

Symmetry, symmetry topological field theory and von Neumann algebra

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ABSTRACT: We study the additivity and Haag duality of the von Neumann algebra of a quantum field theory $\mathcal{T}_{\mathcal{F}}$ with 0-form (and the dual $(d - 2)$ -form) (non)-invertible global symmetry \mathcal{F} . We analyze the symmetric (uncharged) sector von Neumann algebra of $\mathcal{T}_{\mathcal{F}}$ with the inclusion of bi-local and bi-twist operators in it. We establish the connection between the existence of these non-local operators in $\mathcal{T}_{\mathcal{F}}$ and certain properties of the Lagrangian algebra \mathcal{L} of the extended operators in the corresponding symmetry topological field theory (SymTFT). We prove that additivity or Haag duality of the symmetric sector von Neumann algebra is violated when \mathcal{L} satisfies specific criteria, thus generalizing the result of Shao, Sorce and Srivastava to arbitrary dimensions. We further demonstrate the SymTFT construction via concrete examples in two dimensions.

KEYWORDS: Global Symmetries, Topological Field Theories

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1 Introduction

Locality has been the central guiding principle in the searching for the fundamental laws of Nature in modern physics since the era of Einstein. Together with the principle of Quantum Mechanics, it is one of the pillars of Quantum Field Theory [1, 2]. Over the past several decades, physicists have made tremendous efforts to formalize and understand the foundations of QFT. One framework that incorporates locality deeply in its roots is the approach of algebraic QFT [3, 4]. Following the proposal of [5], a \mathbb{C}^* -algebra $\mathcal{A}(R)$, which is taken to be a von Neumann algebra of bounded operators on the Hilbert space \mathcal{H} , is attached to a general region R of spacetime [6]. This approach to QFT, and its modification in the presence of gravity, are particularly useful in recent developments on the algebra of observables in de Sitter space [7, 8]. Nevertheless, in this work, we restrict ourselves to the study of QFT without gravity.

One of the most important features of a QFT is its global symmetry. The observations in the early days [9–23] were coined in the foundational work [24] as *generalized global symmetries*, whose various facets, including higher form symmetries, higher group symmetries, and non-invertible symmetries have been extensively studied in the past decade [25–78]. The interested

readers can also consult the excellent reviews [79–84]. It was found that a particular effective tool for analyzing global symmetries of a QFT is the method of Symmetry Topological Field Theory (SymTFT) [14, 26, 35, 55, 68, 70, 76, 85–92]. For a QFT $\mathcal{T}_{\mathcal{F}}$ with global symmetry \mathcal{F} defined on spacetime M_d , the gist of SymTFT is to separate the symmetry and the dynamics of $\mathcal{T}_{\mathcal{F}}$ at two boundaries of a space homeomorphic to $M_d \times [0, 1]$, where the dynamics of $\mathcal{T}_{\mathcal{F}}$ is encoded by the boundary conditions at the physical boundary on $M_d \times \{1\} \cong M_d$ while the symmetry is encoded by the boundary conditions at $B_{\text{top}} := M_d \times \{0\}$.

Although the von Neumann algebra of observables and the global symmetries of QFT have long been studied mostly parallelly, recent advances have revealed deep connections between them [8, 93–101]. Of particular relevance to the study in this paper is the recent work [102], where the authors explored the connections between the algebraic approach and non-invertible symmetries in 2D. Their analysis focuses on the interplay between the existence of certain non-local operators and certain key locality properties of the symmetric sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}$ (to be defined later) of diagonal RCFT and lattice examples. In this work, we aim to sharpen and generalize the result of [102] to arbitrary dimensions, in which case we find that the SymTFT perspective is particularly useful in proving the existence of the aforementioned specific non-local operators.

A careful reader may have already noticed an important subtlety. On one hand, for an ordinary QFT, the algebraic approach is carried out by assigning an algebra $\mathcal{A}(R)$ of observables in each spacetime region R . The way of this assignment is in principle part of the definition of a QFT and reflects the deep structures and properties of it. On the other hand, the generalized symmetry emphasizes the role of non-local, in particular extended higher-dimensional, operators. Therefore, an important question is whether such non-local operators living in R should be included in the von Neumann algebra $\mathcal{A}(R)$. For this, we will follow the choice made in [102] where the non-local operators are included in $\mathcal{A}(R)$ as long as they do not extend to the boundary of spacetime. This constraint is imposed to exclude the defects that modify the whole Hilbert space rather than being an operator acting on the states in the Hilbert space. One should keep in mind that for a QFT defined alternatively, e.g. by a Hamiltonian or a Lagrangian, there is some freedom in the assignment of $\mathcal{A}(R)$ hence is entitled study the properties of various choices, regardless of whether there is a “correct” one [102].

Another subtlety worth noting is that, unlike the full von Neumann algebra \mathcal{A} , $\mathcal{A}_{\mathcal{F}}$ by itself may not yield a well-defined theory. For example, the torus partition function of \mathbb{Z}_2 -even sector of 2d Ising CFT is

$$Z_{\text{Ising}}^{\text{even}} = \text{Tr} P_+ e^{-\beta \hat{H}} , \quad P_+ = \frac{1}{2} (1 + \hat{U}_{\mathbb{Z}_2}) , \quad (1.1)$$

with P_+ the projection operator and $\hat{U}_{\mathbb{Z}_2}$ the \mathbb{Z}_2 generator. It is obvious that the partition function is not invariant under the modular S -transformation, which brings in states in the \mathbb{Z}_2 -twist sector and cannot be defined on general 2d manifolds. However, the symmetric sector of a theory is still an interesting object to study. It was pointed out in [4] that two crucial properties of the algebra of observables, namely *additivity* and *Haag duality*, cannot hold simultaneously in the local \mathbb{Z}_2 -even sector. This observation motivates the authors of [102] to further study how these two properties are preserved and violated in the symmetric

sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}$ of other 2d QFTs with the global symmetry \mathcal{F} . Motivated by the same reason, in this work we also focus on these properties of $\mathcal{A}_{\mathcal{F}}$.

The paper is organized as follows. We will first state the main results in section 1.1. In section 2, we briefly review the basic notions of von Neumann algebra with an emphasis on its additivity and Haag duality, which are the main properties of von Neumann algebra of interest to us. After that, we define the symmetric sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}$. In section 3, we study the operator content of the symmetric sector von Neumann algebra of $\mathcal{T}_{\mathcal{F}}$ with an emphasis on the uncharged bi-local and the uncharged bi-twist operators (to be defined in that section), both of which will play crucial roles in later discussions. In section 4, we use the SymTFT method to construct the bi-local and the bi-twist operators of $\mathcal{T}_{\mathcal{F}}$ and relate the existence of these operators to certain properties of the *Lagrangian algebra* \mathcal{L} of the corresponding SymTFT. In section 5, we follow the construction of [102] to show that additivity and Haag duality of $\mathcal{A}_{\mathcal{F}}$ are violated in the presence of bi-local and bi-twist operators respectively, hence generalizing the results of [102] to arbitrary dimensions. Combining with the SymTFT construction in section 4, we formulate a theorem relating the violation of additivity and Haag duality of $\mathcal{A}_{\mathcal{F}}$ to the properties of \mathcal{L} . Finally, in section 6, we present several concrete examples of 2d QFT, including the diagonal RCFT, where we prove the equivalence of the results of [102] and ours.

1.1 Main results

In this work, we study a d -dimensional QFT $\mathcal{T}_{\mathcal{F}}$ with (non)-invertible 0-form symmetry \mathcal{F} together with its $(d-2)$ -form dual symmetry and focus on its symmetric sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}(R)$ attached to a general spacetime region R . Namely, we focus on the local and non-local operators living in R that are invariant under the action of any symmetry generator in \mathcal{F} .

Motivated by the 2d cases studied in [102], where the violation of additivity or Haag duality of $\mathcal{A}_{\mathcal{F}}$ is determined by the properties of the symmetry algebra \mathcal{F} , we expect that the possible violation of additivity or Haag duality of $\mathcal{A}_{\mathcal{F}}$ for $\mathcal{T}_{\mathcal{F}}$ in higher dimension can similarly be determined using only the properties of \mathcal{F} . Therefore, it is natural to expect that they can be formulated in the language of SymTFT. The topological boundary condition at B_{top} and the symmetry \mathcal{F} are both determined by an algebraic object \mathcal{L} in the SymTFT which, generalizing the *Lagrangian algebra* for $d=2$, encodes all operators in the SymTFT that can simultaneously end on B_{top} . Thus we will call \mathcal{L} the *Lagrangian algebra* of the SymTFT in general dimensions. The fact that an extended operator $W \in \mathcal{L}$ can end on B_{top} does not forbid it to connect to other non-trivial operators $N_i \in \mathcal{F}$ as depicted in figure 1 for $d=2$. For our purpose, we introduce the subalgebra $\widetilde{\mathcal{L}}$ which includes only the bulk extended operators in \mathcal{L} that cannot connect to any non-trivial element of \mathcal{F} supported on any submanifold of B_{top} .¹ In this work, we will focus only on the 1d and the $(d-1)$ -d extended operators in the bulk which can possibly connect to 1d and $(d-1)$ -d extended objects on B_{top} , respectively. We will also assume that the *Totalitarian principle* holds, i.e. all allowed representations of \mathcal{F} appear in the physical spectrum of $\mathcal{T}_{\mathcal{F}}$. Given the above assumptions, for the symmetric sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}$ our main results are

¹We will define this in section 4.

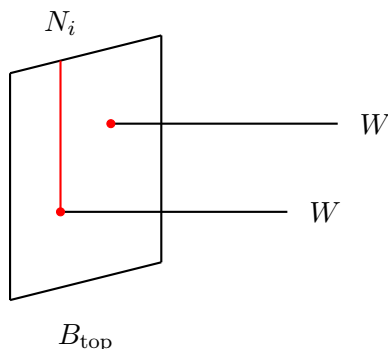


Figure 1. When $d = 2$, a bulk line operator $W \in \mathcal{L}$ can still connect to certain non-trivial line operators $N_i \in \mathcal{F}$ on B_{top} . Here N_i should be a summand of $t(W)$ which is introduced in (4.3).

- The *additivity* of $\mathcal{A}_{\mathcal{F}}$ is violated if $\widetilde{\mathcal{L}} \neq W_0 \oplus U_0$, where W_0 and U_0 are respectively the identity line operator and the identity $(d - 1)$ -dimensional extended operator in the SymTFT.
- The *Haag duality* of $\mathcal{A}_{\mathcal{F}}$ is violated if $\widetilde{\mathcal{L}} \neq \mathcal{L}$.

When restricted to 2d diagonal RCFT, the above two criteria are respectively equivalent to the existence of invertible elements and non-invertible elements in \mathcal{F} , which in turn match the criteria in [102] under which additivity and Haag duality are violated in 2d diagonal RCFT, respectively. We find that, most generally the existence of an invertible element in \mathcal{F} is not sufficient for additivity to be violated, for which we present a counterexample in 2d with $\text{Rep}(S_3)$ symmetry in section 6.

2 Additivity and Haag duality of von Neumann algebra

In this section, we review several basic facts about von Neumann algebras with an emphasis on their additivity and Haag duality. The readers can consult [3, 4, 6, 103]. We study QFT $\mathcal{T}_{\mathcal{F}}$ in spacetime M_d whose global symmetries are characterized by 0-form symmetry algebra \mathcal{F} . Without loss of generality, we assume the spacetime is locally $M_{d-1} \times \mathbb{R}$ with Cauchy surface M_{d-1} .

The von Neumann algebra \mathcal{A} of a QFT is defined to be a weakly closed $*$ -subalgebra of bounded operators $\mathcal{B}(\mathcal{H})$ on the Hilbert space \mathcal{H} of the QFT which contains the unit operator [4]. The *commutant* of a set of bounded operators S is defined to be

$$S' = \{a' \in \mathcal{B}(\mathcal{H}) \mid [a', a] = 0, \forall a \in S\}. \tag{2.1}$$

A useful fact about a von Neumann algebra \mathcal{A} is that it satisfies (Theorem 2.1.4 of [4])

$$\mathcal{A}'' = \mathcal{A}. \tag{2.2}$$

A QFT is characterized by a net of assignments of the algebras

$$R \rightarrow \mathcal{A}(R) \tag{2.3}$$

where R is an open, finitely-extended region of M_d . For a region R , we define R' to be the *causal complement* of R which is the set of all points that are space-like separated from all points of R . The causality structure is expressed by the facts that $\mathcal{A}(R_1)$ commutes with $\mathcal{A}(R_2)$ when R_1 and R_2 are space-like separated, and $\mathcal{A}(R) \subset \mathcal{A}(R'')$ where R'' is the *causal completion* of R . We will restrict ourselves to the assignment $R \rightarrow \mathcal{A}(R)$ for R a subset of a *Cauchy surface* of the spacetime which is M_{d-1} by our assumption.

The von Neumann algebra $\mathcal{A}(M_{d-1})$ has several properties due to the requirement of locality of QFT. Following [102], in this work we shall focus on two of them, *additivity* and *Haag duality* [3]. The von Neumann algebra is said to satisfy additivity if ((III.1.2) of [4])

$$\mathcal{A}(R_1 \cup R_2) = (\mathcal{A}(R_1) \cup \mathcal{A}(R_2))'' \tag{2.4}$$

for two disjoint regions R_1 and R_2 of M_{d-1} .² On the other hand, the von Neumann algebra is said to satisfy Haag duality if ((III.4.9) of [4])

$$\mathcal{A}(R') = \mathcal{A}(R)' \tag{2.5}$$

for a region $R \in M_{d-1}$ and its complement R' .³ As we have said in the introduction, we will follow the choice of [102] to include certain non-local operators in $\mathcal{A}(M_{d-1})$ and check the consequences they lead to for additivity and Haag duality. Roughly speaking, the additivity means that the von Neumann algebra in $R_1 \cup R_2$ is generated by algebras in R_1 and R_2 , and the Haag duality means that the algebra of the causal complement is equal to the commutant of the algebra.

Of particular interest is what we shall call in this work the *symmetric sector von Neumann algebra* defined as

$$\mathcal{A}_{\mathcal{F}}(M_{d-1}) = \{\mathcal{O} \in \mathcal{A}(M_{d-1}) | N\mathcal{O} = \mathcal{O}N, \forall N \in \mathcal{F}\}, \tag{2.6}$$

where we do not distinguish an element N of \mathcal{F} and its representation on the Hilbert space \mathcal{H} of $\mathcal{T}_{\mathcal{F}}$. When the global symmetry on \mathcal{H} is a representation of group G , $\mathcal{A}_G(M_{d-1})$ is nothing but the set of G -neutral operators and forms a closed subalgebra of $\mathcal{A}(M_{d-1})$. Thus $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ can be viewed as a generalization of $\mathcal{A}_G(M_{d-1})$ to arbitrary fusion algebra \mathcal{F} which can contain both invertible and non-invertible elements. We will see in section 5 that additivity and Haag duality of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ can be violated in the presence of certain non-local neutral operators in $\mathcal{A}_{\mathcal{F}}$, whose existence is completely determined by \mathcal{F} . Therefore, our task is to first investigate what types of neutral non-local operators are there in $\mathcal{A}_{\mathcal{F}}$.

3 Operator content of symmetric sector von Neumann algebra

In this section, we study various (non-local) operators in $\mathcal{A}_{\mathcal{F}}$ for the QFT $\mathcal{T}_{\mathcal{F}}$ defined by (2.6) using only the properties of the 0-form symmetry \mathcal{F} , which is most generally described by

²The additivity of \mathcal{A} can be written as $\mathcal{A}(R_1 \vee R_2) = (\mathcal{A}(R_1) \cup \mathcal{A}(R_2))''$ ((III.4.8 of [4])) where $R_1 \vee R_2$ is the causal completion of the union of two causally complete sets R_1 and R_2 . When R_1 and R_2 are space-like separated we have $R_1 \vee R_2 = R_1 \cup R_2$ [104]. Therefore when R_1 and R_2 are two disjoint regions on a Cauchy surface we have $R_1 \vee R_2 = R_1 \cup R_2$, hence (2.4) holds.

³Strictly speaking, this holds only for *diamonds*, i.e. the causal completion of a ball in M_{d-1} [4]. We will see in section 5 that all the regions we consider in this work are indeed of this type.

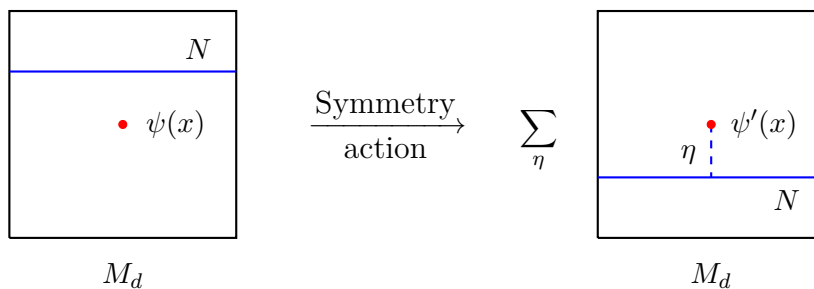


Figure 2. The action of symmetry operator $N \in \mathcal{F}$ on a charged operator. The symmetry can be invertible or non-invertible.

a fusion algebra with fusion rule

$$N_i \times N_j = \sum_k \mathcal{N}_{ij}^k N_k, \quad N_i, N_j, N_k \in \mathcal{F}, \quad (3.1)$$

where $\mathcal{N}_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the fusion coefficients. An object N is invertible if there exists an object $M \in \mathcal{F}$ such that $N \times M = 1$. As the local operators in $\mathcal{A}_{\mathcal{F}}$ are nothing but the uncharged local operators, there exist more non-trivial non-local operators in $\mathcal{A}_{\mathcal{F}}$.

