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This publication URL: https://archive-ouverte.unige.ch/unige:181979

Publication DOI: 10.13097/archive-ouverte/unige:181979

Université de Genève
Départment de Physique Théorique

Faculté de Sciences
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On the arithmetic of resurgent topological strings

Thèse

présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de
Docteur ès sciences, mention physique

par

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de

Rome (Italie)

Thèse N° 5852
Centre d'Impression de l'Université de Genève
Septembre 2024

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

DOCTORAT ÈS SCIENCES, MENTION PHYSIQUE

Thèse de Madame Claudia RELLA

intitulée :

«On the Arithmetic of Resurgent Topological Strings»

La Faculté des sciences, sur le préavis de

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Genève, le 20 septembre 2024

Thèse - 5852 -

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Abstract

Quantizing the mirror curve to a toric Calabi–Yau threefold gives rise to quantum operators whose fermionic spectral traces produce factorially divergent formal power series in the Planck constant and its inverse. These are conjecturally captured by the Nekrasov–Shatashvili and standard topological string free energies, respectively, via the TS/ST correspondence. Based on numerical evidence in the examples of local $\mathbb{P}^2$ and local $\mathbb{F}_0$, we conjecture that their resurgent structures involve peacock patterns of singularities and infinitely many rational Stokes constants, while we obtain an analytic prediction on the asymptotic behavior of the fermionic spectral traces in a WKB double-scaling regime dual to the standard 't Hooft limit. We solve exactly the resurgent structures of the spectral trace of local $\mathbb{P}^2$ at weak and strong coupling and prove closed formulae for the Stokes constants as divisor sum functions and for the perturbative coefficients as values of L-functions. We argue that a full-fledged strong-weak resurgent symmetry is at play, exchanging the perturbative/non-perturbative contributions to the holomorphic and anti-holomorphic blocks in the factorization of the spectral trace. This relies on a global net of relations connecting the perturbative series and the discontinuities in the dual regimes, which is built upon the analytic properties of the L-functions with coefficients given by the Stokes constants and the resurgence of the q-series acting as their generating functions. Then, we show that the latter are holomorphic quantum modular forms and are reconstructed by the median resummation of their asymptotic expansions. This leads us to discuss new perspectives on the resurgence of formal power series and quantum modularity and to introduce the notion of modular resurgence. This thesis is based on the publications [1–3]. The author's graduate work also included the publications [4, 5], which are unrelated to the topics presented here.

Résumé en français

Les séries asymptotiques résurgentes apparaissent naturellement comme des expansions perturbatives dans les théories quantiques. La théorie de la résurgence [6] leur associe de manière unique une collection de corrections non analytiques couplées à un ensemble de nombres complexes, appelés constantes de Stokes, qui capturent des informations sur le comportement à grand ordre de la série asymptotique originale et de ses secteurs non perturbatifs additionnels. Après la proposition originale de [7–9], des preuves de plus en plus nombreuses indiquent que la théorie de la résurgence peut fournir une compréhension systématique des complétions non perturbatives de la théorie des cordes topologiques compactifiées sur une variété torique tridimensionnelle de Calabi–Yau (CY) X.

Rappelons que la courbe miroir de X est décrite par une famille de surfaces de Riemann Σ de genre g_Σ plongées dans $\mathbb{C}^* \times \mathbb{C}^*$. Sa quantification produit des opérateurs de mécanique quantique dont les traces spectrales fermioniques $Z(\boldsymbol{N}, \hbar)$, $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, sont des fonctions bien définies du paramètre de déformation quantique $\hbar \in \mathbb{R}_{>0}$. L'approche de [10], que nous suivons dans cette thèse, explore la résurgence de la corde topologique sur X via les séries numériques factoriellement divergentes obtenues en développant asymptotiquement les traces spectrales fermioniques $Z(\boldsymbol{N}, \hbar)$ pour $\hbar \to \infty$ à $\boldsymbol{N}$ fixe. L'énoncé conjecturel récent de [11, 12], connu sous le nom de correspondance théorie des cordes topologiques/théorie spectrale (TS/ST), implique une dualité forte-faible $\hbar \propto g_s^{-1}$, où g_s est la constante de couplage de la corde, et mène à des formules exactes pour les traces spectrales fermioniques en termes du potentiel grand canonique total de la corde topologique sur X.

Dans cette thèse, reproduisant les travaux [1–3], nous commençons par considérer la limite duale $\hbar \to 0$. Nous effectuons une analyse résurgente numérique de l'expansion perturbative semi-classique de la première trace spectrale fermionique pour deux exemples bien connus de trois-variétés toriques CY, à savoir, le local $\mathbb{P}^2$ et le local $\mathbb{F}_0$, complétant l'étude numérique de [10]. En rassemblant les résultats de [10] et [1], nous décrivons une proposition conjecturale pour la structure résurgente de $\log Z(\boldsymbol{N}, \hbar)$ à $\boldsymbol{N}$ fixé pour $\hbar \to 0$ et $\hbar \to \infty$. Cela implique de manière cruciale des *motifs de paon* de singularités dans le plan de Borel et une infinité de *constantes de Stokes rationnelles*. En utilisant la correspondance TS/ST, les régimes duals en $\hbar$ des traces spectrales fermioniques ci-dessus sont explicitement reliés aux énergies libres des cordes topologiques de Nekrasov–Shatashvili (NS) et conventionnelles, respectivement.

Dans le cas du local $\mathbb{F}_0$, la structure résurgente de la trace spectrale n'est accessible que par des méthodes numériques, et nous révélons la présence de termes de type logarithmique dans les asymptotiques sous-jacents. Dans le cas du local $\mathbb{P}^2$, nous résolvons exactement les structures résurgentes complètes du logarithme de la trace spectrale dans les limites faible et forte en $\hbar$. Les constantes de Stokes sont données par des *fonctions somme de diviseurs*, tandis que leurs séries génératrices sont écrites en termes de symboles q-Pochhammer. De plus, les constantes de Stokes sont les coefficients de deux L-fonctions qui déterminent les coefficients perturbatifs originaux lorsqu'elles sont évaluées en des points entiers. Les propriétés analytiques

et arithmétiques de ces *L-fonctions résurgentes* sous-tendent un réseau global de relations reliant les régimes de couplage faible et fort.

On observe que, contrairement à ce qui se passe pour $\hbar \to \infty$, l'expansion perturbative semi-classique de la trace spectrale des tous les deux local $\mathbb{P}^2$ et $\mathbb{F}_0$ n'a pas un comportement de la forme $\mathrm{e}^{-1/\hbar}$ au premier ordre, ce qui suggère qu'il n'y a pas d'analogue dual de la conjecture du volume de conifold. Nous étendons cette observation à une déclaration générale sur les asymptotiques semi-classiques dominants des traces spectrales fermioniques d'une variété torique CY en dimension trois. Nous étudions le potentiel grand canonique total de la corde topologique dans un *régime de double échelle WKB* approprié associé à la théorie spectrale pour $\hbar \to 0$, qui sélectionne la contribution de l'énergie libre totale de la corde topologique NS. Après un changement approprié de cadre symplectique local dans l'espace des modules, nous obtenons une nouvelle prédiction analytique de la correspondance TS/ST sur le comportement asymptotique WKB des traces spectrales fermioniques.

Revenant aux structures résurgentes exactes de $\log Z(1, \hbar)$ pour le local $\mathbb{P}^2$, nous complétons et améliorons notre nouvelle dualité arithmétique reliant les limites faiblement et fortement couplées en une *symétrie résurgente forte-faible* à part entière, qui échange de manière précise les contributions perturbatives/non perturbatives aux blocs holomorphes et anti-holomorphes dans la factorisation de la trace spectrale. La richesse de la résurgence de la trace spectrale du local $\mathbb{P}^2$ nous conduit à approfondir les propriétés des fonctions génératrices f_0 et f_∞ des constantes de Stokes de couplage faible et fort. D'une part, nous prouvons que f_0 et f_∞ et leurs images sous l'involution de Fricke sont des *fonctions modulaires quantiques holomorphes* pour le sous-groupe de congruence $\Gamma_1(3) \subset \mathsf{SL}_2(\mathbb{Z})$. D'autre part, nous prouvons que le développement asymptotique de f_0 reconstruit la fonction génératrice via la *resommation médiane*. Une affirmation analogue est conjecturée pour f_∞ sur la base de preuves numériques.

Résurgence, sommabilité et modularité ont souvent interagi dans la littérature sur la théorie de Chern–Simons quantique et les invariants quantiques des nœuds et des trois-variétés [13–29], mais il n'existe pas de théorie générale qui les intègre dans un cadre complet. Pourtant, ils se croisent de manière cruciale dans nos résultats, nous donnant un indice sur la manière de combler cette lacune. Nous tirons parti de l'abondante information recueillie à partir de l'analyse approfondie de la résurgence de la trace spectrale du local $\mathbb{P}^2$ et élevons ses caractéristiques cardinales à un paradigme générique de *résurgence modulaire*. Guidés par un ensemble diversifié d'exemples que nous fournissons à titre de preuves, nous caractérisons ces séries formelles divergentes qui produisent des formes modulaires quantiques holomorphes et, simultanément, dont la resommation médiane est efficace. Cela nous conduit naturellement à présenter la notion de *séries résurgentes modulaires*. Enfin, nous montrons qu'un réseau de relations exactes connecte les structures résurgentes de paires de séries résurgentes modulaires et repose sur les rôles fondamentaux joués par les constantes de Stokes, les q-séries agissant comme leurs fonctions génératrices, et les L-fonctions résurgentes correspondantes. Au-delà de l'exemple original de la trace spectrale du local $\mathbb{P}^2$, nos affirmations sont prouvées pour une large classe de séries résurgentes modulaires provenant de la théorie des formes de Maass cuspidale.

Acknowledgements

I am grateful to my supervisor Prof. Marcos Mariño for giving me the opportunity to work on this stimulating subject, for his insightful guidance, and for sharing his passion for our research field. I also thank Prof. Anton Alekseev for offering me his reliable counsel and Prof. Giovanni Bachelet, who supported me at the beginning of my academic journey and has continued to mentor me ever since.

I express my appreciation to Prof. Francis Brown, Prof. Alba Grassi, Prof. Maxim Kontsevich, Prof. Andrew Neitzke, Prof. Mauro Raggi, and Prof. Francesco Riva for our valuable conversations during my graduate studies; and to my collaborators, Dr. Babette Döbrich, Dr. Veronica Fantini, and Prof. Tien-Tien Yu, for our challenging discussions and for initiating me to the nuances of being a researcher.

I am also thankful to my colleagues from the Departments of Mathematics and Theoretical Physics at the University of Geneva, among which Adrián, Bharath, Matijn, Max, Paulo, Pietro, Pranjal, Qianyu, Rahel, Ramon, Rea, Sonia, and Tomás, for their companionship; and to Prof. Vlad Mărgărint and Dr. Donata Passarello, in whom I have found an enriching commonality of experience.

Finally, I am grateful to my family for their consistent support. I thank my father Savino and my mother Stefania for nurturing my curiosity and inciting me to pursue my passions, my brother Antonio for our inspiring chats on science and philosophy, and my partner Thomas for his kindness and comforting encouragement. Last but not least, a heartfelt thanks to my grandparents, Anna, Antonio, Franco, and Iole, for their enduring love and care.

My work was supported by the ERC-SyG project "Recursive and Exact New Quantum Theory" (ReNewQuantum), which received funding from the European Research Council within the European Union's Horizon 2020 research and innovation program under Grant No. 810573, and by the Swiss National Centre of Competence in Research SwissMAP (NCCR 51NF40 The Mathematics of Physics) under Grants No. 182902 and 205607.

Contents

Symbols and Notation

- We take $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ as the sets of the natural, integer, real, and complex numbers. Besides, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{C}' = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$, and $\mathbb{H}_\pm$ are the upper and lower halves of the complex plane, respectively. We will sometimes use the notation $\mathbb{H} = \mathbb{H}_+$. $\mathbb{P}$ is the set of prime numbers.

- $\Re(x)$ and $\Im(x)$ denote the real and imaginary parts of $x \in \mathbb{C}$.

- $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828\ldots$ is Euler's number.

- $\mathsf{SL}_2(\mathbb{Z})$ is the special linear group of 2×2 matrices with coefficients in $\mathbb{Z}$ and unit determinant. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$. We denote by $\mathbb{C}_\gamma := \{y \in \mathbb{C}\colon cy + d \in \mathbb{C}'\}$ the cut place associated with γ.

- Given a commutative ring R and a formal variable z, we denote by $R[\![z]\!]$ and $R\{z\}$ the rings of formal and convergent power series in z with coefficients in R, respectively. The Hadamard product of two formal power series $\phi, \psi \in R[\![z]\!]$ is represented as $\phi \diamond \psi$. The Borel transform of $\phi(z)$ is $\hat{\phi}(\zeta)$, where ζ is a new formal variable. The Borel–Laplace sum of $\phi(z)$ at an angle θ is $s_\theta(\phi)(z)$.

- Take $D \subseteq \mathbb{R}$ and z_0 a condensation point of D. Given two functions $f, g : D \setminus \{z_0\} \to \mathbb{R}$, we adopt the following asymptotic notation in the limit $z \to z_0$:

    - $f = \mathcal{O}(g)$ if $\exists C > 0$ such that $|f(z)| \leq C|g(z)|$ in a punctured neighborhood of z_0;
    - $f = o(g)$ if $\forall \delta > 0$ there is a punctured neighborhood of z_0 where $|f(z)| \leq \delta|g(z)|$.

  The alternative notation $f \ll g$ is sometimes used instead of $f = o(g)$. The asymptotic equivalence $f \sim g$ means $f - g = o(g)$.

- Expressions such as $z \pm \mathrm{i}0$ are intended as limiting values for small imaginary parts going to $0^\pm$, *i.e.*,
$$z \pm \mathrm{i}0 = \lim_{\epsilon \to 0^+} z \pm \mathrm{i}\epsilon\,.$$

- Let functions that agree almost everywhere be identified. We denote by $L^2(\mathbb{R})$ the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ whose squared absolute value is Lebesgue integrable, *i.e.*,
$$\int_{\mathbb{R}} |f(z)|^2 dz < \infty\,.$$
  Given two quantum-mechanical operators A, B acting on $L^2(\mathbb{R})$, their Moyal product is represented as $\mathsf{A} \star \mathsf{B}$, while their canonical commutator is $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}$. A_W denotes the Wigner transform of A.

- $(z;\, q)_m$ is the q-Pochhammer symbol.

- $(zq^\alpha;\, q)_\infty$ is the quantum dilogarithm.

- $\Phi_{\mathsf{b}}(z)$ is Faddeev's quantum dilogarithm.

- ${}_{r+1}\phi_s\left(\begin{smallmatrix} a_0,\, a_1,\, \dots,\, a_r \\ b_1,\, b_2,\, \dots,\, b_s \end{smallmatrix};\, q,\, x\right)$ is the (unilateral) q-hypergeometric series.
- $\mathrm{Li}_n(z)$ is the polylogarithm of order n.
- $B_n(z)$ is the n-th Bernoulli polynomial and $B_n = B_n(0)$ is the n-th Bernoulli number.
- $\Gamma(z)$ is the gamma function and $\Psi(z) = \frac{d}{dz} \log \Gamma(z)$ is the digamma function.
- $\zeta(z)$ is the Riemann zeta function and $\zeta(z, a)$ is the Hurwitz zeta function.
- Let M be an n-dimensional complex manifold. We denote by $H^{p,q}(M)$ for $p, q = 0, \dots, n$ the Dolbeault cohomology groups of M and define its Hodge numbers to be $h^{p,q}(M) = \dim H^{p,q}(M)$.
- $\mathbb{P}^n = \mathbb{CP}^n$ is the n-dimensional complex projective space. Let L^{-1} be the universal line bundle over $\mathbb{P}^n$. Its dual L is called the hyperplane line bundle. We denote by $\mathcal{O}(k) = L^k$ for $k \in \mathbb{Z}$ the holomorphic line bundle over $\mathbb{P}^n$ given by the k-th tensor power of L.

Acronyms

ABJ(M)	Aharony–Bergman–Jafferis(–Maldacena)
BPS	Bogomol'nyi–Prasad–Sommerfield
CFT	Conformal Field Theory
CS	Chern–Simons
CY	Calabi–Yau
GPPV	Gukov–Pei–Putrov–Vafa
LHS	Left-Hand Side
LMO	Le–Murakami–Ohtsuki
NLO	Next-to-Leading Order
NS	Nekrasov–Shatashvili
PB	Padé–Borel
QFT	Quantum Field Theory
RHS	Right-Hand Side
SCFT	Super-Conformal Field Theory
TQFT	Topological Quantum Field Theory
TS/ST	Topological String/Spectral Theory
WKB	Wentzel–Kramers–Brillouin
WRT	Witten–Reshetikhin–Turaev

Introduction

Preface

The A-model topological string theory compactified on a local CY threefold X is defined perturbatively by a worldsheet genus expansion in the string coupling constant g_s and can be expressed in terms of the enumerative invariants of the background geometry. The fixed-genus topological string free energies $F_g(\boldsymbol{t})$, $g \geq 0$, are well-defined expansions in a common region of convergence near the large radius limit $\Re(t_i) \gg 1$ in the moduli space of Kähler structures of X. Yet, at a fixed point $\boldsymbol{t}$ in the convergence region, $F_g(\boldsymbol{t})$ diverges factorially as $(2g)!$ [30], signaling the presence of exponentially small corrections in g_s.

A non-perturbative definition of the topological string on X has been proposed in [11, 12]. By local mirror symmetry, the B-model topological string theory on the mirror $\tilde{X}$ is described by the theory of deformations of its complex structures [7, 31]. This is encoded in algebraic equations parametrizing a family of Riemann surfaces Σ of genus g_Σ embedded in $\mathbb{C}^* \times \mathbb{C}^*$, whose quantization leads naturally to quantum operators acting on the real line. A Planck constant $\hbar \in \mathbb{R}_{>0}$ is introduced as a quantum deformation parameter. The conjectural statement of [11, 12], known as TS/ST correspondence, implies a strong-weak coupling duality $\hbar \propto g_s^{-1}$ and leads to exact formulae for the fermionic spectral traces $Z(\boldsymbol{N}, \hbar)$, $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, of these quantum-mechanical operators. These are well-defined functions of $\hbar$ and are expressed in terms of the total grand potential of the topological string on X. In particular, the A-model total free energies of both the conventional and NS topological strings on X—that is, the generating functions of the Gromov–Witten and refined BPS invariants—are needed to define the total grand potential and can be regarded as non-perturbative corrections of one another in the appropriate regimes [32, 33]. Thus, the TS/ST correspondence provides a way to access the non-perturbative effects associated with the factorial divergence of the topological string perturbation series in the spirit of large-N gauge/string dualities.

After the original proposal of [7–9], growing evidence indicates that the theory of resurgence [6] can be applied to obtain a systematic understanding of the hidden non-perturbative sectors of topological string theory. The machinery of resurgence uniquely associates a divergent formal power series with a collection of non-analytic, exponential-type corrections paired with a non-trivial set of complex numbers, known as Stokes constants, which capture information about the large-order behavior of the original asymptotic series and its additional non-perturbative sectors in a mathematically precise way. For instance, following the work of [34, 35], the resurgent structure of the topological string has recently been investigated through an operator formulation of the BCOV holomorphic anomaly equations [36, 37] satisfied by the conventional closed topological string free energies on toric CY threefolds [38], which has later been extended to arbitrary CY threefolds [39, 40] and Walcher's real topological string [41]. These works produced formal solutions for the multi-instanton trans-series extensions of the free energies that are conjecturally related to the resurgent structure of their perturbative genus expansion in a specific way, although a determination of which of the possible Borel singularities are realized and the values of their Stokes constants are missing. The same methods have been applied to the refined topological string and its NS limit on toric CY threefolds [42–45]. The Stokes constants in these resurgent structures are generally unknown, and only a few have been computed numerically. Yet, growing evidence supports a conjectural identification between them and enumerative invariants counting BPS states of the topological string. The case of the resolved conifold has been studied in [46–49].

Simultaneously, the alternative approach of [10], which we follow and promote in this thesis, investigates the resurgent structure of the topological string on a toric CY threefold via the factorially divergent numerical power series obtained by asymptotically expanding the fermionic spectral traces $Z(\boldsymbol{N}, \hbar)$ at fixed $\boldsymbol{N}$ in the limit $\hbar \to \infty$, corresponding to the weakly interacting regime $g_s \to 0$ of topological string theory according to the strong-weak coupling duality encoded in the TS/ST correspondence.

In this thesis, reproducing the works [1–3], we begin by considering the dual limit $\hbar \to 0$. We perform a numerical resurgent analysis of the semiclassical perturbative expansion of the first fermionic spectral trace for two well-known examples of toric CY threefolds, namely, local $\mathbb{P}^2$ and local $\mathbb{F}_0$, which complements the numerical study of [10]. Putting together the results of [10] and [1], we describe a conjectural proposal for the resurgent structure of $\log Z(\boldsymbol{N}, \hbar)$ at fixed $\boldsymbol{N}$ for both $\hbar \to 0$ and $\hbar \to \infty$. Crucially, this involves, in particular, *peacock patterns* of singularities in the Borel plane and infinite sets of generally calculable and *rational Stokes constants*. Using the TS/ST correspondence, the dual $\hbar$-regimes of the fermionic spectral traces above are explicitly related to the free energies of the NS and conventional topological strings on the target geometry, respectively.

In the case of local $\mathbb{F}_0$, the resurgent structure of the first fermionic spectral trace is only accessible via numerical methods, and we unveil the presence of logarithmic-type terms in the sub-leading asymptotics. However, the power of studying the resurgence of the fermionic spectral traces manifests in the example of local $\mathbb{P}^2$—the benchmark example among all non-compact CY threefolds. Indeed, we solve exactly the complete resurgent structures of the logarithm of the first fermionic spectral trace of local $\mathbb{P}^2$ in both weak and strong limits in $\hbar$, resulting in proven analytic formulae for the Stokes constants. These have a transparent and strikingly simple arithmetic meaning as *divisor sum functions*, while their generating series are known in closed form in terms of q-Pochhammer symbols. In addition, the Stokes constants are the coefficients of two explicit L-functions that determine the original perturbative coefficients when evaluated at integer points. The analytic number-theoretic properties of these *resurgent L-functions* underlie a global net of relations connecting the resurgent structures at weak and strong coupling. Thus, the duality between the weakly and strongly coupled regimes emerges in a new form.

Observe that, in contrast with what occurs in the dual limit $\hbar \to \infty$, the semiclassical perturbative expansion of the first fermionic spectral trace of both local $\mathbb{P}^2$ and local $\mathbb{F}_0$ does not have a global exponential behavior of the form $\mathrm{e}^{-1/\hbar}$ at leading order, thus suggesting that there is no dual analog of the conifold volume conjecture. We extend this observation to a general statement on the dominant semiclassical asymptotics of the fermionic spectral traces of a toric CY threefold. We study the topological string total grand potential in an appropriate *WKB 't Hooft-like regime* associated with the spectral theory for $\hbar \to 0$, which selects the contribution from the total free energy of the refined topological string in the NS limit. After a suitable change of local symplectic frame in the moduli space of the geometry, we obtain a new, non-trivial analytic prediction of the TS/ST correspondence on the WKB asymptotic behavior of the fermionic spectral traces, which implies, in particular, the statement above.

Going back to the exact resurgent structures of $\log Z(1, \hbar)$ for local $\mathbb{P}^2$, we complete and upgrade our novel number-theoretic duality connecting the weakly and strongly coupled limits into a beautiful full-fledged symmetry. We call it the *strong-weak resurgent symmetry*. In a nutshell, it is an exact symmetry between the resurgent structures in the dual regimes building upon the interplay of q-series and L-functions and revolving around the central role played by the Stokes constants. This newly found symmetry acts at the level of the perturbative/non-perturbative contributions to the holomorphic and anti-holomorphic blocks in the factorization of the spectral trace in a precise way. As a side effect, we shed some light on one of the formal dissimilarities between topological strings and complex CS theory pointed out in [10].

The richness of the resurgence of the spectral trace of local $\mathbb{P}^2$ leads us to investigate further the properties of the generating functions of the weak and strong coupling Stokes constants, which we denote by f_0 and f_∞, respectively. On the one hand, we prove that f_0 and f_∞ and their images under Fricke involution are *holomorphic quantum modular forms* of weight zero for the congruence subgroup $\Gamma_1(3) \subset \mathsf{SL}_2(\mathbb{Z})$. Note that the group $\Gamma_1(3)$ is deeply related to the geometry of the problem since the moduli space parametrizing the complex structures of the mirror of local $\mathbb{P}^2$ is the compactification of the quotient $\mathbb{H}/\Gamma_1(3)$, where $\mathbb{H}$ denotes the upper half of the complex plane. Moreover, the generating functions of Gromov–Witten invariants of local $\mathbb{P}^2$ are known to be quasi-modular forms for $\Gamma_1(3)$ [50–52]. On the other hand, we study the summability properties of the asymptotic expansions of f_0 and f_∞, which are explicitly governed by the perturbative series of the logarithm of the spectral trace in the two $\hbar$-regimes. We prove that the asymptotic expansion of f_0 is resummable and, precisely, reconstructs the generating function through the *median resummation*. An analogous statement is conjectured for f_∞ based on numerical evidence.

Resurgence, summability, and modularity are lively domains of research involving different communities, which have so far primarily focused on the study of various examples coming from quantum CS theory and the quantum invariants of knots and 3-manifolds [13–29]. There is currently no general theory that integrates them into a comprehensive framework. Yet, they crucially intersect in our results, giving us a hint on how to address this gap. We take advantage of the abundant amount of information gathered from the in-depth analysis of the resurgence of the spectral trace of local $\mathbb{P}^2$ and lift its cardinal features to a generic paradigm of *modular resurgence*, which relies on the fundamental roles played by the Stokes constants, the q-series acting as their generating functions, and the corresponding resurgent L-functions. We characterize those divergent formal power series that produce holomorphic quantum modular forms and, simultaneously, whose median resummation is effective. This naturally leads us to present the notion of a *modular resurgent series*, whose Borel plane displays a single infinite tower of singularities, the secondary resurgent series are trivial, and the Stokes constants are coefficients of an L-function. The resurgent structure of such a series is called a *modular resurgent structure*.

We show that a global network of exact relations connects the resurgent structures of pairs of modular resurgent series, building upon the analytic properties of the resurgent L-functions. Guided by a diverse and conspicuous set of examples that we provide as supporting evidence, we conjecture that a q-series whose asymptotic series is modular resurgent is a holomorphic quantum modular form and is reconstructed by the median resummation of its asymptotic expansion. Beyond the complete, original example of the spectral trace of local $\mathbb{P}^2$, our statements are proven for a large class of modular resurgent series that originates from the theory of Maass cusp forms.

Although conjectural, our paradigm of modular resurgence is supported by further evidence from the study of quantum invariants of knots and 3-manifolds and combinatorics. The example of the trefoil knot and other families of toric knots have been discussed separately from the resurgence, summability, and modularity perspectives. Evidence of our conjectures is present in the works [13, 53]. Further evidence comes from the study of 3-manifold invariants, such as the WRT invariants. The authors of [14] prove that the Ohtsuki series Z_0 for Seifert fibered homology spheres satisfies the conditions of what we define here as a modular resurgent series. Its median resummation reconstructs the normalized WRT invariants $\tilde{Z}_0$, which are related to the GPPV invariants $\hat{Z}_0$ [54]. Then, the results of [25, 26] show that the GPPV invariants of certain plumbed 3-manifolds are quantum modular forms. From combinatorics, families of q-hypergeometric series [55] provide examples of quantum modular forms, among which the example of σ, σ^* first discussed by Zagier [18, Example 1]. Finally, the resurgent structure of σ, σ^*, studied by Fantini and Kontsevich, appears to be modular [56].

Structure

In Part I, we lay the foundations. Specifically, in Chapter 1, we provide the necessary background on the resurgence of asymptotic series, introducing the notions of Stokes constant, Borel–Laplace sum, discontinuity, and median resummation. We also give a concise exposition of alien calculus and define the Stokes automorphism, which allows us to make sense of the definition of (minimal) resurgent structure. In Chapter 2, we describe what a toric CY threefold is and review the essential ingredients in constructing quantum-mechanical operators and their spectral traces from local mirror symmetry. We revise the definition of the refined topological string theory and its two one-parameter specializations compactified on a toric CY background. The chapter ends with a summary of the recent conjectural correspondence between topological strings and spectral theory, which offers a robust, practical perspective for the computational tasks undertaken in this thesis.

In Part II, we introduce our results on the resurgence of the fermionic spectral traces of toric CY threefolds. Specifically, in Chapter 3, we present a conjectural class of enumerative invariants of topological strings on a toric CY target as Stokes constants of appropriate asymptotic series, which arise naturally in the strongly interacting limit $g_s \to \infty$ of the string, and which can be promoted to resurgent trans-series. The rational invariants studied in this chapter are a natural complement of the conjectural proposal of [10], which addresses the dual weakly coupled limit $g_s \to 0$. Finally, we perform a preliminary numerical study of the resurgent structure of the first fermionic spectral trace of local $\mathbb{P}^2$ and local $\mathbb{F}_0$ for $\hbar \propto g_s^{-1} \to 0$ using the tools of Padé–Borel analysis. In Chapter 4, we present a new analytic prediction on the asymptotic behavior of the fermionic spectral traces of toric CY threefolds in the WKB double-scaling regime in terms of the free energies of the refined topological string in the NS limit. We analyze the example of local $\mathbb{P}^2$ in detail.

In Part III, we propose a novel paradigm linking the resurgence of q-series with the analytic number-theoretic properties of L-functions and quantum modular forms. Specifically, in Chapter 5, we prove an exact and complete solution to the resurgent structure of the logarithm of the spectral trace of local $\mathbb{P}^2$ in both regimes of $\hbar \to 0$ and $\hbar \to \infty$ and uncover a fascinating and rich arithmetic construction. The Stokes constants are the coefficients of L-functions, while a new number-theoretic duality between the weakly and strongly coupled scaling limits emerges. In Chapter 6, we delve into the arithmetic characterization of the dual resurgent structures discovered in the previous chapter and complete it into a full-fledged strong-weak resurgent symmetry. As we take the perspective of the Stokes constants, their generating functions, and their Dirichlet series, we find new exact relations intertwining the dual regimes in $\hbar$. In Chapter 7, we study the quantum modularity properties of the generating functions of the Stokes constants of the logarithm of the spectral trace of local $\mathbb{P}^2$ and prove that they are holomorphic quantum modular functions for $\Gamma_1(3)$. Then, we investigate the summability of their asymptotic expansions employing the notion of median resummation. In Chapter 8, we explore the analytic number-theoretic properties of those resurgent structures that emulate the crucial features of the spectral trace of local $\mathbb{P}^2$ and propose the paradigm of modular resurgence. We define the concept of a modular resurgent series, which we conjecture to have peculiar quantum modularity and summability properties and to satisfy an exact global symmetry generalizing the strong-weak resurgence symmetry of local $\mathbb{P}^2$. Finally, we prove our conjectures for examples originating from the theory of Maass cusp forms. There are four Appendices.

Outlook

The results discussed in this thesis lie at the intersection of multiple research areas and, as such, raise diverse questions and open problems. We now give a short and heterogeneous account of possible directions for future work that we group according to their main subject.

Non-perturbative topological strings

The geometric and physical meaning of the enumerative invariants and non-perturbative sectors of [1, 10] is yet to be understood. However, the remarkable arithmetic fabric underpinning the resurgent properties of the dual $\hbar$-expansions of $\log Z(1, \hbar)$ for local $\mathbb{P}^2$, discovered in [1] and developed in [2] into a global strong-weak resurgent symmetry, paves the way for new insights on the physical interpretation of the Stokes constants.

In the recent works of [38, 42], an operator formulation of the BCOV equations satisfied by the conventional and NS topological string free energies is applied to derive the formal structure of the exact trans-series solutions, although leaving which of the possible Borel singularities are realized and the values of their Stokes constants undetermined. A detailed study of the connection and complementarity between the formalism of [38, 42] and the framework proposed in [1, 10] and further advanced in [2] might help us achieve a more comprehensive understanding of the non-perturbative structure of topological string theory. Let us also mention that other techniques, which do not employ resurgence, have been recently used to address the problem of identifying and counting BPS states of type IIA compactifications on toric CY threefolds [57]. These include generalizations of the WKB approach [58–61] and techniques based on attractor flows [62].

Simultaneously, our results on the quantum modularity of the generating functions suggest a new research direction as they naturally prompt us to investigate the geometric content of the weak and strong coupling Stokes constants in connection to the BPS spectrum of local $\mathbb{P}^2$ and the unique role played by the Fricke involution. In particular, there appear to be promising links with some of the results of [62].

Finally, it would be interesting to extend our results on the first fermionic spectral trace of local $\mathbb{P}^2$ to other toric CY threefolds and possibly study higher-order fermionic spectral traces, where we expect that similar structures might show up, guiding us toward a generalization of the strong-weak resurgent symmetry.

The TS/ST correspondence

We have described how the integral representation of the fermionic spectral traces in the WKB double-scaling regime can be interpreted as a symplectic transformation of the WKB grand potential at large radius. As explained in [50], a change of symplectic frame in the moduli space of the CY X corresponds to an electromagnetic duality transformation in $\mathrm{Sp}(2s, \mathbb{Z})$ of the periods, where $s = b_2(X)$. However, a complete geometric understanding of the effect of the change of symplectic basis of Chapter 4 on the WKB grand potential would require further work, and it would involve, in particular, the use of the specific modular transformation properties of the topological string amplitudes in the NS limit, which have been studied in [63].

In this thesis, we have considered the numerical series obtained from the semiclassical expansion of $Z(\boldsymbol{N}, \hbar)$ at fixed $\boldsymbol{N}$. However, we can rigorously define the perturbative expansion of the fermionic spectral traces in the WKB double-scaling regime directly. An advantage of this formulation is that it allows us to make a precise statement on the TS/ST prediction of the asymptotic behavior of these series. Studying their resurgent structure is a much more complex endeavor than it is for the numerical series because the perturbative coefficients in this 't Hooft-like limit retain a total parametric dependence on the moduli space of the CY. Some work on similar problems of parametric resurgence has been done for the standard 't Hooft regime of conventional topological string theory [64] and in the context of the large-N expansion of gauge theories [30].

Let us mention that the analytic prediction we have presented for the WKB double-scaling regime of the fermionic spectral traces is, in principle, verifiable from a matrix model perspective. Successful quantum-mechanical validation of our complete statement would represent a significant additional piece of evidence in support of the conjecture of [11, 12].

Modular resurgent structures

We would like to broaden the supporting basis for our paradigm of modular resurgence. A bigger pool of examples might help us deepen our understanding of the critical role of the L-functions constructed from the Stokes constants and their functional equation. A complete characterization of these ingredients might, in turn, steer us towards proving our Conjectures 2 and 3.

Moreover, our statements on the median resummation could be placed in a more general context. Indeed, when divergent formal power series arise from the asymptotic expansion of well-defined analytic functions, it is natural to ask which summability method allows us to reconstruct the original functions [65]. It would be interesting to study the effectiveness of summability methods in a broad sense, following the spirit of our Conjectures 2 and 4 and Conjecture 1.1 in [13].

Part I

Fundamentals

Chapter 1

Resurgent asymptotic series

Perturbative computations in quantum theory, particularly in QFT, rely on approximation schemes depending on the existence of a small parameter, which typically plays the role of a coupling constant z. The resulting perturbative expansions are formal power series in z that most often have zero radius of convergence—a fact whose physical meaning and implications were originally pointed out in the works of Dyson [66] in 1952, Bender and Wu [67] in 1971, and Gross and Periwal [68] in 1988.

In this chapter, we review how the divergence of the perturbative series appearing in quantum theories can sometimes be tamed by employing the machinery of resurgence. The resurgent analysis of formal power series with factorial growth unveils a universal mathematical structure involving a set of numerical data called Stokes constants. This non-trivial collection of complex numbers captures information about the large-order behavior of the perturbative coefficients and the additional non-perturbative sectors that are invisible in conventional perturbation theory. We refer to [69–71] for a formal introduction to the resurgence of asymptotic series and to [30, 72] for its application to gauge and string theories.

1.1 Asymptotic expansions and the non-perturbative ambiguity

Let z be a formal variable. The formal power series

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!] \tag{1.1}$$

is called *asymptotic* if there exists a function $f(z)$ such that

$$f(z) - \sum_{n=0}^{N} a_n z^n = o(z^N)\,, \quad z \to 0\,, \tag{1.2}$$

for all $N \in \mathbb{N}_{>0}$. In other words, the partial sum

$$\phi_N(z) = \sum_{n=0}^{N} a_n z^n\,, \tag{1.3}$$

which is obtained by truncating $\phi(z)$ after finitely many terms, approximates $f(z)$ in such a way that the remainder $f(z) - \phi_N(z)$ is much smaller than the last retained term in the limit $z \to 0$. Yet, contrary to the case of convergent series, this remainder does not necessarily cancel as $N \to \infty$ for a fixed choice of z. We say that $\phi(z)$ is the asymptotic expansion of $f(z)$ and write

$$f(z) \sim \phi(z) \,, \quad z \to 0 \,. \tag{1.4}$$

The notion of asymptotic series was contemporarily introduced by Poincaré [73] and Stieltjes [74] in 1886.

Generally, as the truncation order N grows larger, the corresponding partial sum $\phi_N(z)$ grows closer to the function $f(z)$ until it starts diverging for sufficiently large N. A first-approach best estimate of $f(z)$ is then obtained by the *optimal truncation* of its asymptotic expansion—that is, determining for which value of N the partial sum $\phi_N(z)$ minimizes the remainder $f(z) - \phi_N(z)$. In practice, this means truncating the series in the RHS of Eq. (1.1) after the term that is the smallest in absolute value. For concreteness, let us consider the case in which the coefficients of the asymptotic series $\phi(z)$ behave as

$$|a_n| \sim \mathcal{A}^{-n} n! \,, \quad n \gg 1 \,, \tag{1.5}$$

for some constant $\mathcal{A} \in \mathbb{R}_{>0}$. This applies in particular to all examples of interest in this thesis. At fixed $|z|$, the optimal value $N = N^*$, which is assumed to be large, is obtained by minimizing with respect to N the generic term

$$|a_N z^N| \sim \exp\left(N \log N - N - N \log \frac{\mathcal{A}}{|z|} \right) \,, \quad N \gg 1 \,, \tag{1.6}$$

where we have applied Eq. (1.5) and Stirling's approximation for the factorial. Specifically, requiring

$$\partial_N |a_N z^N| \big|_{N=N^*} = 0 \,, \quad N \gg 1 \,, \tag{1.7}$$

implies the choice

$$N^* = \frac{\mathcal{A}}{|z|} \,. \tag{1.8}$$

Note that as $|z|$ increases, the optimal partial sum $\phi_{N^*}(z)$ retains fewer terms of the full asymptotic series $\phi(z)$. Conversely, partial sums retaining more terms are optimal approximations of the function $f(z)$ only for smaller values of $|z|$.

Following the definition in Eq. (1.2), the generic remainder is asymptotic to the first discarded term

$$f(z) - \sum_{n=0}^{N} a_n z^n \sim a_{N+1} z^{N+1} \,, \quad z \to 0 \,. \tag{1.9}$$

Let us now take $N = N^*$. At fixed $|z|$, the quantity $\epsilon(z) = |a_{N^*+1} z^{N^*+1}|$ is a measure of the error associated with the optimal truncation of the asymptotic series $\phi(z)$. It follows straightforwardly from Eq. (1.6) that

$$\epsilon(z) \sim |a_{N^*} z^{N^*}| \sim \mathrm{e}^{-\mathcal{A}/|z|} \,, \quad z \to 0 \,. \tag{1.10}$$

The above error cannot be improved upon using conventional perturbation theory only. In fact, independently of how many terms of its asymptotic expansion are known, the function $f(z)$ can only be reconstructed up to the exponential-type error in the RHS of Eq. (1.10). The latter is non-analytic at $z = 0$ and thus called the *non-perturbative ambiguity*. Note that the constant $\mathcal{A}$ appearing in the large-order behavior of the perturbative coefficients in Eq. (1.5) represents the strength of this ambiguity.

1.2 Basics of the theory of resurgence

An asymptotic expansion does not generally determine the original function uniquely due to the presence of non-analytic terms that are invisible in perturbation theory but whose properties are already hinted at by the large-order growth of the perturbative series. In favorable circumstances, the intrinsic limits of classical asymptotics can be overcome within the framework of resurgence introduced by Écalle [6] in 1981. Let z, ζ be formal variables and fix $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$. The *Borel transform*

$$\mathcal{B}\colon z^{-\alpha}\mathbb{C}[\![z]\!] \to \zeta^{-\alpha}\mathbb{C}\{\zeta\} \tag{1.11}$$

is a map acting on z-monomials as

$$\mathcal{B}[z^{n-\alpha}] := \frac{\zeta^{n-\alpha}}{\Gamma(n-\alpha+1)}\,, \quad n \in \mathbb{Z}_{\geq 0}\,, \tag{1.12}$$

where $\Gamma(n-\alpha+1)$ is the gamma function. The action of the Borel transform above is extended by countable linearity to all formal power series in $z^{-\alpha}\mathbb{C}[\![z]\!]$.

