Quintom Potential from Quantum Anisotropic Cosmological Models

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Additional information is available at the end of the chapter

1. Introduction

At the present time, there are some paradigms to explain the observations for the accelerated expansion of the universe. Most of these paradigms are based on the dynamics of a scalar (quintessence) or multiscalar field (quintom) cosmological models of dark energy, (see the review [1–3]). The main discussion yields over the evolution of these models in the ω-ω' plane (ω is the equation of state parameter of the dark energy) [4–12, 14–19, 21–25]). In the present study we desire to perform our investigation in the case of quintom cosmology, constructed using both quintessence (σ) and phantom (ϕ) fields, mantaining a nonspecific potential form $V(\phi, \sigma)$. There are many works in the literature that deals with this type of problems, but in a general way, and not with a particular ansatz, one that considers dynamical systems [12, 13, 20]. One special class of potentials used to study this behaviour corresponds to the case of the exponential potentials [4, 6, 9, 26, 28] for each field, where the corresponding energy density of a scalar field has the range of scaling behaviors [29, 30], i.e, it scales exactly as a power of the scale factor like, $\rho_\phi \propto a^{-m}$, when the dominant component has an energy density which scales in a similar way. There are other works where other type of potentials are analyzed [1, 9, 15, 19, 20, 23, 24, 31].

How come that we claim that the analysis of general potentials using dynamical systems was made considering particular structures of them, in other words, how can we introduce this mathematical structure within a physical context?. We can partially answer this question, when the Bohmian formalism is introduced, i.e, many of them can be constructed using the Bohm formalism [32–34] of the quantum mechanics under the integral systems premise, which is known as the quantum potential approach. This approach makes possible to identify trajectories associated with the wave function of the universe [32] when we choose the superpotential function as the momenta associated to the coordinate field q^μ. This

investigation was undertaken within the framework of the minisuperspace approximation of quantum theory when we investigate the models with a finite number of degrees of freedom. Considering the anisotropic Bianchi Class A cosmological models from canonical quantum cosmology under determined conditions in the evolution of our universe, and employing the Bohmian formalism, and in particular the Bianchi type I to obtain a family of potentials that correspond to the most probable to model the present day cosmic acceleration. In our analysis, we found this special class of potentials, however these appear mixed.

This work is arranged as follows. In section 2 we present the corresponding Einstein Klein Gordon equation for the quintom model. In section 3, we introduced the hamiltonian apparatus which is applied to Bianchi type I and the Bianchi Type IX in order to construct a master equation for all Bianchi Class A cosmological models with barotropic perfect fluid and cosmological constant. Furthermore, we present the classical equations for Bianchi type I, whose solutions are given in a quadrature form, which are presented in section 5 for particular scalar potentials. In section 4 we present the quantum scheme, where we use the Bohmian formalism and show its mathematical structure, also our approach is presented in a similar way. Our treatment is applied to build the mathematical structure of quintom scalar potentials using the integral systems formalism. For completeness we present the quantum solutions to the Wheeler-DeWitt equation. Moreover, this section represents our main objective for this work, and its where the utmost problem is treated. However it is important emphasize that the quantum potential from Bohm formalism will work as a constraint equation which restricts our family of potentials found. It is well known in the literature that in the Bohm formalism the imaginary part is never determined, however in this work such a problem is solved in order to find the quantum potentials, which is a more important matter for being able to find the classical trajectories, which is shown in section 5, that is devoted to obtain the classical solutions for particular scalar potentials, also we show through graphics how the classical trajectory is projected from its quantum counterpart. Finally we present the time dependence for the Ω, and quintom scalar fields (φ, ς).

2. The model

We begin with the construction of the quintom cosmological paradigm, which requires the simultaneous consideration of two fields, namely one canonical σ and one phantom ϕ and the implication that dark energy will be attributed to their combination. The action of a universe with the constitution of such a two fields, the cosmological term contribution and the matter as perfect fluid content, is

$$\mathcal{L} = \sqrt{-g} \left(R - 2\Lambda - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma - V(\phi, \sigma) \right) + \mathcal{L}_{\text{matter}}, \qquad (1)$$

and the corresponding field equations becomes

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = -\frac{1}{2} \left(\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right)$$
$$+ \frac{1}{2} \left(\nabla_\alpha \sigma \nabla_\beta \sigma - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma \right)$$

$$-\frac{1}{2}g_{\alpha\beta}\,V(\phi,\sigma) - 8\pi GT_{\alpha\beta}, \tag{2}$$

$$g^{\mu\nu}\phi_{,\mu\nu} - g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}\nabla_{\nu}\phi + \frac{\partial V}{\partial\phi} = 0, \qquad \Leftrightarrow \qquad \Box\phi + \frac{\partial V}{\partial\phi} = 0$$

$$g^{\mu\nu}\sigma_{,\mu\nu} - g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}\nabla_{\nu}\sigma - \frac{\partial V}{\partial\sigma} = 0, \qquad \Leftrightarrow \qquad \Box\sigma - \frac{\partial V}{\partial\sigma} = 0,$$

$$T^{\mu\nu}_{;\mu} = 0, \quad \text{with} \quad T_{\mu\nu} = Pg_{\mu\nu} + (P+\rho)u_{\mu}u_{\nu}, \tag{3}$$

here ρ is the energy density, P the pressure, and u_{μ} the velocity, satisfying that $u_{\mu}u^{\mu} = -1$.

3. Hamiltonian approach

Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi Class A cosmological models. The metric has the form

$$ds^2 = -N(t)dt^2 + e^{2\Omega(t)}\,(e^{2\beta(t)})_{ij}\,\omega^i\,\omega^j, \tag{4}$$

where $\beta_{ij}(t)$ is a 3x3 diagonal matrix, $\beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, $\Omega(t)$ is a scalar and ω^i are one-forms that characterize each cosmological Bianchi type model, and obey the form $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$, and C^i_{jk} are structure constants of the corresponding model.

The corresponding metric of the Bianchi type I in Misner's parametrization has the following form

$$ds_I^2 = -N^2dt^2 + e^{2\Omega+2\beta_++2\sqrt{3}\beta_-}dx^2 + e^{2\Omega+2\beta_+-2\sqrt{3}\beta_-}dy^2 + e^{2\Omega-4\beta_+}dz^2, \tag{5}$$

where the anisotropic radii are

$$R_1 = e^{\Omega+\beta_++\sqrt{3}\beta_-}, \qquad R_2 = e^{\Omega+\beta_+-\sqrt{3}\beta_-}, \qquad R_3 = e^{\Omega-2\beta_+}.$$

We use the Bianchi type I and IX cosmological models as toy model to apply the formalism, and write a master equation for all Bianchi Class A models. The lagrangian density (1) for the Bianchi type I is written as (where the overdot denotes time derivative),

$$\mathcal{L}_I = e^{3\Omega}\left[6\frac{\dot{\Omega}^2}{N} - 6\frac{\dot{\beta}_+^2}{N} - 6\frac{\dot{\beta}_-^2}{N} + 6\frac{\dot{\varphi}^2}{N} - 6\frac{\dot{\varsigma}^2}{N} + N\left(-V(\varphi,\varsigma) + 2\Lambda + 16\pi G\rho\right)\right], \tag{6}$$

the fields were re-scaled as $\phi = \sqrt{12}\varphi, \sigma = \sqrt{12}\varsigma$ for simplicity in the calculations.