Generally, for any local operator $\psi(x) \in \mathcal{A}^4$ supported on a Cauchy surface Σ which transforms non-trivially under $N \in \mathcal{F}$, we have

$$U(N)\psi(x) = \rho(N)\psi(x)U(N). \quad (3.2)$$

Without loss of generality, we assume that the local operator is located at $x^0 = 0$ and N on the l.h.s. of (3.2) is supported on a Cauchy surface at $x^0 > 0$ and that on the r.h.s. is supported on a Cauchy surface at $x^0 < 0$ as illustrated in figure 2. In general, the action $\rho(N)$ can be very complicated and does not have to be a linear transformation. Actually, for a non-invertible symmetry, various defects η can be generated when N passes across $\psi(x)$ and all possible configurations need to be summed over. In this sense, $\rho(N)$ in (3.2) is a condensed notation for this complicated action. On the other hand, no defects will be generated when an invertible symmetry generator passes across $\psi(x)$, in which case $\rho(N)$ acts as a linear transformation on $\psi(x)$.

We first consider an invertible element $N \in \mathcal{F}$. Actually, all invertible elements of a general ring \mathcal{F} must form a group $G \subset \mathcal{F}$, and one can assume that there exists local field $\psi_{\mathbf{r}}^i(x)$ living in the vector space $V_{\mathbf{r}}$ of a representation \mathbf{r} of G . Concretely, the symmetry action (3.2) of N can be characterized by

$$\rho(N) \in \text{GL}(V_{\mathbf{r}}) : V_{\mathbf{r}} \rightarrow V_{\mathbf{r}}. \quad (3.3)$$

Therefore, in the superselection sector $V_{\mathbf{r}}$, (3.2) can be written as

$$N\psi_{\mathbf{r}}^i(x) = \rho_{\mathbf{r}}(N)^i_j \psi_{\mathbf{r}}^j(x)N. \quad (i, j = 1, \dots, \dim V_{\mathbf{r}}) \quad (3.4)$$

⁴There is a subtlety that strictly speaking a local operator is unbounded, therefore technically one has to smear it out [105]. Nevertheless this subtlety will not affect the discussions in this work, we therefore insist on calling it a local operator for simplicity.

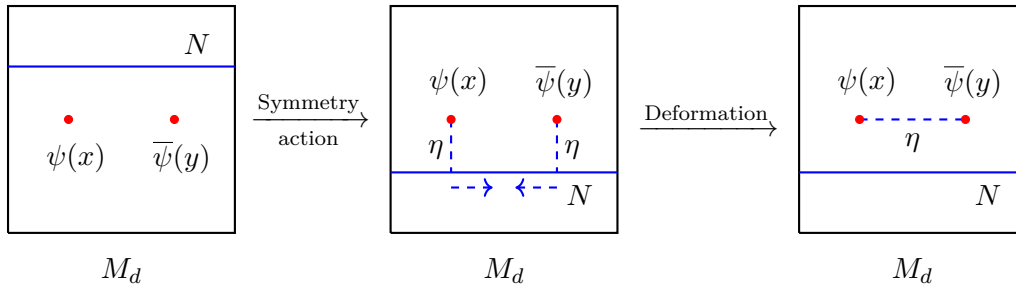


Figure 3. Symmetry action of a non-invertible element on a bi-local operator where a defect line D_ℓ connecting x and y is generated upon deformation.

Using (3.4) and the fact that in an ordinary QFT the existence of a local operator that transforms non-trivially under the action of N also guarantees the existence of its charge conjugation living in the dual vector space $\overline{V}_\mathbf{r}$, one can construct the following bi-local operator

$$\mathrm{tr} \psi_\mathbf{r}(x) \psi_{\overline{\mathbf{r}}}(y) := \psi_\mathbf{r}^i(x) \psi_{\overline{\mathbf{r}}i}(y), \quad (3.5)$$

for $\psi_\mathbf{r}^i(x) \in V_\mathbf{r}$ and $\psi_{\overline{\mathbf{r}}i}(x) \in \overline{V}_\mathbf{r}$. It is not hard to check that

$$N(\mathrm{tr} \psi_\mathbf{r}(x) \psi_{\overline{\mathbf{r}}}(y)) = (\psi_\mathbf{r}^i(x) \rho_\mathbf{r}(N)^j{}_i \rho_{\overline{\mathbf{r}}}(N)^k{}_j \psi_{\overline{\mathbf{r}}k}(y)) N = (\mathrm{tr} \psi_\mathbf{r}(x) \psi_{\overline{\mathbf{r}}}(y)) N. \quad (3.6)$$

Therefore, the bi-local operator $\mathrm{tr} \psi_\mathbf{r}(x) \psi_{\overline{\mathbf{r}}}(y)$ is invariant under the action of $N \in G$. When \mathcal{F} is a group, the local operator $\psi_\mathbf{r}^i(x)$ lives in a representation of \mathcal{F} and the bi-local operator (3.5) is invariant under all action of elements of \mathcal{F} , hence is in $\mathcal{A}_\mathcal{F}(M_{d-1})$.

The situation becomes tricky when there exist non-invertible elements in \mathcal{F} , since a representation of $G \subset \mathcal{F}$ does not fully characterize the transformation of $\psi(x)$ under \mathcal{F} . As illustrated in figure 2, typically under the action of a non-invertible symmetry generator N , a defect line η is generated, which stretches between the local operator $\psi(x)$ and the support of the symmetry operator. Therefore, the bi-local operator $\mathrm{tr} \psi(x) \overline{\psi}(y)$ built upon such $\psi(x)$ cannot be invariant under N , rather a defect line $\eta(\gamma_{xy})$ connecting $\psi(x)$ and $\overline{\psi}(y)$ is generated in the process illustrated in figure 3. Hence, such a bi-local operator is not an element of $\mathcal{A}_\mathcal{F}(M_{d-1})$. A prominent example is the order operator σ with conformal wright $(h, \overline{h}) = (1/16, 1/16)$ in Ising CFT [106–108], which is odd under the invertible \mathbb{Z}_2 -symmetry generator, but transforms non-trivially in the fashion illustrated in figure 2 under the non-invertible object of the symmetry category of Ising CFT. Therefore, it becomes rather tricky to determine if a bi-local operator is in $\mathcal{A}_\mathcal{F}(M_{d-1})$ when \mathcal{F} is non-invertible. We will see in the next section that this can be determined by a simple criterion via the SymTFT method.

Allowing non-local operators in $\mathcal{A}_\mathcal{F}(M_{d-1})$ also opens up the possibility of having more general extended operators beyond bi-local operators. In general, there exist twist operators of the form $\Psi(\partial\Sigma_{d-1})N(\Sigma_{d-1})$ where $N(\Sigma_{d-1})$ is a symmetry generator $N \in \mathcal{F}$ supported on $\Sigma_{d-1} \in M_{d-1}$ and $\Psi(\partial\Sigma_{d-1})$ is a dressing on the boundary $\partial\Sigma_{d-1}$. A prominent example is the operator $\psi(x)L_\epsilon\psi(y)$ in Ising CFT, where ψ is a left-moving fermion with conformal weight $(h, \overline{h}) = (1/2, 0)$ and L_ϵ the invertible \mathbb{Z}_2 -symmetry generator supported on a segment connecting x and y in the 1D space S^1 .⁵ Generalizing such 2d twist operator,

⁵It is argued in [102] that this operator is in $\mathcal{A}_\mathcal{F}(S^1)$ for \mathcal{F} the symmetry category of Ising CFT.

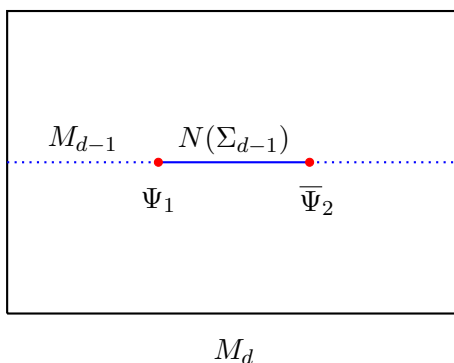


Figure 4. A twist operator supported on $\Sigma_{d-1} \subset M_{d-1}$ with dressing $\Psi_1(\Sigma_{d-2})$ and $\Psi_2(\Sigma'_{d-2})$ at the boundary of Σ_{d-1} .

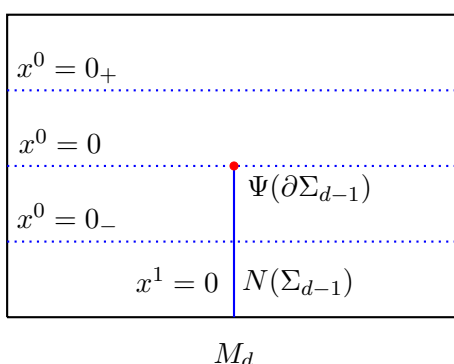


Figure 5. A twist operator supported on Σ with dressing Ψ at $\partial\Sigma$ to ensure consistent termination of N . In this figure x^0 is the coordinate of the vertical direction and x^1 is the coordinate of the horizontal direction. Σ_{d-1} extends from $-\infty$ to 0 along the vertical direction and is put at $x^1 = 0$ with $\partial\Sigma_{d-1}$ its boundary at $x^0 = 0$.

we assume that M_{d-1} is locally $M_{d-2} \times \mathbb{R}$ and consider an element $N \in \mathcal{F}$ supported on $\Sigma_{d-1} \cong M_{d-2} \times (0, 1)$ denoted by $N(\Sigma_{d-1})$. We then dress $N(\Sigma_{d-1})$ by Ψ_1 supported on $\Sigma_{d-2} = M_{d-2} \times \{0\}$ and Ψ_2 supported on $\Sigma'_{d-2} = M_{d-2} \times \{1\}$ to form what we call a *bi-twist* operator $\Psi_1(\Sigma_{d-2})N(\Sigma_{d-1})\Psi_2(\Sigma'_{d-2})$, which is plotted schematically in figure 4. To avoid clustering symbols, we will omit the supports in $\Psi_1(\Sigma_{d-2})$ and $\Psi_2(\Sigma'_{d-2})$ when it does not cause any confusion.

Before checking if there exists any bi-twist operator $\Psi_1 N(\Sigma_{d-1}) \Psi_2$ in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$, we need to check its existence in $\mathcal{A}(M_{d-1})$ first. We will make the discussion general by considering the twist operator $\Psi(\partial\Sigma_{d-1})N(\Sigma_{d-1})$ where Σ_{d-1} is a general simply-connected open subset of M_{d-1} . Physically, $\Psi(\partial\Sigma_{d-1})$ is dressed on $N(\Sigma_{d-1})$ to ensure consistent termination of $N(\Sigma_{d-1})$ at $\partial\Sigma_{d-1}$.

To see how the consistency condition constrains Ψ for given N , we consider the configuration in figure 5. For any local field $\phi(x)$ that is charged under N , we have

$$\begin{aligned} \phi(x^0 = 0_-, x^1 = 0_+, \vec{x}) &= N \cdot \phi(x^0 = 0_-, x^1 = 0_-, \vec{x}), \\ \phi(x^0 = 0_+, x^1 = 0_+, \vec{x}) &= \phi(x^0 = 0_+, x^1 = 0_-, \vec{x}) \end{aligned} \tag{3.7}$$

where $\vec{x} = (x^2, \dots, x^{d-1})$ and the specific representation of N is not specified in order to keep complete generality. If we apply a global symmetry transformation g such that $\phi \rightarrow \phi' = g \cdot \phi$, the boundary conditions will change to

$$\begin{aligned} \phi'(x^0 = 0_-, x^1 = 0_+, \vec{x}) &= N' \cdot \phi'(x^0 = 0_-, x^1 = 0_-, \vec{x}), \\ \phi'(x^0 = 0_+, x^1 = 0_+, \vec{x}) &= \phi'(x^0 = 0_+, x^1 = 0_-, \vec{x}), \end{aligned} \tag{3.8}$$

with $N' = gNg^{-1}$, which implies the tail of the twist operator $\Psi(\partial\Sigma_{d-1})N(\Sigma_{d-1})$ will be transformed to $N'(\Sigma_{d-1})$. When g is an element of the centralizer $C_{\mathcal{F}}(N)$ of N in \mathcal{F} , we have $N = N'$ and Ψ is allowed to transform as a representation of (or more precisely, a module over) $C_{\mathcal{F}}(N)$. One may further induce a module over \mathcal{F} from a module over $C_{\mathcal{F}}(N)$. Hence, for a twist operator $\Psi(\partial\Sigma_{d-1})N(\Sigma_{d-1})$ to exist, we require Ψ live in a module over $C_{\mathcal{F}}(N)$.⁶ For a global symmetry described by a group G , this is equivalent to requiring Ψ live in a vector space carrying a representation of $C_G(N)$ for $N \in G$.

We now investigate the conditions for $\Psi(\partial\Sigma_{d-1})N(\Sigma_{d-1})$ to be in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$. Now we will restrict the discussion to the bi-twist operators $\Psi_1 N(\Sigma_{d-1}) \Psi_2$. For simplicity, we start with $N \in Z(\mathcal{F})$, the set of elements of \mathcal{F} that commute with everything else in \mathcal{F} . To construct a twist operator $\Psi_1 N(\Sigma_{d-1}) \Psi_2$ in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$, we need to dress suitable Ψ_i 's at the boundaries of Σ_{d-1} . As discussed earlier, each Ψ_i is labeled by a module over $C_{\mathcal{F}}(N)$. Since we restrict to $N \in Z(\mathcal{F})$, we have $C_{\mathcal{F}}(N) \cong \mathcal{F}$ hence each Ψ_i is characterized by a module over \mathcal{F} . When \mathcal{F} is invertible, i.e. a group, each Ψ_i transforms linearly under all generators of \mathcal{F} , and we could simply choose Ψ_1 to be in representation \mathbf{r} and Ψ_2 to be in the dual representation $\bar{\mathbf{r}}$. Contracting all the indices by taking the trace like in (3.5) yields $\Psi_1 N(\Sigma_{d-1}) \Psi_2 \in \mathcal{A}_{\mathcal{F}}(M_{d-1})$.

However, the above construction of uncharged bi-twist operators works only for invertible \mathcal{F} with non-trivial center, and does not generalize to non-invertible \mathcal{F} or \mathcal{F} with trivial center. The point, though, is to show that there does exist \mathcal{F} such that $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ contains bi-twist operators. Again, we will see in the next section that uncharged bi-twist operators can be constructed conveniently via the SymTFT method.

4 Symmetric sector von Neumann algebra from SymTFT

The bi-local and the bi-twist operators in the symmetric sector von Neumann algebra discussed in the last section can be studied in a more transparent and unified manner from the perspective of SymTFT. The idea of SymTFT is to attach $\mathcal{T}_{\mathcal{F}}$ with global \mathcal{F} -symmetry on M_d to a topological field theory on $M_d \times [0, 1]$ with two boundaries $B_{\text{top}} \cong M_d \times \{0\}$ and $B_{\text{phys}} \cong M_d \times \{1\}$, as depicted in figure 6. The data of the symmetry and the dynamics of $\mathcal{T}_{\mathcal{F}}$ are separately stored at the topological boundary B_{top} and the physical boundary B_{phys} . A symmetry generator $N \in \mathcal{F}$ is supported on a $(d-1)$ -dimensional submanifold of B_{top} bordant to a Cauchy surface $M_{d-1} \subset B_{\text{phys}}$, and a local operator ψ on B_{phys} is extended into the bulk to be a line operator W stretching between the two boundaries. A symmetry action of N on a local operator ψ in $\mathcal{T}_{\mathcal{F}}$ can be understood from the SymTFT perspective as N

⁶The cautious readers may note that this resembles the data of the *Drinfeld center* of the category of G -graded vector spaces when $\mathcal{F} \cong G$ [109]. This is by no means a coincidence and will be studied in [110].

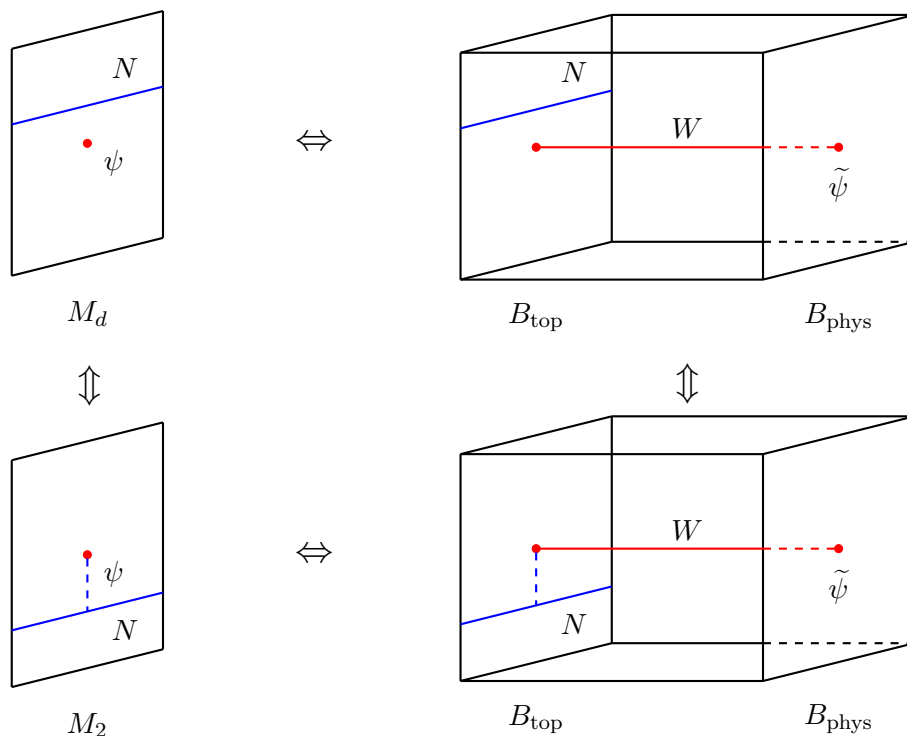


Figure 6. The SymTFT description of a symmetry action of $\mathcal{T}_{\mathcal{F}}$.

on B_{top} passing through the end point of the corresponding W on B_{top} . We anticipate the possibility that a defect connecting N and W on B_{top} can be generated (in particular when \mathcal{F} is non-invertible), which is represented by the dashed line in figure 6 (cf. figure 2).

