inter alia*, the dynamical study of homogeneous cosmological models [173–180, 180–188], the linear growth of large-scale structures [189–197] and astrophysics [198–202].

6.1 Derivation of the $f(R)$ Field Equations

There are three ways in deriving the field equations from the action in $f(R)$ gravity. From the EH action in Eq. (1.1), with some function of R, we get

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m(g_{ab}, \Psi_m)] \ , \tag{6.1}$$

where $\mathcal{L}_m$ is a matter Lagrangian that depends on g_{ab} and matter fields Ψ_m. Unless otherwise stated, in this chapter and the next ones, primes $','' $ etc... are shorthands for first, second, etc... derivatives with respect to the Ricci scalar. We will also use f as a shorthand for $f(R)$.

6.1.1 The Metric Formalism

The first is the standard *metric formalism* in which the field equations are derived by the variation of the action with respect to the metric tensor g_{ab} where it is the only independent variable. In this formalism the affine connection depends on the metric g_{ab}. The variation of the determinant is

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} \ . \tag{6.2}$$

The variation of the Ricci scalar with respect to the inverse metric g^{ab} is given by

$$\begin{aligned} \delta R &= R_{ab}\delta g^{ab} + g^{ab}\delta R_{ab} \ , & (6.3)\\ &= R_{ab}\delta g^{ab} + g^{ab}\left(\nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ca}\right) \ . & (6.4) \end{aligned}$$

In this (metric) formalism the connection can be written as

$$\delta\Gamma^{c(m)}_{ab} = \frac{1}{2}g^{cd}\left(\nabla_a\delta g_{db} + \nabla_b\delta g_{da} - \nabla_d\delta g_{ab}\right) \ . \tag{6.5}$$

Substituting this into Eq. (6.4) we will get

$$\delta R = R_{ab}\delta g^{ab} + g_{ab}\nabla^2\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab} \ . \tag{6.6}$$

Now the variation in the action reads

$$\begin{aligned} \delta S &= \frac{1}{2}\int(\delta f\sqrt{-g} + f\delta\sqrt{-g})d^4x \ , & (6.7)\\ &= \frac{1}{2}\int(f'\delta R\sqrt{-g} - \frac{1}{2}\delta\sqrt{-g}g_{ab}\delta g^{ab}f)d^4x \ , & (6.8)\\ &= \frac{1}{2}\int\sqrt{-g}\left[f'\left(R_{ab}\delta g^{ab} + g_{ab}\nabla^2\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab}\right) - \frac{1}{2}g_{ab}\delta g^{ab}\delta f\right]d^4x \ . & (6.9) \end{aligned}$$

Using integration by parts, we get

$$\delta S = \frac{1}{2}\int\sqrt{-g}\delta g^{ab}\left[f'R_{ab} - \frac{1}{2}g_{ab}f + \left(g_{ab}\nabla^2 - \nabla_a\nabla_b\right)f'\right]d^4x \ . \tag{6.10}$$

Since the action remains invariant under variations,

$$f'R_{ab} - \frac{1}{2}f(R)g_{ab} + \left(g_{ab}\nabla^2 - \nabla_a\nabla_b\right)f' = T^m_{ab} \ , \tag{6.11}$$

where T^m_{ab} is the energy-momentum tensor of the matter fields defined by the variational derivative of $\mathcal{L}_m$ in terms of g^{ab}:

$$T^m_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{ab}} \ , \tag{6.12}$$

and satisfying the continuity equation

$$\nabla^a T^m_{ab} = 0 \ . \tag{6.13}$$

The trace of Eq. (6.11) is given by

$$3\nabla^2 f' + f'R - 2f = T \ , \tag{6.14}$$

where $T = g^{ab}T^m_{ab}$. The first term in Eq. (6.14) is a propagating scalar degree of freedom $\psi \equiv f'$ and the trace equation determines the dynamics of the scalar field [168]. It is due to the introduction of fourth-order derivatives of the metric in the last two terms of the LHS of Eq. (6.11) that this formalism is sometimes referred to as a fourth-order theory of gravity. We will use this formalism for the rest of the thesis.

6.1.2 The Palatini Formalism

The second way of deriving the $f(R)$ field equations is the *Palatini formalism* in which both g_{ab} and Γ^c_{gd} are treated as independent variables, and hence involves varying the action with respect to both variables [203]. Varying the action (6.1) with respect to g_{ab} yields

$$f' R_{ab}(\Gamma) - \frac{1}{2} f g_{ab} = T^m_{ab} \; , \tag{6.15}$$

where $R_{ab}(\Gamma)$ is the Ricci tensor corresponding to the connections Γ^c_{gd} and it is different from the Ricci tensor calculated in terms of metric connections R_{ab}. The trace of (6.15) gives

$$f' R - 2f = T \; . \tag{6.16}$$

Here the Ricci scalar $R(T)$ is different from the Ricci scalar $R(g)$ in the metric formalism and is given by

$$R(T) = R(g) + \frac{3}{2[f'(R(T))]^2}[\nabla_a f'(R(T))][\nabla^a f'(R(T))] + \frac{3}{f'(R(T))}\nabla^2 f'(R(T)) \; . \tag{6.17}$$

The variation of the action with respect to the connection leads to the following field equations:

$$\begin{aligned} G_{ab} &= \frac{T_{ab}}{f'} - \frac{1}{2}(R(T) - \frac{f}{f'})g_{ab} + \frac{1}{f'}(\nabla_a\nabla_b - g_{ab}\nabla^2)f' \\ &\quad - \frac{3}{2(f')^2}\left[\nabla_a f' \nabla_b f' - \frac{1}{2} g_{ab}\nabla_c f' \nabla^c f'\right] \; . \end{aligned} \tag{6.18}$$

They are the same field equations as those obtained via the metric formalism but this will no longer hold for $f(R)$ theories whose Lagrangians are non-linear terms in R. Therefore the action in Palatini $f(R)$ is equivalent to

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\varphi R(T) - U(\varphi)\right] + \int d^4x \mathcal{L}_m(g_{ab}, \Psi_m) \; , \tag{6.19}$$

where

$$\varphi = f'(R(T)), \quad U = \frac{R(T)f'(R(T)) - f(R(T))}{2} \; . \tag{6.20}$$

From Eq. (6.16) we note that $\partial_\varphi U = R/(2)$, and therefore

$$4U - 2\varphi\partial_\varphi U = T \; . \tag{6.21}$$

Now using the relation (6.17) we can re-write the action (6.19) as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\varphi R(g) + \frac{3}{4}\frac{1}{\varphi}(\nabla\varphi)^2 - U(\varphi)\right] + \int d^4x \mathcal{L}_m(g_{ab}, \Psi_m) \; . \tag{6.22}$$

We note that there are no second-order covariant derivatives of f', and hence the Palatini formalism is sometimes known as a first-order approach [204].

These two approaches give rise to different field equations for a non-linear Lagrangian density in R, while for the GR action they are identical with each other.

6.1.3 The Metric-affine Formalism

The matter part of the action (6.15) depends on the affine connection and hence introduces a torsion associated with matter. If this connection has its standard geometrical meaning the resulting theory will be a metric-affine theory of gravity [205]. The field equations (6.11) can be written in the following form:

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} = T_{ab} \;, \tag{6.23}$$

where the effective EMT of standard matter is given by

$$\tilde{T}^m_{ab} \equiv \frac{T^m_{ab}}{f'} \;, \tag{6.24}$$

and the *curvature fluid* contribution to the total EMT is

$$T^R_{ab} \equiv \frac{1}{f'}\left[g_{ab}(f - Rf')/2 + \nabla_a\nabla_b f' - g_{ab}\nabla^2 f'\right] \;. \tag{6.25}$$

Since $\nabla^a G_{ab} = 0$ for the total fluid, it follows that the total energy-momentum is also conserved: $\nabla^a T_{ab} = 0$ and for the standard matter $\nabla^a T^{(m)}_{ab} = 0$. It is straightforward to see that

$$\nabla^a T^R_{ab} = \frac{f''}{f'^2}\tilde{T}^m_{ab}\nabla^a R \,, \tag{6.26}$$

$$\nabla^a \tilde{T}^m_{ab} = \frac{\nabla^a T^{(m)}_{ab}}{f'} - \frac{f''}{f'^2}T^{(m)}_{ab}\nabla^a R \,. \tag{6.27}$$

This theory is not yet a well explored one. But it is important to note once more that metric-affine gravity is a modification of GR that follows its geometrical spirit but relaxes its simplifying assumptions [205].

6.2 The Dynamics of $f(R)$ Gravity

The effective total energy-momentum tensor described by

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \tag{6.28}$$

sources the following thermodynamical quantities:

$$\begin{aligned} \rho &= \tilde{\rho}_m + \rho_R \;, \\ p &= \tilde{p}_m + p_R \;, \\ q_a &= \tilde{q}^m_a + q^R_a \;, \\ \pi_{ab} &= \tilde{\pi}^m_{ab} + \pi^R_{ab}, \end{aligned} \tag{6.29}$$

with

$$\tilde{\rho}_m = \frac{\rho_m}{f'}, \quad \tilde{p}_m = \frac{p_m}{f'}, \quad \tilde{q}^m_a = \frac{q_a}{f'}, \quad \tilde{\pi}^m_{ab} = \frac{\pi^m_{ab}}{f'} \tag{6.30}$$

representing the effective energy density, isotropic pressure, heat flux and anisotropic pressure terms for standard matter. In a perfect fluid the quantities q^m_a and π^m_{ab} both vanish in an FLRW background and the effective background energy density and isotropic pressure of the curvature fluid are given by

$$\rho_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R}\right], \tag{6.31}$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R}\right] \;, \tag{6.32}$$

with an effective equation of state [199]

$$w_R \equiv -\frac{3f - 3Rf' + 6f''\ddot{R} + 6f''\dot{R}^2 + 4\Theta f''\dot{R}}{3f - 3Rf' + 6\Theta f''\dot{R}} \ . \tag{6.33}$$

There are also other ways of defining the effective equation of state w_R in a way that it mimics the equation of state of dark energy [206–210]. The energy conservation equations for the matter and curvature-fluid components read

$$\dot{\rho}_m = -\Theta(\rho_m + p_m), \tag{6.34}$$

$$\dot{\rho}_R = -\Theta(\rho_R + p_R) + \rho_m \frac{f''\dot{R}}{f'^2} \ , \tag{6.35}$$

thus showing energy exchange between the two components due to the coupling between these equations. The Friedmann and Raychaudhuri equations for these theories are generalised as

$$\Theta^2 = 3(\tilde{\rho}_m + \rho_R) - \tfrac{3}{2}\tilde{R} = 3\frac{\rho_m}{f'} + \frac{3}{2}\left(R - \frac{f}{f'}\right) - 3\Theta\dot{R}\frac{f''}{f'} - \frac{9k}{a^2} \ , \tag{6.36}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\tilde{\rho}_m + 3\tilde{p}_m) - \tfrac{1}{2}(\rho_R + 3p_R) \ , \tag{6.37}$$

$$= -\frac{1}{3}\Theta^2 - \frac{1}{2f'}\left(2\rho_m - f - 2\Theta\dot{R}f''\right) \ . \tag{6.38}$$

The generalised Friedmann equations in a flat FLRW read

$$3f'H^2 = (\rho + 3p) + \frac{1}{2}(f'R - f) - 3H\dot{f}' \ , \tag{6.39}$$

$$-2f'\dot{H} = (\rho + 4p) + \ddot{f}' - H\dot{f}' \ . \tag{6.40}$$

The Ricci scalar is given by

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) \ , \tag{6.41}$$

whereas the 3-Ricci curvature scalar is

$$\tilde{R} = 2\left(\rho - \frac{1}{3}\Theta^2\right) = 6k/a^2 \ . \tag{6.42}$$

6.3 The Viability of $f(R)$ Models

Over the last few decades, several models of $f(R)$ gravitation have been proposed [174, 200, 211–217]. However, most of these models suffer from a number of problems such as matter instability [218–220], the instability of cosmological perturbations [189, 190, 221], the absence of the matter era [222], and the inability to satisfy local gravity constraints [55]. Amendola et al [223] derived some conditions for cosmological viability of $f(R)$ models. Viable $f(R)$ dark energy models need to satisfy [200, 204, 215, 224]:

1. Avoiding the appearance of ghost particles. This corresponds to putting the constraint

$$f' > 0 \ \ \forall R \tag{6.43}$$

to the $f(R)$ model in question. This ensures that the graviton energy is positive and hence that gravity remain attractive [215].

2. Avoiding the matter instability. This requires the condition

$$f'' > 0 \ \ \forall R \gg f'' \ . \tag{6.44}$$

Any violation of this condition gives rise to the negative mass squared ($M^2 < 0$, M being a characteristic mass scale) and tachyonic scalaron field.

3. The early universe was governed by a GR-like law of gravitation (as evidenced by BBN and CMB constraints), *i.e.*,

$$\lim_{R\to\infty} \frac{f}{R} = 1 \Rightarrow f' < 1 \ , \tag{6.45}$$

implying together with Condition (2) that f' must monotonically asymptote to 1 from below.

4. A less stringent condition is also that

$$|f' - 1| \ll 1 \ , \tag{6.46}$$

at recent epochs but this is not a necessary condition for the ongoing cosmic acceleration.

5. For a stable late-time de Sitter-type expansion

$$\frac{f'}{f''} > R \ \forall R > 0 \ . \tag{6.47}$$

6.4 Some $f(R)$ Models

In this section we briefly discuss some of the most common models of $f(R)$ gravitation that have gained some level of attention in recent years.

6.4.1 R^2 Model

The first model of inflation proposed by Starobinsky [225] showed that R^2 corrections in the standard GR gravitational action of the form

$$f = R + \beta R^2, \quad (\beta > 0) \tag{6.48}$$

where $\beta \equiv \frac{1}{6M^2}$ can lead to accelerated expansion. In the slow-roll approximation for inflation, the the inflaton field $\phi \gg M$ and the slow-roll parameter is given by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{36\beta H^2} \ . \tag{6.49}$$

The number of e-foldings in this model can be given by

$$N \simeq \frac{1}{2\epsilon} \ , \tag{6.50}$$

whereas reheating around $\phi \simeq 0$ results in the inflaton potential

$$V(\phi) \simeq \frac{1}{12\beta}\phi^2 \ . \tag{6.51}$$

This model is very consistent with the thermal temperature anisotropies observed in the CMB and thus it can be an alternative to the scalar field models of inflation [226].

[1] $\beta = 1$ has been used for this figure.

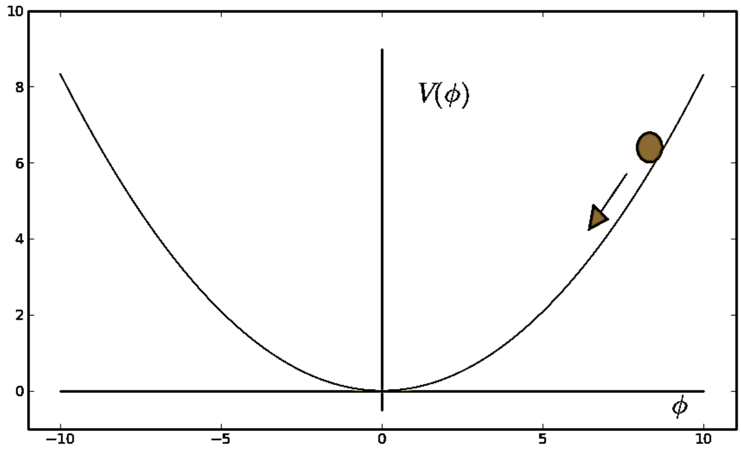


Figure 6.1: Scalar field rolling down to the minimum of the potential well equivalent[1] to the $R+\beta R^2$.