The momenta are defined as $\Pi_{q^i} = \frac{\partial\mathcal{L}}{\partial\dot{q}^i}$, where $q^i = (\beta_\pm, \Omega, \varphi, \varsigma)$ are the coordinates fields.

$$\Pi_\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} = \frac{12e^{3\Omega}\dot{\Omega}}{N}, \qquad \rightarrow \dot{\Omega} = \frac{N\Pi_\Omega}{12}e^{-3\Omega}$$

$$\Pi_\pm = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_\pm} = -12\frac{e^{3\Omega}\dot{\beta}_\pm}{N}, \qquad \rightarrow \dot{\beta}_\pm = -\frac{N\Pi_\pm}{12}e^{-3\Omega} \qquad (7)$$

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 12\frac{e^{3\Omega}\dot{\varphi}}{N}, \qquad \rightarrow \dot{\varphi} = \frac{N\Pi_\varphi}{12}e^{-3\Omega}$$

$$\Pi_\varsigma = \frac{\partial \mathcal{L}}{\partial \dot{\varsigma}} = -12\frac{e^{3\Omega}\dot{\varsigma}}{N}, \qquad \rightarrow \dot{\varsigma} = -\frac{N\Pi_\varsigma}{12}e^{-3\Omega}.$$

Writing (6) in canonical form, $\mathcal{L}_{canonical} = \Pi_q \dot{q} - N\mathcal{H}$ and substituting the energy density for the barotropic fluid, we can find the Hamiltonian density $\mathcal{H}$ in the usual way

$$\mathcal{H}_I = \frac{e^{-3\Omega}}{24}\left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega}\left\{24V(\varphi,\varsigma) - 48\left(\Lambda + 8\pi GM_\gamma e^{-3(\gamma+1)\Omega}\right)\right\}\right]. \quad (8)$$

For the Bianchi type IX we have the Lagrangian and Hamiltonian density, respectively

$$\mathcal{L}_{IX} = e^{3\Omega}\left[6\frac{\dot{\Omega}^2}{N} - 6\frac{\dot{\beta}_+^2}{N} - 6\frac{\dot{\beta}_-^2}{N} + 6\frac{\dot{\varphi}^2}{N} - 6\frac{\dot{\varsigma}^2}{N} + N\left(-V(\varphi,\varsigma) + 2\Lambda + 16\pi G\rho\right)\right]$$
$$+Ne^{-2\Omega}\left\{\frac{1}{2}\left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+}\right)\right.$$
$$\left. - \left(e^{-2\beta_+ + 2\sqrt{3}\beta_-} + e^{-2\beta_+ - 2\sqrt{3}\beta_-} + e^{4\beta_+}\right)\right\}\right], \quad (9)$$

$$\mathcal{H}_{IX} = \frac{e^{-3\Omega}}{24}\left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega}\left\{24V(\varphi,\varsigma) - 48\left(\Lambda + 8\pi GM_\gamma e^{-3(\gamma+1)\Omega}\right)\right\}\right]$$
$$-24e^{4\Omega}\left\{\frac{1}{2}\left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+}\right)\right.$$
$$\left. - \left(e^{-2\beta_+ + 2\sqrt{3}\beta_-} + e^{-2\beta_+ - 2\sqrt{3}\beta_-} + e^{4\beta_+}\right)\right\}\right], \quad (10)$$

where we have used the covariant derivative of (3), obtaining the relation

$$3\dot{\Omega}\rho + 3\dot{\Omega}p + \dot{\rho} = 0, \quad (11)$$

whose solution becomes

$$\rho = M_\gamma e^{-3(1+\gamma)\Omega}. \quad (12)$$

where M_γ is an integration constant, in this sense we have all Bianchi Class A cosmological models, and their corresponding Hamiltonian density becomes

$$\mathcal{H}_A = \mathcal{H}_I - \frac{1}{24} e^{-3\Omega} U_A(\Omega, \beta_\pm), \tag{13}$$

where the gravitational potential can be seen in table I, in particular, the Bianchi Type IX is

$$U_{IX}(\Omega, \beta_\pm) = 12 e^{4\Omega} \left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{4\beta_+} \right.$$
$$\left. -2 \left\{ e^{4\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-} + e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right).$$

Considering the inflationary phenomenon $\gamma = -1$, the Hamiltonian density is

$$\mathcal{H}_{IX} = \frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right.$$
$$-24 e^{4\Omega} \left\{ \frac{1}{2} \left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} \right) \right.$$
$$\left. \left. - \left(e^{-2\beta_+ + 2\sqrt{3}\beta_-} + e^{-2\beta_+ - 2\sqrt{3}\beta_-} + e^{4\beta_+} \right) \right\} \right], \tag{14}$$

where $\lambda_{\text{eff}} = 48(\Lambda + 8\pi G M_{-1})$.

The equation (13) can be considered as a master equation for all Bianchi Class A cosmological models in this formalism, where $U(\Omega, \beta_\pm)$ is the potential term of the cosmological model under consideration, it can be read in table 1.

Bianchi type	Hamiltonian density $\mathcal{H}$
I	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right]$
II	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right.$ $\left. -12 e^{4\Omega} e^{4\beta_+ + 4\sqrt{3}\beta_-} \right]$
VI$_{-1}$	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right.$ $\left. -48 e^{4\Omega} e^{4\beta_+} \right]$
VII$_0$	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{eff} \right\} \right.$ $\left. -12 e^{4\Omega} \left(e^{4\beta_+ + 4\sqrt{3}\beta_-} - e^{4\beta_+} + e^{4\beta_+ - 4\sqrt{3}\beta_-} \right) \right]$
VIII	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right.$ $-12 e^{4\Omega} \left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} \right.$ $\left. \left. +2 \left\{ e^{4\beta_+} - e^{-2\beta_+ - 2\sqrt{3}\beta_-} - e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right) \right]$
IX	$\frac{e^{-3\Omega}}{24} \left[\Pi_\Omega^2 - \Pi_\varsigma^2 - \Pi_+^2 - \Pi_-^2 + \Pi_\varphi^2 + e^{6\Omega} \left\{ 24V(\varphi, \varsigma) - \lambda_{\text{eff}} \right\} \right.$ $-12 e^{4\Omega} \left(e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} \right.$ $\left. \left. +2 \left\{ e^{4\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-} + e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right) \right]$

Table 1. Hamiltonian density for the Bianchi Class A models in the quintom approach for the inflationary phenomenon.