To be concrete, we will first study 2d theory with symmetry \mathcal{F} , whose corresponding 3d SymTFT was first introduced in [111, 112]. The set of line operators (anyons) was studied in detail and shown to be in one-to-one correspondence with irreducible representations of \mathcal{F} [113–116]. This fact plays an important role in the following discussion. We will review various types of (extended) operators in the SymTFT and discuss the conditions for them to be in the symmetric sector von Neumann algebra $\mathcal{A}_{\mathcal{F}}(M_1)$. We will then generalize the discussion to $d > 2$.

4.1 Uncharged operators from SymTFT

Local operators. We begin with the local operators of $\mathcal{T}_{\mathcal{F}}$. A generic line operator in its corresponding SymTFT can be written as

$$L_{\alpha} = n_{\alpha} W_{\alpha}, \quad (n_{\alpha} \in \mathbb{Z}_{\geq 0}) \tag{4.1}$$

where W_{α} is a line operator in the bulk SymTFT as shown in figure 6. The coefficient n_{α} will be explained later. A line operator L_{α} is in a *Lagrangian algebra* \mathcal{L} , i.e. $L_{\alpha} \in \mathcal{L}$ (and $W_{\alpha} \in \mathcal{L}$), if it is allowed to end on B_{top} [117], which in turn means the corresponding W_{α}

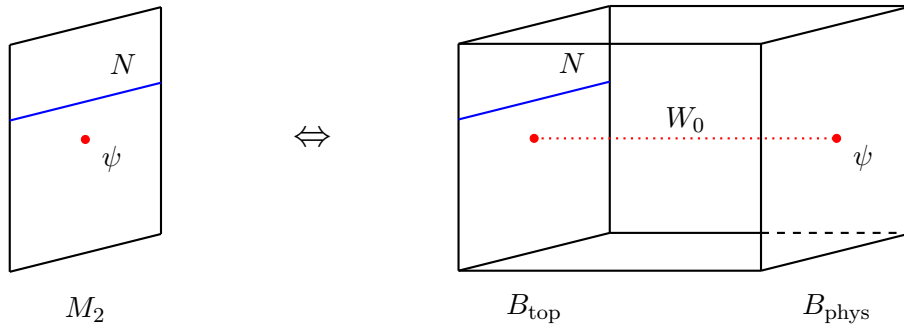


Figure 7. An uncharged local operator seen from the SymTFT perspective.

is terminable at a point on B_{top} .⁷ More precisely, one can write \mathcal{L} as

$$\mathcal{L} = \bigoplus_{\alpha} L_{\alpha}, \quad (4.2)$$

where L_{α} 's mutually commute with each other and satisfy certain technical conditions [117, 118]. In particular, the identity line operator W_0 in the bulk is always in the direct sum (4.2) with $n_0 = 1$. Inequivalent topological boundary conditions at B_{top} are in one-to-one correspondence with inequivalent choices of the Lagrangian algebras of the SymTFT. Moreover, L_{α} corresponds to a local operator of $\mathcal{T}_{\mathcal{F}}$ upon interval compactification.

By definition, the local operators in $\mathcal{A}_{\mathcal{F}}(M_1)$ are the ones invariant under the actions of all $N \in \mathcal{F}$ on B_{top} . From the SymTFT perspective, such an uncharged local operator is always represented by the identity line W_0 connecting the operator to B_{top} as shown in figure 7. Equivalently, we say that the endpoint of W_0 at B_{top} furnishes a trivial representation of \mathcal{F} . Hence, an uncharged local operator is essentially trivial from the SymTFT perspective.

Twist operators. For a given \mathcal{L} , a line W_{α} is not terminable on the B_{top} if $L_{\alpha} = n_{\alpha}W_{\alpha} \notin \mathcal{L}$, $\forall n_{\alpha} > 0$ [42]. Rather, a “tail” labeled by $N \in \mathcal{F}$ along γ on B_{top} going to infinity must be attached to such W_{α} as shown in figure 8. Such an operator $\psi N(\gamma)$ in $\mathcal{T}_{\mathcal{F}}$ on M_2 , or its SymTFT incarnation $\tilde{\psi}W_{\alpha}N(\gamma)$ in $M_2 \times [0, 1]$, is called a *twist operator*. Since the boundary condition at infinity is changed due to N , a twist operator should actually be thought of as an operator of a different theory characterized by the boundary condition at infinity prescribed by N . Therefore, all twist operators with $N(\gamma)$ stretching to infinity must be excluded from the von Neumann algebra $\mathcal{A}(M_1)$ of $\mathcal{T}_{\mathcal{F}}$, since it is simply not an operator acting on the Hilbert space of $\mathcal{T}_{\mathcal{F}}$.

Despite the fact that twist operators with $N(\gamma)$ extending to infinity are not in $\mathcal{A}(M_1)$, they are still very important because we can truncate $N(\gamma)$ and dress it at the point of truncation in a consistent manner as discussed in section 2. Before discussing such truncation from the SymTFT perspective, we note that it is possible that a line in the bulk extends to inequivalent tails on B_{top} . Conversely, the same tail can also be attached to different lines

⁷A line W_{α} is said to be not terminable on B_{top} when it has to connect to a defect line, which will later be called a “tail”, on B_{top} extending to infinity. We will encounter this case in a moment.

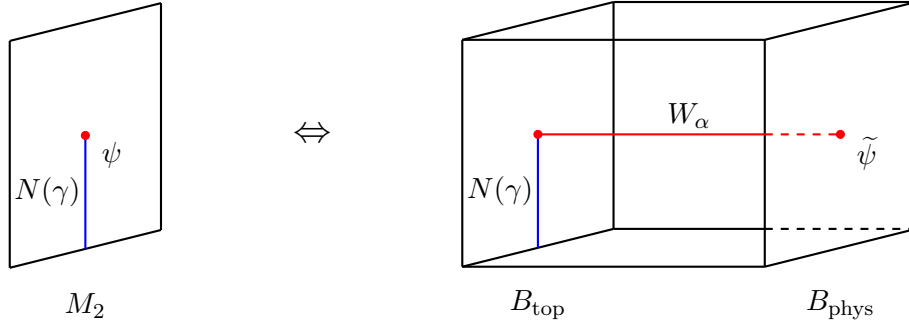


Figure 8. A twist operator in the SymTFT picture. For the configuration on the right we have $N(\gamma) \in t(W_\alpha)$.

in the bulk. For a line W_α that can extend to different tails in B_{top} , we write

$$t(W_\alpha) = \bigoplus_{N_i \in \mathcal{F}} a_i N_i, \quad (a_i \in \mathbb{Z}_{\geq 0}) \quad (4.3)$$

where each N_i (more precisely, $N_i(\gamma)$) is a tail to which W can attach, and the physical meaning of a_i will be explained in a moment. We note that in the notation (4.3), $L \in \mathcal{L}$ if the identity of \mathcal{F} , or a multiple of it, is a summand in $t(L)$. For W_α with $t(W_\alpha) = aN$ and W_β with $t(W_\beta) = bM$, we have

$$\text{Hom}_{\mathcal{F}}(t(W_\alpha), t(W_\beta)) = \begin{cases} \mathbb{C}^{ab}, & (N = M) \\ \emptyset, & (N \neq M) \end{cases} \quad (4.4)$$

where ab is number of ways that $t(W_\alpha)$ can be joined to $t(W_\beta)$ when $N = M$. To see why, we first consider two tails aN and N on B_{top} . The coefficient a means there is an a -fold ambiguity for choosing an N on B_{top} , which in turn implies that there are a inequivalent ways to join aN with N .⁸ Now it should be clear that for W_α with tail aN and W_β with tail bN , there are ab inequivalent ways to join them. Therefore, for generic W_α with tail $\bigoplus a_i N_i$ and W_β with tail $\bigoplus b_i N_i$, we have

$$\text{Hom}_{\mathcal{F}}(t(W_\alpha), t(W_\beta)) = \bigoplus_i \mathbb{C}^{a_i b_i}, \quad (4.5)$$

with total dimension $\sum_i a_i b_i$. Physically, $\sum_i a_i b_i$ is the total number of inequivalent ways that $t(W_\alpha)$ can be joined with $t(W_\beta)$ on B_{top} , where each $a_i b_i$ is the number of ways that i^{th} component of the two tails can be joined.

Physically, for $t(W_\alpha) = \bigoplus_i a_i N_i$ and $t(W_\beta) = \bigoplus_i b_i N_i$, when $a_i b_i$ is non-zero for certain i , W_α can be joined to W_β through N_i . Hence, to check if W_α and W_β can be joined on B_{top} , we need to look for the corresponding tails represented in terms of the elements of \mathcal{F} . As (4.4) counts the number of inequivalent ways the tails can be joined on B_{top} , it is natural to look for its SymTFT version in terms of W 's. For this, we have [117–121]

$$\text{Hom}_{\mathcal{F}}(t(W_\alpha), t(W_\beta)) = \text{Hom}_{\text{SymTFT}}(W_\alpha, W_\beta \otimes \mathcal{L}). \quad (4.6)$$

⁸The coefficient in (4.1) is interpreted similarly.

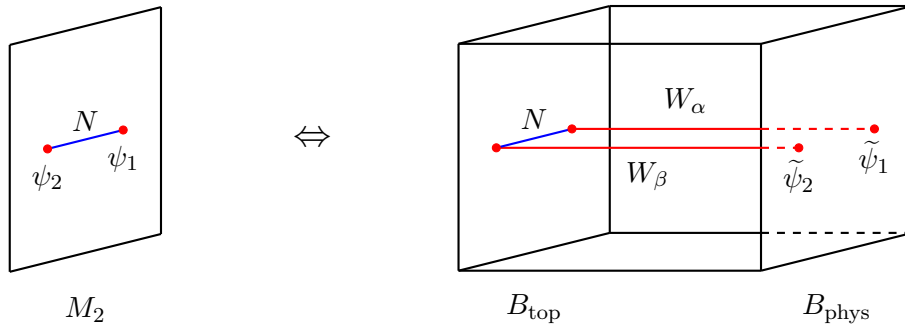


Figure 9. The bi-twist operator in the SymTFT picture. When $N = 1$ is the identity line, the operator becomes a bi-local operator in $\mathcal{T}_{\mathcal{F}}$.

Physically, while the l.h.s. counts the number of ways that two tails can be joined on B_{top} , the r.h.s. of the above equation simply counts the times of appearances of W_{α} in the fusion $W_{\beta} \otimes \mathcal{L}$, and the symmetry \mathcal{F} is implicitly determined by the choice of \mathcal{L} .

A terminable line $W_{\alpha} \in \mathcal{L}$ on B_{top} (with its endpoint $\tilde{\psi}(x)$ at B_{phys}) may become a twist operator $\tilde{\psi}W_{\alpha}N(\gamma)$ under the action of an element of \mathcal{F} as illustrated in figure 6. When this is the case, one says that $\psi(x)$ together with $\psi(x)N(\gamma)$ for all possible N 's which can be obtained via symmetry actions in \mathcal{F} form a multiplet in $\mathcal{T}_{\mathcal{F}}$.

Bi-twist and bi-local operators. Let us consider two lines W_{α} and W_{β} in the bulk. As discussed previously, they can be connected on B_{top} by a line N iff

$$\text{Hom}_{\text{SymTFT}}(W_{\alpha}, W_{\beta} \times \mathcal{L}) \neq \emptyset. \quad (4.7)$$

Therefore, when (4.7) holds, W_{α} and W_{β} together with the line connecting them on B_{top} can form a bi-twist operator as shown in figure 9. In particular, when $N = 1$, or equivalently when $W_{\alpha}, W_{\beta} \in \mathcal{L}$, the endpoints of W_{α} and W_{β} form a bi-local operator in 2D. Similar to a single local operator, a generic bi-local or bi-twist operator can become a different bi-local or bi-twist operator under a symmetry action (cf. figure 6). Since both the bi-twist and bi-local operators have finite volume support and do not change the boundary condition at infinity, we choose to include them in $\mathcal{A}(M_1)$ and the same choice has been made in [102].

We now look for bi-twist and bi-local operators that are in $\mathcal{A}_{\mathcal{F}}(M_1)$. The construction of such operators, which was a bit cumbersome and subject to many edge cases in section 2 from the perspective of the physical QFT living on the boundary, becomes transparent from the SymTFT perspective in the bulk. We consider connecting $\tilde{\psi}_1$ with $\tilde{\psi}_2$ by a line W_{α} in the bulk, assuming such a line exists. Then after dragging and projecting W_{α} onto B_{top} , the composite operator $\tilde{\psi}_1W_{\alpha}\tilde{\psi}_2$ must commute with all \mathcal{F} on B_{top} , since we can simply drag it back into the bulk to make it not touch anything on B_{top} . The operators are shown in figure 10 and are referred to as patch operators [122–126].

Recall that when W_{α} can be connected to different tails N_i on B_{top} , we write $t(W_{\alpha}) = \bigoplus a_i N_i$ where a_i 's specify the ambiguities for each N_i . This means one can connect W_{α} to $W_{\bar{\alpha}}$ (the overline indicates orientation-reversal) by an N_i on B_{top} . However, one generically cannot drag $W_{\alpha}N_iW_{\bar{\alpha}}$ into the bulk in the manner discussed previously. Otherwise, $W_{\alpha}N_iW_{\bar{\alpha}}$ can be

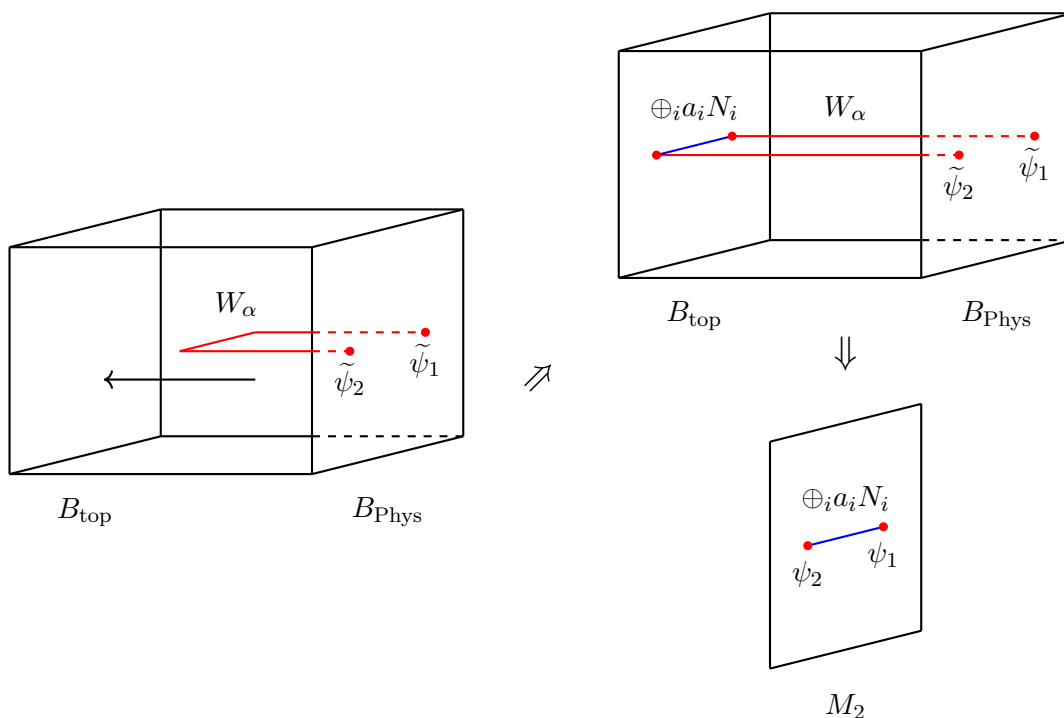


Figure 10. A set of symmetric bi-twist (or bi-local depending on whether $\oplus_i a_i N_i \propto 1$) operators can be constructed by bending the bulk operator W_α and dragging it onto B_{top} .

dragged into the bulk then dragged back onto B_{top} with a different N_j connecting them where the transformation from N_i to N_j is realized by a symmetry action in \mathcal{F} . Hence $W_\alpha N_i W_{\bar{\alpha}}$ is non-invariant under an action in \mathcal{F} , which contradicts the fact that any bi-twist operator which can be freely dragged into the bulk must commute with all actions in \mathcal{F} . Therefore, to ensure that a boundary line can be freely dragged into the bulk, we must connect W_α and $W_{\bar{\alpha}}$ by the full $\oplus_i a_i N_i$ as illustrated in figure 10. In other words, an uncharged bi-twist/bi-local operator is realized by the operator $W_\alpha (\oplus_i a_i N_i) W_{\bar{\alpha}}$ in the SymTFT in the bulk. These operators are therefore by definition in $\mathcal{A}_{\mathcal{F}}(M_1)$ and will play important roles in section 5.

4.2 Generalization to higher dimensions

In this section, we generalize the above construction to higher dimensions, where the distinction between bi-local operators and bi-twist operators becomes manifest. While the bi-local and the bi-twist operators can be constructed in essentially the same manner as illustrated in figure 10, one has to be careful about the dimensionality of various operators. For example, when \mathcal{F} is a 0-form symmetry, while an uncharged bi-local can still be constructed by a line in the bulk ending on B_{top} , a bi-twist operator should be viewed as a $(d - 1)$ -dimensional hypersurface in the bulk joined with a $(d - 1)$ -dimensional tail on B_{top} .