6.4.2 $1/R^n$ Model

The idea that the cosmic acceleration today may have originated from some modified theory of gravity have been proposed by Carroll et al [227], as one of the earliest geometrical alternatives to dark energy by introducing a Lagrangian of the form

$$f = R - \frac{\beta^{2(n+1)}}{R^n} \ , \quad (\beta > 0, n > 0) \ , \tag{6.52}$$

where β here is a constant with dimensions of mass. The total effective fluid is due to matter instability and hence the accelerated expansion requires

$$w_R = -1 + \frac{2(n+2)}{3(2n+1)(n+1)} < -\frac{1}{3} \ , \tag{6.53}$$

and can be achieved with carefully chosen values of β and n.There is a general understanding that this model does not possess a standard matter-dominated epoch because of a large coupling between dark energy and dark matter [177, 189, 190, 228].

6.4.3 R^n Models

Among the most widely studied $f(R)$ models of gravitation are the power-law $f(R)$ models whose Lagrangian densities take the form

$$f = \beta R^n \ , \tag{6.54}$$

where $\beta = \beta(n)$ is a running coupling constant chosen in such a way that $\beta = 1$ for GR, *i.e.*, for $n = 1$. This model has been used to obtain spherically symmetric solutions for galaxy clustering, and it is shown that the rotation curves of spiral galaxies shows a good agreement with the observational data at n=1.7 [229].

6.4.4 $\alpha R + \beta R^n$ Model

This is a class of models whose Lagrangian densities look like

$$f = \alpha R + \beta R^n \ . \tag{6.55}$$

It is considered as a generalisation of both the R^n and the GR actions since $\alpha = 0$ reduces to the R^n and $\alpha = 1$, $\beta = 0$ reduces to GR. Although this toy model is currently used a lot as an alternative solution to inflationary and dark energy cosmology, it is known to violate the $f'' > 0$ condition, and is not considered a viable candidate [55, 227].

6.4.5 Starobinsky Models

These are models of the form [230]

$$f = R + \beta R_c \left[\left(1 + R^2/R_c^2\right)^{-n} - 1 \right] , \quad (\beta > 0, n > 0) , \tag{6.56}$$

with positive free parameters β, n. R_c roughly corresponding to the order of the curvature scale of the present-day value, R_0. An interesting aspect of these models is that in the limiting extreme curvature regimes,

$$\lim_{R/R_c \to 0} f = R , \tag{6.57}$$

$$\lim_{R/R_c \to \infty} f \simeq R - \beta R_c \equiv R - 2\Lambda . \tag{6.58}$$

Thus at low-curvature regimes, there is no existence of the cosmological constant in the model whereas there is an effective cosmological constant term $\Lambda \equiv \beta R_c/2$ that can mimic dark energy at late times.

Starobinsky shows that for some limiting values of n and $R/R_c \gg 1$ in the de Sitter solution $R = const.$, one can obtain a cosmic expansion history indistinguishable from that can be obtained in the Concordance Model.

6.4.6 Hu-Sawicki Models

Hu and Sawicki [231] proposed a Lagrangian density of the form

$$f = R - \frac{\beta_1 m^2 \left(R/m^2\right)^n}{1 + \beta_2 \left(R/m^2\right)^n} , \tag{6.59}$$

where β_1, β_2 and n are the free parameters of the model and

$$m^2 = \frac{\rho_m}{3} , \tag{6.60}$$

is a curvature scale that depends on the value of the matter energy density. It is interesting to note that this model also introduces no cosmological constant term for low-curvature regimes, since

$$\lim_{R/m^2 \to 0} f = R ,$$

whereas for high curvature regimes, one can recover an effective cosmological constant at present epoch because in this limiting case

$$\lim_{R/m^2 \to \infty} f \simeq R - \frac{\beta_1}{\beta_2} m^2 + \frac{\beta_1}{\beta_2^2} m^2 \left(\frac{R}{m^2}\right)^{-n} .$$

These models have more recently shown growing popularity as viable $f(R)$ gravitational solutions.

6.4.7 Appleby-Battye Models

These are models that can be parametrised as [232, 233]

$$f = R + R_c \log\left[e^{-\beta} + \left(1 - e^{-\beta}\right) e^{-R/R_c}\right] \,, \tag{6.61}$$

where $\beta > 0$ is a dimensionless constant and R_c scales as the present-day value of Λ. These models, like the Starobinsky and Hu-Sawicki models, can be shown to mimic ΛCDM for large R and avoid the cosmological constant in the limiting $R = 0$ regime

$$\lim_{R/R_c \to 0} f = R \,, \tag{6.62}$$

$$\lim_{R/Rc \to \infty} f \simeq R - \beta R_c \,. \tag{6.63}$$

6.5 Scalar-Tensor Representation

Before we close this review chapter, it is worth pointing out that there exists an equivalence between $f(R)$ and Scalar-Tensor theories of gravity [204, 234–237]. If one replaces the $f(R)$ action with one containing a standard matter non-minimally coupled with a classical scalar field ϕ defined as

$$\phi \equiv f' - 1 \,, \tag{6.64}$$

the resulting action can be written as

$$\mathcal{A}_\phi = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(\phi(R)) + \mathcal{L}_m], \tag{6.65}$$

and the corresponding field equations can be thought of as the field equations of a classical canonical scalar field ϕ with a potential $V(\phi)$ given by

$$(1+\phi)G_{ab} = T^m_{ab} + \frac{1}{2}g_{ab}(f - (1+\phi)R) + \nabla_b\nabla_a\phi - g_{ab}\nabla_c\nabla^c\phi \,. \tag{6.66}$$

In this representation, the scalar field ϕ will have an EMT given by

$$T^\phi_{ab} = \frac{1}{1+\phi}\left[\frac{1}{2}g_{ab}(f - (1+\phi)R) + \nabla_b\nabla_a\phi - g_{ab}\nabla_c\nabla^c\phi\right], \tag{6.67}$$

and satisfies the Klein-Gordon equation (KGE)

$$\nabla_a\nabla^a\phi - \frac{1}{3}\left[2f - (1+\phi)R + (\rho_m - 3p_m)\right] = 0 \,. \tag{6.68}$$

Defining the scalar field potential via the equation

$$\frac{dV}{d\phi} \equiv \frac{1}{3}\left[2f - (1+\phi)R\right] = \frac{dV}{dR}\frac{dR}{d\phi} \,, \tag{6.69}$$

and using the trace equation

$$\frac{1}{3}T^a{}_a = \frac{T}{3} = \frac{1}{3}(\rho_m - 3p_m) \,, \tag{6.70}$$

the KGE can be rewritten as

$$\nabla_a\nabla^a\phi - \frac{dV}{d\phi} - \frac{1}{3}(\rho_m - 3p_m) = 0 \,. \tag{6.71}$$

The scalar potential evolves according to

$$\frac{dV}{dR} = \frac{1}{3}\left[2f - (1+\phi)R\right] f'' \; . \tag{6.72}$$

Using this correspondence, one can recast the evolution equations in $f(R)$ gravity into respective cosmological equations in Scalar-Tensor theories, and vice versa. For example, in [52], we used the equivalence between $f(R)$ gravity and the Brans-Dicke subclass of scalar-tensor theories, and treating the Chaplygin gas as a scalar field model in a universe without conventional matter forms, we reconstructed the Lagrangian densities for the $f(R)$ gravitational action.

Chapter 7

Irrotational-fluid Cosmologies in $f(R)$ Gravity

Consistency is contrary to nature, contrary to life. The only completely consistent people are dead.

Aldous Huxley

As an application of the preceding chapter, several classes of cosmological models with irrotational fluid flows and where the underlying theory of gravitation is $f(R)$-gravity are investigated in this chapter. Using the $1+3$ covariant decomposition formalism, the integrability conditions describing a consistent evolution of the linearized field equations of shear-free dust universes are presented. We also derive consistency relations of models with more severe constraints, such as non-expanding spacetimes as well as those spacetimes with vanishing gravito-magnetic or gravito-electric components of the Weyl tensor.

In order to understand the dynamics of nonlinear fluid flows in $f(R)$ theories, it is important to understand the relationship between their Newtonian and general relativistic limits. This is relevant both in the physics of gravitational collapse and the late nonlinear stages of cosmic structure formation [108, 238–242]. The differential properties of time-like geodesics describe the fluid flows in cosmology [31, 191, 243]. The expansion Θ, shear (distortion) σ_{ab}, rotation (vorticity) ω^a, and acceleration A_a of the four-velocity field u^a tangent to the fluid flowlines describe kinematics of such fluid flows. The generalised field equations, *i.e*, governing the fluid flows, are obtained by contracting the Ricci identities along and orthogonal to u^a.

7.1 The $1+3$ Covariant Description

The covariant approach in cosmology is an excellent framework for studying cosmological perturbations, and has been primarily developed to analyze the evolution of linear perturbations of Friedmann-Lemaître-Robertson-Walker (FLRW) models in general relativity (GR) [244–246]. The formalism is based on threading spacetimes via covariantly defined variables with respect to partial frames. A fundamental observer slices spacetime into time and space. Unlike in the standard gauge-invariant perturbation formalism (based on the foliation of a background spacetime with hypersurfaces and perturbing away from it), the covariant approach starts from the theory and reduces to linearities in a particular background and has the main advantage that no unphysical gauge modes appear here.

In the $1+3$ covariant decomposition, the four-velocity u^a defines the *covariant time derivative* for any tensor $S^{a..b}{}_{c..d}$ along an observer's worldlines:

$$\dot{S}^{a..b}{}_{c..d} = u^e \nabla_e S^{a..b}{}_{c..d} \,. \tag{7.1}$$

The projection tensor into the tangent 3-spaces orthogonal to u^a is given by

$$h_{ab} \equiv g_{ab} + u_a u_b \,, \tag{7.2}$$

and is used to define the fully orthogonally *projected covariant derivative* for any tensor $S^{a..b}{}_{c..d}$:

$$\tilde{\nabla}_e S^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q_d h^r{}_e \nabla_r S^{f..g}{}_{p..q} \,, \tag{7.3}$$

with total projection on all the free indices. The orthogonally *projected symmetric trace-free* (PSTF) part of vectors and rank-2 tensors is defined as

$$V^{\langle a\rangle} = h^a_b V^b \,, \; S^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \tfrac{1}{3}h^{ab}h_{cd}\right] S^{cd} \,, \tag{7.4}$$

and the volume element for the rest spaces orthogonal to u^a is given by [31, 247, 248]

$$\epsilon_{abc} = u^d \eta_{dabc} = -\sqrt{|g|}\delta^0{}_{[a}\,\delta^1{}_b \delta^2{}_c \delta^3{}_{d]} u^d \Rightarrow \epsilon_{abc} = \epsilon_{[abc]}, \; \epsilon_{abc}u^c = 0, \tag{7.5}$$

where η_{abcd} is the 4-dimensional volume element with the properties

$$\eta_{abcd} = \eta_{[abcd]} = 2\epsilon_{ab[c}u_{d]} - 2u_{[a}\epsilon_{b]cd}. \tag{7.6}$$

We define the covariant spatial divergence and curl of vectors and rank-2 tensors as [249]

$$\mathrm{div}V = \tilde{\nabla}^a V_a \,, \qquad (\mathrm{div}S)_a = \tilde{\nabla}^b S_{ab} \,, \tag{7.7}$$
$$\mathrm{curl}\, V_a = \epsilon_{abc}\tilde{\nabla}^b V^c \,, \quad \mathrm{curl}\, S_{ab} = \epsilon_{cd(a}\tilde{\nabla}^c S_{b)}{}^d \,. \tag{7.8}$$

The first covariant derivative of u^a can be split into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c, \tag{7.9}$$

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a}u_{b\rangle}$ and $\omega^a \equiv \epsilon^{abc}\tilde{\nabla}_b u_c$. The *Weyl conformal curvature tensor* C_{abcd} is defined as [31, 248]

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]} \tag{7.10}$$

and can be split into its "electric" and "magnetic" parts, respectively, as

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{7.11}$$

E_{ab} and H_{ab} represent the free gravitational field [31], enabling gravitational action at a distance, *i.e*, tidal forces and gravitational waves, and influence the motion of matter and radiation through the geodesic deviation for timelike and null vector fields, respectively.

Cosmological quantities that vanish in the background spacetime are considered to be first-order and gauge-invariant by virtue of the Stewart-Walker lemma [111]. In a multi-component fluid universe filled with standard matter fields (dust, radiation, etc) and curvature contributions, the total energy density, isotropic and anisotropic pressures and heat flux are as given in Eq. (6.29) [192] where the linearised thermodynamic quantities for the curvature fluid component are defined as

$$\rho_R = \frac{1}{f'}\left[\tfrac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] \,, \tag{7.12}$$

$$p_R = \frac{1}{f'}\left[\tfrac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \tfrac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R\right)\right] , \tag{7.13}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \tfrac{1}{3}f''\Theta\tilde{\nabla}_a R\right] , \tag{7.14}$$

$$\pi^R_{ab} = \frac{f''}{f'}\left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}\right] . \tag{7.15}$$

Applying the $1+3$ covariant decomposition on the Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^{e} = 0 , \quad (\nabla_a\nabla_b - \nabla_b\nabla_a)u_c = R_{abc}{}^{d}u_d \tag{7.16}$$

for the total fluid 4-velocity u^a, the following linearised propagation (evolution) and constraint equations are obtained [192, 240]:

$$\dot{\rho}_m = -(\rho_m + p_m)\Theta - \tilde{\nabla}^a q^m_a , \tag{7.17}$$

$$\dot{\rho}_R = -(\rho_R + p_R)\Theta + \frac{\rho_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a , \tag{7.18}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\rho + 3p) + \tilde{\nabla}_a A^a , \tag{7.19}$$

$$\dot{q}^m_a = -\tfrac{4}{3}\Theta q^m_a - \rho_m A_a , \tag{7.20}$$

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\rho_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b\pi^R_{ab} , \tag{7.21}$$

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b A^c , \tag{7.22}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} , \tag{7.23}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta E_{ab} - \tfrac{1}{2}(\rho + p)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} - \tfrac{1}{6}\Theta\pi_{ab} , \tag{7.24}$$

$$\dot{H}_{ab} = -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi_{b\rangle}{}^{d} , \tag{7.25}$$

$$(C^1)_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \epsilon_{abc}\tilde{\nabla}^b\omega^c + q_a = 0 , \tag{7.26}$$

$$(C^2)_{ab} := \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^{d} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0 , \tag{7.27}$$

$$(C^3)_a := \tilde{\nabla}^b H_{ab} + (\rho + p)\omega_a + \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b q^c = 0 , \tag{7.28}$$

$$(C^4)_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\rho + \tfrac{1}{3}\Theta q_a = 0 , \tag{7.29}$$

$$(C^5) := \tilde{\nabla}^a\omega_a = 0 , \tag{7.30}$$

$$(C^6)_a := \tilde{\nabla}_a p_m + (\rho_m + p_m)A_a = 0 . \tag{7.31}$$

The evolution equations propagate consistent initial data on some initial $(t = t_0)$ hypersurface S_0 uniquely along the (generally future-directed) reference timelike congruence whereas the constraints restrict the initial data to be specified. For consistency, the constraint equations must remain satisfied on any hypersurface S_t for all comoving time t .

7.2 Irrotational Spacetimes

Consistency analyses of the field equations for different models where integrability conditions, combinations of the Bianchi identities and their consequences, arise from imposing external restrictions have been made over the years [240, 241, 249–253]. We are going to explore some [sub]classes of irrotational cosmological models and show how these models put restrictions on the possible forms of the underlying $f(R)$ gravitational theory.