3.1. Classical field equation for Bianchi type I

On the other hand, the Einstein field equations (2,3) for the Bianchi type I, are

$$3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\beta}_+^2}{N^2} - 3\frac{\dot{\beta}_-^2}{N^2} = 8\pi G\rho - 3\frac{\dot{\varphi}^2}{N^2} + 3\frac{\dot{\varsigma}^2}{N^2} + \Lambda + \frac{V(\varphi,\varsigma)}{2}, \tag{15}$$

$$2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\Omega}\dot{\beta}_+}{N^2} - 3\sqrt{3}\frac{\dot{\Omega}\dot{\beta}_-}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} - \frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} + \frac{\dot{\beta}_+\dot{N}}{N^3} - \sqrt{3}\frac{\ddot{\beta}_-}{N^2} + 3\frac{\dot{\beta}_-^2}{N^2}$$
$$+\sqrt{3}\frac{\dot{\beta}_-\dot{N}}{N^3} = -8\pi GP + 3\frac{\dot{\varphi}^2}{N^2} - 3\frac{\dot{\varsigma}^2}{N^2} + \Lambda + \frac{V(\varphi,\varsigma)}{2}, \tag{16}$$

$$2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} - 3\frac{\dot{\Omega}\dot{\beta}_+}{N^2} + 3\sqrt{3}\frac{\dot{\Omega}\dot{\beta}_-}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} - \frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} + \frac{\dot{\beta}_+\dot{N}}{N^3} + \sqrt{3}\frac{\ddot{\beta}_-}{N^2} + 3\frac{\dot{\beta}_-^2}{N^2}$$
$$-\sqrt{3}\frac{\dot{\beta}_-\dot{N}}{N^3} = -8\pi GP + 3\frac{\dot{\varphi}^2}{N^2} - 3\frac{\dot{\varsigma}^2}{N^2} + \Lambda + \frac{V(\varphi,\varsigma)}{2}, \tag{17}$$

$$2\frac{\ddot{\Omega}}{N^2} + 3\frac{\dot{\Omega}^2}{N^2} + 6\frac{\dot{\Omega}\dot{\beta}_+}{N^2} - 2\frac{\dot{\Omega}\dot{N}}{N^3} + 2\frac{\ddot{\beta}_+}{N^2} + 3\frac{\dot{\beta}_+^2}{N^2} - 2\frac{\dot{\beta}_+\dot{N}}{N^3}$$
$$+3\frac{\dot{\beta}_-^2}{N^2} = -8\pi GP + 3\frac{\dot{\varphi}^2}{N^2} - 3\frac{\dot{\varsigma}^2}{N^2} + \Lambda + \frac{V(\varphi,\varsigma)}{2}, \tag{18}$$

$$-3\frac{\dot{\Omega}\dot{\varsigma}}{N^2} + \frac{\dot{N}\dot{\varsigma}}{N^3} - \frac{\ddot{\varsigma}}{N^2} + \frac{\partial V(\varsigma,\varphi)}{\partial\varsigma} = 0, \tag{19}$$

$$-3\frac{\dot{\Omega}\dot{\varphi}}{N^2} + \frac{\dot{N}\dot{\varphi}}{N^3} - \frac{\ddot{\varphi}}{N^2} - \frac{\partial V(\varsigma,\varphi)}{\partial\varphi} = 0, \tag{20}$$

which can be written as

$$8\pi GP - \Lambda + \frac{1}{2}\left(-6\varphi'^2 + 6\varsigma'^2 - V(\varphi,\varsigma)\right) = -\frac{2}{3}\frac{a''}{a} - 3H^2, \tag{21}$$

$$8\pi G\rho + \Lambda + \frac{1}{2}\left(-6\varphi'^2 + 6\varsigma'^2 + V(\varphi,\varsigma)\right) = 3H^2, \tag{22}$$

$$-3\Omega'\varsigma' - \varsigma'' - \frac{\partial V(\varsigma,\varphi)}{\partial\varsigma} = 0, \tag{23}$$

$$-3\Omega'\varphi' - \varphi'' + \frac{\partial V(\varsigma,\varphi)}{\partial\varphi} = 0, \tag{24}$$

where H^2 is defined as $H^2 = H_1 H_2 + H_1 H_3 + H_2 H_3$, $a = R_1 R_2 R_3$, and $H_i = \frac{\dot{R}_i}{R_i}$. We have done the time transformation $\frac{d}{d\tau} = \frac{d}{Ndt} = \prime$. Adding (21) and (22) we arrive

$$-\frac{a''}{a} = 12\pi G\left[\rho + \rho_\varphi + \rho_\varsigma + P + P_\varphi + P_\varsigma\right], \tag{25}$$

where

$$P_\varphi = \frac{1}{16\pi G}\left(-6\varphi'^2 - V(\varphi,\varsigma)|_\varsigma\right), \qquad P_\varsigma = \frac{1}{16\pi G}\left(6\varsigma'^2 - V(\varphi,\varsigma)|_\varphi\right),$$

$$\rho_\varphi = \frac{1}{16\pi G}\left(-6\varphi'^2 + V(\varphi,\varsigma)|_\varsigma\right), \qquad \rho_\varsigma = \frac{1}{16\pi G}\left(6\varsigma'^2 + V(\varphi,\varsigma)|_\varphi\right),$$

which are useful when we study the behavior of dynamical systems. Additionally we can introduce the total quintom energy density and pressure as:

$$\rho_{DE} = \rho_\varsigma + \rho_\varphi, \qquad P_{DE} = P_\varsigma + P_\varphi, \qquad P_{DE} = \omega_{DE}\rho_{DE} \tag{26}$$

where

$$\omega_{DE} = \frac{6\dot\varsigma^2 - 6\dot\varphi^2 - V(\varsigma,\varphi)}{6\dot\varsigma^2 - 6\dot\varphi^2 + V(\varsigma,\varphi)} \tag{27}$$

To solve the set of differential equation $(\beta_\pm, \Omega, \varphi, \varsigma)$ we begin with the equations (16, 17) where we obtain the relation between the functions β_- and Ω as

$$\beta_- = \beta_0 \int e^{-3\Delta\Omega}d\tau, \tag{28}$$

similar to equations (17,18) we find

$$\beta_+ = \beta_1 \int e^{-3\Delta\Omega}d\tau, \tag{29}$$

then there is the relation between the anisotropic functions $\beta_- = \beta_2\beta_+$ with $\beta_2 = \frac{\beta_0}{\beta_1}$.

For separable potentials, equations (24,23) can be solved in some cases in terms of the Ω function, then, using equation (15) we can obtain in a quadrature form, the structure of Ω as

$$\int \frac{d\Omega}{\sqrt{h(\Omega)}} = \Delta\tau, \tag{30}$$

where the function $h(\Omega)$ has the corresponding information of all functions presented in this equation (15). For instance, when the potential $V(\varphi,\varsigma)$ becomes null or constant, the formalism is like the one formulated by Sáez and Ballester in 1986 [35] because both field are equivalent, see equations (24,23), where

$$\varsigma' = \varsigma_0 e^{-3\Delta\Omega}, \qquad \varsigma(\tau) = \varsigma_0 \int e^{-3\Delta\Omega(\tau)}d\tau + \varsigma_1, \tag{31}$$

$$\varphi' = \varphi_0 e^{-3\Delta\Omega}, \qquad \varphi(\tau) = \varphi_0 \int e^{-3\Delta\Omega(\tau)}d\tau + \varphi_1,, \tag{32}$$

and the function $h(\Omega)$ is

$$h(\Omega) = \frac{\frac{8\pi G M_\gamma}{3} e^{-3(1+\gamma)\Omega} + \left(-\varphi_0^2 + \varsigma_0^2\right) e^{-6\Delta\Omega} + \frac{\Lambda_{eff}}{3}}{1 - 9\left(1 + \beta_2^2\right)\beta_1^2 e^{-6\Delta\Omega}}, \tag{33}$$

where $\Lambda_{eff} = \Lambda + V_0/2$. For particular values in the γ parameter, the equation (30) has a solution. This formalism was studied by one of the author and collaborators, in the FRW and Bianchi type Class A cosmological models, [36–38].