The BF -SymTFT proposed in [68] provides a natural framework for the construction of such operators in general dimension, at least for invertible symmetries. To be concrete, we focus on QFT $\mathcal{T}_{\text{SU}(n)}$ with non-anomalous $\text{SU}(n)$ flavor symmetry on a d -dimensional

manifold M_d . The SymTFT action in the bulk is an $SU(n)$ BF -action

$$S = \int_{M_{d+1}} \text{tr} B \wedge F, \tag{4.8}$$

for $M_{d+1} \cong M_d \times [0, 1]$ and the QFT lives on $B_{\text{phys}} \cong M_d \times \{1\}$. To construct a bi-local operator from the SymTFT perspective, we consider a Wilson line $W_{\mathbf{r}}$ connecting two local operators on B_{phys}

$$\text{tr} \tilde{\psi}_1(x) W_{\mathbf{r}}(\gamma_{xy}) \tilde{\psi}_2(y) \equiv \tilde{\psi}_1(x) \mathcal{P} \exp \left(i \int_{\gamma_{xy}} A \right) \tilde{\psi}_2(y), \tag{4.9}$$

for a path γ_{xy} connecting x and y in the bulk. Since $\mathcal{P} \exp \left(i \int_{\gamma_{xy}} A \right)$ with fixed x and y is topological in M_{d+1} , it can be freely restricted onto or dragged away from B_{top} , therefore such an operator must become an uncharged bi-local operator $\text{tr}(\psi_1(x)\psi_2(y))$ after shrinking the interval (cf. (3.5)).

The construction of a bi-twist operator with codimension-1 support is a bit more tricky. The bulk topological operator in this case is $U_{[\alpha]}(\Sigma_{d-1})$ with codimension-2 support Σ_{d-1} in the bulk, which depends on a conjugacy class $[\alpha]$ of $SU(n)$. Without reviewing the details of [68], the key observation is that a (hyper)surface operator of this type that depends on a single element of $SU(n)$ cannot be topological and hence cannot be freely dragged into the bulk. Therefore, unless α takes certain special values, e.g. $\alpha \in Z(SU(n))$ which is the center element of $SU(n)$, the tail of $U_{[\alpha]}(\Sigma_{d-1})$ on B_{top} cannot commute with all elements of $SU(n)$. Note that there is no contradiction here since any $\alpha \in Z(G)$ is the only representative of its own conjugacy class. However, when $\alpha \notin Z(SU(n))$, one has to include all elements in its conjugacy class to construct the topological operator $U_{[\alpha]}(\Sigma_{d-1})$ in order for it to be freely moved away from B_{top} hence is a candidate in $\mathcal{A}_{SU(n)}(M_{d-1})$.

More precisely, this topological operator can be written explicitly as [68]

$$U_{[\alpha]}(\Sigma_{d-1}) = \int dg \exp \left(i \int_{\Sigma_{d-1}} (\tilde{\alpha}, B) \right), \tag{4.10}$$

where dg is the Haar measure of G and $\tilde{\alpha}$ is the parallel transport of $g\alpha(p)g^{-1}$ for a covariantly constant α at an arbitrary $p \in \Sigma_{d-1}$ by the holonomy of A . One may recognize that this is a concrete realization of (4.3), where the integral plays the role of the direct sum in (4.3). We then terminate such $U_{[\alpha]}(\Sigma_{d-1})$ on B_{phys} with suitable $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ at the two boundaries of Σ_{d-1} to obtain a bi-twist operator $\text{tr} \Psi_1 N(\Sigma_{d-1}) \Psi_2 \in \mathcal{A}_{SU(n)}(M_{d-1})$ after projecting to $\mathcal{T}_{SU(n)}$ (cf. figure 4). In this work, we will not explicitly give a full construction of bi-twist operators in the presence of non-invertible symmetries; rather, we just note that the construction should be similar to the one given above.

In the above discussion, we assumed the topological boundary B_{top} supports the 0-form $SU(n)$ flavor symmetry. If we gauge the non-anomalous $SU(n)$ flavor symmetry⁹ in $\mathcal{T}_{SU(n)}$, a $(d-2)$ -form $\text{Rep}(SU(n))$ non-invertible symmetry whose charged objects are $(d-2)$ -dimensional extended operator will emerge. In the SymTFT picture, the gauging

⁹The $SU(n)$ symmetry is gauged by summing over flat $SU(n)$ -connections on M_d .

corresponds to switching to another topological boundary B'_{top} with boundary condition such that only $(d - 1)$ -dimensional extended operator $U_{[\alpha]}$ can end on B'_{top} . The Wilson line operators W in the bulk will become the generators $(d - 2)$ -form $\text{Rep}(\text{SU}(n))$ symmetry on B'_{top} . The roles of bi-local operators and bi-twist operators are exchanged after the gauging. The extended operator $U_{[\alpha]}(\Sigma_{d-1})$ terminated at $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ on B_{phys} become a bi-local operator $\text{tr} \tilde{\Psi}_1 \tilde{\Psi}_2 \in \mathcal{A}_{\text{Rep}(\text{SU}(n))}(M_{d-1})$ and is uncharged under the $(d - 2)$ -form symmetry after projecting to $\mathcal{T}_{\text{Rep}(\text{SU}(n))}$. On the other hand, the local operators $\tilde{\psi}_1(x), \tilde{\psi}_2(y)$ on B_{phys} connected by a bulk Wilson line operators $W_{\mathbf{r}}(\gamma_{xy})$ will project to the bi-twist operator $\psi_1(x) \mathcal{P} \exp \left(i \int_{\gamma_{xy}} A \right) \psi_2(y) \in \mathcal{A}_{\text{Rep}(\text{SU}(n))}(M_{d-1})$ after projecting to $\mathcal{T}_{\text{Rep}(\text{SU}(n))}$.

We can also gauge only the center subgroup $Z(\text{SU}(n)) = \mathbb{Z}_n$ in $\mathcal{T}_{\text{SU}(n)}$, the resulting symmetry would be a mixture between a 0-form $\text{PSU}(n)$ symmetry and a $(d - 2)$ -form \mathbb{Z}_n symmetry. In the SymTFT picture, only the Wilson line operators $W_{\mathbf{r}}$ whose representation \mathbf{r} is neutral under the \mathbb{Z}_N center can end on the new topological boundary. They give rise to the local operators $\psi(x)$ charged under the $\text{PSU}(n)$ symmetry after projecting to $\mathcal{T}_{\text{PSU}(n) \times \mathbb{Z}_n^{(d-2)}}$. The $(d - 1)$ -dimensional extended operator $U_{[\alpha]}$ with the conjugacy class $[\alpha] \in Z(\text{SU}(n))$ can also end on the topological boundary, and they give rise to the $(d - 2)$ -dimensional extended operators $\Psi(\Sigma_{d-2})$ charged under the $(d - 2)$ -form \mathbb{Z}_n symmetry. Therefore, we have two kinds of symmetric bi-local (extended) operators in $\mathcal{T}_{\text{PSU}(n) \times \mathbb{Z}_n^{(d-2)}}$. Other Wilson line operators W' and extended operators U' cannot end on the topological boundary, and they should connect to 1-dimensional or $(d - 1)$ -dimensional tails. The tails are generators of the $(d - 2)$ -form \mathbb{Z}_n symmetry and 0-form $\text{PSU}(n)$ symmetry on the topological boundary. As a consequence, we also have two kinds of symmetric bi-twist (extended) operators in $\mathcal{T}_{\text{PSU}(n) \times \mathbb{Z}_n^{(d-2)}}$ after such \mathbb{Z}_n -gauging.

In general, one can study a d -dimensional theory $\mathcal{T}_{\mathcal{F}}$ whose \mathcal{F} contains 0-form and/or $(d - 2)$ -form symmetries. In the $(d + 1)$ -dimensional SymTFT, we can define the topological boundary B_{top} via choosing the collection of line operators W_{α} and $(d - 1)$ -dimensional extended operator U_{μ} in the bulk that can end simultaneously on B_{top} and write formally as (cf. (4.2))

$$\mathcal{L} = \left(\bigoplus_{\alpha} W_{\alpha} \right) \oplus \left(\bigoplus_{\mu} U_{\mu} \right). \tag{4.11}$$

If \mathcal{F} happens to have other p -form symmetries in it, one must also include other compatible endable extended operators in the SymTFT to \mathcal{L} .

As a final remark, note that we see a $(d - 2)$ -form symmetry arise as the consequence of gauging 0-form symmetry \mathcal{F} . On the other hand, if \mathcal{F} is a pure 0-form symmetry, then the local operators are not allowed to transform to twist operators under non-invertible symmetry action when $d > 2$, since the tail of the twist operators are 1-dimensional which implies the $(d - 2)$ -form symmetry should coexist in the theory. In the following discussion, we will allow the presence of $(d - 2)$ -form symmetry in \mathcal{F} , which can interact non-trivially with the 0-form symmetry to allow the transition from local operators to twist operators. We will denote the generator of $(d - 2)$ -form symmetry by $\eta(\gamma)$ for a line γ , the fact that it can attach to a non-invertible 0-form generator $N(M_{d-1})$ supported on M_{d-1} implies

$$\eta(\gamma)N(M_{d-1}) = N(M_{d-1})\eta(\gamma) = N(M_{d-1}), \quad (\gamma \subset M_{d-1}) \tag{4.12}$$

such that $\eta(\gamma)$ can be absorbed into $N(M_{d-1})$. Multiply $\overline{N}(M_{d-1})$ on both side and assume $1 \in N\overline{N}$, one has $\eta(\gamma) \in N(M_{d-1})\overline{N}(M_{d-1})$ for any $\gamma \subset M_{d-1}$. Therefore, one would expect

$$N(M_{d-1})\overline{N}(M_{d-1}) = \mathcal{C}(M_{d-1}), \tag{4.13}$$

where $\mathcal{C}(M_{d-1})$ is known as a condensation of $\eta(\gamma)$ on M_{d-1} . Notice that (4.13) does not contradict with $1 \in N\overline{N}$ since the identity element 1 is also an ingredient of \mathcal{C} . For example, when M_{d-1} is a compact manifold and $\eta(\gamma)$ generates an \mathbb{Z}_k $(d-2)$ -form symmetry, we can write

$$\mathcal{C}(M_{d-1}) = \bigoplus_{\gamma \in H_1(M_{d-1}, \mathbb{Z}_k)} \eta(\gamma), \tag{4.14}$$

which is known as 1-gauging the $(d-3)$ -form symmetry along M_{d-1} [127].

In the literature, similar N -defects are constructed as the duality defects of the theory under the gauging of a p -form symmetry, and they are symmetries of the theory only when it is self-dual. For example, the $2d$ Ising model at temperature T is mapped to itself with dual temperature T' under the Kramers-Wannier (KW) transformation, or \mathbb{Z}_2 -gauging. The KW duality defect N becomes a symmetry only at the critical temperature T_c . Duality defects in $4d$ on gauging 1-form symmetries are also studied in [49, 50, 69, 70, 128, 129]. In the following, we will assume the existence of such N as a non-invertible symmetry and not delve into the details of its construction.

4.3 Relation to the non-SymTFT construction and a sub-algebra of the Lagrangian algebra

It should be clear from our construction in section 3 that $\mathcal{A}_{\mathcal{F}}(M_1)$ contains a bi-local operator whenever there exists a local operator $\psi(x)$ that does not become a twist operator connecting to a defect line under any non-intertible element of \mathcal{F} . The SymTFT construction presented in section 4 makes this construction sharper by encapsulating the above somewhat complicated condition into a simpler one constraining the property of the Lagrangian algebra \mathcal{L} .

Recall that a charged local operator $\psi(x)$ corresponds to a terminable line $W \in \mathcal{L}$ with one end on B_{top} and the other end on $\tilde{\psi}(x)$. Requiring W be terminable means $t(W) = a_0 1 \oplus (\oplus_{i \neq 0} a_i N_i)$ with non-zero a_0 . If $t(W) \neq a_0 1$, there must exist certain non-invertible symmetries, say $M \in \mathcal{F}$, upon which the identity tail 1 can be transformed to another N_i of $t(W)$. Therefore, after projection to $\mathcal{T}_{\mathcal{F}}$, $\psi(x)$ cannot be invariant under M as it must become a twist operator joined to N_i defect line. Therefore, in order to have a bi-local operator in $\mathcal{A}_{\mathcal{F}}(M_1)$, we must require that there exists $L \in \mathcal{L}$ such that $t(L) \propto 1$. We define $\tilde{\mathcal{L}} \subset \mathcal{L}$ to be the subset containing all $L \in \mathcal{L}$ such that $t(L) \propto 1$ on B_{top} . Therefore, the necessary and sufficient condition for $\mathcal{A}_{\mathcal{F}}(M_1)$ containing a bi-local operator is $\tilde{\mathcal{L}} \neq W_0$, which means it contains non-trivial elements other than the bulk identity.

On the other hand, there always exist symmetric bi-twist operators in $\mathcal{A}_{\mathcal{F}}(M_1)$. Clearly, for any bulk line operator $W' \notin \tilde{\mathcal{L}}$ with $t(W') \neq a_0 1$, it can be transformed into a twist operator connecting to a defect line on B_{top} . We then construct the lines $\tilde{\psi}_1 W'$ and $\tilde{\psi}_2 W'$ in M_3 and connect the lines by $t(W') = \oplus_i a'_i N_i$ on B_{top} . Since this operator can be freely dragged into M_3 , it must be uncharged under all symmetry actions in \mathcal{F} . After projection

to \mathcal{F} , the two operators ψ_1 and ψ_2 connecting by $t(W') = \oplus_i a'_i N_i$ transform under two representations of \mathcal{F} that dual to each other, since they are lifted to W' and its orientation reversal in the bulk SymTFT.

In higher dimension, we similarly define $\widetilde{\mathcal{L}} \subset \mathcal{L}$ to be the subset containing all line operators W and $(d-1)$ -dimensional extended operators U in \mathcal{L} that cannot attach to non-trivial 1-dimensional and $(d-1)$ -dimensional tails on B_{top} . Then $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ contains bi-local (extended) operators if $\widetilde{\mathcal{L}} \neq W_0 \oplus U_0$, which means it contains operators other than the trivial line and $(d-1)$ -dimensional extended operators. On the other hand, one can always construct symmetric bi-twist operators using line operators $W' \notin \widetilde{\mathcal{L}}$ or $(d-1)$ -dimensional extended operators $U' \notin \widetilde{\mathcal{L}}$, connecting two local operators or two $(d-2)$ -dimensional extended operators on B_{phys} as discussed in section 4.2.

In summary, we introduce $\widetilde{\mathcal{L}} \subset \mathcal{L}$ to be the subset generated by all line operators $W \in \mathcal{L}$ and $(d-1)$ -dimensional extended operator $U \in \mathcal{L}$ that cannot join to any non-trivial tail on B_{top} . Thus, any local operator on B_{phys} that is the endpoint of W cannot be transformed to a twist operator under any symmetry action in \mathcal{F} . A similar argument holds for any $(d-2)$ -dimensional extended operator on B_{phys} that is the ending locus of U . Clearly, $\widetilde{\mathcal{L}}$ is closed under fusion.

5 Additivity and Haag duality in the presence of Bi-local and Bi-twist operators

We are now ready to discuss the implications of the operator content of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ on certain properties of it. In particular, we focus on additivity and Haag duality of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$. We anticipate the case where \mathcal{F} is a mixture of 0-form and $(d-2)$ -form symmetry as discussed at the end of section 4.2, which allows the transition between local operators and twist operators under non-invertible symmetries. In general, we will consider four kinds of bi-local and bi-twist operators in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$.

- Bi-local operators $\text{tr} \psi_1(x) \psi_2(y)$;
- Bi-local extended operators $\text{tr} \Psi_1(\Sigma_{d-2}) \Psi_2(\Sigma'_{d-2})$;
- Bi-twist operators $\text{tr} \psi_1(x) \eta(\gamma_{xy}) \psi_2(y)$;
- Bi-twist extended operators $\text{tr} \Psi_1(\Sigma_{d-2}) N(\Sigma_{d-1}) \Psi_2(\Sigma'_{d-2})$ with $\partial \Sigma_{d-1} = \Sigma_{d-2} \cup \overline{\Sigma'_{d-2}}$.

Following [102], we will show the following holds in general, regardless of the details of the dynamics of \mathcal{F} :

- Additivity of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated whenever $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ contains bi-local operators $\text{tr} \psi_1(x) \psi_2(y)$ or bi-local $(d-2)$ -dimensional extended operators $\text{tr} \Psi_1(\Sigma_{d-2}) \Psi_2(\Sigma'_{d-2})$.
- Haag duality of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated whenever $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ contains bi-twist operators $\text{tr} \Psi_1(\Sigma_{d-2}) N(\Sigma_{d-1}) \Psi_2(\Sigma'_{d-2})$ or $\text{tr} \psi_1(x) \eta(\gamma_{xy}) \psi_2(y)$ that commute with all components of bi-local (extended) operators in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$.

We will first show the above holds in general dimensions, then we will specialize to 2d, where the conditions in the above claim can be made concrete using the Lagrangian algebra explicitly.

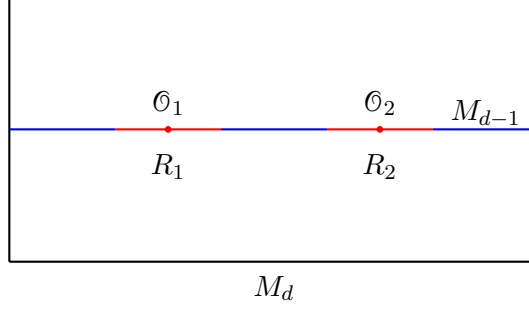


Figure 11. Putting a bi-local operator on M_{d-1} . Here the operator \mathcal{O} can be either local operator $\psi(x)$ or extended operator $\Psi(\Sigma_{d-2})$.