Irrotational fluid flows admit geodesic timelike congruences with vanishing vorticity

$$\omega_a = 0 \tag{7.32}$$

and characterise potential cosmological models for the late universe and gravitational collapse. For a barotropic irrotational matter fluid with the equation of state $p_m = w\rho_m$, the evolution equations(7.17)-(7.25) for this class of spacetimes can be rewritten as:

$$\dot{\rho}_m = -(1+w)\rho_m\Theta - \tilde{\nabla}^a q^m_a \,, \tag{7.33}$$
$$\dot{\rho}_R = -(\rho_R + p_R)\Theta + \frac{\rho_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a \,, \tag{7.34}$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2f'}(1+3w)\rho_m - \tfrac{1}{2}\left(\rho_R + 3p_R\right) + \tilde{\nabla}_a A^a \,, \tag{7.35}$$
$$\dot{q}^m_a = -\tfrac{4}{3}\Theta q^m_a - \rho_m A_a \,, \tag{7.36}$$
$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\rho_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} \,, \tag{7.37}$$
$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} \,, \tag{7.38}$$
$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta E_{ab} - \tfrac{1}{2}\left(\rho + p\right)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} - \tfrac{1}{6}\Theta\pi_{ab} \,, \tag{7.39}$$
$$\dot{H}_{ab} = -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi_{b\rangle}{}^d \,, \tag{7.40}$$

and are constrained by the following equations:

$$(C^{1*})_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + q_a = 0 \,, \tag{7.41}$$
$$(C^{2*})_{ab} := \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d - H_{ab} = 0 \,, \tag{7.42}$$
$$(C^{3*})_a := \tilde{\nabla}^b H_{ab} + \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b q^c = 0 \,, \tag{7.43}$$
$$(C^{4*})_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\rho + \tfrac{1}{3}\Theta q_a = 0 \,, \tag{7.44}$$
$$(C^{5*})_a := w\tilde{\nabla}_a\rho_m + (1+w)\rho_m A_a = 0 \,, \tag{7.45}$$
$$(C^{6*})_a := \epsilon_{abc}\tilde{\nabla}^b A^c = 0 \implies A_a = \tilde{\nabla}_a\psi \text{ for some scalar } \psi \,. \tag{7.46}$$

The new constraint(7.46) arises as a result of our irrotational restriction. To check for temporal consistency, we propagate this constraint to obtain

$$\left(\epsilon_{abc}\tilde{\nabla}^b A^c\right)^{\cdot} = 0 \,, \tag{7.47}$$

which is an identity. On the other hand, taking the curl of this constraint, one obtains

$$\begin{aligned} curl(curl(A_a)) \quad &= \tilde{\nabla}_a\left(\tilde{\nabla}^b A_b\right) - \tilde{\nabla}^2 A_a + \tfrac{2}{3}\left(\rho - \tfrac{1}{3}\Theta^2\right)A_a && (7.48)\\ &= \tilde{\nabla}_a\left(\tilde{\nabla}^b\tilde{\nabla}_b\psi\right) - \tilde{\nabla}^2\tilde{\nabla}_a\psi + \tfrac{2}{3}\left(\rho - \tfrac{1}{3}\Theta^2\right)\tilde{\nabla}_a\psi && (7.49)\\ &= \tilde{\nabla}_a\left(\tilde{\nabla}^2\psi\right) - \tilde{\nabla}^2\left(\tilde{\nabla}_a\psi\right) + \tfrac{2}{3}\left(\rho - \tfrac{1}{3}\Theta^2\right)\tilde{\nabla}_a\psi = 0 \,, && (7.50)\end{aligned}$$

which is another identity by virtue of Eq's. (C.4) and(C.12).

7.2.1 Dust Spacetimes

Pure dust spacetimes are characterised by

$$w = 0 = p_m \,, q^m_a = 0 = A_a \,, \pi^m_{ab} = 0 \,, \tag{7.51}$$

and the linearised evolution and constraint equations read:

$$\dot{\rho}_d = -\rho_d\Theta \; , \tag{7.52}$$
$$\dot{\rho}_R = -(\rho_R + p_R)\Theta + \frac{\rho_d f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a \; , \tag{7.53}$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2f'}\rho_d - \tfrac{1}{2}\left(\rho_R + 3p_R\right) \; , \tag{7.54}$$
$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\rho_d f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} \; , \tag{7.55}$$
$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi^R_{ab} \; , \tag{7.56}$$
$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}^R_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta E_{ab} - \tfrac{1}{2}\left(\frac{\rho_d}{f'} + \rho_R + p_R\right)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a} q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab} \; , \tag{7.57}$$
$$\dot{H}_{ab} = -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^{R\ d}_{b\rangle} \; , \tag{7.58}$$

$$(C^{1d})_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + q^R_a = 0 \; , \tag{7.59}$$
$$(C^{2d})_{ab} := \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d - H_{ab} = 0 \; , \tag{7.60}$$
$$(C^{3d})_a := \tilde{\nabla}^b H_{ab} + \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b q^c_R = 0 \; , \tag{7.61}$$
$$(C^{4d})_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} - \tfrac{1}{3f'}\tilde{\nabla}_a\rho_m - \tfrac{1}{3}\tilde{\nabla}_a\rho_R + \tfrac{1}{3}\Theta q^R_a = 0 \; . \tag{7.62}$$

Notice here that no new constraints appear.

7.2.1.1 Shear-free Spacetimes

Over the years, the role of shear in GR and the special nature of shear-free cases in particular have been studied [204, 243, 254–256]. Gödel showed [254] that shear-free time-like geodesics of some spatially homogeneous universes cannot expand and rotate simultaneously and this result was later generalized by Ellis [243] to include inhomogeneous cases of shear-free time-like geodesics. Goldberg and Sachs, on the other hand, showed [255] that shear-free null geodesic congruences *in vacuo* require an algebraically special Weyl tensor, a result later generalized by Robinson and Schild [256] to include non-vanishing, but special, forms of the Ricci tensor.

An interesting aspect of these shear-free solutions is that they do not hold in Newtonian gravitation theory [257–259], although Newtonian theory is a limiting case of GR under special circumstances, namely at low-speed relative motion of matter with no gravito-magnetic effects, *i.e*, vanishing magnetic part of the Weyl tensor, and hence no gravitational waves. Now if we turn off the shear, i.e., if we set

$$\sigma_{ab} = 0 \tag{7.63}$$

in the above propagation equations, we get Eq.(7.56) turning into a new constraint

$$(C^{5d})_{ab} := E_{ab} - \tfrac{1}{2}\pi^R_{ab} = 0 \; , \tag{7.64}$$

the temporal and spatial consistencies of which have to be checked. It is interesting to note here that, unlike for shear-free dust spacetimes in GR, the electric component of the Weyl tensor does not vanish because of the non-vanishing contribution of the anisotropic pressure π^R_{ab}. However, we see from Eq. (7.60) that H_{ab} identically vanishes, thus resulting in another constraint from Eq. (7.58):

$$(C^{6d})_{ab} := \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} - \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c\pi^{d\ d}_{b\rangle} = 0 \; , \tag{7.65}$$

which is an identity by virtue of Eq. (7.64). From(7.61), we see that q^R_a is irrotational, and therefore can be written as the gradient of a scalar:

$$q_a^R = \tilde{\nabla}_a \phi \, . \tag{7.66}$$

However, we already know from Eq. (7.59) that $q_a^R = \frac{2}{3}\tilde{\nabla}_a\Theta$. One can therefore conclude that in irrotational and shear-free dust spacetimes,

$$\phi = \tfrac{2}{3}\Theta + C \, , \tag{7.67}$$

for some spatially constant scalar C. We can rewrite this dynamical constraint on the expansion history, using Eq. (7.14) in (7.66), as

$$\tfrac{2}{3} f'\tilde{\nabla}_a\Theta + \left(f''\dot{R} - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} = 0 \, . \tag{7.68}$$

For GR, i.e., $f = R \, , f' = 1 \, , f'' = f''' = 0$, we obtain a spatially constant expansion since

$$\tilde{\nabla}_a\Theta = 0 \, , \tag{7.69}$$

which is trivially true for the class of models under consideration. Now to check for temporal consistency of Eq. (7.64), we take the time derivative of both sides of this equation to obtain the relation

$$\dot{\pi}^R_{ab} + \tfrac{2}{3}\Theta\pi^R_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a} q^R_{b\rangle} = 0 \, , \tag{7.70}$$

which, using q_a^R and π^R_{ab} as defined by Eq's. (7.14) and(7.15), can be rewritten as

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + \tfrac{3f''}{2f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R} = 0 \, . \tag{7.71}$$

Thus irrotational shear-free dust spacetimes governed by $f(R)$ gravitational physics evolve consistently if Eq. (7.71) is satisfied. Note that Eq. (7.71) becomes an identity in the GR limit. Now by taking the curl of the above equation, we obtain

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}R + \tfrac{3f''}{2f'}\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\dot{R} = 0 \, , \tag{7.72}$$

which is an identity by virtue of Eq. (C.3). Thus the temporal consistency condition given by Eq. (7.71) is satisfied on any initial hypersurface. Moreover, from Eq's. (7.65) and (7.72), we can conclude that *all irrotational shear-free dust spacetimes in $f(R)$-gravity are spatially consistent.*

A further restriction one can make for such shear-free spacetimes is turning off E_{ab}. This is a case of vanishing Weyl tensor (since $H_{ab} = 0$ by virtue of Eq. (7.60)), resulting in a locally conformally flat metric. For such a class of cosmological models Eq. (7.57) changes to the (linearised) constraint

$$\tilde{\nabla}_{\langle a} q^R_{b\rangle} = 0 = \frac{1}{f'}\left[\left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R}\right] \, . \tag{7.73}$$

Eq. (7.56) implies

$$\pi^R_{ab} = 0 = \frac{f''}{f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R \, , \tag{7.74}$$

which, for $f'' \neq 0$, leads to the conclusion that

$$\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R = 0 \, . \tag{7.75}$$

Using this and the relation (C.1), we see that Eq. (7.73) becomes an identity. Thus the *linearised $f(R)$ field equations in irrotational and shear-free dust spacetimes with vanishing Weyl tensor are consistent.*

7.2.1.2 Dust Solutions with $divH = 0$

The vanishing of the divergence of a non-zero H_{ab} is a necessary condition for gravitational radiation [260–262]. Here we analyse the consistency of divergence-free GM ($\tilde{\nabla}^b H_{ab} = 0$) scenarios in an effort to understand the nature of gravitationally radiating irrotational dust spacetimes. We see that there are no new constraints arising as a result of imposing a divergence-free H_{ab} to the field equations, but as in the shear-free case discussed above, Eq. (7.61) implies that q^R_a satisfies Eq. (7.66) whereas Eq. (7.67) generalises to

$$\tilde{\nabla}_a \phi = \tfrac{2}{3}\tilde{\nabla}_a \Theta - \tilde{\nabla}^b \sigma_{ab} \ . \tag{7.76}$$

Another interesting subclass of these models arises if both H_{ab} and E_{ab} are divergence-free. Described as "purely radiative" dust spacetimes [263], such models should satisfy the additional modified constraint

$$\tilde{\nabla}_a \rho_m + f'\tilde{\nabla}_a \rho_R + f'\Theta q^R_a - \tfrac{3f'}{2}\tilde{\nabla}^b \pi^R_{ab} = 0 \tag{7.77}$$

as a result of Eq. (7.62). One can see from this result that purely radiative irrotational dust spacetimes in GR where $f = R$ should be spatially homogeneous with $\tilde{\nabla}_a \rho_m = 0$.

The so-called *Newtonian-like* spacetimes are described by the vanishing of the GM component of the Weyl tensor [264]. Thus if the Weyl tensor is to have a purely GE component, then we notice from Eq. (7.60) that a curl-free shear is required, *i.e.*,

$$H_{ab} = 0 \implies \epsilon_{cd(a}\tilde{\nabla}^c \sigma_{b)}{}^d = 0 \ , \tag{7.78}$$

and the constraint(7.65) is obtained from Eq. (7.58). Although such models are known to be of limited applicability in GR-based cosmology, there are some interesting features in $f(R)$ cosmologies.

7.2.1.3 Purely Gravito-magnetic Spacetimes

These are models with vanishing gravito-electric component of the Weyl tensor and are referred to as *anti-Newtonian*[1] models because they are considered to be the most extreme of non-Newtonian gravitational models [264]. The only anti-Newtonian solutions in GR are the FLRW spacetimes [241, 264], but a recent covariant consistency analysis [267] has shown that linearised anti-Newtonian universes are permitted by some models of $f(R)$ gravity.

As can be seen from the set of equations(7.52)-(7.62), no new constraint equations arise as a result of vanishing E_{ab}. This is because of the non-vanishing of π^R_{ab} for generic $f(R)$ models; but in the GR limiting case Eq. (7.57) would have turned into a new constraint since $\pi^R_{ab} = 0$.

7.2.2 Non-expanding Spacetimes

In this section we explore theoretical cases where the background spacetime is not expanding, i.e., $\Theta = 0$ to analyse the kind of [in]consistencies one would obtain if a universe with such properties existed. In this very special case, the linearised evolution equations (7.33)-(7.40) become

$$\dot{\rho}_m = -\tilde{\nabla}^a q^m_a \ , \tag{7.79}$$

$$\dot{q}^m_a = \frac{w}{1+w}\tilde{\nabla}_a \rho_m \ , \tag{7.80}$$

$$\dot{\rho}_R = \frac{\rho_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a \ , \tag{7.81}$$

$$\dot{q}^R_a = \frac{\rho_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} \ , \tag{7.82}$$

$$\dot{\sigma}_{ab} = -E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a} A_{b\rangle} \ , \tag{7.83}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \tfrac{1}{2}(\rho + p)\,\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a} q_{b\rangle} \ , \tag{7.84}$$

[1] An earlier use of the word 'anti-Newtonian' exists [265, 266], where the word is used to refer to an earlier stage of the Universe when the dimension of irregularity exceeds the cosmological (Hubble) horizon.

$$\dot{H}_{ab} = -\epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi_{b\rangle}{}^d \,, \tag{7.85}$$

whereas the revised constraint equations are given by

$$(C^{1s})_a := \tilde{\nabla}^b\sigma_{ab} + q_a = 0 \,, \tag{7.86}$$
$$(C^{2s})_{ab} := \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d - H_{ab} = 0 \,, \tag{7.87}$$
$$(C^{3s})_a := \tilde{\nabla}^b H_{ab} + \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b q^c = 0 \,, \tag{7.88}$$
$$(C^{4s})_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\rho = 0 \,, \tag{7.89}$$
$$(C^{5s})_a := w\tilde{\nabla}_a\rho_m + (1+w)\rho_m A_a = 0 \,, \tag{7.90}$$
$$(C^{6s}) := \tilde{\nabla}_a A^a - \tfrac{1}{2f'}(1+3w)\rho_m - \tfrac{1}{2}\left(\rho_R + 3p_R\right) = 0 \,. \tag{7.91}$$

Eq. (7.80) has been obtained by using Eq. (7.45) into Eq. (7.36) whereas Eq. (7.91) arises from Eq. (7.35), showing that in the non-expanding case the Raychaudhuri (acceleration) equation changes into a constraint.