4. Quantum approach

On the Wheeler-DeWitt (WDW) equation there are a lot of papers dealing with different problems, for example in [39], they asked the question of what a typical wave function for the universe is. In Ref. [40] there appears an excellent summary of a paper on quantum cosmology where the problem of how the universe emerged from big bang singularity can no longer be neglected in the GUT epoch. On the other hand, the best candidates for quantum solutions become those that have a damping behavior with respect to the scale factor, in the sense that we obtain a good classical solution using the WKB approximation in any scenario in the evolution of our universe [41, 42]. Our goal in this paper deals with the problem to build the appropriate scalar potential in the inflationary scenario.

The Wheeler-DeWitt equation for this model is achieved by replacing $\Pi_{q^\mu} = -i\partial_{q^\mu}$ in (8). The factor $e^{-3\Omega}$ may be a factor ordered with $\hat{\Pi}_\Omega$ in many ways. Hartle and Hawking [41] have suggested what might be called a semi-general factor ordering which in this case would order $e^{-3\Omega}\hat{\Pi}_\Omega^2$ as

$$-e^{-(3-Q)\Omega}\partial_\Omega e^{-Q\Omega}\partial_\Omega = -e^{-3\Omega}\partial_\Omega^2 + Q e^{-3\Omega}\partial_\Omega, \tag{34}$$

where Q is any real constant that measure the ambiguity in the factor ordering in the variable Ω. In the following we will assume this factor ordering for the Wheeler-DeWitt equation, which becomes

$$\Box\Psi + Q\frac{\partial\Psi}{\partial\Omega} + e^{6\Omega}U(\Omega, \beta_\pm, \varphi, \varsigma, \lambda_{\text{eff}})\Psi = 0, \tag{35}$$

where $\Box = -\frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\varsigma^2} - \frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\beta_-^2} + \frac{\partial^2}{\partial\beta_+^2}$ is the d'Alambertian in the coordinates $q^\mu = (\Omega, \varsigma, \beta_\pm, \varphi)$. In the following, we introduce the main idea of the Bohm formalism, and why we choose the phase in the wave function to be real and not imaginary.

4.1. Mathematical structure in the Bohm formalism

In this section we will explain how the quantum potential approach or as is also known, the Bohm formalism [34], works in the context of quantum cosmology. For the cases that will be object of our investigation in the sections to come, it is sufficient to consider the simplest model, for which the whole quantum dynamics resides in the single equation,

$$\mathcal{H}\psi = \left(g^{\mu\nu}\nabla_\mu\nabla_\nu - V(q^\mu)\right)\psi = 0, \tag{36}$$

where the metric may be q^μ dependent. The ψ is called the wave function of the universe, and we consider that ψ has the following traditional decomposition

$$\psi = R(q^\mu)\, e^{\frac{i}{\hbar}S(q^\mu)}, \tag{37}$$

with R and S as real functions. Inserting (37) into (36), we obtain two equations corresponding to the real and imaginary parts, respectively, which are

$$\Box R - R\left[\frac{1}{\hbar^2}(\nabla S)^2 + V\right] = \Box R - R\left[H(S)\right] = 0, \tag{38}$$

$$2\nabla R \cdot \nabla S + R\Box S = 0, \tag{39}$$

when we consider the problem of factor ordering, usually in cosmological problems, as we indicated in the beginning of this section, equation (34), must be included as linear term of $Q\frac{\partial\psi}{\partial q}$, where Q is a real parameter that measures the ambiguity in this factor ordering. So, the equations (38,39) are written as

$$\Box R + Q\frac{\partial R}{\partial q} - R\left[\frac{1}{\hbar^2}(\nabla S)^2 + V\right] = 0, \tag{40}$$

$$2\nabla R \cdot \nabla S + R\Box S + R\frac{\partial S}{\partial q} = 0, \tag{41}$$

where q is a single field coordinate.

We assume that the wave function ψ is a solution of equation (36), and thus this equation is equally satisfied. Considering the Hamilton-Jacobi analysis, we can identify the equation (40) as the most important equation of this treatment, because with this equation we can derive the time dependence, and thus, it serves as the evolutionary equation in this formalism. Following the Hamilton-Jacobi procedure, the Π_q momenta is related to the superpotential function S, as $\Pi_{q^\mu} = \frac{\partial S}{\partial q^\mu}$, which are related with the classical momenta (8) written in the previous section, thus,

$$\frac{dq^\mu}{dt} = g^{\mu\nu}\frac{\delta H(S)}{\delta\frac{\partial S}{\partial q^\nu}}, \tag{42}$$

which defines the trajectory q^μ in terms of the phase of the wave function S. We substitute this equation into (40), and we find (using $\dot{q}^\mu = \frac{dq^\mu}{dt}$ and $\hbar = 1$),

$$\left[\Box R + Q\frac{\partial R}{\partial q}\right] = R\left[g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + V\right]. \tag{43}$$

Therefore we see that the quantum evolution differs from the classical one only by the presence of the quantum potential term

$$\left[\Box R + Q\frac{\partial R}{\partial q}\right]$$

on the left-hand side of the equation of motion. Since we assume that the wave function is known, the quantum potential term is also known.

In the next subsection we will choose the $\psi = We^{-S}$ ansatz for the wave function, it was first remarked by Kodama [43, 44] that the solutions to the Wheeler-DeWitt (WDW) equation in the formulation of Arnowitt-Deser and Misner (ADM) and the Ashtekar formulation (in the connection representation) are related by $\psi_{ADM} = \psi_A e^{\pm i\Phi_A}$, where Ψ_A is the homogeneous specialization for the generating functional of the canonical transformation between ADM variables to Ashtekar's, [45]. This function was calculated explicitly for the diagonal Bianchi type IX model by Kodama, who also found $\Psi_A = const$ as a solution, and Ψ_A is pure imaginary, for a certain factor ordering, one expects a solution of the form $\psi = We^{\pm\Phi}$, where W is a constant, and $\Phi = i\Phi_A$. In fact this type of solution has been found for the diagonal Bianchi Class A cosmological models [46, 47], but W in some cases is a function, as we will see in our present study.