5.1 Violation of additivity and Haag duality in general dimensions

Recall from section 4.3 that there exist bi-local operators $\text{tr} \psi_1(x) \psi_2(y) \in \mathcal{A}_{\mathcal{F}}(M_{d-1})$ or bi-local extended operators $\text{tr} \Psi_1(\Sigma_{d-2}) \Psi_2(\Sigma'_{d-2})$ when $\widetilde{\mathcal{L}} \neq \mathcal{L}$. When this is the case, we pick two disjoint regions R_1 and R_2 on the Cauchy surface M_{d-1} and put one of the component operators of the bi-local operator, say $\psi_1(x)$ or $\Psi_1(\Sigma_{d-2})$, in R_1 while the other one $\psi_2(y)$ or $\Psi_2(\Sigma'_{d-2})$ in R_2 as illustrated in figure 11. While $\psi_1(x), \Psi_1(\Sigma_{d-2}) \notin \mathcal{A}_{\mathcal{F}}(R_1)$ and $\psi_2(y), \Psi_2(\Sigma'_{d-2}) \notin \mathcal{A}_{\mathcal{F}}(R_2)$ as both are charged operators under N , we have

$$\text{tr} \psi_1(x) \psi_2(y) \in \mathcal{A}_{\mathcal{F}}(R_1 \cup R_2), \quad \text{tr} \Psi_1(\Sigma_{d-2}) \Psi_2(\Sigma'_{d-2}) \in \mathcal{A}_{\mathcal{F}}(R_1 \cup R_2), \quad (5.1)$$

due to (3.6) (or visually as figure 9 with $N = 1$). On the other hand, we have [102]

$$\text{tr} \psi_1(x) \psi_2(y) \notin (\mathcal{A}_{\mathcal{F}}(R_1) \cup \mathcal{A}_{\mathcal{F}}(R_2))'', \quad \text{tr} \Psi_1(\Sigma_{d-2}) \Psi_2(\Sigma'_{d-2}) \notin (\mathcal{A}_{\mathcal{F}}(R_1) \cup \mathcal{A}_{\mathcal{F}}(R_2))'', \quad (5.2)$$

the additivity of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated. The reason is that we can construct a bi-twist operator with two endpoint operators lying in the blue region, and the symmetry generator connecting the two operators covers either R_1 or R_2 . This operator is an element of the commutant $(\mathcal{A}_{\mathcal{F}}(R_1) \cup \mathcal{A}_{\mathcal{F}}(R_2))'$, and it does not commute with the above bi-local operators in general. By the discussion in section 4.3, this means that additivity of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated whenever $\widetilde{\mathcal{L}} \neq W_0 \oplus U_0$.

On the other hand, Haag duality can potentially be violated by the configuration in figure 12 where $R \cup R' = M_{d-1}$ and $R \cap R' = \emptyset$. Let us first focus on a bi-twist extended operator $\Psi_1(\Sigma_{d-2}) N(\Sigma_{d-1}) \Psi_2(\Sigma'_{d-2})$ plotted in figure 12, which is by definition not in $\mathcal{A}_{\mathcal{F}}(R)$ since its support is not entirely within R . Hence by (2.5) Haag duality will be violated if $\Psi_1(\Sigma_{d-2}) N(\Sigma_{d-1}) \Psi_2(\Sigma'_{d-2})$ happens to commute with everything in $\mathcal{A}_{\mathcal{F}}(R')$, i.e. if it is in $\mathcal{A}_{\mathcal{F}}(R)'$.

Clearly, $\Psi_1(\Sigma_{d-2}) N(\Sigma_{d-1}) \Psi_2(\Sigma'_{d-2}) \notin \mathcal{A}_{\mathcal{F}}(R)'$ when \mathcal{F} is invertible, i.e. when \mathcal{F} is a group. To see this, we note that for \mathcal{F} to be really a symmetry of $\mathcal{T}_{\mathcal{F}}$, there must exist at least one charged local operator that transforms non-trivially under \mathcal{F} . Thus one can form a bi-local operator $\text{tr} \psi_{\mathbf{r}}(x) \psi_{\overline{\mathbf{r}}}(y)$ whose components $\psi_{\mathbf{r}}(x)$ and $\psi_{\overline{\mathbf{r}}}(y)$ both transform non-trivially under the action of N but is itself in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$. We then put $\psi_{\mathbf{r}}(x)$ in the middle R' region and $\psi_{\overline{\mathbf{r}}}(y)$ in the leftmost R' region in figure 12. Clearly, we have

$$\text{tr} \psi_{\mathbf{r}}(x) \psi_{\overline{\mathbf{r}}}(y) \in \mathcal{A}_{\mathcal{F}}(R') \quad (5.3)$$

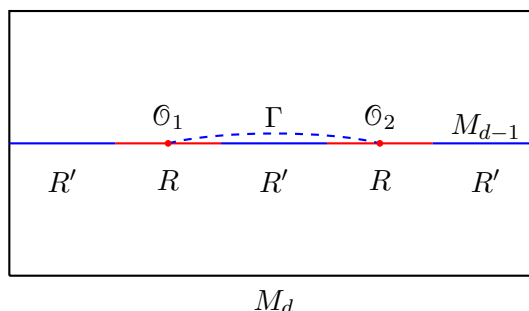


Figure 12. Putting a bi-twist (extended) operator $\mathcal{O}_1\Gamma\mathcal{O}_2$ on M_{d-1} . Here $\mathcal{O}_1\Gamma\mathcal{O}_2$ can be either $\psi_1(x)\eta(\gamma_{xy})\psi_2(y)$ or $\Psi_1(\Sigma_{d-2})N(\Sigma_{d-1})\Psi_2(\Sigma'_{d-2})$. Note that though the tail Γ connecting two operators is represented by the dashed curve slightly away from M_{d-1} for clarity, we indeed have $\Gamma \subset M_{d-1}$.

but since the charged local component $\psi_r(x)$ does not commute with $N(\Sigma_{d-1})$, we have

$$\Psi_1(\Sigma_{d-2})N(\Sigma_{d-1})\Psi_2(\Sigma'_{d-2}) \notin \mathcal{A}_{\mathcal{F}}(R')'. \tag{5.4}$$

Therefore, when \mathcal{F} is invertible, Haag duality of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ cannot be violated by any bi-twist extended operators. The argument also holds for bi-twist operator $\psi_1(x)\eta(\gamma_{xy})\psi_2(y)$ where we can consider putting the bi-local extended operator $\text{tr}\Psi(\Sigma_{d-2})\bar{\Psi}_{d-2}(\Sigma_{d-2})$ in different R' region.

It should now be clear that in order for Haag duality to potentially be violated in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$, \mathcal{F} has to be non-invertible. For example, to violate Haag duality by a twist operator $\Psi_1(\Sigma_{d-1})N(\Sigma_{d-2})\Psi_2(\Sigma'_{d-1})$ we have to ensure that any uncharged bi-local operator $\text{tr}\psi_1(x)\psi_2(y) \in \mathcal{A}_{\mathcal{F}}(M_{d-1})$ and bi-local extended operator $\text{tr}\Psi_1(\Sigma_{d-2})\Psi_2(\Sigma'_{d-2})$ have both $\psi_1(x)$, $\Psi_1(\Sigma_{d-2})$ and $\psi_2(y)$, $\Psi_2(\Sigma'_{d-2})$ invariant under $N(\Sigma)$, otherwise we would have $\Psi_1(\Sigma_{d-1})N(\Sigma_{d-2})\Psi_2(\Sigma'_{d-1}) \notin \mathcal{A}_{\mathcal{F}}(R')'$ hence not violating Haag duality. Obviously, one possibility is that there are no non-trivial bi-local and bi-local extended operators in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ at all. This in turn means that any local and extended operator will become a twist operator upon the action of a non-invertible element of \mathcal{F} . Hence, it implies that $\widetilde{\mathcal{L}} = W_0 \oplus U_0$.

More generally, when $\widetilde{\mathcal{L}} \neq W_0 \oplus U_0$ and $\widetilde{\mathcal{L}} \neq \mathcal{L}$ to ensure \mathcal{F} has non-invertible 0-form generators which can generate tails acting on the local (extended) operators. There are bi-local (extended) operators in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$, which can be easily seen from the SymTFT construction in section 4. The question is then whether there exists an $N \in \mathcal{F}$ such that any local component of a bi-local (extended) operator in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is invariant under the action of N . Recall that, by definition, for any $\text{tr}\psi_1(x)\psi_2(y) \in \mathcal{A}_{\mathcal{F}}(M_{d-1})$, the component local operators $\psi_1(x)$ and $\psi_2(x)$ can never become a twist operator under any symmetry action in \mathcal{F} . In other words, a component local operator of an uncharged bi-local operator can at most undergo a linear transformation under any symmetry action in \mathcal{F} . Let us say it undergoes a linear transformation

$$\psi_1(x) \rightarrow M(\Sigma_{d-1})\psi_1(x) = \rho(M)\psi_1(x)M(\Sigma_{d-1}), \tag{5.5}$$

under the action of certain 0-form symmetry $M \in \mathcal{F}$ and similarly for $\psi_2(y)$. Let us choose a non-invertible $M \in \mathcal{F}$ and consider the action of $\overline{M}M$ with \overline{M} the orientation reversal

of M , we have

$$\psi_1(x) \rightarrow \bar{M}(\Sigma_{d-1})M(\Sigma_{d-1})\psi_1(x) = \rho(M)\rho(\bar{M})\psi_1(x)\bar{M}(\Sigma_{d-1})M(\Sigma_{d-1}), \quad (5.6)$$

where the fusion between $\bar{M}(\Sigma_{d-1})$ with $M(\Sigma_{d-1})$ generally gives

$$\bar{M}(\Sigma_{d-1})M(\Sigma_{d-1}) = n_0 1 \oplus \left(\bigoplus_N N(\Sigma_{d-1}) \right), \quad (5.7)$$

where $N(\Sigma_{d-1})$ on the r.h.s. can include genuine $(d-1)$ -dimensional generator and also line operators $\eta(\gamma)$ condensing on Σ_{d-1} , as discussed in section 4.2. Alternatively, we can first fuse \bar{M} and M and obtains

$$\bar{M}(\Sigma)M(\Sigma)\psi_1(x) = \left(n_0 1 \oplus \left(\bigoplus_N N(\Sigma_{d-1}) \right) \right) \psi_1(x) = \psi_1(x)n_0 1 \oplus \left(\bigoplus_N \rho(N)\psi_1(x)N(\Sigma) \right). \quad (5.8)$$

Compared to the r.h.s. of (5.6), we immediately have

$$\rho(M)\rho(\bar{M}) = \rho(N) = 1, \quad \forall N \in \bar{M}M, \quad (5.9)$$

which holds for any local operators $\psi_1(x)$ that cannot become a twist operator. Therefore, if there exists a bulk operator U such that $N \in t(U)$ and $N \in \bar{M}M$ for some non-invertible symmetry M , we can construct the bi-twist operator $\Psi_1(\Sigma_{d-2})N(\Sigma_{d-1})\Psi_2(\Sigma'_{d-2})$ which commutes with $\text{tr} \psi_1(x)\psi_2(y)$ for the configuration in figure 12. Notice that the above discussion is independent of the dimension of the operator and can be generalized to any charged objects \mathcal{O} as long as M is non-invertible such that (5.7) holds. In particular, it also applies to the components of the bi-local extended operator $\text{tr} \Psi_1(\Sigma_{d-2})\Psi_2(\Sigma'_{d-2}) \in \mathcal{A}_{\mathcal{F}}(M_{d-1})$.

In SymTFT, the set of operators in \mathcal{L} gives natural candidates of the bi-twist operators in $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ commuting with $\text{tr} \psi_1(x)\psi_2(y)$ and $\text{tr} \Psi_1(\Sigma_{d-2})\Psi_2(\Sigma'_{d-2})$. Since $\tilde{\mathcal{L}} \neq \mathcal{L}$, we can always pick a bulk line operator $W' \in \mathcal{L}$ or $(d-1)$ -dimensional extended operator $U' \in \mathcal{L}$ such that $W', U' \notin \tilde{\mathcal{L}}$, and construct a symmetric bi-twist (extended) operator as illustrated in section 4. After projection to $\mathcal{T}_{\mathcal{F}}$, the former gives a bi-twist local operator connected by line operators η' , and the latter gives a bi-twist extended operator connected by $(d-1)$ -dimensional generators N' . Since both η' and N' are tails that can end on some non-invertible generator M , they belong to $M\bar{M}$ and commute with all operators constructed by $\tilde{\mathcal{L}}$ as we have just shown.

As a result, we can rephrase the two statements at the beginning of this section in the language of SymTFT. Given a topological boundary B_{top} defined by the set of endable operators (Lagrangian algebra) \mathcal{L} , we have

- Additivity of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated if $\tilde{\mathcal{L}} \neq W_0 \oplus U_0$.
- Haag duality of $\mathcal{A}_{\mathcal{F}}(M_{d-1})$ is violated if $\tilde{\mathcal{L}} \neq \mathcal{L}$.

Here $\tilde{\mathcal{L}} \subset \mathcal{L}$ is the subset of operators in \mathcal{L} that cannot join to any non-trivial tail on B_{top} , and W_0, U_0 are respectively the identity line operator and the identity $(d-1)$ -dimensional extended operator.

where d_α is the quantum dimension of W_α . It turns out that $\dim \mathcal{L}$ is related to the total quantum dimension $D_{\mathcal{X}(\mathcal{F})}$ of the SymTFT $\mathcal{X}(\mathcal{F})$ as

$$\dim \mathcal{L} = D_{\mathcal{X}(\mathcal{F})}, \tag{5.13}$$

where the quantum dimension $D_{\mathcal{X}(\mathcal{F})}$ is defined by

$$D_{\mathcal{X}(\mathcal{F})} = \sqrt{\sum_{W_\alpha \in \mathcal{X}(\mathcal{F})} d_\alpha^2}. \tag{5.14}$$

Second, all the line operators $W_\alpha \in \mathcal{L}$ are bosonic.

Recall that the modular S -matrix is given by

$$S_{\alpha\beta} = \frac{1}{D_{\mathcal{X}(\mathcal{F})}} \sum_\gamma \mathcal{N}_{\alpha\beta}^\gamma \frac{\theta_\gamma}{\theta_\alpha \theta_\beta} d_\gamma, \tag{5.15}$$

where θ is the topological spin of W , which is the diagonal element of the modular T -matrix. If $W_\alpha \in \widetilde{\mathcal{L}}$, the fusion $W_\alpha \times \mathcal{L}$ should only contains elements inside \mathcal{L} , otherwise there exists a $W_\gamma \notin \mathcal{L}$ such that

$$\text{Hom}_{\mathcal{F}}(W_\gamma, W_\alpha) = \text{Hom}_{\text{SymTFT}}(W_\gamma, W_\alpha \times \mathcal{L}) \neq 0, \tag{5.16}$$

where we use (4.6). That means $t(W_\alpha)$ shares common elements with $t(W_\gamma)$ on B_{top} . Since W_γ must project to the non-trivial generator in \mathcal{F} on the B_{top} , we have a contradiction. Therefore $W_\gamma \in W_\alpha \times W_\mu \in \mathcal{L}$ is also bosonic and we can set all topological spin $\theta_\alpha, \theta_\mu$ and θ_γ equal to one and get

$$S_{\alpha\beta} = \frac{1}{D_{\mathcal{X}(\mathcal{F})}} \sum_\gamma \mathcal{N}_{\alpha\beta}^\gamma d_\gamma = \frac{d_\alpha d_\beta}{D_{\mathcal{X}(\mathcal{F})}}, \tag{5.17}$$

where we use the identity

$$d_\alpha d_\mu = \sum_\gamma \mathcal{N}_{\alpha\mu}^\gamma d_\gamma. \tag{5.18}$$

Therefore we have

$$\frac{S_{\alpha\beta}^* S_{00}}{S_{0\alpha} S_{0\beta}} = \frac{d_\alpha d_\beta}{D_{\mathcal{X}(\mathcal{F})}} \times \frac{S_{00}}{S_{0\alpha}} \times \frac{S_{00}}{S_{0\beta}} \times S_{00}^{-1} = 1, \tag{5.19}$$

where we use $S_{0\alpha}/S_{00} = d_\alpha$ and $S_{00} = D_{\mathcal{X}(\mathcal{F})}^{-1}$.

6 Examples

In this section, we will study three types of 2D examples and illustrate the analysis of the violation of additivity and Haag duality from the SymTFT perspective. We will begin with the simplest abelian group \mathbb{Z}_N and then move to general non-abelian groups and investigate S_3 as a concrete example. Then, we will consider the diagonal RCFT whose symmetries are generated by Verlinde lines and recover the criteria in [102].

6.1 \mathbb{Z}_N symmetry

The first example is the 2D anomalous free \mathbb{Z}_N theory $\mathcal{T}_{\mathbb{Z}_N}$. The SymTFT is the 3d BF-theory with level- N

$$S = \frac{N}{2\pi} \int B \wedge dA, \tag{6.1}$$

where we have a pair of U(1)-valued gauge field A and B . The anyons are given by the gauge invariant Wilson line operators $W_{(\alpha,\beta)}[\Gamma]$

$$W_{(\alpha,\beta)}[\Gamma] = \exp\left(i \oint_{\Gamma} \alpha A + \beta B\right), \quad (\alpha, \beta \in \mathbb{Z}_N) \tag{6.2}$$

with $\alpha, \beta = 0, 1, \dots, N-1$ and Γ is a 1-cycle. The fusion rule is simply

$$W_{(\alpha,\beta)} \times W_{(\alpha',\beta')} = W_{(\alpha+\alpha',\beta+\beta')}. \tag{6.3}$$

The Lagrangian algebras can be found by finding the maximal commuting set of operators. We have two basic Lagrangian algebras, the Dirichlet one and the Neumann one

$$\mathcal{L}_{\text{Dir}} = \bigoplus_{\alpha \in \mathbb{Z}_N} W_{(\alpha,0)}, \quad \mathcal{L}_{\text{Neu}} = \bigoplus_{\beta \in \mathbb{Z}_N} W_{(0,\beta)}. \tag{6.4}$$

If N is not prime, for any positive integers P, Q satisfying $N = PQ$ and $\text{gcd}(P, Q) \neq 1$, one has others Lagrangian algebras

$$\mathcal{L}_{P,Q} = \bigoplus_{\alpha \in \mathbb{Z}_Q, \beta \in \mathbb{Z}_P} W_{(P\alpha, Q\beta)}, \tag{6.5}$$

and the Dirichlet (Neumann) algebra is the special case when $P = 1$ ($Q = 1$).