7.2.2.1 Dust Solutions

In the case of dust

$$A_a = 0 = q^m_a \,, \tag{7.92}$$

the active gravitational mass $\rho + 3p = 0$, because of Eq. (7.91). Since Eq. (7.79) implies $\rho_d(t) =$ constant, we notice that

$$\rho_R + 3p_R = \text{const} \tag{7.93}$$

as well. From the definitions of Eqs. (7.12) and (7.13) for ρ_R and p_R and the *trace equation*

$$3f''\ddot{R} + 3\dot{R}^2 f''' + 3\Theta\dot{R}f'' - 3f''\tilde{\nabla}^2 R - Rf' + 2f - \rho_m + 3p_m = 0 \,, \tag{7.94}$$

we conclude that Eq. (7.93) implies[2]

$$f - 2f''\tilde{\nabla}^2 R = \text{const} \,. \tag{7.95}$$

Thus any nonrotating and noexpanding dust spacetime in $f(R)$ cosmology should have a gravitational Lagrangian that satisfies Eq. (7.95).

7.2.2.2 Shear-free Solutions

As mentioned in the previous subsection, shear-free assumptions in cosmology result in many interesting and at times intriguing properties. Despite the limited applicability of such assumptions in standard cosmology - not least because the Universe is known, beyond any reasonable doubt, to be expanding - it is interesting to explore the different mathematical constraints one obtains if $f(R)$ is the basic gravitational physics behind such cosmology. If we make the shear-free assumption, the propagation equation (7.83) turns into the constraint

$$(C^{7s})_{ab} := E_{ab} - \tfrac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} \,, \tag{7.96}$$

whereas the constraint equations Eqs. (7.86) and(7.87) imply $q_a = 0$ and $H_{ab} = 0$. This means that Eq. (7.84) reduces to

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = 0 \,. \tag{7.97}$$

[2] In GR, where $f(R) = R$, this translates into stating the obvious result that a constant ρ_d implies a constant R since $f'' = 0$.

If we differentiate the new constraint equation (7.96) with respect to cosmic time, and solve simultaneously with Eq. (7.97) we obtain

$$\dot{E}_{ab} - (\tilde{\nabla}_{\langle a}A_{b\rangle})^{\cdot} = 0 \, . \tag{7.98}$$

On the other hand, if we take the gradient of Eq. (7.96) and solve simultaneously with Eq. (7.89), we obtain

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{6}\tilde{\nabla}_a \rho - \tfrac{1}{2}\tilde{\nabla}^b\tilde{\nabla}_{\langle a}A_{b\rangle} = 0 \, . \tag{7.99}$$

Moreover, the curl condition of Eq. (7.96) is identically satisfied by virtue of Eq's. (7.85),(7.46) and(C.3).

If we consider the special case of dust

$$A_a = 0 = q_a^m \, , \pi_{ab}^m = 0 \, , \tag{7.100}$$

in this shear-free setting, then Eq. (7.98) implies $E_{ab} =$ const in time. However, since E_{ab} is related to π^R_{ab} via Eq. (7.96), then π^R_{ab} = const as well. This dictates that, because of Eq. (7.15), the term $\frac{f''}{f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R$, be constant in cosmic time. It is also no coincidence that Eq. (7.95) is recovered for this subclass as a result of Eq. (7.91). Another interesting point to note about non-expanding, shear-free dust spacetimes is that since $q_a = 0 \implies q^R_a = 0$, we are dictated by Eq. (7.14) to conclude

$$\left(f''\tilde{\nabla}_a R\right)^{\cdot} = 0 \, , \tag{7.101}$$

thus putting a constraint on the form of the viable $f(R)$ gravitational action that describes such spacetimes. We notice that Eq. (7.101) is an identity in GR.

A completely general covariant analysis of irrotational fluids in $f(R)$ cosmology requires taking nonlinear effects into account. In the case of purely radiative irrotational dust spacetimes, the consistency requirement implies that such models need not be homogeneous, unlike their GR counterparts.

The introduction of integrability conditions in $f(R)$ gravitational models, and their conservations should these conditions be constraints, provides useful insight into the forms of the $f(R)$ action. It also helps us explore the existence and nature of some universe models that would otherwise not exist under the stricter requirement of the action involving the EH Lagrangian of GR.

Chapter 8

Chaplygin-gas Solutions of $f(R)$ Gravity

You cannot apply mathematics as long as words still becloud reality.

Hermann Weyl

In the scramble for the understanding of the nature of dark matter and dark energy, it has recently been suggested that the change of behavior of the missing energy density might be regulated by the change in the equation of state of the background fluid instead of the form of the potential [268, 269]. The Chaplygin Gas (CG) model in cosmology is one of the most profound candidates for this suggestion. For quite sometime now, the CG model has been considered as another alternative to the cosmological FLRW universe models with a perfect fluid equation of state with a negative pressure [270, 271]. The model provides an interesting features of the cosmic expansion history consistent with a smooth transition between an inflationary phase, the matter-dominated decelerating era, and then late-time accelerated de Sitter phase of cosmic expansion can be achieved [50, 272, 273]; the expansion of the Universe as alternative to the cosmological constant [274]. It predicts also that the expanding universe will continue expanding and this could in principle be observed.

8.1 Chaplygin Gas in FLRW Models

The Chaplygin gas model in FLRW background provides a cosmic expansion history with a universe filled with an exotic background fluid: the Chaplygin gas. The model consists of a universe that transits from a decelerating matter-dominated phase to a late-time accelerated one. However, in its intermediate stages, it behaves as a mixture of a cosmological constant and a perfect fluid obeying the $p = w\rho$ equation of state. The resulting evolution of the Universe is not in disagreement with the current observation of cosmic acceleration [50, 270, 272, 273].

8.1.1 Original and Generalised CG Model

The CG model was introduced first by Chaplygin [275] as a model for aerodynamical studies. Chaplygin assumed an ideal fluid of gas in stabilised motion where all effects of external forces are be neglected in order to get a steady irrotational flow of the gas, where vorticity formation is avoidable. It is necessary to consider the pressure as a function of the density. So it is convenient to take the equation of state in the form

$$p = -\frac{A}{\rho^{\alpha}} , \tag{8.1}$$

where p and ρ are respectively pressure and energy density in a comoving reference frame with $\rho > 0$, and A and α are positive constants. $\alpha = 1$ is constant of the ratio of specific fluctuation, $i.e$, heat[1]. A more generalised CG equation of state is obtained when $0 \leq \alpha \leq 1$ [276–278]. Then under homogeneity considerations, the relativistic energy conservation equation in the context of FRW cosmology substituting for the equation of state yields an expression for the density in terms of the scale factor a given as

$$\rho = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}} , \tag{8.2}$$

where B here is a positive integration constant. When $\alpha = 1$ this solution takes us back to the original Chaplygin gas. This simple and elegant model smoothly interpolates between matter phase, for small value of a ($a^6 \ll B/A$) the energy density is approximated by

$$\rho \sim \frac{\sqrt{B}}{a^3} , \tag{8.3}$$

which clearly corresponds to a dust-like dominated phase. For large values of the cosmological radius a, it follows that

$$p \simeq -\rho \Rightarrow \quad \rho \sim \sqrt{A} \Rightarrow \quad p \sim -\sqrt{A} , \tag{8.4}$$

which corresponds to an empty universe with a cosmological constant $\sqrt{A}$ and that is a de Sitter universe, through an intermediate regime described by the equation of state for stiff matter $p = \rho$. The interesting point, however, is that such an evolution is accounted by using only one fluid. There is a possibility of interpreting the model as a "quintessential" model [50, 269, 279][2] The critical density for such model is

$$\rho_c = (A + B)^{\frac{1}{1+\alpha}} , \tag{8.5}$$

whereas the Hubble parameter given through the Friedmann equation by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho = \frac{1}{3}\left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}} . \tag{8.6}$$

This model automatically leads to an asymptotic phase where the equation of state is dominated by a cosmological constant $\sqrt{A}$. Subsequently, it has been shown that this model admits, under conditions, an inhomogeneous generalisation, which can be regarded as a unification of dark matter and dark energy models [272].

We can express the perturbed pressure as

$$\delta p = \frac{A}{\rho^2}\delta\rho , \tag{8.7}$$

and for the density contrast we have

$$\delta = \frac{\delta\rho}{\rho} \propto t^{2/3} . \tag{8.8}$$

The most important point of the evolution of density perturbations in a universe dominated by the Chaplygin gas is that a universe dominated by the Chaplygin gas admits an initial phase of growing perturbations, with the same rate as in the dust case of the cosmological standard model, from which it follows decreasing oscillations, which asymptotically go to zero [280]. Furthermore, the model predicts an increasing value for the effective cosmological constant [50], $i.e$, in the context of a Chaplygin cosmology, once an expanding universe starts accelerating it cannot decelerate any more, a fact that we seem to be observing today.

For open or flat Chaplygin cosmologies ($k = -1, 0$), the Universe always evolves from a decelerating to an accelerating epoch. For the closed Chaplygin cosmological models ($k = 1$), for static

[1] The CG equation of state (EoS) its also obtainable from Nambu-Goto action for d-branes moving in a $(d+2)$-dimensional spacetime and that by using the observational coordinates in the light-cone gauge [270].

[2] A quintessence field is a scalar field with standard kinetic term, which is minimally coupled to gravity.

Einstein universes, the solution yields

$$B = \frac{2}{3\sqrt{3A}} \; . \tag{8.9}$$

The generalised Chaplygin gas cosmological model, with no additional fluid components, is compatible with structure formation and large scale structure only for α sufficiently small ($\alpha < 10^{-5}$), in which case it is indistinguishable from the ΛCDM model.

The GCG of Eq. (8.1) is important to cosmology and it is one of the promising candidates to explain the present accelerated expansion of the universe with simple model unifying dark matter and dark energy [272, 274, 280] as manifestations of a single cosmic fluid [281–284].

8.1.2 Modified and Extended CG Models

The modified Chaplygin gas (MCG) model is often used often to describe the acceleration phase of the Universe from the radiation era to the ΛCDM model. It includes a matter term [276, 285]

$$p = A\rho - \frac{B}{\rho^{\alpha}} \; , \quad 0 \leq \alpha \leq 1 \; , \tag{8.10}$$

where $0 < A < 1/3$, $0 < \alpha < 1$ and B is a positive constant. It has been shown that $A = 1/3$ is the best fitted value to describe evolution of the Universe from radiation regime to the Λ-cold dark matter regime [286]. The energy density for such a model is given by

$$\rho = \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}}\right]^{\frac{1}{1+\alpha}} \; , \tag{8.11}$$

where C is an arbitrary integration constant. This model is a more appropriate choice to have constant negative pressure at low energy density and high pressure at high energy density. We can use this equation of state to describe low-surface brightness galaxies which are supposed to be dominated by dark matter [287]. The MCG equation of state is a suitable description for an ordinary linear barotropic fluid [288]. While there are other barotropic fluids with equation of state being quadratic and higher orders. For example, recently the model has been extended [289–291] so that the resulting equation of state can also recover a barotropic fluids with higher orders

$$p_c = \sum_{i=1}^{n} A_i \rho_c^i - \frac{B}{\rho^{\alpha}} \; , \tag{8.12}$$

where p_c and ρ_c are the pressure and energy density of the extended Chaplygin gas which is the unification of the dark matter and dark energy. There is no general solution to this approach. But if we reduce $n = 1$ the above expression recovers the standard MCG. Barotropic fluids with quadratic equation of state can be recovered by setting $n = 2$, reducing Eq. (8.12) to

$$p_c = A_1\rho_c + A_2\rho_c^2 - \frac{B}{\rho_c^{\alpha}} \; , \tag{8.13}$$

where A_1, A_2, B and α are positive constants [292]. These models give us the second-order solution [293]

$$\rho_c = \left[\frac{B}{1+A_1} + \frac{C}{a^{3(1+A_1)(1+\alpha)}} e^{-(1+\alpha)(1+A_1)f(\rho_c)}\right]^{\frac{1}{1+\alpha}} \; , \tag{8.14}$$

where $C = (1 + A_1)^{-1}$ and $f(\rho_c)$ is a function of the critical energy density ρ_c given by [293]

$$\begin{aligned} f(\rho_c) &= \frac{A_2\rho_c}{(1+A_1)^2} - \frac{BA_2\rho_c}{(1+\alpha)(1+A_1)^2((1+A_1)\rho_c^{1+\alpha}-B)} \\ &+A_2\int \frac{B(2+\alpha)}{(1+\alpha)(1+A_1)^2((1+A_1)\rho_c^{1+\alpha}-B)}d\rho_c\,. \end{aligned} \tag{8.15}$$

Note that in case of $A_2 = 0$, we have a vanishing $f(\rho_c)$. With higher order n, we will recover a higher-order barotropic fluid. From this model numerically, increasing n decreases the value of the scale factor a and hence decreases the value of the energy density. The evolution of the scale factor corresponding to $n = 1$ is faster than the case with $n = 2$, and the Hubble expansion parameter and dark energy density are decreasing with n. Different orders of n have been studied in [293–295], and it has been shown that by choosing appropriate values of constant parameters, the model has more agreement with observational data than ΛCDM.

8.1.3 Generalised and Modified Cosmic Chaplygin Gas Models

The generalized cosmic Chaplygin gas (GCCG) models are those Chaplyging gas models that admit the equation of state given by [296]

$$p = -\rho^{-\alpha}\left[C + (\rho^{1+\alpha} - C)^{-\omega}\right]\,, \tag{8.16}$$

where

$$C = \frac{A}{1+\omega} - 1\,, \tag{8.17}$$

with A a constant which now can take on both positive and negative values, and $0 > \omega > -l$, l being a positive definite constant which can take on values larger than unity. In the special case when $\omega = 0$ one can write that $C = A - 1$. The speciality of this model is stability so the theory is free from unphysical behaviours even when the vacuum fluid satisfies the phantom energy condition [286]. The above equation satisfies the following conditions:

i) it becomes a de Sitter fluid at late time and when $\omega = -1$,

ii) it reduces to $p = \omega\rho$ in the limit that the Chaplygin parameter $A \to 0$,

iii) it reduces to the equation of state of current Chaplygin unified dark matter models at high energy density,

iv) the evolution of density perturbations becomes free from any pathological behaviour of the matter power spectrum for physically reasonable values of the involved parameters at late times.

By integrating the cosmic conservation law for energy we get for the energy density

$$\rho(a) = \left[C + \left(1 + \frac{B}{a^{3(1+\alpha)(1+\omega)}}\right)^{\frac{1}{1+\omega}}\right]^{\frac{1}{1+\alpha}}\,, \tag{8.18}$$

where B is a positive integration constant. B shows the effect of Chaplygin gas, and the cosmic effect represented by ω. A further extension of the CG model is called modified cosmic Chaplygin gas (MCCG) [286, 297], where the EOS is further generalized to

$$p = \gamma\rho - \frac{1}{\rho^\alpha}\left[\frac{B}{1+\omega} - 1 + \left(\rho^{1+\alpha} - \frac{B}{1+\omega} + 1\right)^{-\omega}\right]\,. \tag{8.19}$$

Here B and γ could be both positive or negative constants, and $-l < \omega < 0$ where l is a positive definite constant with values larger than unity. Here also, $0 < \alpha \leq 1$, and the case where $\omega = 0$

gives the equation of state corresponding to the MCG. If we then put $\gamma = 0$, the equation of state corresponding to the GCG is recovered. We can also reach back to the simplest case where $\alpha = 1$, the Chaplygin gas's original equation of state.

Chaplygin gas models have been studied in flat Friedmann models, in terms of the recently proposed "statefinder [3]" parameters [298], dimensionless parameters that allow us to characterise the properties of dark energy in a model-independent manner. It has also been shown that the simple flat Friedmann model with Chaplygin gas can equivalently be described in terms of a homogeneous minimally coupled scalar field ϕ, which has been used in a variety of inflationary models in describing the transition from the quasi-exponential expansion of the early universe to a power law expansion in order to understand the present acceleration of the Universe [50, 51].