Let $\phi(z)$ be a *Gevrey-1* asymptotic series of the form

$$\phi(z) = z^{-\alpha}\sum_{n=0}^{\infty} a_n z^n \in z^{-\alpha}\mathbb{C}[\![z]\!]\,, \quad |a_n| \leq \mathcal{A}^{-n} n! \quad n \gg 1\,, \tag{1.13}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ and $\mathcal{A} \in \mathbb{R}_{>0}$, as before. Its Borel transform, which we denote for simplicity

$$\hat{\phi}(\zeta) := \mathcal{B}[\phi](\zeta)\,, \tag{1.14}$$

is explicitly given by

$$\hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k-\alpha+1)}\zeta^{k-\alpha} \in \zeta^{-\alpha}\mathbb{C}\{\zeta\} \tag{1.15}$$

and is analytic in an open neighborhood of $\zeta = 0$ of radius $\mathcal{A}$. When extended to the whole complex ζ-plane, known as *Borel plane*, $\hat{\phi}(\zeta)$ shows a (possibly infinite) set of singularities $\zeta_\omega \in \mathbb{C}$, which we label by the index $\omega \in \Omega$. A ray in the Borel plane that starts at the origin and passes through a singularity ζ_ω is called a *Stokes ray* and is denoted by

$$\mathcal{C}_{\theta_\omega} = e^{i\theta_\omega}\mathbb{R}_{\geq 0}\,, \quad \theta_\omega = \arg(\zeta_\omega)\,. \tag{1.16}$$

The Borel plane is partitioned into sectors bounded by the Stokes rays, and the Borel transform converges to a generally different holomorphic function in each sector.

We recall that the Gevrey-1 asymptotic series $\phi(z)$ is called *resurgent* if its Borel transform $\hat{\phi}(\zeta)$ can be endlessly analytically continued. Namely, for every $L > 0$, there is a finite set of points Ω_L in the Riemann surface of $\zeta^{-\alpha}$ such that $\hat{\phi}(\zeta)$ can be analytically continued along any path that avoids Ω_L and has length at most L. If, additionally, its Borel transform has only simple poles and logarithmic branch points, then it is called *simple resurgent*. Let us assume that the formal power series $\phi(z)$ in Eq. (1.13) is simple resurgent. If the singularity ζ_ω is a simple pole, the local expansion of the Borel transform in Eq. (1.15) around $\zeta = \zeta_\omega$ has the form

$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi i(\zeta - \zeta_\omega)} + \text{regular in } \zeta - \zeta_\omega\,, \tag{1.17}$$

where $S_\omega \in \mathbb{C}$ is the *Stokes constant* at ζ_ω. Whereas, if the singularity ζ_ω is a logarithmic branch point, the local expansion of the Borel transform around it has the form

$$\hat{\phi}(\zeta) = -\frac{S_\omega}{2\pi \mathrm{i}} \log(\zeta - \zeta_\omega)\hat{\phi}_\omega(\zeta - \zeta_\omega) + \text{regular in } \zeta - \zeta_\omega \,, \tag{1.18}$$

where again $S_\omega \in \mathbb{C}$ is the corresponding Stokes constant. If we introduce the variable $\xi = \zeta - \zeta_\omega$, then the function

$$\hat{\phi}_\omega(\xi) = \sum_{k=0}^{\infty} \hat{a}_{k,\omega} \xi^{k-\beta} \in \xi^{-\beta}\mathbb{C}\{\xi\} \,, \tag{1.19}$$

where $\beta \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$, is locally analytic at $\xi = 0$ and can be regarded as the Borel transform of the Gevrey-1 asymptotic series

$$\phi_\omega(z) = z^{-\beta} \sum_{n=0}^{\infty} a_{n,\omega} z^n \in z^{-\beta}\mathbb{C}[\![z]\!], \qquad a_{n,\omega} = \Gamma(n - \beta + 1)\, \hat{a}_{n,\omega} \,. \tag{1.20}$$

Note that the value of the Stokes constant S_ω depends on the normalization of $\phi_\omega(z)$.

If the analytic continuation of the Borel transform $\hat{\phi}(\zeta)$ in Eq. (1.15) does not grow too fast at infinity[1], its Laplace transform at an arbitrary angle θ in the Borel plane gives the *Borel–Laplace sum* of the original, divergent formal power series $\phi(z)$ at angle θ, which is denoted by $s_\theta(\phi)(z)$. Explicitly,

$$s_\theta(\phi)(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-\zeta} \hat{\phi}(\zeta z) \, d\zeta = z^{-1} \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-\zeta/z} \hat{\phi}(\zeta) \, d\zeta \,, \tag{1.21}$$

whose asymptotics near the origin reconstructs $\phi(z)$. If the Borel–Laplace sum in Eq. (1.21) exists in some region of the complex z-plane, we say that the series $\phi(z)$ is Borel–Laplace summable along the direction θ. Note that the Borel–Laplace sum inherits the sectorial structure of the Borel transform. It is a locally analytic function with discontinuities across the special rays identified by

$$\arg(z) = \arg(\zeta_\omega) \,, \quad \omega \in \Omega \,. \tag{1.22}$$

Remark 1.2.1. *It is sometimes useful to adopt a different convention for the Borel and Laplace transforms of the Gevrey-1 asymptotic series in Eq. (1.13). In particular, when explicitly specified in the text, we will use the alternative definitions*

$$\hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k-\alpha)} \zeta^{k-\alpha-1} \,, \tag{1.23a}$$

$$s_\theta(\phi)(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-\zeta/z} \hat{\phi}(\zeta) \, d\zeta = z \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-\zeta} \hat{\phi}(\zeta z) \, d\zeta \,, \tag{1.23b}$$

and refer to Eqs. (1.15) and (1.21) everywhere else in this thesis. Note that the above definition for the Borel transform is the first derivative in z of the one in Eq. (1.15), and the Laplace transform, which is its formal inverse, is simply changed accordingly. All the statements presented in this chapter are unaltered by the switch of conventions. Yet, the logarithmic singularities of the Borel transform in Eq. (1.15) are simple poles of the Borel transform in Eq. (1.23a).

[1] Roughly, we require that the Borel transform grows at most exponentially in an open sector of the Borel plane containing the angle θ.

The discontinuity across an arbitrary direction θ is the difference between the Borel–Laplace sums along two rays in the complex z-plane that lie slightly above and below θ. Namely,

$$\operatorname{disc}_\theta \phi(z) = s_{\theta_+}(\phi)(z) - s_{\theta_-}(\phi)(z) = \int_{\mathcal{C}_{\theta_+} - \mathcal{C}_{\theta_-}} \mathrm{e}^{-\zeta} \hat{\phi}(\zeta z)\, d\zeta\,, \tag{1.24}$$

where $\theta_\pm = \theta \pm \epsilon$ for some small positive angle ϵ and

$$\mathcal{C}_{\theta_\pm} = \mathrm{e}^{\mathrm{i}\theta_\pm} \mathbb{R}_{\geq 0} \tag{1.25}$$

are the corresponding rays in the Borel plane. By the standard contour deformation

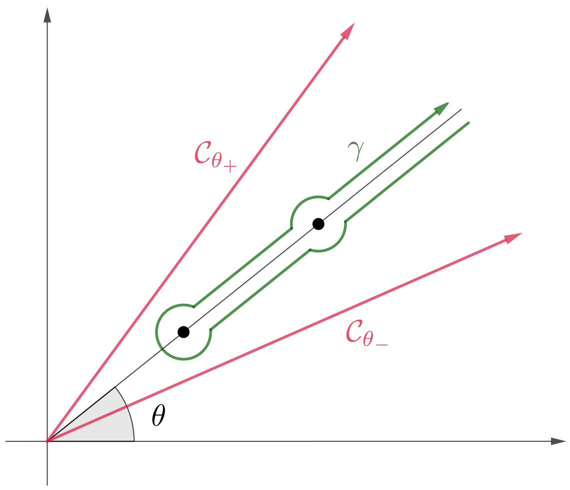


Figure 1.1: The difference between the Laplace transforms along the paths $\mathcal{C}_{\theta_\pm}$, which lie slightly above and below the angle θ in the complex ζ-plane, is deformed into an integral along the contour γ, which picks up the non-analytic contributions from the singularities along the direction θ.

argument, we can deform the integration contour $\mathcal{C}_{\theta_+} - \mathcal{C}_{\theta_-}$ in Eq. (1.24) into a path γ encircling each and all of the singularities in the direction θ, as shown in Fig. 1.1, and obtain that the two lateral Borel–Laplace sums differ by exponentially small terms—one for each of the contributing singularities. More precisely, if the Borel transform $\hat{\phi}(z)$ has only simple poles, then we have that

$$\operatorname{disc}_\theta \phi(z) = \sum_{\omega \in \Omega_\theta} S_\omega \mathrm{e}^{-\zeta_\omega / z}\,, \tag{1.26}$$

where the index $\omega \in \Omega_\theta$ labels the singularities ζ_ω such that $\arg(\zeta_\omega) = \theta$ and the complex numbers S_ω are the same Stokes constants that appear in Eq. (1.17). Similarly, if there are only logarithmic branch points, we have that

$$\operatorname{disc}_\theta \phi(z) = \sum_{\omega \in \Omega_\theta} S_\omega \mathrm{e}^{-\zeta_\omega / z} s_{\theta_-}(\phi_\omega)(z)\,, \tag{1.27}$$

where again the complex numbers S_ω are the Stokes constants appearing in Eq. (1.18), while $\phi_\omega(z)$ is the formal power series in Eq. (1.20).

We have seen how the non-perturbative corrections to the asymptotic series $\phi(z)$ in Eq. (1.13) resurge from the local behavior of its Borel transform at the singularities ζ_ω, $\omega \in \Omega$, in the complex ζ-plane. At a fixed angle θ, the discontinuity $\mathrm{disc}_\theta \phi(z)$ receives a contribution from each Borel singularity $\zeta_\omega \in \Omega_\theta$ consisting of the Stokes constant $S_\omega \in \mathbb{C}$ and the asymptotic series $\phi_\omega(z)$ in Eq. (1.20). This connection between the perturbative (*zero-instanton*) and the non-perturbative (*higher-instanton*) information finds an alternative realization in the *large-order relations* satisfied by the perturbative coefficients of the original asymptotic series.[2] In particular, the large-n behavior of the coefficients a_n, $n \in \mathbb{N}$, is controlled at leading order by the local behavior of the Borel transform at the singularity closest to the origin—following Eq. (1.17) in the case of a simple pole and Eq. (1.18) in the case of a logarithmic branch point. The subdominant singularities contribute with exponentially small corrections in the large-n limit. We show an explicit application in Section 3.2.2.

The effectiveness of the median resummation

In studying the summability properties of factorially divergent formal power series that arise as asymptotic limits of well-defined holomorphic functions, it is natural to ask which summability method is efficient in reproducing the original function from its asymptotic expansion. In quantum CS theory, non-perturbative invariants, such as the WRT invariant for knots and the Kashaev invariant for 3-manifolds, are conjecturally reconstructed by the *median resummation* of their asymptotic expansions at roots of unity [13, Conjecture 1]. As we show in [2], more evidence of the effectiveness of the median resummation comes from studying the spectral traces of the quantum operators obtained by quantizing the mirror curve to a toric CY threefold. We recall that the median resummation of the Gevrey-1 asymptotic series $\phi(z)$ in Eq. (1.13) across an arbitrary direction θ in the complex z-plane is the average of the two lateral Borel–Laplace sums $s_{\theta_\pm}(\phi)(z)$, that is,

$$\mathcal{S}^{\mathrm{med}}_\theta \phi(z) = \frac{s_{\theta_+}(\phi)(z) + s_{\theta_-}(\phi)(z)}{2} \,, \tag{1.28}$$

which is an analytic function for $\arg(z) \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$. Equivalently, we can write

$$\mathcal{S}^{\mathrm{med}}_\theta \phi(z) = \begin{cases} s_{\theta_-}(\phi)(z) + \frac{1}{2}\,\mathrm{disc}_\theta \phi(z) \,, & \Re\left(\mathrm{e}^{-\mathrm{i}\theta_-} z\right) > 0 \,, \\ \\ s_{\theta_+}(\phi)(z) - \frac{1}{2}\,\mathrm{disc}_\theta \phi(z) \,, & \Re\left(\mathrm{e}^{-\mathrm{i}\theta_+} z\right) > 0 \,, \end{cases} \tag{1.29}$$

where $\Re\left(\mathrm{e}^{-\mathrm{i}\theta} z\right) > 0$. The expression above highlights how the discontinuities can be suitably interpreted as corrections to the standard Borel–Laplace transform.

1.3 Alien calculus and the Stokes automorphism

As described in the previous section, the domain of convergence of the analytically continued Borel–Laplace sum of a resurgent asymptotic series is generally partitioned into sectors governed by the singularity structure of the Borel transform. Different sectorial solutions are found after each discontinuous jump across a Stokes ray and must be glued

[2] This observation was first made by Bender and Wu [67] in 1971 in the specific case of the quartic anharmonic oscillator.

together to produce a full non-perturbative completion of the original asymptotic series defined throughout the complex plane. To do so, we resort to the notions of alien derivative and Stokes automorphism. We briefly recall here the basics of *alien calculus* used in this thesis.

Once more, let $\phi(z)$ be a resurgent asymptotic series and $\hat{\phi}(\zeta)$ its Borel transform. We denote by ζ_ω, $\omega \in \Omega_\theta$, the singularities of $\hat{\phi}(\zeta)$ that lie on the same Stokes line at an angle

$$\theta = \arg(\zeta_\omega) \tag{1.30}$$

in the complex ζ-plane. For simplicity, we will now number the singularities along the given Stokes ray according to their increasing distance from the origin. Let us fix a value $\omega = r \in \mathbb{N}_{\neq 0}$. When analytically continuing $\hat{\phi}(\zeta)$ from the origin to the singularity ζ_r along the direction in Eq. (1.30), each singularity ζ_i, $i = 1, \ldots, r-1$, must be avoided by either passing slightly above or slightly below it. This creates ambiguity in the prescription. We label by $\epsilon_i = \pm 1$ the two choices and introduce the notation

$$\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r}(\zeta) \tag{1.31}$$

to indicate that the analytic continuation is performed in such a way that the singularity ζ_i is encircled above or below according to the value of ϵ_i for $i = 1, \ldots, r-1$. Suppose that the local expansion of the Borel transform at $\zeta = \zeta_r$ has the form

$$\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r}(\zeta) = -\frac{1}{2\pi \mathrm{i}\xi} c^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r} - \frac{\log(\xi)}{2\pi \mathrm{i}} \hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{r;\,\zeta_1,\ldots,\zeta_r}(\xi) + \ldots\,, \tag{1.32}$$

where $\xi = \zeta - \zeta_r$, the dots denote regular terms in ξ, $c^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r}$ is a complex number, and $\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{r;\,\zeta_1,\ldots,\zeta_r}(\xi)$ is the germ of an analytic function at $\xi = 0$. The *alien derivative* at the singularity ζ_r acts on the formal power series $\phi(z)$ as

$$\Delta_{\zeta_r}\phi(z) = \sum_{\epsilon_1,\ldots,\epsilon_{r-1}} \frac{p(\epsilon)!q(\epsilon)!}{r!} \left(c^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r} + \mathcal{B}^{-1}\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{r;\,\zeta_1,\ldots,\zeta_r}(z) \right)\,, \tag{1.33}$$

where $\mathcal{B}^{-1}$ denotes the formal inverse of the Borel transform and $p(\epsilon)$, $q(\epsilon)$ are the number of times that ± 1 occur in the set $\{\epsilon_1, \ldots, \epsilon_{r-1}\}$, respectively. Furthermore, $\Delta_\zeta \phi(z) = 0$ if $\zeta \in \mathbb{C}$ is not a singular point in the Borel plane of $\phi(z)$.

Example 1.3.1. *Let us consider the simple example of*

$$\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r}(\zeta) = -\frac{\log(\xi)}{2\pi \mathrm{i}} S_r + \ldots\,, \tag{1.34}$$

where $S_r \in \mathbb{C}$ is a constant that depends only on the choice of the singularity ζ_r. Since the inverse Borel transform acts trivially on numbers, we have that

$$\Delta_{\zeta_r}\phi(z) = S_r \sum_{\epsilon_1,\ldots,\epsilon_{r-1}} \frac{p(\epsilon)!q(\epsilon)!}{r!} = S_r \sum_{p=0}^{r-1} \frac{p!(r-1-p)!}{r!} \binom{r-1}{p} = S_r\,. \tag{1.35}$$

Observe that in the case of

$$\hat{\phi}^{\epsilon_1,\ldots,\epsilon_{r-1}}_{\zeta_1,\ldots,\zeta_r}(\zeta) = -\frac{1}{2\pi \mathrm{i}\xi} S_r + \ldots\,, \tag{1.36}$$

where $S_r \in \mathbb{C}$ is again a number which depends only on the choice of the singularity ζ_r, the alien derivative at ζ_r acts on $\phi(z)$ according to the same formula in Eq. (1.35).

Note that the alien derivative Δ_ζ, $\zeta \in \mathbb{C}$, is a derivation in the algebra of resurgent functions. In particular, it satisfies the expected Leibniz rule when acting on a product, that is,

$$\Delta_\zeta \left(\phi_1(z)\phi_2(z)\right) = \left(\Delta_\zeta\phi_1(z)\right)\phi_2(z) + \phi_1(z)\left(\Delta_\zeta\phi_2(z)\right) , \tag{1.37}$$

where $\phi_1(z)$, $\phi_2(z)$ are two given resurgent formal power series. As a consequence, the alien derivative also acts naturally on exponentials. Namely,

$$\Delta_\zeta \mathrm{e}^{\phi(z)} = \sum_{k=0}^{\infty} \frac{1}{k!}\Delta_\zeta \phi^k(z) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\phi^{k-1}(z)\Delta_\zeta\phi(z) = \mathrm{e}^{\phi(z)}\Delta_\zeta\phi(z) \,. \tag{1.38}$$

Let us now go back to the definition of the discontinuity across the direction θ in Eq. (1.24) and assume that all singularities on the Stokes ray $\mathcal{C}_\theta$ are of logarithmic type. If we regard the lateral Borel–Laplace sums as operators acting on formal power series, the *Stokes automorphism* $\mathfrak{S}_\theta$ at the angle θ is defined by the composition

$$s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta \,, \tag{1.39}$$

and the discontinuity formula in Eq. (1.27) has the equivalent form

$$\mathfrak{S}_\theta(\phi) = \phi + \sum_{\omega\in\Omega_\theta} S_\omega \mathrm{e}^{-\zeta_\omega/z}\phi_\omega \,. \tag{1.40}$$

Moreover, the Stokes automorphism can be written as

$$\mathfrak{S}_\theta = \exp\left(\sum_{\omega\in\Omega_\theta} \mathrm{e}^{-\zeta_\omega/z}\Delta_{\zeta_\omega}\right) , \tag{1.41}$$

where Δ_{ζ_ω} is again the alien derivative associated to the singularity ζ_ω, $\omega \in \Omega_\theta$.

Minimal resurgent structures

As we have shown in Section 1.2, moving from one formal power series to a new one encoded in the singularity structure of its Borel transform, we eventually build a whole family of new asymptotic series ϕ_ω, $\omega \in \Omega$, from the single input in Eq. (1.13). We can repeat the procedure with each new series obtained this way, assuming all series are simple resurgent. For each $\omega \in \Omega$, let us define the *basic trans-series*

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z}\phi_\omega(z) \,, \tag{1.42}$$

so that its Borel–Laplace sum at the angle θ is given by

$$s_\theta(\Phi_\omega)(z) = \mathrm{e}^{-\zeta_\omega/z}s_\theta(\phi_\omega)(z) \,. \tag{1.43}$$

The corresponding Stokes automorphism acts on the basic trans-series as

$$\mathfrak{S}_\theta(\Phi_\omega) = \Phi_\omega + \sum_{\omega'\in\Omega_\theta} S_{\omega\omega'}\Phi_{\omega'} \,, \tag{1.44}$$

where $S_{\omega\omega'} \in \mathbb{C}$ is the Stokes constant of the secondary asymptotic series $\phi_\omega(z)$ at the singularity $\zeta_{\omega'}$ such that $\arg(\zeta_{\omega'}) = \theta$. Observe that in the case of all simple poles, the same definitions and considerations above apply after imposing that $\phi_\omega(z) = 1$, $\omega \in \Omega$.

The (minimal) *resurgent structure* associated with $\phi(z)$ is the smallest collection of basic trans-series that resurge from it and form a closed set under Stokes automorphisms. It is introduced by Gu and Mariño in [10] and is denoted by

$$\mathfrak{B}_\phi = \{\Phi_\omega(z)\}_{\omega\in\bar{\Omega}} \, , \quad \bar{\Omega} \subseteq \Omega \, . \tag{1.45}$$

We stress that $\mathfrak{B}_\phi$ does not necessarily include all the basic trans-series arising from $\phi(z)$. As pointed out in [10], an example is provided by complex CS theory on the complement of a hyperbolic knot.

Example 1.3.2. *In CS theory with a complex gauge group, the perturbative partition function is obtained by expanding the Feynman path integral around a flat connection on the base three-manifold M. The result is a formal power series. If M is the complement of a hyperbolic knot in the three-sphere, there are at least two distinct flat connections, including the abelian and geometric connections. These are associated with two different asymptotic series ϕ_0, ϕ_g, which can be completed to their minimal resurgent structures $\mathfrak{B}_{\phi_0}, \mathfrak{B}_{\phi_g}$, respectively. It follows from the results of [15, 29, 75–78] that $\phi_0 \notin \mathfrak{B}_{\phi_g}$ and $\mathfrak{B}_{\phi_g} \subsetneq \mathfrak{B}_{\phi_0}$.*

Finally, we construct the (possibly infinite-dimensional) matrix of Stokes constants

$$\mathcal{S}_\phi = \{S_{\omega\omega'}\}_{\omega,\omega'\in\bar{\Omega}} \, , \tag{1.46}$$

which is indexed by the distinct basic trans-series in the minimal resurgent structure of $\phi(z)$ and incorporates additional information about the non-analytic corrections to the original asymptotic series.

Chapter 2

Calabi–Yau geometries and the topological string

In the groundbreaking work of Candelas, Horowitz, Strominger, and Witten [79] in 1985, compact CY threefolds were identified as potential candidates for the compactification of ten-dimensional $O(32)$ and $E_8 \times E_8$ supergravity and superstring theory with unbroken $\mathcal{N} = 1$ supersymmetry in four space-time dimensions—indicating that CY manifolds could lead to a phenomenological model of string theory. According to the results of [79], the geometry and topology of the manifold determine important features of particle physics as we know it. For example, the number of generations of the low-energy theory in four dimensions equals half the absolute value of the Euler characteristic of the compact CY threefold. While CY geometries have played an important role in guiding our understanding of string compactifications, string-theoretic developments have paved the way for some of the most fascinating results in the study of the CY moduli space and mirror symmetry [80–83].

In this chapter, we review the two-way connection between the spectral theory of quantum-mechanical operators and the topological string theory [84, 85] on toric (hence, non-compact) CY threefolds, which builds upon notions of quantization and local mirror symmetry [86, 87] and has recently found an explicit formulation in the conjectural statement of [11, 12, 88], known as the TS/ST correspondence. This leads to exact formulae for the fermionic spectral traces of the quantum operators constructed from the quantization of the mirror curve in terms of the enumerative invariants of the geometry and provides a non-perturbative realization of the topological string.

2.1 Geometric setup and local mirror symmetry

It was conjectured by Calabi [89, 90] in 1954 and proved by Yau [91, 92] in 1976 that a compact Kähler manifold with vanishing first Chern class possesses a Ricci-flat Kähler metric. Many different definitions of CY manifolds have been studied since. We adopt the following.

Definition 2.1.1. *A CY n-fold is an n-dimensional complex manifold that admits a Kähler metric with vanishing Ricci curvature.*

In this thesis, we fix the complex dimension to be $n = 3$ and assume the manifold to be simply connected. The Hodge numbers $h^{p,q} = h^{p,q}(X)$ for $p, q = 0, 1, 2, 3$ of a CY threefold

X are constrained by several non-trivial relations. Consequently, all but the integers $h^{1,1}$ and $h^{2,1}$ are explicitly determined and the Hodge diamond takes the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{2.1}$$

The Euler characteristic $\chi = \chi(X)$ of X simplifies accordingly and is given by

$$\chi = 2\left(h^{1,1} - h^{2,1}\right) . \tag{2.2}$$

Importantly, $h^{1,1}$ and $h^{2,1}$ classify the infinitesimal deformations of the Kähler and complex structures of X, respectively. Moreover, there exists a second, topologically different CY threefold $\tilde{X}$ such that the Dolbeault cohomology groups $H^{1,1}$ and $H^{2,1}$ satisfy

$$H^{1,1}(\tilde{X}) = H^{2,1}(X) , \quad H^{2,1}(\tilde{X}) = H^{1,1}(X) , \tag{2.3}$$

that is, the Kähler and complex structure moduli are exchanged. Hence,

$$h^{1,1}(\tilde{X}) = h^{2,1}(X) , \quad h^{2,1}(\tilde{X}) = h^{1,1}(X) , \quad \chi(\tilde{X}) = -\chi(X) . \tag{2.4}$$

This symmetry within the pair of *mirror manifolds* X and $\tilde{X}$ is the basic instance of *mirror symmetry*. We refer the reader to [80–82] for details.

Remark 2.1.1. *The existence of mirror symmetry was originally hinted at by physical considerations. After restricting to the compactified component of the target space, superstring theory is described by an $\mathcal{N} = 2$ SCFT in two dimensions, whose ring of chiral primary fields encodes the cohomology ring of the CY [93]. The correspondence is, however, not unique. The presence of two distinct chiral primary rings of the same SCFT implies the existence of two different CY threefolds related by the exchange $h^{1,1} \leftrightarrow h^{2,1}$ and composing a* mirror pair. *Both experimental verifications and first-principle constructions of mirror pairs have been performed since [94, 95].*

Note that our definition of CY manifold applies to the local (or non-compact) case as well. A local CY manifold can be interpreted as an open neighborhood in a compact one. For example, a rigid two-sphere embedded in a compact CY threefold is locally isomorphic to the total space of the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, which is a non-compact CY threefold known as the *resolved conifold* and belongs to a special class of manifolds.

Definition 2.1.2. *An n-dimensional* toric variety *X is an algebraic variety containing the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$ as a dense open subset and admitting a natural action $\mathbb{T} \times X \to X$.*

All projective and weighted projective spaces are toric, and so is the resolved conifold. More generally, we define a *toric* CY n-fold as an n-dimensional smooth toric variety embedded in projective space that satisfies the CY condition in Definition 2.1.1. All toric

CY manifolds are non-compact[1] and thus do not fit the original program of [79]. Yet, they too have some physically interesting applications—for example, in the context of topological string theory [97, 98].

We will now delve into the specifics of the toric case. Let X be a toric CY threefold and $\boldsymbol{t} = (t_1, \ldots, t_s)$ be the *complexified* Kähler moduli of X, where

$$s = b_2(X) = h^{1,1}(X) \tag{2.5}$$

is the second Betti number of X. When applied to the local case, mirror symmetry simplifies considerably. *Local mirror symmetry* [86, 87] pairs X with a mirror CY threefold $\tilde{X}$ in such a way that the theory of variations of complex structures of the mirror $\tilde{X}$ is encoded in an algebraic equation of the form

$$W(\mathrm{e}^x, \mathrm{e}^y) = 0\,, \tag{2.6}$$

which describes a Riemann surface Σ embedded in $\mathbb{C}^* \times \mathbb{C}^*$, called the *mirror curve* to the toric CY threefold X, and determines the B-model topological string theory on $\tilde{X}$ [7, 31]. We denote the genus of Σ by g_Σ. The complex deformation parameters of $\tilde{X}$ can be divided into g_Σ true moduli of the geometry, denoted by $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_{g_\Sigma})$, and $r_\Sigma = s - g_\Sigma$ mass parameters, denoted by $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{r_\Sigma})$ [99, 100]. They are related to the *Batyrev coordinates* $\boldsymbol{z} = (z_1, \ldots, z_s)$ of $\tilde{X}$ by[2]

$$-\log z_i = \sum_{j=1}^{g_\Sigma} C_{ij}\mu_j + \sum_{k=1}^{r_\Sigma} \alpha_{ik} \log \xi_k\,, \quad i = 1, \ldots, s\,, \tag{2.7}$$

where the constant coefficients C_{ij}, α_{ik} are determined by the toric data of X, and the chemical potentials μ_j are defined by

$$\kappa_j = \mathrm{e}^{\mu_j}\,, \quad j = 1, \ldots, g_\Sigma\,. \tag{2.8}$$

At the same time, the complex moduli of the mirror $\tilde{X}$ are related to the Kähler parameters of X via the *mirror map*

$$-t_i(\boldsymbol{z}) = \log z_i + \Pi_i(\boldsymbol{z})\,, \quad i = 1, \ldots, s\,, \tag{2.9}$$

where $\Pi_i(\boldsymbol{z})$ is a power series in $\boldsymbol{z}$ with finite radius of convergence. Together with Eq. (2.7), it implies that

$$t_i(\boldsymbol{\mu}, \boldsymbol{\xi}) = \sum_{j=1}^{g_\Sigma} C_{ij}\mu_j + \sum_{k=1}^{r_\Sigma} \alpha_{ik} \log \xi_k + \mathcal{O}(\mathrm{e}^{-\mu_j})\,, \quad i = 1, \ldots, s\,, \tag{2.10}$$

where we have introduced the vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)$.

Following a choice of symplectic basis $\{A_i,\, B_i\}_{i=1,\ldots,s}$ of one-cycles on the spectral curve Σ, the classical periods of the meromorphic differential one-form

$$\lambda = y(x)dx\,, \tag{2.11}$$

[1] Compact CY manifolds can be implemented as hypersurfaces in compact toric manifolds using Batyrev's reflexive polytopes [96], although this is not relevant to the results discussed in this thesis.

[2] We can choose the Batyrev coordinates so that the first g_Σ correspond to true moduli and the remaining r_Σ correspond to mass parameters.

where the function $y(x)$ is locally defined by Eq. (2.6), satisfy

$$t_i(\boldsymbol{z}) \propto \oint_{A_i} \lambda\,, \quad \partial_{t_i} F_0(\boldsymbol{z}) \propto \oint_{B_i} \lambda\,, \quad i = 1, \dots, s\,. \tag{2.12}$$

The function $F_0(\boldsymbol{z})$ is the classical prepotential of the geometry [101], which represents the genus-zero amplitude of the B-model topological string on $\tilde{X}$, that is, the generating function of the genus-zero Gromov–Witten invariants of X composed with the mirror map. Its quantization is often studied employing a set of constraints, known as *holomorphic anomaly equations*, satisfied by the higher-genus amplitudes $F_g(\boldsymbol{z})$, $g \geq 1$, which are the generating functions of the higher-genus Gromov–Witten invariants of X after the mirror map is applied [37].

Toric del Pezzo CY threefolds

For simplicity, we will often consider the case in which the mirror curve to a CY threefold has genus one. A large class of such manifolds is given by the *toric (almost) del Pezzo CY threefolds*, which are defined as the total space of the canonical line bundle K_S on a toric (almost) del Pezzo surface S, that is,

$$X = K_S \to S\,, \tag{2.13}$$

also called *local* S. Since the mirror curve to X is elliptic, the mirror manifold $\tilde{X}$ has one true complex modulus κ and $s-1$ mass parameters ξ_k, $k = 1, \dots, s-1$. Correspondingly, Eq. (2.6) admits a single canonical parametrization

$$O_S(x, y) + \kappa = 0\,. \tag{2.14}$$

Toric del Pezzo surfaces are classified by reflexive polyhedra in two dimensions [99, 102]. The polyhedron Δ_S associated with the toric del Pezzo surface S is the convex hull of the origin together with a set of two-dimensional vectors

$$\nu^{(i)} = (\nu_1^{(i)}, \nu_2^{(i)})\,, \quad i = 1, \dots, s+2\,. \tag{2.15}$$

Once extended into the vectors

$$\bar{\nu}^{(i)} = (1, \nu_1^{(i)}, \nu_2^{(i)})\,, \quad i = 1, \dots, s+2\,, \tag{2.16}$$

they satisfy the relations

$$\sum_{i=0}^{s+2} Q_i^\alpha \bar{\nu}^{(i)} = 0\,, \quad \alpha = 1, \dots, s\,, \tag{2.17}$$

where $\bar{\nu}^{(0)} = (1, 0, 0)$ and $Q_i^\alpha \in \mathbb{Z}$ is an entry of the charge matrix of the geometry[3]. Moreover, the function $O_S(x, y)$ appearing in the mirror curve to X in Eq. (2.14) has the form

$$O_S(x, y) = \sum_{i=1}^{s+2} \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} y + f_i(\boldsymbol{\xi})\right)\,, \tag{2.18}$$

[3]A toric CY threefold X can be physically interpreted as the moduli space of vacua for the complex scalars ϕ_i, $i = 1, \dots, s+2$, of chiral superfields in a 2-dimensional supersymmetric σ-model [103]. These fields transform as $\phi_i \to \mathrm{e}^{\mathrm{i} Q_i^\alpha \theta_\alpha} \phi_i$ under the gauge group $U(1)^s$.

where $f_i(\boldsymbol{\xi})$ is a function of the mass parameters of the mirror. Examples of toric del Pezzo surfaces are the projective space $\mathbb{P}^2$, the Hirzebruch surfaces $\mathbb{F}_n$ for $n = 0, 1, 2$, and the blowups of $\mathbb{P}^2$ at n points, usually denoted by $\mathcal{B}_n$, for $n = 1, 2, 3$.

Finally, when the mass parameters satisfy appropriate symmetry conditions, the relation between the Batyrev coordinate z of $\tilde{X}$ and its modulus κ simplifies to

$$z = 1/\kappa^r \,, \tag{2.19}$$

where the value of the constant r is determined by the geometry. For example, such symmetry restrictions trivially apply to local $\mathbb{P}^2$, for which $r = 3$, while they correspond to imposing $\xi = 1$ for local $\mathbb{F}_0$, which has in this case $r = 2$. We stress that, in what follows, we will take X as a generic toric CY threefold, and the genus-one case will only be considered when explicitly stated.

2.2 Quantum operators from mirror curves

The mirror curve Σ in Eq. (2.6) has a parametric dependence on the complex deformation parameters of $\tilde{X}$ and can be identified with the family of *canonical forms*

$$O_j(x, y) + \kappa_j = 0 \,, \quad j = 1, \ldots, g_\Sigma \,, \tag{2.20}$$

where $O_j(x, y)$ is a polynomial in the exponential variables e^x, $\mathrm{e}^y \in \mathbb{C}^*$ whose coefficients depend on the mass parameters $\boldsymbol{\xi}$. Different canonical forms are related by $\mathsf{SL}_2(\mathbb{Z})$-transformations and global translations in x, $y \in \mathbb{C}$. Therefore, dropping the explicit dependence on x, y to simplify the notation, we can write

$$O_j + \kappa_j = P_{jm} \left(O_m + \kappa_m \right) \,, \quad j, m = 1, \ldots, g_\Sigma \,, \tag{2.21}$$

where P_{jm} is an overall monomial of the form e^{ax+by} for some $a, b \in \mathbb{Z}$. Note that this is equivalent to[4]

$$O_j = O_j^{(0)} + \sum_{m \neq j} P_{jm} \kappa_m \,, \quad j = 1, \ldots, g_\Sigma \,, \tag{2.22}$$

where $O_j^{(0)}$ is a new polynomial in e^x, e^y with $\boldsymbol{\xi}$-dependent coefficients.

Following [11, 12, 88], the functions $O_j(x, y)$ appearing in Eq. (2.20) can be quantized by making an appropriate choice of reality conditions for the variables $x, y \in \mathbb{C}$, promoting x, y to self-adjoint Heisenberg operators x, y on the real line satisfying the commutation relation

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar \,, \tag{2.23}$$

and applying the standard Weyl prescription for ordering ambiguities. They are thus uniquely associated with g_Σ different self-adjoint *quantum-mechanical operators* O_j, $j = 1, \ldots, g_\Sigma$, acting on $L^2(\mathbb{R})$. The mass parameters $\boldsymbol{\xi}$ become parameters of the operators O_j, and a Planck constant $\hbar \in \mathbb{R}_{>0}$ is introduced as a quantum deformation parameter. Consequently, the classical mirror map $t_i(\boldsymbol{z})$ in Eq. (2.9) is promoted to a *quantum mirror map* $t_i(\boldsymbol{z}, \hbar)$ given by

$$-t_i(\boldsymbol{z}, \hbar) = \log z_i + \Pi_i(\boldsymbol{z}, \hbar), \,, \quad i = 1, \ldots, s \,, \tag{2.24}$$

[4]The function $O_j^{(0)}$ is interpreted as an unperturbed version of O_j. The latter is recovered when the relevant corrections encoded in the complex deformation parameters κ_m, $m \neq j$, are added to $O_j^{(0)}$ with weights given by the corresponding monomials P_{jm}.

which reproduces the conventional mirror map in the semiclassical limit $\hbar \to 0$ and is determined as an A-period of a quantum-corrected version of the differential λ in Eq. (2.11) obtained via the all-orders, perturbative WKB approximation [104, 105]. We remark that this quantization scheme can be applied to a toric del Pezzo CY threefold. In this case, there is a single self-adjoint operator O_S acting on $L^2(\mathbb{R})$, whose inverse is denoted by $\rho_S = \mathsf{O}_S^{-1}$.

Let us now introduce the quantum operators P_{jm}, $\mathsf{O}_j^{(0)}$ associated with the functions P_{jm}, $O_j^{(0)}$ appearing in Eqs. (2.21) and (2.22), respectively. Note that they satisfy the transformations

$$\mathsf{P}_{jm} = \mathsf{P}_{mj}^{-1}\,, \quad \mathsf{P}_{jm} = \mathsf{P}_{ji}^{1/2}\mathsf{P}_{im}\mathsf{P}_{ji}^{1/2}\,, \quad \mathsf{O}_j^{(0)} = \mathsf{P}_{jm}^{-1/2}\mathsf{O}_m^{(0)}\mathsf{P}_{jm}^{1/2}\,, \quad j \neq m\,, \tag{2.25}$$

while we fix $\mathsf{P}_{jj} = 1$ by convention. We then quantize the relation in Eq. (2.21), which gives

$$\mathsf{O}_j + \kappa_j = \mathsf{P}_{jm}^{1/2}\left(\mathsf{O}_m + \kappa_m\right)\mathsf{P}_{jm}^{1/2}\,, \quad j, m = 1, \ldots, g_\Sigma\,, \tag{2.26}$$

and define the inverse operators

$$\rho_j = \mathsf{O}_j^{-1}\,, \quad \rho_j^{(0)} = \left(\mathsf{O}_j^{(0)}\right)^{-1}\,, \quad j = 1, \ldots, g_\Sigma\,, \tag{2.27}$$

again acting on $L^2(\mathbb{R})$. It was conjectured in [11, 12], and rigorously proved in [106] for a large number of toric del Pezzo CY threefolds—including the examples considered in this thesis, that the inverse operators ρ_j are positive-definite and of *trace class*, therefore possessing discrete, positive spectra, provided that the mass parameters $\boldsymbol{\xi}$ of the mirror manifold satisfy suitable reality and positivity conditions.