Begin with the topological boundary B_{Dir} where the anyons $W_{(\alpha,0)}$ can end on that, it supports a \mathbb{Z}_N symmetry generated by $W_{(0,1)}$. The Neumann boundary B_{Neu} is obtained by gauging the \mathbb{Z}_N symmetry generated by $W_{(0,1)}$ and the corresponding Lagrangian algebra becomes \mathcal{L}_{Neu} . The role between $W_{(\alpha,0)}$ and $W_{(0,\beta)}$ are exchanged on B_{Neu} such that $W_{(0,\alpha)}$ can end on the boundary while $W_{(1,0)}$ serves as symmetry generator of the dual \mathbb{Z}_N symmetry. Moreover, if we only gauge a subgroup of $\mathbb{Z}_P \in \mathbb{Z}_N$ we will obtain the topological boundary $B_{(P,Q)}$ whose Lagrangian algebra is $\mathcal{L}_{(P,Q)}$. The symmetry supported on $B_{(P,Q)}$ is $\mathbb{Z}_P \times \mathbb{Z}_Q$ with a mixed anomaly between the two factors, generated separately by $W_{(1,0)}$ and $W_{(0,1)}$.

For all of them, one finds

$$\widetilde{\mathcal{L}}_{\text{Dir}} = \mathcal{L}_{\text{Dir}}, \quad \widetilde{\mathcal{L}}_{\text{Neu}} = \mathcal{L}_{\text{Neu}}, \quad \widetilde{\mathcal{L}}_{P,Q} = \mathcal{L}_{P,Q}. \tag{6.6}$$

Therefore, the additivity is violated, and the Haag duality is preserved. It matches our result for invertible \mathcal{F} , i.e. when \mathcal{F} is a group.

6.2 S_3 symmetry

For a general discrete group G , the elements of its SymTFT $\mathcal{L}(G)$ are described by the pair $\alpha = (A, r)$ where A labels the conjugacy class $Cl(G)$ of G and r labels the irreducible representation under the centralizer group of $Cl(G)$.

| \otimes | W_0 | W_1 | W_2 | W_3 | W_4 | W_5 | W_6 | W_7 |
|-----------|-------|-------|-----------------------------|--|--|-----------------------------|-----------------------------|-----------------------------|
| W_0 | W_0 | W_1 | W_2 | W_3 | W_4 | W_5 | W_6 | W_7 |
| W_1 | W_1 | W_0 | W_2 | W_4 | W_3 | W_5 | W_6 | W_7 |
| W_2 | W_2 | W_2 | $W_0 \oplus W_1 \oplus W_2$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_6 \oplus W_7$ | $W_5 \oplus W_7$ | $W_5 \oplus W_6$ |
| W_3 | W_3 | W_4 | $W_3 \oplus W_4$ | $W_0 \oplus W_2 \oplus W_5$ $\oplus W_6 \oplus W_7$ | $W_1 \oplus W_2 \oplus W_5$ $\oplus W_6 \oplus W_7$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ |
| W_4 | W_4 | W_3 | $W_3 \oplus W_4$ | $W_1 \oplus W_2 \oplus W_5$ $\oplus W_6 \oplus W_7$ | $W_0 \oplus W_2 \oplus W_5$ $\oplus W_6 \oplus W_7$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ |
| W_5 | W_5 | W_5 | $W_6 \oplus W_7$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_0 \oplus W_1 \oplus W_5$ | $W_7 \oplus W_2$ | $W_6 \oplus W_2$ |
| W_6 | W_6 | W_6 | $W_5 \oplus W_7$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_7 \oplus W_2$ | $W_0 \oplus W_1 \oplus W_6$ | $W_5 \oplus W_2$ |
| W_7 | W_7 | W_7 | $W_5 \oplus W_6$ | $W_3 \oplus W_4$ | $W_3 \oplus W_4$ | $W_6 \oplus W_2$ | $W_5 \oplus W_2$ | $W_0 \oplus W_1 \oplus W_7$ |

Table 1. The fusion table of $\mathcal{L}(S_3)$.

Let us consider the non-anomalous order-3 cyclic group $S_3 = \{e, a, a^2, b, ab, a^2b\}$. a and b are the generators of \mathbb{Z}_3 and \mathbb{Z}_2 satisfying $a^3 = b^2 = 1$ and the constraint $ba = a^2b$. There are three conjugacy classes represented by $[e], [a], [b]$

$$[e] = \{e\}, \quad [a] = \{a, a^2\}, \quad [b] = \{b, ab, a^2b\}, \quad (6.7)$$

and the centralizer group for e, a, b are separately

$$C(e) = S_3, \quad C(a) = \mathbb{Z}_3, \quad C(b) = \mathbb{Z}_2, \quad (6.8)$$

where \mathbb{Z}_3 and \mathbb{Z}_2 are generated by a and b separately. Recall that the irreducible representations for \mathbb{Z}_N are characterized by an \mathbb{Z}_N -integer k with

$$r_k(g) = e^{\frac{2\pi i}{N} k}, \quad (g \text{ is the generator of } \mathbb{Z}_N) \quad (6.9)$$

and the irreducible representations for S_3 contain the trivial representation r_+ , sign representation r_- and the 2-dimensional standard representation r_E . There are in total 8 line operators labeled by

$$([e], r_+), ([e], r_-), ([e], r_E), ([b], r_0), ([b], r_1), ([a], r_0), ([a], r_1), ([a], r_2) \quad (6.10)$$

and we shall label them as W_0, \dots, W_7 in the following. The fusion rule is summarized in table 1.

There are four Lagrangian algebras given by [117]

$$\begin{aligned} \mathcal{L}_1 &= W_0 \oplus W_1 \oplus 2W_2, & \mathcal{L}_2 &= W_0 \oplus W_3 \oplus W_5, \\ \mathcal{L}_3 &= W_0 \oplus W_1 \oplus 2W_5, & \mathcal{L}_4 &= W_0 \oplus W_2 \oplus W_3. \end{aligned} \quad (6.11)$$

For each Lagrangian algebra \mathcal{L}_i , we have the corresponding topological boundary B_i , and they are related as follows. Begin with \mathcal{L}_1 where W_0, W_1, W_2 can end on the topological boundary B_1 , it supports a S_3 symmetry whose generators will be identified later. If we gauge S_3 on B_1 , we will obtain B_2 characterized by \mathcal{L}_2 , and the underlying symmetry is $\text{Rep}(S_3)$. Alternatively, we can also gauge $\mathbb{Z}_3 \in S_3$ or $\mathbb{Z}_2 \in S_3$, and we will get separately B_3 and B_4 whose Lagrangian algebras are \mathcal{L}_3 and \mathcal{L}_4 . The symmetries supported on B_3 and B_4 are still S_3 and $\text{Rep}(S_3)$.

| W_α | $ W_\alpha \times \mathcal{L}_1\rangle$ |
|------------|---|
| W_0 | $ 1, 1, 2, 0, 0, 0, 0, 0\rangle$ |
| W_1 | $ 1, 1, 2, 0, 0, 0, 0, 0\rangle$ |
| W_2 | $ 2, 2, 4, 0, 0, 0, 0, 0\rangle$ |
| W_3 | $ 0, 0, 0, 3, 3, 0, 0, 0\rangle$ |
| W_4 | $ 0, 0, 0, 3, 3, 0, 0, 0\rangle$ |
| W_5 | $ 0, 0, 0, 0, 0, 2, 2, 2\rangle$ |
| W_6 | $ 0, 0, 0, 0, 0, 2, 2, 2\rangle$ |
| W_7 | $ 0, 0, 0, 0, 0, 2, 2, 2\rangle$ |

Table 2. The fusion between line operators with \mathcal{L}_1

In order to determine the tails of each line operator on the topological boundary, we need to consider the homomorphism introduced in (4.6) and thus consider the fusion $W_\alpha \times \mathcal{L}_i$. It is convenient to label each line operator W_α using 8-dimensional orthogonal basis such that

$$|W_\alpha\rangle \equiv \underbrace{|0, \dots, 0, 1, 0, \dots\rangle}_{\text{The } (\alpha + 1)\text{-th component is 1}} \tag{6.12}$$

and one can write

$$\begin{aligned} |\mathcal{L}_1\rangle &= |1, 1, 2, 0, 0, 0, 0, 0\rangle, & |\mathcal{L}_2\rangle &= |1, 0, 0, 1, 0, 1, 0, 0\rangle, \\ |\mathcal{L}_3\rangle &= |1, 1, 0, 0, 0, 2, 0, 0\rangle, & |\mathcal{L}_4\rangle &= |1, 0, 1, 1, 0, 0, 0, 0\rangle. \end{aligned} \tag{6.13}$$

Then the dimension of homomorphism in $\mathcal{X}(S_3)$ is $\dim \text{Hom}_{\mathcal{X}(S_3)}(W_\alpha, W_\beta) = \langle W_\alpha | W_\beta \rangle$, and the dimension of homomorphism in $\mathcal{F}(B_i)$ is

$$\dim \text{Hom}_{\mathcal{F}(B_i)}(t(W_\alpha), t(W_\beta)) = \langle W_\alpha | W_\beta \times \mathcal{L}_i \rangle. \tag{6.14}$$

We will mainly focus on \mathcal{L}_1 and \mathcal{L}_2 , and the discussion of \mathcal{L}_3 and \mathcal{L}_4 are totally parallel to \mathcal{L}_1 and \mathcal{L}_2 .

Topological boundary B_1 : S_3 -Symmetry. One can work out the fusion between each line operator W_α with \mathcal{L}_1 as summarized in table 2. One can see $2W_0, 2W_1, W_2$ are all the same after fusing \mathcal{L}_1 , and one can write

$$\text{Hom}_{\mathcal{F}(B_1)}(t(W_\alpha), t(2W_0)) = \text{Hom}_{\mathcal{F}(B_1)}(t(W_\alpha), t(2W_1)) = \text{Hom}_{\mathcal{F}(B_1)}(t(W_\alpha), t(W_2)), \tag{6.15}$$

for any $W_\alpha \in \mathcal{X}(S_3)$. That indicates $2W_0, 2W_1, W_2$ project to the same elements on the topological boundary. Since $t(W_0)$ is the identity element in $\mathcal{F}(\mathcal{L}_1)$, we can conclude

$$t(W_0) = t(W_1) = 1, \quad t(W_2) = 1 \oplus 1. \tag{6.16}$$

Similarly, we also have the following identifications

$$t(W_3) = t(W_4), \quad t(W_5) = t(W_6) = t(W_7), \tag{6.17}$$

on the topological boundary. We also need to determine whether they are simple or not at the boundary. Using (4.6) it is easy to find

$$\dim \text{Hom}_{\mathcal{F}(\mathcal{L}_1)}(t(W_3), t(W_3)) = 3, \quad \dim \text{Hom}_{\mathcal{F}(\mathcal{L}_1)}(t(W_5), t(W_5)) = 2, \tag{6.18}$$

| W_α | $ W_\alpha \times \mathcal{L}_2\rangle$ |
|------------|---|
| W_0 | $ 1, 0, 0, 1, 0, 1, 0, 0\rangle$ |
| W_1 | $ 0, 1, 0, 0, 1, 1, 0, 0\rangle$ |
| W_2 | $ 0, 0, 1, 1, 1, 0, 1, 1\rangle$ |
| W_3 | $ 1, 0, 1, 2, 1, 1, 1, 1\rangle$ |
| W_4 | $ 0, 1, 1, 1, 2, 1, 1, 1\rangle$ |
| W_5 | $ 1, 1, 0, 1, 1, 2, 0, 0\rangle$ |
| W_6 | $ 0, 0, 1, 1, 1, 0, 1, 1\rangle$ |
| W_7 | $ 0, 0, 1, 1, 1, 0, 1, 1\rangle$ |

Table 3. The fusion between line operators with \mathcal{L}_2

which indicates W_3 is composed of three simple elements and W_5 is composed of two simple elements on the boundary, and we can write

$$t(W_3) = \beta_1 \oplus \beta_2 \oplus \beta_3, \quad t(W_5) = \alpha_1 \oplus \alpha_2. \quad (6.19)$$

Moreover, the fusion rules between W_3 and W_5 in table 1 implies the following fusion rules on B_1

$$\begin{aligned} (\alpha_1 \oplus \alpha_2) \times (\alpha_1 \oplus \alpha_2) &= 1 \oplus 1 \oplus \alpha_1 \oplus \alpha_2, \\ (\beta_1 \oplus \beta_2 \oplus \beta_3)^2 &= 3 \oplus \alpha_1 \oplus \alpha_2, \\ (\beta_1 \oplus \beta_2 \oplus \beta_3) \times (\alpha_1 \oplus \alpha_2) &= 2\beta_1 \oplus 2\beta_2 \oplus 2\beta_3. \end{aligned} \quad (6.20)$$

The first one is satisfied if we think of α_1, α_2 generating a \mathbb{Z}_3 symmetry and identify $\alpha_1 = a, \alpha_2 = a^2$. If we further identify

$$\beta_1 = b, \quad \beta_2 = ab, \quad \beta_3 = a^2b, \quad (6.21)$$

where b is the \mathbb{Z}_2 generator satisfying $ba = a^2b$, then they solve the second and third fusion rules. Therefore, we can deduce the symmetry $\mathcal{F}(\mathcal{L}_1)$ on B_1 is S_3 and W_0, W_3, W_5 are conjugacy classes $[e], [a], [b]$ when restricting to the boundary.

Since the fusions between W_0, W_1, W_2 with \mathcal{L}_1 are all inside \mathcal{L}_1 , we have $\widetilde{\mathcal{L}}_1 = \mathcal{L}_1$ so that the additivity is violated but the Haag duality is preserved.

Topological boundary $B_2 : \text{Rep}(S_3)$. One can work out the fusion between each line operator W_α with \mathcal{L}_2 as summarized in table 3. And we should identify the following line operators on the topological boundary

$$\begin{aligned} t(W_2) &= t(W_6) = t(W_7), & t(W_5) &= t(W_0) + t(W_1), \\ t(W_3) &= t(W_0) + t(W_2), & t(W_4) &= t(W_1) + t(W_2). \end{aligned} \quad (6.22)$$

We can pick three generators $t(W_0), t(W_1), t(W_2)$. Here $t(W_0) = 1$ is the identity, and from (4.6) we can read the multiplicity

$$\dim \text{Hom}_{\mathcal{F}(\mathcal{L}_2)}(t(W_1), t(W_1)) = 1, \quad \dim \text{Hom}_{\mathcal{F}(\mathcal{L}_2)}(t(W_2), t(W_2)) = 1, \quad (6.23)$$

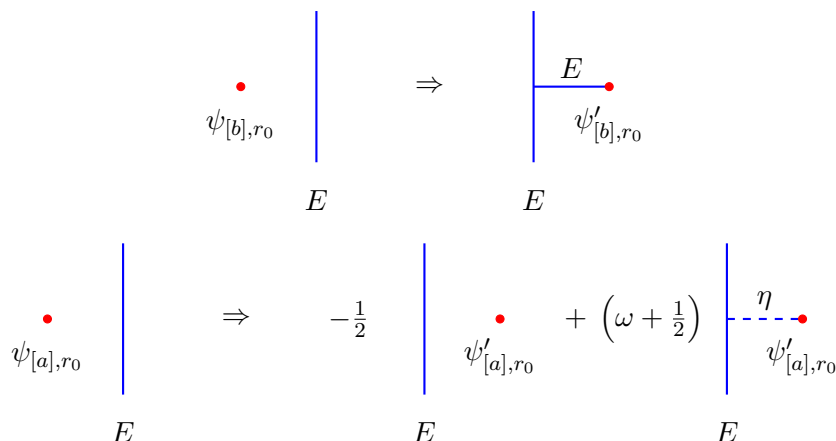


Figure 13. The local operators can transform to twist operators under the E -transformation in $\text{Rep}(S_3)$. Here $\omega = e^{2\pi i/3}$ is the 3rd-root of unity.

therefore all of the three line operators are simple on the boundary. Let us denote $t(W_1) = \eta$ and $t(W_2) = E$, the fusion rules between W_1 and W_2 in table 1 further implies

$$\eta \times \eta = 1, \quad \eta \times E = E, \quad E \times E = 1 \oplus \eta \oplus E, \tag{6.24}$$

which are exactly the fusion rules of $\text{Rep}(S_3)$: 1 is the trivial representation r_+ , η and E are the sign representation r_- and the 2-dimensional representation r_E .

Since we have $t(W_3) = 1 \oplus E$ and $t(W_5) = 1 \oplus \eta$, therefore $\widetilde{\mathcal{L}}_2 = W_0$. One may conclude that the additivity is preserved, but the Haag duality is violated. Indeed, if we denote the local operators represented by W_3 and W_5 as $\psi_{[b],r_0}$ and $\psi_{[a],r_0}$, they will transform under the non-invertible E -symmetry according to figure 13 [118]. Since both $\psi_{[b],r_0}$ and $\psi_{[a],r_0}$ form multiplets with twist operators, one cannot build a symmetric bi-local operator to violate the additivity. On the other hand, we can easily construct a symmetric bi-twist operator using bulk lines so that the Haag duality is violated. Notice that this is an example where we have an invertible symmetry $\eta \in \text{Rep}(S_3)$ but the additivity is not violated.

6.3 Diagonal RCFT

Finally, we will revisit the diagonal RCFT whose symmetry \mathcal{F} is generated by the Verlinde lines, and show that the two conditions are the same as those in [102]. The local primary operators of the diagonal RCFT are denoted as $\phi_{\alpha,\alpha^*}(z, \bar{z})$ with conformal weight (h_α, h_α) . They are one-to-one correspondent to the Verlinde lines L_α in the diagonal RCFT with the topological spin $\theta_\alpha = \exp(2\pi i h_\alpha)$.

