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Entropies and correlations in classical and quantum systems

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To our friend Gaetano Vilasi
on the occasion of his 70th birthday.

Summary. — We present a review of entropy properties for classical and quantum systems including Shannon entropy, von Neumann entropy, Rényi entropy, and Tsallis entropy. We discuss known and new entropic and information inequalities for classical and quantum systems, both composite and noncomposite. We demonstrate matrix inequalities associated with the entropic subadditivity and strong subadditivity conditions and give a new inequality for matrix elements of unitary matrices.

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1. – Introduction

Statistical properties of classical and quantum systems are described within the framework of the probability-distribution formalism in classical domain (see, for example, [1]) and the density matrices [2, 3] in quantum domain. The paradigmatic characteristics of classical states determined by a probability distribution is Shannon entropy [4], and its quantum counterpart associated with a quantum-state density matrix is von Neumann entropy [5]. Entropy characterizes a degree of disorder in systems with fluctuating physical observables. The smaller the Shannon entropy, the larger the order in the system.

Analogously, the interpretation of the order in mixed state of quantum systems is related to the value of von Neumann entropy.

For multipartite systems, relations between entropies of a composite system and its subsystems characterize a degree of correlations of the subsystem observables.

For quantum systems, the phenomenon of entanglement [6] is a specific particularity of strong quantum correlations, some aspects of which can be characterized by the values of entropies of the subsystem states [7].

For classical systems, the generalizations of the notion of Shannon entropy, which is a functional of the probability distribution, *i.e.*, the number corresponding to the probability distribution, was suggested by Rényi [8] and Tsallis [9]. The entropies they introduced are functionals of the probability distributions which, in addition, depend on the parameter λ. For a limit value of the λ-parameter, the entropies are converted into the Shannon entropy. Analogous generalizations of the von Neumann entropy are quantum entropies introduced for quantum systems [8, 9]. The properties of quantum-system entropies and specific entropic inequalities for composite-system states were discussed in [10-18]. Some properties of the entropies were considered in [16, 18, 19]. The general properties of the probability vectors were presented in [20].

The aim of this paper is to present a review of entropic and information characteristics of classical and quantum states, in particular, taking into account the tomographic-probability approach for describing quantum states [21, 22]. Another goal of the present paper is to obtain some new matrix inequalities using the properties of hidden correlations [23-27] of observables associated with degrees of freedom of noncomposite systems analogous to correlations of degrees of freedom in multipartite systems known to be characterized by entropic inequalities for the composite-system density matrix.

This paper is organized as follows.

In sect. **2**, we discuss the probability distributions (probability vectors) and Shannon entropy. We consider the Rényi classical entropy in sect. **3** and study the classical Tsallis entropy and its relation to Shannon and Rényi entropies in sect. **4**. In sect. **5**, we pay our attention at quantum entropies, *i.e.*, von Neumann entropy and its generalizations, and in sect. **6** we switch to the tomographic distributions of qudit states and their entropic properties. We discuss the hidden correlations of single qudit states in sect. **7** and consider qudit thermal states in sect. **8**. We employ the density-matrix entropic inequalities for obtaining new relations for matrix elements of unitary matrices in sect. **9** and present our conclusions and prospectives in sect. **10**.

2. – Probability vector and Shannon entropy

We recall some simple properties of probability distributions [1, 24].

Given a random variable, for example, random position $-\infty < x < \infty$ of a particle. The probability density $P(x) \geq 0$, $\int P(x)\,\mathrm{d}x = 1$, *e.g.*, a normal probability distribution $P(x,\sigma,\bar{x}) = (2\pi\sigma)^{-1/2}\exp[-(x-\bar{x})^2]/2\sigma$, which is nonnegative function of random position x and two parameters — position mean $\bar{x}$ and dispersion σ can be considered as Gaussian state. Observables determined as functions of random position x like $f(x)$ are associated with such statistical characteristics as moments $\langle f^k(x)\rangle = \int f^k(x)P(x,\sigma,\bar{x})\,\mathrm{d}x$.

Now we consider a particular case of the particle position which can take only a finite number of values x_j, $j = 1, 2, \ldots, N$. With the probability N-vector $\vec{P}$ with components $P(j)$, we associate the probability distribution $P(x_j) \equiv P(j) \geq 0$, the nonnegative function with the normalization condition $\sum_{j=1}^{N} P(j) = 1$. With integer (or random variable) j, we associate random position x_j. We call the probability vector $\vec{P}$ the state of the system. Statistical properties of observables $f(x_j) \equiv f(j)$ are associated with moments $\langle f^k(j)\rangle = \sum_{j=1}^{k} f^k(j)P(j)$.

One can introduce functionals, *e.g.*, Shannon entropy

$$H_{\vec{P}} = -\sum_{j=1}^{N} P(j)\ln P(j). \tag{1}$$

The entropy has the minimum value $H_{\vec{P}0} = 0$ if all components of $\vec{P}$ except a single one are equal to zero, and it is maximal if all components of the probability vector are equal: $P(j) = 1/N$.