The model can be re-expressed as flat Friedman universes containing a scalar field with particular self-interaction potentials [299,300]; in other words, a very light scalar field ϕ whose effective potential $V(\phi)$ leads to an accelerated phase at the late stages of the Universe [271] by constructing models where the matter responsible for such behaviour is also represented by a scalar field [301, 302].

In [50, 303]a homogeneous scalar field $\phi(t)$ and a potential $V(\phi)$ have been shown to describe Chaplygin cosmology. An extended work is done by [304] with a modified CG. Moreover, the Chaplygin gas is the only gas known to admit a supersymmetric generalisation [305].

8.2 Chaplygin Gas as $f(R)$ Gravity?

In the remainder of the chapter, we are going to study models of $f(R)$ gravity which, when we impose the Chaplygin gas equations of state (EoS) to their effective pressure and energy density, produce viable exact solutions that reduce to the ΛCDM scenario in the approximate cosmological limits.

8.2.1 Constant Ricci-Curvature Scenarios

Analogous to the matter EoS $p_m = w_m\rho_m$, where w_m is the matter EoS parameter, we can define the EoS for the curvature fluid as $p_R = w_R\rho_R$.

For FLRW spacetimes, the Ricci scalar R is given by

$$R = 2\dot{\Theta} + \frac{4}{3}\Theta^2 \ , \tag{8.20}$$

where Θ is the cosmic expansion parameter related to the cosmological scale factor $a(t)$ and the Hubble parameter $H(t)$ via the equations

$$\Theta \equiv 3\frac{\dot{a}(t)}{a(t)} = 3H(t) \ . \tag{8.21}$$

Solving for $\dot{\Theta}$ one can rewrite the above equation as

$$\dot{\Theta} = \frac{R}{2}\left(1 - \frac{4}{3R}\Theta^2\right) \ . \tag{8.22}$$

If the Ricci scalar varies slowly, *i.e.*, if R is almost constant, the solution of this ordinary differential

[3]The statefinder diagnostic trajectories in the plane {s,r} are constructed from the scale factor $a(t)$ and their derivatives up to third order as

$$r = \frac{\dddot{a}}{aH^3} \quad \text{and} \quad s = -\frac{r-1}{3(\frac{a\ddot{a}}{\dot{a}^2} - \frac{1}{2})} \ .$$

equation (o.d.e) takes the form

$$\Theta = \frac{1}{2}\sqrt{3R}\tanh\left[\sqrt{\frac{R}{3}}(t-t_0)\right] , \tag{8.23}$$

for some constant of integration t_0 that can be taken to be the time at the commencement of the inflationary phase of expansion. Solving for the cosmological scale factor gives

$$a(t) = a_0\sqrt{\cosh\left[\sqrt{\frac{R}{3}}(t_0-t)\right]} . \tag{8.24}$$

For simplicity we set $t_0 \simeq 0$. During steady-state exponential expansion in a de Sitter spacetime (such as during inflation or late-time evolution), the approximation $\dot{\Theta} \to 0$ results in

$$R = \frac{4}{3}\Theta^2 = \text{const} , a(t) = a_0 e^{\frac{1}{3}\Theta t} , \tag{8.25}$$

and the matter content evolves according to

$$\rho_m = \rho_0\left(\frac{a(t)}{a_0}\right)^{-3(1+w_m)} . \tag{8.26}$$

During such an exponentially expanding cosmic evolution phase, for an initial energy density ρ_0, we see that the matter energy density decays exponentially:

$$\rho_m = \rho_0 e^{-(1+w_m)\Theta t} . \tag{8.27}$$

8.2.2 Chaplygin Gas Solutions in $f(R)$ Gravity

8.2.2.1 Original and Generalized Chaplygin Gases

In the original treatment, the negative pressure associated with the Chaplygin gas models is related to the (positive) energy density through the EoS

$$p = -\frac{A}{\rho^\alpha} \tag{8.28}$$

for positive constant A and $\alpha = 1$. But this was later generalized [50, 306] to include $0 \leq \alpha \leq 1$. One of the first cosmological interpretations of such a fluid model was given in [302] where for flat universes, Eq. (8.28) corresponds to a viscosity term that is inversely proportional to dust energy density. Ever since the discovery of cosmic acceleration, however, both the original and generalized Chaplygin gas models have been extensively studied as alternatives to dark energy and/or unified dark energy and dark matter models (see, e.g., [50, 269, 272, 274, 307, 308]).

Now if we consider the background curvature energy density and isotropic pressure terms defined in Eqs. (7.12) and (7.13) above, in the constant-curvature limiting case, we have

$$\rho_R = \frac{1}{4}\left[R(f'+1) - 2f\right] = -p_R . \tag{8.29}$$

This equation of state, with an effective EoS parameter $w_R = -1$, provides the condition for an exponential (accelerated) expansion with a constant Hubble parameter. The energy density ρ_R (with its negative pressure p_R) remains constant and can be interpreted as playing the role of the cosmological constant Λ.

Considering the *curvature fluid* as a manifestation of the Chaplygin gas with the EoS (8.28), we obtain

$$p_R = -\rho_R = -\frac{A}{\rho_R^\alpha} \,, \tag{8.30}$$

which, using Eq. (8.29), leads to the o.d.e

$$R\frac{f(R)}{dR} - 2f(R) + R = 4A^{\frac{1}{\alpha+1}} \,. \tag{8.31}$$

Solving this ordinary differential equation yields

$$f(R) = R + C_1 R^2 - 2A^{\frac{1}{\alpha+1}} \tag{8.32}$$

for an arbitrary (integration) constant C_1. We note that the ΛCDM solution $f(R) = R - 2\Lambda$ is already a particular solution with $C_1 = 0$ and $A = \Lambda^{\alpha+1}$. In particular, if $\alpha = 0$, then $A = \Lambda$, from which, going back to Eq. (8.30), one concludes $\rho_R = \Lambda$.

If we include the linearized Laplacian term in Eqs. (7.12) and (7.13) and use the eigenvalue $-\frac{k^2}{a^2}$ of the covariantly defined Laplace-Beltrami operator $\tilde{\nabla}^2$ on (almost) FLRW spacetimes

$$\tilde{\nabla}^2 R = -\frac{k^2}{a^2} R \tag{8.33}$$

for a comoving wavenumber k, we obtain the second-order o.d.e

$$B^2 R f''(R) - R f'(R) + 2f(R) - R + 4A^{\frac{1}{\alpha+1}} = 0 \,, \tag{8.34}$$

where here we have defined

$$B^2 \equiv \frac{4(2+3\alpha)}{3(1+\alpha)} \frac{k^2}{a^2} \,. \tag{8.35}$$

The solution of Eq. (8.34) is given, for arbitrary constants C_2, C_3, by

$$f(R) = R + C_2\left[R^2 - 2RB^2\right] + C_3\left[(R^2 - 2RB^2)Ei\left(1, -\frac{R}{B^2}\right) + (R - B^2)B^2 e^{\frac{R}{B^2}}\right] - 2A^{\frac{1}{\alpha+1}} \,, \tag{8.36}$$

which should reduce to the quadratic solution (8.32) for negligible values of B^2, *i.e.*, for small first-order contributions to the energy density and pressure terms.

8.2.2.2 Modified Chaplygin Gas

On the other hand, if one considers the modified Chaplygin gas (MCG) EoS [51, 131, 286, 289, 290, 306, 309, 310]

$$p_R = \gamma\rho_R - \frac{A}{\rho_R^\alpha} \,, \tag{8.37}$$

then the resulting $f(R)$ model generalizes to

$$f(R) = R + C_4 R^2 - 2\left(\frac{A}{\gamma+1}\right)^{\frac{1}{\alpha+1}} \,, \tag{8.38}$$

where C_4 is an arbitrary integration constant.

The ΛCDM solution is a limiting case of this generalized model when $C_4 = 0$ and $A = (\gamma+1)\Lambda^{\alpha+1}$. In particular, if $\alpha = 0 = \gamma$, then $A = \Lambda$.

Following similar arguments as in the preceding subsection, if we include the linearized Laplacian contributions to the energy density and pressure, we get Eq. (8.34) generalized to

$$B^2 R f''(R) - R f'(R) + 2f(R) - R + 4\left(\frac{A}{\gamma+1}\right)^{\frac{1}{\alpha+1}} = 0 \,, \tag{8.39}$$

the solution of which can be given by

$$f(R) = R + C_5\left[R^2 - 2RB^2\right] + C_6\left[(R^2 - 2RB^2)Ei\left(1, -\frac{R}{B^2}\right) + (R - B^2)B^2 e^{\frac{R}{B^2}}\right] - 2\left(\frac{A}{\gamma+1}\right)^{\frac{1}{\alpha+1}}, \tag{8.40}$$

for an arbitrary integration constants C_5 and C_6. This solution obviously generalizes Solutions (8.32),(8.36) and (8.38) and should reduce to the quadratic solution (8.32) for vanishingly small B^2 values. In [311], it has been shown that any quadratic Lagrangian leading to an isotropic, homogeneous cosmological model takes the form

$$f(R) = R - 2\Lambda - \frac{1}{6}\beta R^2\,, \tag{8.41}$$

where β is an arbitrary, real constant. If we keep only the quadratic solution in (8.40), *i.e.*, if we set $C_6 = 0$, the Lagrangian (8.41) corresponds to the choice

$$C_5 = -\frac{1}{6}\beta\,, B = 0\,, A = (\gamma+1)\Lambda^{\alpha+1}\,. \tag{8.42}$$

Another interesting fact worth pointing out here is that the condition for the existence of a maximally symmetric vacuum solution in $f(R)$ gravity [311]

$$R_0 f'(R_0) = 2f(R_0) \tag{8.43}$$

leads to the quadratic solution resulting in the constraint

$$R_0\left(1 - 2C_5B^2\right) - 4\left(\frac{A}{\gamma+1}\right)^{\frac{1}{\alpha+1}} = 0\,. \tag{8.44}$$

The corresponding GR de Sitter, anti-de Sitter and Minkowski solutions $R_0 = 4\Lambda$ (respectively for $\Lambda > 0\,, \Lambda < 0$ and $\Lambda = 0$) are obtained when $C_5B^2 = 0$ and $A = (\gamma+1)\Lambda^{\alpha+1}$.

The resulting solutions are generally quadratic in the Ricci scalar, but have appropriate ΛCDM solutions in limiting cases. These solutions, given appropriate initial conditions, can be potential candidates for scalar field-driven early universe expansion, *i.e.*, inflation, and dark energy-driven late-time cosmic acceleration. Of course, to take this model seriously one should have a good fundamental reason to believe such fluid exists in the real universe.

8.2.2.3 Modified Generalized Chaplygin Gas

The so-called *modified generalized Chaplygin gas* (mGCG) model is described by a barotropic equation of state of the form [312, 313]

$$p = \beta\rho - (1+\beta)\frac{A}{\rho^\alpha}\,. \tag{8.45}$$

Models of $f(R)$ gravity that satisfy the condition (8.29), at the same time mimicking the mGCG, can be shown to be governed by the same equation as (8.31) and admit the same solutions (8.32), provided $\beta \neq -1$. On the other hand, if linearized Laplacian terms are included, then the corresponding differential equation in $f(R)$ generalizes to

$$D^2Rf''(R) - Rf'(R) + 2f(R) - R + 4A^{\frac{1}{\alpha+1}} = 0\,, \tag{8.46}$$

where we have defined

$$D^2 \equiv \frac{4\left[2 + 3\alpha + 3\beta(1+\alpha)\right]}{3(1+\alpha)(1+\beta)}\frac{k^2}{a^2}\,. \tag{8.47}$$

Worthy of note is that this equation and its solution

$$\begin{aligned} f(R) \quad &= \quad R + C_7 \left[R^2 - 2RD^2\right] + C_8 \left[(R^2 - 2RD^2)Ei\left(1, -\frac{R}{D^2}\right) + (R - D^2)D^2 e^{\frac{R}{D^2}}\right] \\ &\quad -2\left(\frac{A}{\gamma+1}\right)^{\frac{1}{\alpha+1}} , \end{aligned} \tag{8.48}$$

reduce to their *generalized* counterparts of Eqs. (8.34) and (8.36) when $\beta = 0$.

Chapter 9

Conclusions and Future Outlook

> It would be interesting to find out
> what goes on in that moment
> when someone looks at you and
> draws all sorts of conclusions.
>
> Malcolm Gladwell

We humans are very eager to observe and understand what is in the Universe, and that is exactly what makes the study of our cosmos very interesting. In cosmology, we distinguish between the observable universe for which by definition we have data and a universe which includes regions we cannot directly influence or experiment on. The inference of the geometrical properties of the universe from the observable universe cannot be achieved without hypothesis and philosophical prejudices.

The accurate determination of cosmological distances is the most important probe in cosmology. Observing the Universe and measuring many of the cosmological observables can allow us to determine the cosmological parameters which, in turn, allow us to study and track the history of the whole Universe and predict its future and in between study its evolution. In this thesis we calculated the observables in the so-called *lightcone gauge* adapted to the observational coordinates. We developed this gauge by perturbing the lightcone, and reproducing a new linear perturbed gauge to satisfy us up to first-order calculations of the observables. The calculations of the observables in the new gauge introduced was much easier than the ones we usually obtain in the standard gauge. We then used this perturbed gauge to compute the fractional perturbation of galaxy number counts, which is truly measured in large galaxy surveys. In the later calculation we did not have to worry about the spatial positions of the galaxy because we only observe galaxy redshifts and sky positions on the lightcone background.

Ever since Hubble's observations of the 1920s that changed our worldview once and for all, it has become a textbook fact that the Universe is expanding. One of the important observables in cosmology is the measure of the rate of this expansion of the Universe. Towards the turn of this century, a totally unexpected discovery challenged our worldview once again: the Universe is not only expanding but this expansion happens at an accelerated rate. Since Einstein's field equations based on General Relativity do not predict this (in fact the expansion should have been slowing down at the present epoch due to matter domination), this discovery came as a surprise and left many rushing to find possible hypotheses. One of such hypotheses is that the Universe is filled with an invisible form of energy called dark energy. Other propositions came in the form of modifications to the underlying theory of gravity. In this thesis we discussed one of such possible modifications, $f(R)$ gravity. This theory of gravity is one of many candidates studied to solve the current accelerated rate of expansion. The theory has its gravitational action chosen to include generic functions of the Ricci scalar, and therefore comes with an the extra degree of freedom that can explain different cosmic expansion scenarios, including accelerated ones. In particular, we studied $f(R)$ gravitational models that allow irrotational fluid flows. The integrability conditions describing a consistent evolution of the linearized field equations of shear-free dust universes were studied, as well as the consistency relations of models with more severe constraints, such as non-expanding spacetimes as well as those spacetimes with vanishing gravito-magnetic or gravito-electric components of the Weyl tensor.

We also studied the possibility of the $f(R)$ theory of gravity acting as an exotic curvature fluid with negative pressure such as a Chaplygin gas. Assuming a temporally constant Ricci scalar, the resulting $f(R)$ solutions mimicking the Chaplygin gas equations of state are generally quadratic in the Ricci scalar, but have appropriate ΛCDM solutions as their limiting cases. These solutions, given appropriate initial conditions, can be potential candidates for scalar field-driven early universe expansion (inflation) and dark energy-driven late-time cosmic acceleration.

The whole thesis was divided into three parts. **Part I** covered the literature review and the general theoretical foundation of what would follow as calculations of the cosmological observables in **Part II**. **Part III** focused on a class of alternative cosmological models, $f(R)$ theories of gravity, and their cosmological underpinnings.