The Verlinde lines generate a non-invertible symmetry \mathcal{F} , and the SymTFT $\mathcal{L}(\mathcal{F})$ of \mathcal{F} is the quantum double defined by

$$\mathcal{L}(\mathcal{F}) = \mathcal{F} \boxtimes \widetilde{\mathcal{F}}, \tag{6.25}$$

where the bulk anyons $W_{(\alpha,\beta)}$ are labeled by a pair (α, β) as

$$W_{(\alpha,\beta)} \equiv L_\alpha \boxtimes L_{\bar{\beta}}. \tag{6.26}$$

In the SymTFT picture, the local primary operators $\phi_{\alpha,\alpha^*}(z, \bar{z})$ are considered as the anyon $W_{(\alpha,\alpha)}$ emitting from an operator in the physical boundary and ending on the topological boundary B_{top} . Therefore, the Lagrangian algebra for a diagonal RCFT is

$$\mathcal{L}_{\text{Diag}} = \bigoplus_{\alpha} W_{(\alpha,\alpha)} = \bigoplus_{\alpha} L_{\alpha} \boxtimes L_{\bar{\alpha}}. \quad (6.27)$$

On the other hand, the non-diagonal anyons $W_{(\alpha,\beta)}$ cannot end on B_{top} and will connect to non-trivial Verlinde lines on the topological boundary. They will lead to the twist primaries $\phi_{\alpha,\beta^*}(z, \bar{z})$ with $\alpha \neq \beta$, which is attached to some Verlinde line L_{γ} in the diagonal RCFT.

The subalgebra $\widetilde{\mathcal{L}}_{\text{Diag}}$ is given by the subset of diagonal anyon $W_{(\alpha,\alpha)}$ satisfying

$$W_{(\alpha,\alpha)} \times \mathcal{L}_{\text{Diag}} \in \mathcal{L}_{\text{Diag}}, \quad (6.28)$$

so that it cannot transit to any non-trivial Verlinde lines on the boundary according to (4.6). In particular, the fusion between $W_{\alpha,\alpha}$ with $W_{\bar{\alpha},\bar{\alpha}} \in \mathcal{L}_{\text{Diag}}$ implies L_{α} is invertible

$$L_{\alpha} \times L_{\bar{\alpha}} = 1, \quad (6.29)$$

otherwise there will be non-diagonal anyons like $W_{(0,\gamma)}$ and $W_{(\gamma,0)}$ generated in the fusion $W_{(\alpha,\alpha)} \times W_{(\bar{\alpha},\bar{\alpha})}$. One can also prove $W_{(\alpha,\alpha)} \in \widetilde{\mathcal{L}}_{\text{Diag}}$ is equivalent to $L_{\alpha} \times L_{\bar{\alpha}} = 1$. Suppose the latter is true, which means the fusion coefficient satisfies

$$N_{\alpha\bar{\alpha}}^{\beta} = \begin{cases} 1, & (\beta = 0) \\ 0, & (\beta \neq 0) \end{cases} \quad (6.30)$$

Notice that the fusion coefficient satisfies the identity

$$\sum_f N_{af}^d N_{bc}^f = \sum_e N_{ab}^e N_{ec}^d. \quad (6.31)$$

Setting $a = \bar{\alpha}, b = \alpha$ one has

$$\sum_f N_{\bar{\alpha}f}^d N_{\alpha c}^f = \delta_c^d, \quad (6.32)$$

where we used $N_{0c}^d = \delta_c^d$. Then sum over d we have

$$\sum_f \left(\sum_d N_{\bar{\alpha}f}^d \right) N_{\alpha c}^f = 1, \quad (6.33)$$

which holds for any c . The point is one cannot have $L_{\bar{\alpha}} \times L_f = 0$ since that would imply $L_f = 0$ by multiplying L_{α} on both side, therefore one must have $\sum_d N_{\bar{\alpha}f}^d \in \mathbb{Z}_{\geq 1}$ for any f . Since the fusion coefficients $N_{\alpha c}^f$ are also non-negative integers, that implies there exist a $\gamma(c)$ depending on c such that

$$\sum_d N_{\bar{\alpha}\gamma(c)}^d = 1, \quad N_{\alpha c}^f = \begin{cases} 1, & f = \gamma(c), \\ 0, & f \neq \gamma(c). \end{cases} \quad (6.34)$$

Thus we arrive at

$$L_\alpha \times L_c = L_{\gamma(c)}, \tag{6.35}$$

which means the fusion between L_α with any other Verlinde line L_c will map that into a single Verlinde line $L_{\gamma(c)}$, and one has

$$W_{(\alpha,\alpha)} \times W_{(\beta,\beta)} = W_{(\beta(c),\beta(c))} \in \mathcal{L}_{\text{Diag}} \quad \Rightarrow \quad W_{(\alpha,\alpha)} \times \mathcal{L}_{\text{Diag}} \in \mathcal{L}_{\text{Diag}}. \tag{6.36}$$

Therefore $\widetilde{\mathcal{L}}_{\text{Diag}}$ contains all the diagonal anyons $W_{(\alpha,\alpha)} = L_\alpha \boxtimes L_{\bar{\alpha}}$ with invertible Verlinde lines L_α

$$\widetilde{\mathcal{L}}_{\text{Diag}} = \bigoplus_{\text{invertible } L_\alpha} L_\alpha \boxtimes L_{\bar{\alpha}}. \tag{6.37}$$

Let us denote $C_{\text{ab}} \in \mathcal{F}$ as the set generated by the invertible Verlinde lines, then the additivity is violated if $\widetilde{\mathcal{L}}_{\text{Diag}} \neq L_0 \boxtimes L_0$, or equivalently C_{ab} contains Verlinde lines other than L_0 . That indicates \mathcal{F} contains non-trivial invertible symmetries. On the other hand, the Haag duality is violated when $\widetilde{\mathcal{L}}_{\text{Diag}} \neq \mathcal{L}_{\text{Diag}}$, or equivalently when $C_{\text{ab}} \subsetneq \mathcal{F}$. Which means \mathcal{F} contains non-invertible symmetry generators.

7 Conclusion and discussion

In this work, we generalize the result of [102] by exploiting the power of SymTFT in the analysis of the global symmetries of the QFT. The many edge cases when attempting the construction of uncharged non-local operators purely by analyzing the symmetry algebra of QFT as exemplified in section 3 are once-and-for-all resolved by analyzing the Lagrangian algebra of the corresponding SymTFT in a simple and unified way as shown in section 4. In section 5 we summarize and prove the conditions for the violation of additivity and Haag duality of the von Neumann algebra as certain constraints on the Lagrangian algebra, which offers an immediate generalization to any dimension. We then provide concrete examples in the SymTFT language to cross-check our statements in section 6.

The key observation is that the violation of additivity or Haag duality of the von Neumann algebra can be detected when restricting the von Neumann algebra to its uncharged sector and including in it certain types of non-local operators. The operators we consider are either bi-local or bi-twist operators, the former is an uncharged pair of charged local operators (or extended operators each of which projects to a local operator when projecting its support down to a point) while the latter is an uncharged combination of two charged operators connected by an operator labeled by an element of \mathcal{F} supported on a manifold whose two boundaries support the aforementioned two charged operators. The neutrality of such non-local operators can then be determined conveniently by analyzing the corresponding operator in the SymTFT as graphically displayed in figure 9 and 10. We define the notion of the tail of a bulk operator to facilitate the analysis and the set $\widetilde{\mathcal{L}} \subset \mathcal{L}$ of the bulk operators that cannot join to non-trivial tails on the topological boundary. Using the segmentation of the Cauchy surface proposed in [102], we summarize and prove the condition for the violation of additivity as $\widetilde{\mathcal{L}} \neq W_0 \oplus U_0$ and for the violation of Haag duality as $\widetilde{\mathcal{L}} \neq \mathcal{L}$.

An interesting future direction would be to generalize the current framework to spacetime symmetries. Recently, the author of [131] has made progress on incorporating spacetime symmetry in SymTFT, resulting in symmetry-enriched topological (SET) order. The challenge lies in the construction of the non-local operators in such SymTFT with spacetime symmetry data and we would like to attempt this in future works.

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References

- [1] S. Weinberg, *The quantum theory of fields. Vol. 1: Foundations*, Cambridge University Press (2005) [[DOI:10.1017/CB09781139644167](https://doi.org/10.1017/CB09781139644167)] [[INSPIRE](#)].
- [2] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press (2013) [[DOI:10.1017/CB09781139644174](https://doi.org/10.1017/CB09781139644174)] [[INSPIRE](#)].
- [3] R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, *J. Math. Phys.* **5** (1964) 848 [[INSPIRE](#)].
- [4] R. Haag, *Local Quantum Physics*, Springer, Berlin (1996) [[DOI:10.1007/978-3-642-61458-3](https://doi.org/10.1007/978-3-642-61458-3)] [[INSPIRE](#)].
- [5] I.E. Segal, *Postulates for general quantum mechanics*, *Annals of Mathematics* **48** (1947) 930.
- [6] R. Haag and B. Schroer, *Postulates of Quantum Field Theory*, *J. Math. Phys.* **3** (1962) 248.
- [7] V. Chandrasekaran, R. Longo, G. Penington and E. Witten, *An algebra of observables for de Sitter space*, *JHEP* **02** (2023) 082 [[arXiv:2206.10780](https://arxiv.org/abs/2206.10780)] [[INSPIRE](#)].
- [8] E. Witten, *Algebras, regions, and observers*, *Proc. Symp. Pure Math.* **107** (2024) 247 [[arXiv:2303.02837](https://arxiv.org/abs/2303.02837)] [[INSPIRE](#)].

- [9] M.G. Alford and J. March-Russell, *New order parameters for nonAbelian gauge theories*, *Nucl. Phys. B* **369** (1992) 276 [INSPIRE].
- [10] M.G. Alford and J. March-Russell, *Discrete gauge theories*, *Int. J. Mod. Phys. B* **5** (1991) 2641 [INSPIRE].
- [11] M. Bucher, K.-M. Lee and J. Preskill, *On detecting discrete Cheshire charge*, *Nucl. Phys. B* **386** (1992) 27 [hep-th/9112040] [INSPIRE].
- [12] M.G. Alford, K.-M. Lee, J. March-Russell and J. Preskill, *Quantum field theory of nonAbelian strings and vortices*, *Nucl. Phys. B* **384** (1992) 251 [hep-th/9112038] [INSPIRE].
- [13] Z. Nussinov and G. Ortiz, *A symmetry principle for topological quantum order*, *Annals Phys.* **324** (2009) 977 [cond-mat/0702377] [INSPIRE].
- [14] E. Witten, *AdS/CFT correspondence and topological field theory*, *JHEP* **12** (1998) 012 [hep-th/9812012] [INSPIRE].
- [15] D.S. Freed, G.W. Moore and G. Segal, *The Uncertainty of Fluxes*, *Commun. Math. Phys.* **271** (2007) 247 [hep-th/0605198] [INSPIRE].
- [16] T. Pantev and E. Sharpe, *GLSM's for Gerbes (and other toric stacks)*, *Adv. Theor. Math. Phys.* **10** (2006) 77 [hep-th/0502053] [INSPIRE].
- [17] T. Pantev and E. Sharpe, *String compactifications on Calabi-Yau stacks*, *Nucl. Phys. B* **733** (2006) 233 [hep-th/0502044] [INSPIRE].
- [18] T. Pantev and E. Sharpe, *Notes on gauging noneffective group actions*, [hep-th/0502027] [INSPIRE].
- [19] S. Hellerman et al., *Cluster decomposition, T-duality, and gerby CFT's*, *Adv. Theor. Math. Phys.* **11** (2007) 751 [hep-th/0606034] [INSPIRE].
- [20] S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, [hep-th/0612073] [INSPIRE].
- [21] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115 [arXiv:1305.0318] [INSPIRE].
- [22] A. Kapustin and R. Thorngren, *Higher Symmetry and Gapped Phases of Gauge Theories*, *Prog. Math.* **324** (2017) 177 [arXiv:1309.4721] [INSPIRE].
- [23] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and Duality*, *JHEP* **04** (2014) 001 [arXiv:1401.0740] [INSPIRE].
- [24] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].
- [25] F. Albertini, M. Del Zotto, I. García Etxebarria and S.S. Hosseini, *Higher Form Symmetries and M-theory*, *JHEP* **12** (2020) 203 [arXiv:2005.12831] [INSPIRE].
- [26] F. Apruzzi et al., *Symmetry TFTs from String Theory*, *Commun. Math. Phys.* **402** (2023) 895 [arXiv:2112.02092] [INSPIRE].
- [27] F. Apruzzi, S. Schafer-Nameki, L. Bhardwaj and J. Oh, *The Global Form of Flavor Symmetries and 2-Group Symmetries in 5d SCFTs*, *SciPost Phys.* **13** (2022) 024 [arXiv:2105.08724] [INSPIRE].
- [28] F. Apruzzi, L. Bhardwaj, D.S.W. Gould and S. Schafer-Nameki, *2-Group symmetries and their classification in 6d*, *SciPost Phys.* **12** (2022) 098 [arXiv:2110.14647] [INSPIRE].

- [29] F. Apruzzi, *Higher form symmetries TFT in 6d*, *JHEP* **11** (2022) 050 [[arXiv:2203.10063](#)] [[INSPIRE](#)].
- [30] F. Apruzzi, M. van Beest, D.S.W. Gould and S. Schäfer-Nameki, *Holography, 1-form symmetries, and confinement*, *Phys. Rev. D* **104** (2021) 066005 [[arXiv:2104.12764](#)] [[INSPIRE](#)].
- [31] F. Apruzzi, I. Bah, F. Bonetti and S. Schafer-Nameki, *Noninvertible Symmetries from Holography and Branes*, *Phys. Rev. Lett.* **130** (2023) 121601 [[arXiv:2208.07373](#)] [[INSPIRE](#)].
- [32] F. Apruzzi, F. Bonetti, D.S.W. Gould and S. Schafer-Nameki, *Aspects of categorical symmetries from branes: SymTFTs and generalized charges*, *SciPost Phys.* **17** (2024) 025 [[arXiv:2306.16405](#)] [[INSPIRE](#)].
- [33] B.S. Acharya et al., *Junctions, edge modes, and G_2 -holonomy orbifolds*, *Beijing J. Pure Appl. Math.* **1** (2024) 273 [[arXiv:2304.03300](#)] [[INSPIRE](#)].
- [34] V. Bashmakov, M. Del Zotto, A. Hasan and J. Kaidi, *Non-invertible symmetries of class S theories*, *JHEP* **05** (2023) 225 [[arXiv:2211.05138](#)] [[INSPIRE](#)].
- [35] M. van Beest, D.S.W. Gould, S. Schafer-Nameki and Y.-N. Wang, *Symmetry TFTs for 3d QFTs from M-theory*, *JHEP* **02** (2023) 226 [[arXiv:2210.03703](#)] [[INSPIRE](#)].
- [36] P. Benetti Genolini and L. Tizzano, *Instantons, symmetries and anomalies in five dimensions*, *JHEP* **04** (2021) 188 [[arXiv:2009.07873](#)] [[INSPIRE](#)].
- [37] O. Bergman, Y. Tachikawa and G. Zafrir, *Generalized symmetries and holography in ABJM-type theories*, *JHEP* **07** (2020) 077 [[arXiv:2004.05350](#)] [[INSPIRE](#)].
- [38] L. Bhardwaj and S. Schäfer-Nameki, *Higher-form symmetries of 6d and 5d theories*, *JHEP* **02** (2021) 159 [[arXiv:2008.09600](#)] [[INSPIRE](#)].
- [39] L. Bhardwaj, *2-Group symmetries in class S*, *SciPost Phys.* **12** (2022) 152 [[arXiv:2107.06816](#)] [[INSPIRE](#)].
- [40] L. Bhardwaj, L.E. Bottini, S. Schafer-Nameki and A. Tiwari, *Non-invertible higher-categorical symmetries*, *SciPost Phys.* **14** (2023) 007 [[arXiv:2204.06564](#)] [[INSPIRE](#)].
- [41] L. Bhardwaj and S. Schafer-Nameki, *Generalized charges, part I: Invertible symmetries and higher representations*, *SciPost Phys.* **16** (2024) 093 [[arXiv:2304.02660](#)] [[INSPIRE](#)].
- [42] L. Bhardwaj and S. Schafer-Nameki, *Generalized charges, part II: Non-invertible symmetries and the symmetry TFT*, *SciPost Phys.* **19** (2025) 098 [[arXiv:2305.17159](#)] [[INSPIRE](#)].
- [43] L. Bhardwaj, M. Hubner and S. Schafer-Nameki, *1-form Symmetries of 4d $N=2$ Class S Theories*, *SciPost Phys.* **11** (2021) 096 [[arXiv:2102.01693](#)] [[INSPIRE](#)].
- [44] L. Bhardwaj, M. Bullimore, A.E.V. Ferrari and S. Schafer-Nameki, *Anomalies of generalized symmetries from solitonic defects*, *SciPost Phys.* **16** (2024) 087 [[arXiv:2205.15330](#)] [[INSPIRE](#)].
- [45] L. Bhardwaj, S. Schafer-Nameki and J. Wu, *Universal Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200143 [[arXiv:2208.05973](#)] [[INSPIRE](#)].
- [46] L. Bhardwaj, S. Schafer-Nameki and A. Tiwari, *Unifying constructions of non-invertible symmetries*, *SciPost Phys.* **15** (2023) 122 [[arXiv:2212.06159](#)] [[INSPIRE](#)].
- [47] L. Bhardwaj, L.E. Bottini, S. Schafer-Nameki and A. Tiwari, *Non-invertible symmetry webs*, *SciPost Phys.* **15** (2023) 160 [[arXiv:2212.06842](#)] [[INSPIRE](#)].
- [48] N. Braeger, V. Chakrabhavi, J.J. Heckman and M. Hübner, *Generalized symmetries of nonsupersymmetric orbifolds*, *Phys. Rev. D* **111** (2025) 066015 [[arXiv:2404.17639](#)] [[INSPIRE](#)].