The Shannon entropy characterizes a degree of the order in the system. If the particle position x takes discrete values, which we denote as $x_{j_1,j_2,\ldots,j_n}$, one can introduce the probability distribution $P(x)$, using the notation $P(x_{j_1,j_2,\ldots,j_k}) \equiv P(j_1, j_2, \ldots, j_k)$, where $j_1 = 1, 2, \ldots, n_1$, $j_2 = 1, 2, \ldots, n_2$, and $j_k = 1, 2, \ldots, n_k$. The nonnegative function $P(j_1, j_2, \ldots, j_k)$ is normalized

$$\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\cdots\sum_{j_k=1}^{n_k} P(j_1, j_2, \ldots, j_k) = 1. \tag{2}$$

One can interpret the function $P(j_1, j_2, \ldots, j_k)$ as a joint probability distribution of k random variables $j_1, j_2, \ldots, j_k$. The probability vector $\vec{P}$ associated with the function $P(j_1, j_2, \ldots, j_k)$ determines the Shannon entropy

$$H_{\vec{P}} = -\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\cdots\sum_{j_k=1}^{n_k} P(j_1, j_2, \ldots, j_k)\ln P(j_1, j_2, \ldots, j_k). \tag{3}$$

Also for $s < k$, we introduce the marginals (vectors $\vec{\mathcal{P}}$)

$$\mathcal{P}(j_1, j_2, \ldots, j_s) = \sum_{j_{s+1}=1}^{n_{s+1}}\sum_{j_{s+2}=1}^{n_{s+2}}\cdots\sum_{j_k=1}^{n_k} P(j_1, j_2, \ldots, j_k). \tag{4}$$

The marginal probability distributions determine the Shannon entropies associated with the probability distributions of subsystems of the particle positions, *e.g.*,

$$H_{\vec{\mathcal{P}}} = -\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\cdots\sum_{j_s=1}^{n_s} \mathcal{P}(j_1, j_2, \ldots, j_s)\ln \mathcal{P}(j_1, j_2, \ldots, j_s). \tag{5}$$

There are obvious relations between the functionals $H_{\vec{P}}$ and $H_{\vec{\mathcal{P}}}$. For example, it is clear that the degree of disorder in the system is larger than the degree of disorder in the subsystem; this means that the mathematical inequality holds

$$H_{\vec{P}} \geq H_{\vec{\mathcal{P}}}. \tag{6}$$

Inequality (6) can be checked for an arbitrary set of nonnegative numbers $P(x_j)$, $j = 1, 2, \ldots, N$, $N = n_1 n_2 \cdots n_k$ independently of the interpretation of the numbers as the probability distribution of a position.

If $N = n_1 n_2$, there exists an entropic inequality (subadditivity condition); see, for example, [1]

$$H_{\vec{P}} \leq H_{\vec{\mathcal{P}}_1} + H_{\vec{\mathcal{P}}_2}, \tag{7}$$

where vector $\vec{P}$ has N components

$$P(1,1), P(1,2), \ldots, P(1,n_2), P(2,1), \ldots, P(2,2), P(2,n_2), \ldots, P(n_1-1,n_2), P(n_1,n_2),$$

vector $\vec{\mathcal{P}}_1$ has n_1 components, and vector $\vec{\mathcal{P}}_2$ has n_2 components,

$$\mathcal{P}_1(j_1) = \sum_{j_2=1}^{n_2} P(j_1, j_2), \qquad \mathcal{P}_2(j_2) = \sum_{j_1=1}^{n_1} P(j_1, j_2). \tag{8}$$

Inequality (7) is interpreted as the relation for Shannon entropy of bipartite system $H(1,2) \equiv H_{\vec{P}}$ and entropies of its two subsystems $H(1) \equiv H_{\vec{\mathcal{P}}_1}$ and $H(2) \equiv H_{\vec{\mathcal{P}}_2}$ of the form

$$I = H(1) + H(2) - H(1,2) \geq 0; \tag{9}$$

here I is the mutual information.

At the absence of correlations in bipartite system, *i.e.*, for the probability distribution of the product form

$$P(j_1, j_2) = \mathcal{P}_1(j_1)\mathcal{P}_2(j_2), \tag{10}$$

the mutual information $I = 0$.

There is another inequality connecting Shannon entropies for a system with three subsystems.

For a given probability distribution or the probability vector $\vec{P}$ with N components $P_1, P_2, \ldots, P_N$ labeled as $P_{j_1 j_2 j_3}$, where $j_1 = 1, 2, \ldots, n_1$, $j_2 = 1, 2, \ldots, n_2$, $j_3 = 1, 2, \ldots, n_3$, and $N = n_1 n_2 n_3$, one can introduce three extra probability vectors $\vec{\mathcal{P}}_{12}$, $\vec{\mathcal{P}}_{23}$, and $\vec{\mathcal{P}}_2$. The components of vectors $\vec{\mathcal{P}}_{12}$, $\vec{\mathcal{P}}_{23}$, and $\vec{\mathcal{P}}_2$ labeled as $(\mathcal{P}_{12})_{j_1 j_2}$, $(\mathcal{P}_{23})_{j_2 j_3}$, and $(\mathcal{P}_2)_{j_2}$, respectively, read

$$(\mathcal{P}_{12})_{j_1 j_2} = \sum_{j_3=1}^{n_3} P_{j_1 j_2 j_3}, \quad (\mathcal{P}_{23})_{j_2 j_3} = \sum_{j_1=1}^{n_1} P_{j_1 j_2 j_3}, \quad (\mathcal{P}_2)_{j_2} = \sum_{j_1=1}^{n_1} \sum_{j_3=1}^{n_3} P_{j_1 j_2 j_3},$$

The inequality called the strong subadditivity condition in terms of Shannon entropies is

$$H_{\vec{P}} + H_{\vec{\mathcal{P}}_2} \leq H_{\vec{\mathcal{P}}_{12}} + H_{\vec{\mathcal{P}}_{23}}. \tag{11}$$