In Chapter 1, we introduced the literature review on General Relativity and some cosmological observables that are of importance to the development of the thesis.

In Chapter 2, we presented some basic background information about the GR solution and the relation to the standard models of the Universe. We explained the perturbation theory and its necessity to the evolution of the Universe, and we calculated the observables in light of the perturbed first-order FLRW models.

In Chapter 3, we introduced the observational coordinates set that is adapted to our past-lightcone. Since we are the only observers in the Universe, we do our observation from one point at the vertices of our lightcone. We derived the observational spacetime metric. In particular, we were able to identify the way to fix the gauge properly, by showing the relations between the perturbations of spacetime in observational coordinates and those perturbations in the standard metric approach. We derived its dynamical equations for the perturbations in observational coordinates. We decomposed the equations into spherical harmonics in order to simplify the system and produce functions suitable for the extraction of observational quantities, and checked for consistency with the Einstein field equations degrees of freedom. We introduced the GLC gauge and presented the key differences between the GLC gauge and the observational coordinates gauge and hence the PLG gauge.

In Chapter 4, we presented the observables of spacetime in the lightcone gauge using the observational coordinates. We verified the observables on the perturbed lightcone gauge with those obtained in the standard perturbed gauge. The result was accurate and satisfactory to our expectations. We showed the advantage of the method developed in our PLG gauge, namely that the observable relations are simpler than in the standard formalism (not involving integro-differential equations) and it can be extended to second order more easily.

In Chapter 5, we calculated the overdensity regions using the PLG gauge and the results were remarkable. We did a verification with the standard gauge as well.

After introducing the rationale for considering alternative gravitational models, we presented the generalized forms of the field equations of $f(R)$ gravity in Chapter 6. We gave an overview of the $1+3$ covariant spacetime splitting formalism as applied to $f(R)$ theories and gave the propagation and constraint equations relating the kinematical quantities arising from the covariant decomposition, first for the full nonlinear case and then in the linearised regime.

In Chapter 7, in the framework of these new gravitational theories, we outlined the general conditions that any $f(R)$ model should satisfy for non-rotational dust universes for it to be a viable candidate for explaining observations on all cosmological scales. We looked at the consistency relations of linearized perturbations of FLRW universes with irrotational fluid flows arising as a result of imposing special restrictions to the field equations. We showed that linearized shear-free dust models have a vanishing gravito-magnetic component of the Weyl tensor. The case of vanishing full Weyl tensor in linearised $f(R)$ field equations was also explored, as well as those models with purely gravito-magnetic spacetimes. A subclass of gravito-magnetic models are those in which the divergence of H_{ab} is zero, a necessary condition for emission of gravitational waves. In GR, it is known that these models are homogeneous dust FLRW universes. We showed that the homogeneity condition is not necessary in $f(R)$ gravity. Lastly, we derived an integrability condition for non-rotating and no-expanding dust spacetimes in $f(R)$ gravity.

In Chapter 8, we explored exact $f(R)$ gravity solutions that mimic Chaplygin-gas inspired ΛCDM cosmology for the so-called original, generalized, modified and generalized modified Chaplygin gas equations of state. The resulting solutions are generally quadratic in the Ricci scalar, but have appropriate ΛCDM solutions as their limiting cases. These solutions, given appropriate initial conditions, can be potential candidates for scalar field-driven early universe expansion (inflation) and dark energy-driven late-time cosmic acceleration. The solutions discussed here are based on a slowly-changing Ricci curvature assumption. For future work more realistic solutions should relax this assumption, and consider higher-order corrections. Numerical computations require more physically motivated initial conditions, currently not fully understood.

Appendices

Appendix A

Some Useful Relations on Part I

A.1 The Perturbed FLRW Metric

The perturbed FLRW metric and the inverse metric in a conformal time:

$$g_{00} = -a^2(1+2\phi)\ , \quad g^{00} = -1/a^2 + 2\phi/a^2\ , \tag{A.1}$$
$$g_{0i} = a^2 B_i\ , \quad g^{0i} = -1/a^2 B^i\ , \tag{A.2}$$
$$g_{ij} = a^2(\delta_{ij} + 2C_{ij})\ , \quad g^{ij} = \delta^{ij}/a^2 - 2C^{ij}/a^2\ . \tag{A.3}$$

The Christoffer symbols up to first order:

$$\Gamma^0{}_{00} = a'/a + \phi'\ , \tag{A.4}$$
$$\Gamma^0{}_{0j} = D_j\phi + a'/aB_j\ , \tag{A.5}$$
$$\Gamma^0{}_{jk} = (a'/a - 2a'/a\phi)\delta_{jk} + C'_{jk} + 2a'/aC_{jk} - B_{(j|k)}\ , \tag{A.6}$$
$$\Gamma^i{}_{00} = a'/aB^i + B'^i + D^i\phi\ , \tag{A.7}$$
$$\Gamma^i{}_{0j} = a'/a\delta^i_j + C'^i_j + \frac{1}{2}(B^i_{|j} - B^{|i}_j)\ , \tag{A.8}$$
$$\Gamma^i{}_{jk} = \partial_k C^i_j - \partial^i C_{jk} + \partial_j C^i_k - a'/a\delta_{jk}B^i\ . \tag{A.9}$$

A.2 Spherical Harmonic Decomposition

The spherically symmetric spatial metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 = \gamma_{IJ}dx^I dx^J, \tag{A.10}$$

where

$$\gamma^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}\ , \tag{A.11}$$

and the axial metric, which is a function of just θ and ϕ, is given by

$$\varepsilon_I{}^J = \begin{pmatrix} 0 & \frac{1}{\sin\theta} \\ -\sin\theta & 0 \end{pmatrix}\ , \tag{A.12}$$

where
$$\varepsilon^I{}_J = \gamma^{Ik}\gamma_{Jl}\varepsilon_k{}^l, \quad \text{and} \quad \varepsilon_I{}^J\varepsilon_J{}^K = -\gamma_I{}^k \ . \tag{A.13}$$

A.2.1 Properties of the Harmonics Decomposition

The spherical harmonic decomposition is a conventional way to decompose a spatial tensor field δQ into components which transform irreducibly under translations and rotation components, where these components evolve independently. This is achieved by splitting the spacetime into a 2 × 2 manifold, indicating 2-dimensional spherically symmetric surfaces and 2-dimensional spherical harmonic functions $Y^{lm}(\phi, \theta)$.

$$\text{Scalars} : X(w, y, X^k) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\left[X^{lm}(w, y)Y^{lm}(X^k)\right] , \tag{A.14}$$

$$\text{Vectors} : X_I(w, y, X^k) = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\left[X^{lm}(w, y)Y_I^{lm}(X^k) + \overline{X}^{lm}(w, y)\overline{Y}_I^{lm}(X^k)\right] , \tag{A.15}$$

$$\begin{aligned}\text{Tensors} : X_{IJ}(w, y, X^k) = {} & \frac{1}{2}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} X^T_{lm}(w, y)\gamma_{IJ}Y^{lm}(X^k) \\ & + \sum_{l=2}^{+\infty}\sum_{m=-l}^{l}\left[X^{lm}(w, y)Y_{IJ}^{lm}(X^k) + \overline{X}^{lm}(w, y)\overline{Y}_{IJ}^{lm}(X^k)\right] .\end{aligned} \tag{A.16}$$

A.2.1.1 Polar Decomposition

The $(Y^{lm}, Y_I^{lm}, Y_{IJ}^{lm})$ represent the polar parts; for scalar, vector and tensor respectively. For the scalar polar part we get
$$\overline{\nabla}^2 Y^{lm} = \gamma_{IJ}\overline{\nabla}^I\overline{\nabla}^J Y^{lm} = -l(l+1)Y^{lm} , \tag{A.17}$$
$$\varepsilon_{IJ}\overline{\nabla}^I\overline{\nabla}^J Y^{lm} = 0 . \tag{A.18}$$

We can re-write the polar vector part as

$$Y_I^{lm} = \partial_I Y^{lm} , \tag{A.19}$$

where we can say

$$Y_\theta^{lm} = \partial_\theta Y^{lm} , \tag{A.20}$$
$$Y_\phi^{lm} = \partial_\phi Y^{lm} . \tag{A.21}$$

The polar vector divergence can be given as

$$\overline{\nabla}^I Y_I^{lm} = \Delta Y^{lm} = -l(l+1)Y^{lm} . \tag{A.22}$$

Then for the polar tensor part we have

$$Y_{IJ}^{lm} = \overline{\nabla}_I\partial_J Y^{lm} + \frac{l(l+1)}{2}\gamma_{IJ}Y^{lm} , \tag{A.23}$$

this means

$$Y_{\theta\theta}^{lm} = \partial_\theta^2 Y^{lm} + \frac{l(l+1)}{2}Y^{lm} , \tag{A.24}$$
$$Y_{\phi\phi}^{lm} = \partial_\phi^2 Y^{lm} + \sin\theta\cos\theta\partial_\theta Y^{lm} + \frac{l(l+1)}{2}\sin^2\theta Y^{lm} , \tag{A.25}$$

$$Y_{\theta\phi}^{lm} = \partial_{\theta\phi}^2 Y^{lm} - \frac{\cos\theta}{\sin\theta}\partial_\phi Y^{lm} . \tag{A.26}$$

The polar tensor part is trace-free

$$\gamma^{IJ} Y_{IJ}^{lm} = 0 , \tag{A.27}$$

and given this fact we can have

$$Y^{lm\,I}{}_{I} = \gamma^{\theta\theta} Y_{\theta\theta}^{lm} + \gamma^{\phi\phi} Y_{\phi\phi}^{lm} = 0 , \tag{A.28}$$

$$\left[\partial_\theta^2 Y^{lm} + \frac{1}{\sin^2\theta}\partial_\phi^2 Y^{lm} + \frac{l(l+1)}{2} Y^{lm} + \frac{l(l+1)}{2} Y^{lm} + \frac{\cos\theta}{\sin\theta}\partial_\theta Y^{lm}\right] = 0 , \tag{A.29}$$

$$\left[\partial_\theta^2 Y^{lm} + \frac{1}{\sin^2\theta}\partial_\phi^2 Y^{lm} + \frac{\cos\theta}{\sin\theta}\partial_\theta Y^{lm}\right] = -l(l+1) Y^{lm} . \tag{A.30}$$

Also note the polar tensor divergence

$$\overline{\nabla}^I Y_{IJ}^{lm} = \overline{\nabla}^2 Y_J^{lm} + \frac{l(l+1)}{2}\overline{\nabla}_J Y^{lm} = \frac{2 - l(l+1)}{2} Y_J^{lm} , \tag{A.31}$$

whereas

$$\Delta Y_I^{lm} = \overline{\nabla}^2 Y_I^{lm} = (1 - l(l+1)) Y_I^{lm} , \tag{A.32}$$

from which one gets

$$\Delta Y_\phi = \left[\partial_\theta^2 Y_\phi^{lm} + \frac{1}{\sin^2\theta}\partial_\phi^2 Y_\phi^{lm} + \frac{\cos\theta}{\sin\theta}\partial_\theta Y_\phi^{lm}\right] = (1 - l(l+1)) Y_\phi^{lm} , \tag{A.33}$$

$$\Delta Y_\theta = \left[\partial_\theta^2 Y_\theta^{lm} + \frac{1}{\sin^2\theta}\partial_\phi^2 Y_\theta^{lm} + \frac{\cos\theta}{\sin\theta}\partial_\theta Y_\theta^{lm}\right] = (1 - l(l+1)) Y_\theta^{lm} , \tag{A.34}$$

and for polar tensor

$$\Delta Y_{IJ}^{lm} = (4 - l(l+1)) Y_{IJ}^{lm} . \tag{A.35}$$

A.2.1.2 Axial Decomposition

The $(\overline{Y}_I^{lm}, \overline{Y}_{IJ}^{lm})$ represent the axial part for vectors and tensors respectively and there is no axial part for scalars. We can re-define the axial vector part as

$$\overline{Y}_I^{lm} = \varepsilon_I{}^J \partial_J Y^{lm} , \tag{A.36}$$

where

$$\overline{Y}_\theta^{lm} = \frac{1}{\sin\theta}\partial_\phi Y^{lm} , \tag{A.37}$$

$$\overline{Y}_\phi^{lm} = -\sin\theta\,\partial_\theta Y^{lm} . \tag{A.38}$$

The axial vector divergence is given as

$$\overline{\nabla}^I \overline{Y}_I^{lm} = \overline{\nabla}^I \varepsilon_I{}^J Y_J^{lm} = \gamma^{Im}\overline{\nabla}_m \varepsilon_I{}^J \overline{\nabla}_J Y^{lm} , \tag{A.39}$$

$$= \frac{1}{\sin\theta}\left[\partial_{\theta\phi}^2 Y^{lm} - \frac{\cos\theta}{\sin\theta}\partial_\phi Y^{lm}\right] - \frac{1}{\sin\theta}\left[\partial_{\theta\phi}^2 Y^{lm} - \frac{\cos\theta}{\sin\theta}\partial_\phi Y^{lm}\right] = 0 . \tag{A.40}$$

For axial tensors we get

$$\overline{Y}_{IJ} = 2\overline{\nabla}_{(I}\overline{Y}_{J)} = -2\varepsilon^k{}_{(I}\overline{\nabla}_{J)}\overline{\nabla}_k Y , \tag{A.41}$$

where

$$\overline{Y}_{\theta\theta} = \frac{2}{\sin\theta}\left[\partial^2_{\theta\phi}Y - \frac{\cos\theta}{\sin\theta}\partial_\phi Y\right] , \tag{A.42}$$

$$\overline{Y}_{\phi\phi} = -2\sin\theta\left[\partial^2_{\theta\phi}Y - \frac{\cos\theta}{\sin\theta}\partial_\phi Y\right] , \tag{A.43}$$

$$\overline{Y}_{\theta\phi} = -\sin\theta\left[\partial^2_{\theta}Y - \frac{1}{\sin^2\theta}\partial^2_\phi Y - \frac{\cos\theta}{\sin\theta}\partial_\theta Y\right] . \tag{A.44}$$

The axial tensor is also trace-free

$$\gamma^{IJ}\overline{Y}^{lm}_{IJ} = 0 , \tag{A.45}$$

which implies that

$$\partial_\theta \overline{Y}^{lm}_\theta + \frac{1}{\sin^2\theta}\partial_\phi \overline{Y}^{lm}_\phi + \frac{\cos\theta}{\sin\theta}\overline{Y}^{lm}_\theta = 0 . \tag{A.46}$$

The axial tensor divergence is

$$D^I\overline{Y}^{lm}_{IJ} = -2D^I\varepsilon^k{}_{(I}D_{J)}Y^{lm}_k = -[1-l(l+1)][\gamma^{\theta\theta}\overline{\nabla}_\theta Y\delta^\phi_J - \gamma^{\phi\phi}\overline{\nabla}_\phi Y\delta^\theta_J] . \tag{A.47}$$