4.2. Our treatment

Using the ansatz

$$\Psi = e^{\pm a_1\beta_+}e^{\pm a_2\beta_-}\,\Xi(\Omega,\varsigma,\varphi), \tag{44}$$

the WDW equation is read as

$$\left[\Box + Q\frac{\partial}{\partial\Omega} + e^{6\Omega}U(\varphi,\varsigma,\lambda_{\text{eff}}) + c^2\right]\Xi = 0, \tag{45}$$

where $c^2 = a_1^2 + a_2^2$ and now $\Box$ is written in the reduced coordinates $\ell^\mu = (\Omega,\varsigma,\varphi)$

We find that the WDW equation is solved when we choose an ansatz similar to the one employed in the Bohmian formalism of quantum mechanics [34], so we make the following Ansatz for the wave function

$$\Xi(\ell^\mu) = W(\ell^\mu)e^{-S(\ell^\mu)}, \tag{46}$$

where $S(\ell^\mu)$ is known as the superpotential function, and W is the amplitude of probability that is employed in Bohmian formalism [34]. Then (45) transforms into

$$\Box W - W\Box S - 2\nabla W \cdot \nabla S - Q\frac{\partial W}{\partial\Omega} + QW\frac{\partial S}{\partial\Omega} + W\left[(\nabla S)^2 - \mathcal{U}\right] = 0, \tag{47}$$

where now, $\Box = G^{\mu\nu}\frac{\partial^2}{\partial\ell^\mu\partial\ell^\nu}$, $\nabla W \cdot \nabla \Phi = G^{\mu\nu}\frac{\partial W}{\partial\ell^\mu}\frac{\partial\Phi}{\partial\ell^\nu}$, $(\nabla)^2 = G^{\mu\nu}\frac{\partial}{\partial\ell^\mu}\frac{\partial}{\partial\ell^\nu} = -(\frac{\partial}{\partial\varsigma})^2 + (\frac{\partial}{\partial\Omega})^2 + (\frac{\partial}{\partial\varphi})^2$, with $G^{\mu\nu} = \text{diag}(-1,1,1)$, $\mathcal{U} = e^{6\Omega}U(\varsigma,\varphi,\lambda_{\text{eff}}) + c^2$ is the potential term of the cosmological model under consideration.

Eq (47) can be written as the following set of partial differential equations

$$(\nabla S)^2 - \mathcal{U} = 0, \tag{48a}$$

$$\Box W - Q\frac{\partial W}{\partial \Omega} = 0 \tag{48b}$$

$$W\left(\Box S - Q\frac{\partial S}{\partial \Omega}\right) + 2\nabla W \cdot \nabla S = 0, . \tag{48c}$$

The first two equations correspond to the real part in a separated way, also, the first equation is called the Einstein-Hamilton-Jacobi equation (EHJ), and the third equation is the imaginary part, such as the equations presented in previous section (40, 41).

Following the references [32, 33], first, we shall choose to solve Eqs. (48a) and (48c), whose solutions at the end will have to fulfill Eq. (48b), which will play the role of a constraint equation.

Taking the ansatz

$$S(\Omega, \varsigma, \varphi) = \frac{e^{3\Omega}}{\mu}g(\varphi)h(\varsigma) + c\left(b_1\Omega + b_2\Delta\varphi + b_3\Delta\varsigma\right), \tag{49}$$

where $\Delta\varphi = \varphi - \varphi_0$, $\Delta\varsigma = \varsigma - \varsigma_0$ with φ_0 and ς_0 as constant scalar fields, and b_i as arbitrary constants. Then, Eq (48a) is transformed as

$$e^{6\Omega}\left[\frac{h^2}{\mu^2}\left(\frac{dg}{d\varphi}\right)^2 - \frac{g^2}{\mu^2}\left(\frac{dh}{d\varsigma}\right)^2 + \frac{9}{\mu^2}g^2h^2 - U(\varphi,\varsigma,\lambda_{\text{eff}})\right]$$
$$+\frac{6ce^{3\Omega}}{\mu}\left[b_1gh + \frac{b_2}{3}h\frac{dg}{d\varphi} - \frac{b_3}{3}g\frac{dh}{d\varsigma}\right] + c^2\left(b_1^2 + b_2^2 - b_3^2 - 1\right) = 0. \tag{50}$$

At this point we question ourselves how to solve this equation in relation to the constant c, implying the behavior of the universe with the anisotropic parameter $\beta_\pm$.

1. When we consider this equation as an expansion in powers of e^{Ω}, then each term is null in a separated way, but maintaining that the constant $c \neq 0$,

$$b_1^2 - b_3^2 + b_2^2 - 1 = 0, \tag{51}$$

$$b_1gh + \frac{b_2}{3}h\frac{dg}{d\varphi} - \frac{b_3}{3}g\frac{dh}{d\varsigma} = 0, \tag{52}$$

$$h^2\left(\frac{dg}{d\varphi}\right)^2 - g^2\left(\frac{dh}{d\varsigma}\right)^2 + 9g^2h^2 - U(\varphi,\varsigma,\lambda_{\text{eff}}) = 0, \tag{53}$$

these set of equations do not have solutions in closed form, because the first equation is not satisfied. So, it is necessary to take c=0, implying that the wave function in the anisotropic coordinates have an oscilatory and hyperbolic behavior

2. For the case c=0, we have the following.

   The constants a_i are related as $a_2 = \pm ia_1$, hence the wave function corresponding to the anisotropic behavior becomes $e^{\pm a_1\beta_+ \pm ia_1\beta_-}$, i.e, one part goes as oscillatory in the anisotropic parameter.

4.3. Mathematical structure of potential fields

To solve the Hamilton-Jacobi equation (48a)

$$-\left(\frac{\partial S}{\partial \varsigma}\right)^2 + \left(\frac{\partial S}{\partial \Omega}\right)^2 + \left(\frac{\partial S}{\partial \varphi}\right)^2 = e^{6\alpha}U(\varphi,\varsigma,\lambda_{\text{eff}})$$

we propose that the superpotential function has the form

$$S = \frac{e^{3\Omega}}{\mu}g(\varphi)h(\varsigma),\tag{54}$$

and the potential

$$U = g^2 h^2 \left[a_0 G(g) + b_0 H(h)\right],\tag{55}$$

where $g(\varphi)$, $h(\varsigma)$, $G(g)$ and $H(h)$ are generic functions of the arguments, which will be determined under this process. When we introduce the ansatz in (48a) we find the following master equations for the fields (φ,ς), (here $c_1 = \mu a_0$ and $c_0 = \mu_0 b_0$)

$$d\varphi = \pm\frac{dg}{g\sqrt{p^2 + c_1 G}}, \qquad \text{with} \quad p^2 = v^2 - \frac{9}{2},\tag{56a}$$

$$d\varsigma = \pm\frac{dh}{h\sqrt{\ell^2 - c_0 H}}, \qquad \text{with} \quad \ell^2 = v^2 + \frac{9}{2},\tag{56b}$$

where v is the separation constant.

For a particular structure of functions G and H we can solve the $g(\varphi)$ and $h(\varsigma)$ functions, and then use the expression for the potential term (55) over again to find the corresponding scalar potential that leads to an exact solution to the Hamilton-Jacobi equation (48a). Some examples are shown in Tables 2 and 3.