One can write this inequality for N nonnegative numbers in an explicit form, *i.e.*,

$$
\begin{aligned}
(12)\qquad & -\sum_{j=1}^{N} P_j \ln P_j - \sum_{j=2}^{n_2}\left(\sum_{j=1}^{n_1}\sum_{j=3}^{n_3} P_{j_1j_2j_3}\right)\ln\left(\sum_{j=1}^{n_1}\sum_{j=3}^{n_3} P_{j_1j_2j_3}\right) \leq \\
& -\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}\left(\sum_{j_3=1}^{n_3} P_{j_1j_2j_3}\right)\ln\left(\sum_{j_3=1}^{n_3} P_{j_1j_2j_3}\right) \\
& -\sum_{j_2=1}^{n_2}\sum_{j_3=1}^{n_3}\left(\sum_{j_1=1}^{n_1} P_{j_1j_2j_3}\right)\ln\left(\sum_{j_1=1}^{n_1} P_{j_1j_2j_3}\right).
\end{aligned}
$$

Inequality (12) is valid for an arbitrary set of positive numbers $P_1, P_2, \dots P_N$ such that $\sum_{j=1}^{N} P_j = 1$. We recall that numbers $P_{j_1j_2j_3}$ are the same numbers P_j labeled, in view of the invertible map of integers j and triples of integers $j_1j_2j_3$ employed in [24].

3. – Rényi entropy

For the probability N-vector $\vec{P}$, one can introduce the parameter-dependent entropy, *e.g.*, Rényi entropy [8]

$$
R_{\vec{P}}(\lambda) = \frac{1}{1-\lambda}\ln\left(\sum_{j=1}^{N} P_j^{\lambda}\right). \qquad (13)
$$

In the limit $\lambda \to 1$, the Rényi entropy converts to the Shannon entropy

$$
R_{\vec{P}}(1) = H_{\vec{P}}. \qquad (14)
$$

The Rényi entropy contains more information on the properties of the probability vector $\vec{P}$ since it depends on extra parameters.

4. – Tsallis entropy

The other entropy is Tsallis entropy, which for given probability vector $\vec{P}$ reads

$$
T_{\vec{P}}(\lambda) = \frac{1}{1-\lambda}\left(\sum_{j=1}^{N} P_j^{\lambda} - 1\right) = -\sum_{j=1}^{N} P_j^{\lambda}\,\frac{P_j^{1-\lambda}-1}{1-\lambda}. \qquad (15)
$$

In the limit $\lambda \to 1$, the Tsallis entropy converts to the Shannon entropy

$$
T_{\vec{P}}(\lambda = 1) = H_{\vec{P}}. \qquad (16)
$$

Both entropies satisfy the relations

$$
T_{\vec{P}}(\lambda) = \frac{1}{1-\lambda}\left\{\exp\left[(1-\lambda)R_{\vec{P}}(\lambda)\right] - 1\right\}, \qquad (17)
$$

$$
R_{\vec{P}}(\lambda) = \frac{1}{1-\lambda}\left\{\exp\left[T_{\vec{P}}(\lambda)(1-\lambda)\right] + 1\right\}. \qquad (18)
$$

For small values of the number $(1-\lambda)R_{\vec{P}}(\lambda)$, relations (17) and (18) yield

$$T_{\vec{P}}(\lambda) \approx R_{\vec{P}}(\lambda). \tag{19}$$

One can characterize the difference of two probability vectors $\vec{P}$ and $\vec{S}$ by the nonnegative relative Tsallis entropy; it reads (see, *e.g.*, [28, 29])

$$D_\lambda(\vec{P} \mid \vec{S}) = -\sum_j P_j \ln_\lambda \frac{S_j}{P_j}, \quad \ln_\lambda(x) = (x^{1-\lambda}-1)(1-\lambda). \tag{20}$$

For $\lambda \to 1$, the nonnegative relative Tsallis entropy converts to the nonnegative relative Shannon entropy,

$$D_1(\vec{P} \mid \vec{S}) = \sum_j (P_j \ln P_j - P_j \ln S_j) \geq 0. \tag{21}$$

For $\vec{P} = \vec{S}$, relative entropies (20) and (21) are equal to zero.

5. – Quantum entropies

In quantum mechanics, the system states, both mixed and pure, are described by density operator $\hat{\rho}$ acting in the Hilbert space $\mathcal{H}$. The density matrix ρ_{jk} [2, 3], being determined by the operator as $\rho_{jk} = \langle j \mid \hat{\rho} \mid k\rangle$ is Hermitian matrix with nonnegative eigenvalues and unit trace. The pure states have the density operators which are rank-one projectors, *i.e.*, $\hat{\rho}^2 = \hat{\rho}$ and the purity parameter $\mathrm{Tr}\,\hat{\rho}^2 = 1$. The mixed states have the purity parameter $\mu = \mathrm{Tr}\,\hat{\rho}^2 < 1$.

The von Neumann entropy S of the quantum system is determined as

$$S = -\mathrm{Tr}\,(\hat{\rho}\ln\hat{\rho}) = -\sum_j P_j \ln P_j, \tag{22}$$

where P_j are eigenvalues of the matrix ρ. Equation (22) provides generalization of the Shannon entropy to the quantum-system states. For all pure states, the von Neumann entropy is equal to zero.