The axial vector harmonics also obey

$$\Delta\overline{Y}^{lm}_I = \overline{\nabla}^2\overline{Y}^{lm}_I = (1-l(l+1))\overline{Y}^{lm}_I , \tag{A.48}$$

from which one gets

$$\Delta\overline{Y}_\phi = \left[\partial^2_\theta\overline{Y}^{lm}_\phi + \frac{1}{\sin^2\theta}\partial^2_\phi\overline{Y}^{lm}_\phi + \frac{\cos\theta}{\sin\theta}\partial_\theta\overline{Y}^{lm}_\phi\right] = (1-l(l+1))\overline{Y}^{lm}_\phi , \tag{A.49}$$

$$\Delta\overline{Y}_\theta = \left[\partial^2_\theta\overline{Y}^{lm}_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi\overline{Y}^{lm}_\theta + \frac{\cos\theta}{\sin\theta}\partial_\theta\overline{Y}^{lm}_\theta\right] = (1-l(l+1))\overline{Y}^{lm}_\theta , \tag{A.50}$$

and for tensors

$$\Delta\overline{Y}^{lm}_{IJ} = (4-l(l+1))\overline{Y}^{lm}_{IJ} . \tag{A.51}$$

Appendix B

Some Useful Relations on Part II

B.1 Zero-order Coordinates Transformation

We can see now how the background changes under the PLG transformation. The Jacobian map will look like

$$\frac{\partial x^{\hat{\mu}}}{\partial x^{\nu}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\hat{\mu}}_{\nu} + \delta^{\hat{\mu}}_{0}\delta^{1}_{\nu} \ , \tag{B.1}$$

whereas the inverse matrix is given by

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\nu}}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\mu}_{\hat{\nu}} - \delta^{\mu}_{0}\delta^{1}_{\hat{\nu}} \ . \tag{B.2}$$

Then at the background level, the line element becomes,

$$ds^2 = a^2(w-y)(-dw^2 + 2dwdy + S^2(y)d\Omega^2) \ , \tag{B.3}$$

and the 4-velocity is given by

$$u^{\mu} = \frac{1}{a(w-y)}\delta^{\mu}_{0} \ . \tag{B.4}$$

The scale factor $a(\eta)$ transforms as $a(\eta) = a(w-y)$, and $S(\chi) = S(y)$, *i.e*, they are the same spacetime functions but expressed in the new coordinates. The vector field $\mathbf{k}$ is given by

$$k_{\mu} = a(w)\delta^{0}_{\mu} \quad , \quad k^{\nu} = (a(w)/a^2(w-y))\delta^{\nu}_{1} \ . \tag{B.5}$$

Therefore

$$\frac{dy}{d\nu} = a(w)/a^2(w-y) \to 1/a(w), \quad \text{at C} \ . \tag{B.6}$$

B.1.1 The Observables at Zeroth-order

The observable quantities are given by the relations

1. The Redshift:
$$1 + z = a(w)/a(w - y) \ . \tag{B.7}$$

2. The Area Distance:
$$r_A^2 = S^2(y)a^2(w)/(1 + z)^2 = a^2(w - y)S^2(y) \ . \tag{B.8}$$

3. The Luminosity Distance:
$$d_L = (1 + z)^2 r_A^2 = a^2(w)S^2(y) \ . \tag{B.9}$$

4. The Number Count
$$dN = f_m n(w - y) r_A^2 \ d\Omega \ a(w)(1 + z)^{-1} dy = f_m n(w - y)a^3(w - y)S^2(y) \ d\Omega dy \ . \tag{B.10}$$

B.2 Relations on the Observational Metric

From the observational metric we can write the complete perturbed metric components to first order as:

$$\begin{aligned}
g_{ww} &= -a^2(w-y)\,(1-\delta\alpha) \ , && \text{(B.11)}\\
g_{wy} &= a^2(w-y)\,(1+\delta\beta) \ , && \text{(B.12)}\\
g_{yy} &= 0 \ , && \text{(B.13)}\\
g_{wI} &= a^2(w-y)\,(v_I) \ , && \text{(B.14)}\\
g_{IJ} &= a^2(w-y)\,(\Omega_{IJ}+H_{IJ}) \ , && \text{(B.15)}\\
g_{yI} &= 0 \ . && \text{(B.16)}
\end{aligned}$$

The contravariant metric tensor follows from the constraint to the required order:

$$g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda \ , \tag{B.17}$$

and that gives

$$\begin{aligned}
g^{ww} &= 0 \ , && \text{(B.18)}\\
g^{wy} &= a^{-2}(w-y)^{-1}\,(1-\delta\beta) \ , && \text{(B.19)}\\
g^{wI} &= 0 \ , && \text{(B.20)}\\
g^{yy} &= a^{-2}(w-y)^{-1}\,(1-(\delta\alpha+2\delta\beta)) \ , && \text{(B.21)}\\
g^{IJ} &= a^{-2}(w-y)^{-1}\left(\Omega^{IJ}-H^{IJ}\right) \ , && \text{(B.22)}\\
g^{yI} &= -a^{-2}(w-y)^{-1}\Omega^{IJ}v_J \ . && \text{(B.23)}
\end{aligned}$$

Using the definition of the Christoffel symbols given by Eq. (2) we can have the components of the affine connection coefficients calculated to first order as

$$\begin{aligned}
\Gamma^w_{ww} &= 2\frac{\partial_w a}{a} + \frac{\partial_y a}{a}(1-\delta\beta-\delta\alpha) + \partial_w\delta\beta - \frac{1}{2}\partial_y\delta\alpha \ , && \text{(B.24)}\\
\Gamma^w_{wI} &= -\frac{a'}{a}v_I - \frac{1}{2}\partial_y v_I + \frac{1}{2}\partial_I\delta\beta \ , && \text{(B.25)}\\
\Gamma^w_{IJ} &= \frac{\partial_y a}{a}(-\Omega_{IJ} - H_{IJ} + \Omega_{IJ}\delta\beta) - \frac{1}{2}\partial_y\Omega_{IJ} - \frac{1}{2}\partial_y H_{IJ} + \frac{1}{2}\epsilon\delta\beta\partial_y\Omega_{IJ} \ , && \text{(B.26)}\\
\Gamma^y_{yy} &= 2\frac{\partial_y a}{a} + \partial_y\delta\beta \ , && \text{(B.27)}\\
\Gamma^y_{wy} &= \frac{\partial_y a}{a}(-1+\delta\alpha+\delta\beta) + \frac{1}{2}\partial_y\delta\alpha \ , && \text{(B.28)}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^y_{Iy} &= \frac{\partial_y a}{a}(v_I - \Omega^{Jk}\Omega_{Ik}v_J) + \frac{1}{2}\partial_y v_I - \frac{1}{2}\Omega^{Jk}\partial_y\Omega_{Ik}v_J + \frac{1}{2}\partial_I\delta\beta \;, && \text{(B.29)}\\
\Gamma^y_{Iw} &= -\frac{\partial_w a}{a}\Omega_{IJ}\Omega^{IJ}v_I - \frac{\partial_y a}{a}v_I - \frac{1}{2}\partial_y v_I + \frac{1}{2}\partial_I\delta\alpha + \frac{1}{2}\partial_I\delta\beta \;, && \text{(B.30)}\\
\Gamma^y_{ww} &= \frac{\partial_w a}{a}(1-\delta\alpha-\delta\beta) + \frac{\partial_y a}{a}(1-2\delta\alpha-2\delta\beta) + \frac{1}{2}\partial_w\delta\alpha + \partial_w\delta\beta - \frac{1}{2}\partial_y\delta\alpha \;, && \text{(B.31)}\\
\Gamma^y_{IJ} &= \frac{1}{2}\partial_I v_J + \frac{1}{2}\partial_J v_I - \frac{\partial_w a}{a}(\Omega_{IJ} + \epsilon H_{IJ} - \Omega_{IJ}\delta\beta) - \frac{\partial_y a}{a}(\Omega_{IJ} + H_{IJ} - \Omega_{IJ} \\
&\quad \delta\alpha - 2\Omega_{IJ}\delta\beta) - \frac{1}{2}\partial_w H_{IJ} - \frac{1}{2}\partial_y H_{IJ} - \frac{1}{2}\partial_y\Omega_{IJ} + \frac{1}{2}\partial_y\Omega_{IJ}\delta\alpha \\
&\quad +\partial_y\Omega_{IJ}\delta\beta + \frac{1}{2}\Omega^{kJ}v_J\partial_I\Omega_{kJ} \;, && \text{(B.32)}\\
\Gamma^I_{kJ} &= \frac{\partial_y a}{a}\Omega^{Im}\Omega_{kJ}v_m + \frac{1}{2}\Omega^{Im}v_m\partial_y\Omega_{kJ} + \frac{1}{2}\Omega^{mI}\partial_k\Omega_{mJ} + \frac{1}{2}\Omega^{mI}\partial_k H_{mJ} \\
&\quad +\frac{1}{2}\Omega^{mI}\partial_J\Omega_{mk} + \frac{1}{2}\Omega^{mI}\partial_J H_{mk} - \frac{1}{2}\Omega^{mI}\partial_m\Omega_{kJ} - \frac{1}{2}\Omega^{mI}\partial_m H_{kJ} \\
&\quad -\frac{1}{2}H^{mI}\partial_k\Omega_{mJ} - \frac{1}{2}H^{mI}\partial_J\Omega_{mk} + \frac{1}{2}H^{mI}\partial_m\Omega_{kJ} \;, && \text{(B.33)}\\
\Gamma^I_{ww} &= -\frac{\partial_y a}{a}\Omega^{IJ}v_J + \Omega^{IJ}\partial_w v_J - \frac{1}{2}\Omega^{IJ}\partial_J\delta\alpha \;, && \text{(B.34)}\\
\Gamma^I_{wy} &= \frac{\partial_y a}{a}\Omega^{IJ}v_J + \frac{1}{2}\Omega^{IJ}\partial_y v_J - \frac{1}{2}\Omega^{IJ}\partial_J\delta\beta \;, && \text{(B.35)}\\
\Gamma^I_{wJ} &= \frac{\partial_w a}{a}(\Omega^{kI}\Omega_{kJ} + \Omega^{kI}H_{kJ} - \Omega_{kJ}H^{kI}) + \frac{1}{2}\Omega^{kI}\partial_w H_{kJ} + \frac{1}{2}\Omega^{kI}\partial_J v_k \\
&\quad -\frac{1}{2}\Omega^{kI}\partial_k v_J \;, && \text{(B.36)}\\
\Gamma^I_{yJ} &= \frac{\partial_y a}{a}(\Omega^{kI}\Omega_{kJ} + \Omega^{kI}H_{kJ} - \Omega_{kJ}H^{kI}) + \frac{1}{2}\Omega^{kI}\partial_y\Omega_{kJ} + \frac{1}{2}\Omega^{kI}\partial_y H_{kJ} \\
&\quad -\frac{1}{2}H^{kI}\partial_y\Omega_{kJ} \;, && \text{(B.37)}
\end{aligned}
$$

where

$$\Gamma^w_{wy} = \Gamma^w_{Iy} = \Gamma^w_{yy} = \Gamma^I_{yy} = 0 \;. \tag{B.38}$$

The Riemann tensor is defined by

$$R^\mu_{\nu\gamma\beta} = \Gamma^\mu_{\nu\beta,\gamma} - \Gamma^\mu_{\nu\gamma,\beta} + \Gamma^\alpha_{\nu\beta}\Gamma^\mu_{\gamma\alpha} - \Gamma^\delta_{\nu\gamma}\Gamma^\mu_{\beta\delta} \;, \tag{B.39}$$

whereas the Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor:

$$R_{\mu\nu} = g^{\gamma\beta}R_{\gamma\mu\beta\nu} \;, \tag{B.40}$$

and the Ricci scalar is given as

$$R = R^\mu{}_\mu \;. \tag{B.41}$$

B.3 Derivatives and Integrals

B.3.1 Commuting Partial Derivatives

Here is an attempt at finding a way to commute partial derivatives and integrals that we used in our calculations. Assume we want to calculate $\partial_\eta X$ where X is first order and is written:

$$X = \int Y d\lambda \;. \tag{B.42}$$

Then we have

$$\frac{d}{d\lambda}X = Y \Leftrightarrow (\partial_\chi + \partial_\eta)\, X = a^2 Y \; . \tag{B.43}$$

Therefore,

$$(\partial_\chi + \partial_\eta)\, \partial_\eta X = a^2 \partial_\eta Y + 2a^2 \mathcal{H} Y \; , \tag{B.44}$$

or equivalently,

$$\frac{d}{d\lambda}\partial_\eta X = \partial_\eta Y + 2\mathcal{H} Y \; . \tag{B.45}$$

Hence,

$$\partial_\eta \int Y d\lambda = \int \left[\partial_\eta Y + 2\mathcal{H} Y\right] d\lambda \; . \tag{B.46}$$

Similarly:

$$\partial_\chi \int Y d\lambda = \int \left[\partial_\chi Y\right] d\lambda \; . \tag{B.47}$$

Using these two relations and integrating by parts, we recover

$$\int \frac{d}{d\lambda} Y d\lambda = Y \; . \tag{B.48}$$

Finally,

$$\partial_I \int Y d\lambda = \int \partial_I Y d\lambda \; . \tag{B.49}$$

B.3.2 Integration Formulae

Using integration by parts we can convert the double integrals over time into a single integral, using a regular function $f(\eta)$. We are going to integrate by parts the following function:

$$\int_{\eta_s}^{\eta_o} (\eta - \eta_s) f(\eta) d\eta, \tag{B.50}$$

Choosing the auxiliary functions

$$u = \eta - \eta_s \quad , \quad dv = f(\eta) d\eta, \tag{B.51}$$

$$du = d\eta \quad , \quad v = \int_{\eta_s}^{\eta} f(\eta') d\eta' \tag{B.52}$$

then we can write

$$\begin{aligned}
\int_{\eta_s}^{\eta_o} (\eta - \eta_s) f(\eta) d\eta \;&=\; \left((\eta - \eta_s) \int_{\eta_s}^{\eta} f(\eta) d\eta \right)_{n_s}^{n_o} - \int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta} f(\eta') d\eta', && \text{(B.53)}\\
&=\; (\eta_o - \eta_s) \int_{\eta_s}^{\eta_o} f(\eta) d\eta - \int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta} f(\eta') d\eta', && \text{(B.54)}\\
&=\; \int_{\eta_s}^{\eta_o} d\eta \left[\int_{\eta_s}^{\eta} f(\eta) d\eta + \int_{\eta}^{\eta_o} f(\eta) d\eta \right] - \int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta} f(\eta') d\eta', && \text{(B.55)}\\
&=\; \int_{\eta_s}^{\eta_o} d\eta (\eta - \eta_s) f(\eta) + \int_{\eta_s}^{\eta_o} d\eta (\eta_o - \eta) f(\eta) d\eta - \int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta} f(\eta') d\eta', && \text{(B.56)}\\
0 \;&=\; \int_{\eta_s}^{\eta_o} d\eta (\eta_o - \eta) f(\eta) - \int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta} f(\eta') d\eta' \; . && \text{(B.57)}
\end{aligned}$$

Thus

$$\int_{\eta_s}^{\eta_o} d\eta(\eta_o-\eta)f(\eta)=\int_{\eta_s}^{\eta_o} d\eta\int_{\eta_s}^{\eta} f(\eta')d\eta'. \tag{B.58}$$

or

$$\int_{\eta_s}^{\eta_o} d\eta(\eta-\eta_s)f(\eta)=\int_{\eta_s}^{\eta_o} d\eta\int_{\eta_s}^{\eta} f(\eta')d\eta'. \tag{B.59}$$

We can also use the following from Eq. (B.55):

$$\int_{\eta_s}^{\eta_o} f(\eta)d\eta=\frac{1}{\eta_o-\eta_s}\left[\int_{\eta_s}^{\eta_o} d\eta(\eta-\eta_s)f(\eta)+\int_{\eta_s}^{\eta_o} d\eta\int_{\eta_s}^{\eta} f(\eta')d\eta'\right]. \tag{B.60}$$