As shown in [12, 88], there is a consistent way to define a *generalized spectral (or Fredholm) determinant* $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$ associated with the set of trace-class operators ρ_j, $j = 1, \ldots, g_\Sigma$, or equivalently with the toric CY threefold X. Namely, we introduce the auxiliary trace-class operators

$$\mathsf{A}_{jm} = \rho_j^{(0)}\mathsf{P}_{jm}\,, \quad j, m = 1, \ldots, g_\Sigma\,, \tag{2.28}$$

and fix an arbitrary value of the index j. Then,

$$\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar) = \det\left(1 + \kappa_1\mathsf{A}_{j1} + \cdots + \kappa_{g_\Sigma}\mathsf{A}_{jg_\Sigma}\right)\,, \tag{2.29}$$

which is independent of the choice of the index j thanks to the property

$$\mathsf{A}_{jm} = \mathsf{P}_{ji}^{-1/2}\mathsf{A}_{im}\mathsf{P}_{ji}^{1/2}\,. \tag{2.30}$$

which follows from Eqs. (2.25) and (2.28). Note that the generalized spectral determinant in Eq. (2.29) can be identified with the standard Fredholm determinant of the quantum operator $\kappa_1\mathsf{A}_{j1} + \cdots + \kappa_{g_\Sigma}\mathsf{A}_{jg_\Sigma}$, which is conjecturally positive definite and of trace class [107], and is then an entire function on the moduli space of complex structures of $\tilde{X}$ parametrized by $\boldsymbol{\kappa}$. In particular, the power series expansion of the analytically continued spectral determinant $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$ around the point $\boldsymbol{\kappa} = 0$, known as the *orbifold point*, is well-defined. Namely, we can write

$$\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)\, \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}}\,, \tag{2.31}$$

where $\boldsymbol{N} = (N_1, \ldots, N_{g_\Sigma}) \in \mathbb{N}^{g_\Sigma}$ and the coefficient functions $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$ are the *fermionic spectral traces* of the toric CY threefold X. We also fix

$$Z(0, \ldots, 0, \boldsymbol{\xi}, \hbar) = 1 \,. \tag{2.32}$$

Classical results in Fredholm theory [107–109] provide an explicit determinant expression for the fermionic spectral traces. Let us introduce the non-negative integer

$$N = \sum_{i=1}^{g_\Sigma} N_i \tag{2.33}$$

and define the matrix

$$R_j(x_m, x_n) = A_{jk}(x_m, x_n) \quad \text{if} \quad \sum_{i=1}^{k-1} N_i < m \leq \sum_{i=1}^{k} N_i \,, \tag{2.34}$$

where $A_{jk}(x_m, x_n) = \langle x_m | \mathsf{A}_{jk} | x_n \rangle$ represents the integral kernel of the operator A_{jk} for $m, n = 1, \ldots, N$. Then, the fermionic spectral trace $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$ is given by the multi-cut matrix model integral[5]

$$Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar) = \frac{1}{N_1! \cdots N_{g_\Sigma}!} \int \det_{m,n} \left(R_j(x_m, x_n) \right) d^N x \,, \tag{2.35}$$

which does not depend on the choice of the index $1 \leq j \leq g_\Sigma$. Finally, note that the spectral determinants of the individual operators ρ_j in Eq. (2.27) are uniquely determined by the generalized spectral determinant $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$ as

$$\det(1 + \kappa_j \rho_j) = \frac{\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)}{\Xi(\kappa_1, \ldots, \kappa_{j-1}, 0, \kappa_{j+1}, \ldots, \kappa_{g_\Sigma}, \boldsymbol{\xi}, \hbar)} \,, \quad j = 1, \ldots, g_\Sigma \,, \tag{2.36}$$

which also implies that

$$\det(1 + \kappa_j \rho_j^{(0)}) = \Xi(0, \ldots, 0, \kappa_j, 0, \ldots, 0, \boldsymbol{\xi}, \hbar) \,, \quad j = 1, \ldots, g_\Sigma \,. \tag{2.37}$$

Recall that the spectral determinant of the positive-definite and trace-class operator ρ_j can be alternatively written as the infinite product[6]

$$\det(1 + \kappa_j \rho_j) = \prod_{n=0}^{\infty} \left(1 + \kappa_j \mathrm{e}^{-E_n^{(j)}} \right) \,, \tag{2.38}$$

where $\mathrm{e}^{-E_n^{(j)}}$ for $n \in \mathbb{N}$ are the eigenvalues of the operator. Thus, it follows from Eq. (2.36) that the vanishing space of $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$, which is a codimension one submanifold of the moduli space, fully determines the spectrum of eigenvalues $\mathrm{e}^{-E_n^{(j)}}$ of each of the quantum operators ρ_j as a function of the complex moduli κ_m, $m \neq j$.

[5] The connection between fermionic spectral traces and matrix models has been developed in [110, 111].

[6] Note that the spectral determinant of ρ_j can be physically interpreted as the grand canonical partition function of an ideal Fermi gas where the one-particle problem has energy levels $E_n^{(j)}$.

2.3 Basics of topological string theory

Let X be a CY threefold with $s = b_2(X)$. The *total free energy* of the *A-model conventional topological string* compactified on X is formally given by the generating series[7]

$$F^{\mathrm{WS}}(\boldsymbol{t}, g_s) = \sum_{g\geq 0} F_g(\boldsymbol{t})\, g_s^{2g-2}\,, \tag{2.39}$$

where g_s is the topological string coupling constant and $F_g(\boldsymbol{t})$ is the free energy at fixed worldsheet genus $g \geq 0$ as a function of the flat coordinates t_i, $i = 1, \ldots, s$, on the moduli space of Kähler structures of X. In a common region of convergence near the so-called *large radius limit* $\Re(t_i) \gg 1$, the topological string amplitudes $F_g(\boldsymbol{t})$ are well-defined formal power series in e^{-t_i}. Explicitly,

$$F_0(\boldsymbol{t}) = \frac{1}{6} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{\boldsymbol{d}} N_0^{\boldsymbol{d}}\, \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}}\,, \tag{2.40a}$$

$$F_1(\boldsymbol{t}) = \sum_{i=1}^{s} b_i t_i + \sum_{\boldsymbol{d}} N_1^{\boldsymbol{d}}\, \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}}\,, \tag{2.40b}$$

$$F_g(\boldsymbol{t}) = C_g + \sum_{\boldsymbol{d}} N_g^{\boldsymbol{d}}\, \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}}\,, \quad g \geq 2\,, \tag{2.40c}$$

where the degree vector $\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s$ represents a class in the two-homology group $\mathrm{H}_2(X, \mathbb{Z})$ and $N_g^{\boldsymbol{d}} \in \mathbb{Q}$ is the *Gromov–Witten invariant* of X at genus g and degree $\boldsymbol{d}$. The coefficients a_{ijk}, b_i are cubic and linear couplings characterizing the perturbative genus-zero and genus-one amplitudes, while C_g is the so-called constant map contribution [36, 37]. Yet, at a fixed point $\boldsymbol{t}$ in the common region of convergence, the numerical series $F_g(\boldsymbol{t})$ appear to diverge factorially as $(2g)!$ [30, 112], and the total free energy $F^{\mathrm{WS}}(\boldsymbol{t}, g_s)$ does not, in general, define a function of $\boldsymbol{t}$ and g_s. It can, however, be resummed order by order in e^{-t_i} and at all orders in g_s, yielding the expression [113]

$$F^{\mathrm{WS}}(\boldsymbol{t}, g_s) = \frac{1}{6g_s^2} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} b_i t_i + \sum_{g\geq 2} C_g g_s^{2g-2} + F^{\mathrm{GV}}(\boldsymbol{t}, g_s)\,, \tag{2.41}$$

where $F^{\mathrm{GV}}(\boldsymbol{t}, g_s)$ is given by the formal power series

$$F^{\mathrm{GV}}(\boldsymbol{t}, g_s) = \sum_{g\geq 0} \sum_{\boldsymbol{d}} \sum_{w=1}^{\infty} n_g^{\boldsymbol{d}} \frac{1}{w} \left(2 \sin \frac{w g_s}{2}\right)^{2g-2} \mathrm{e}^{-w\boldsymbol{d}\cdot\boldsymbol{t}}\,, \tag{2.42}$$

where $n_g^{\boldsymbol{d}} \in \mathbb{Z}$ is the *Gopakumar–Vafa invariant* of X at genus g and degree $\boldsymbol{d}$.

When the target CY threefold X is toric, the topological string partition function can be engineered as a special limit of the instanton partition function of Nekrasov [114]. A more general theory, known as *refined topological string theory* [115–117], is constructed by splitting the string coupling constant g_s into two independent parameters as

$$g_s^2 = -\epsilon_1 \epsilon_2\,, \tag{2.43}$$

[7] The superscript WS in Eq. (2.39) stands for worldsheet.

where ϵ_1, ϵ_2 correspond to the two equivariant rotations of the space-time $\mathbb{C}^2$. Together with g_s, a second coupling constant

$$\hbar = \epsilon_1 + \epsilon_2 \tag{2.44}$$

is introduced and identified with the quantum deformation parameter that appears in the quantization of the spectral curve in Eq. (2.6) in the mirror B-model. The total free energy of the A-model refined topological string on X at large radius has a double perturbative expansion in g_s and $\hbar$ of the form [118–120]

$$F(\boldsymbol{t}, \epsilon_1, \epsilon_2) = \sum_{g,n \geq 0} F_{g,n}(\boldsymbol{t})\, g_s^{2g-2}\, \hbar^{2n}\,, \tag{2.45}$$

from which the genus expansion of the standard topological string in Eq. (2.39) is recovered in the limit $g_s = \epsilon_1 = -\epsilon_2$, and we have that

$$F_g(\boldsymbol{t}) = F_{g,0}(\boldsymbol{t})\,, \quad g \geq 0\,. \tag{2.46}$$

Another remarkable one-parameter specialization of the refined theory is obtained when one of the two equivariant parameters ϵ_1, ϵ_2 is set to zero and the other is kept finite, *e.g.*, $\epsilon_2 \to 0$ while $\hbar = \epsilon_1$ is fixed, which is known as the *NS limit* [121]. Since the refined total free energy in Eq. (2.45) has a simple pole in this regime, the NS total free energy is defined as the one-parameter generating series[8]

$$F^{\mathrm{NS}}(\boldsymbol{t}, \hbar) = \lim_{\epsilon_2 \to 0} -\epsilon_2 F(\boldsymbol{t}, \epsilon_1, \epsilon_2) = \sum_{n \geq 0} F_n^{\mathrm{NS}}(\boldsymbol{t})\, \hbar^{2n-1}\,, \tag{2.47}$$

where the NS topological amplitude at fixed order n in $\hbar$ is given by

$$F_n^{\mathrm{NS}}(\boldsymbol{t}) = F_{0,n}(\boldsymbol{t})\,, \quad n \geq 0\,. \tag{2.48}$$

In the refined framework, the Gopakumar–Vafa invariants are generalized to a wider set of integer enumerative invariants, called the *refined BPS invariants* [122, 123]. We denote them by $N^{\boldsymbol{d}}_{j_{\mathrm{L}}, j_{\mathrm{R}}}$, where $j_{\mathrm{L}}, j_{\mathrm{R}}$ are two non-negative half-integers and $\boldsymbol{d}$ is the degree vector. The perturbative expansion at large radius of the NS total free energy is expressed as the generating functional

$$\begin{aligned} F^{\mathrm{NS}}(\boldsymbol{t}, \hbar) =& \frac{1}{6\hbar} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \hbar \sum_{i=1}^{s} b_i^{\mathrm{NS}} t_i \\ &+ \sum_{j_{\mathrm{L}}, j_{\mathrm{R}}} \sum_{\boldsymbol{d}} \sum_{w=1}^{\infty} N^{\boldsymbol{d}}_{j_{\mathrm{L}}, j_{\mathrm{R}}} \frac{\sin \frac{\hbar w}{2}(2j_{\mathrm{L}}+1) \sin \frac{\hbar w}{2}(2j_{\mathrm{R}}+1)}{2w^2 \sin^3 \frac{\hbar w}{2}} \mathrm{e}^{-w \boldsymbol{d} \cdot \boldsymbol{t}}\,, \end{aligned} \tag{2.49}$$

which reproduces Eq. (2.47) when expanded in powers of $\hbar$. The coefficients a_{ijk} are the same ones that appear in Eq. (2.42), while the constants b_i^{NS} can be obtained via mirror symmetry [116, 117].

Furthermore, we have that $F_0^{\mathrm{NS}}(\boldsymbol{t}) = F_0(\boldsymbol{t})$, and the higher-order NS free energies are given by the perturbative WKB quantum corrections to the classical prepotential [104, 105]—namely, $F^{\mathrm{NS}}(\boldsymbol{t}, \hbar)$ can be identified with the quantum prepotential associated with

[8] The superscript NS in Eq. (2.47) stands for Nekrasov–Shatashvili.

the quantum-deformed version of the classical B-period in Eq. (2.12). However, the perturbative WKB quantization condition determined by the NS topological string alone proves insufficient to fully solve the spectral problem associated with the quantum operators ρ_j, $j = 1, \ldots, g_\Sigma$, defined in Eq. (2.27). An *exact quantization condition* [11, 124] is obtained by adding an infinite series of complex instanton corrections [125, 126], which are non-perturbative in $\hbar$ and encoded in the standard topological string. Conversely, the non-perturbative sector in g_s of $F^{\mathrm{WS}}(\boldsymbol{t}, g_s)$ is captured by the NS limit of the refined topological string. Thus, the worldsheet and NS total free energies in Eqs. (2.41) and (2.49) can be regarded as non-perturbative corrections of one another in the appropriate regimes [32, 33].

The total grand potential

We recall that the *total grand potential* of topological string theory on X is defined as the sum [127]

$$J(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = J^{\mathrm{WS}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) + J^{\mathrm{WKB}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)\,. \tag{2.50}$$

The *worldsheet grand potential* is obtained from the generating functional of Gopakumar–Vafa invariants of X in Eq. (2.42) as

$$J^{\mathrm{WS}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}\boldsymbol{t}(\hbar) + \pi \mathrm{i}\boldsymbol{B}, \frac{4\pi^2}{\hbar}\right)\,, \tag{2.51}$$

where $\boldsymbol{t}(\hbar)$ is the quantum mirror map in Eq. (2.24), and $\boldsymbol{B}$ is a constant vector determined by the geometry, called *B-field*, whose presence has the effect of introducing a sign $(-1)^{\boldsymbol{w}\boldsymbol{d}\cdot\boldsymbol{B}}$ in the generating series in Eq. (2.42). The all-genus worldsheet generating functional above encodes the non-perturbative contributions in $\hbar$ due to complex instantons, which are contained in the standard topological string. Note that there is a *strong-weak coupling duality* between the spectral theory of the operators arising from the quantization of the mirror curve Σ and the standard topological string theory on X. Namely,

$$g_s = \frac{4\pi^2}{\hbar}\,. \tag{2.52}$$

The *WKB grand potential* is obtained from the generating functional of refined BPS invariants in Eq. (2.49) as

$$\begin{aligned} J^{\mathrm{WKB}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = &\sum_{i=1}^{s} \frac{t_i(\hbar)}{2\pi}\frac{\partial F^{\mathrm{NS}}(\boldsymbol{t}(\hbar), \hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi}\frac{\partial}{\partial \hbar}\left(\frac{1}{\hbar}F^{\mathrm{NS}}(\boldsymbol{t}(\hbar), \hbar)\right) \\ &+ \frac{2\pi}{\hbar}\sum_{i=1}^{s} b_i t_i(\hbar) + A(\boldsymbol{\xi}, \hbar)\,, \end{aligned} \tag{2.53}$$

where the derivative with respect to $\hbar$ in the second term in the RHS does not act on the $\hbar$-dependence of the quantum mirror map $\boldsymbol{t}(\hbar)$. The coefficients b_i are the same ones appearing in Eq. (2.41), while the function $A(\boldsymbol{\xi}, \hbar)$ is not known in closed form for arbitrary geometries, although it has been conjectured in many examples. The all-orders WKB generating functional above contains the perturbative corrections in $\hbar$ to the quantum-mechanical spectral problem associated with X, which are captured by the NS topological string on X.

The total grand potential can be expressed as a formal power series expansion in the large radius limit $\Re(t_i) \gg 1$ with the structure

$$J(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = \frac{1}{12\pi\hbar} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} \left(\frac{2\pi}{\hbar} b_i + \frac{\hbar}{2\pi} b_i^{\mathrm{NS}} \right) t_i + \mathcal{O}(\mathrm{e}^{-t_i}, \mathrm{e}^{-2\pi t_i/\hbar}) , \quad (2.54)$$

where the infinitesimally small corrections in e^{-t_i}, $\mathrm{e}^{-2\pi t_i/\hbar}$ have $\hbar$-dependent coefficients. Observe that the trigonometric functions that enter the total grand potential through Eqs. (2.41) and (2.49) have double poles for $\hbar \in \pi\mathbb{Q}$. Yet, a cancellation mechanism guarantees convergence of the sum [127]. Rigorous results on the convergence properties of the expansion in Eq. (2.54) are missing. However, extensive evidence suggests that it is analytic in a neighborhood of $t_i \to \infty$ when $\hbar$ is real [127, 128], while it appears to inherit the divergent behavior of the generating functionals in Eqs. (2.41) and (2.49) for arbitrary complex values of $\hbar$.

2.4 From topological strings to quantum operators and back

Based on the previous insights of [125–127, 129–131], a conjectural duality has been recently proposed in [11, 12], relating the topological string theory on a toric CY threefold to the spectral theory of the quantum-mechanical operators coming from the quantization of its mirror curve. This is known as the *TS/ST correspondence* and is now supported by an increasing amount of evidence obtained in applications to concrete examples [106, 110, 111, 132, 133]. We refer to the detailed review in [134] and references therein.

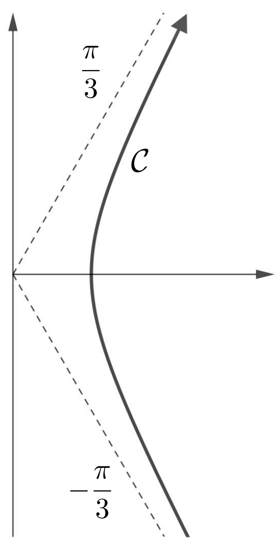


Figure 2.1: The integration contour $\mathcal{C}$ in the complex plane of the chemical potential μ. Note that $\mathcal{C}$ is the standard contour used for the integral representation of the Airy function.

The main conjectural statement of the TS/ST correspondence provides exact expressions for the spectral determinant and the fermionic spectral traces of the collection of self-adjoint operators ρ_j, $j = 1, \ldots, g_\Sigma$, introduced in Eq. (2.27) in terms of the standard and NS topological string amplitudes on X. More precisely, the spectral determinant $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$ defined in Eq. (2.29) satisfies

$$\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar) = \sum_{\boldsymbol{n} \in \mathbb{Z}^{g_\Sigma}} \exp(J(\boldsymbol{\mu} + 2\pi \mathrm{i} \boldsymbol{n}, \boldsymbol{\xi}, \hbar)) = \mathrm{e}^{J(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)} \Theta(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) \,, \tag{2.55}$$

where the sum over $\boldsymbol{n} \in \mathbb{Z}^{g_\Sigma}$ produces a periodic function in the chemical potentials μ_j that can be equivalently recast by factoring out a *quantum-deformed Riemann theta function*[9] $\Theta(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)$. It follows that the fermionic spectral traces $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$, $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, which have been introduced in Eq. (2.31), are determined by the orbifold expansion of the topological string theory on X. Note that the expression in the RHS of Eq. (2.55) can be interpreted as a well-defined large-μ_j expansion in powers of $\mathrm{e}^{-\mu_j}$, $\mathrm{e}^{-2\pi \mu_j / \hbar}$. Indeed, the total grand potential and the quantum theta function appear to have a common region of convergence in a neighborhood of the limit $\mu_j \to \infty$, which corresponds to the large radius point in the moduli space of X. However, being the spectral determinant an entire function of $\boldsymbol{\kappa}$, the conjecture in Eq. (2.55) implies that such a product in the RHS is, indeed, entire in $\boldsymbol{\mu}$. Moreover, Eqs. (2.55) and (2.31) lead to an integral formula for the fermionic spectral traces as appropriate residues at the origin $\boldsymbol{\kappa} = 0$. Namely, [11, 12, 129]

$$Z_X(\boldsymbol{N}, \boldsymbol{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma} \exp(J(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) - \boldsymbol{N} \cdot \boldsymbol{\mu}) \,, \tag{2.56}$$

where the integration contour along the imaginary axis can be suitably deformed to make the integral convergent. For example, the appropriate choice in genus one turns out to be the contour $\mathcal{C}$ going from $\mathrm{e}^{-\mathrm{i}\pi/3}\infty$ to $\mathrm{e}^{\mathrm{i}\pi/3}\infty$ in the complex plane of the chemical potential, as shown in Fig. 2.1. Because of the trace-class property of the quantum operators ρ_j, the fermionic spectral traces $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$ are well-defined functions of $\hbar \in \mathbb{R}_{>0}$. Besides, although being initially defined only for non-negative integer values of N_j, the Airy-type integral in Eq. (2.56) naturally extends them to entire functions of $\boldsymbol{N} \in \mathbb{C}^{g_\Sigma}$ [128]. In what follows, for simplicity, we will drop from our notation the explicit dependence on $\boldsymbol{\xi}$ of $\Xi(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar)$ and $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$.

[9] The quantum theta function $\Theta(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar)$ becomes a classical theta function in the so-called maximally supersymmetric case of $\hbar = 2\pi$ [11].

Part II

Resurgence of the spectral theory

Chapter 3

Topological strings beyond perturbation theory

Let X be a toric CY threefold. In this chapter, building on the recent works of [10, 38, 42, 75, 76, 135], we apply the machinery of Chapter 1 to the construction of Chapter 2 and conjecture the resurgent structure of asymptotic series that arise naturally as perturbative expansions in the strongly coupled regime of the topological string on X—this involves a collection of rational Stokes constants representing a new class of enumerative invariants of the manifold. An analogous conjecture applies to the weakly coupled regime.

Although the definition of the proposed invariants as Stokes constants of appropriate perturbative expansions in topological string theory does not rely on the existence of a non-perturbative completion for the asymptotic series of interest, taking the perspective of the fermionic spectral traces significantly facilitates the computational tasks undertaken in this thesis, since they can be expressed as matrix integrals and factorized into holomorphic/anti-holomorphic blocks. After introducing the geometry of the two simplest and best-known examples of toric CY threefolds with genus-one mirror curve, that is, local $\mathbb{P}^2$ and local $\mathbb{F}_0$, we compute the all-orders semiclassical perturbative expansions of their first fermionic spectral traces and perform a preliminary numerical investigation of their resurgent structures. In the case of local $\mathbb{P}^2$, the numerical analysis anticipates features of the full analytic solution derived in Chapter 5. In the case of local $\mathbb{F}_0$, the resurgent structure is currently accessible via numerical methods only, and we unveil the presence of logarithmic-type terms in the sub-leading asymptotics. We reproduce [1, Sections 3 and 5] and part of [1, Section 4].

3.1 The resurgent structures of the spectral traces

Let us go back to the fermionic spectral traces $Z(\boldsymbol{N}, \hbar)$, $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, in Eq. (2.35) and perturbatively expand them in the limits $\hbar \to 0$ and $\hbar \to \infty$ with $\boldsymbol{N}$ fixed. We construct the two families of asymptotic expansions

$$\log Z(\boldsymbol{N}, \hbar) \sim \phi_{\boldsymbol{N}}(\hbar) \qquad \text{for } \hbar \to 0\,, \tag{3.1a}$$

$$\log Z(\boldsymbol{N}, \hbar) \sim \psi_{\boldsymbol{N}}(\hbar^{-1}) \qquad \text{for } \hbar \to \infty\,, \tag{3.1b}$$

indexed by the vector of non-negative integers $\boldsymbol{N}$. Under the assumption that the formal power series in Eq. (3.1) are Gevrey-1 and simple resurgent, which is explicitly verified in the examples considered in this thesis, the theory of resurgence can be applied to give us access to the non-analytic sectors that are hidden in perturbation theory. The fundamental notions and tools of resurgence are described in Chapter 1. To perform a resurgent analysis of the fermionic spectral traces, we consider their analytic continuation to $\hbar \in \mathbb{C}'$.

Remark 3.1.1. *In its original formulation, the TS/ST correspondence of [11, 12], and therefore also the statement in Eq. (2.56), applies to* $\hbar \in \mathbb{R}_{>0}$*. The issue of the complexification of* $\hbar$*, or, equivalently, of* g_s*, in the context of the TS/ST correspondence, has been addressed in various studies since [136–139]. It follows from the work of [106] that the integral kernels of a large class of examples of toric CY threefolds, including the ones considered in this thesis, admit a well-defined analytic continuation to* $\hbar \in \mathbb{C}'$*. Moreover, the analytically continued fermionic spectral traces obtained in this way match the natural analytic continuation of the total grand potential of the topological string to complex values of* $\hbar$ *such that* $\Re(\hbar) > 0$*. The TS/ST correspondence is, then, still applicable. Moreover, in light of the strong-weak duality between the topological string coupling constant* g_s *and the quantum deformation parameter* $\hbar$ *in Eq. (2.52), the regimes of* $\hbar \to 0$ *and* $\hbar \to \infty$ *of the spectral theory correspond to the strong and weak coupling limits of the dual topological string theory, respectively, via the TS/ST correspondence. We will discuss in Chapter 4 how the formal power series in Eq. (3.1) can be independently defined from the topological string via the integral in Eq. (2.56).*

Building on numerical evidence obtained in some concrete genus-one examples, it was conjectured by Gu and Mariño in [10] that the Stokes constants appearing in the resurgent structure of the asymptotic series $\exp(\psi_{\boldsymbol{N}}(\hbar^{-1}))$ for $\boldsymbol{N}$ fixed, that is, the perturbative expansion in g_s of the fermionic spectral traces directly, are non-trivial integer invariants of the geometry related to the counting of BPS states. The same resurgent machine advocated in [10] is applied here to the dual asymptotic series $\phi_{\boldsymbol{N}}(\hbar)$ and $\psi_{\boldsymbol{N}}(\hbar^{-1})$ for $\boldsymbol{N}$ fixed. More precisely, we formulate a conjectural proposal for the resurgent structure of the asymptotic series in Eq. (3.1) and describe a universal construction underlying the non-perturbative corrections to the strong and weak coupling perturbative expansions of the fermionic spectral traces for toric CY threefolds. Explicit results in support of the conjecture are obtained in genus-one examples. Let us now detail our statements following [1, Section 3.2].

For fixed $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, each of the formal power series in Eq. (3.1a), which emerge in the semiclassical limit $\hbar \to 0$ of the spectral theory, is associated with a *minimal resurgent structure* $\mathfrak{B}_{\phi_{\boldsymbol{N}}}$ of the form

$$\mathfrak{B}_{\phi_{\boldsymbol{N}}} = \{\Phi_{\sigma,n;\boldsymbol{N}}(\hbar)\}_{\sigma=0,\dots,l_0;\, n\in\mathbb{Z}} \,, \tag{3.2}$$

where the non-negative integer l_0 and the complex constant $\mathcal{A}_0$ depend on the choice of $\boldsymbol{N}$ and on the geometry, while

$$\Phi_{\sigma,n;\boldsymbol{N}}(\hbar) = \phi_{\sigma;\boldsymbol{N}}(\hbar) \mathrm{e}^{-n\frac{\mathcal{A}_0}{\hbar}} \,. \tag{3.3}$$

Namely, for each value of $\sigma \in \{0, \dots, l_0\}$, a Gevrey-1 asymptotic series $\phi_{\sigma;\boldsymbol{N}}(\hbar)$ resurges from the original perturbative expansion

$$\phi_{\boldsymbol{N}}(\hbar) = \phi_{0;\boldsymbol{N}}(\hbar) \,. \tag{3.4}$$

The singularities of the Borel transform $\hat{\phi}_{\sigma;\boldsymbol{N}}(\zeta)$ are located along infinite towers in the complex ζ-plane so that every two singularities in the same tower are spaced by an integer multiple of $\mathcal{A}_0$. The global arrangement of the complete set of Borel singularities of $\mathfrak{B}_{\phi_{\boldsymbol{N}}}$ is known as a *peacock pattern*[1]. See Fig. 3.1 for a schematic illustration. In this way, each asymptotic series $\phi_{\sigma;\boldsymbol{N}}(\hbar)$ gives rise to an infinite family of basic trans-series $\Phi_{\sigma,n;\boldsymbol{N}}(\hbar)$, $n \in \mathbb{Z}$. For fixed $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, the corresponding infinite-dimensional matrix of Stokes

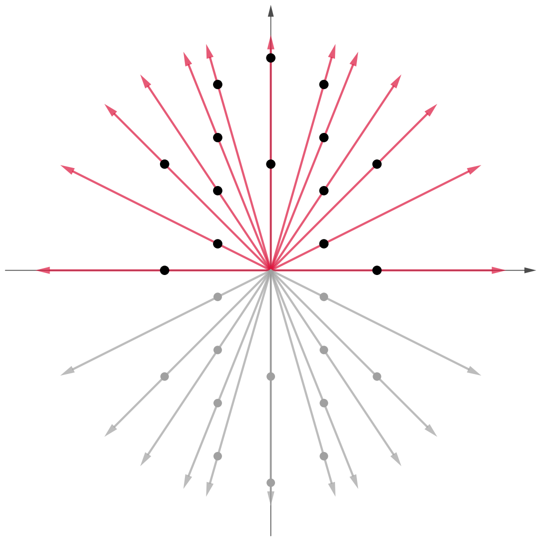


Figure 3.1: Infinite towers of singularities and the peacock arrangement of Stokes rays in the complex Borel plane.

constants $\mathcal{S}_{\phi_{\boldsymbol{N}}}$ can be written as

$$\mathcal{S}_{\phi_{\boldsymbol{N}}} = \{S_{\sigma,\sigma',n;\boldsymbol{N}} \in \mathbb{Q}\}_{\sigma,\sigma'=0,\ldots,l_0;\, n\in\mathbb{Z}}\,, \tag{3.5}$$

after fixing a canonical normalization of the asymptotic series $\phi_{\sigma;\boldsymbol{N}}(\hbar)$. We conjecture that the Stokes constants $S_{\sigma,\sigma',n;\boldsymbol{N}}$ are closely related to non-trivial sequences of integers, thus representing a new class of *enumerative invariants* of the topological string on X. Furthermore, they can be naturally organized as the coefficients of generating functions that are expressible in the form of *q-series* and are uniquely determined by the original perturbative expansion in Eq. (3.1a). Schematically,

$$S_{\sigma\sigma';\boldsymbol{N}}(q) = \sum_{n\in\mathbb{Z}} S_{\sigma\sigma',n;\boldsymbol{N}}\, q^n\,. \tag{3.6}$$

The conjectural proposal for each of the formal power series in Eq. (3.1b), which emerge in the strongly coupled limit $\hbar \to \infty$ of the spectral theory, is entirely analogous. At fixed $\boldsymbol{N} \in \mathbb{N}^{g_\Sigma}$, the minimal resurgent structure $\mathfrak{B}_{\psi_{\boldsymbol{N}}}$ is given by

$$\mathfrak{B}_{\psi_{\boldsymbol{N}}} = \{\Psi_{\sigma,n;\boldsymbol{N}}(\hbar^{-1})\}_{\sigma=0,\ldots,l_\infty;\, n\in\mathbb{Z}}\,, \tag{3.7}$$

where again $l_\infty \in \mathbb{Z}_{>0}$ and $\mathcal{A}_\infty \in \mathbb{C}$ depend on the choice of $\boldsymbol{N}$ and on the geometry, while

$$\Psi_{\sigma,n;\boldsymbol{N}}(\hbar^{-1}) = \psi_{\sigma;\boldsymbol{N}}(\hbar^{-1})\mathrm{e}^{-n\mathcal{A}_\infty\hbar}\,, \tag{3.8}$$

[1]Peacock patterns are typically observed in theories controlled by a quantum curve in exponentiated variables, including complex Chern–Simons theory on the complement of a hyperbolic knot [75, 76].

resulting in a dual peacock arrangement of singularities in the complex Borel plane. The infinite-dimensional matrix of Stokes constants $\mathcal{S}_{\psi_{\boldsymbol{N}}}$ is instead

$$\mathcal{S}_{\psi_{\boldsymbol{N}}} = \{R_{\sigma,\sigma',n;\boldsymbol{N}} \in \mathbb{Q}\}_{\sigma,\sigma'=0,\dots,l_\infty;\, n\in\mathbb{Z}} \,, \tag{3.9}$$

after fixing the normalization of the asymptotic series $\psi_{\sigma;\boldsymbol{N}}(\hbar^{-1})$. Once more, the Stokes constants $R_{\sigma,\sigma',n;\boldsymbol{N}}$ provide a new non-trivial class of enumerative invariants of X and can be naturally organized into the generating q-series

$$R_{\sigma\sigma';\boldsymbol{N}}(q) = \sum_{n\in\mathbb{Z}} R_{\sigma\sigma',n;\boldsymbol{N}}\, q^n \,, \tag{3.10}$$

which are uniquely determined by the original perturbative expansion in Eq. (3.1b).

We do not yet have a direct physical or geometric interpretation of the proposed invariants. However, the exact solution to the resurgent structures of the first fermionic spectral trace of local $\mathbb{P}^2$ in both limits $\hbar \to 0$ and $\hbar \to \infty$, which is presented in Chapter 5, sheds some light on this problem. When looking at the logarithm of the spectral trace $\mathrm{Tr}(\rho_{\mathbb{P}^2})$, the Stokes constants have a manifest and strikingly simple arithmetic meaning as divisor sum functions. The perturbative coefficients are encoded in well-known L-functions, while the duality between the weakly and strongly coupled scaling regimes emerges in number-theoretic form. Further developments are presented in Chapters 6 and 7. On the other hand, the Stokes constants for the exponentiated series appear to be generally complex numbers with no manifest interpretation, although expressible in terms of the Stokes constants of the series in Eq. (3.1) through a closed partition-theoretic formula. Nonetheless, as we will show in the concrete examples of Chapter 5, the analytic solution to the resurgent structures of the series in Eq. (3.1) can be translated into results on the corresponding exponentiated series $\exp(\phi_{\boldsymbol{N}}(\hbar))$ and $\exp(\psi_{\boldsymbol{N}}(\hbar^{-1}))$ by applying the tools of alien calculus that we have briefly recalled in Section 1.3.

Remark 3.1.2. *Let us stress that the TS/ST statement in Eq.* (2.56) *and the strong-weak coupling duality in Eq.* (2.52) *map the semiclassical $\hbar$-series in Eq.* (3.1a) *into a dual perturbative $\hbar$-expansion that is described by the NS limit of the refined topological string on X. Analogously, the asymptotic series for $\hbar \to \infty$ in Eq.* (3.1b) *is associated with a dual perturbative expansion in the weakly coupled regime $g_s \to 0$ that is captured by the conventional topological string on the same geometry. Notably, the standard and NS topological string free energies can be regarded as non-perturbative corrections of one another in the appropriate regimes [32, 33]. The interplay of the two one-parameter specializations of the refined topological string theory on a local CY threefold with the perturbative and non-perturbative contributions to the fermionic spectral traces in the dual limits in $\hbar$ is further discussed in Chapter 4 and Section 6.2.*

In what follows, we introduce and explore the two best-known examples of toric CY threefolds with genus-one mirror curve, that is, local $\mathbb{P}^2$ and local $\mathbb{F}_0$.

3.2 The example of local $\mathbb{P}^2$

The simplest example of a toric del Pezzo CY threefold is the total space of the canonical line bundle over the projective surface $\mathbb{P}^2$, that is,

$$X = \mathcal{O}(-3) \to \mathbb{P}^2 \,, \tag{3.11}$$

also known as *local* $\mathbb{P}^2$. The polyhedron $\Delta_{\mathbb{P}^2}$ associated with $\mathbb{P}^2$ is the convex hull of the origin together with the two-dimensional vectors

$$\nu^{(1)} = (1,0)\,, \quad \nu^{(2)} = (0,1)\,, \quad \nu^{(3)} = (-1,-1)\,, \tag{3.12}$$

as shown in Fig. 3.2. Correspondingly, the moduli space of complex structures of the mirror manifold $\tilde{X}$ is encoded in the one-parameter family of elliptic curves

$$O_{\mathbb{P}^2}(x,y) = \mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + \kappa = 0\,, \quad x, y \in \mathbb{C}\,, \tag{3.13}$$

where κ is the one true complex modulus of $\tilde{X}$, and there are no mass parameters. Eq. (3.13) describes a genus-one Riemann surface Σ embedded in $\mathbb{C}^* \times \mathbb{C}^*$, which is the mirror curve to local $\mathbb{P}^2$.

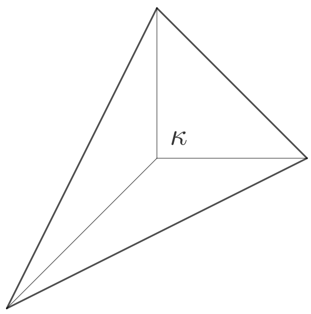


Figure 3.2: Toric diagram of local $\mathbb{P}^2$. We show the vectors in Eq. (3.12) and the polyhedron $\Delta_{\mathbb{P}^2}$. The complex modulus κ corresponds to the internal vertex of the diagram.

In the parametrization given by the Batyrev coordinate

$$z = 1/\kappa^3\,, \tag{3.14}$$

the *Picard–Fuchs differential equation* associated with the mirror of local $\mathbb{P}^2$ is [50]

$$\mathcal{L}_{\mathbb{P}^2}\mathcal{P} = \left[\Theta^3 - 3z(3\Theta+1)(3\Theta+2)\Theta\right]\mathcal{P} = 0\,, \tag{3.15}$$

where $\Theta = z d/dz$ and $\mathcal{P}$ is the full period vector of the meromorphic one-form λ. The Picard–Fuchs differential operator $\mathcal{L}_{\mathbb{P}^2}$ has three singular points, which are the *large radius point* at $z = 0$, the *conifold point* at $z = -1/27$, and the *orbifold point* at $1/z = 0$. In addition, by looking at the monodromy data of the Picard–Fuchs differential equation, the moduli space of complex structures of $\tilde{X}$ can be identified with the compactified quotient $\overline{\mathbb{H}/\Gamma_1(3)}$, where $\mathbb{H}$ denotes the upper half of the complex plane and $\Gamma_1(3)$ is the *congruence subgroup*

$$\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \,\middle|\, a, d \equiv_3 1\,, c \equiv_3 0 \right\}\,, \tag{3.16}$$

which is generated by the elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\,. \tag{3.17}$$

We refer to [51, Section 10.1] for a detailed discussion.

The canonical Weyl quantization scheme of [11] applied to the function $O_{\mathbb{P}^2}(x, y)$ in Eq. (3.13), together with an appropriate choice of reality conditions for the variables $x, y \in \mathbb{C}$, produces the self-adjoint quantum operator[2]

$$\mathsf{O}_{\mathbb{P}^2}(\mathsf{x}, \mathsf{y}) = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}} , \tag{3.18}$$

acting on $L^2(\mathbb{R})$, where x, y are self-adjoint Heisenberg operators satisfying the commutation relation $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$. A Planck constant $\hbar \in \mathbb{R}_{>0}$ is introduced as a quantum deformation parameter. It was conjectured in [11, 12] and later proven in [106] that the inverse operator

$$\rho_{\mathbb{P}^2} = \mathsf{O}_{\mathbb{P}^2}^{-1} \tag{3.19}$$

is positive-definite and of trace class. The fermionic spectral traces $Z_{\mathbb{P}^2}(N, \hbar)$, where N is a non-negative integer, are well-defined functions of $\hbar \in \mathbb{R}_{>0}$ and can be computed explicitly as multi-dimensional matrix model integrals [110]. Specifically,

$$\begin{aligned} Z_{\mathbb{P}^2}(N, \hbar) &= \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} (-1)^{\epsilon(\sigma)} \int \prod_{i=1}^{N} \rho_{\mathbb{P}^2}(x_i, x_{\sigma(i)}) \, d^N x \\ &= \frac{1}{N!} \int \det \left(\rho_{\mathbb{P}^2}(x_i, x_j) \right) \, d^N x \, , \end{aligned} \tag{3.20}$$

where $\mathcal{S}_N$ is the N-th permutation group, $\epsilon(\sigma)$ is the signature of the element $\sigma \in \mathcal{S}_N$, and $\rho_{\mathbb{P}^2}(x_i, x_j)$ denotes the *integral kernel* of the operator $\rho_{\mathbb{P}^2}$. This is known explicitly as [110]

$$\rho_{\mathbb{P}^2}(x, \, y) = \frac{\mathrm{e}^{\pi \mathsf{b}(x+y)/3}}{2\mathsf{b} \cosh(\pi(x-y)/\mathsf{b} + \mathrm{i}\pi/6)} \frac{\Phi_{\mathsf{b}}(y + \mathrm{i}\mathsf{b}/3)}{\Phi_{\mathsf{b}}(x - \mathrm{i}\mathsf{b}/3)} \, , \tag{3.21}$$

where b is related to $\hbar$ by

$$2\pi \mathsf{b}^2 = 3\hbar \tag{3.22}$$

and Φ_{b} denotes Faddeev's quantum dilogarithm. See Appendix B for its definition and fundamental properties. Note that the integral kernel in Eq. (3.21) is well-defined for

$$\hbar \in \mathbb{C}' = \mathbb{C} \setminus \mathbb{R}_{\leq 0} \, , \tag{3.23}$$

since Φ_{b} can be analytically continued to all values of b such that $\mathsf{b}^2 \notin \mathbb{R}_{\leq 0}$. Consequently, the same holds for the fermionic spectral traces in Eq. (3.20).

In the rest of this section, we will perform a numerical study of the resurgent structure of the first fermionic spectral trace

$$Z_{\mathbb{P}^2}(1, \hbar) = \mathrm{Tr}(\rho_{\mathbb{P}^2}) \tag{3.24}$$

in the semiclassical limit $\hbar \to 0$.

[2]The spectral theory of $\mathsf{O}_{\mathbb{P}^2}$ has been studied numerically in [131].