Classical Rényi and Tsallis entropies also have quantum generalization. For example, quantum Rényi entropy reads

$$R_\rho(\lambda) = \frac{1}{1-\lambda}\ln \mathrm{Tr}\,(\hat{\rho}^\lambda) = \frac{1}{1-\lambda}\ln\sum_j P_j^\lambda, \tag{23}$$

and quantum Tsallis entropy is (see, *e.g.*, [24])

$$T_\rho(\lambda) = -\mathrm{Tr}\left(\hat{\rho}\frac{\hat{\rho}^{\lambda-1}-1)}{1-\lambda}\right). \tag{24}$$

As a limit, both entropies have a von Neumann entropy

$$\lim_{\lambda\to1} T_\rho(\lambda) = \lim_{\lambda\to1} R_\rho(\lambda) = S. \tag{25}$$

The relative Tsallis entropy has the nonnegativity property, *i.e.*,

$$D_\lambda(\hat\rho \mid \hat\sigma) = \frac{1 - \mathrm{Tr}\left(\hat\rho^\lambda \hat\sigma^{1-\lambda}\right)}{1-\lambda} \geq 0. \tag{26}$$

In the limit $\lambda \to 1$, we arrive at the nonnegativity of quantum relative entropy [30]

$$U(\hat\rho \mid \hat\sigma) = \mathrm{Tr}\left(\hat\rho \ln \hat\rho - \hat\rho \ln \hat\sigma\right) \geq 0. \tag{27}$$

6. – Quantum tomographic entropies

For the spin-j systems (qudits), the states $\hat\rho$ can be described in terms of spin tomograms $w(m \mid \vec n)$ [31-34],

$$w(m \mid \vec n) = \mathrm{Tr}\, \hat\rho \mid m, \vec n\rangle\langle m, \vec n \mid, \tag{28}$$

where $m = -j, -j+1, \ldots, j$ is the spin projection on the direction $\vec n$. The spin tomogram $w(m \mid \vec n)$ is the probability distribution of the spin projections, *i.e.*, it is nonnegative and satisfies the normalization condition

$$\sum_{m=-j}^{j} w(m \mid \vec n) = 1. \tag{29}$$

For the state $\hat\rho(1, 2, \ldots, N)$ of a composite system of N spins $j_1, j_2, \ldots, j_N$, the tomogram is a joint probability distribution $w(m_1, m_2, \ldots, m_N \mid \vec n_1, \vec n_2, \ldots, \vec n_N)$,

$$\begin{aligned} w(m_1, m_2, \ldots, m_N \mid \vec n_1, \vec n_2, \ldots, \vec n_N) &= \mathrm{Tr}\left\{\hat\rho(1, 2, \ldots, N)\right. \\ \times \mid m_1, m_2, \ldots, m_N, \vec n_1, \vec n_2, \ldots, \vec n_N\rangle &\left.\langle m_1, m_2, \ldots, m_N, \vec n_1, \vec n_2, \ldots, \vec n_N \mid\right\}, \end{aligned}$$

where m_k is the spin projection of the k-th spin j_k onto the direction $\vec n_k$.

The fact that quantum states can be identified with classical probability distributions provides the possibility to reconstruct the density operator of the system states [33-35], and for quantum tomograms one can use the relationships known for classical probabilities. Thus, the tomographic Shannon entropy of quantum state reads

$$H(\vec n) = -\sum_{m=-j}^{j} w(m \mid \vec n) \ln w(m \mid \vec n). \tag{30}$$

Tomographic entropies have all properties of entropies associated with classical probabilities and simultaneously describe quantum states. In view of this fact, the relationships like equalities and inequalities for tomographic entropies reflect the properties of quantum states.

For example, the subadditivity condition for tomographic Shannon entropy of the two-qudit system

$$H(\vec n_1, \vec n_2) = -\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} w(m_1, m_2 \mid \vec n_1, \vec n_2) \ln w(m_1, m_2 \mid \vec n_1, \vec n_2), \tag{31}$$

which provides the nonnegativity condition for tomographic information, along with its value

$$I_t = -H(\vec{n}_1,\vec{n}_2) + H_1(\vec{n}_1) + H_2(\vec{n}_2) \geq 0, \tag{32}$$

where $H_1(\vec{n}_1)$ and $H_2(\vec{n}_2)$ are tomographic entropies of the first and second qudits, respectively,

$$H_1(\vec{n}_1) = -\sum_{m_1=-j_1}^{j_1} w_1(m_1 \mid \vec{n}_1)\ln w_1(m_1 \mid \vec{n}_1), \tag{33}$$

$$H_2(\vec{n}_2) = -\sum_{m_2=-j_2}^{j_2} w_2(m_2 \mid \vec{n}_2)\ln w_2(m_2 \mid \vec{n}_2), \tag{34}$$

characterizes a degree of quantum correlations in the bipartite qudit system. The tomograms $w_1(m_1 \mid \vec{n}_1)$ and $w_2(m_2 \mid \vec{n}_2)$ are marginals of the joint probability distribution $w(m_1,m_2 \mid \vec{n}_1,\vec{n}_2)$. An analogous new inequality for tomographic-probability distribution of two qudits can be written for Tsallis entropy [24].

The strong subadditivity condition [14] for a system of three qudits can be also written for tomogram $w(m_1,m_2,m_3 \mid \vec{n}_1,\vec{n}_2.\vec{n}_3)$; it reads

$$H(\vec{n}_1,\vec{n}_2,\vec{n}_3) + H_2(\vec{n}_2) \leq H_{12}(\vec{n}_1,\vec{n}_2) + H_{23}(\vec{n}_2,\vec{n}_3), \tag{35}$$

where tomographic entropies of the qudit system and its subsystems are

$$\begin{aligned}
H(\vec{n}_1,\vec{n}_2,\vec{n}_3) &= -\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3} w(m_1,m_2,m_3 \mid \vec{n}_1,\vec{n}_2.\vec{n}_3)\\
&\quad\times \ln w(m_1,m_2,m_3 \mid \vec{n}_1,\vec{n}_2.\vec{n}_3),\\
H_{12}(\vec{n}_1,\vec{n}_2) &= -\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} w_{12}(m_1,m_2 \mid \vec{n}_1,\vec{n}_2)\ln w_{12}(m_1,m_2 \mid \vec{n}_1,\vec{n}_2),\\
H_2(\vec{n}_2) &= -\sum_{m_2=-j_2}^{j_2} w_2(m_2 \mid \vec{n}_2)\ln w_2(m_2 \mid \vec{n}_2),\\
H_{23}(\vec{n}_2,\vec{n}_3) &= -\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3} w_{23}(m_2,m_3 \mid \vec{n}_2,\vec{n}_3)\ln w_{23}(m_2,m_3 \mid \vec{n}_2,\vec{n}_3).
\end{aligned}$$