Thereby, the superpotential $S(\Omega,\varphi)$ is known, and the possible quintom potentials are shown in table 3

To solve (48c) we assume that

$$W = e^{[\eta(\Omega)+\xi(\varphi)+\lambda(\varsigma)]},\tag{57}$$

and introducing the corresponding superpotential function S (54) into the equation (48c), it follow the equation

H(h)	h(ς)	G(g)	g(φ)
0	$h_0 e^{\pm\ell\Delta\varsigma}$	0	$g_0 e^{\pm p\Delta\varphi}$
H_0	$h_0 e^{\pm\sqrt{\ell^2 - c_0 H_0}\Delta\varsigma}$	G_0	$g_0 e^{\pm\sqrt{p^2 + c_1 G_0}\Delta\varphi}$
$H_0 h^{-2}$	$\frac{\sqrt{c_0 H_0}}{\ell}\cosh[\ell\Delta\varsigma]$	$G_0 g^{-2}$	$\frac{\sqrt{c_1 G_0}}{p}\sinh[p\Delta\varphi]$
$H_0 h^{-n}\ (n\neq 2)$	$\left[\frac{c_0 H_0}{\ell^2}\cosh^2\left(\frac{n\ell\Delta\varsigma}{2}\right)\right]^{1/n}$	$G_0 g^{-n}\ (n\neq 2)$	$\left[\frac{c_1 G_0}{p^2}\sinh^2\left(\frac{np\Delta\varphi}{2}\right)\right]^{1/n}$
$H_0\,\ln h$	$e^{u(\varsigma)},\quad u(\varsigma)=\dfrac{\ell^2 - \left(\frac{c_0 H_0}{2}\Delta\varsigma\right)^2}{c_0 H_0}$	$G_0\,\ln g$	$e^{v(\varphi)},\quad v(\varphi)=\dfrac{-p^2 + \left(\frac{c_1 G_0}{2}\Delta\varphi\right)^2}{c_1 G_0}$
$H_0(\ln h)^2$	$e^{r(\varsigma)},\quad r(\varsigma)=\dfrac{\ell}{\sqrt{c_0 H_0}}\sin\left(\sqrt{c_0 H_0}\,\Delta\varsigma\right)$	$G_0(\ln g)^2$	$e^{\omega(\varphi)},\quad \omega(\varphi)=\dfrac{p}{\sqrt{c_1 G_0}}\sinh\left(\sqrt{c_1 G_0}\,\Delta\varphi\right)$

Table 2. Some exact solutions to eqs. (56a,56b), where n is any real number, G_0 and H_0 are arbitrary constants.

$U(\varphi,\varsigma)$	Relation between all constants
0	$\ell^2(s - p^2 - 3k - 9)^2 - p^2(s - \ell^2)^2$ $+\ell^2 p^2(k^2 - Q^2) = 0$
$U_0 e^{\pm 2[\sqrt{\ell^2 - c_0 H_0}\Delta\varsigma + \sqrt{p^2 + c_1 G_0}\Delta\varphi]}$	$(\ell^2 - c_0 H_0)(s - p^2 - 3k - 9 - c_1 G_0)^2$ $-(p^2 + c_1 G_0)(s - \ell^2 + c_0 H_0)^2$ $+(\ell^2 - c_0 H_0)(p^2 + c_1 G_0)(k^2 - Q^2) = 0$
$U_0 \sinh^2(p\Delta\varphi) + U_1\cosh^2(\ell\Delta\varsigma)$	$k(k - 6) = Q^2,\quad s = \ell^2,$ $6k(9 + p^2) + 9Q^2 - p^4 + (\ell^2 - 9)^2 = 0$
$b_0 H_0\left[\frac{c_1 G_0}{p^2}\sinh^2\left(\frac{np\Delta\varphi}{2}\right)\right]^{\frac{2}{n}}\left[\frac{c_0 H_0}{\ell^2}\cosh^2\left(\frac{n}{2}\ell\Delta\varsigma\right)\right]^{\frac{2-n}{n}} +$ $a_0 G_0\left[\frac{c_1 G_0}{p^2}\sinh^2\left(\frac{np\Delta\varphi}{2}\right)\right]^{\frac{2-n}{n}}\left[\frac{c_0 H_0}{\ell^2}\cosh^2\left(\frac{n}{2}\ell\Delta\varsigma\right)\right]^{\frac{2}{n}}$	quantum constraint is not satisfied
$e^{2u(\varsigma)+2v(\varphi)}\left[b_0 H_0 u(\varsigma) + a_0 G_0\, v(\varphi)\right]$	quantum constraint is not satisfied
$e^{2r(\varsigma)+2\omega(\varphi)}\left[b_0 H_0 r^2 + a_0 G_0\omega^2\right]$	quantum constraint is not satisfied

Table 3. The corresponding quintom potentials that emerge from quantum cosmology in direct relation with the table (2). Also we present the relation between all constant that satisfy the eqn. (48b).

$$\frac{1}{g}\frac{d^2 g}{d\varphi^2} + \frac{2}{g}\frac{dg}{d\varphi}\frac{d\xi}{d\varphi} + 9 - \frac{1}{h}\frac{d^2 h}{d\varsigma^2} - \frac{2}{h}\frac{d\lambda}{d\varsigma}\frac{dh}{d\varsigma} + 6\frac{d\eta}{d\Omega} - 3Q = 0, \tag{58}$$

and using the method of separation of variables, we arrive to a set of ordinary differential equations for the functions $\eta(\Omega)$, $\xi(\varphi)$ and $\lambda(\varsigma)$ (however, this decomposition is not unique, because it depend as we put the constants in the equations).

$$2\frac{d\eta}{d\Omega} - Q = k, \tag{59}$$

$$\frac{d^2 g}{d\varphi^2} + 2\frac{dg}{d\varphi}\frac{d\xi}{d\varphi} = [s - 3(k + 3)]g, \tag{60}$$

$$\frac{d^2 h}{d\varsigma^2} + 2\frac{dh}{d\varsigma}\frac{d\lambda}{d\varsigma} = sh, \tag{61}$$

whose solutions in the generic fields g and h are

$$\eta(\Omega) = \frac{Q+k}{2}\Omega,$$

$$\lambda(\varsigma) = \frac{s}{2}\int \frac{d\varsigma}{\partial_\varsigma(\ln h)} - \frac{1}{2}\int \frac{\frac{d^2h}{d\varsigma^2}}{\partial_\varsigma h}d\varsigma,$$

$$\xi(\varphi) = \left(\frac{s}{2} - \frac{3k}{2} - \frac{9}{2}\right)\int \frac{d\varphi}{\partial_\varphi(\ln g)} - \frac{1}{2}\int \frac{\frac{d^2g}{d\varphi^2}}{\partial_\varphi g}d\varphi,$$

then

$$W = e^{\frac{s}{2}\int\left(\frac{d\varsigma}{\partial_\varsigma(\ln h)} + \frac{d\varphi}{\partial_\varphi(\ln g)}\right)}e^{-\frac{1}{2}\int\left(\frac{\frac{d^2h}{d\varsigma^2}}{\partial_\varsigma h}d\varsigma + \frac{\frac{d^2g}{d\varphi^2}}{\partial_\varphi g}d\varphi\right)}e^{\frac{k}{2}\left(\Omega - 3\int\frac{d\varphi}{\partial_\varphi(\ln g)}\right)}e^{\frac{1}{2}\left(Q\Omega - 9\int\frac{d\varphi}{\partial_\varphi(\ln g)}\right)}. \qquad (62)$$