3.2.1 Computing the semiclassical expansion

Let us apply the phase-space formulation of quantum mechanics to obtain the WKB expansion of the trace of the inverse operator $\rho_{\mathbb{P}^2}$ at NLO in $\hbar \to 0$, starting from the explicit expression of the operator $\mathsf{O}_{\mathbb{P}^2}$ in Eq. (3.18) and following Appendix A. For simplicity, we denote by O_W, ρ_W the Wigner transforms of the operators $\mathsf{O}_{\mathbb{P}^2}$, $\rho_{\mathbb{P}^2}$, respectively. The Wigner transform of $\mathsf{O}_{\mathbb{P}^2}$ is obtained by performing the integration in Eq. (A.1) directly. As we show in Example A.0.1, this simply gives the classical function

$$O_{\mathrm{W}} = \mathrm{e}^{x} + \mathrm{e}^{y} + \mathrm{e}^{-x-y} \, . \tag{3.25}$$

Substituting it into Eqs. (A.11a) and (A.11b), we have

$$\mathcal{G}_2 = -\frac{\hbar^2}{4} \left[\mathrm{e}^{x+y} + \mathrm{e}^{-x} + \mathrm{e}^{-y}\right] + \mathcal{O}(\hbar^4) \, , \tag{3.26a}$$

$$\mathcal{G}_3 = -\frac{\hbar^2}{4} \mathrm{e}^{-2(x+y)} \left[-3\mathrm{e}^{2x+2y} + \mathrm{e}^{x+3y} + \mathrm{e}^{3x+4y} + \mathrm{e}^{x} + (x \leftrightarrow y)\right] + \mathcal{O}(\hbar^4) \, , \tag{3.26b}$$

where $(x \leftrightarrow y)$ indicates the symmetric expression after exchanging the variables x and y. It follows from Eq. (A.12) that the Wigner transform of $\rho_{\mathbb{P}^2}$ up to order $\hbar^2$ is then given by

$$\rho_{\mathrm{W}} = \frac{1}{O_{\mathrm{W}}} - \frac{9\hbar^2}{4} \frac{1}{O_{\mathrm{W}}^4} + \mathcal{O}(\hbar^4) \, . \tag{3.27}$$

We note that the same result can be obtained by solving Eq. (A.15) order by order in powers of $\hbar^2$. Integrating Eq. (3.27) over phase space, as in Eq. (A.3), we obtain the NLO perturbative expansion in $\hbar$ of the trace, that is,

$$\begin{aligned} \mathrm{Tr}(\rho_{\mathbb{P}^2}) &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} \rho_{\mathrm{W}} \, dx dy \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} \frac{1}{O_{\mathrm{W}}} \, dx dy - \frac{9\hbar}{8\pi} \int_{\mathbb{R}^2} \frac{1}{O_{\mathrm{W}}^4} \, dx dy + \mathcal{O}(\hbar^4) \, , \end{aligned} \tag{3.28}$$

and evaluating the integrals explicitly, we find

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{\Gamma\left(\frac{1}{3}\right)^3}{6\pi\hbar} \left\{1 - \frac{\hbar^2}{72} + \mathcal{O}(\hbar^4)\right\} \, , \tag{3.29}$$

where $\Gamma(z)$ denotes the Gamma function. We stress that the phase-space formalism adopted above provides, in principle, the perturbative expansion of $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ at all orders in $\hbar$ by systematically extending all intermediate computations beyond order $\hbar^2$. It is not, however, the most practical path.

Under the assumption that $\Re(\mathsf{b}) > 0$, the spectral trace of local $\mathbb{P}^2$ has the integral representation [110]

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathsf{b}} \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathsf{b} x/3} \frac{\Phi_{\mathsf{b}}(x + \mathsf{i}\mathsf{b}/3)}{\Phi_{\mathsf{b}}(x - \mathsf{i}\mathsf{b}/3)} \, dx \, , \tag{3.30}$$

which is an analytic function of $\hbar \in \mathbb{C}'$. The integral above can be evaluated using the integral Ramanujan formula or analytically continuing x to the complex domain, completing the integration contour from above, and summing over residues, which yields the closed formula [106]

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathsf{b}} \mathrm{e}^{-\frac{\pi \mathrm{i}}{36}(12c_{\mathsf{b}}^2+4\mathsf{b}^2-3)} \frac{\Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{\mathrm{i}\mathsf{b}}{3}\right)^2}{\Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{2\mathrm{i}\mathsf{b}}{3}\right)} = \frac{1}{\mathsf{b}} \left| \Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{\mathrm{i}\mathsf{b}}{3}\right)\right|^3 , \tag{3.31}$$

where $c_{\mathsf{b}} = \mathrm{i}(\mathsf{b}+\mathsf{b}^{-1})/2$. Moreover, following [10], the expression in Eq. (3.31) can be factorized into a product of $q, \tilde{q}$-*series* by applying the infinite product representation in Eq. (B.9). Namely, we have that

$$\Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{\mathrm{i}\mathsf{b}}{3}\right) = \frac{(q^{2/3};\, q)_\infty}{(w^{-1};\, \tilde{q})_\infty} , \quad \Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{2\mathrm{i}\mathsf{b}}{3}\right) = \frac{(q^{1/3};\, q)_\infty}{(w;\, \tilde{q})_\infty} , \tag{3.32}$$

where $(xq^\alpha;\, q)_\infty$ is the quantum dilogarithm defined in Eq. (B.1). Therefore,

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathsf{b}} \mathrm{e}^{-\frac{\pi\mathrm{i}}{36}\mathsf{b}^2 + \frac{\pi\mathrm{i}}{12}\mathsf{b}^{-2} + \frac{\pi\mathrm{i}}{4}} \frac{(q^{2/3};\, q)_\infty^2}{(q^{1/3};\, q)_\infty} \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty^2} , \tag{3.33}$$

where we have introduced

$$q = \mathrm{e}^{2\pi\mathrm{i}\mathsf{b}^2} = \mathrm{e}^{3\mathrm{i}\hbar} , \quad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\mathsf{b}^2} = \mathrm{e}^{-4\pi^2\mathrm{i}/(3\hbar)} , \quad w = \mathrm{e}^{2\pi\mathrm{i}/3} . \tag{3.34}$$

Note that the factorization into *holomorphic/anti-holomorphic blocks* in Eq. (3.33) is not symmetric in q, $\tilde{q}$. We further assume that $\Im(\mathsf{b}^2) > 0$, and therefore $\Im(\hbar) > 0$, which implies $|q|, |\tilde{q}| < 1$, so that the $q, \tilde{q}$-series converge.

Let us consider the formula in Eq. (3.33) and derive its all-orders perturbative expansion in the limit $\hbar \to 0$. The $\tilde{q}$-series giving the anti-holomorphic block in the factorized expression in Eq. (3.33) converge for $\hbar \to 0$. In particular, they contribute via the constant factor

$$\frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty^2} \sim \frac{1-w}{(1-w^{-1})^2} = \frac{-\mathrm{i}}{\sqrt{3}} . \tag{3.35}$$

Applying the asymptotic expansion formula for the quantum dilogarithm in Eq. (B.19b) with the choice of $\alpha = 1/3, 2/3$ and recalling the identities [140]

$$\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}\Gamma(1/3)^{-1} , \quad B_{2n+1} = 0, \quad B_n(1/3) = (-1)^n B_n(2/3) , \tag{3.36}$$

where $n \in \mathbb{N}$, we have that

$$\begin{aligned} 2\log(q^{2/3};\, q)_\infty - \log(q^{1/3};\, q)_\infty \sim & -\frac{\pi\mathrm{i}}{12}\mathsf{b}^{-2} - \frac{1}{2}\log(-2\pi\mathrm{i}\mathsf{b}^2) + \log\left(3\frac{\Gamma(1/3)^3}{(2\pi)^{3/2}}\right) \\ & + \frac{\pi\mathrm{i}}{36}\mathsf{b}^2 - 3\sum_{n=1}^{\infty}(2\pi\mathrm{i}\mathsf{b}^2)^{2n}\frac{B_{2n}B_{2n+1}(2/3)}{2n(2n+1)!} , \end{aligned} \tag{3.37}$$

where $B_n(z)$ is the n-th Bernoulli polynomial, $B_n = B_n(0)$ is the n-th Bernoulli number, and $\Gamma(z)$ is the gamma function. We note that the terms of order b^2 and b^{-2} cancel with the opposite contributions from the exponential in Eq. (3.33) so that there is no global exponential pre-factor. However, the logarithmic term in b^2 gives a global pre-factor of the form $1/\mathsf{b}^2$ after the exponential expansion. Substituting Eqs. (3.35) and (3.37) into

Eq. (3.33), and using that $2\pi\mathsf{b}^2 = 3\hbar$, we obtain the *all-orders semiclassical expansion* of the spectral trace of local $\mathbb{P}^2$ in the form[3]

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) \sim \frac{\Gamma(1/3)^3}{6\pi\hbar} \exp\left(3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}B_{2n+1}(2/3)}{2n(2n+1)!}(3\hbar)^{2n}\right) . \tag{3.38}$$

We comment that $\omega_2 = \Gamma(1/3)^3/4\pi$ is the real half-period of the Weierstrass elliptic function in the equianharmonic case, which corresponds to the elliptic invariants $g_2 = 0$ and $g_3 = 1$, while the other half-period is $\omega_1 = \mathrm{e}^{\pi\mathrm{i}/3}\omega_2$ [142]. The formula in Eq. (3.38) allows us to compute the coefficients in the asymptotic expansion of $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ in the limit $\hbar \to 0$ at arbitrarily high order. Discarding the global pre-factor in Eq. (3.38), the perturbative coefficients are rational numbers with alternating signs. The first few terms are

$$1 - \frac{\hbar^2}{72} + \frac{23\hbar^4}{51840} - \frac{491\hbar^6}{11197440} + \frac{1119703\hbar^8}{112870195200} - \frac{71569373\hbar^{10}}{17878638919680} + \mathrm{O}(\hbar^{12}) , \tag{3.39}$$

which confirms our analytic calculation at NLO in Eq. (3.29).

3.2.2 Exploratory tests of higher instanton sectors

Let us denote by

$$\upsilon(\hbar) = \sum_{n=0}^{\infty} \bar{a}_{2n}\hbar^{2n} \in \mathbb{Q}[\![\hbar]\!] \tag{3.40}$$

the formal power series appearing in the RHS of Eq. (3.38) after exponentiation, which is simply related to the semiclassical perturbative expansion of the spectral trace of local $\mathbb{P}^2$ by

$$\mathrm{Tr}(\rho_{\mathbb{P}^2})\frac{6\pi\hbar}{\Gamma(1/3)^3} \sim \upsilon(\hbar) . \tag{3.41}$$

A numerical investigation of the perturbative expansion of the spectral trace of local $\mathbb{P}^2$ in the dual limit $\hbar \to \infty$ has been carried out by Gu and Mariño in [10].

Let us truncate the infinite sum in Eq. (3.40) to a very high but finite order $d \gg 1$. We denote the resulting $\mathbb{Q}$-polynomial by $\upsilon_d(\hbar)$ and its Borel transform by $\hat{\upsilon}_d(\zeta)$. Explicitly,

$$\upsilon_d(\hbar) = \sum_{n=0}^{d} \bar{a}_{2n}\hbar^{2n} \in \mathbb{Q}[\hbar] , \quad \hat{\upsilon}_d(\zeta) = \sum_{n=0}^{d} \frac{\bar{a}_{2n}}{(2n)!}\zeta^{2n} \in \mathbb{Q}[\zeta] , \tag{3.42}$$

where the coefficients $\bar{a}_{2n}$, $1 \le n \le d$, are computed by Taylor expanding the exponential in the RHS of Eq. (3.38). The first few terms of $\upsilon_d(\hbar)$ are shown in Eq. (3.39). It is straightforward to verify that the perturbative coefficients satisfy the factorial growth

$$\bar{a}_{2n} \sim (-1)^n(2n)!\left(\frac{4\pi^2}{3}\right)^{-2n} , \quad n \gg 1 . \tag{3.43}$$

The formal power series $\upsilon(\hbar)$ is then a Gevrey-1 asymptotic series. We assume that $\hbar \in \mathbb{C}'$ and perform a full numerical *Padé–Borel analysis* [143–145] in the complex ζ-plane. Let d be even. We compute the singular points of the *diagonal Padé approximant* of order $d/2$

[3]The formula in Eq. (3.38) has also been obtained in [141].

of the truncated Borel expansion $\hat{v}_d(\zeta)$, which we denote by $\hat{v}_d^{\mathrm{PB}}(\zeta)$, and we observe two dominant complex conjugate branch points at $\zeta = \pm 4\pi^2 \mathrm{i}/3$ and their respective arcs of accumulating spurious poles mimicking two branch cuts along the positive and negative imaginary axis. Let us now introduce the complex numbers

$$\mathcal{A}_0 = \frac{4\pi^2}{3}\,, \quad \zeta_n = \mathcal{A}_0 \mathrm{i} n\,, \quad n \in \mathbb{Z}_{\neq 0}\,. \tag{3.44}$$

A zoom-in to the poles of the Padé approximant in the bottom part of the upper-half complex ζ-plane is shown in the leftmost plot in Fig. 3.3.

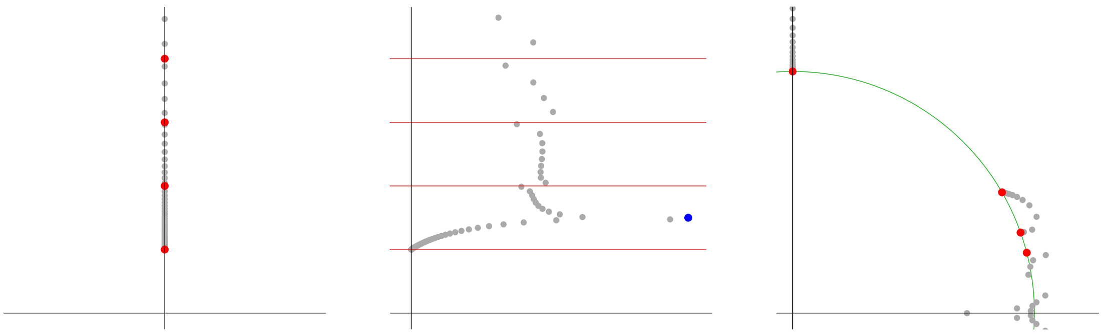


Figure 3.3: In the leftmost plot, a zoom-in to the poles of $\hat{v}_d^{\mathrm{PB}}(\zeta)$ (in gray) in the bottom part of the upper-half complex ζ-plane. We show the first few integer multiples of the dominant pole (in red). In the central plot, a zoom-in to the poles of $\hat{v}_{d,\mathrm{probe}}^{\mathrm{PB}}(\zeta)$ (in gray) in the bottom part of the upper-half complex ζ-plane, including the test charge singularity at $\zeta = \zeta_{\mathrm{probe}}$ (in blue). We show the horizontal lines intersecting the positive imaginary axis at the first few physical branch points (in red). In the rightmost plot, a zoom-in to the poles of $\hat{v}_d^{\mathrm{PCB}}(\eta)$ (in gray) in the upper-right quarter of the complex η-plane. We show the first few physical branch points (in red) and the unit circle (in green). The plots are obtained with $d = 300$.

To reveal the presence of subdominant singularities hidden by the unphysical Padé poles, we apply the *potential theory* interpretation of Padé approximation [146, 147]. More concretely, to test the presence of a suspected NLO branch point at $\zeta = \zeta_2$, we add by hand the *test charge singularity*[4]

$$\hat{v}_{\mathrm{probe}}(\zeta) = (\zeta - \zeta_{\mathrm{probe}})^{-1/5}\,, \tag{3.45}$$

where $\zeta_{\mathrm{probe}} = \mathcal{A}_0\,(1/14 + 3\mathrm{i}/2)$, to the truncated sum $\hat{v}_d(\zeta)$. We then compute the $(d/2)$-diagonal Padé approximant

$$\hat{v}_{d,\mathrm{probe}}^{\mathrm{PB}}(\zeta) = (\hat{v}_d + \hat{v}_{\mathrm{probe}})^{\mathrm{PB}}(\zeta)\,. \tag{3.46}$$

The resulting Padé–Borel poles distribution is distorted, as shown in the central plot in Fig. 3.3. The targeted true branch point at $\zeta = \zeta_2$ is now clearly visible, and so it is the singular term at $\zeta = \zeta_{\mathrm{probe}}$. The other true branch points at $\zeta = \zeta_3, \zeta_4$ are outlined.

[4]The specific value of the complex number ζ_{probe}, dictating the position of the test charge, is chosen to maximize the qualitative visibility of the true branch points.

We can further improve the precision of the Padé extrapolation and analytic continuation of the truncated Borel series $\hat{\psi}_d(\zeta)$ using *conformal maps*. Namely, we perform the change of variable

$$\zeta = \mathcal{A}_0 \frac{2\eta}{1-\eta^2}\,, \quad \eta \in \mathbb{C}\,, \tag{3.47}$$

which maps the cut Borel ζ-plane

$$\mathbb{C} \setminus \{(-\infty\mathrm{i}, \zeta_{-1}] \cup [\zeta_1, +\infty\mathrm{i})\} \tag{3.48}$$

into the interior of the unit disk $|\eta| < 1$. The dominant branch points $\zeta = \zeta_{\pm 1}$ are mapped into $\eta = \pm\mathrm{i}$, while the point at infinity is mapped into $\eta = \pm 1$. Correspondingly, the branch cuts $(-\infty\mathrm{i}, \zeta_{-1}]$ and $[\zeta_1, +\infty\mathrm{i})$ along the imaginary ζ-axis split each one into two identical copies that lie onto the two lower-half and upper-half quarters of the unit circle in the η-plane, respectively. The *inverse conformal map* is explicitly given by

$$\eta = \pm\sqrt{\frac{-1+\sqrt{1+(\zeta/\mathcal{A}_0)^2}}{1+\sqrt{1+(\zeta/\mathcal{A}_0)^2}}}\,, \quad \zeta \in \mathbb{C}\,. \tag{3.49}$$

We compute the $(d/2)$-diagonal Padé approximant of the conformally mapped Borel expansion, which we denote by $\hat{\upsilon}_d^{\mathrm{PCB}}(\eta)$. Its singularities in the complex η-plane are shown in the rightmost plot in Fig. 3.3. We observe two symmetric arcs of spurious poles emanating from $\eta = \pm\mathrm{i}$ along the imaginary axis in opposite directions. The smaller arcs of poles jumping along the unit circle towards the real axis represent the Padé boundary of convergence joining the conformal map images of the repeated singularities at $\zeta = \zeta_n$, $n \in \mathbb{Z}_{\neq 0}$. We will show in Section 5.1 that the pattern of singularities suggested by the numerical analysis performed here is confirmed analytically.

Let us now test the resurgent structure of the asymptotic series $\upsilon(\hbar)$ in Eq. (3.40). We recall that the local expansions of the Borel transform $\hat{\upsilon}(\zeta)$ in the neighborhoods of its dominant singularities at $\zeta = \zeta_{\pm 1}$ are governed by the *one-instanton* perturbative corrections $\upsilon_{\pm 1}(\hbar)$ and the corresponding Stokes constants $\bar{S}_{\pm 1}$. Let us consider the standard functional ansatz

$$\upsilon_1(\hbar) = \sum_{k=0}^{\infty} c_k \hbar^{k-\beta} \in \hbar^{-\beta}\mathbb{C}[\![\hbar]\!]\,, \tag{3.50}$$

where the coefficient c_k can be interpreted as the $(k+1)$-loop contribution around the one-instanton configuration in the upper-half Borel ζ-plane, and $\beta \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ is the so-called *characteristic exponent*. We fix the normalization condition $c_0 = 1$ and further assume that $\upsilon_{-1}(\hbar) = \upsilon_1(\hbar)$ and $\bar{S}_{-1} = \bar{S}_1$. By Cauchy's integral theorem, the large-n asymptotics of the perturbative coefficients $\bar{a}_{2n}$ in Eq. (3.42) is controlled at leading order by

$$\bar{a}_{2n} \sim \frac{(-1)^n \bar{S}_1}{\pi\mathrm{i}} \frac{\Gamma(2n+\beta)}{\mathcal{A}_0^{2n+\beta}} \sum_{k=0}^{\infty} \frac{c_k \mathcal{A}_0^k}{\prod_{j=1}^{k}(2n+\beta-j)}\,, \quad n \gg 1\,. \tag{3.51}$$

An independent numerical estimate of $\mathcal{A}_0$ is obtained from the convergence of the sequences

$$4n^2 \frac{\bar{a}_{2n}}{\bar{a}_{2n+2}} = |\mathcal{A}_0|^2 + \mathcal{O}(1/n)\,, \tag{3.52a}$$

$$\frac{|\mathcal{A}_0|^2 \bar{a}_{2n+2}}{4n^2 \bar{a}_{2n}} + \frac{4n^2 \bar{a}_{2n-2}}{|\mathcal{A}_0|^2 \bar{a}_{2n}} = 2\cos(2\theta_{\mathcal{A}_0}) + \mathcal{O}(1/n)\,, \tag{3.52b}$$

which give the absolute value $|\mathcal{A}_0| \approx 4\pi^2/3$ and the phase $\theta_{\mathcal{A}_0} \approx 0$, as expected. Analogously, a numerical estimate of the characteristic exponent β is obtained from the convergence of the sequence

$$\left(\frac{\mathcal{A}_0^2}{4n^2}\frac{\bar{a}_{2n+2}}{\bar{a}_{2n}} - 1\right) 2n = 1 + 2\beta + \mathcal{O}(1/n) \,, \tag{3.53}$$

which gives $\beta \approx 0$. Then, we estimate the Stokes constant $\bar{S}_1$ as the large-n limit of the sequence[5]

$$\pi\mathrm{i}(-1)^n \frac{\mathcal{A}_0^{2n}}{\Gamma(2n)}\bar{a}_{2n} = \bar{S}_1 + \mathcal{O}(1/n) \,, \tag{3.54}$$

which gives $\bar{S}_1 \approx 3\sqrt{3}\mathrm{i}$. Let us proceed to systematically extract the coefficients c_k, $k \in \mathbb{N}$. We first Taylor expand the quotients appearing in the RHS of Eq. (3.51) in the large-n limit and rearrange them to give

$$\bar{a}_{2n} \sim \frac{(-1)^n \bar{S}_1}{\pi\mathrm{i}} \frac{\Gamma(2n)}{\mathcal{A}_0^{2n}} \sum_{i=0}^{\infty} \frac{\mu_i}{(2n)^i} \,, \quad n \gg 1 \,, \tag{3.55}$$

where the new coefficients μ_i are expressed in closed form as

$$\mu_0 = c_0 = 1 \,, \quad \mu_i = \sum_{k=1}^{i} c_k \mathcal{A}_0^k \left(\sum_{\substack{1 \le m_1, \dots, m_k \le i \\ m_1 + \dots + m_k = i}} \prod_{j=1}^{k} j^{m_j - 1} \right) \,, \quad i \in \mathbb{N}_{\neq 0} \,. \tag{3.56}$$

We define the sequence Q_{2n}, $n \in \mathbb{N}$, such that

$$Q_{2n} = \pi\mathrm{i}(-1)^n \frac{\mathcal{A}_0^{2n}}{\bar{S}_1 \Gamma(2n)} \bar{a}_{2n} \sim \sum_{i=0}^{\infty} \frac{\mu_i}{(2n)^i} \,, \quad n \gg 1 \,, \tag{3.57}$$

and we obtain a numerical estimate of the coefficients μ_i, $i \in \mathbb{N}$, as the large-n limits of the recursively-defined sequences

$$Q_{2n}^{(1)} = 2n(Q_{2n} - 1) = \mu_1 + \mathcal{O}(1/n) \,, \tag{3.58a}$$

$$Q_{2n}^{(i)} = 2n(Q_{2n}^{(i-1)} - \mu_{i-1}) = \mu_i + \mathcal{O}(1/n) \,, \quad i \in \mathbb{N}_{>0} \,. \tag{3.58b}$$

We substitute the numerical values for the coefficients μ_i in their explicit relations with the coefficients c_k in Eq. (3.56), and term-by-term we find in this way that

$$c_{2k} \approx \bar{a}_{2k} \,, \quad c_{2k+1} \approx 0 \,, \quad k \in \mathbb{N} \,, \tag{3.59}$$

that is, the coefficients of the one-instanton asymptotic series $\upsilon_1(\hbar)$ in Eq. (3.50) identically correspond to the coefficients of the original perturbative series $\upsilon(\hbar)$ in Eq. (3.40). We conclude that $\upsilon_1(\hbar) = \upsilon(\hbar)$, as confirmed by the analytic solution in Section 5.1. We comment that the numerical convergence of the large-n limits of all sequences above has been accelerated using Richardson transforms.[6]

[5] Using the first 300 perturbative coefficients, the numerical estimate for the first Stokes constant agrees with the exact value from Section 5.1 up to 32 digits.

[6] Consider a sequence $\{s_n\}$, $n \in \mathbb{N}$, with the asymptotics $s_n \sim \sum_{i=0}^{\infty} \frac{\mu_i}{n^i}$ for $n \gg 1$. Its N-th Richardson transform is $\mathcal{R}^{(N)}(s_n) = \sum_{m=0}^{N} \frac{s_{n+m}(n+m)^N(-1)^{m+N}}{m!(N-m)!}$. As the first $N-1$ sub-leading tails are removed, the transformed sequence behaves as $\mathcal{R}^{(N)}(s_n) \sim \mu_0 + \mathcal{O}(n^{-(N+1)})$ in the large-n limit.

Finally, we perform a numerical test of the discontinuity. More precisely, let us rotate the complex ζ-plane by an angle of $-\pi/2$ to move the branch cuts of the Borel transform $\hat{\upsilon}(\zeta)$ to the real axis. The corresponding change of variable is $z = -\mathrm{i}\zeta$. We analogously rotate the complex $\hbar$-plane and introduce the variable $x = -\mathrm{i}\hbar$. We fix a small positive angle $\epsilon \ll 1$ and a small positive value of $x \ll 1$. We compute numerically the lateral Borel resummations across the positive real axis as[7]

$$s_{\pm}^{\mathrm{PB}}(\upsilon)(x) = \mathrm{e}^{\pm \mathrm{i}\epsilon} \int_0^{\infty} \hat{\upsilon}_d^{\mathrm{PB}}(z\mathrm{e}^{\pm \mathrm{i}\epsilon} x)\, \mathrm{e}^{-z\mathrm{e}^{\pm \mathrm{i}\epsilon}} dz\,, \tag{3.60}$$

where $\hat{\upsilon}_d^{\mathrm{PB}}(z)$ is the diagonal Padé approximant of order $d/2$ of the truncated Borel series $\hat{\upsilon}_d(z)$. The corresponding discontinuity is evaluated as

$$\mathrm{disc}^{\mathrm{PB}}(\upsilon)(x) = s_{+}^{\mathrm{PB}}(\upsilon)(x) - s_{-}^{\mathrm{PB}}(\upsilon)(x) = 2\mathrm{i}\Im\left(s_{+}^{\mathrm{PB}}(\upsilon)(x)\right)\,. \tag{3.61}$$

Assuming that all *higher-order instanton* perturbative series $\upsilon_n(x)$ are trivially equal to $\upsilon(x)$, as it is found to be true in the exact analysis of Section 5.1, the Stokes constants $\bar{S}_n$, $n \in \mathbb{Z}_{>0}$, are estimated numerically through the recursive relation

$$\mathrm{disc}^{\mathrm{PB}}(\upsilon)(x)\, \mathrm{e}^{n\mathcal{A}_0/x} - s_{-}^{\mathrm{PB}}(\upsilon)(x) \sum_{k=1}^{n-1} \bar{S}_k \mathrm{e}^{(n-k)\mathcal{A}_0/x} = s_{-}^{\mathrm{PB}}(\upsilon)(x)\bar{S}_n + \mathcal{O}(\mathrm{e}^{-\mathcal{A}_0/x})\,, \tag{3.62}$$

which reproduces the exact formula in Eq. (5.75). Analogously, we obtain a numerical estimate of the Stokes constants associated with the branch points on the negative real axis in the rotated complex z-plane. We remark that all numerical computations described here for $\upsilon(\hbar)$ are straightforwardly and successfully applied to the asymptotic series $\phi(\hbar) = \log \upsilon(\hbar)$ in Eq. (5.1)—once more in full agreement with our analytic solution in Section 5.1.

3.3 The example of local $\mathbb{F}_0$

A second example of a toric del Pezzo CY threefold is the total space of the canonical line bundle over the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, that is,

$$X = \mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1\,, \tag{3.63}$$

also known as *local* $\mathbb{F}_0$, which has been extensively studied due to its close relation to ABJ(M) theory [127, 128, 148, 149]. The Hirzebruch surface $\mathbb{F}_0$ is identified by the polyhedron $\Delta_{\mathbb{F}_0}$, which is the convex hull of the origin together with the two-dimensional vectors

$$\nu^{(1)} = (1,0)\,, \quad \nu^{(2)} = (-1,0)\,, \quad \nu^{(3)} = (0,1)\,, \quad \nu^{(4)} = (0,-1)\,, \tag{3.64}$$

as shown in Fig. 3.4, and by the only non-trivial function

$$f_2(\xi) = \log(\xi)\,, \tag{3.65}$$

where ξ is the mass parameter. Correspondingly, the moduli space of complex structures of the mirror manifold $\tilde{X}$ is encoded in the two-parameter family of elliptic curves

$$O_{\mathbb{F}_0}(x,y) = \mathrm{e}^x + \xi\mathrm{e}^{-x} + \mathrm{e}^y + \mathrm{e}^{-y} + \kappa = 0\,, \quad x, y \in \mathbb{C}\,, \tag{3.66}$$

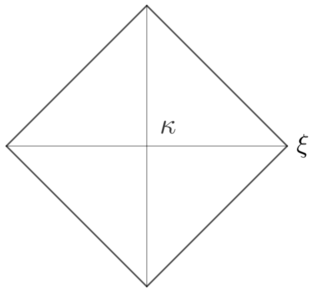


Figure 3.4: Toric diagram of local $\mathbb{F}_0$. We show the vectors in Eq. (3.64) and the polyhedron $\Delta_{\mathbb{F}_0}$. The complex modulus κ corresponds to the internal vertex of the diagram, while the mass parameter ξ corresponds to one of the four external vertices.

where κ is the one true complex modulus of $\tilde{X}$. Eq. (3.66) describes a genus-one Riemann surface Σ embedded in $\mathbb{C}^* \times \mathbb{C}^*$, which is the mirror curve to local $\mathbb{F}_0$.

For simplicity, we impose the condition[8] $\xi = 1$, which implies the relation

$$z = 1/\kappa^2 \, . \tag{3.67}$$

In the parametrization given by the Batyrev coordinate z, the *Picard–Fuchs differential equation* associated with the mirror of local $\mathbb{F}_0$ is thus [50]

$$\mathcal{L}_{\mathbb{F}_0}\mathcal{P} = \left[\Theta^3 - 4z(2\Theta+1)^2\Theta\right]\mathcal{P} = 0\,, \tag{3.68}$$

where again $\Theta = zd/dz$ and $\mathcal{P}$ is the full period vector of the meromorphic one-form λ. The Picard–Fuchs differential operator $\mathcal{L}_{\mathbb{F}_0}$ has three singular points, which are the *large radius point* at $z = 0$, the *conifold point* at $z = 1/16$, and the *orbifold point* at $1/z = 0$. Besides, by looking at the monodromy data of the Picard–Fuchs differential equation, the moduli space of complex structures of $\tilde{X}$ can be identified with the compactified quotient $\overline{\mathbb{H}/\Gamma_1(4)}$, where $\mathbb{H}$ denotes the upper half of the complex plane and $\Gamma_1(4)$ is the *congruence subgroup*

$$\Gamma_1(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \,\middle|\, a, d \equiv_4 1\,, c \equiv_4 0 \right\} . \tag{3.69}$$

The canonical Weyl quantization scheme of [11] applied to the function $O_{\mathbb{F}_0}(x,y)$ in Eq. (3.66) with $\xi = 1$, together with an appropriate choice of reality conditions for the variables $x, y \in \mathbb{C}$, produces the self-adjoint quantum operator

$$\mathsf{O}_{\mathbb{F}_0}(\mathsf{x},\mathsf{y}) = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}\,, \tag{3.70}$$

acting on $L^2(\mathbb{R})$, where x, y are self-adjoint Heisenberg operators satisfying the commutation relation $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$. A Planck constant $\hbar \in \mathbb{R}_{>0}$ is introduced as a quantum deformation parameter. It was conjectured in [11, 12] and later proven in [106] that the inverse operator

$$\rho_{\mathbb{F}_0} = \mathsf{O}_{\mathbb{F}_0}^{-1} \tag{3.71}$$

[7]Following [143, 144], the numerical precision of the lateral Borel resummations can be improved with the use of conformal maps.

[8]We remark that the TS/ST correspondence is expected to hold for arbitrary values of the mass parameters, as suggested by the evidence provided in [88].

is positive-definite and of trace class. The fermionic spectral traces $Z_{\mathbb{F}_0}(N, \hbar)$, where N is a non-negative integer, are well-defined functions of $\hbar \in \mathbb{R}_{>0}$ and can be computed explicitly as multi-dimensional matrix model integrals [111]. Analogously to Eq. (3.20), we have that

$$\begin{aligned} Z_{\mathbb{F}_0}(N, \hbar) &= \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} (-1)^{\epsilon(\sigma)} \int \prod_{i=1}^{N} \rho_{\mathbb{F}_0}(x_i, x_{\sigma(i)}) \, d^N x \\ &= \frac{1}{N!} \int \det \left(\rho_{\mathbb{F}_0}(x_i, x_j) \right) \, d^N x \,, \end{aligned} \tag{3.72}$$

where again $\mathcal{S}_N$ is the N-th permutation group, $\epsilon(\sigma)$ is the signature of the element $\sigma \in \mathcal{S}_N$, and $\rho_{\mathbb{F}_0}(x_i, x_j)$ denotes the *integral kernel* of the operator $\rho_{\mathbb{F}_0}$. This is known explicitly as [111]

$$\rho_{\mathbb{F}_0}(x_1, \, x_2) = \frac{\mathrm{e}^{\pi \mathsf{b}(x_1 + x_2)/2}}{2\mathsf{b} \cosh(\pi(x_1 - x_2)/\mathsf{b})} \frac{\Phi_{\mathsf{b}}(x_1 + \mathrm{i}\mathsf{b}/4)}{\Phi_{\mathsf{b}}(x_1 - \mathrm{i}\mathsf{b}/4)} \frac{\Phi_{\mathsf{b}}(x_2 + \mathrm{i}\mathsf{b}/4)}{\Phi_{\mathsf{b}}(x_2 - \mathrm{i}\mathsf{b}/4)} \,, \tag{3.73}$$

where b is related to $\hbar$ by

$$\pi \mathsf{b}^2 = \hbar \,, \tag{3.74}$$

and Φ_{b} is Faddeev's quantum dilogarithm. As in the case of local $\mathbb{P}^2$, the integral kernel in Eq. (3.73), and therefore also the fermionic spectral traces in Eq. (3.72), can be analytically continued to $\hbar \in \mathbb{C}'$.

In the rest of this section, we will perform a numerical study of the resurgent structure of the first fermionic spectral trace

$$Z_{\mathbb{F}_0}(1, \hbar) = \mathrm{Tr}(\rho_{\mathbb{F}_0}) \tag{3.75}$$

in the semiclassical limit $\hbar \to 0$.

3.3.1 Computing the semiclassical expansion

As we have done for local $\mathbb{P}^2$ in Section 3.2.1, we apply the phase-space formulation of quantum mechanics to obtain the WKB expansion of the trace of the inverse operator $\rho_{\mathbb{F}_0}$ at NLO in $\hbar \to 0$, starting from the explicit expression of the operator $\mathsf{O}_{\mathbb{F}_0}$ in Eq. (3.70) and following Appendix A. For simplicity, we denote by O_W, ρ_W the Wigner transforms of the operators $\mathsf{O}_{\mathbb{F}_0}$, $\rho_{\mathbb{F}_0}$, respectively. The Wigner transform of $\mathsf{O}_{\mathbb{F}_0}$ is obtained by performing the integration in Eq. (A.1). As we show in Example A.0.1, this simply gives the classical function

$$O_{\mathrm{W}} = \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{y} + \mathrm{e}^{-y} \,. \tag{3.76}$$

Substituting it into Eqs. (A.11a) and (A.11b), we have

$$\mathcal{G}_2 = -\frac{\hbar^2}{4} \left[\mathrm{e}^{x+y} + \mathrm{e}^{-x+y} + \mathrm{e}^{x-y} + \mathrm{e}^{-x-y} \right] + \mathcal{O}(\hbar^4) \,, \tag{3.77a}$$

$$\mathcal{G}_3 = -\frac{\hbar^2}{4} \left[\mathrm{e}^{2x+y} + \mathrm{e}^{2x-y} + \mathrm{e}^{-2x+y} + \mathrm{e}^{-2x-y} - 2\mathrm{e}^{x} - 2\mathrm{e}^{-x} + (x \leftrightarrow y) \right] + \mathcal{O}(\hbar^4) \,, \tag{3.77b}$$

where $(x \leftrightarrow y)$ indicates the symmetric expression after exchanging the variables x and y. It follows from Eq. (A.12) that the Wigner transform of $\rho_{\mathbb{F}_0}$ up to order $\hbar^2$ is then given by

$$\rho_{\mathrm{W}} = \frac{1}{O_{\mathrm{W}}} - \hbar^2 \frac{1}{O_{\mathrm{W}}^3} + \mathcal{O}(\hbar^4) \,. \tag{3.78}$$

We note that the same result can be obtained by solving Eq. (A.15) order by order in powers of $\hbar^2$. Integrating Eq. (3.78) over phase space, as in Eq. (A.3), we obtain the NLO perturbative expansion in $\hbar$ of the trace, that is,

$$\begin{aligned}\mathrm{Tr}(\rho_{\mathbb{F}_0}) &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}\rho_{\mathrm{W}}\,dxdy \\ &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}\frac{1}{O_{\mathrm{W}}}\,dxdy - \frac{\hbar}{2\pi}\int_{\mathbb{R}^2}\frac{1}{O_{\mathrm{W}}^3}\,dxdy + \mathcal{O}(\hbar^4)\,,\end{aligned} \tag{3.79}$$

and evaluating the integrals explicitly, we find

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) = \frac{\pi}{4\hbar}\left\{1 - \frac{\hbar^2}{64} + \mathcal{O}(\hbar^4)\right\}. \tag{3.80}$$

We stress that the phase-space formalism adopted above provides, in principle, the perturbative expansion of $\mathrm{Tr}(\rho_{\mathbb{F}_0})$ at all orders in $\hbar$ by systematically extending all intermediate computations beyond order $\hbar^2$. However, as for the case of local $\mathbb{P}^2$, a more efficient way to extract the perturbative coefficients is described below.

Under the assumption that $\Re(\mathsf{b}) > 0$, the spectral trace of local $\mathbb{F}_0$ has the integral representation [111]

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) = \frac{1}{2\mathsf{b}}\int_{\mathbb{R}} \mathrm{e}^{\pi\mathsf{b}x}\frac{\Phi_{\mathsf{b}}(x+\mathrm{i}\mathsf{b}/4)^2}{\Phi_{\mathsf{b}}(x-\mathrm{i}\mathsf{b}/4)^2}\,dx\,, \tag{3.81}$$

which is a well-defined analytic function of $\hbar \in \mathbb{C}'$. The integral in Eq. (3.81) can be evaluated explicitly by analytically continuing x to the complex domain, closing the integration contour from above, and applying Cauchy's residue theorem. The resulting expression for $\mathrm{Tr}(\rho_{\mathbb{F}_0})$ is the sum of products of *holomorphic/anti-holomorphic blocks* written in terms of $q, \tilde{q}$-*series*, respectively. Namely, we have that [10]

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) = -\frac{\mathrm{i}}{2}\left(G(q)\tilde{g}(\tilde{q}) + 8\mathsf{b}^{-2}g(q)\tilde{G}(\tilde{q})\right), \tag{3.82}$$

where we have introduced

$$q = \mathrm{e}^{2\pi\mathrm{i}\mathsf{b}^2} = \mathrm{e}^{2\mathrm{i}\hbar}\,, \quad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}/\mathsf{b}^2} = \mathrm{e}^{-2\pi^2\mathrm{i}/\hbar}\,. \tag{3.83}$$

Note that the factorization in Eq. (3.82) is not symmetric in q, $\tilde{q}$ as the holomorphic and anti-holomorphic blocks are given by different series. More precisely, they are

$$g(q) = \sum_{m=0}^{\infty}\frac{(q^{1/2};\,q)_m^2}{(q;\,q)_m^2}q^{m/2} = {}_2\phi_1\left(\begin{matrix}q^{1/2}, & q^{1/2}\\ & q\end{matrix};\,q,\,q^{1/2}\right), \tag{3.84a}$$

$$G(q) = \sum_{m=0}^{\infty}\frac{(q^{1/2};\,q)_m^2}{(q;\,q)_m^2}q^{m/2}\left(1 + 4\sum_{s=1}^{\infty}\frac{q^{s(m+1/2)}}{1+q^{s/2}}\right), \tag{3.84b}$$

$$\tilde{g}(\tilde{q}) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1;\,\tilde{q})_m^2}{(\tilde{q};\,\tilde{q})_m^2}(-\tilde{q})^m = \frac{1}{2}{}_2\phi_1\left(\begin{matrix}-1, & -1\\ & \tilde{q}\end{matrix};\,\tilde{q},\,-\tilde{q}\right), \tag{3.84c}$$

$$\tilde{G}(\tilde{q}) = \sum_{m=0}^{\infty}\frac{(-\tilde{q};\,\tilde{q})_m^2}{(\tilde{q};\,\tilde{q})_m^2}(-1)^m\left(\sum_{s=0}^{\infty}\frac{\tilde{q}^{(2s+1)(m+1)}}{1-\tilde{q}^{2s+1}}\right), \tag{3.84d}$$

where $(x;q)_m$ denotes the q-shifted factorial defined in Eq. (B.3), and ${}_{r+1}\phi_s$ is the q-hypergeometric series defined in Eq. (B.5). As in the case of local $\mathbb{P}^2$, we assume that $\Im(\mathsf{b}^2) > 0$, and therefore $\Im(\hbar) > 0$, so that $|q|, |\tilde{q}| < 1$ and the $q, \tilde{q}$-series converge.