Here, tomograms $w_{12}(m_1,m_2 \mid \vec{n}_1,\vec{n}_2)$, $w_2(m_2 \mid \vec{n}_2)$, and $w_{23}(m_2,m_3 \mid \vec{n}_2,\vec{n}_3)$ are marginals of the joint probability distribution $w(m_1,m_2,m_3 \mid \vec{n}_1,\vec{n}_2,\vec{n}_3)$. In spite of the fact that inequality (35) is classical entropic inequality, it characterizes quantum correlations in composite qudit systems.

Tomographic entropic inequalities for bipartite (32) and tripartite (35) qudit systems correspond to the presence of quantum correlations of qudit observables of the subsystems. The inequalities can be checked experimentally, *e.g.*, in experiments with superconducting circuits where the density matrix of qudit states is measured [36-38].

We can write the subadditivity condition for Tsallis entropy in the form

$$\begin{aligned}(36)\qquad &\frac{1}{1-\lambda}\left\{\left(\sum_{m_1=-j_1}^{j_1} w_1^\lambda(m_1\mid\vec{n}_1)-1\right)+\left(\sum_{m_2=-j_2}^{j_2} w_2^\lambda(m_2\mid\vec{n}_2)-1\right)\right\}\geq\\ &\frac{1}{1-\lambda}\left\{\left(\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} w^\lambda(m_1,m_2\mid\vec{n}_1,\vec{n}_2)-1\right)\right\}.\end{aligned}$$

In the limit $\lambda\to 1$, inequality (36) converts to the Shannon tomographic entropic inequality discussed above. Since Rényi entropy is a function of Tsallis entropy, inequality (36) provides some relationships for Rényi entropy.

7. – Hidden correlations in single qudit systems

It was found (see, for example, [23, 24]) that the subadditivity condition and the strong subadditivity condition exist for single qudit states as well. We demonstrate these inequalities written for tomograms of single-qudit states on the examples of qudit with $j = 3/2$ and qudit with $j = 7/2$, respectively. Tomogram of the qudit state with $j = 3/2$, *i.e.*, $w(m\mid\vec{n})$, $m = -3/2,\ -1/2,\ 1/2,\ 3/2$ can be considered as a probability distribution $P_{11}(\vec{n})\equiv w(3/2\mid\vec{n})$, $P_{12}(\vec{n})\equiv w(1/2\mid\vec{n})$, $P_{21}(\vec{n})\equiv w(-1/2\mid\vec{n})$, and $P_{22}(\vec{n})\equiv w(-3/2\mid\vec{n})$ of two "artificial" qubits. Such a consideration [23,24,36] provides the possibility to introduce marginals for each "artificial" qubits, respectively:

$$(37)\qquad \Omega_+^{(1)}(\vec{n}) = P_{11}(\vec{n})+P_{12}(\vec{n}),\qquad \Omega_-^{(1)}(\vec{n}) = P_{21}(\vec{n})+P_{22}(\vec{n}),$$

$$(38)\qquad \Omega_+^{(2)}(\vec{n}) = P_{11}(\vec{n})+P_{21}(\vec{n}),\qquad \Omega_-^{(2)}(\vec{n}) = P_{12}(\vec{n})+P_{22}(\vec{n}).$$

One can apply the subadditivity condition for Tsallis entropy to these probability distributions written in terms of tomogram and arrive at the following inequality:

$$\begin{aligned}(39)\quad &\frac{1}{1-\lambda}\left[\left(\Omega_+^{(1)}(\vec{n})\right)^\lambda+\left(\Omega_-^{(1)}(\vec{n})\right)^\lambda-1\right]+\frac{1}{1-\lambda}\left[\left(\Omega_+^{(2)}(\vec{n})\right)^\lambda+\left(\Omega_-^{(2)}(\vec{n})\right)^\lambda-1\right]\geq\\ &\frac{1}{1-\lambda}\left[w^\lambda(3/2\mid\vec{n})+w^\lambda(1/2\mid\vec{n})+w^\lambda(-1/2\mid\vec{n})+w^\lambda(-3/2\mid\vec{n})-1\right].\end{aligned}$$

We can rewrite this inequality in terms of qudit probabilities and obtain

$$\begin{aligned}(40)\quad &\frac{1}{1-\lambda}\left\{[(w(3/2\mid\vec{n})+w(1/2\mid\vec{n})]^\lambda+[w(-1/2\mid\vec{n})+w(-3/2\mid\vec{n})]^\lambda-1\right\}+\\ &\frac{1}{1-\lambda}\left\{[(w(3/2\mid\vec{n})+w(-1/2\mid\vec{n})]^\lambda+[w(1/2\mid\vec{n})+w(-3/2\mid\vec{n})]^\lambda-1\right\}\geq\\ &\frac{1}{1-\lambda}\left[w^\lambda(3/2\mid\vec{n})+w^\lambda(1/2\mid\vec{n})+w^\lambda(-1/2\mid\vec{n})+w^\lambda(-3/2\mid\vec{n})-1\right].\end{aligned}$$