B.4 Some Useful Derivation for Sec. 4.4

Using Eq. (4.12) we have

$$\begin{aligned}\nabla_{\perp i}\nabla^{\perp i}E &= (-n^in^j\nabla_i\nabla_j-\gamma^{ij}\nabla_{\perp i}n_j\nabla_\parallel+\gamma^{ij}\nabla_i\nabla_j)E\ , && \text{(B.61)}\\ &= (-n^in^j\nabla_i\nabla_j-\gamma^{ij}(\nabla_i-n_i\nabla_\parallel)n_j\nabla_\parallel+\gamma^{ij}\nabla_i\nabla_j)E\ , && \text{(B.62)}\\ &= (-n^in^j\nabla_i\nabla_j-\gamma^{ij}\nabla_in_jn^j\nabla_j+\gamma^{ij}n_i(n^i\nabla_i)n_j(n^j\nabla_j)+\gamma^{ij}\nabla_i\nabla_j)E=0\ , && \text{(B.63)}\end{aligned}$$

and

$$\begin{aligned}\nabla_{\perp i}F^{\perp i} &= (\nabla_i-n_i\nabla_\parallel)(F^i-n^iF_\parallel)\ , && \text{(B.64)}\\ &= \nabla_iF^i-\nabla_in^iF_\parallel-n_i\nabla_\parallel F^i+n_i\nabla_\parallel n^iF_\parallel=0\ . && \text{(B.65)}\end{aligned}$$

From Eq. (4.22) and Eq. (4.24) we can write

$$h^{\perp i}_{\perp i}=h^i_i-h_\parallel(n^in_i-\frac{1}{2}N^i{}_i)-2h_{\perp|(i}n^{i)}=0\ . \tag{B.66}$$

We can also simplify and write

$$\begin{aligned}&\frac{1}{2}\nabla_{\perp i}n^jn^k\nabla_jh_k{}^{\perp i}+\frac{1}{2}\nabla_{\perp i}n^jn^k\nabla_kh_j{}^{\perp i}+\nabla_{\perp i}n^jh'_j{}^{\perp i}-\frac{1}{2}\nabla_{\perp i}\nabla^{\perp i}h_{jk}n^jn^k\\ &=\nabla_{\perp i}n^jn^k\nabla_kh_j{}^{\perp i}+(\nabla_{\perp i}n^j)h'_j{}^{\perp i}-\frac{1}{2}(\nabla^2-n^in^j\nabla_i\nabla_j-\gamma^{ij}\nabla_{\perp i}n_j\nabla_\parallel)h_{lk}n^ln^k\ , && \text{(B.67)}\\ &=\nabla_{\perp i}n_jn^k\nabla_k\frac{1}{2}(h^i_j-h_\parallel(\cancelto{0}{n^in_j}-1/2\cancelto{0}{N^i{}_j})-\cancelto{0}{h^{\perp i}{}_{\perp j}})+\frac{1}{\chi}n_in^j\gamma^i_j(\gamma^j_i-n_in^j)h'_j{}^lN^i_l\\ &\quad-\frac{1}{2}(\nabla^2-n^in^j\nabla_i\nabla_j-\gamma^{ij}(\nabla_i-n_i\nabla_\parallel)n_j\nabla_\parallel)h_{lk}n^ln^k\ , && \text{(B.68)}\\ &=\frac{1}{2}(\nabla_i-n_i\nabla_\parallel)n_jn^k\nabla_kh^i_j+\frac{2}{\chi}n_in^jh'_j{}^lN^i_l-\frac{1}{2}(\nabla^2-n^in^j\nabla_i\nabla_j-\gamma^{ij}\nabla_in_j\nabla_\parallel\\ &\quad+\gamma^{ij}n_i\nabla_\parallel n_j\nabla_\parallel)h_{lk}n^ln^k\ , && \\ & && \text{(B.69)}\\ &=\cancel{-\frac{1}{2}n_in_j\nabla_ln^ln^k}\nabla_kh^i_j+\frac{2}{\chi}n_ln^jh'_j{}^l-\frac{1}{2}\left(2\nabla^2-\cancel{2n^in^j\nabla_i\nabla_j}\right)h_{lk}n^ln^k\ , && \text{(B.70)}\\ &=\frac{2}{\chi}n^in^jh'_{ij}-\nabla^2h_{ij}n^in^j\ , && \text{(B.71)}\end{aligned}$$

and the following terms can be re-written as

$$\cancelto{0}{\nabla_{\perp i}n^j\nabla_j\overline{B}^{\perp i}}-\nabla_{\perp i}\nabla^{\perp i}\overline{B}_jn^j+\cancelto{0}{\nabla_{\perp i}\overline{B}^{\perp i\prime}}=-(\nabla^2-\nabla^2_\parallel-\frac{2}{\chi}\nabla_\parallel)\overline{B}_jn^j\ , \tag{B.72}$$

$$= -\nabla^2 \overline{B}_j n^j + \nabla_\parallel (\frac{d}{d\eta} - \nabla\eta)\overline{B}_j n^j + \frac{1}{\chi}\nabla_\parallel)\overline{B}_j n^j + \frac{1}{\chi}\nabla_\parallel)\overline{B}_i n^i \ , \tag{B.73}$$

$$= -\nabla^2 \overline{B}_j n^j + \cancelto{0}{\frac{d}{d\eta}\nabla_\parallel \overline{B}^\parallel} - \nabla_\parallel \overline{B}'_j n^j + \frac{1}{\chi}\nabla_\parallel \overline{B}_j n^j + \frac{1}{\chi}\nabla_\parallel \overline{B}_i n^i \ , \tag{B.74}$$

$$= -\nabla^2 \overline{B}_i n^i - \frac{1}{2}\nabla_i n^i \overline{B}'_\parallel - \frac{1}{2}\nabla_j n^j \overline{B}'_\parallel + \frac{2}{\chi}\nabla_{(j}\overline{B}_{i)} n^i n^j \ , \tag{B.75}$$

$$= -\nabla^2 \overline{B}_i n^i - \nabla_{(i}\overline{B}'_{j)} n^i n^j + \frac{2}{\chi}\nabla_{(i}\overline{B}_{j)} n^i n^j \ . \tag{B.76}$$

Moreover, we note

$$\begin{aligned}
&2\nabla_{\perp i} n^j \nabla_{(j} F^{\perp i)\prime} + \nabla_{\perp i} n^j n^k \nabla_k \nabla_{(j} F^{\perp i)} - \nabla_{\perp i}\nabla^{\perp i}\nabla_{(k}F_{j)} n^j n^k + \nabla_{\perp i} n^j n^k \nabla_j \nabla_{(k} F^{\perp i)} \\
&= \cancelto{0}{\nabla_{\perp i} n^j \nabla_j F^{\perp i\prime}} + \nabla_{\perp i} n_j \nabla^{\perp i} F^{j\prime} + \cancelto{0}{\nabla_{\perp i} n^j n^k \nabla_k \nabla_j F^{\perp i}} - \cancel{\nabla_{\perp i}\nabla^{\perp i}\nabla_k F_j n^j n^k} \\
&\quad + \cancel{\nabla_{\perp i}\nabla^{\perp i}\nabla_k F_j n^j n^k} \ ,
\end{aligned} \tag{B.77}$$

$$= \nabla_{\perp i} n_j \nabla^{\perp i} F^{j\prime} \ , \tag{B.78}$$

$$= (\nabla^2 - \nabla_\parallel^2 - \frac{2}{\chi}\nabla_\parallel) n_j F^{j\prime} \ , \tag{B.79}$$

$$= \nabla^2 n_j F^{j\prime} - \nabla_\parallel (\frac{d}{d\eta} - \nabla\eta) n_j F^{j\prime} - \frac{2}{\chi}\nabla_\parallel n_j F^{j\prime} \ , \tag{B.80}$$

$$= \nabla^2 n_j F^{j\prime} - \cancelto{0}{\nabla_\parallel \frac{d}{d\eta} F^{\parallel\prime}} + \nabla_\parallel \nabla\eta F^{j\prime} n_j - \frac{2}{\chi}\nabla_\parallel n_j F^{j\prime} \ , \tag{B.81}$$

$$= \nabla^2 n_j F^{j\prime} + \nabla_\parallel F^{j\prime\prime} n_j - \frac{2}{\chi}\nabla_\parallel n_j F^{j\prime} \ , \tag{B.82}$$

$$= \nabla^2 n_j F^{j\prime} + \frac{1}{2}\nabla_\parallel F^{j\prime\prime} n_j + \frac{1}{2}\nabla_\parallel F^{i\prime\prime} n_i - \frac{1}{\chi}\nabla_\parallel n_j F^{j\prime} - \frac{1}{\chi}\nabla_\parallel n_i F^{i\prime} \ , \tag{B.83}$$

$$= \nabla^2 n_j F^{j\prime} + \nabla_{(i}F_{j)}{}^{\prime\prime} n^j n^i - \frac{2}{\chi}\nabla_{(i}F_{j)}{}^{\prime} n^j n^i \ , \tag{B.84}$$

where Eq. (4.5) has been made use of.

B.5 Some Useful Expressions for Computing H^T

The following facts/relations have been used in our calculations of Chapter 5:

$$\begin{aligned}
&\frac{1}{2}\nabla_{\perp i} n^j n^k \partial_j h_k{}^{\perp i} + \frac{1}{2}\nabla_{\perp i} n^j n^k \partial_k h_j{}^{\perp i} + \nabla_{\perp i} n^j h'_j{}^{\perp i} - \frac{1}{2}\nabla_{\perp i}\nabla^{\perp i} h_{jk} n^j n^k \\
&= \nabla_{\perp i} n^j n^k \nabla_k h_j{}^{\perp i} + \nabla_{\perp i} n^j h'_j{}^{\perp i} - \frac{1}{2}\nabla_{\perp i}\nabla^{\perp i} h_{jk} n^j n^k \ ,
\end{aligned} \tag{B.85}$$

$$= \cancelto{0}{(\nabla_i + n_i \nabla_\parallel)[n^j n^k \nabla_k h_j{}^{\perp i} + n^j h'_j{}^{\perp i}]} - \frac{1}{2}\nabla_{\perp i}\nabla^{\perp i} h_{jk} n^j n^k \ , \tag{B.86}$$

$$= -\frac{1}{2}\nabla_{\perp i}\nabla^{\perp i} h_{jk} n^j n^k \ , \tag{B.87}$$

$$\equiv -\frac{1}{2}(\partial_\theta^2 + \cot\theta\partial\theta + 1/\sin^2\theta\partial_\phi^2) h_{\chi\chi} n^\chi n^\chi \ , \tag{B.88}$$

$$
\begin{aligned}
&2\nabla_{\perp i} n^j \nabla_{(j} F^{\perp i)'} + \nabla_{\perp i} n^j n^k \partial_k \partial_{(j} F^{\perp i)} - \nabla_{\perp i} \nabla^{\perp i} \partial_{(k} F_{j)} n^j n^k + \nabla_{\perp i} n^j n^k \partial_j \partial_{(k} F^{\perp i)} \\
&= \cancelto{0}{\nabla_{\perp i} n^j \nabla_j F^{\perp i'}} + \nabla_{\perp i} n_j \nabla^{\perp i} F^{j'} + \cancelto{0}{\nabla_{\perp i} n^j n^k \nabla_k \nabla_j F^{\perp i}} - \cancel{\nabla_{\perp i} \nabla^{\perp i} \partial_k F_j n^j n^k} \\
&\quad + \cancel{\nabla_{\perp i} \nabla^{\perp i} \nabla_k F_j n^j n^k} \ , \qquad \text{(B.89)} \\
&= \nabla_{\perp i} \nabla^{\perp i} F^{j'} n_j \ , \qquad \text{(B.90)}
\end{aligned}
$$

Or we can say (B.91)

$$
\equiv (\partial_\theta^2 + \cot\theta \partial\theta + 1/\sin^2\theta \partial_\phi^2) n^\chi F'_\chi \ , \qquad \text{(B.92)}
$$

$$
\begin{aligned}
\cancelto{0}{\nabla_{\perp i} n^j \nabla_j \overline{B}^{\perp i}} - \nabla_{\perp i} \nabla^{\perp i} \overline{B}_j n^j + \cancelto{0}{\nabla_{\perp i} \overline{B}^{\perp i'}} &= -\nabla_{\perp i} \nabla^{\perp i} \overline{B}_j n^j \ , \qquad \text{(B.93)} \\
&\equiv -(\partial_\theta^2 + \cot\theta \partial\theta + 1/\sin^2\theta \partial_\phi^2) \overline{B}_\chi n^\chi \ . \qquad \text{(B.94)}
\end{aligned}
$$

Appendix C

Some Useful Relations on Part III

C.1 Linearised Differential Identities

For all scalars f, vectors V_a and tensors that vanish in the background, $S_{ab} = S_{\langle ab\rangle}$, the following linearised identities hold [192, 240, 249]:

$$
\begin{aligned}
\left(\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}f\right)^{\cdot} &= \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{f} - \tfrac{2}{3}\Theta\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}f + \dot{f}\tilde{\nabla}_{\langle a}A_{b\rangle}\ , && \text{(C.1)}\\
\epsilon^{abc}\tilde{\nabla}_b\tilde{\nabla}_c f &= 0\ , && \text{(C.2)}\\
\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}f &= \epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{(b}\tilde{\nabla}^{d)}f = \epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^d f = 0\ , && \text{(C.3)}\\
\tilde{\nabla}^2\left(\tilde{\nabla}_a f\right) &= \tilde{\nabla}_a\left(\tilde{\nabla}^2 f\right) + \tfrac{1}{3}\tilde{R}\tilde{\nabla}_a f\ , && \text{(C.4)}\\
\left(\tilde{\nabla}_a f\right)^{\cdot} &= \tilde{\nabla}_a\dot{f} - \tfrac{1}{3}\Theta\tilde{\nabla}_a f + \dot{f}A_a\ , && \text{(C.5)}\\
\left(\tilde{\nabla}_a S_{b\cdots}\right)^{\cdot} &= \tilde{\nabla}_a\dot{S}_{b\cdots} - \tfrac{1}{3}\Theta\tilde{\nabla}_a S_{b\cdots}\ , && \text{(C.6)}\\
\left(\tilde{\nabla}^2 f\right)^{\cdot} &= \tilde{\nabla}^2\dot{f} - \tfrac{2}{3}\Theta\tilde{\nabla}^2 f + \dot{f}\tilde{\nabla}^a A_a\ , && \text{(C.7)}\\
\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}V_c &= -\tfrac{1}{6}\tilde{R}V_{[a}h_{b]c}\ , && \text{(C.8)}\\
\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}S^{cd} &= -\tfrac{1}{3}\tilde{R}S_{[a}{}^{(c}h_{b]}{}^{d)}\ , && \text{(C.9)}\\
\tilde{\nabla}^a\left(\epsilon_{abc}\tilde{\nabla}^b V^c\right) &= 0\ , && \text{(C.10)}\\
\tilde{\nabla}_b\left(\epsilon^{cd\langle a}\tilde{\nabla}_c S^{b\rangle}_d\right) &= \tfrac{1}{2}\epsilon^{abc}\tilde{\nabla}_b\left(\tilde{\nabla}_d S^d_c\right)\ , && \text{(C.11)}\\
\mathrm{curlcurl}V_a &= \tilde{\nabla}_a\left(\tilde{\nabla}^b V_b\right) - \tilde{\nabla}^2 V_a + \tfrac{2}{3}\left(\mu - \tfrac{1}{3}\Theta^2\right)V_a\ , && \text{(C.12)}
\end{aligned}
$$

where $\tilde{R} \equiv 2\left(\mu - \frac{1}{3}\Theta^2\right)$ is the 3-curvature scalar.

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