Let us consider the integral representation in Eq. (3.81) and perturbatively expand it in the limit $\hbar \to 0$. We perform the change of variable $y = 2\pi \mathsf{b} x$ and write it equivalently as

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) = \frac{1}{4\pi\mathsf{b}^2} \int_{\mathbb{R}} \exp\left(\frac{y}{2} + 2\log\Phi_{\mathsf{b}}\left(\frac{y + \mathrm{i}\pi\mathsf{b}^2/2}{2\pi\mathsf{b}}\right) - 2\log\Phi_{\mathsf{b}}\left(\frac{y - \mathrm{i}\pi\mathsf{b}^2/2}{2\pi\mathsf{b}}\right)\right) dy\,. \tag{3.85}$$

Recall that the asymptotic expansion formula in Eq. (B.18) yields

$$\log\Phi_{\mathsf{b}}\left(\frac{y \pm \mathrm{i}\pi\mathsf{b}^2/2}{2\pi\mathsf{b}}\right) \sim \sum_{k=0}^{\infty} (2\pi\mathrm{i}\mathsf{b}^2)^{2k-1} \frac{B_{2k}(1/2)}{(2k)!} \mathrm{Li}_{2-2k}(-\mathrm{e}^{y \pm \mathrm{i}\pi\mathsf{b}^2/2}) \tag{3.86}$$

for $\mathsf{b} \to 0$, where $B_n(z)$ is the n-th Bernoulli polynomial and $\mathrm{Li}_n(z)$ is the polylogarithm of order n. We eliminate the remaining b-dependence in Eq. (3.86) by expanding the polylogarithms around $\mathsf{b} \to 0$. More precisely, using the well-known formula

$$\frac{\partial \mathrm{Li}_s(\mathrm{e}^{\mu})}{\partial \mu} = \mathrm{Li}_{s-1}(\mathrm{e}^{\mu})\,, \quad s, \mu \in \mathbb{C}\,, \tag{3.87}$$

we obtain the Taylor expansion

$$\mathrm{Li}_{2-2k}(-\mathrm{e}^{y \pm \mathrm{i}\pi\mathsf{b}^2/2}) \sim \sum_{m=0}^{\infty} \frac{1}{m!} \left(\pm\frac{\mathrm{i}\pi\mathsf{b}^2}{2}\right)^m \mathrm{Li}_{2-2k-m}(-\mathrm{e}^{y})\,, \quad k \geq 0\,. \tag{3.88}$$

Let us denote the exponent of the integrand in Eq. (3.85) by $V(y, \mathsf{b})$. After recombining the terms in the nested expansions in Eqs. (3.86) and (3.88), we derive a well-defined and fully-determined perturbative series in b^2, which is

$$V(y, \mathsf{b}) \sim \frac{y}{2} + 2 \sum_{k,m=0}^{\infty} (2\pi\mathrm{i}\mathsf{b}^2)^{2k+m-1} \frac{B_{2k}(1/2)}{4^m m!(2k)!} \mathrm{Li}_{2-2k-m}(-\mathrm{e}^{y})[1 - (-1)^m]\,. \tag{3.89}$$

Note that the quantity $[1-(-1)^m]$ is zero for even values of m. Therefore, introducing the notation $m = 2q+1$ and $p = k+q$, we have that

$$V(y, \mathsf{b}) \sim \frac{y}{2} + \sum_{p=0}^{\infty} (2\pi\mathrm{i}\mathsf{b}^2)^{2p} \mathrm{Li}_{1-2p}(-e^{y}) \sum_{q=0}^{p} \frac{B_{2p-2q}(1/2)}{4^{2q}(2q+1)!(2p-2q)!}\,. \tag{3.90}$$

This formula can be further simplified by using the symmetry and translation identities for the Bernoulli polynomials. Namely,

$$B_n(1-z) = (-1)^n B_n(z)\,, \quad B_n(z+v) = \sum_{k=0}^{n} \binom{n}{k} B_k(z) v^{n-k}\,, \tag{3.91}$$

where $n \in \mathbb{N}$ and $z, v \in \mathbb{C}$. In particular, choosing $z = 1/2$, $v = 1/4$, and $n = 2p+1$, a simple computation shows that

$$\frac{4}{(2p+1)!} B_{2p+1}(3/4) = \sum_{q=0}^{p} \frac{B_{2p-2q}(1/2)}{4^{2q}(2q+1)!(2p-2q)!}\,. \tag{3.92}$$

Substituting Eq. (3.92) into Eq. (3.90) and separating the contribution from

$$\mathrm{Li}_1(-\mathrm{e}^y) = -\log(1+\mathrm{e}^y)\,, \tag{3.93}$$

we finally obtain the perturbative expansion

$$V(y,\mathsf{b}) \sim \frac{y}{2} - \log(1+\mathrm{e}^y) + 4\sum_{p=1}^{\infty}(2\pi\mathrm{i}\mathsf{b}^2)^{2p}\frac{B_{2p+1}(3/4)}{(2p+1)!}\mathrm{Li}_{1-2p}(-e^y)\,, \tag{3.94}$$

where only one infinite sum remains. We remark that the term of order b^{-2} in Eq. (3.94) vanishes, that is, the potential $V(y,\mathsf{b})$ does not have a critical point around which to perform a saddle point approximation of the integral in Eq. (3.85). Moreover, for each $p \geq 1$, the coefficient function of $(2\pi\mathrm{i}\mathsf{b}^2)^{2p}$ can be written explicitly as a rational function in the variable $t = \mathrm{e}^y$ with coefficients in $\mathbb{Q}$. Indeed, the polylogarithm of negative integer order is

$$\mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}}\sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle z^{n-k}\,, \quad n \in \mathbb{Z}_{>0}\,, \quad z \in \mathbb{C}\,, \tag{3.95}$$

where $\left\langle {n \atop k} \right\rangle$ are the Eulerian numbers. Applying Eq. (3.95) to Eq. (3.94), we find that

$$V(y,\,\mathsf{b}) \sim \frac{1}{2}\log(t) - \log(1+t) + \sum_{p=1}^{\infty}\tilde{\mathsf{b}}^{4p}\frac{P_p(t)}{(1+t)^{2p}}\,, \tag{3.96}$$

where we have introduced the new variable

$$\tilde{\mathsf{b}}^2 = 2\pi\mathrm{i}\mathsf{b}^2\,, \tag{3.97}$$

while $P_p(t)$ is a $\mathbb{Q}$-polynomial in t of degree $2p-1$. Explicitly,

$$P_p(t) = 4\frac{B_{2p+1}(3/4)}{(2p+1)!}\sum_{m=1}^{2p-1}(-1)^m\left\langle {2p-1 \atop 2p-1-m} \right\rangle t^m\,, \quad p \geq 1\,. \tag{3.98}$$

We will now Taylor expand the exponential $\mathrm{e}^{V(y,\,\mathsf{b})}$ in the limit $\mathsf{b} \to 0$ and obtain a second perturbative b-series, whose coefficients are identified $\mathbb{Q}$-rational functions in t and can be explicitly integrated term-by-term to give the all-orders $\hbar$-expansion of the spectral trace in Eq. (3.85). In particular, we find that

$$\begin{aligned}\frac{1+t}{t^{1/2}}\,\mathrm{e}^{V(y,\,\mathsf{b})} &\sim 1 + \sum_{r=1}^{\infty}\frac{1}{r!}\left(\sum_{p=1}^{\infty}\tilde{\mathsf{b}}^{4p}\frac{P_p(t)}{(1+t)^{2p}}\right)^r \\ &= 1 + \sum_{k=1}^{\infty}\frac{\tilde{\mathsf{b}}^{4k}}{(1+t)^{2k}}\sum_{m\in\mathcal{P}(k)}\frac{1}{|m|!}\binom{|m|}{N_1,\ldots,N_k}P_{m_1}(t)\cdots P_{m_{|m|}}(t)\,,\end{aligned} \tag{3.99}$$

where $\mathcal{P}(k)$ is the set of all partitions $m = (m_1,\ldots,m_{|m|})$ of the positive integer k, $|m|$ denotes the length of the partition, and $N_i \in \mathbb{N}$ is the number of times that the positive integer $i \in \mathbb{Z}_{>0}$ is repeated in the partition m. Note that $\sum_{i=1}^{k} N_i = |m|$. The asymptotic expansion in Eq. (3.99) can be written in a more compact form as

$$\mathrm{e}^{V(y,\,\mathsf{b})} \sim \frac{t^{1/2}}{1+t}\left(1 + \sum_{k=1}^{\infty}\tilde{\mathsf{b}}^{4k}\frac{P'_k(t)}{(1+t)^{2k}}\right)\,, \tag{3.100}$$

where $P'_k(t)$ is a new $\mathbb{Q}$-polynomial in t of degree $2k-1$, which is given explicitly by

$$P'_k(t) = \sum_{m\in\mathcal{P}(k)} \frac{1}{N_1!\cdots N_k!} P_{m_1}(t)\cdots P_{m_{|m|}}(t) = \sum_{n=1}^{2k-1} c_{k,n} t^n\,, \quad k \geq 1\,. \tag{3.101}$$

The numbers $c_{k,n} \in \mathbb{Q}$ are directly determined by the coefficients of the polynomials $P_p(t)$, $p \geq 1$, in Eq. (3.98) via the exponential expansion formula above. Let us now substitute Eqs. (3.100) and (3.101) into the integral representation for the spectral trace in Eq. (3.85), which gives

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) \sim \frac{1}{4\pi\mathsf{b}^2}\int_{\mathbb{R}} \frac{\mathrm{e}^{y/2}}{1+\mathrm{e}^y} dy + \frac{1}{4\pi\mathsf{b}^2}\sum_{k=1}^{\infty}(2\pi\mathrm{i}\mathsf{b}^2)^{2k}\sum_{n=1}^{2k-1} c_{k,n}\int_{\mathbb{R}}\frac{\mathrm{e}^{y/2+ny}}{(1+\mathrm{e}^y)^{1+2k}}dy\,. \tag{3.102}$$

Evaluating the integrals in terms of gamma functions and using the property

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}\,, \quad n \in \mathbb{N}\,, \tag{3.103}$$

we obtain that

$$\begin{aligned}\int_{\mathbb{R}} \frac{\mathrm{e}^{y/2+ny}}{(1+\mathrm{e}^y)^{1+2k}}dy &= \frac{\Gamma\left(\frac{1}{2}+2k-n\right)\Gamma\left(\frac{1}{2}+n\right)}{\Gamma(1+2k)} \\ &= \pi\frac{(2n-1)!!(4k-2n-1)!!}{4^k(2k)!}\,, \quad k,n\geq 0\,.\end{aligned} \tag{3.104}$$

Therefore, the *all-orders semiclassical expansion* of the spectral trace of local $\mathbb{F}_0$ can be written explicitly as

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) \sim \frac{1}{4\mathsf{b}^2}\left(1+\sum_{k=1}^{\infty}\mathsf{b}^{4k}(-1)^k\frac{\pi^{2k}}{(2k)!}\sum_{n=1}^{2k-1}c_{k,n}(2n-1)!!(4k-2n-1)!!\right)\,. \tag{3.105}$$

Note that, substituting $\pi\mathsf{b}^2 = \hbar$ and factoring out $\pi/4\hbar$, Eq. (3.105) proves that the resulting asymptotic series in $\hbar^2$ has coefficients in $\mathbb{Q}$ of alternating signs. We describe below how to numerically implement our algorithmic procedure to efficiently compute the coefficients in the asymptotic expansion of $\mathrm{Tr}(\rho_{\mathbb{F}_0})$ in the limit $\hbar \to 0$ up to very high order. Discarding the global pre-factor in Eq. (3.105), the first few terms are

$$1 - \frac{\hbar^2}{64} + \frac{19\hbar^4}{49152} - \frac{1013\hbar^6}{47185920} + \frac{814339\hbar^8}{338228674560} - \frac{449996063\hbar^{10}}{974098582732800} + \mathrm{O}(\hbar^{12})\,, \tag{3.106}$$

which confirms our analytic calculation at NLO in Eq. (3.80).

Comments on the algorithmic implementation

In order to obtain several hundreds of terms of the perturbative series in $\hbar$ for the first fermionic spectral trace of local $\mathbb{F}_0$ reasonably fast, we write a numerical algorithm that reproduces the analytic procedure described above. We comment here briefly on some technical details. Let us denote by $d_{\max}$ the maximum order in b to be computed numerically. We will work in the variable $\tilde{\mathsf{b}}$ defined in Eq. (3.97). After removing all non-rational factors from the intermediate steps and truncating all calculations at the power $\tilde{\mathsf{b}}^{d_{\max}+1}$

at every step, the computational complexity is dominated by the heavy multiplication of large multivariate polynomials, which is required at the early stage of the exponential expansion, and by the manipulation of the special functions that appear at the last stage of the integral evaluation. We implement our algorithm as the following two-step process.

(1) Starting from the series in Eq. (3.96), removing by hand the factor $1/(1+t)^{2p}$, $p \geq 1$, and truncating at order $d_{\max}+1$ in $\tilde{\mathsf{b}}$, we introduce the polynomial

$$\varphi_1(\tilde{\mathsf{b}}, t) = \sum_{p=1}^{d_{\max}/4} \tilde{\mathsf{b}}^{4p} P_p(t) \,, \tag{3.107}$$

where $P_p(t)$ is defined in Eq. (3.98), and its coefficients are computed explicitly. Then, we apply the variable redefinition

$$\tilde{\mathsf{b}} = t^{d_{\max}/2} \,, \tag{3.108}$$

which transforms the two-variables polynomial in Eq. (3.107) into a polynomial in the single variable t, which we denote by $\varphi_1(t)$. Note that there is a one-to-one map between the coefficients of $\varphi_1(\tilde{\mathsf{b}}, t)$ and the coefficients of $\varphi_1(t)$. This allows us to perform the exponential expansion in Eq. (3.99) in the univariate polynomial ring $\mathbb{Q}[t]$, instead of the bivariate polynomial ring $\mathbb{Q}[t][\mathsf{b}]$ without loss of information. Truncating at order

$$b_{\max} + 1 = (d_{\max}+1) d_{\max}/2 \,, \tag{3.109}$$

we denote the resulting polynomial after the exponential expansion as

$$\varphi_2(t) = \mathrm{e}^{\varphi_1(t)} = 1 + \sum_{m=1}^{b_{\max}} C_m t^m \,, \tag{3.110}$$

where $C_m \in \mathbb{Q}$ is known numerically.

(2) Note that $\varphi_2(t)$ corresponds to the series in brackets in Eq. (3.100). Indeed, we can write

$$\begin{aligned} \varphi_2(t) &= 1 + \sum_{k=1}^{d_{\max}/4} \left(t^{d_{\max}/2}\right)^{4k} \sum_{n=1}^{2k-1} c_{k,n} t^n \\ &= 1 + \sum_{k=1}^{d_{\max}/4} \sum_{n=1}^{2k-1} c_{k,n} t^{2k d_{\max}+n} \,, \end{aligned} \tag{3.111}$$

where $c_{k,n} \in \mathbb{Q}$ is defined in Eq. (3.101). Comparing Eqs. (3.110) and (3.111), we find that

$$c_{k,n} = C_{2k d_{\max}+n} \,, \tag{3.112}$$

for all $1 \leq n \leq 2k-1$ and $k \geq 1$. Therefore, we extract the polynomial $P'_k(t)$ in Eq. (3.100) by selecting the monomials of order t^m in Eq. (3.110) such that

$$2k d_{\max} + 1 \leq m \leq 2k d_{\max} + 2k - 1 \,. \tag{3.113}$$

Finally, the numerical coefficient of the term $\tilde{\mathsf{b}}^{4k}$, $k \geq 1$, in the perturbative expansion of $\mathrm{Tr}(\rho_{\mathbb{F}_0})$, after removing by hand its global pre-factor, is given by the finite sum

$$\sum_{n=1}^{2k-1} C_{2k d_{\max}+n} I(k,\, n) \,, \tag{3.114}$$

where $I(k, n) \in \pi\mathbb{Q}$ denotes the numerical result of the pre-evaluated integral in Eq. (3.104). We stress that numerical integration is not necessary.

3.3.2 Exploratory tests of higher instanton sectors

Let us denote by

$$\upsilon(\hbar) = \sum_{n=0}^{\infty} \bar{a}_{2n} \hbar^{2n} \in \mathbb{Q}[\![\hbar]\!] \tag{3.115}$$

the formal power series appearing in the RHS of Eq. (3.105), which is simply related to the semiclassical perturbative expansion of the spectral trace of local $\mathbb{F}_0$ by

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) \frac{4\hbar}{\pi} \sim \upsilon(\hbar) \,. \tag{3.116}$$

A numerical investigation of the perturbative expansion of the spectral trace of local $\mathbb{F}_0$ in the dual limit $\hbar \to \infty$ has been carried out by Gu and Mariño in [10].

Let us truncate the series in Eq. (3.115) to a very high but finite order $d \gg 1$ and denote the resulting $\mathbb{Q}$-polynomial by $\upsilon_d(\hbar)$ and its Borel transform by $\hat{\upsilon}_d(\zeta)$. Explicitly,

$$\upsilon_d(\hbar) = \sum_{n=0}^{d} \bar{a}_{2n} \hbar^{2n} \in \mathbb{Q}[\hbar] \,, \quad \hat{\upsilon}_d(\zeta) = \sum_{n=0}^{d} \frac{\bar{a}_{2n}}{(2n)!} \zeta^{2n} \in \mathbb{Q}[\zeta] \,, \tag{3.117}$$

where the coefficients $\bar{a}_{2n}$, $1 \leq n \leq d$, have been computed numerically as described in Section 3.3.1. The first few terms of $\upsilon_d(\hbar)$ are shown in Eq. (3.106). It is straightforward to verify that the perturbative coefficients satisfy the factorial growth

$$\bar{a}_{2n} \sim (-1)^n (2n)! (2\pi^2)^{-2n} \,, \quad n \gg 1 \,, \tag{3.118}$$

and we conclude that $\upsilon(\hbar)$ is a Gevrey-1 asymptotic series. As we have done in Section 3.2.2 for the local $\mathbb{P}^2$ geometry, we assume that $\hbar \in \mathbb{C}'$ and apply the machinery of *Padé–Borel approximation* [143–145] to the truncated series $\hat{\upsilon}_d(\zeta)$ to extrapolate the complex singularity structure of the exact analytically continued Borel function $\hat{\upsilon}(\zeta)$. Let d be even. We compute the singular points of the diagonal Padé approximant of order $d/2$ of the truncated Borel expansion $\hat{\upsilon}_d(\zeta)$, which we denote by $\hat{\upsilon}_d^{\mathrm{PB}}(\zeta)$, and we observe two dominant complex conjugate branch points at $\zeta = \pm 2\pi^2\mathrm{i}$, which match the leading divergent growth of the perturbative coefficients in Eq. (3.118). Two symmetric arcs of complex conjugate spurious poles accumulate at the dominant singularities, mimicking two branch cuts that emanate straight from the branch points along the positive and negative imaginary axis in opposite directions. Let us now introduce the complex numbers

$$\mathcal{A} = 2\pi^2 \,, \quad \zeta_n = n 2\pi^2 \mathrm{i} \,, \quad n \in \mathbb{Z}_{\neq 0} \,. \tag{3.119}$$

A zoom-in to the poles of the Padé approximant in the bottom part of the upper-half complex ζ-plane is shown in the leftmost plot in Fig. 3.5. Note that there might be subdominant true singularities of $\hat{\upsilon}_d(\zeta)$ that are obscured by the unphysical Padé poles representing the dominant branch cuts. In particular, educated by the example of local $\mathbb{P}^2$, we might guess that the dominant branch points at $\zeta = \zeta_{\pm 1}$ are repeated at integer multiples of $\mathcal{A}$ along the imaginary axis to form a discrete tower of singularities. To reveal their presence, we apply again the interpretation of Padé approximation in terms

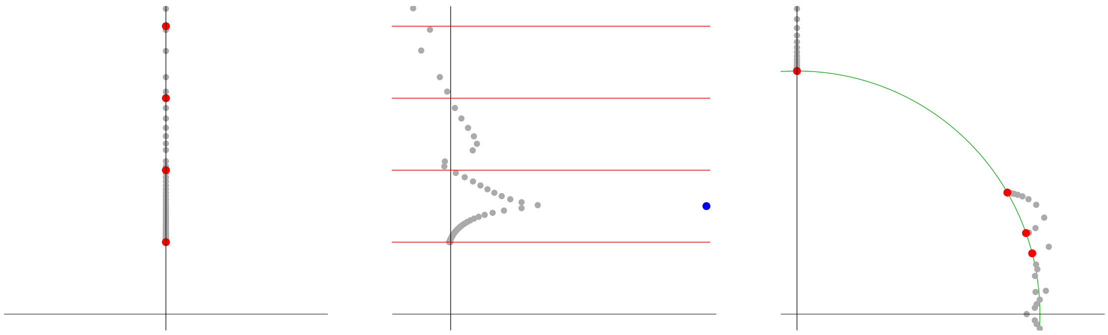


Figure 3.5: In the leftmost plot, a zoom-in to the poles of $\hat{\upsilon}_d^{\mathrm{PB}}(\zeta)$ (in gray) in the bottom part of the upper-half complex ζ-plane. We show the first few integer multiples of the dominant pole (in red). In the central plot, a zoom-in to the poles of $\hat{\upsilon}_{d,\mathrm{probe}}^{\mathrm{PB}}(\zeta)$ (in gray) in the bottom part of the upper-half complex ζ-plane, including the test charge singularity at $\zeta = \zeta_{\mathrm{probe}}$ (in blue). We show the horizontal lines intersecting the positive imaginary axis at the first few physical branch points (in red). In the rightmost plot, a zoom-in to the poles of $\hat{\upsilon}_d^{\mathrm{PCB}}(\eta)$ (in gray) in the upper-right quarter of the complex η-plane. We show the first few physical branch points (in red) and the unit circle (in green). The plots are obtained with $d = 300$.

of electrostatic potential theory [146, 147]. More concretely, to test the presence of the suspected NLO branch point at $\zeta = \zeta_2$, we add by hand the singular term

$$\hat{\upsilon}_{\mathrm{probe}}(\zeta) = (\zeta - \zeta_{\mathrm{probe}})^{-1/5} \,, \tag{3.120}$$

where $\zeta_{\mathrm{probe}} = \mathcal{A}\,(1/20 + 3\mathrm{i}/2)$, to the truncated sum $\hat{\upsilon}_d$. We then compute the $(d/2)$-diagonal Padé approximant

$$\hat{\upsilon}_{d,\mathrm{probe}}^{\mathrm{PB}}(\zeta) = (\hat{\upsilon}_d + \hat{\upsilon}_{\mathrm{probe}})^{\mathrm{PB}}(\zeta) \,. \tag{3.121}$$

The resulting Padé–Borel poles distribution is distorted, as shown in the central plot in Fig. 3.5, but the genuine physical singularities of $\hat{\upsilon}_d(\zeta)$ did not move. Consequently, the targeted true branch point at $\zeta = \zeta_2$ is now clearly visible, and so it is the test charge singularity at $\zeta = \zeta_{\mathrm{probe}}$. The other true branch points at $\zeta = \zeta_3, \zeta_4$ are outlined.

We can further improve the precision of the Padé extrapolation and analytic continuation of the truncated Borel series $\hat{\upsilon}_d(\zeta)$ using conformal maps. Namely, we perform the change of variable

$$\zeta = \mathcal{A}\frac{2\eta}{1-\eta^2} \,, \quad \eta \in \mathbb{C} \,, \tag{3.122}$$

which maps the cut Borel ζ-plane

$$\mathbb{C} \setminus \{(-\infty\mathrm{i}, \zeta_{-1}] \cup [\zeta_1, +\infty\mathrm{i})\} \tag{3.123}$$

into the interior of the unit disk $|\eta| < 1$. The dominant branch points $\zeta = \zeta_{\pm 1}$ are mapped into $\eta = \pm\mathrm{i}$, while the point at infinity is mapped into $\eta = \pm 1$. Correspondingly, the branch cuts $(-\infty\mathrm{i}, \zeta_{-1}]$ and $[\zeta_1, +\infty\mathrm{i})$ along the imaginary ζ-axis split each one into two identical copies that lie onto the two lower-half and upper-half quarters of the unit circle in the η-plane, respectively. The inverse conformal map is explicitly given by

$$\eta = \pm\sqrt{\frac{-1+\sqrt{1+(\zeta/\mathcal{A})^2}}{1+\sqrt{1+(\zeta/\mathcal{A})^2}}}\,, \quad \zeta \in \mathbb{C}\,. \tag{3.124}$$

We compute the $(d/2)$-diagonal Padé approximant of the conformally mapped Borel expansion, which we denote by $\hat{\upsilon}_d^{\mathrm{PCB}}(\eta)$. Its singularities in the complex η-plane are shown in the rightmost plot in Fig. 3.5. We observe two symmetric arcs of spurious poles emanating from $\eta = \pm\mathrm{i}$ along the imaginary axis in opposite directions. The smaller arcs of poles jumping along the unit circle towards the real axis represent the Padé boundary of convergence joining the conformal map images of the repeated singularities at $\zeta = \zeta_n$, $n \in \mathbb{Z}_{\neq 0}$. Note that the composition of Padé approximation and conformal maps naturally solves the problem of hidden singularities by separating the repeated branch points into different accumulation points on the unit circle in the conformally mapped complex η-plane. Thus, our numerical analysis motivates the following ansatz. The singularities of the exact Borel series $\hat{\upsilon}(\zeta)$ are logarithmic branch points at $\zeta = \zeta_n$, $n \in \mathbb{Z}_{\neq 0}$. We remark that the complex singularity pattern unveiled here is entirely analogous to what has been found in Section 3.2.2 for local $\mathbb{P}^2$ and is a particularly simple example of the peacock pattern described in Section 3.1. However, it turns out that the resurgent structure of the asymptotic series in Eq. (3.115) is more complicated than what has been observed in other examples.

Let us return to the factorization formula for the spectral trace of local $\mathbb{F}_0$ in Eq. (3.82). In the semiclassical limit $\hbar \to 0$, we have that $\tilde{q} = \mathrm{e}^{-2\pi^2\mathrm{i}/\hbar} \to 0$ as well, and the anti-holomorphic blocks $\tilde{g}(\tilde{q})$ and $\tilde{G}(\tilde{q})$ in Eqs. (3.84c) and (3.84d) contribute trivially at leading order as

$$\tilde{g}(\tilde{q}) \sim \frac{1}{2}\,, \quad \tilde{G}(\tilde{q}) \sim \tilde{q}\,. \tag{3.125}$$

On the other hand, the asymptotics of the holomorphic blocks $g(q)$ and $G(q)$ in Eqs. (3.84a) and (3.84b) for $q = \mathrm{e}^{2\mathrm{i}\hbar} \to 1$ depends a priori on the ray in the complex $\hbar$-plane along which the limit $\hbar \to 0$ is taken. We introduce the variable $\tau = \hbar/\pi$, such that $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and write

$$\tau = \mathrm{e}^{\mathrm{i}\alpha}/N\,, \quad \alpha \in \mathbb{R}\,, \quad N \in \mathbb{N}\,. \tag{3.126}$$

Taking $N \to \infty$ while keeping α fixed, the formula in Eq. (3.82) gives the semiclassical asymptotics

$$\mathrm{Tr}(\rho_{\mathbb{F}_0}) \sim -\frac{\mathrm{i}}{4}G(q) - \frac{\mathrm{i}4}{\tau}\mathrm{e}^{-2\pi\mathrm{i}/\tau}g(q)\,. \tag{3.127}$$

Note that the contribution of $g(q)$ is suppressed by the exponentially small factor $\mathrm{e}^{-2\pi^2\mathrm{i}/\hbar}$ corresponding to the one-instanton sector, yielding that $\mathrm{Tr}(\rho_{\mathbb{F}_0}) \sim -\mathrm{i}G(q)/4$ at leading order. We expect then to be able to recover the perturbative series $\upsilon(\hbar)$ in Eq. (3.115) from the *radial asymptotic behavior* of $G(q)$. Let us show that this is indeed the case.

We test the expected functional form

$$G\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) \sim C\tau^{-1}\left(1 + \sum_{n=1}^{\infty} c_n\tau^n\right)\,, \tag{3.128}$$

where $C \in \mathbb{C}$ and $c_n \in \mathbb{R}$. We fix $0 < \alpha < \pi/2$ and take $N \to \infty$. A numerical estimate of the overall constant C is obtained from the convergence of the sequences

$$\Re\left(\tau G\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right)\right) = \Re(C) + \mathcal{O}\left(1/N\right) , \tag{3.129a}$$

$$\Im\left(\tau G\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right)\right) = \Im(C) + \mathcal{O}\left(1/N\right) , \tag{3.129b}$$

which give $C \approx \mathrm{i}$, and we proceed to systematically extract the coefficients c_n in Eq. (3.128) as the large-N limits of the recursively-built sequences

$$\Re\left(\frac{1}{\tau^n}\left(-\mathrm{i}\tau G\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) - \sum_{j=0}^{n-1} c_j \tau^j\right)\right) = c_n + \mathcal{O}\left(1/N\right) , \quad n \in \mathbb{N}_{>0} , \tag{3.130}$$

where $c_0 = 1$. We obtain, in this way, the high-precision numerical estimates

$$c_{2n} \approx \pi^{2n} \bar{a}_{2n} , \quad c_{2n+1} \approx 0 , \quad n \in \mathbb{N} , \tag{3.131}$$

where $\bar{a}_{2n} \in \mathbb{Q}$ are the coefficients of the perturbative series $\upsilon(\hbar)$ in Eq. (3.115), as expected. The numerical convergence of the large-N limits of all sequences above has been accelerated with the help of Richardson transforms.

Let us now move on to the sub-dominant term in the RHS of Eq. (3.127) and determine the leading-order radial asymptotics of the q-hypergeometric series $g(q)$, for which we do not have an independent prediction. We observe that the standard functional ansatz

$$g\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) \sim C_1 \mathrm{e}^{C_2/\tau} \tau^b , \tag{3.132}$$

where $C_1, C_2 \in \mathbb{C}$ and $b \in \mathbb{R}$, fails our numerical tests, hinting at a possible logarithmic-type behavior. We formulate the new ansatz[9]

$$g\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) \sim C_1 \log(\tau) + C_2 , \tag{3.133}$$

where $C_1, C_2 \in \mathbb{C}$, and we test it as follows. Again, let us fix $0 < \alpha < \pi/2$. We define the numerical sequences

$$R_N = \Re(g(q)) , \quad I_N = \Im(g(q)) , \quad N \in \mathbb{N} , \tag{3.134}$$

which satisfy

$$R_N \sim -\Re(C_1)\log(N) + \Re(C_2) - \alpha\Im(C_1) , \tag{3.135a}$$

$$I_N \sim -\Im(C_1)\log(N) + \Im(C_2) + \alpha\Re(C_1) , \tag{3.135b}$$

for $N \gg 1$. We find that $\Re(C_1) = -1/\pi$ from the convergence of the large-N relation

$$S_N^{(1)} = N(R_{N+1} - R_N) \sim -\Re(C_1) . \tag{3.136}$$

Analogously, we compute the large-N limit of the sequence

$$S_N^{(2)} = R_N + \Re(C_1)\log(N) \sim \Re(C_2) - \alpha\Im(C_1) , \tag{3.137}$$

which turns out to be independent of α, yielding $\Im(C_1) = 0$ and $\Re(C_2) = 0.9225325\ldots$. Finally, the estimate $\Im(C_2) = 1/2$ is obtained from the convergence of the sequence

$$S_N^{(3)} = I_N - \alpha\Re(C_1) \sim \Im(C_2) , \tag{3.138}$$

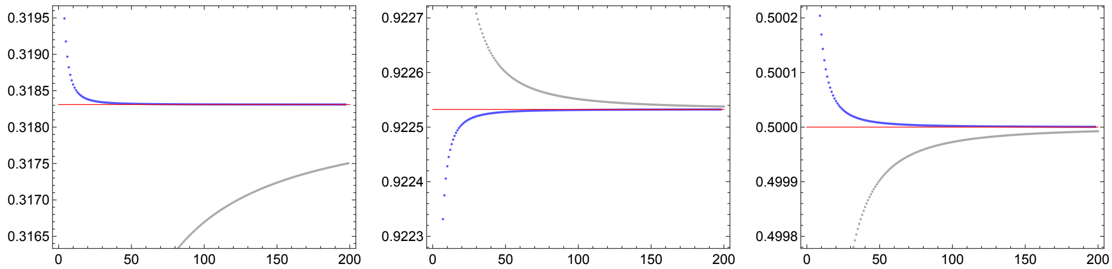


Figure 3.6: The sequences $S_N^{(1)}$ (left), $S_N^{(2)}$ (center), and $S_N^{(3)}$ (right) are shown in gray, their second Richardson transforms in blue, and their estimated asymptotic limits in red. The plots are obtained with $\alpha = \pi/3$ and N up to 200.

for $N \gg 1$. Again, all sequences are accelerated using Richardson transforms, as shown in the plots in Fig. 3.6. We conclude that the q-series $g(q)$ has the proposed leading-order asymptotics in Eq. (3.133). More precisely, we have that

$$g\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) \sim -\frac{1}{\pi}\log(\tau) + B + \frac{\mathrm{i}}{2}\,, \tag{3.139}$$

where $B = 0.9225325\ldots$. Note that the results that we have obtained from the radial asymptotic analysis of $g(q)$ and $G(q)$ are effectively independent of the choice of angle $0 < \alpha < \pi/2$.

We remark that, as a consequence of Eq. (3.127), the holomorphic block $g(q)$ contains sub-dominant, exponentially suppressed perturbative corrections to the semiclassical perturbative expansion of the first spectral trace of local $\mathbb{F}_0$, which is itself captured by the q-series $G(q)$. It follows then that the one-instanton resurgent contribution to the discontinuity, which we have denoted before by $\upsilon_1(\hbar)$, has a leading-order logarithmic behavior of the functional form in Eq. (3.133). Indeed, a straightforward independent test shows that the standard guess for $\upsilon_1(\hbar)$ in Eq. (3.50) fails since the corresponding numerical extrapolation of the one-instanton coefficients does not converge—in agreement with the radial asymptotics prediction. As a next step, we can simply extend the leading-order behavior in Eq. (3.133) to a full perturbative expansion captured by a complete ansatz of the form

$$g\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) \sim C_1 \log(\tau)\left(1 + \sum_{n=1}^{\infty} d_n \tau^n\right) + C_2\left(1 + \sum_{n=1}^{\infty} e_n \tau^n\right)\,, \tag{3.140}$$

where $d_n, e_n \in \mathbb{C}$, whose numerical investigation we leave for future work.

[9]Note that the leading asymptotics of the standard hypergeometric function ${}_2F_1(1/2, 1/2; 1; \mathrm{e}^{\pi\mathrm{i}\tau})$ in the limit $\tau \to 0$ is known to be $-\frac{1}{\pi}\log(-\frac{\pi\mathrm{i}}{16}\tau)$.

Chapter 4

A new prediction of the TS/ST correspondence

Let us go back to the general framework of Chapter 2 and consider the topological string on a toric CY threefold X with $s = b_2(X)$. As the moduli of X and its mirror $\tilde{X}$ are allowed to vary, local mirror symmetry gives access to the global structure of the fibration of the topological string theory over the moduli space. Each of the overlapping open patches covering the moduli space is parametrized by a different set of canonical flat coordinates, which defines a local symplectic frame and is associated with different expressions for the free energies. In the geometric formalism of [50], a change of symplectic frame corresponds to an electromagnetic duality transformation in $\mathrm{Sp}(2s, \mathbb{Z})$ of the periods, while the topological string amplitudes in different frames are related by a formal Fourier–Laplace transform.

In this chapter, we review the construction of the standard 't Hooft regime associated with the weakly coupled limit $g_s \to 0$ of the topological string and the geometric interpretation of the corresponding specialization of the integral in Eq. (2.56). Then, we show how a dual, WKB double-scaling regime associated with the limit $\hbar \to 0$ can be similarly introduced so that the symplectic transformation encoded in the corresponding specialization of the integral in Eq. (2.56) can be interpreted as a change of symplectic frame in the moduli space of X. We obtain, in this way, a new analytic prediction for the semiclassical asymptotics of the fermionic spectral traces in terms of the free energies of the refined topological string in the NS limit, which allows us to propose a non-trivial analytic test of the conjecture of [11, 12]. We reproduce [1, Section 6].

4.1 The standard double-scaling regime

Due to the functional forms of the worldsheet and WKB grand potentials in Eqs. (2.51) and (2.53), respectively, there are appropriate scaling regimes in the coupling constants in which only one of the two components effectively contributes to the total grand potential in Eq. (2.50). In the *standard double-scaling limit* [125, 150, 151]

$$\hbar \to \infty\,, \quad \mu_j \to \infty\,, \quad \frac{\mu_j}{\hbar} = \zeta_j \;\text{ fixed}\,, \quad j = 1, \ldots, g_\Sigma\,, \tag{4.1a}$$

$$m_k = \xi_k^{2\pi/\hbar} \;\text{ fixed}\,, \quad k = 1, \ldots, r_\Sigma\,, \tag{4.1b}$$

the quantum mirror map in Eq. (2.24) becomes trivial. Simultaneously, the total grand potential in Eq. (2.50) has the asymptotic genus expansion

$$J^{\text{'t Hooft}}(\boldsymbol{\zeta}, \boldsymbol{m}, \hbar) = \sum_{g=0}^{\infty} J_g(\boldsymbol{\zeta}, \boldsymbol{m})\, \hbar^{2-2g}\,, \tag{4.2}$$

where the fixed-genus 't Hooft grand potential $J_g(\boldsymbol{\zeta}, \boldsymbol{m})$ are given by

$$J_0(\boldsymbol{\zeta}, \boldsymbol{m}) = \frac{1}{16\pi^4}\hat{F}_0(\boldsymbol{T}) + \frac{1}{4\pi^2}\sum_{i=1}^{s} b_i^{\text{NS}} T_i + A_0(\boldsymbol{m})\,, \tag{4.3a}$$

$$J_1(\boldsymbol{\zeta}, \boldsymbol{m}) = \hat{F}_1(\boldsymbol{T}) + A_1(\boldsymbol{m})\,, \tag{4.3b}$$

$$J_g(\boldsymbol{\zeta}, \boldsymbol{m}) = (4\pi^2)^{2g-2}(\hat{F}_g(\boldsymbol{T}) - C_g) + A_g(\boldsymbol{m})\,, \quad g \geq 2\,, \tag{4.3c}$$

under the assumption that the function $A(\boldsymbol{\xi}, \hbar)$, which is defined in Eq. (2.53), has the formal power series expansion

$$A(\boldsymbol{\xi}, \hbar) = \sum_{g=0}^{\infty} A_g(\boldsymbol{m})\, \hbar^{2-2g}\,. \tag{4.4}$$

The constants C_g are the ones occurring in Eq. (2.40), while $\hat{F}_g(\boldsymbol{t})$ is the standard topological string amplitude at genus g after the B-field has been turned on, as in Eq. (2.51). We have introduced the $\hbar$-independent *rescaled Kähler parameters*

$$T_i(\boldsymbol{\zeta}, \boldsymbol{m}) = \frac{2\pi}{\hbar} t_i(\boldsymbol{\mu}, \boldsymbol{\xi})\,, \quad i = 1, \ldots, s\,, \tag{4.5}$$

which satisfy[1]

$$T_i(\boldsymbol{\zeta}, \boldsymbol{m}) = 2\pi \sum_{j=1}^{g_\Sigma} C_{ij}\zeta_j + \sum_{k=1}^{r_\Sigma} \alpha_{ik} \log m_k\,, \quad i = 1, \ldots, s\,. \tag{4.6}$$

Except for three polynomial terms, the WKB grand potential in Eq. (2.53) only contributes non-perturbatively to the 't Hooft grand potential in Eq. (4.2).

Due to the TS/ST correspondence, there is a related 't Hooft limit for the fermionic spectral traces that extracts the all-genus expansion of the conventional topological string on X. Namely,

$$\hbar \to \infty\,, \quad N_j \to \infty\,, \quad \frac{N_j}{\hbar} = \lambda_j \ \text{fixed}\,, \quad j = 1, \ldots, g_\Sigma\,. \tag{4.7}$$

The saddle-point evaluation of the integral in Eq. (2.56) in the double-scaling regime in Eq. (4.7), which is performed using the standard 't Hooft expansion in Eq. (4.2) for the total grand potential, represents a symplectic transformation from the large radius point to the so-called *maximal conifold point*[2] in the moduli space of Kähler structures of X [12, 88]. Specifically, it follows from the geometric formalism of [50] that the 't Hooft parameters λ_j are flat coordinates on the moduli space corresponding to the maximal conifold frame of

[1] We neglect the exponentially small corrections $\mathcal{O}(\mathrm{e}^{-\hbar\zeta_j}/\hbar)$.