In the limit $\lambda \to 1$, inequality (40) converts to the Shannon tomographic entropic inequality

$$
\begin{aligned}
(41)\qquad & -\ [w(3/2\mid\vec{n})+w(1/2\mid\vec{n})]\ln[w(3/2\mid\vec{n})+w(1/2\mid\vec{n})]\\
& -\ [w(-1/2\mid\vec{n})+w(-3/2\mid\vec{n})]\ln[w(-1/2\mid\vec{n})+w(-3/2\mid\vec{n})]\\
& -\ [w(3/2\mid\vec{n})+w(-1/2\mid\vec{n})]\ln[w(3/2\mid\vec{n})+w(-1/2\mid\vec{n})]\\
& -\ [w(1/2\mid\vec{n})+w(-3/2\mid\vec{n})]\ln[w(1/2\mid\vec{n})+w(-3/2\mid\vec{n})]\geq\\
& -w(3/2\mid\vec{n})\ln[w(3/2\mid\vec{n})]-w(1/2\mid\vec{n})\ln[w(1/2\mid\vec{n})]\\
& -w(-1/2\mid\vec{n})\ln[w(-1/2\mid\vec{n})]-w(-3/2\mid\vec{n})\ln[w(-3/2\mid\vec{n})]\,.
\end{aligned}
$$

The difference of the left- and right-hand sides of this inequality can be interpreted as Shannon information related to correlations of the two "artificial" qubits. This new inequality can be checked experimentally.

The physical meaning of two "artificial" qubits associated to the qudit with $j = 3/2$ is as follows.

The states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$ of two "artificial" qubits correspond to the states $|\,3/2,\vec{n}\rangle$, $|\,1/2,\vec{n}\rangle$,$|\,-1/2,\vec{n}\rangle$, and $|\,-3/2,\vec{n}\rangle$, respectively.

The state of the first "artificial" qubit $|\uparrow,\vec{n}_1\rangle$ corresponds to the event where measurements give the value of spin-3/2 projections on the direction $\vec{n}$ equal either to $+3/2$ or to $+1/2$.

The state of the first "artificial" qubit $|\downarrow,\vec{n}_1\rangle$ corresponds to the event where measurements give the value of spin-3/2 projections on the direction $\vec{n}$ equal either to $-3/2$ or to $-1/2$.

The second artificial qubit states $|\uparrow,\vec{n}_2\rangle$ and $|\downarrow,\vec{n}_2\rangle$ correspond to two pairs of the qudit projections on the direction $\vec{n}$, namely, $(+3/2,-1/2)$ and $(+1/2,-3/2)$, respectively.

In a similar way, we can obtain the entropic inequalities for arbitrary single qudit states, which are tomographic entropic subadditivity and strong subadditivity conditions.

In Appendix A, we present the strong subadditivity condition for the tomogram of a single-qudit state with $j = 7/2$.

8. – Entropy of the system thermal state

In this section, we consider a system with Hamiltonian $\hat{H}$ acting in the Hilbert space $\mathcal{H}$. We assume that the system is in thermal state; it is a specific state with the density operator

$$\hat{\rho}(\beta)=Z^{-1}(\beta)\exp(-\beta\hat{H}),\tag{42}$$

where $Z(\beta)$ is the partition function, the parameter $\beta = T^{-1}$, and T is the temperature. The energy of the system in the state $\hat{\rho}(\beta)$ is defined as

$$E(\beta)=\langle\hat{H}\rangle=\mathrm{Tr}\left[Z^{-1}(\beta)\left(\exp(-\beta\hat{H})\right)\hat{H}\right].\tag{43}$$

The von Neumann entropy $S(\beta)$ of the system in the state (42) is equal to

$$S(\beta)=-\mathrm{Tr}\,[\hat{\rho}(\beta)\ln\hat{\rho}(\beta)]=\mathrm{Tr}\left[Z^{-1}(\beta)\left(\exp(-\beta\hat{H})\right)\left(\beta\hat{H}+\ln Z(\beta)\right)\right].\tag{44}$$

From eqs. (42)–(44) follows that for the system state with the density operator $\hat{\rho}(\beta)$ there exists the relation between entropy $S(\beta)$ and energy $E(\beta)$, namely,

$$S(\beta) - \beta E(\beta) = \ln Z(\beta). \tag{45}$$

In [39], the inequality relating entropy and energy of arbitrary qudit systems was obtained; it looks as follows:

$$E + S \leq \ln Z(\beta = -1), \tag{46}$$

where the entropy of the system in the state $\hat{\rho}$ reads $S = -\mathrm{Tr}\,(\hat{\rho} \ln \hat{\rho})$, and the energy of the system is $E = \mathrm{Tr}(\hat{H}\hat{\rho})$, with $\hat{H}$ being the qudit-system Hamiltonian. The function $Z(\beta) = \mathrm{Tr}\,\exp(-\beta\hat{H})$ in (46) is the partition function. This relation was obtained in view of the nonnegativity condition for the relative entropy by comparing the density operator $\hat{\rho}$ and the density operator $Z^{-1}(\beta)\exp(-\beta\hat{H})$ of the thermal state.

We can see that the thermal state provides equality (45); this means that in the thermal state the sum of dimensionless energy and entropy is determined by the partition function at the point $\beta = -1$.

9. – Some inequalities for unitary matrices

In this section, we use entropic inequalities to obtain some new relations for matrix elements of unitary matrices.

For the density matrix $\rho(1,2)$ of the bipartite-system state, one has the Araki-Lieb inequality [12] for the von Neumann entropies of the system and its subsystems

$$-\mathrm{Tr}\,(\rho(1,2)\ln\rho(1,2)) \geq \left|\mathrm{Tr}\,(\rho(1)\ln\rho(1)) - \mathrm{Tr}\,(\rho(2)\ln\rho(2))\right|, \tag{47}$$

where $\rho(1)$ and $\rho(2)$ are the density matrices of the subsystem states $\rho(1) = \mathrm{Tr}_2(\rho(1,2))$ and $\rho(2) = \mathrm{Tr}_1(\rho(1,2))$.