In a similar way, the constraint (48b) can be written as

$$\partial_\varphi^2\xi + (\partial_\varphi\xi)^2 - \partial_\varsigma^2\lambda - (\partial_\varsigma\lambda)^2 + \frac{k^2 - Q^2}{4} = 0, \qquad (63)$$

or in other words (here $\mu_0 = s - 3(3 + \kappa)$)

$$2\frac{\partial_\varsigma^3 h}{\partial_\varsigma h} - 2\frac{\partial_\varphi^3 g}{\partial_\varphi g} + 4sh\frac{\partial_\varsigma^2 h}{(\partial_\varsigma h)^2} - 4\mu_0 g\frac{\partial_\varphi^2 g}{(\partial_\varphi g)^2} - 3\frac{(\partial_\varsigma^2 h)^2}{(\partial_\varsigma h)^2} + 3\frac{(\partial_\varphi^2 g)^2}{(\partial_\varphi g)^2} - \frac{s^2 h^2}{(\partial_\varsigma h)^2}$$

$$+ \frac{\mu_0^2 g^2}{(\partial_\varphi g)^2} - 2s + 2\mu_0 + k^2 - Q^2 = 0.$$

when we use the different cases presented in table (2), the following relations between all constants were found, which we present in the same table II with the quintom potentials. So, the quantum solutions for each potential scalar fields are presented in quadrature form, using the equations (46, 54) and (57).

Thereby, under canonical quantization we were able to determine a family of potentials that are the most probable to characterize the inflation phenomenon in the evolution of our universe. The exact quantum solutions to the Wheeler-DeWitt equation were found using the Bohmian scheme [34] of quantum mechanics where the ansatz to the wave function $\Psi(\ell^\mu) = e^{a_1\beta_+ + ia_i\beta_-}W(\ell^\mu)e^{-S(\ell^\mu)}$ includes the superpotential function which plays an important role in solving the Hamilton-Jacobi equation. It was necessary to study the classical behavior in order to know when the Universe evolves from a quintessence dominated phase to a phantom dominated phase crossing the $w_{eff} = -1$ dividing line, as a transient stage. Also, this family of potentials can be studied within the dynamical systems framework to obtain useful information about the asymptotic properties of the model and give a classification of which ones are in agreement with the observational data [48]

5. Classical solutions a la WKB

For our study, we shall make use of a semi-classical approximation to extract the dynamics of the WDW equation. The semi-classical limit of the WDW equation is achieved by taking $\Psi = e^{-S}$, and imposing the usual WKB conditions on the superpotential function S, namely

$$\left(\frac{\partial S}{\partial q}\right)^2 >> \frac{\partial^2 S}{\partial q^2}$$

Hence, the WDW equation, under the particular factor ordering Q = 0, becomes exactly the afore-mentioned EHJ equation (48a) (this approximation is equivalent to a zero quantum potential in the Bohmian interpretation of quantum cosmology [53]). The EHJ equation is also obtained if we introduce the following transformation on the canonical momenta $\Pi_q \rightarrow \partial_q S$ in Eq. (8) and then Eq. (8) provides the classical solutions of the Einstein.Klein.Gordon (EKG) equations. Moreover, for the particular cases shown in Table 1, the classical solutions of the EKG, in terms of $q(\tau)$, arising from Eqs. (8) and (50) are given by

$$gh = 4\mu \frac{d\Omega}{d\tau}, \qquad \frac{d\varphi}{\partial_\varphi Lng} + \frac{d\varsigma}{\partial_\varsigma Lnh} = 0, \tag{64}$$

the second equation appears in the W function (57), then the W is simplified by, we also have the corresponding relation with the time τ

$$d\tau = 12\mu \frac{1}{h} \frac{d\varphi}{\partial_\varphi g}, \qquad d\tau = -12\mu \frac{1}{g} \frac{d\varsigma}{\partial_\varsigma h}. \tag{65}$$

In the following subsection, we will give details about the solutions corresponding to some of the scalar potential shown in Table (3).

To recover the solutions for the anisotropic function $\beta_\pm$, (28,29) we need to extend the superpotential function $\mathcal{S} = S - a_1\beta_+ - ia_2\beta_-$, remember that these functions were used in the ansatz for the wave function (44) in order to simplify the WDW equation (45). With this extension, we has

$$\frac{\partial \mathcal{S}}{\partial \beta_\pm} = -b_i = \text{constants}$$

and using the corresponding momenta (8), we obtain the corresponding solutions written in quadrature form in equations (28,29). In this subsection we calculate the solution for the Ω function, thence the classical solution will be complete.

5.0.1. Free wave function

This particular case corresponds to an null potential function $U(\varphi, \varsigma)$, (see first line in Table (3)). The particular exact solution for the wave function Ξ becomes

$$\Xi(\Omega,\varphi,\varsigma) = e^{\pm\frac{s}{2}\left(\frac{\Delta\varsigma}{\ell}+\frac{\Delta\varphi}{p}\right)}e^{\pm\frac{1}{2}(\ell\Delta\varsigma+p\Delta\varphi)}e^{\frac{\Omega}{2}(k+Q)\pm(3k-9)\frac{\Delta\varphi}{2p}} \operatorname{Exp}\left[-g_0h_0e^{3\Omega\pm\ell\Delta\varsigma\pm p\Delta\varphi}\right], \quad (66)$$
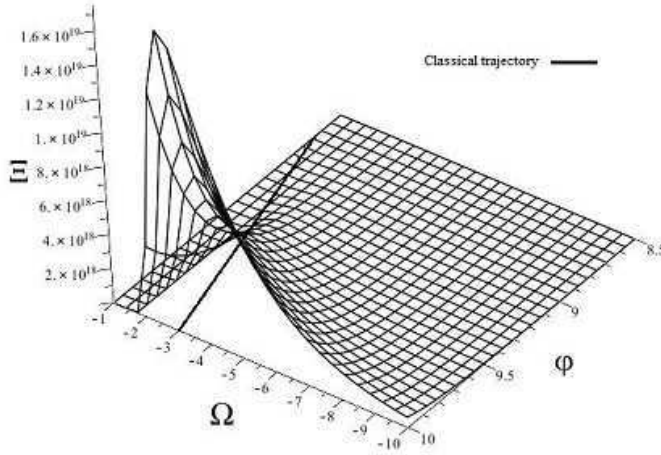