[2] The maximal conifold point can be defined as the unique point in the conifold locus of moduli space where its connected components intersect transversally.

the geometry. In particular, the fermionic spectral traces in Eq. (2.35) have the asymptotic genus expansion

$$\log Z^{\text{'t Hooft}}(\boldsymbol{N}, \boldsymbol{\xi}, \hbar) = \sum_{g=0}^{\infty} \mathcal{F}_g(\boldsymbol{\lambda}, \boldsymbol{m})\, \hbar^{2-2g} \,, \tag{4.8}$$

where the coefficient $\mathcal{F}_g(\boldsymbol{\lambda}, \boldsymbol{m})$ can be interpreted as the genus-g free energy of the standard topological string on X in the maximal conifold frame. We remark that, as a consequence of Eq. (4.8), the fermionic spectral traces provide a well-defined, non-perturbative completion of the conventional topological string theory on X.

4.2 The WKB double-scaling regime

We examine a second scaling limit of the total grand potential, which is dual to the standard 't Hooft limit in Eq. (4.1). Namely, in the semiclassical regime

$$\hbar \to 0\,, \quad \mu_j \;\; \text{fixed}\,, \quad j = 1, \ldots, g_\Sigma\,, \tag{4.9a}$$

$$\xi_k \;\; \text{fixed}\,, \quad k = 1, \ldots, r_\Sigma\,, \tag{4.9b}$$

the quantum mirror map $\boldsymbol{t}(\boldsymbol{z}, \hbar)$ becomes classical by construction as it reduces to the formula in Eq. (2.10). The total grand potential in Eq. (2.50) retains only the contribution of the WKB grand potential in Eq. (2.53) coming from the NS limit of the refined topological string on X and can be formally expanded in powers of $\hbar$ as

$$J^{\text{WKB}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \hbar) = \sum_{n=0}^{\infty} J_n(\boldsymbol{\mu}, \boldsymbol{\xi})\, \hbar^{2n-1} \,, \tag{4.10}$$

where the fixed-order WKB grand potentials $J_n(\boldsymbol{\mu}, \boldsymbol{\xi})$ are given by

$$J_0(\boldsymbol{\mu}, \boldsymbol{\xi}) = \sum_{i=1}^{s} \frac{t_i}{2\pi} \frac{\partial F_0^{\text{NS}}(\boldsymbol{t})}{\partial t_i} - \frac{1}{\pi} F_0^{\text{NS}}(\boldsymbol{t}) + 2\pi \sum_{i=1}^{s} b_i t_i + A_0(\boldsymbol{\xi}) \,, \tag{4.11a}$$

$$J_n(\boldsymbol{\mu}, \boldsymbol{\xi}) = \sum_{i=1}^{s} \frac{t_i}{2\pi} \frac{\partial F_n^{\text{NS}}(\boldsymbol{t})}{\partial t_i} + \frac{(2n-2)}{2\pi} F_n^{\text{NS}}(\boldsymbol{t}) + A_n(\boldsymbol{\xi}) \,, \quad n \geq 1 \,, \tag{4.11b}$$

under the assumption that the function $A(\boldsymbol{\xi}, \hbar)$, which appears in Eq. (2.53), has the asymptotic expansion[3]

$$A(\boldsymbol{\xi}, \hbar) = \sum_{n=0}^{\infty} A_n(\boldsymbol{\xi})\, \hbar^{2n-1} \,. \tag{4.12}$$

The constants b_i are the same ones that occur in Eq. (2.41), while $F_n^{\text{NS}}(\boldsymbol{t})$ is the NS topological string amplitude of order n defined in Eq. (2.47). Note that the worldsheet grand potential in Eq. (2.51) does not contribute to the semiclassical perturbative expansion of the total grand potential, but it contains explicit non-perturbative effects in $\hbar$.

Given the integral representation in Eq. (2.56), we can define a second 't Hooft-like limit for the fermionic spectral traces $Z(\boldsymbol{N}, \boldsymbol{\xi}, \hbar)$ that corresponds to the semiclassical regime

[3] Both assumptions on $A(\boldsymbol{\xi}, \hbar)$ in Eqs. (4.4) and (4.12) can be easily tested in examples.

for the total grand potential in Eq. (4.9) and is dual to the standard 't Hooft limit in Eq. (4.7). We refer to it as the *WKB double-scaling regime.* Namely,

$$\hbar \to 0\,, \quad N_j \to \infty\,, \quad N_j \hbar = \sigma_j \text{ fixed}\,, \quad j = 1, \ldots, g_\Sigma\,. \tag{4.13}$$

Let us consider the case of a toric del Pezzo CY threefold X for simplicity, which has $g_\Sigma = 1$. The following arguments can then be straightforwardly generalized to the case of arbitrary genus. Using the 't Hooft-like expansion in Eq. (4.10) for the total grand potential, the integral formula in Eq. (2.56) in the WKB double-scaling regime in Eq. (4.13) becomes

$$Z(N, \boldsymbol{\xi}, \hbar) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} d\mu \, \exp\left(J^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) - \frac{1}{\hbar}\mu\sigma \right)\,, \tag{4.14}$$

where $\mathcal{C}$ is an integration contour going from $\mathrm{e}^{-\mathrm{i}\pi/3}\infty$ to $\mathrm{e}^{+\mathrm{i}\pi/3}\infty$ in the complex plane of the chemical potential, as shown in Fig. 2.1. Let us introduce the functions

$$S(\mu, \boldsymbol{\xi}, \sigma) = \mu\sigma - J_0(\mu, \boldsymbol{\xi})\,, \tag{4.15a}$$

$$\bar{Z}(\mu, \boldsymbol{\xi}, \hbar) = \exp\left(\sum_{n=1}^{\infty} J_n(\mu, \boldsymbol{\xi}) \hbar^{2n-1} \right)\,, \tag{4.15b}$$

and write Eq. (4.14) equivalently as

$$Z(N, \boldsymbol{\xi}, \hbar) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} d\mu \, \mathrm{e}^{-\frac{1}{\hbar} S(\mu, \boldsymbol{\xi}, \sigma)} \, \bar{Z}(\mu, \boldsymbol{\xi}, \hbar)\,. \tag{4.16}$$

We identify the critical point $\mu = \mu^*$ of the integrand by solving the classical relation

$$\left. \frac{\partial J_0(\mu, \boldsymbol{\xi})}{\partial \mu} \right|_{\mu = \mu^*} = \sigma\,, \tag{4.17}$$

which gives σ as a function of μ^*, and vice-versa. Evaluating the integral in Eq. (4.16) via saddle-point approximation around μ^* in the limit $\hbar \to 0$, we obtain the formal power series expansion

$$Z(N, \boldsymbol{\xi}, \hbar) = \exp\left(\sum_{n=0}^{\infty} \mathcal{J}_n(\sigma, \boldsymbol{\xi}) \hbar^{2n-1} \right)\,. \tag{4.18}$$

The leading-order contribution is given by the Legendre transform

$$\mathcal{J}_0(\sigma, \boldsymbol{\xi}) = J_0(\mu^*, \boldsymbol{\xi}) - \sigma\mu^*\,, \tag{4.19}$$

where the saddle point μ^* is expressed as a function of σ through Eq. (4.17), the NLO correction is given by the one-loop approximation to the integral in Eq. (4.16), that is,

$$\mathcal{J}_1(\sigma, \boldsymbol{\xi}) = J_1(\mu^*, \boldsymbol{\xi}) - \frac{1}{2} \log\left(2\pi \frac{\partial^2 J_0(\mu^*, \boldsymbol{\xi})}{\partial \mu^2} \right)\,, \tag{4.20}$$

and the higher-order contributions $\mathcal{J}_n(\sigma, \boldsymbol{\xi})$, $n \geq 1$, can be computed systematically by summing over higher-loop Feynman diagrams. Note that differentiating the formula in Eq. (4.19) gives

$$\frac{\partial \mathcal{J}_0(\sigma, \boldsymbol{\xi})}{\partial \sigma} = -\mu^*\,. \tag{4.21}$$

4.3 Interpreting the change of frame

For concreteness, let us consider the example of the local $\mathbb{P}^2$ geometry described in Section 3.2. We fix the parametrization of moduli space given by the Batyrev coordinate z in Eq. (3.14) and consider the Picard–Fuchs differential equation in Eq. (3.15). The classical periods of the meromorphic one-form λ in Eq. (2.11) are the zeros of the Picard–Fuchs differential operator $\mathcal{L}_{\mathbb{P}^2}$. Solving Eq. (3.15) locally around the large radius point $z = 0$ and using, for instance, the Frobenius method, produces a trivial constant solution and the two non-trivial independent solutions

$$w_1(z) = \begin{cases} \log(-z) + \tilde{w}_1(z)\,, & -1/27 < z < 0\,, \\ \log(z) + \tilde{w}_1(z)\,, & z > 0\,, \end{cases} \tag{4.22a}$$

$$w_2(z) = \begin{cases} \log(-z)^2 + 2\tilde{w}_1(z)\log(-z) + \tilde{w}_2(z)\,, & -1/27 < z < 0\,, \\ \log(z)^2 + 2\tilde{w}_1(z)\log(z) + \tilde{w}_2(z)\,, & z > 0\,, \end{cases} \tag{4.22b}$$

where $\tilde{w}_1(z), \tilde{w}_2(z)$ are the formal power series

$$\begin{aligned} \tilde{w}_1(z) &= 3\sum_{j=1}^{\infty}(-z)^j \frac{(3j-1)!}{(j!)^3} \\ &= -6z + 45z^2 - 560z^3 + \frac{17325z^4}{2} - \frac{756756z^5}{5} + \dots\,, \end{aligned} \tag{4.23a}$$

$$\begin{aligned} \tilde{w}_2(z) &= 18\sum_{j=1}^{\infty}(-z)^j \frac{(3j-1)!}{(j!)^3} \sum_{n=j+1}^{3j-1} \frac{1}{n} \\ &= -18z + \frac{423z^2}{2} - 2972z^3 + \frac{389415z^4}{8} - \frac{21981393z^5}{25} + \dots\,. \end{aligned} \tag{4.23b}$$

Note that these series expansions converge for $|z| < 1/27$ and the formulae in Eq. (2.12) become

$$t = -w_1(z)\,, \quad \partial_t F_0 = \frac{w_2(z)}{6}\,, \tag{4.24}$$

with a suitable choice of normalization.

Let us derive exact expressions in terms of special functions for the analytic continuation of the classical periods at large radius. We remark that such closed formulae have already been computed in the well-known context of the standard 't Hooft regime [110]. However, this involves the conventional topological free energies after the B-field has been turned on, which is taken into account by implementing the change of sign $z \mapsto -z$. Using the explicit results of [110] and reversing the effects of the change of sign in z, we find that the classical periods of local $\mathbb{P}^2$ at large radius can be analytically continued and resummed in closed form in the two distinct regions $-1/27 < z < 0$ and $z > 0$ of moduli space. Choosing appropriately the branch of the logarithm function, the first period $w_1(z)$ is given by

$$w_1(z) = \begin{cases} \log(-z) - 6z\, {}_4F_3\left(1, 1, \tfrac{4}{3}, \tfrac{5}{3};\, 2, 2, 2;\, -27z\right)\,, & -1/27 < z < 0\,, \\ \log(z) - 6z\, {}_4F_3\left(1, 1, \tfrac{4}{3}, \tfrac{5}{3};\, 2, 2, 2;\, -27z\right)\,, & z > 0\,, \end{cases} \tag{4.25}$$

while the second period $w_2(z)$ is given by

$$w_2(z) = \begin{cases} -\frac{5\pi^2}{3} + \frac{3}{\pi\sqrt{3}} G^{3,2}_{3,3}\left(\begin{matrix} 1/3, & 2/3, & 1 \\ 0, & 0, & 0 \end{matrix}; -27z\right), & -1/27 < z < 0\,, \\ -\frac{2\pi^2}{3} + \frac{3}{\pi\sqrt{3}} G^{3,2}_{3,3}\left(\begin{matrix} 1/3, & 2/3, & 1 \\ 0, & 0, & 0 \end{matrix}; -27z\right) & \\ \quad -2\pi\mathrm{i}\log(z) + 12\pi\mathrm{i} z\, {}_4F_3\left(1, 1, \frac{4}{3}, \frac{5}{3};\, 2, 2, 2;\, -27z\right), & z > 0\,, \end{cases} \tag{4.26}$$

where ${}_4F_3$ is a generalized hypergeometric function and $G^{3,2}_{3,3}$ is a Meijer G-function. We show the analytically continued periods in Eqs. (4.25) and (4.26) together with their series expansions at the origin in Eqs. (4.22a) and (4.22b) in the plots in Fig. 4.1. Let us comment

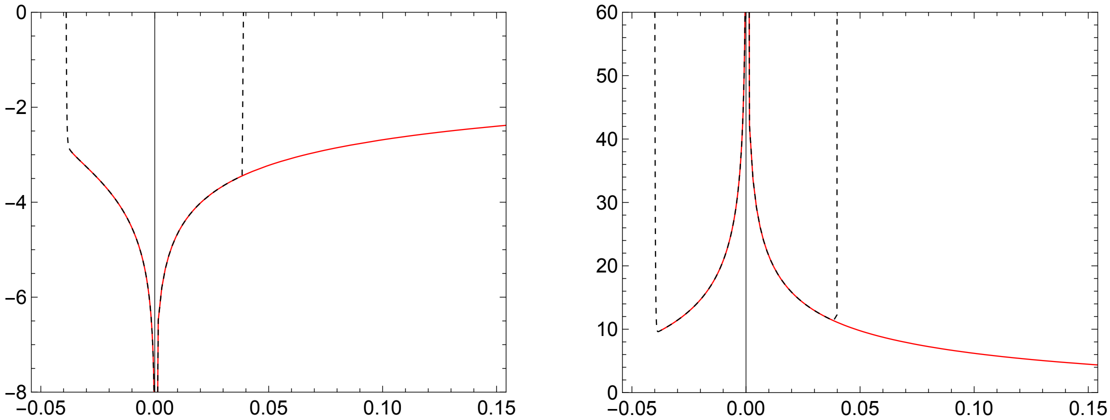


Figure 4.1: The functions $w_1(z)$ and $w_2(z)$ in Eqs. (4.25) and (4.26) (in solid red) on the left and on the right, respectively, and their large radius expansions in Eqs. (4.22a) and (4.22b) (in dashed black), for $-1/27 < z < 0$ and $z > 0$. The formal power series in Eq. (4.23) are truncated at $j = 200$.

briefly on their behavior at the critical points of the geometry. Both functions $w_1(z)$, $w_2(z)$ have a vertical asymptote at the large radius point $z = 0$, where they approach $\mp\infty$ from both sides, respectively. In the orbifold limit $z \to +\infty$, both periods have a horizontal asymptote corresponding to

$$\lim_{z\to+\infty} w_1(z) = 0\,, \quad \lim_{z\to+\infty} w_2(z) = -\frac{2\pi^2}{3}\,. \tag{4.27}$$

Finally, at the conifold point $z = -1/27$, they reach the finite limits

$$\lim_{z\to-\frac{1}{27}^+} w_1(z) = -3\log(3) + \frac{2}{9}\,{}_4F_3\left(1, 1, \frac{4}{3}, \frac{5}{3};\, 2, 2, 2;\, 1\right) = -2.9075935\ldots\,, \tag{4.28a}$$

$$\lim_{z\to-\frac{1}{27}^+} w_2(z) = \pi^2 = 9.8696044\ldots\,. \tag{4.28b}$$

We show the derivatives $\partial_z w_1(z)$, $\partial_z w_2(z)$ together with their series expansions at the origin in the plots in Fig. 4.2. Note that the function $w_1(z)$ is singular at the conifold point, while $w_2(z)$ is not, and we have that

$$\lim_{z\to -\frac{1}{27}^+} \partial_z w_1(z) = -\infty\,, \qquad \lim_{z\to -\frac{1}{27}^+} \partial_z w_2(z) = 36\sqrt{3}\pi\,. \tag{4.29}$$

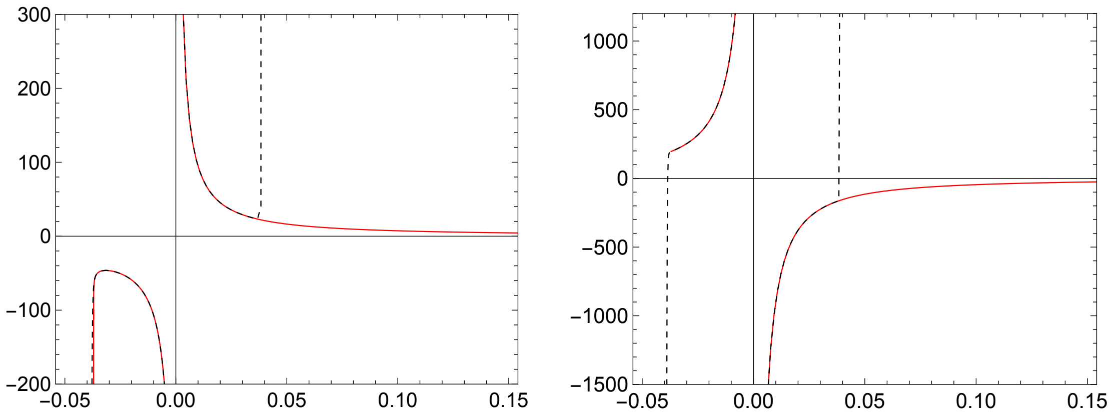


Figure 4.2: The derivatives $\partial_z w_1(z)$ and $\partial_z w_2(z)$ of the periods in Eqs. (4.25) and (4.26) (in solid red) on the left and on the right, respectively, and their large radius expansions following Eq. (4.22) (in dashed black), for $-1/27 < z < 0$ and $z > 0$. The formal power series in Eq. (4.23) are truncated at $j = 200$.

We recall that the function $A(\hbar)$ appearing in Eq. (2.53) is conjectured in closed form for the local $\mathbb{P}^2$ geometry. Namely, [125]

$$A(\hbar) = \frac{3}{4}A_c(\hbar/\pi) - \frac{1}{4}A_c(3\hbar/\pi)\,, \tag{4.30}$$

where the function $A_c(\hbar)$ can be expressed as [152, 153]

$$A_c(\hbar) = \frac{2\zeta(3)}{\pi^2\hbar}\left(1 - \frac{\hbar^3}{16}\right) + \frac{\hbar^2}{\pi^2}\int_0^\infty \frac{x}{e^{\hbar x}-1}\log\left(1 - e^{-2x}\right)dx\,. \tag{4.31}$$

It follows that the perturbative expansion of $A(\hbar)$ in the limit $\hbar \to 0$ satisfies the functional form in Eq. (4.12). More precisely,

$$A(\hbar) = \frac{4\zeta(3)}{3\pi\hbar} + \frac{\hbar}{8\pi} + \frac{\hbar^3}{2880\pi} - \frac{\hbar^5}{604800\pi} + \frac{\hbar^7}{33868800\pi} + \mathcal{O}(\hbar^9)\,. \tag{4.32}$$

Substituting the values $A_0 = 4\zeta(3)/3\pi$ and $b = 1/12$ in Eq. (4.11a), the leading-order WKB grand potential can be written as

$$J_0(\mu) = \frac{t}{2\pi}\partial_t F_0(t) - \frac{1}{\pi}F_0(t) + \frac{\pi}{6}t + \frac{4\zeta(3)}{3\pi}\,, \tag{4.33}$$

where $F_0(t) = F_0^{\rm NS}(t)$ is the genus-zero topological free energy at large radius. Integrating Eq. (4.24) and fixing the integration constant appropriately, we have that [87]

$$F_0(t) = \frac{t^3}{18} + 3e^{-t} - \frac{45}{8}e^{-2t} + \frac{244}{9}e^{-3t} - \frac{12333}{64}e^{-4t} + \frac{211878}{125}e^{-5t} + \mathcal{O}\left(e^{-6t}\right)\,. \tag{4.34}$$

We can now apply Eqs. (4.17), (4.24), and (4.33) to express the 't Hooft parameter σ as a function of z. Recalling that the Kähler modulus t is related to the chemical potential by $t = 3\mu$, we find that

$$\sigma = \frac{3}{2\pi}\left(t\,\partial_t^2 F_0(t) - \partial_t F_0(t)\right) + \frac{\pi}{2} = \frac{1}{4\pi}\left(\frac{w_1(z)\,\partial_z w_2(z)}{\partial_z w_1(z)} - w_2(z)\right) + \frac{\pi}{2}\,. \tag{4.35}$$

Observe that the modular coordinate $\partial_t^2 F_0(z)$ also appears in the study of the modular properties and BPS spectrum of local $\mathbb{P}^2$ [51, 52, 62]. Using the exact formulae for $w_1(z)$, $w_2(z)$ in Eqs. (4.25) and (4.26), we determine the explicit dependence of σ on the Batyrev coordinate $z \in (-1/27, 0) \cup (0, +\infty)$, which is shown in the plot in Fig. 4.3. The

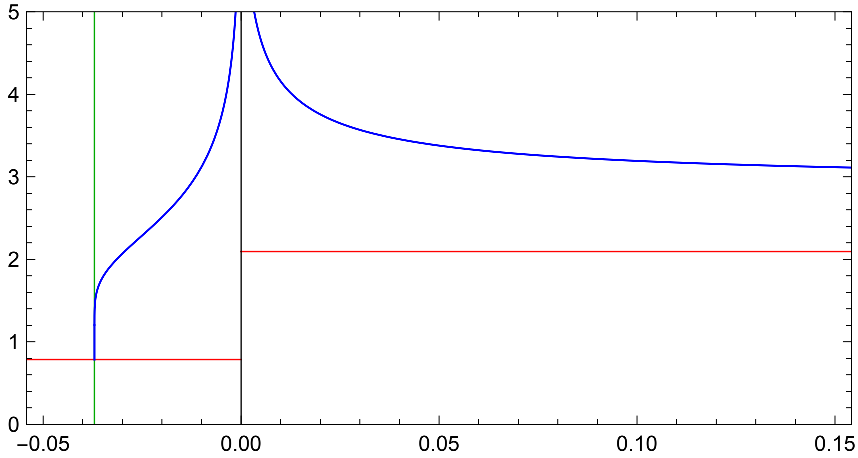


Figure 4.3: The 't Hooft parameter σ (in solid blue) in Eq. (4.35) as a function of the Batyrev coordinate $z \in (-1/27, 0) \cup (0, +\infty)$. We show the horizontal lines corresponding to $\sigma = \pi/4, 2\pi/3$ (in solid red) and the vertical line at $z = -1/27$ (in solid green).

two distinct analytic continuations of the classical periods correspond to two distinct analyticity regions of the 't Hooft parameter, whose behavior at the critical points is captured by the limits

$$\lim_{z \to -\frac{1}{27}^+} \sigma(z) = \frac{\pi}{4}\,, \qquad \lim_{z \to 0^\pm} \sigma(z) = +\infty\,, \qquad \lim_{z \to +\infty} \sigma(z) = \frac{2\pi}{3}\,. \tag{4.36}$$

Note that, analogously to the function $w_1(z)$, also σ is singular at the conifold point of moduli space. In particular, we have that

$$\lim_{z \to -\frac{1}{27}^+} \partial_z \sigma(z) = +\infty\,. \tag{4.37}$$

We stress that σ is strictly positive for all values of z considered here.

4.4 Analysis of the leading-order behavior

Let us now go back to the formula in Eq. (4.18) and the new analytic prediction of the TS/ST correspondence contained in it. The asymptotic behavior of the fermionic spectral traces $Z(N, \boldsymbol{\xi}, \hbar)$ in the WKB double-scaling regime in Eq. (4.13) is determined by the

WKB grand potential of the topological string theory, that is, by the total free energy of the refined topological string in the NS limit, after the transformation of local symplectic frame which is encoded in the integral in Eq. (4.14). We can write Eq. (4.18) at leading order as

$$Z(N,\boldsymbol{\xi},\hbar) = \exp\left(\mathcal{J}_0(\sigma,\boldsymbol{\xi})\frac{1}{\hbar} + \mathcal{O}(\hbar)\right) , \tag{4.38}$$

where $\mathcal{J}_0(\sigma,\boldsymbol{\xi})$ is a highly non-trivial function of the 't Hooft parameter σ and is given explicitly by the Legendre transform in Eq. (4.19). In what follows, we will drop from our notation the explicit dependence on the mass parameters $\boldsymbol{\xi}$ for simplicity. We stress that the parametric dependence of the functional coefficients in Eq. (4.18) on the moduli space of Kähler structures of X makes directly studying the full resurgent structure of the given asymptotic expansion a much more difficult task than it is to investigate the numerical series $\phi_N(\hbar)$ in Eq. (3.1a) at fixed $N \in \mathbb{N}$—as we do in this thesis. Nonetheless, we can perform a detailed analysis of the dominant contribution in Eq. (4.38).

Let us consider the example of local $\mathbb{P}^2$ once more. Using Eqs. (4.24), (4.33), and (4.35), we find that

$$\begin{aligned}\mathcal{J}_0(\sigma) &= -\frac{t^2}{2\pi}\partial_t^2 F_0(t) + \frac{t}{\pi}\partial_t F_0(t) - \frac{1}{\pi}F_0(t) + \frac{4\zeta(3)}{3\pi} \\ &= -\frac{1}{6\pi}w_1(z)\left(\frac{w_1(z)\partial_z w_2(z)}{2\partial_z w_1(z)} + w_2(z)\right) - \frac{1}{\pi}F_0(t) + \frac{4\zeta(3)}{3\pi} ,\end{aligned} \tag{4.39}$$

and we apply Eqs. (4.25) and (4.26) to express the classical periods as analytic functions of $z \in (-1/27, 0) \cup (0, +\infty)$, as before. If we consider the large radius expansion of the genus-zero topological free energy $F_0(t)$, which is given in Eq. (4.34), we find that $\mathcal{J}_0(\sigma)$ has a vertical asymptote for $z \to 0^{\pm}$, where it approaches $-\infty$ from both sides, it remains negative for positive values of z, while it reaches zero at the value $z = z_0 = -0.0232698\ldots$, corresponding to $\sigma = \sigma_0 = 2.36250\ldots$, and it becomes positive for values of z smaller than z_0. We show the resulting explicit dependence of $\mathcal{J}_0(\sigma)$ on z near the large radius point in the plot in Fig. 4.4.

We can similarly access the asymptotic behavior of $\mathcal{J}_0(\sigma)$ at the orbifold point in moduli space, where the 't Hooft parameter σ approaches the natural limit of $2\pi/3$. Recall that the Batyrev coordinate z is related to the true complex structure parameter κ by $z = \kappa^{-3}$. We express the analytically continued classical periods $w_1(z)$, $w_2(z)$ in Eqs. (4.25) and (4.26) for $z > 0$ as functions of κ. Substituting them into Eq. (4.24) and integrating over the Kähler modulus t, we find the orbifold expansion of the classical prepotential $F_0(\kappa)$. Precisely,

$$F_0(\kappa) = \frac{4\zeta(3)}{3} - \frac{\pi^2\Gamma(\frac{1}{3})}{9\Gamma(\frac{2}{3})^2}\kappa + \frac{\pi\Gamma(\frac{1}{3})^2}{6\sqrt{3}\Gamma(\frac{2}{3})^4}\kappa^2 + \frac{\pi^2\Gamma(\frac{2}{3})^3}{18\Gamma(\frac{1}{3})}\kappa^3 + \mathcal{O}(\kappa^4) , \tag{4.40}$$

after fixing the integration constant appropriately [11]. As a consequence of the orbifold limits in Eq. (4.27) and the orbifold expansion in Eq. (4.40), the formula in Eq. (4.39) gives the leading-order asymptotics

$$\mathcal{J}_0\left(\sigma \to \frac{2\pi}{3}\right) \sim J_0\left(\mu \to 0\right) \sim -\frac{1}{\pi}F_0(\kappa) + \frac{4\zeta(3)}{3\pi} = 0 + \mathcal{O}(\kappa) , \tag{4.41}$$

that is, the dominant term in the WKB expansion of $\log Z(N,\hbar)$ for local $\mathbb{P}^2$, as given in Eq. (4.38), vanishes at the orbifold point $\kappa = 0$ of the moduli space, where it matches the order-zero WKB grand potential in Eq. (4.33).

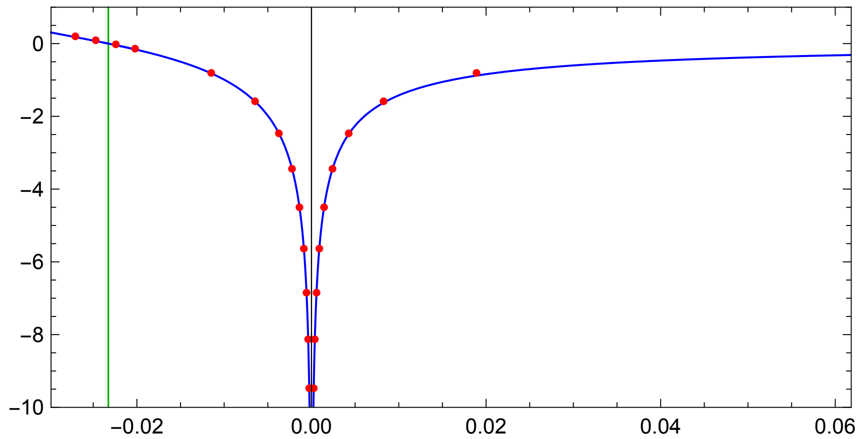


Figure 4.4: The transformed order-zero WKB grand potential $\mathcal{J}_0(\sigma)$ (in solid blue) in Eq. (4.39) as a function of the Batyrev coordinate $z \approx 0$. We show the vertical line at $z = z_0$ (in solid green). We truncate the large radius expansion of $F_0(t)$ in Eq. (4.34) at order $\mathcal{O}\left(\mathrm{e}^{-15t}\right)$. We show the numerical data obtained from the sequences in Eq. (4.48) (as red dots) for $k = 7$, N up to 80, and $\sigma = 2.2, 2.3, 2.4, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5, 7$. The convergence is accelerated with a Richardson transform of order 2.

Let us point out that, after expanding the conjectural formula for the spectral determinant in Eq. (2.55) at leading order in the semiclassical regime in Eq. (4.9), we obtain that

$$\log \Xi(\kappa, \hbar) \sim J^{\mathrm{WKB}}(\mu, \hbar)\,, \quad \hbar \to 0\,, \quad \mu \;\text{ fixed}\,, \tag{4.42}$$

while the generalized theta function $\Theta(\mu, \hbar)$ gives sub-leading, oscillatory corrections to the dominant asymptotics. Since the fixed-N fermionic spectral traces are initially defined as the functional coefficients in the orbifold expansion of $\Xi(\kappa, \hbar)$ in Eq. (2.31), the statements in Eqs. (4.41) and (4.42) refute a global behavior of the form $\mathrm{e}^{-1/\hbar}$ in their semiclassical asymptotic expansion—contrary to what occurs in the limit $\hbar \to \infty$, as observed in [10] and in Section 5.2. This new prediction of the TS/ST correspondence is explicitly verified for the first fermionic spectral trace of both local $\mathbb{P}^2$ and local $\mathbb{F}_0$ in Sections 3.2.1 and 3.3.1, thus providing a first successful quantum-mechanical test of the analytic formulation encoded in Eq. (4.18). We comment that obtaining analytic results from Eqs. (2.31) and (2.55) is generally a difficult task, as it requires knowledge of the analytic continuation of the total grand potential to the orbifold frame of the geometry for the values of $\hbar$ of interest.

We conclude by presenting here a numerical test of consistency of the analytic prediction in Eq. (4.39) at large radius for local $\mathbb{P}^2$. It is well-known that, by expanding the total grand potential $J(\mu, \hbar)$ in Eq. (2.56) in the large-μ limit, the fermionic spectral traces $Z(N, \hbar)$, $N \in \mathbb{N}$, can be decomposed into an infinite sum of *Airy functions* and their derivatives. Namely, [11]

$$\exp\left(J(\mu, \hbar)\right) = \exp\left(\frac{C(\hbar)}{3}\mu^3 + B(\hbar)\mu + A(\hbar)\right) \sum_{l,n} a_{l,n} \mu^n \mathrm{e}^{-l\mu}\,, \tag{4.43}$$

where $a_{l,n} \in \mathbb{C}$, and the sum runs over $n \in \mathbb{N}$ and $l = 3p + 6\pi q/\hbar$ for $p, q \in \mathbb{N}$. We have denoted by $A(\hbar)$ the same function that appears in Eq. (2.53) and have introduced

$$B(\hbar) = \frac{\pi}{2\hbar} - \frac{\hbar}{16\pi}\,, \quad C(\hbar) = \frac{9}{4\pi\hbar}\,. \tag{4.44}$$

Under some assumptions on the convergence of the expansion in Eq. (4.43), one can substitute it into the TS/ST statement in Eq. (2.56) and perform the integration term-by-term, which yields [11]

$$Z(N,\hbar) = \frac{\mathrm{e}^{A(\hbar)}}{C(\hbar)^{1/3}} \sum_{l,n} a_{l,n} \left(-\frac{\partial}{\partial N}\right)^n \mathrm{Ai}\left(\frac{N+l-B(\hbar)}{C(\hbar)^{1/3}}\right)\,, \tag{4.45}$$

where $\mathrm{Ai}(x)$ is the Airy function, and the indices l, n are defined as above. The leading-order behavior in the so-called *M-theory limit*, which is for $N \to \infty$ and $\hbar$ fixed, is given by the Airy function

$$Z(N,\hbar) \sim \mathrm{Ai}\left(\frac{N-B(\hbar)}{C(\hbar)^{1/3}}\right)\,, \quad N \gg 1\,, \tag{4.46}$$

while all additional terms in the RHS of Eq. (4.45) are exponentially small corrections. Note that the series in Eq. (4.45) is convergent and allows us to obtain highly accurate numerical estimates of the fermionic spectral traces [134]. More precisely, we truncate it to the finite sum $Z^{(k)}(N,\hbar)$, where $k \in \mathbb{N}$ denotes the number of terms that have been retained. Since we are interested in the WKB double-scaling regime in Eq. (4.13), let us fix a value of the 't Hooft parameter $\sigma \in \mathbb{R}_{>0}$ and take

$$\hbar = \sigma/N\,, \quad N \in \mathbb{N}\,. \tag{4.47}$$

Thus, the sequence of numerical approximations $Z^{(k)}(N,\sigma/N)$ tends to the true function $Z(N,\hbar)$ for $k, N \to \infty$ for each choice of σ. Note, however, that we can only access in this way those values of σ that correspond to the large radius frame in moduli space, that is, $z \approx 0$. As we have found in Section 4.3, this amounts to $\sigma \gg 1$. We obtain a numerical estimate of the transformed order-zero WKB grand potential $\mathcal{J}_0(\sigma)$ in Eq. (4.39) near the large radius point from the convergence of the sequence

$$\frac{\sigma}{N}\log\left(Z^{(k)}\left(N,\frac{\sigma}{N}\right)\right) \sim \mathcal{J}_0(\sigma)\,, \quad N \gg 1\,, \tag{4.48}$$

with k fixed, which is accelerated with the help of Richardson transforms. The resulting numerical data correctly captures the analytic behavior described by the formula in Eq. (4.39) at large radius, including the change of sign of $\mathcal{J}_0(\sigma)$ occurring at $z = z_0$, as shown for a selection of points in the plot in Fig. 4.4. As expected, the agreement increases as the points get closer to $z = 0$ and improves systematically by taking larger values of $k \in \mathbb{N}$. The two z-coordinates corresponding to each choice of σ are obtained by inverting the relation in Eq. (4.35).

Part III

Resurgence, L-functions, and quantum modularity

Chapter 5

The exact resurgent structures of local $\mathbb{P}^2$

We return to the crucial example of local $\mathbb{P}^2$introduced in Section 3.2 and take the perspective of the fermionic spectral traces once more. The computational advantage of this choice was highlighted by the numerical study of Section 3.2.2. Yet, we can say much more.

In this chapter, we solve exactly the resurgent structure of the semiclassical $\hbar$-expansion of the logarithm of the spectral trace of local $\mathbb{P}^2$, which unveils a fascinating arithmetic construction. The Stokes constants are given by explicit divisor sum functions, which can be expressed as the Dirichlet convolution of simple arithmetic functions, and they are encoded in a generating series written in terms of q-Pochhammer symbols. The perturbative coefficients are captured by special values of a known L-function, which admits a notable factorization as the product of a Riemann zeta function and a Dirichlet L-function, corresponding to the Dirichlet factors in the decomposition of the Stokes constants. Analogously, we present a complete analytic solution to the resurgent structure of the perturbative series arising in the dual strongly coupled limit $\hbar \to \infty$, previously studied numerically by Gu and Mariño in [10]. We prove that the Stokes constants are manifestly related to their semiclassical analogs, the mapping being realized by a simple exchange of divisors, while the perturbative coefficients are special values of the same L-function above after unitary shifts in the arguments of its factors. A new number-theoretic duality between the weakly and strongly coupled regimes emerges. We reproduce part of [1, Section 4].

5.1 The limit $\hbar \to 0$

Let us denote by $\phi(\hbar)$ the formal power series appearing in the exponent in Eq. (3.38). Namely,

$$\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n} \hbar^{2n} \in \mathbb{Q}[\![\hbar]\!] \,, \tag{5.1}$$

whose coefficients a_{2n}, $n \in \mathbb{Z}_{>0}$, are

$$a_{2n} = (-1)^{n-1} \frac{B_{2n} B_{2n+1}(2/3)}{2n(2n+1)!} 3^{2n+1} \,. \tag{5.2}$$

Note that $\phi(\hbar)$ is simply related to the semiclassical perturbative expansion of the logarithm of the spectral trace of local $\mathbb{P}^2$ by

$$\log \mathrm{Tr}(\rho_{\mathbb{P}^2}) - 3\log\Gamma(1/3) + \log(6\pi\hbar) \sim \phi(\hbar)\,. \tag{5.3}$$

We refer the reader to Section 3.2.1 for the detailed derivation of the above formula. Recall that a preliminary numerical analysis of the resurgent structure of $\phi(\hbar)$ was performed in Section 3.2.2. In this section, we perform an independent analytic study, which cross-checks and confirms our numerical investigation and fully determines the exact resurgent structure of the asymptotic series in Eq. (5.1). In doing so, we uncover a rich fabric of startling number-theoretic properties that lead us to the novel results of Section 5.3 and the subsequent chapters.

5.1.1 Resumming the Borel transform

We recall that the Bernoulli polynomials have the asymptotic behavior [154]

$$B_{2n}(z) \sim 2(-1)^{n-1}\cos(2\pi z)\frac{(2n)!}{(2\pi)^{2n}}\,, \tag{5.4a}$$

$$B_{2n+1}(z) \sim 2(-1)^{n-1}\sin(2\pi z)\frac{(2n+1)!}{(2\pi)^{2n+1}}\,, \tag{5.4b}$$

for $n \gg 1$. It follows that the coefficients of $\phi(\hbar)$ satisfy the factorial growth

$$|a_{2n}| \sim (2n)!\mathcal{A}_0^{-2n} \quad n \gg 1\,, \quad \mathcal{A}_0 = \frac{4\pi^2}{3}\,, \tag{5.5}$$

and $\phi(\hbar)$ is indeed a Gevrey-1 asymptotic series. Its Borel transform

$$\hat{\phi}(\zeta) = 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}B_{2n+1}(2/3)}{2n(2n)!(2n+1)!}(3\zeta)^{2n} \in \mathbb{Q}\{\zeta\} \tag{5.6}$$

is the germ of an analytic function at the origin in the complex ζ-plane.

Proposition 5.1.1. *Using the definition in Eq. (C.2), we interpret the Borel transform $\hat{\phi}(\zeta)$ in Eq. (5.6) as the* Hadamard product

$$\hat{\phi}(\zeta) = (f \diamond g)(\zeta)\,, \tag{5.7}$$

where the formal power series $f(\zeta)$ and $g(\zeta)$, which are defined as

$$f(\zeta) = \sum_{n=1}^{\infty}\frac{B_{2n+1}(2/3)}{(2n+1)!}\zeta^{2n}\,, \tag{5.8a}$$

$$g(\zeta) = 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n)!}(3\zeta)^{2n}\,, \tag{5.8b}$$

have finite radius of convergence at $\zeta = 0$ and can be resummed explicitly as[1]

$$f(\zeta) = \frac{1}{2+4\cosh(\zeta/3)} - \frac{1}{6}\,, \quad |\zeta| < 2\pi\,, \tag{5.9a}$$

$$g(\zeta) = -3\log\left(\frac{2}{3\zeta}\sin\left(\frac{3\zeta}{2}\right)\right)\,, \quad |\zeta| < 2\pi/3\,. \tag{5.9b}$$

[1] We impose that $f(0) = g(0) = 0$ in order to eliminate the removable singularities of $f(\zeta)$, $g(\zeta)$ at the origin.