For the pure state, the entropy of the system state is equal to zero; this means that $\mathrm{Tr}\,(\rho(1)\ln\rho(1)) = \mathrm{Tr}\,(\rho(2)\ln\rho(2))$, and it is related to the fact that the nonzero eigenvalues of matrices $\rho(1)$ and $\rho(2)$ coincide.

Now we employ this circumstance while considering the matrix $\rho = uPu^\dagger$, where P is rank-1 projector and u is unitary matrix. If all matrix elements of the matrix P except P_{11} are equal to zero, then the matrix ρ corresponds to pure state with zero entropy. In [24], it was shown that matrices $\rho(1)$ and $\rho(2)$ are obtained by the following recipe.

If matrix ρ is the $N \times N$-matrix, where $N = mn$, it can be presented in a block form with m^2 blocks ρ_{jk} $(j,k = 1,2,\ldots,m)$. Then the matrix $\rho(1)$ is the $m \times m$-matrix with matrix elements $\mathrm{Tr}\,\rho_{jk}$. The matrix $\rho(2)$ is the $n \times n$-matrix expressed as a sum of blocks, $\rho(2) = \sum_{j=1}^{m}\rho_{jj}$. The matrices $\rho(1)$ and $\rho(2)$ are nonnegative Hermitian ones with unit trace, if the initial matrix ρ is nonnegative Hermitian matrix with $\mathrm{Tr}\,\rho = 1$.

Now for $N = nm = 6 = 2 \cdot 3$, we construct explicitly matrices $\rho(1)$ and $\rho(2)$ for the initial 6×6-matrix ρ given in the block form, $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$, where blocks ρ_{11}, ρ_{12}, ρ_{21}, and ρ_{22} are 3×3-matrices $\rho_{jk} = \begin{pmatrix} R_{11}^{jk} & R_{12}^{jk} & R_{13}^{jk} \\ R_{21}^{jk} & R_{22}^{jk} & R_{23}^{jk} \\ R_{31}^{jk} & R_{32}^{jk} & R_{33}^{jk} \end{pmatrix}$. Then the recipe formulated provides the 2×2-matrix $\rho(1)=\sum_{s=1}^{2}\begin{pmatrix} R_{ss}^{11} & R_{ss}^{12}\\ R_{ss}^{21} & R_{ss}^{22}\end{pmatrix}$ and the 3×3-matrix $\rho(2)=\sum_{k=1}^{3}\begin{pmatrix} R_{11}^{kk} & R_{12}^{kk} & R_{13}^{kk}\\ R_{21}^{kk} & R_{22}^{kk} & R_{23}^{kk}\\ R_{31}^{kk} & R_{32}^{kk} & R_{33}^{kk}\end{pmatrix}$. According to the Araki-Lieb inequality, the nonzero eigenvalues of matrices $\rho(1)$ and $\rho(2)$ are equal, and this fact provides the relations for matrix elements of the unitary 6×6-matrix u.

We take a column of the unitary matrix u and denote its matrix elements as $u_{j1}=z_1$, $u_{j2}=z_2$, $u_{j3}=z_3$, $u_{j4}=z_4$, $u_{j5}=z_5$, and $u_{j6}=z_6$. The matrix

$$\rho(1)=\begin{pmatrix} |z_1|^2+|z_2|^2+|z_3|^2 & z_1z_4^*+z_2z_5^*+z_3z_6^*\\ z_1^*z_4+z_2^*z_5+z_3^*z_6 & |z_4|^2+|z_5|^2+|z_6|^2\end{pmatrix}$$

has nonnegative eigenvalues λ_1 and λ_2, and the matrix

$$\rho(2)=\begin{pmatrix} |z_1|^2+|z_4|^2 & z_1z_2^*+z_4z_5^* & z_1z_3^*+z_4z_6^*\\ z_1^*z_2+z_4^*z_5 & |z_2|^2+|z_5|^2 & z_2z_3^*+z_5z_6^*\\ z_1^*z_3+z_4^*z_6 & z_2^*z_3+z_5^*z_6 & |z_3|^2+|z_6|^2\end{pmatrix}$$

has the same eigenvalues λ_1 and λ_2 plus $\lambda_3=0$; this means that $\det\rho(2)=0$. The eigenvalues λ_1 and λ_2 of the matrices $\rho(1)$ and $\rho(2)$ read

$$\lambda_{1,2}=\frac{1}{2}\pm\frac{1}{2}\left[1-4\left(a(1-a)-|b|^2\right)\right]^{1/2},$$

where $a=|z_1|^2+|z_2|^2+|z_3|^2$ and $b=z_1z_4^*+z_2z_5^*+z_3z_6^*$.

Analogous relations are valued for column (rows) of arbitrary unitary $N\times N$-matrices, and the properties of matrices $\rho(1)$ and $\rho(2)$ are expressed as the equality

$$\mathrm{Tr}\left(\rho(1)\ln\rho(1)\right)=\mathrm{Tr}\left(\rho(2)\ln\rho(2)\right).$$

Explicit expressions for matrices ρ, $\rho(1)$, and $\rho(2)$ in terms of matrix elements u_{11}, u_{21}, u_{31}, u_{41}, u_{51}, and u_{61} of the unitary 6×6-matrix are given in Appendix B.

10. – Conclusions

To conclude, we point out the main results of our work.

We reviewed the properties of classical and quantum entropies including deformed entropies. We considered entropic inequalities like the subadditivity and strong subadditivity conditions and applied these inequalities to tomographic-probability distributions of single qudit states. The entropic inequalities for qudit tomograms are new quantum inequalities characterizing correlations in systems of qudits. Employing the structure of entropic inequalities we obtained some new nonlinear relations for matrix elements of unitary matrices. The relations obtained can be checked experimentally, *e.g.*, in experiments with superconducting qudits [36-38].