Figure 1. Exact wave function for the free case, i.e. for $U(\varphi,\varsigma) = 0$. The wave function (66) is peaked around the classical trajectory $\Delta\Omega \pm \frac{3}{p}\Delta\varphi = v_0 = $ const, which is the solid line shown on the $\{\Omega,\varphi\}$ plane. For this case $v_0 = -1$ on equation (68a)

the classical trajectory implies that $\frac{\Delta\varsigma}{\ell} + \frac{\Delta\varphi}{p} = 0$, then there is the relation between the fields φ and ς, as

$$\Delta\varsigma = -\frac{\ell}{p}\Delta\varphi.$$

So, this wavefunction can be written in terms of φ and Ω solely,

$$\Xi(\Omega,\varphi) = e^{\frac{k}{2}\left(\Delta\Omega\pm\frac{3}{p}\Delta\varphi\right)+\frac{Q}{2}\Delta\Omega} \operatorname{Exp}\left[-g_0h_0e^{3\left(\Omega\pm\frac{3}{p}\Delta\varphi\right)}\right], \quad (67)$$

Using the equantion (65), we find the *classical trajectory* on the $\{\Omega,\varphi\}$ plane as

$$\Delta\Omega \pm \frac{3}{p}\Delta\varphi = v_0 = \text{const}, \quad (68a)$$

$$\Delta\varphi = \operatorname{Ln}\left[\frac{3g_0h_0}{4}\Delta\tau\right]^{\pm\frac{p}{9}}, \quad (68b)$$

$$\Delta\varsigma = \text{Ln}\left[\frac{3g_0 h_0}{4}\Delta\tau\right]^{\mp\frac{\ell}{9}}, \tag{68c}$$

$$\Delta\Omega = \text{Ln}\left[\frac{3g_0 h_0}{4}\Delta\tau\right]^{\frac{1}{3}}, \tag{68d}$$

which corresponds to constant phase of second exponential in the W function. The behavior of the scale factor correspond at stiff matter epoch in the evolution on the universe.

5.0.2. Exponential scalar potential

For an exponential scalar potential, see second line in Table (3), the exact solution of the WDW equation is similar to the last one, only we redefine the constants,

$$\ell \to \sqrt{\ell^2 - c_1 G_0}, \qquad p \to \sqrt{p^2 + c_0 H_0}$$

5.0.3. Hyperbolic scalar potential

This case corresponds to third line in Table ($U(\varphi,\varsigma) = U_0 \sinh^2(p\Delta\varphi) + U_1 \cosh^2(\ell\Delta\varsigma)$, the wave function for this is

$$Y = e^{-\frac{1}{2}\text{Ln}|\cosh(\ell\Delta\varsigma)\sinh(p\Delta\varphi)|}e^{\frac{k}{2}\left[\Delta\Omega - \frac{3}{p^2}\text{Ln}|\cosh(p\Delta\varphi)|\right]}e^{\frac{1}{2}\left(Q\Delta\Omega - \frac{9}{p^2}\text{Ln}|\cosh(p\Delta\varphi)|\right)}e^{\left[-g_0 h_0 e^{3\Omega}\sinh(p\Delta\varphi)\cosh(\ell\Delta\varsigma)\right]} \tag{69}$$

and the classical trajectory in the plane $\{\Omega, \varphi\}$ reads as

$$\Delta\Omega - \frac{3}{p^2}\text{Ln}|\cosh(p\Delta\varphi)| = \epsilon = \text{const.} \tag{70}$$

Using the second equation in (64), we find the relation between the quintom fields

$$\Delta\varsigma = \frac{1}{\ell}\text{arcsinh}\left[\frac{F_0}{\cosh^{\frac{1}{p^2}}(p\Delta\varphi)}\right]^{\ell^2},$$

then, the time dependence of the quintom fields only were possible to write in quadrature form, having the following structure

$$\sqrt{\frac{c_0 H_0}{c_1 G_0}}\frac{\Delta\tau}{12\ell} = \int \frac{d\varphi}{\cosh^{\frac{9}{p^2}}(p\Delta\varphi)\sqrt{F_0^{2\ell^2} + \cosh^{\frac{2\ell^2}{p^2}}(p\Delta\varphi)}}. \tag{71}$$
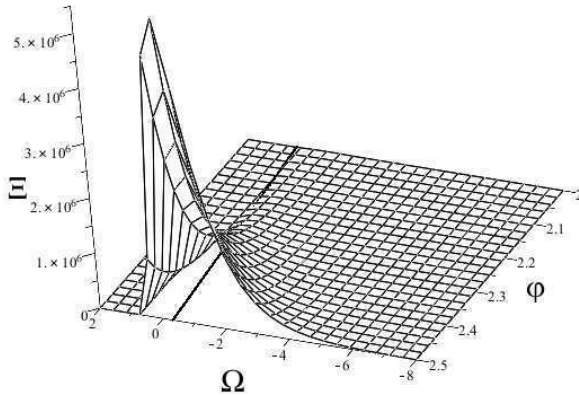
Figure 2.   Wave Function for the potential $U(\varphi, \varsigma) = U_0 \sinh^2(p\Delta\varphi) + U_1 \cosh^2(\ell\Delta\varsigma)$. The wave function (69) is peaked around the classical trajectory $\Delta\Omega - \frac{3}{p^2} \mathrm{Ln}|\cosh(p\Delta\varphi)| = \epsilon$, which is the solid line shown on the $\{\Omega, \varphi\}$ plane. For this case $\epsilon = 2.4$ on equation (70)

6. Conclusions

In summary, we presented the corresponding Einstein Klein Gordon equation for the quintom model, which is applied to the Bianchi Type I cosmological model including as a matter content a barotropic perfect fluid and cosmological constant, and the classical solutions are given in a quadrature form for null and constant scalar potentials, these solutions are related to the Sáez-Ballester formalism, [35–38].

The quantum scheme in the Bohmian formalism and its mathematical structure, and our approach were applied to the Bianchi type I cosmological model in order to build the mathematical structure of quintom scalar potentials using the integral systems formalism.

Also, we presented the quantum solutions to the Wheeler-DeWitt equation, which is the main equation to be solved and such a subject is our principal objective in this work to obtain the family of scalar potential in the inflation phenomenon.

We emphasize that the quantum potential from the Bohm formalism will work as a constraint equation which restricts our family of potentials found, see Table (3), [33].

It is well known in the literature that in the Bohm formalism the imaginary part is never determined, however in this work such a problem has been solved in order to find the quantum potentials, which was a more important matter for being able to find the classical trajectories, which were showed through graphics how the classical trajectory is projected from its quantum counterpart. We include some steps how we solve the imaginary like equation (48c) when we found the superpotential function S (54) and particular ansatz for the function W, being the equation (58), and using the separation variables method we find

the set of equations that is necessary to solve. Also we give some explanation why this decomposition in not unique.

Finally, we do the comment that solution to the equation (50), when we write this equation as a quadratic equation, $C_2 x^2 + C_1 x + C_0 = 0$ with $x = e^{\Omega}$, is not possible because the set of equations that appear , only a subset have solution in closed form. The algebraic one is not fulfill, making that the c parameter become null. In forthcoming work we will analyze under the scheme of dynamical systems, the relation between the corresponding critical points and its stability properties with these classical solutions for separable potentials obtained in this work [54].

7. Acknowledgments

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