Proof. The Bernoulli polynomials with argument 2/3 are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n(2/3)}{n!}\zeta^n = \frac{\zeta e^{2\zeta/3}}{e^{\zeta}-1}\,, \quad |\zeta| < 2\pi\,. \tag{5.10}$$

We apply the hyperbolic identities

$$\frac{e^{\zeta}-1}{2e^{\zeta/2}} = \sinh(\zeta/2)\,, \quad e^{\zeta/6} = \sinh(\zeta/6) + \cosh(\zeta/6)\,, \tag{5.11}$$

and we take the odd part of both sides of Eq. (5.10). We obtain in this way that

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}(2/3)}{(2n+1)!}\zeta^{2n+1} = \frac{\zeta}{2}\frac{\sinh(\zeta/6)}{\sinh(\zeta/2)}\,, \quad |\zeta| < 2\pi\,. \tag{5.12}$$

Using the sum-of-arguments and the half-argument identities for $\sinh(\zeta/3+\zeta/6)$ and $\coth(\zeta/6)$, respectively, the formula in Eq. (5.12) together with Eq. (5.8a) yields the statement in Eq. (5.9a). Let us consider Eq. (C.5a) for $a = 1$ and apply the identity in Eq. (5.64a) for $\zeta(2n,1) = \zeta(2n)$, $n \geq 1$. We find that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}}{2n(2n)!}(3\zeta)^{2n} = \log\left(\Gamma\left(1+\frac{3\zeta}{2\pi}\right)\Gamma\left(1-\frac{3\zeta}{2\pi}\right)\right)\,, \quad |\zeta| < 2\pi/3\,, \tag{5.13}$$

and the statement in Eq. (5.9b) then follows from Eq. (5.8b) and Euler's reflection formula for the gamma function, that is,

$$\Gamma(1+x)\,\Gamma(1-x) = \frac{\pi x}{\sin(\pi x)}\,, \quad x \in \mathbb{C}\setminus\mathbb{Z}\,, \tag{5.14}$$

with the choice $x = 3\zeta/2\pi$. □

After being analytically continued to the whole complex plane, the function $f(\zeta)$ has poles of order one along the imaginary axis at

$$\mu_k^{\pm} = 2\pi i(\pm 1 + 3k)\,, \quad k \in \mathbb{Z}\,, \tag{5.15}$$

while the function $g(\zeta)$ has logarithmic branch points along the real axis at

$$\nu_m = \frac{2\pi}{3}m\,, \quad m \in \mathbb{Z}_{\neq 0}\,. \tag{5.16}$$

We illustrate the singularities of $f(\zeta)$, $g(\zeta)$ in the Borel plane in Fig. 5.1 on the left.

Proposition 5.1.2. *The Borel transform $\hat{\phi}(\zeta)$ in Eq. (5.6) can be expressed as*

$$\hat{\phi}(\zeta) = -\frac{3\sqrt{3}}{2\pi}\sum_{k\in\mathbb{Z}}\frac{1}{1+3k}\log\left(\frac{4\pi i(1+3k)}{3\zeta}\sin\left(\frac{3\zeta}{4\pi i(1+3k)}\right)\right)\,, \tag{5.17}$$

which is a well-defined, exact function of ζ.

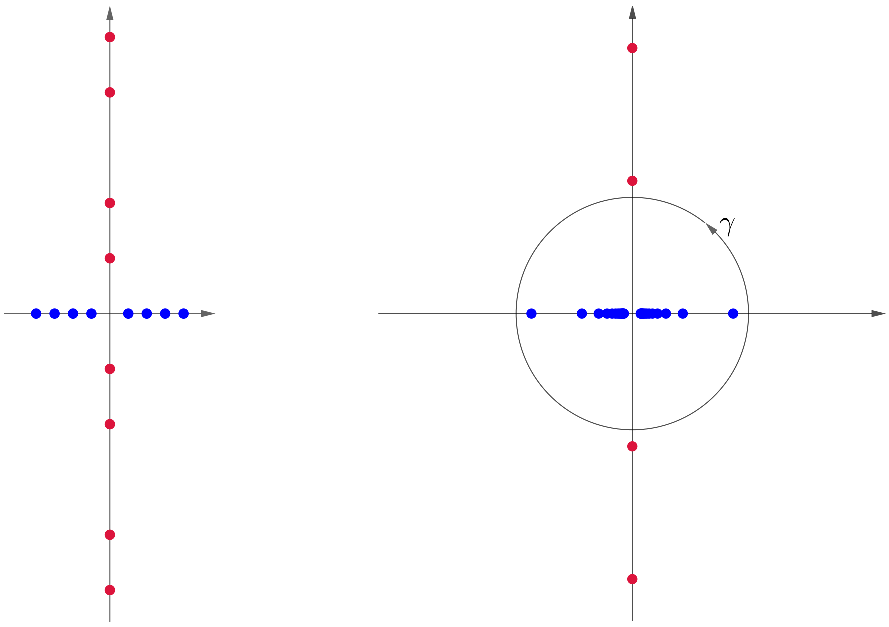


Figure 5.1: On the left, the first few singularities of $f(\zeta)$ (in red) and $g(\zeta)$ (in blue) in the complex ζ-plane. On the right, the contour γ and the first few singularities of $f(s)$ (in red) and $g(\zeta/s)$ (in blue) in the complex s-plane with reference values $r = 5.5$ and $\zeta = 10$.

Proof. We will now apply Hadamard's multiplication theorem [155, 156]. We refer to Appendix C for a short introduction. Let γ be a circle in the complex s-plane centered at the origin $s = 0$ with radius $0 < r < 2\pi$. As a consequence of the Hadamard decomposition in Eq. (5.7), the Borel transform can be written as the integral

$$\begin{aligned} \hat{\phi}(\zeta) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma f(s) g(\zeta/s) \frac{\mathrm{d}s}{s} \\ &= -\frac{3}{4\pi \mathrm{i}} \int_\gamma \left(\frac{1}{1 + 2\cosh(s/3)} - \frac{1}{3} \right) \log\left(\frac{2s}{3\zeta} \sin\left(\frac{3\zeta}{2s} \right) \right) \frac{\mathrm{d}s}{s} \,, \end{aligned} \tag{5.18}$$

for $|\zeta| < 2\pi r/3$. For such values of ζ, the function $s \mapsto g(\zeta/s)$ has logarithmic branch points at $s = \zeta/\nu_m$, $m \in \mathbb{Z}_{\neq 0}$, which sit inside the contour of integration γ and accumulate at the origin, and no singularities for $|s| > r$. The function $f(s)$ has simple poles at the points $s = \mu_k^\pm$ with residues

$$\operatorname*{Res}_{s=2\pi\mathrm{i}(\pm 1+3k)} f(s) = \mp \frac{\sqrt{3}\mathrm{i}}{2} \,, \quad k \in \mathbb{Z} \,. \tag{5.19}$$

We illustrate the singularities of $f(s), g(\zeta/s)$ in the complex s-plane in Fig. 5.1 on the right. By Cauchy's residue theorem, the integral in Eq. (5.18) can be evaluated by summing the residues at the poles of the integrand that lie outside γ, thus allowing us to express the Borel transform as an exact function of ζ. More precisely, we find the desired analytic formula

$$\begin{aligned}\hat{\phi}(\zeta) &= -\sum_{k\in\mathbb{Z}} \operatorname*{Res}_{s=2\pi\mathrm{i}(1+3k)} f(s)g(\zeta/s)\frac{1}{s} - \sum_{k\in\mathbb{Z}} \operatorname*{Res}_{s=2\pi\mathrm{i}(-1+3k)} f(s)g(\zeta/s)\frac{1}{s} \\ &= -\frac{3\sqrt{3}}{2\pi}\sum_{k\in\mathbb{Z}}\frac{1}{1+3k}\log\left(\frac{4\pi\mathrm{i}(1+3k)}{3\zeta}\sin\left(\frac{3\zeta}{4\pi\mathrm{i}(1+3k)}\right)\right).\end{aligned} \tag{5.20}$$

The convergence of the infinite sum in the second line of Eq. (5.20) can be easily verified by, *e.g.*, the limit comparison test. □

Corollary 5.1.3. *The singularities of the Borel transform* $\hat{\phi}(\zeta)$ *in Eq. (5.17) are logarithmic branch points located along the imaginary axis at*

$$\zeta_{k,m} = \mu_k^+\nu_m = \frac{4\pi^2\mathrm{i}}{3}(1+3k)m\,, \quad k\in\mathbb{Z}\,, \quad m\in\mathbb{Z}_{\neq 0}\,, \tag{5.21}$$

which we write equivalently as

$$\zeta_n = \mathcal{A}_0\mathrm{i}n\,, \quad n\in\mathbb{Z}_{\neq 0}\,, \tag{5.22}$$

where $\mathcal{A}_0 = 4\pi^2/3$ *as before—that is, the branch points lie at all non-zero integer multiples of the two complex conjugate dominant singularities at* $\pm 4\pi^2\mathrm{i}/3$*, as illustrated in Fig. 5.2.*

This is the simplest occurrence of the *peacock pattern* of singularities described in Section 3.1. There are two Stokes lines at the angles $\pm\pi/2$. Note that the analytic

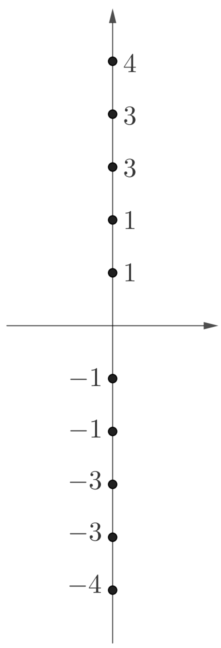


Figure 5.2: The first few singularities of the Borel transform of the asymptotic series $\phi(\hbar)$, defined in Eq. (5.1), and the associated integer constants $\alpha_n \in \mathbb{Z}_{\neq 0}$, defined in Eq. (5.30).

expression for the Borel transform in Eq. (5.17) is explicitly simple resurgent, and thus we expect its local singular behavior to be of the form in Eq. (1.18).

Corollary 5.1.4. *The local expansion of the Borel transform $\hat{\phi}(\zeta)$ in Eq. (5.17) at $\zeta = \zeta_n$, $n \in \mathbb{Z}_{\neq 0}$, is given by*

$$\hat{\phi}(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \dots \,, \tag{5.23}$$

where $S_n \in \mathbb{C}$ is the Stokes constant.

Proof. The local expansion of $\hat{\phi}(\zeta)$ around the logarithmic singularity $\zeta = \zeta_n$ is obtained by summing the contributions from all pairs $(k, m) \in \mathbb{Z} \times \mathbb{Z}_{\neq 0}$ such that $n = (1 + 3k)m$. There are finitely many such pairs of integers, and we collect them into a set I_n for each $n \in \mathbb{Z}_{\neq 0}$. For a fixed value of $k \in \mathbb{Z}$, we denote the corresponding term in the sum in Eq. (5.17) by

$$f_k(\zeta) = -\frac{3\sqrt{3}}{2\pi(1+3k)} \log\left(\frac{4\pi \mathrm{i}(1+3k)}{3\zeta} \sin\left(\frac{3\zeta}{4\pi \mathrm{i}(1+3k)}\right)\right) . \tag{5.24}$$

We expand it locally around $\zeta = \zeta_{k,m}$ for every choice of $m \in \mathbb{Z}_{\neq 0}$ and obtain

$$f_k(\zeta) = -\frac{s_{k,m}}{2\pi \mathrm{i}} \log(\zeta - \zeta_{k,m}) + \dots \,, \tag{5.25}$$

where the dots denote regular terms in $\zeta - \zeta_{k,m}$, and $s_{k,m}$ is a complex number. Since $\zeta_n = \zeta_{k,m}$ for all $(k, m) \in I_n$, it follows that the local expansion of $\hat{\phi}(\zeta)$ at $\zeta = \zeta_n$ is given by

$$\hat{\phi}(\zeta) = \sum_{k \in \mathbb{Z}} f_k(\zeta) = -\frac{S_n}{2\pi \mathrm{i}} \log(\zeta - \zeta_n) + \dots \,, \tag{5.26}$$

where the Stokes constant S_n is the finite sum

$$S_n = \sum_{(k,m) \in I_n} s_{k,m} \,. \tag{5.27}$$

□

It follows from Corollary 5.1.4 that the locally analytic function that resurges at $\zeta = \zeta_n$ is simply

$$\hat{\phi}_n(\zeta - \zeta_n) = 1 \,, \quad n \in \mathbb{Z}_{\neq 0} \,. \tag{5.28}$$

We observe that the Laplace transform in Eq. (1.21) acts trivially on constants, and thus we also have that

$$\phi_n(\hbar) = 1 \,, \quad n \in \mathbb{Z}_{\neq 0} \,, \tag{5.29}$$

that is, there are no perturbative contributions from the higher-order instanton sectors. Moreover, the procedure above allows us to derive analytically all the Stokes constants. After being suitably normalized, the Stokes constants S_n are rational numbers and are simply related to an interesting *sequence of integers* α_n, $n \in \mathbb{Z}_{\neq 0}$. In particular, we find that

$$S_1 = 3\sqrt{3}\mathrm{i} \,, \quad S_n = S_1 \frac{\alpha_n}{n} \quad n \in \mathbb{Z}_{\neq 0,1} \,, \tag{5.30a}$$

$$\alpha_n = -\alpha_{-n} \,, \quad \alpha_n \in \mathbb{Z}_{>0} \quad n \in \mathbb{Z}_{>0} \,. \tag{5.30b}$$

Explicitly, the first several integer constants α_n, $n > 0$, are

$$1, 1, 3, 3, 4, 3, 8, 5, 9, 4, 10, 9, 14, 8, 12, 11, 16, 9, 20, 12, \dots \,. \tag{5.31}$$

The pattern of singularities in the Borel plane and the associated $\alpha_n \in \mathbb{Z}_{\neq 0}$ are shown in Fig. 5.2.

5.1.2 Closed formulae for the Stokes constants

We will now present and prove a series of exact arithmetic formulae for the Stokes constants S_n of the asymptotic series $\phi(\hbar)$, defined in Eq. (5.1), and the related integer constants α_n, defined in Eq. (5.30), for $n \in \mathbb{Z}_{\neq 0}$. We begin by showing that both sequences $\{S_n\}$ and $\{\alpha_n\}$ define explicit *divisor sum functions.*

Theorem 5.1.5. *The normalized Stokes constant S_n/S_1, where $S_1 = 3\sqrt{3}\mathrm{i}$, is determined by the positive integer divisors of $n \in \mathbb{Z}_{\neq 0}$ according to the closed formula*

$$\frac{S_n}{S_1} = \sum_{\substack{d|n \\ d\equiv_3 1}} \frac{1}{d} - \sum_{\substack{d|n \\ d\equiv_3 2}} \frac{1}{d}\,, \tag{5.32}$$

which implies that $S_n = S_{-n}$ and $S_n/S_1 \in \mathbb{Q}_{>0}$.

Proof. Let us denote by D_n the set of positive integer divisors of n. We recall that n satisfies the factorization property

$$n = (1+3k)m\,, \quad k \in \mathbb{Z}\,, \quad m \in \mathbb{Z}_{\neq 0}\,. \tag{5.33}$$

It follows that either

$$m = \frac{n}{d}\,, \quad k = \frac{d-1}{3}\,, \tag{5.34}$$

where $d \in D_n$ such that $d-1$ is divisible by 3, or

$$m = -\frac{n}{d}\,, \quad k = -\frac{d+1}{3}\,, \tag{5.35}$$

where $d \in D_n$ such that $d+1$ is divisible by 3. In the first case of $d \equiv_3 1$, substituting the values of k, m from Eq. (5.34) into Eqs. (5.24) and (5.21), we find that the contribution to the Stokes constant S_n coming from the local expansion of $f_k(\zeta)$ around $\zeta_{k,m}$ is simply $s_{k,m} = 3\sqrt{3}\mathrm{i}/d$. In the second case of $d \equiv_3 2$, substituting the values of k, m from Eq. (5.35) into Eqs. (5.24) and (5.21), we find that the contribution to the Stokes constant S_n coming from the local expansion of $f_k(\zeta)$ around $\zeta_{k,m}$ is simply $s_{k,m} = -3\sqrt{3}\mathrm{i}/d$. Finally, for any divisor $d \in D_n$ which is a multiple of 3, neither $d-1$ or $d+1$ are divisible by 3, which implies that the choice $m = \pm n/d$ is not allowed, and the corresponding contribution is $s_{k,m} = 0$. Putting everything together and using Eq. (5.27), we find the desired statement. □

We note that the arithmetic formula for the Stokes constants in Eq. (5.32) can be written equivalently as a closed expression for the integer constants α_n, $n \in \mathbb{Z}_{\neq 0}$. Namely,

$$\alpha_n = \sum_{\substack{d|n \\ \frac{n}{d}\equiv_3 1}} d - \sum_{\substack{d|n \\ \frac{n}{d}\equiv_3 2}} d\,, \tag{5.36}$$

which implies that $\alpha_n = -\alpha_{-n}$ and $\alpha_n \in \mathbb{Z}_{>0}$ for all $n > 0$. Two corollaries then follow straightforwardly from Theorem 5.1.5.

Proposition 5.1.6. *The positive integer constants α_n, $n \in \mathbb{Z}_{>0}$, satisfy the closed formulae*

$$\alpha_{p_1^{e_1}} = \frac{p_1^{e_1+1}-1}{p_1-1}\,, \quad \alpha_{p_2^{e_2}} = \frac{p_2^{e_2+1}+(-1)^{e_2}}{p_2+1}\,, \quad \alpha_{p_3^{e_3}} = p_3^{e_3}\,, \tag{5.37}$$

where $e_i \in \mathbb{N}$, and $p_i \in \mathbb{P}$ are prime numbers such that $p_i \equiv_3 i$ for $i = 1, 2, 3$. Moreover, they obey the multiplicative property

$$\alpha_n = \prod_{p \in \mathbb{P}} \alpha_{p^e}\,, \quad n = \prod_{p \in \mathbb{P}} p^e\,, \quad e \in \mathbb{N}\,. \tag{5.38}$$

Proof. The three closed formulae follow directly from Eq. (5.36). Explicitly, let $n = p^e$ with $p \in \mathbb{P}$ and $e \in \mathbb{N}$. We have that

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 1}} d = p^e\,, \quad \sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 2}} d = 0\,, \quad \text{if} \quad p \equiv 0 \mod 3\,, \tag{5.39a}$$

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 1}} d = \sum_{i=0}^{e} p^i = \frac{p^{e+1}-1}{p-1}\,, \quad \sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 2}} d = 0\,, \quad \text{if} \quad p \equiv 1 \mod 3\,, \tag{5.39b}$$

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 1}} d = \sum_{i=0}^{\lfloor e/2 \rfloor} p^{e-2i}\,, \quad \sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 2}} d = \sum_{i=0}^{\lfloor e/2 \rfloor} p^{e-(2i+1)}\,, \quad \text{if} \quad p \equiv 2 \mod 3\,. \tag{5.39c}$$

Let us now prove the multiplicity property. We will prove a slightly stronger statement. We write $n = pq$ for $p, q \in \mathbb{Z}_{>0}$ coprimes. We choose a positive integer divisor $d|n$, and we write $d = st$ where $s|p$ and $t|q$. Consider two cases:

(1) Suppose that $n/d \equiv_3 1$. Then, either $p/s \equiv_3 q/t \equiv_3 1$, or $p/s \equiv_3 q/t \equiv_3 2$, and therefore

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 1}} d = \sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 1}} s \sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 1}} t + \sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 2}} s \sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 2}} t\,. \tag{5.40}$$

(2) Suppose that $n/d \equiv_3 2$. Then, either $p/s \equiv_3 1$ and $q/t \equiv_3 2$, or $p/s \equiv_3 2$ and $q/t \equiv_3 1$, and therefore

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 2}} d = \sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 1}} s \sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 2}} t + \sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 2}} s \sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 1}} t\,. \tag{5.41}$$

Substituting Eqs. (5.40) and (5.41) into Eq. (5.36), we find that

$$\alpha_n = \left(\sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 1}} s - \sum_{\substack{s|p \\ \frac{p}{s} \equiv_3 2}} s \right) \left(\sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 1}} t - \sum_{\substack{t|q \\ \frac{q}{t} \equiv_3 2}} t \right) = \alpha_p \alpha_q\,, \tag{5.42}$$

which proves that the sequence α_n, $n \in \mathbb{Z}_{>0}$, defines a *multiplicative arithmetic function.* Note that the proof breaks if p, q are not coprimes since the formulae above lead, in general, to overcounting the contributions coming from common factors. Therefore, the sequence α_n is not totally multiplicative. Note that the sequence of normalized Stokes constants S_n/S_1, $n \in \mathbb{Z}_{>0}$, is also a multiplicative arithmetic function. ☐

Corollary 5.1.7. *The positive integer constants α_n, $n \in \mathbb{Z}_{>0}$, are encoded in the generating function*

$$\sum_{n=1}^{\infty} \alpha_n x^n = \sum_{m=1}^{\infty} \frac{m x^m}{1 + x^m + x^{2m}} \,. \tag{5.43}$$

Proof. We denote by $f(x)$ the generating function in the RHS of Eq. (5.43). We note that

$$f(x) = f_1(x) - f_2(x) \,, \tag{5.44}$$

where the functions $f_1(x)$, $f_2(x)$ are defined by

$$f_1(x) = \sum_{m=1}^{\infty} \frac{m x^m}{1 - x^{3m}} \,, \quad f_2(x) = \sum_{m=1}^{\infty} \frac{m x^{2m}}{1 - x^{3m}} \,. \tag{5.45}$$

The formula in Eq. (5.43) follows from the stronger statement

$$\sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 1}} d = \frac{1}{n!} \frac{\mathrm{d}^n f_1(0)}{\mathrm{d}x^n} \,, \quad \sum_{\substack{d|n \\ \frac{n}{d} \equiv_3 2}} d = \frac{1}{n!} \frac{\mathrm{d}^n f_2(0)}{\mathrm{d}x^n} \,, \quad n \in \mathbb{Z}_{>0} \,. \tag{5.46}$$

We will now prove this claim for the function $f_1(x)$. The case of $f_2(x)$ is proven analogously. Let us denote by

$$f_{1,m}(x) = \frac{m x^m}{1 - x^{3m}} \,, \quad m \in \mathbb{N}_{\neq 0} \,, \tag{5.47}$$

and consider the derivative $\mathrm{d}^n f_{1,m}(x)/\mathrm{d}x^n$ for fixed m. We want to determine its contributions to $\mathrm{d}^n f_1(0)/\mathrm{d}x^n$. Since we are interested in those terms that survive after taking $x = 0$, we look for the monomials of order x^{dm-n}, where $d|n$, in the numerator of $\mathrm{d}^n f_{1,m}(x)/\mathrm{d}x^n$, and we take $m = n/d$. More precisely, deriving a-times the factor $m x^m$ and $(n-a)$-times the factor $(1 - x^{3m})^{-1}$, we have the term

$$\binom{n}{a} \frac{\mathrm{d}^a (m x^m)}{\mathrm{d}x^a} \frac{\mathrm{d}^{n-a} (1 - x^{3m})^{-1}}{\mathrm{d}x^{n-a}} \,, \quad a \in \mathbb{N}_{\neq 0} \,. \tag{5.48}$$

Recall that the generalized binomial theorem for the geometric series yields

$$\frac{\mathrm{d}^{n-a} (1 - x^{3m})^{-1}}{\mathrm{d}x^{n-a}} = \sum_{k=0}^{\infty} \frac{(3mk)!}{(3mk - n + a)!} x^{3mk-n+a} \,. \tag{5.49}$$

Substituting Eq. (5.49) into Eq. (5.48) and performing the derivation, we have

$$n! \sum_{k=0}^{\infty} m \binom{m}{m-a} \binom{3mk}{3mk - n + a} x^{(1+3k)m-n} \,. \tag{5.50}$$

It follows then that the only non-zero term at fixed $m \in \mathbb{N}_{\neq 0}$ comes from the values of $k \in \mathbb{N}$ and $a \in \mathbb{N}_{\neq 0}$ such that $(1 + 3k)m = n$ and $a = m$, which implies in turn that $m|n$ and $n/m \equiv_3 1$. Finally, summing the non-trivial contributions over m gives precisely

$$\frac{\mathrm{d}^n f_1(0)}{\mathrm{d}x^n} = \sum_{\substack{m|n \\ \frac{n}{m} \equiv_3 1}} n! m \binom{m}{0} \binom{n-m}{0} = n! \sum_{\substack{m|n \\ \frac{n}{m} \equiv_3 1}} m \,. \tag{5.51}$$

□

A third notable consequence of Theorem 5.1.5 is that the Stokes constants S_n, $n \in \mathbb{Z}_{>0}$, can be naturally organized as coefficients of a generating function given by *quantum dilogarithms*.

Corollary 5.1.8. *The Stokes constants S_n, $n \in \mathbb{Z}_{>0}$, are encoded in the generating function*

$$\sum_{n=1}^{\infty} S_n x^n = -\mathrm{i}\pi - 3\log \frac{(w;\, x)_\infty}{(w^{-1};\, x)_\infty}\,, \quad |x| < 1\,, \tag{5.52}$$

where $w = \mathrm{e}^{2\pi\mathrm{i}/3}$.

Proof. We apply the definition of the quantum dilogarithm in Eq. (B.1) and Taylor expand the logarithm function for $|x| < 1$. We obtain in this way that

$$\log(w;\, x)_\infty = \sum_{m=0}^{\infty} \log(1 - w x^m) = -\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \frac{x^{mk}}{k} w^k\,, \tag{5.53}$$

and therefore also

$$\log \frac{(w;\, x)_\infty}{(w^{-1};\, x)_\infty} = -\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \frac{x^{mk}}{k}(w^k - w^{-k})\,. \tag{5.54}$$

We observe that

$$w^k - w^{-k} = \mathrm{e}^{2\pi\mathrm{i}k/3} - \mathrm{e}^{-2\pi\mathrm{i}k/3} = \begin{cases} 0 & \text{for} \quad k \equiv 0 \mod 3 \\ \mathrm{i}\sqrt{3} & \text{for} \quad k \equiv 1 \mod 3 \\ -\mathrm{i}\sqrt{3} & \text{for} \quad k \equiv 2 \mod 3 \end{cases}. \tag{5.55}$$

Substituting Eq. (5.55) into Eq. (5.54), and performing the change of variable $n = mk$, we find

$$\log \frac{(w;\, x)_\infty}{(w^{-1};\, x)_\infty} = -\mathrm{i}\sqrt{3}\sum_{n=1}^{\infty} x^n \left(\sum_{\substack{d|n \\ d\equiv_3 1}} \frac{1}{d} - \sum_{\substack{d|n \\ d\equiv_3 2}} \frac{1}{d} \right) - \sum_{k=1}^{\infty} \frac{1}{k}(w^k - w^{-k})\,, \tag{5.56}$$

where the last term in the RHS is simply resummed to

$$-\sum_{k=1}^{\infty} \frac{1}{k}(w^k - w^{-k}) = \log(1-w) - \log(1-w^{-1}) = \log(-w) = -\frac{\pi\mathrm{i}}{3}\,. \tag{5.57}$$

Substituting the arithmetic formula for the Stokes constants in Eq. (5.32) into the expression in Eq. (5.56), we obtain the desired statement. □

Choosing $x = \tilde{q} = \mathrm{e}^{-\mathcal{A}_0\mathrm{i}/\hbar}$, Corollary 5.1.8 directly implies an exact expression in terms of $\tilde{q}$-series for the *discontinuity* of the asymptotic series $\phi(\hbar)$ across the positive imaginary axis, which borders the only two distinct Stokes sectors in the upper half of the Borel plane. Namely, following the definition in Eq. (1.27) and using Eq. (5.29), we have that

$$\mathrm{disc}_{\pi/2}\phi(\hbar) = s_+(\phi)(\hbar) - s_-(\phi)(\hbar) = \sum_{n=1}^{\infty} S_n \mathrm{e}^{-n\mathcal{A}_0\mathrm{i}/\hbar}\,, \tag{5.58}$$

where $s_{\pm}(\phi)(\hbar)$ are the lateral Borel resummations at angles $\pi/2 \pm \epsilon$ with $\epsilon \ll 1$, which lie slightly above and below the Stokes line along the positive imaginary axis, respectively. Substituting Eq. (5.52) into Eq. (5.58), we obtain the exact formula

$$\mathrm{disc}_{\pi/2}\phi(\hbar) = -\mathrm{i}\pi - 3\log(w;\, \tilde{q})_{\infty} + 3\log(w^{-1};\, \tilde{q})_{\infty}\,. \tag{5.59}$$

We stress that $(w;\, \tilde{q})_{\infty}$ and $(w^{-1};\, \tilde{q})_{\infty}$ are the same $\tilde{q}$-series which appear in the *anti-holomorphic block* of $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ in Eq. (3.33).

5.1.3 Exact large-order relations

We provide here an alternative closed formula for the perturbative coefficients a_{2n}, $n \in \mathbb{Z}_{>0}$, of the asymptotic series $\phi(\hbar)$ in Eq. (1.20), which highlights an interesting link to analytic number theory. We recall that the large-n asymptotics of the coefficients is controlled at leading order by the singular behavior of the Borel transform $\hat{\phi}(\zeta)$ in the neighborhood of its dominant complex conjugate singularities $\zeta_{\pm 1}$, which is encoded in the local expansion in Eq. (5.23). We have then

$$a_{2n} \sim \frac{\Gamma(2n)}{\pi\mathrm{i}(\mathcal{A}_0\mathrm{i})^{2n}} S_1\,, \quad n \gg 1\,, \tag{5.60}$$

where again $\mathcal{A}_0 = 4\pi^2/3$ and $S_1 = 3\sqrt{3}\mathrm{i}$. By systematically including the contributions from all sub-dominant singularities in the Borel plane, the leading asymptotics can be upgraded to an *exact large-order relation*, which is[2]

$$a_{2n} = \frac{\Gamma(2n)}{\pi\mathrm{i}(\mathcal{A}_0\mathrm{i})^{2n}} \sum_{m=1}^{\infty} \frac{S_m}{m^{2n}}\,, \quad n \in \mathbb{Z}_{>0}\,, \tag{5.61}$$

where the Stokes constant S_m is written in Eq. (5.30). Notably, up to the simple prefactors above, the perturbative coefficients of $\phi(\hbar)$ are given by the *Dirichlet series* encoding the weak coupling Stokes constants evaluated at even integer points.

Theorem 5.1.9. *The Stokes constants S_m, $m \in \mathbb{Z}_{>0}$, satisfy the exact relations*

$$\sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} = 3\sqrt{3}\mathrm{i}\frac{\zeta(2n)}{3^{2n+1}} \left(\zeta\left(2n+1, \frac{1}{3}\right) - \zeta\left(2n+1, \frac{2}{3}\right) \right)\,, \quad n \in \mathbb{Z}_{>0}\,, \tag{5.62}$$

where $\zeta(z)$ is the Riemann zeta function and $\zeta(z,a)$ is the Hurwitz zeta function.

Proof. Substituting the original expression for the perturbative coefficients a_{2n}, $n \in \mathbb{Z}_{>0}$, in Eq. (5.2) into the exact large-order relation in Eq. (5.61), we have that

$$\sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} = -3\pi\mathrm{i}(2\pi)^{4n} \frac{B_{2n}B_{2n+1}(2/3)}{(2n)!(2n+1)!}\,. \tag{5.63}$$

Using the known identities

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}\,, \tag{5.64a}$$

[2] The formula in Eq. (5.61) has also been obtained in [157].

$$B_{2n+1}(2/3) = -B_{2n+1}(1/3)\,, \quad B_{2n+1}(z) = -(2n+1)\zeta(-2n,z)\,, \tag{5.64b}$$

the formula in Eq. (5.63) becomes

$$\sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} = -3\pi \mathrm{i}(-1)^n (2\pi)^{2n} \frac{\zeta(2n)}{(2n)!} \left(\zeta\left(-2n, \frac{2}{3}\right) - \zeta\left(-2n, \frac{1}{3}\right)\right) . \tag{5.65}$$

We recall that the Hurwitz zeta function satisfies the functional equation

$$\zeta\left(1-z, \frac{a}{b}\right) = \frac{2\Gamma(z)}{(2\pi b)^z} \sum_{j=1}^{b} \zeta\left(z, \frac{j}{b}\right) \cos\left(\frac{\pi z}{2} - \frac{2\pi j a}{b}\right) , \tag{5.66}$$

for integers $1 \leq a \leq b$, which gives in particular

$$\begin{aligned} \zeta\left(-2n, \frac{2}{3}\right) - \zeta\left(-2n, \frac{1}{3}\right) =& (-1)^n \frac{2\sqrt{3}(2n)!}{(6\pi)^{2n+1}} \left(\zeta\left(2n+1, \frac{2}{3}\right)\right. \\ &\left. - \zeta\left(2n+1, \frac{1}{3}\right)\right) . \end{aligned} \tag{5.67}$$

Substituting Eq. (5.67) into Eq. (5.65), we obtain the desired statement. □

Note that the exact expression in Eq. (5.62) can be written equivalently in terms of the integer constants α_m, $m \in \mathbb{Z}_{\neq 0}$. Namely,

$$\sum_{m=1}^{\infty} \frac{\alpha_m}{m^{2n+1}} = \frac{\zeta(2n)}{3^{2n+1}} \left(\zeta\left(2n+1, \frac{1}{3}\right) - \zeta\left(2n+1, \frac{2}{3}\right)\right) , \quad n \in \mathbb{Z}_{>0} . \tag{5.68}$$

Remark 5.1.1. *The formula in Eq.* (5.62) *hints at a fascinating connection to the* analytic theory of L-functions. *Namely, let us point out that the series in the LHS of Eq.* (5.62) *belongs to the family of* Dirichlet series *[158]. As a consequence of Corollary 5.1.6, the sequence of complex numbers* $\{S_m\}_{m\in\mathbb{Z}_{>0}}$ *defines a bounded multiplicative arithmetic function. Therefore, the corresponding Dirichlet series satisfies an expansion as an* Euler product *indexed by the set of prime numbers* $\mathbb{P}$*, that is,*

$$\sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} = \prod_{p\in\mathbb{P}} \sum_{e=0}^{\infty} \frac{S_{p^e}}{p^{e(2n)}} , \quad n \in \mathbb{Z}_{>0} , \tag{5.69}$$

which proves that the given Dirichlet series is, indeed, an L-series. *Similarly, the same statements apply to the Dirichlet series associated with the sequence of integers* $\{\alpha_m\}_{m\in\mathbb{Z}_{>0}}$*, which appears in the LHS of Eq.* (5.68)*. We will further explore these ideas in Section 5.3.*

5.1.4 Exponentiating with alien calculus

We will now translate our analytic solution to the resurgent structure of the asymptotic series $\phi(\hbar)$ in Eq. (5.1) into results on the original, exponentiated perturbative series in Eq. (3.38). Again, we adopt the notation of Section 3.2.2. In particular, we write

$$\upsilon(\hbar) = \mathrm{e}^{\phi(\hbar)} = \exp\left(3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n} B_{2n+1}(2/3)}{2n(2n+1)!} (3\hbar)^{2n}\right) \in \mathbb{Q}[\![\hbar]\!] , \tag{5.70}$$

which is also a Gevrey-1 asymptotic series. Its Borel transform $\hat{\upsilon}(\zeta)$ inherits from $\hat{\phi}(\zeta)$ the same pattern of singularities in Eq. (5.22). Namely, there are infinitely many and discrete logarithmic branch points located along the imaginary axis of the complex ζ-plane at $\zeta_n = \mathcal{A}_0 \mathrm{i}n$, $n \in \mathbb{Z}_{\neq 0}$, where $\mathcal{A}_0 = 4\pi^2/3$ as before. We denote by $s_{\pm}(\upsilon)(\hbar)$ the lateral Borel resummations at the angles $\pi/2 \pm \epsilon$ with $\epsilon \ll 1$, which lie slightly above and below the Stokes line along the positive imaginary axis, respectively. Let us apply Eqs. (1.39) and (1.41) and expand the exponential operator defining the *Stokes automorphism*. We find that

$$
\begin{aligned}
s_+(\upsilon)(\hbar) &= s_- \circ \mathfrak{S}_{\pi/2}(\upsilon)(\hbar) = s_- \circ \exp\left(\sum_{n=1}^{\infty} \mathrm{e}^{-\zeta_n/\hbar} \Delta_{\zeta_n}\right)(\upsilon)(\hbar) \\
&= s_-(\upsilon)(\hbar) + \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{n_1,\ldots,n_r=1}^{\infty} \mathrm{e}^{-(\zeta_{n_1}+\cdots+\zeta_{n_r})/\hbar} s_-\left(\Delta_{\zeta_{n_1}} \cdots \Delta_{\zeta_{n_r}} \upsilon\right)(\hbar) \\
&= s_-(\upsilon)(\hbar) + \sum_{k=1}^{\infty} \mathrm{e}^{-\zeta_k/\hbar} \sum_{p \in \mathcal{P}(k)} \frac{1}{r!} \binom{r}{N_1, \ldots, N_k} s_-\left(\Delta_{\zeta_{n_1}} \cdots \Delta_{\zeta_{n_r}} \upsilon\right)(\hbar) \, ,
\end{aligned} \tag{5.71}
$$

where Δ_{ζ_n} is the *alien derivative* associated with the singularity ζ_n, $n \in \mathbb{Z}_{>0}$, whose definition and basic properties are summarized in Section 1.3, and $\mathcal{P}(k)$ is the set of all partitions of the positive integer k. A partition $p \in \mathcal{P}(k)$ of length $|p| = r \in \mathbb{N}_{\neq 0}$ has the form $p = (n_1, \ldots, n_r)$ with $1 \leq n_1 \leq \cdots \leq n_r \leq k$ such that $n_1 + \cdots + n_r = k$. We denote by $N_i \in \mathbb{N}$ the number of times the positive integer $i \in \mathbb{N}_{\neq 0}$ is repeated in the partition p. Note that $\sum_{i=1}^{k} N_i = r$.

Since Eq. (1.35) directly applies to the asymptotic series $\phi(\hbar)$, the action of the alien derivative Δ_{ζ_n} on $\phi(\hbar)$ simplifies to give precisely the Stokes constant at the singularity ζ_n, that is,[3]

$$
\Delta_{\zeta_n} \phi(\hbar) = S_n \, , \tag{5.72}
$$

where S_n is written explicitly in Eq. (5.30), while the formula in Eq. (1.38) becomes

$$
\Delta_{\zeta_n} \upsilon(\hbar) = \Delta_{\zeta_n} \mathrm{e}^{\phi(\hbar)} = S_n \upsilon(\hbar) \, , \tag{5.73}
$$

and, therefore, we have

$$
\Delta_{\zeta_{n_1}} \cdots \Delta_{\zeta_{n_r}} \upsilon(\hbar) = S_{n_1} \cdots S_{n_r} \upsilon(\hbar) \, , \quad r \in \mathbb{N}_{\neq 0} \, . \tag{5.74}
$$

Substituting Eq. (5.74) into the last line of Eq. (5.71), we obtain that

$$
s_+(\upsilon)(\hbar) = s_-(\upsilon)(\hbar) + s_-(\upsilon)(\hbar) \sum_{k=1}^{\infty} \mathrm{e}^{-\zeta_k/\hbar} \bar{S}_k \, , \tag{5.75}
$$

where the asymptotic series $\upsilon_k(\hbar)$, which resurges from $\upsilon(\hbar)$ at the singularity ζ_k, is simply

$$
\upsilon_k(\hbar) = \upsilon(\hbar) \, , \quad k \in \mathbb{Z}_{>0} \, , \tag{5.76}
$$

and the Stokes constant $\bar{S}_k \in \mathbb{C}$ of $\upsilon(\hbar)$ at the singularity ζ_k is fully determined by the Stokes constants of $\phi(\hbar)$ via the closed formula

$$
\bar{S}_k = \sum_{p \in \mathcal{P}(k)} \frac{1}{r!} \binom{r}{N_1, \ldots, N_k} S_{n_1} \cdots S_{n_r} \, , \quad k \in \mathbb{Z}_{>0} \, . \tag{5.77}
$$

[3] Let us stress that the output of alien derivation on a formal power series has, in general, a more complex dependence on the Stokes constants. For more details, see Section 1.3.

We stress that the sum over partitions in Eq. (5.77) is finite, and thus all the Stokes constants of the original perturbative series $\upsilon(\hbar)$ are known exactly. More precisely, the discontinuity formula in Eq. (5.75) solves the resurgent structure of $\upsilon(\hbar)$ analytically. Note that the instanton sectors associated with the symmetric singularities along the negative imaginary axis are analytically derived from the resurgent structure of $\phi(\hbar)$ by applying the same computations above to the discontinuity of $\upsilon(\hbar)$ across the angle $3\pi/2$. We find straightforwardly that, if we define $\mathcal{P}(k) = \mathcal{P}(|k|)$ when $k < 0$, the formulae in Eqs. (5.76) and (5.77) hold for all values of $k \in \mathbb{Z}_{\neq 0}$. In particular, we have that $\bar{S}_k = \bar{S}_{-k}$.