- [49] Y. Choi et al., *Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions*, *Commun. Math. Phys.* **402** (2023) 489 [[arXiv:2204.09025](#)] [[INSPIRE](#)].
- [50] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Global Symmetries in the Standard Model*, *Phys. Rev. Lett.* **129** (2022) 161601 [[arXiv:2205.05086](#)] [[INSPIRE](#)].
- [51] M. Cvetič, J.J. Heckman, M. Hübner and E. Torres, *0-form, 1-form, and 2-group symmetries via cutting and gluing of orbifolds*, *Phys. Rev. D* **106** (2022) 106003 [[arXiv:2203.10102](#)] [[INSPIRE](#)].
- [52] M. Cvetič, M. Dierigl, L. Lin and H.Y. Zhang, *Higher-form symmetries and their anomalies in M-/F-theory duality*, *Phys. Rev. D* **104** (2021) 126019 [[arXiv:2106.07654](#)] [[INSPIRE](#)].
- [53] M. Cvetič, J.J. Heckman, M. Hübner and E. Torres, *Fluxbranes, generalized symmetries, and Verlinde's metastable monopole*, *Phys. Rev. D* **109** (2024) 046007 [[arXiv:2305.09665](#)] [[INSPIRE](#)].
- [54] M. Cvetič, J.J. Heckman, M. Hübner and E. Torres, *Generalized symmetries, gravity, and the swampland*, *Phys. Rev. D* **109** (2024) 026012 [[arXiv:2307.13027](#)] [[INSPIRE](#)].
- [55] M. Cvetič et al., *Cornering relative symmetry theories*, *Phys. Rev. D* **111** (2025) 085026 [[arXiv:2408.12600](#)] [[INSPIRE](#)].
- [56] M. Cvetič, J.J. Heckman, M. Hübner and C. Murdia, *Metric isometries, holography, and continuous symmetry operators*, *Phys. Rev. D* **112** (2025) 106020 [[arXiv:2501.17911](#)] [[INSPIRE](#)].
- [57] M. Del Zotto, I. García Etxebarria and S. Schafer-Nameki, *2-Group Symmetries and M-Theory*, *SciPost Phys.* **13** (2022) 105 [[arXiv:2203.10097](#)] [[INSPIRE](#)].
- [58] M. Del Zotto, I. García Etxebarria and S.S. Hosseini, *Higher form symmetries of Argyres-Douglas theories*, *JHEP* **10** (2020) 056 [[arXiv:2007.15603](#)] [[INSPIRE](#)].
- [59] M. Del Zotto et al., *Higher symmetries of 5D orbifold SCFTs*, *Phys. Rev. D* **106** (2022) 046010 [[arXiv:2201.08372](#)] [[INSPIRE](#)].
- [60] M. Etheredge, I. Garcia Etxebarria, B. Heidenreich and S. Rauch, *Branes and symmetries for $\mathcal{N} = 3$ S-folds*, *JHEP* **09** (2023) 005 [[arXiv:2302.14068](#)] [[INSPIRE](#)].
- [61] I. García Etxebarria, *Branes and Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200154 [[arXiv:2208.07508](#)] [[INSPIRE](#)].
- [62] I. García Etxebarria and N. Iqbal, *A goldstone theorem for continuous non-invertible symmetries*, *JHEP* **09** (2023) 145 [[arXiv:2211.09570](#)] [[INSPIRE](#)].
- [63] S. Gukov, P.-S. Hsin and D. Pei, *Generalized global symmetries of T[M] theories. Part I*, *JHEP* **04** (2021) 232 [[arXiv:2010.15890](#)] [[INSPIRE](#)].
- [64] J.J. Heckman, M. Hübner, E. Torres and H.Y. Zhang, *The Branes Behind Generalized Symmetry Operators*, *Fortsch. Phys.* **71** (2023) 2200180 [[arXiv:2209.03343](#)] [[INSPIRE](#)].
- [65] J.J. Heckman, M. Hübner and C. Murdia, *On the holographic dual of a topological symmetry operator*, *Phys. Rev. D* **110** (2024) 046007 [[arXiv:2401.09538](#)] [[INSPIRE](#)].
- [66] C.-T. Hsieh, Y. Tachikawa and K. Yonekura, *Anomaly Inflow and p-Form Gauge Theories*, *Commun. Math. Phys.* **391** (2022) 495 [[arXiv:2003.11550](#)] [[INSPIRE](#)].
- [67] M. Hubner, D.R. Morrison, S. Schafer-Nameki and Y.-N. Wang, *Generalized Symmetries in F-theory and the Topology of Elliptic Fibrations*, *SciPost Phys.* **13** (2022) 030 [[arXiv:2203.10022](#)] [[INSPIRE](#)].

- [68] Q. Jia et al., *Symmetry Topological Field Theory for Flavor Symmetry*, [arXiv:2503.04546](#) [[INSPIRE](#)].
- [69] J. Kaidi, K. Ohmori and Y. Zheng, *Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories*, *Phys. Rev. Lett.* **128** (2022) 111601 [[arXiv:2111.01141](#)] [[INSPIRE](#)].
- [70] J. Kaidi, K. Ohmori and Y. Zheng, *Symmetry TFTs for Non-invertible Defects*, *Commun. Math. Phys.* **404** (2023) 1021 [[arXiv:2209.11062](#)] [[INSPIRE](#)].
- [71] J. Kaidi, G. Zafrir and Y. Zheng, *Non-invertible symmetries of $\mathcal{N} = 4$ SYM and twisted compactification*, *JHEP* **08** (2022) 053 [[arXiv:2205.01104](#)] [[INSPIRE](#)].
- [72] Y. Lee, K. Ohmori and Y. Tachikawa, *Matching higher symmetries across Intriligator-Seiberg duality*, *JHEP* **10** (2021) 114 [[arXiv:2108.05369](#)] [[INSPIRE](#)].
- [73] R. Liu, R. Luo and Y.-N. Wang, *Higher-matter and Landau-Ginzburg theory of higher-group symmetries*, *SciPost Phys.* **18** (2025) 052 [[arXiv:2406.03974](#)] [[INSPIRE](#)].
- [74] D.R. Morrison, S. Schafer-Nameki and B. Willett, *Higher-Form Symmetries in 5d*, *JHEP* **09** (2020) 024 [[arXiv:2005.12296](#)] [[INSPIRE](#)].
- [75] J. Tian and Y.-N. Wang, *5D and 6D SCFTs from \mathbb{C}^3 orbifolds*, *SciPost Phys.* **12** (2022) 127 [[arXiv:2110.15129](#)] [[INSPIRE](#)].
- [76] J. Tian and Y.-N. Wang, *A tale of bulk and branes: Symmetry TFT of 6D SCFTs from IIB/F-theory*, *JHEP* **03** (2025) 085 [[arXiv:2410.23076](#)] [[INSPIRE](#)].
- [77] J. Tian and X. Wang, *Higher form and higher group symmetries via mirror symmetry*, *JHEP* **10** (2025) 075 [[arXiv:2503.09967](#)] [[INSPIRE](#)].
- [78] M. Del Zotto, J.J. Heckman, D.S. Park and T. Rudelius, *On the Defect Group of a 6D SCFT*, *Lett. Math. Phys.* **106** (2016) 765 [[arXiv:1503.04806](#)] [[INSPIRE](#)].
- [79] S. Schafer-Nameki, *ICTP lectures on (non-)invertible generalized symmetries*, *Phys. Rept.* **1063** (2024) 1 [[arXiv:2305.18296](#)] [[INSPIRE](#)].
- [80] L. Bhardwaj et al., *Lectures on generalized symmetries*, *Phys. Rept.* **1051** (2024) 1 [[arXiv:2307.07547](#)] [[INSPIRE](#)].
- [81] R. Luo, Q.-R. Wang and Y.-N. Wang, *Lecture notes on generalized symmetries and applications*, *Phys. Rept.* **1065** (2024) 1 [[arXiv:2307.09215](#)] [[INSPIRE](#)].
- [82] P.R.S. Gomes, *An introduction to higher-form symmetries*, *SciPost Phys. Lect. Notes* **74** (2023) 1 [[arXiv:2303.01817](#)] [[INSPIRE](#)].
- [83] C. Cordova, T.T. Dumitrescu, K. Intriligator and S.-H. Shao, *Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond*, in the proceedings of the *Snowmass 2021*, Seattle, U.S.A., July 17–26 (2022) [[arXiv:2205.09545](#)] [[INSPIRE](#)].
- [84] S.-H. Shao, *What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries*, in the proceedings of the *Theoretical Advanced Study Institute in Elementary Particle Physics 2023: Aspects of Symmetry*, Boulder, U.S.A., June 05–30 (2023) [[arXiv:2308.00747](#)] [[INSPIRE](#)].
- [85] D.S. Freed and C. Teleman, *Relative quantum field theory*, *Commun. Math. Phys.* **326** (2014) 459 [[arXiv:1212.1692](#)] [[INSPIRE](#)].
- [86] L. Kong et al., *Algebraic higher symmetry and categorical symmetry — a holographic and entanglement view of symmetry*, *Phys. Rev. Res.* **2** (2020) 043086 [[arXiv:2005.14178](#)] [[INSPIRE](#)].

- [87] D. Gaiotto and J. Kulp, *Orbifold groupoids*, *JHEP* **02** (2021) 132 [[arXiv:2008.05960](#)] [[INSPIRE](#)].
- [88] F. Baume et al., *SymTrees and Multi-Sector QFTs*, *Phys. Rev. D* **109** (2024) 106013 [[arXiv:2310.12980](#)] [[INSPIRE](#)].
- [89] J. Chen, W. Cui, B. Haghighat and Y.-N. Wang, *SymTFTs and duality defects from 6d SCFTs on 4-manifolds*, *JHEP* **11** (2023) 208 [[arXiv:2305.09734](#)] [[INSPIRE](#)].
- [90] J.J. Heckman and M. Hübner, *Celestial Topology, Symmetry Theories, and Evidence for a NonSUSY D3-Brane CFT*, *Fortsch. Phys.* **73** (2025) 2400270 [[arXiv:2406.08485](#)] [[INSPIRE](#)].
- [91] N. Braeger, V. Chakrabhavi, J.J. Heckman and M. Hübner, *Generalized symmetries of non-SUSY and discrete torsion string backgrounds*, *SciPost Phys.* **19** (2025) 160 [[arXiv:2504.10484](#)] [[INSPIRE](#)].
- [92] J.J. Heckman, M. Hübner and C. Murdia, *Symmetry Theories, Wigner’s Function, Compactification, and Holography*, [arXiv:2505.23887](#) [[INSPIRE](#)].
- [93] D. Harlow and H. Ooguri, *Symmetries in quantum field theory and quantum gravity*, *Commun. Math. Phys.* **383** (2021) 1669 [[arXiv:1810.05338](#)] [[INSPIRE](#)].
- [94] H. Casini, M. Huerta, J.M. Magan and D. Pontello, *Entropic order parameters for the phases of QFT*, *JHEP* **04** (2021) 277 [[arXiv:2008.11748](#)] [[INSPIRE](#)].
- [95] H. Casini and J.M. Magan, *On completeness and generalized symmetries in quantum field theory*, *Mod. Phys. Lett. A* **36** (2021) 2130025 [[arXiv:2110.11358](#)] [[INSPIRE](#)].
- [96] V. Benedetti, H. Casini and J.M. Magan, *Generalized symmetries and Noether’s theorem in QFT*, *JHEP* **08** (2022) 304 [[arXiv:2205.03412](#)] [[INSPIRE](#)].
- [97] V. Benedetti, H. Casini and J.M. Magan, *Charges in the UV completion of neutral electrodynamics*, *JHEP* **06** (2023) 095 [[arXiv:2212.11291](#)] [[INSPIRE](#)].
- [98] V. Benedetti, H. Casini and J.M. Magan, *ABJ anomaly as a U(1) symmetry and Noether’s theorem*, *SciPost Phys.* **18** (2025) 041 [[arXiv:2309.03264](#)] [[INSPIRE](#)].
- [99] V. Benedetti et al., *Modular invariance as completeness*, *Phys. Rev. D* **110** (2024) 125004 [[arXiv:2408.04011](#)] [[INSPIRE](#)].
- [100] V. Benedetti, H. Casini and J.M. Magan, *Selection rules for RG flows of minimal models*, *Phys. Rev. D* **111** (2025) 065024 [[arXiv:2412.16587](#)] [[INSPIRE](#)].
- [101] X. Yu and H.Y. Zhang, *Von Neumann subfactors and non-invertible symmetries*, *SciPost Phys.* **19** (2025) 154 [[arXiv:2504.05374](#)] [[INSPIRE](#)].
- [102] S.-H. Shao, J. Sorce and M. Srivastava, *Additivity, Haag duality, and non-invertible symmetries*, *JHEP* **08** (2025) 009 [[arXiv:2503.20863](#)] [[INSPIRE](#)].
- [103] H. Araki, *Mathematical theory of quantum fields*, Oxford University Press (1999), https://books.google.co.kr/books/about/Mathematical_Theory_of_Quantum_Fields.html?id=VpVmi5iyaYoC.
- [104] H. Casini, *The quantum logic of causal sets*, *Class. Quant. Grav.* **19** (2002) 6389 [[gr-qc/0205013](#)] [[INSPIRE](#)].
- [105] J. Sorce, *Notes on the type classification of von Neumann algebras*, *Rev. Math. Phys.* **36** (2024) 2430002 [[arXiv:2302.01958](#)] [[INSPIRE](#)].

- [106] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line*, *Nucl. Phys. B* **495** (1997) 533 [[cond-mat/9612187](#)] [[INSPIRE](#)].
- [107] V.B. Petkova and J.B. Zuber, *Generalized twisted partition functions*, *Phys. Lett. B* **504** (2001) 157 [[hep-th/0011021](#)] [[INSPIRE](#)].
- [108] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-Wannier duality from conformal defects*, *Phys. Rev. Lett.* **93** (2004) 070601 [[cond-mat/0404051](#)] [[INSPIRE](#)].
- [109] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, American Mathematical Society (2015) [[DOI:10.1090/surv/205](#)].
- [110] Q. Jia et al., *Categorical Continuous Symmetry*, [arXiv:2509.13170](#) [[INSPIRE](#)].
- [111] V.G. Turaev and O.Y. Viro, *State sum invariants of 3-manifolds and quantum 6 j-symbols*, *Topology* **31** (1992) 865 [[INSPIRE](#)].
- [112] J.W. Barrett and B.W. Westbury, *Invariants of piecewise linear three manifolds*, *Trans. Am. Math. Soc.* **348** (1996) 3997 [[hep-th/9311155](#)] [[INSPIRE](#)].
- [113] M. Izumi, *The structure of sectors associated with Longo-Rehren inclusions. I: General theory*, *Commun. Math. Phys.* **213** (2000) 127 [[INSPIRE](#)].
- [114] D.E. Evans and T. Gannon, *The exoticness and realisability of twisted Haagerup-Izumi modular data*, *Commun. Math. Phys.* **307** (2011) 463 [[arXiv:1006.1326](#)] [[INSPIRE](#)].
- [115] M. Mueger, *From subfactors to categories and topology II: The quantum double of tensor categories and subfactors*, *J. Pure Appl. Algebra* **180** (2001) 159 [[math/0111205](#)] [[INSPIRE](#)].
- [116] Y.-H. Lin, M. Okada, S. Seifnashri and Y. Tachikawa, *Asymptotic density of states in 2d CFTs with non-invertible symmetries*, *JHEP* **03** (2023) 094 [[arXiv:2208.05495](#)] [[INSPIRE](#)].
- [117] I. Cong, M. Cheng and Z. Wang, *Topological Quantum Computation with Gapped Boundaries*, [arXiv:1609.02037](#) [[INSPIRE](#)].
- [118] L. Bhardwaj, L.E. Bottini, D. Pajer and S. Schäfer-Nameki, *Gapped phases with non-invertible symmetries: (1+1)d*, *SciPost Phys.* **18** (2025) 032 [[arXiv:2310.03784](#)] [[INSPIRE](#)].
- [119] L. Kong, *Anyon condensation and tensor categories*, *Nucl. Phys. B* **886** (2014) 436 [[arXiv:1307.8244](#)] [[INSPIRE](#)].
- [120] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, *Fields Inst. Commun.* **39** (2003) 25 [[math/0106050](#)] [[INSPIRE](#)].
- [121] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators 1. Partition functions*, *Nucl. Phys. B* **646** (2002) 353 [[hep-th/0204148](#)] [[INSPIRE](#)].
- [122] W. Ji and X.-G. Wen, *Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions*, *Phys. Rev. Res.* **2** (2020) 033417 [[arXiv:1912.13492](#)] [[INSPIRE](#)].
- [123] X.-C. Wu, W. Ji and C. Xu, *Categorical Symmetries at Criticality*, [arXiv:2012.03976](#) [[DOI:10.1088/1742-5468/ac08fe](#)] [[INSPIRE](#)].
- [124] A. Chatterjee and X.-G. Wen, *Symmetry as a shadow of topological order and a derivation of topological holographic principle*, *Phys. Rev. B* **107** (2023) 155136 [[arXiv:2203.03596](#)] [[INSPIRE](#)].
- [125] A. Chatterjee, W. Ji and X.-G. Wen, *Emergent generalized symmetry and maximal symmetry topological order*, *Phys. Rev. B* **112** (2025) 115142 [[arXiv:2212.14432](#)] [[INSPIRE](#)].

- [126] K. Inamura and X.-G. Wen, *2+1D symmetry-topological-order from local symmetric operators in 1+1D*, [arXiv:2310.05790](#) [INSPIRE].
- [127] K. Roumpedakis, S. Seifnashri and S.-H. Shao, *Higher Gauging and Non-invertible Condensation Defects*, *Commun. Math. Phys.* **401** (2023) 3043 [[arXiv:2204.02407](#)] [INSPIRE].
- [128] Y. Choi et al., *Noninvertible duality defects in 3+1 dimensions*, *Phys. Rev. D* **105** (2022) 125016 [[arXiv:2111.01139](#)] [INSPIRE].
- [129] A. Antinucci et al., *The holography of non-invertible self-duality symmetries*, *JHEP* **03** (2025) 052 [[arXiv:2210.09146](#)] [INSPIRE].
- [130] M. Barkeshli, P. Bonderson, M. Cheng and Z. Wang, *Symmetry Fractionalization, Defects, and Gauging of Topological Phases*, *Phys. Rev. B* **100** (2019) 115147 [[arXiv:1410.4540](#)] [INSPIRE].
- [131] S.D. Pace, Ö.M. Aksoy and H.T. Lam, *Spacetime symmetry-enriched SymTFT: from LSM anomalies to modulated symmetries and beyond*, [arXiv:2507.02036](#) [INSPIRE].