Appendix A

We present here the strong subadditivity condition for tomogram $w(m \mid \vec{n})$ of the qudit state with $j = 7/2$, where $m = 7/2, 5/2, \ldots, -7/2$; it reads

$$
\begin{aligned}
&- \sum_{m=-7/2}^{7/2} \omega(m \mid \vec{n}) \ln \omega(m \mid \vec{n}) \\
&- \left(\sum_{m>0} \omega(m \mid \vec{n})\right) \ln \left(\sum_{m'>0} \omega(m' \mid \vec{n})\right) - \left(\sum_{m<0} \omega(m \mid \vec{n})\right) \ln \left(\sum_{m'<0} \omega(m' \mid \vec{n})\right) \leq \\
&- \left(w(7/2 \mid \vec{n}) + w(5/2 \mid \vec{n})\right) \ln \left(w(7/2 \mid \vec{n}) + w(5/2 \mid \vec{n})\right) \\
&- \left(w(3/2 \mid \vec{n}) + w(1/2 \mid \vec{n})\right) \ln \left(w(3/2 \mid \vec{n}) + w(1/2 \mid \vec{n})\right) \\
&- \left(w(-1/2 \mid \vec{n}) + w(-3/2 \mid \vec{n})\right) \ln \left(w(-1/2 \mid \vec{n}) + w(-3/2 \mid \vec{n})\right) \\
&- \left(w(-5/2 \mid \vec{n}) + w(-7/2 \mid \vec{n})\right) \ln \left(w(-5/2 \mid \vec{n}) + w(-7/2 \mid \vec{n})\right) \\
&- \left(w(7/2 \mid \vec{n}) + w(-1/2 \mid \vec{n})\right) \ln \left(w(7/2 \mid \vec{n}) + w(-1/2 \mid \vec{n})\right) \\
&- \left(w(5/2 \mid \vec{n}) + w(-3/2 \mid \vec{n})\right) \ln \left(w(5/2 \mid \vec{n}) + w(-3/2 \mid \vec{n})\right) \\
&- \left(w(3/2 \mid \vec{n}) + w(-5/2 \mid \vec{n})\right) \ln \left(w(3/2 \mid \vec{n}) + w(-5/2 \mid \vec{n})\right) \\
&- \left(w(1/2 \mid \vec{n}) + w(-7/2 \mid \vec{n})\right) \ln \left(w(1/2 \mid \vec{n}) + w(-7/2 \mid \vec{n})\right).
\end{aligned}
$$

This entropic inequality written for noncomposite quantum system is a new inequality which can be checked experimentally. It reflects quantum correlations between different groups of spin projections of the results of measurements. At the absence of correlation the inequality converts to the equality.

Appendix B

Given a unitary 6×6-matrix u_{jk}. We construct the matrix ρ

$$
\rho = \begin{pmatrix}
|u_{11}|^2 & u_{11}u_{21}^* & u_{11}u_{31}^* & u_{11}u_{41}^* & u_{11}u_{51}^* & u_{11}u_{61}^* \\
u_{21}u_{11}^* & |u_{21}|^2 & u_{21}u_{31}^* & u_{21}u_{41}^* & u_{21}u_{51}^* & u_{21}u_{61}^* \\
u_{31}u_{11}^* & u_{31}u_{21}^* & |u_{31}|^2 & u_{31}u_{41}^* & u_{31}u_{51}^* & u_{31}u_{61}^* \\
u_{41}u_{11}^* & u_{41}u_{21}^* & u_{41}u_{31}^* & |u_{41}|^2 & u_{41}u_{51}^* & u_{41}u_{61}^* \\
u_{51}u_{11}^* & u_{51}u_{21}^* & u_{51}u_{31}^* & u_{51}u_{41}^* & |u_{51}|^2 & u_{51}u_{61}^* \\
u_{61}u_{11}^* & u_{61}u_{21}^* & u_{61}u_{31}^* & u_{61}u_{41}^* & u_{61}u_{51}^* & |u_{61}|^2
\end{pmatrix}
$$

and two matrices $\rho(1)$

$$
\rho(1) = \begin{pmatrix}
|u_{11}|^2 + |u_{21}|^2 + |u_{31}|^2 & u_{11}u_{41}^* + u_{21}u_{51}^* + u_{31}u_{61}^* \\
u_{41}u_{11}^* + u_{51}u_{21}^* + u_{61}u_{31}^* & |u_{41}|^2 + |u_{51}|^2 + |u_{61}|^2
\end{pmatrix}
$$

and $\rho(2)$

$$
\rho(2) = \begin{pmatrix}
|u_{11}|^2 + |u_{41}|^2 & u_{11}u_{21}^* + u_{41}u_{51}^* & u_{11}u_{31}^* + u_{41}u_{61}^* \\
u_{21}u_{11}^* + u_{51}u_{41}^* & |u_{21}|^2 + |u_{51}|^2 & u_{21}u_{31}^* + u_{51}u_{61}^* \\
u_{31}u_{11}^* + u_{61}u_{41}^* & u_{31}u_{21}^* + u_{61}u_{51}^* & |u_{31}|^2 + |u_{61}|^2
\end{pmatrix}.
$$

They satisfy the nonlinear equations for equal eigenvalues of matrices $\rho(1)$ and $\rho(2)$.

Different nonlinear equations and their applications were discussed in [40]. The solutions in terms of unitary matrices of nonlinear matrix equations can be obtained for $N \times N$-matrices ρ.

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