Let us point out that the Stokes constants $\bar{S}_k$, $k \in \mathbb{Z}_{\neq 0}$, are generally complex numbers. However, we can say something more. The discontinuity formula in Eq. (5.59) can be directly exponentiated to give an exact generating function in terms of known $\tilde{q}$-series for the Stokes constants $\bar{S}_k$. Namely, we find that

$$\sum_{k=1}^{\infty} \bar{S}_k \tilde{q}^k = \mathrm{e}^{-\mathrm{i}\pi} \frac{(w^{-1};\, \tilde{q})_\infty^3}{(w;\, \tilde{q})_\infty^3} \,, \tag{5.78}$$

where $w = \mathrm{e}^{2\pi\mathrm{i}/3}$. As a consequence of the q-binomial theorem, the quotient of $\tilde{q}$-series in the RHS of Eq. (5.78) can be expanded in powers of $\tilde{q}$, and the resulting numerical coefficients are combinations of integers, related to the enumerative combinatorics of counting partitions, and complex numbers, arising as integer powers of the complex constants w, w^{-1}. Explicitly, the first several Stokes constants $\bar{S}_k$, $k > 0$, are

$$3\sqrt{3}\mathrm{i}, -\frac{27}{2} + \frac{3\sqrt{3}\mathrm{i}}{2}, -\frac{27}{2} - \frac{21\sqrt{3}\mathrm{i}}{2}, -18\sqrt{3}\mathrm{i}, 27 - 30\sqrt{3}\mathrm{i}, \frac{189}{2} - \frac{51\sqrt{3}\mathrm{i}}{2}, 162 + 15\sqrt{3}\mathrm{i}, \dots . \tag{5.79}$$

We stress that a special simplification occurs when factoring out the contribution from $S_1 = 3\sqrt{3}\mathrm{i}$. More precisely, if we divide the discontinuity formula in Eq. (5.59) by S_1 and take the exponential of both sides, we find a new generating series, that is,

$$\sum_{k=1}^{\infty} \bar{S}'_k \tilde{q}^k = \mathrm{e}^{-\frac{\pi}{3\sqrt{3}}} \left(\frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty} \right)^{\frac{\mathrm{i}}{\sqrt{3}}} \,, \tag{5.80}$$

where the new constants $\bar{S}'_k$ are, notably, rational numbers. Let us stress that these rational Stokes constants $\bar{S}'_k$ appear naturally in the resurgent study of the normalized perturbative series $\phi'(\hbar) = \phi(\hbar)/3\sqrt{3}\mathrm{i}$ after exponentiation. Explicitly, the first several values of $\bar{S}'_k$, $k > 0$, are

$$1, 1, \frac{5}{3}, \frac{13}{6}, \frac{83}{30}, \frac{299}{90}, \frac{419}{90}, \frac{409}{72}, \frac{23137}{3240}, \frac{138761}{16200}, \dots . \tag{5.81}$$

Finally, we note that the numbers $k!\bar{S}'_k$, $k \in \mathbb{Z}_{>0}$, define a sequence of positive integers, which is

$$1, 2, 10, 52, 332, 2392, 23464, 229040, 2591344, 31082464, \dots . \tag{5.82}$$

5.2 The limit $\hbar \to \infty$

Let us now go back to the exact formula for the spectral trace of local $\mathbb{P}^2$ in Eq. (3.33) and derive its all-orders perturbative expansion in the dual limit $\hbar \to \infty$. In the strong-weak coupling duality of Eq. (2.52) between the spectral theory of the quantum operator $\rho_{\mathbb{P}^2}$ and the standard topological string theory on local $\mathbb{P}^2$, this regime corresponds to the weakly interacting limit $g_s \to 0$ of the topological string. The resurgent structure of the asymptotic series obtained by perturbatively expanding $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ for $\hbar \to \infty$ has been studied numerically in [10]. We will show here how the same procedure we presented in Section 5.1 for the semiclassical limit $\hbar \to 0$ can be straightforwardly applied to the dual case. We obtain, in this way, a fully analytic solution to the resurgent structure of the spectral trace at strong coupling and unveil remarkable number-theoretic relations that are dual to the ones discovered in Section 5.1.

5.2.1 Computing the perturbative series

Let us start by applying the known asymptotic expansion formula for the quantum dilogarithm in Eq. (B.19a) to the anti-holomorphic block in Eq. (3.33) and explicitly evaluate the special functions that appear. We recall that

$$\log(1-w) - 2\log(1-w^{-1}) = -\frac{\pi \mathrm{i}}{2} - \frac{1}{2}\log(3)\,, \tag{5.83a}$$

$$\mathrm{Li}_2(w) - 2\mathrm{Li}_2(w^{-1}) = \frac{\pi^2}{18} + \mathrm{i}V\,, \tag{5.83b}$$

$$\mathrm{Li}_0(w) - 2\mathrm{Li}_0(w^{-1}) = \frac{1}{2} + 3\sqrt{3}\mathrm{i}B_1(2/3)\,, \tag{5.83c}$$

where we have defined $V = 2\Im\left(\mathrm{Li}_2(\mathrm{e}^{\pi\mathrm{i}/3})\right)$ and $w = \mathrm{e}^{2\pi\mathrm{i}/3}$, as before. For integer $n \geq 2$, the dilogarithm functions give

$$\begin{aligned}\mathrm{Li}_{2-2n}(w) - 2\mathrm{Li}_{2-2n}(w^{-1}) &= \sum_{s=1}^{\infty} \frac{1}{s^{2-2n}}(w^s - 2w^{-s}) \\ &= -\,3^{2n-2}\left[\zeta(2-2n) + \left(\frac{1}{2} + \frac{3\sqrt{3}\mathrm{i}}{2}\right)\zeta\left(2-2n, \frac{1}{3}\right)\right. \\ &\quad \left. + \left(\frac{1}{2} - \frac{3\sqrt{3}\mathrm{i}}{2}\right)\zeta\left(2-2n, \frac{2}{3}\right)\right]\,.\end{aligned} \tag{5.84}$$

Using the identity

$$\zeta\left(2-2n, \frac{1}{3}\right) + \zeta\left(2-2n, \frac{2}{3}\right) \propto \zeta(2-2n) = 0\,, \quad n \in \mathbb{Z}_{>1}\,, \tag{5.85}$$

and the formulae in Eq. (5.64b), the expression in Eq. (5.84) simplifies to

$$\mathrm{Li}_{2-2n}(w) - 2\mathrm{Li}_{2-2n}(w^{-1}) = 3^{2n-1}\sqrt{3}\mathrm{i}\frac{B_{2n-1}(2/3)}{2n-1}\,. \tag{5.86}$$

Substituting Eqs. (5.83) and (5.86) into the formula in Eq. (B.19a), we find that the anti-holomorphic block contributes in the limit $\hbar \to \infty$ as

$$\begin{aligned}\log(w;\tilde{q})_\infty - 2\log(w^{-1};\tilde{q})_\infty \sim &-\frac{\pi\mathrm{i}}{12}\mathsf{b}^{-2} - \frac{\pi\mathrm{i}}{4} - \frac{1}{4}\log(3) + \left(\frac{\pi\mathrm{i}}{36} - \frac{V}{2\pi}\right)\mathsf{b}^2 \\ &- \sqrt{3}\mathrm{i}\sum_{n=1}^{\infty}(6\pi\mathrm{i}\mathsf{b}^{-2})^{2n-1}\frac{B_{2n}B_{2n-1}(2/3)}{(2n-1)(2n)!}\,,\end{aligned} \tag{5.87}$$

while the holomorphic block contributes trivially as

$$\frac{(q^{2/3};\, q)_\infty^2}{(q^{1/3};\, q)_\infty} \sim 1\,. \tag{5.88}$$

Remark 5.2.1. *We note that the terms of order b^2 and b^{-2} in Eq. (5.87) only partially cancel with the opposite contributions from the exponential in Eq. (3.33), leaving the exponential factor*

$$Tr(\rho_{\mathbb{P}^2}) \sim \exp\left(-\frac{3V}{g_s}\right), \quad g_s \to 0\,, \tag{5.89}$$

which proves the statement of the conifold volume conjecture[4] *in the special case of local $\mathbb{P}^2$. A dominant exponential of the form in Eq. (5.89) was already found numerically in [10] for both local $\mathbb{P}^2$ and local $\mathbb{F}_0$ in the limit $g_s \to 0$. However, we show in this thesis that, for the same geometries, the perturbative expansion in the limit $\hbar \to 0$ of the first fermionic spectral trace does not have such a global exponential pre-factor, being dominated by a leading term of order $\hbar^{-1}$. This suggests no analog of the conifold volume conjecture in the semiclassical regime.*

Substituting Eqs. (5.87) and (5.88) into Eq. (3.33), and using $2\pi\mathsf{b}^2 = 3\hbar$, we obtain the *all-orders perturbative expansion* for $\hbar \to \infty$ of the spectral trace of local $\mathbb{P}^2$, that is,

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \sqrt{\frac{2\pi}{3^{5/2}\hbar}}\mathrm{e}^{-\frac{3V}{4\pi^2}\hbar}\exp\left(\sqrt{3}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)}\left(\frac{4\pi^2}{\hbar}\right)^{2n-1}\right), \tag{5.90}$$

which allows us to compute the coefficients in the asymptotic expansion of $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ at strong coupling up to arbitrarily high order. Discarding the global pre-factor above, the perturbative coefficients belong to $\mathbb{Q}[\pi, \sqrt{3}]$. The first few terms are

$$1 + \frac{\pi^2}{6\sqrt{3}\hbar} + \frac{\pi^4}{216\hbar^2} - \frac{59\pi^6}{19440\sqrt{3}\hbar^3} - \frac{251\pi^8}{1399680\hbar^4} + \frac{23687\pi^{10}}{58786560\sqrt{3}\hbar^5} + \mathrm{O}(\hbar^{-6})\,. \tag{5.91}$$

5.2.2 Resumming the Borel transform

Let us introduce the parameter

$$\tau = -\frac{1}{\mathsf{b}^2} = -\frac{2\pi}{3\hbar} \tag{5.92}$$

and note that we can write $\tilde{q} = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $q = \mathrm{e}^{-2\pi\mathrm{i}/\tau}$. We denote by $\psi(\tau)$ the formal power series appearing in the exponent in Eq. (5.90). Namely,

$$\psi(\tau) = \sum_{n=1}^{\infty} b_{2n}\tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][\![\tau]\!]\,, \tag{5.93}$$

whose coefficients b_{2n}, $n \in \mathbb{Z}_{>0}$, are

$$b_{2n} = (-1)^n\sqrt{3}\frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)}(6\pi)^{2n-1}\,. \tag{5.94}$$

[4] The conifold volume conjecture for toric CY manifolds has been tested in examples of genus one and two in [12, 88, 110, 111, 159].

Note that $\psi(\tau)$ is simply related to the perturbative expansion in the limit $\hbar \to \infty$ of the logarithm of the spectral trace of local $\mathbb{P}^2$ by

$$\log \mathrm{Tr}(\rho_{\mathbb{P}^2}) - \frac{V}{2\pi\tau} - \frac{1}{2}\log(\tau) + \frac{3}{4}\log(3) - \frac{\pi \mathrm{i}}{2} \sim \psi(\tau)\,. \tag{5.95}$$

As a consequence of the known asymptotic behavior of the Bernoulli polynomials in Eq. (5.4), we obtain that the coefficients of $\psi(\tau)$ satisfy the factorial growth

$$|b_{2n}| \leq (2n)!\mathcal{A}_\infty^{-2n} \quad n \gg 1\,, \quad \mathcal{A}_\infty = \frac{2\pi}{3} = \frac{\mathcal{A}_0}{2\pi}\,, \tag{5.96}$$

and $\psi(\tau)$ is a Gevrey-1 asymptotic series. Its Borel transform is given by

$$\hat{\psi}(\zeta) = \sqrt{3}\sum_{n=1}^{\infty}(-1)^n \frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)(2n-1)!}(6\pi\zeta)^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}]\{\zeta\}\,, \tag{5.97}$$

which is the germ of an analytic function at the origin in the complex ζ-plane.

Proposition 5.2.1. *Using the definition in Eq.* (C.2)*, we interpret the Borel transform* $\hat{\psi}(\zeta)$ *in Eq.* (5.97) *as the* Hadamard product

$$\hat{\psi}(\zeta) = (f \diamond g)(\zeta)\,, \tag{5.98}$$

where the formal power series $f(\zeta)$ *and* $g(\zeta)$*, which are defined as*

$$f(\zeta) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\zeta^{2n-1}\,, \tag{5.99a}$$

$$g(\zeta) = \sum_{n=1}^{\infty}(-1)^n \frac{B_{2n-1}(2/3)}{(2n-1)(2n-1)!}\sqrt{3}(6\pi)^{2n-1}\zeta^{2n-1}\,, \tag{5.99b}$$

have finite radius of convergence at $\zeta = 0$ *and can be resummed explicitly as*[5]

$$f(\zeta) = -\frac{1}{2\zeta}\left(2 - \zeta \coth\left(\frac{\zeta}{2}\right)\right)\,, \quad |\zeta| < 2\pi\,, \tag{5.100a}$$

$$g(\zeta) = \frac{3}{2}\log\left(\frac{\cos(\pi/6 + \pi\zeta)}{\cos(\pi/6 - \pi\zeta)}\right)\,, \quad |\zeta| < 1/3\,. \tag{5.100b}$$

Proof. The Bernoulli numbers are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!}\zeta^n = \frac{\zeta}{2}\left(\coth\left(\frac{\zeta}{2}\right) - 1\right)\,, \quad |\zeta| < 2\pi\,. \tag{5.101}$$

Taking the even part of both sides of Eq. (5.101) and multiplying by $1/\zeta$, together with the formula in Eq. (5.99a), yields the statement in Eq. (5.100a). Let us apply the second identity in Eq. (5.151b) to the power series in the RHS of Eq. (5.99b) and use the functional

[5] We impose that $f(0) = g(0) = 0$ in order to eliminate the removable singularities of $f(\zeta)$, $g(\zeta)$ at the origin.

equation for the Hurwitz zeta function in Eq. (5.66) for $\zeta(2-2n, 2/3)$, $n \geq 2$. We find that

$$g(\zeta) = -\pi\sqrt{3}\zeta + 3\sum_{n=2}^{\infty}\left(\zeta\left(2n-1, \frac{2}{3}\right) - \zeta\left(2n-1, \frac{1}{3}\right)\right)\frac{\zeta^{2n-1}}{2n-1}\,. \tag{5.102}$$

Let us now use the formula in Eq. (C.5b) for $a = 2/3, 1/3$ and recall the known identity

$$\Psi(2/3) - \Psi(1/3) = \frac{\pi}{\sqrt{3}}\,, \tag{5.103}$$

where $\Psi(a)$ denotes the digamma function. We obtain in this way that

$$g(\zeta) = \frac{3}{2}\log\left(\frac{\Gamma(2/3-\zeta)\Gamma(1/3+\zeta)}{\Gamma(2/3+\zeta)\Gamma(1/3-\zeta)}\right)\,, \quad |\zeta| < 1/3\,. \tag{5.104}$$

After we apply Euler's reflection formula in Eq. (5.14) with $x = 1/3+\zeta, 1/3-\zeta$ and use the trigonometric identities

$$\cos(\pi/6 \pm \pi\zeta) = \sin(\pi/3 \mp \pi\zeta)\,, \tag{5.105}$$

the formula in Eq. (5.104) then yields the statement in Eq. (5.100b). □

After being analytically continued to the whole complex ζ-plane, the function $f(\zeta)$ has poles of order one along the imaginary axis at

$$\mu_m = 2\pi \mathrm{i} m\,, \quad m \in \mathbb{Z}_{\neq 0}\,, \tag{5.106}$$

while the function $g(\zeta)$ has logarithmic branch points along the real axis at

$$\nu_k^- = -\frac{1}{3} + 2k\,, \quad \nu_k^+ = \frac{2}{3} + 2k\,, \quad k \in \mathbb{Z}\,. \tag{5.107}$$

Proposition 5.2.2. *The Borel transform $\hat{\psi}(\zeta)$ in Eq. (5.97) can be expressed as*[6]

$$\hat{\psi}(\zeta) = -\frac{3}{2\pi \mathrm{i}}\sum_{m\in\mathbb{Z}_{\neq 0}}\frac{1}{m}\log\left(\cos\left(\frac{\pi}{6} + \frac{\zeta}{2\mathrm{i}m}\right)\right)\,, \tag{5.108}$$

which is a well-defined, exact function of ζ.

Proof. We consider a circle γ in the complex s-plane with center $s = 0$ and radius $0 < r < 2\pi$ and apply Theorem C.0.1. The Borel transform can be written as the integral

$$\begin{aligned}\hat{\psi}(\zeta) &= \frac{1}{2\pi \mathrm{i}}\int_\gamma f(s)g(\zeta/s)\frac{\mathrm{d}s}{s} \\ &= -\frac{3}{4\pi \mathrm{i}}\int_\gamma \frac{2 - s\coth(s/2)}{s}\log\left(\frac{\cos(\pi/6+\pi\zeta/s)}{\cos(\pi/6-\pi\zeta/s)}\right)\frac{\mathrm{d}s}{s}\,,\end{aligned} \tag{5.109}$$

for $|\zeta| < r/3$. For such values of ζ, the function $s \mapsto g(\zeta/s)$ has logarithmic branch points at $s = \zeta/\nu_k^\pm$, $k \in \mathbb{Z}$, which sit inside the contour of integration γ and accumulate at the

[6] We stress that each of the infinite sums giving the Borel transforms in Eqs. (5.17) and (5.108) can be straightforwardly written as the logarithm of an infinite product.

origin, and no singularities for $|s| > r$. The function $f(s)$ has simple poles at the points $s = \mu_m$ with residues

$$\operatorname*{Res}_{s=2\pi i m} f(s) = 1\,, \quad m \in \mathbb{Z}_{\neq 0}\,. \tag{5.110}$$

By Cauchy's residue theorem, the integral in Eq. (5.109) can be evaluated by summing the residues at the poles of the integrand which lie outside γ, allowing us to express the Borel transform as an exact function of ζ. More precisely, we find the desired analytic formula

$$\begin{aligned}\hat{\psi}(\zeta) &= -\sum_{m\in\mathbb{Z}_{\neq 0}} \operatorname*{Res}_{s=2\pi i m} f(s) g(\zeta/s)\frac{1}{s} \\ &= -\frac{3}{2\pi i}\sum_{m\in\mathbb{Z}_{\neq 0}} \frac{1}{m}\log\left(\cos\left(\frac{\pi}{6}+\frac{\zeta}{2im}\right)\right)\,.\end{aligned} \tag{5.111}$$

The convergence of the infinite sum in the RHS of Eq. (5.111) can be easily verified by, *e.g.*, the limit comparison test. □

Corollary 5.2.3. *The singularities of the Borel transform $\hat{\psi}(\zeta)$ in Eq. (5.108) are logarithmic branch points located along the imaginary axis at*

$$\eta^-_{k,m} = \nu^-_{-k}\mu_{-m} = \frac{2\pi i}{3}(1+6k)m\,, \quad \eta^+_{k,m} = \nu^+_{-k}\mu_{-m} = \frac{2\pi i}{3}(-2+6k)m\,, \tag{5.112}$$

for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\neq 0}$, which we write equivalently as

$$\eta_n = \mathcal{A}_\infty i n\,, \quad n \in \mathbb{Z}_{\neq 0}\,, \tag{5.113}$$

where $\mathcal{A}_\infty = 2\pi/3$ as before—that is, the branch points lie at all non-zero integer multiples of the two complex conjugate dominant singularities at $\pm 2\pi i/3$, as illustrated in Fig. 5.3.

Analogously to the dual case of $\hbar \to 0$, there are only two Stokes lines at the angles $\pm\pi/2$. Moreover, the analytic expression in Eq. (5.108) is explicitly simple resurgent.

Corollary 5.2.4. *The local expansion of the Borel transform $\hat{\psi}(\zeta)$ in Eq. (5.108) at $\zeta = \eta_n$, $n \in \mathbb{Z}_{\neq 0}$, is given by*

$$\hat{\psi}(\zeta) = -\frac{R_n}{2\pi i}\log(\zeta - \eta_n) + \dots\,, \tag{5.114}$$

where $R_n \in \mathbb{C}$ is the Stokes constant.

Proof. The local expansion around the logarithmic singularity $\zeta = \eta_n$ is obtained by summing the contributions from all pairs $(k, m) \in \mathbb{Z} \times \mathbb{Z}_{\neq 0}$ such that $n = (-1 + 6k)m$ or $n = (2 + 6k)m$. Let us collect such pairs of integers into the finite sets I^+_n, I^-_n for each $n \in \mathbb{Z}_{\neq 0}$, accordingly. For a fixed value of $m \in \mathbb{Z}_{\neq 0}$, we denote the corresponding term in the sum in Eq. (5.108) by

$$f_m(\zeta) = -\frac{3}{2\pi i m}\log\left(\cos\left(\frac{\pi}{6}+\frac{\zeta}{2im}\right)\right)\,, \tag{5.115}$$

whose expansion around $\zeta = \eta^\pm_{k,m}$, for fixed $k \in \mathbb{Z}$, is given by

$$f_m(\zeta) = -\frac{s^\pm_{k,m}}{2\pi i}\log(\zeta - \eta^\pm_{k,m}) + \dots\,, \tag{5.116}$$

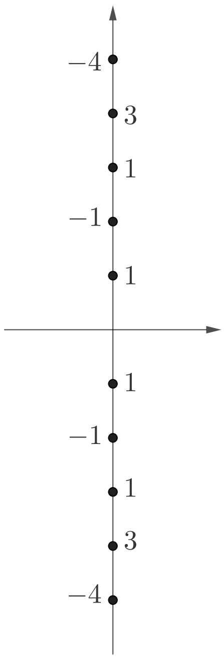


Figure 5.3: The first few singularities of the Borel transform of the asymptotic series $\psi(\tau)$, defined in Eq. (5.93), and the associated integer constants $\beta_n \in \mathbb{Z}_{\neq 0}$, defined in Eq. (5.121).

where the dots denote regular terms in $\zeta - \eta^{\pm}_{k,m}$, and $s^{\pm}_{k,m}$ is a complex number. Since $\eta_n = \eta^{\pm}_{k,m}$ for all $(k,m) \in I^{\pm}_n$, it follows that the local expansion of $\hat{\psi}(\zeta)$ at $\zeta = \eta_n$ is again given by

$$\hat{\psi}(\zeta) = \sum_{m \in \mathbb{Z}_{\neq 0}} f_m(\zeta) = -\frac{R_n}{2\pi \mathrm{i}} \log(\zeta - \eta_n) + \dots \,, \tag{5.117}$$

where the Stokes constant R_n is now the finite sum

$$R_n = \sum_{(k,m) \in I^+_n} s^+_{k,m} + \sum_{(k,m) \in I^-_n} s^-_{k,m} \,. \tag{5.118}$$

□

It follows from Corollary 5.2.4 that the locally analytic function that resurges at $\zeta = \eta_n$ is trivially

$$\hat{\psi}_n(\zeta - \eta_n) = 1 \,, \quad n \in \mathbb{Z}_{\neq 0} \,, \tag{5.119}$$

which also implies that

$$\psi_n(\tau) = 1 \,, \quad n \in \mathbb{Z}_{\neq 0} \,. \tag{5.120}$$

Once again, the procedure above allows us to derive analytically all the Stokes constants. In the limit $\hbar \to \infty$, the Stokes constants R_n are rational numbers, and they too are simply related to an interesting *sequence of integers* β_n, $n \in \mathbb{Z}_{\neq 0}$. In particular, we find that

$$R_1 = 3 \,, \quad R_n = R_1 \frac{\beta_n}{n} \quad n \in \mathbb{Z}_{\neq 0,1} \,, \tag{5.121a}$$

$$\beta_n = \beta_{-n}\,, \quad \beta_n \in \mathbb{Z}_{\neq 0} \quad n \in \mathbb{Z}_{>0}\,. \tag{5.121b}$$

Explicitly, the first several integer constants β_n, $n > 0$, are

$$1, -1, 1, 3, -4, -1, 8, -5, 1, 4, -10, 3, 14, -8, -4, 11, -16, -1, 20, -12, \ldots\,. \tag{5.122}$$

The pattern of singularities in the Borel plane and the associated $\beta_n \in \mathbb{Z}_{\neq 0}$ are shown in Fig. 5.3.

5.2.3 Closed formulae for the Stokes constants

The exact resummation of the Borel transform in Eq. (5.108) allows us to obtain and prove a series of exact arithmetic formulae for the Stokes constants R_n of the asymptotic series $\psi(\tau)$, defined in Eq. (5.93), and the related integer constants β_n, defined in Eq. (5.121), for $n \in \mathbb{Z}_{\neq 0}$. These new arithmetic statements are manifestly dual to the analogous formulae presented in Section 5.1.2 for the semiclassical limit of $\hbar \to 0$. Besides, their proofs follow very similar arguments. Let us begin by showing that the sequences $\{R_n\}$ and $\{\beta_n\}$ define appropriate *divisor sum functions*.

Theorem 5.2.5. *The normalized Stokes constant R_n/R_1, where $R_1 = 3$, is determined by the positive integer divisors of $n \in \mathbb{Z}_{\neq 0}$ according to the closed formula*

$$\frac{R_n}{R_1} = \sum_{\substack{d|n \\ d\equiv_3 1}} \frac{d}{n} - \sum_{\substack{d|n \\ d\equiv_3 2}} \frac{d}{n}\,, \tag{5.123}$$

which implies that $R_n = -R_{-n}$ and $R_n/R_1 \in \mathbb{Q}_{\neq 0}$.

Proof. Let us denote by D_n the set of positive integer divisors of n. We recall that n satisfies one of the two factorization properties

$$n = (1+6k)m \quad \text{or} \quad n = (-2+6k)m\,, \quad k \in \mathbb{Z}\,, \quad m \in \mathbb{Z}_{\neq 0}\,. \tag{5.124}$$

It follows that one of four cases applies. Namely,

$$m = \frac{n}{d}\,, \quad k = \frac{d-1}{6}\,, \quad \text{if} \quad d \equiv 1 \mod 6\,, \tag{5.125a}$$
$$m = -\frac{n}{d}\,, \quad k = -\frac{d-2}{6}\,, \quad \text{if} \quad d \equiv 2 \mod 6\,, \tag{5.125b}$$
$$m = \frac{n}{d}\,, \quad k = \frac{d+2}{6}\,, \quad \text{if} \quad d \equiv 4 \mod 6\,, \tag{5.125c}$$
$$m = -\frac{n}{d}\,, \quad k = -\frac{d+1}{6}\,, \quad \text{if} \quad d \equiv 5 \mod 6\,, \tag{5.125d}$$

where $d \in D_n$. In both cases of Eqs. (5.125a) and (5.125c), which represent together the congruence class of $d \equiv_3 1$, substituting the given values of k, m into Eqs. (5.115) and (5.112), we find that the contribution to the Stokes constant R_n coming from the local expansion of $f_m(\zeta)$ around $\eta^{\pm}_{k,m}$ is simply $s^{\pm}_{k,m} = 3d/n$. Furthermore, in both cases of Eqs. (5.125b) and (5.125d), which populate the congruence class of $d \equiv_3 2$, substituting the given values of k, m into Eqs. (5.115) and (5.112), we find that the contribution to the Stokes constant R_n coming from the local expansion of $f_m(\zeta)$ around $\eta^{\pm}_{k,m}$ is simply $s^{\pm}_{k,m} = -3d/n$. Finally, for any divisor $d \in D_n$ which is a multiple of 3, neither $d \pm 1$ or

$d \pm 2$ are divisible by 6, which implies that the choice $m = \pm n/d$ is not allowed, and the corresponding contribution is $s^{\pm}_{k,m} = 0$. Putting everything together and using Eq. (5.118), we find the desired statement. □

The arithmetic formula for the Stokes constants in Eq. (5.123) is equivalent to

$$\beta_n = \sum_{\substack{d|n \\ d\equiv_3 1}} d - \sum_{\substack{d|n \\ d\equiv_3 2}} d \,, \tag{5.126}$$

which implies that $\beta_n = \beta_{-n}$ and $\beta_n \in \mathbb{Z}_{\neq 0}$ for all $n > 0$, as expected. Note the strikingly simple *arithmetic symmetry* between the formulae in Eqs. (5.32) and (5.36) and the formulae in Eqs. (5.123) and (5.126). More precisely, the Stokes constants R_n in the limit $\hbar \to \infty$ are obtained from the Stokes constants S_n in the semiclassical limit $\hbar \to 0$ via the exchange of divisors $d \mapsto n/d$ in the arguments of the sums. As before, two corollaries follow straightforwardly from Theorem 5.2.5.

Corollary 5.2.6. *The integer constants β_n, $n \in \mathbb{Z}_{>0}$, satisfy the closed formulae*

$$\beta_{p_1^{e_1}} = \frac{p_1^{e_1+1}-1}{p_1-1} \,, \quad \beta_{p_2^{e_2}} = \frac{(-1)^{e_2} p_2^{e_2+1}+1}{p_2+1} \,, \quad \beta_{p_3^{e_3}} = 1 \,, \tag{5.127}$$

where $e_i \in \mathbb{N}$, and $p_i \in \mathbb{P}$ are prime numbers such that $p_i \equiv_3 i$ for $i = 1, 2, 3$. Moreover, they obey the multiplicative property

$$\beta_n = \prod_{p \in \mathbb{P}} \beta_{p^e} \,, \quad n = \prod_{p \in \mathbb{P}} p^e \,, \quad e \in \mathbb{N} \,. \tag{5.128}$$

Proof. The three closed formulae follow directly from Eq. (5.126). Explicitly, let $n = p^e$ with $p \in \mathbb{P}$ and $e \in \mathbb{N}$. We have that

$$\sum_{\substack{d|n \\ d\equiv_3 1}} d = 1 \,, \quad \sum_{\substack{d|n \\ d\equiv_3 2}} d = 0 \,, \quad \text{if} \quad p \equiv 0 \mod 3 \,, \tag{5.129a}$$

$$\sum_{\substack{d|n \\ d\equiv_3 1}} d = \sum_{i=0}^{e} p^i = \frac{p^{e+1}-1}{p-1} \,, \quad \sum_{\substack{d|n \\ d\equiv_3 2}} d = 0 \,, \quad \text{if} \quad p \equiv 1 \mod 3 \,, \tag{5.129b}$$

$$\sum_{\substack{d|n \\ d\equiv_3 1}} d = \sum_{i=0}^{\lfloor e/2 \rfloor} p^{2i} \,, \quad \sum_{\substack{d|n \\ d\equiv_3 2}} d = \sum_{i=0}^{\lfloor e/2 \rfloor} p^{2i+1} \,, \quad \text{if} \quad p \equiv 2 \mod 3 \,. \tag{5.129c}$$

Let us now prove the multiplicity property. We will prove a slightly stronger statement. We write $n = pq$ for $p, q \in \mathbb{Z}_{>0}$ coprimes. We choose a positive integer divisor $d|n$, and we write $d = st$ where $s|p$ and $t|q$. Consider two cases:

(1) Suppose that $d \equiv_3 1$. Then, either $s \equiv_3 t \equiv_3 1$, or $s \equiv_3 t \equiv_3 2$, and therefore

$$\sum_{\substack{d|n \\ d\equiv_3 1}} d = \sum_{\substack{s|p \\ s\equiv_3 1}} s \sum_{\substack{t|q \\ t\equiv_3 1}} t + \sum_{\substack{s|p \\ s\equiv_3 2}} s \sum_{\substack{t|q \\ t\equiv_3 2}} t \,. \tag{5.130}$$

(2) Suppose that $d \equiv_3 2$. Then, either $p/s \equiv_3 1$ and $q/t \equiv_3 2$, or $p/s \equiv_3 2$ and $q/t \equiv_3 1$, and therefore

$$\sum_{\substack{d|n \\ d\equiv_3 2}} d = \sum_{\substack{s|p \\ s\equiv_3 1}} s \sum_{\substack{t|q \\ t\equiv_3 2}} t + \sum_{\substack{s|p \\ s\equiv_3 2}} s \sum_{\substack{t|q \\ t\equiv_3 1}} t \,. \tag{5.131}$$

Substituting Eqs. (5.130) and (5.131) into Eq. (5.126), we find that

$$\beta_n = \left(\sum_{\substack{s|p \\ s\equiv_3 1}} s - \sum_{\substack{s|p \\ s\equiv_3 2}} s \right) \left(\sum_{\substack{t|q \\ t\equiv_3 1}} t - \sum_{\substack{t|q \\ t\equiv_3 2}} t \right) = \beta_p \beta_q \,, \tag{5.132}$$

which proves that the sequence β_n, $n \in \mathbb{Z}_{>0}$, defines a *multiplicative arithmetic function*. Note that the proof breaks if p, q are not coprimes since the formulae above lead, in general, to overcounting the contributions coming from common factors. Therefore, the sequence β_n is not totally multiplicative. Note that the sequence of normalized Stokes constants R_n/R_1, $n \in \mathbb{Z}_{>0}$, is also a multiplicative arithmetic function. □

Corollary 5.2.7. *The integer constants β_n, $n \in \mathbb{Z}_{>0}$, are encoded in the generating function*

$$\sum_{n=1}^{\infty} \beta_n x^n = \sum_{m=1}^{\infty} \frac{x^m (1 - x^{2m})}{(1 + x^m + x^{2m})^2} \,. \tag{5.133}$$

Proof. We denote by $f(x)$ the generating function in the RHS of Eq. (5.133). We note that

$$f(x) = f_1(x) - f_2(x) \,, \tag{5.134}$$

where the functions $f_1(x), f_2(x)$ are defined by

$$f_1(x) = \sum_{m=0}^{\infty} \frac{(3m+1)x^{3m+1}}{1 - x^{3m+1}} \,, \quad f_2(x) = \sum_{m=0}^{\infty} \frac{(3m+2)x^{3m+2}}{1 - x^{3m+2}} \,. \tag{5.135}$$

The formula in Eq. (5.133) follows from the stronger statement

$$\sum_{\substack{d|n \\ d\equiv_3 1}} d = \frac{1}{n!} \frac{\mathrm{d}^n f_1(0)}{\mathrm{d}x^n} \,, \quad \sum_{\substack{d|n \\ d\equiv_3 2}} d = \frac{1}{n!} \frac{\mathrm{d}^n f_2(0)}{\mathrm{d}x^n} \,, \quad n \in \mathbb{Z}_{>0} \,. \tag{5.136}$$

We will now prove this claim for the function $f_1(x)$. The case of $f_2(x)$ is proven analogously. Let us denote by

$$f_{1,m}(x) = \frac{(3m+1)x^{3m+1}}{1 - x^{3m+1}} \,, \quad m \in \mathbb{N} \,, \tag{5.137}$$

and consider the derivative $\mathrm{d}^n f_{1,m}(x)/\mathrm{d}x^n$ for fixed m. We want to determine its contributions to $\mathrm{d}^n f_1(0)/\mathrm{d}x^n$. Since we are interested in those terms that survive after taking $x = 0$, we look for the monomials of order $x^{d(3m+1)-n}$, where $d|n$, in the numerator of $\mathrm{d}^n f_{1,m}(x)/\mathrm{d}x^n$, and we take $3m+1 = n/d$. More precisely, let us introduce the parameter $q = 3m+1$ for simplicity of notation. Deriving a-times the factor qx^q and $(n-a)$-times the factor $(1-x^q)^{-1}$, we have the term

$$\binom{n}{a} \frac{\mathrm{d}^a (qx^q)}{\mathrm{d}x^a} \frac{\mathrm{d}^{n-a}(1-x^q)^{-1}}{\mathrm{d}x^{n-a}} \,, \quad a \in \mathbb{N}_{\neq 0} \,. \tag{5.138}$$

Recall that the generalized binomial theorem for the geometric series yields

$$\frac{\mathrm{d}^{n-a}(1-x^q)^{-1}}{\mathrm{d}x^{n-a}} = \sum_{k=0}^{\infty} \frac{(qk)!}{(qk-n+a)!} x^{qk-n+a} \,. \tag{5.139}$$

Substituting Eq. (5.139) into Eq. (5.138) and performing the derivation, we find

$$n! \sum_{k=0}^{\infty} q \binom{q}{q-a} \binom{qk}{qk-n+a} x^{(1+k)q-n} \,. \tag{5.140}$$

It follows then that the only non-zero term at fixed $m \in \mathbb{N}$ comes from the values of $k \in \mathbb{N}$ and $a \in \mathbb{N}_{\neq 0}$ such that $(1+k)q = n$ and $a = q = 3m+1$, which implies in turn that $q|n$ with $q \equiv_3 1$. Finally, summing the non-trivial contributions over q gives precisely

$$\frac{\mathrm{d}^n f_1(0)}{\mathrm{d}x^n} = \sum_{\substack{q|n \\ q\equiv_3 1}} n! q \binom{q}{0} \binom{n-q}{0} = n! \sum_{\substack{q|n \\ q\equiv_3 1}} q \,. \tag{5.141}$$

□

Finally, Theorem 5.2.5 implies that the Stokes constants R_n, $n \in \mathbb{Z}_{>0}$, can be naturally organized as coefficients of a generating function given by *quantum dilogarithms*.

Corollary 5.2.8. *The Stokes constants R_n, $n \in \mathbb{Z}_{>0}$, are encoded in the generating function*

$$\sum_{n=1}^{\infty} R_n x^{n/3} = 3 \log \frac{(x^{2/3};\, x)_\infty}{(x^{1/3};\, x)_\infty} \,, \quad |x| < 1 \,. \tag{5.142}$$

Proof. We apply the definition of the quantum dilogarithm in Eq. (B.1) and Taylor expand the logarithm function for $|x| < 1$. We obtain in this way that

$$\log(x^{2/3};\, x)_\infty = \sum_{k=0}^{\infty} \log(1 - x^{2/3+k}) = -\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{x^{2m/3+mk}}{m} \,, \tag{5.143}$$

and therefore also

$$\log \frac{(x^{2/3};\, x)_\infty}{(x^{1/3};\, x)_\infty} = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x^{(1+3k)m/3} - \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} x^{(2+3k)m/3} \,. \tag{5.144}$$

Renaming $n = (1+3k)m$ and $n = (2+3k)m$ in the first and second terms of the RHS, respectively, we find

$$\log \frac{(x^{2/3};\, x)_\infty}{(x^{1/3};\, x)_\infty} = \sum_{n=1}^{\infty} x^{n/3} \left(\sum_{\substack{d|n \\ d\equiv_3 1}} \frac{d}{n} - \sum_{\substack{d|n \\ d\equiv_3 2}} \frac{d}{n} \right) \,. \tag{5.145}$$

Substituting the arithmetic formula for the Stokes constants in Eq. (5.123) into the expression in Eq. (5.145), we obtain the desired statement. □

Choosing $x = q = \mathrm{e}^{-2\pi \mathrm{i}\tau^{-1}}$, Corollary 5.2.8 directly provides an exact q-series expression for the *discontinuity* of the asymptotic series $\psi(\tau)$ across the positive imaginary axis, which borders the only two distinct Stokes sectors in the upper half of the Borel plane. Namely, following the definition in Eq. (1.27) and using Eq. (5.120), we find that

$$\mathrm{disc}_{\pi/2}\psi(\tau) = s_+(\psi)(\tau) - s_-(\psi)(\tau) = \sum_{n=1}^{\infty} R_n \mathrm{e}^{-n\mathcal{A}_\infty \mathrm{i}\tau} \,, \tag{5.146}$$

where $s_\pm(\psi)(\tau)$ are the lateral Borel resummations at angles $\pi/2 \pm \epsilon$ with $\epsilon \ll 1$, which lie slightly above and below the Stokes line along the positive imaginary axis, respectively. Substituting Eq. (5.142) into Eq. (5.146), we obtain the exact formula

$$\mathrm{disc}_{\pi/2}\psi(\tau) = 3\log(q^{2/3};\, q)_\infty - 3\log(q^{1/3};\, q)_\infty \,, \tag{5.147}$$

which is dual to the discontinuity formula in Eq. (5.59). We stress that the q-series $(q^{2/3};\, q)_\infty$ and $(q^{1/3};\, q)_\infty$ occur in the *holomorphic block* of $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ in Eq. (3.33)[7].

Remark 5.2.2. *The $q, \tilde{q}$-series in the factorized expression of the spectral trace of local $\mathbb{P}^2$ encode the perturbative information in one asymptotic limit in $\hbar$, while being invisible to perturbation theory in the dual limit. Simultaneously, the same $q, \tilde{q}$-series supply the discontinuities in the opposite regimes. Thus, the holomorphic block resurges from the asymptotics of the anti-holomorphic one, and vice versa. This two-way exchange of perturbative/non-perturbative content between the holomorphic/anti-holomorphic blocks is one of several exact cross-relations connecting the weak and strong coupling resurgent structures and paving the way towards the formulation of a global strong-weak resurgent symmetry, which we will describe in Section 6.2.*

5.2.4 Exact large-order relations

Following the same arguments of Section 5.1.3, we provide here a number-theoretic characterization of the perturbative coefficients b_{2n}, $n \in \mathbb{N}_{\neq 0}$, of the asymptotic series $\psi(\tau)$ in Eq. (5.93). Once again, we upgrade the large-n asymptotics of the coefficients by systematically including the contributions from all sub-dominant singularities in the Borel plane and obtain in this way the *exact large-order relation*

$$b_{2n} = \frac{\Gamma(2n-1)}{\pi \mathrm{i}(\mathcal{A}_\infty \mathrm{i})^{2n-1}} \sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} \,, \quad n \in \mathbb{Z}_{>0} \,, \tag{5.148}$$

where the Stokes constant R_m is written in Eq. (5.121) and $\mathcal{A}_\infty = 2\pi/3$ as before. Remarkably, up to the simple prefactors above, the perturbative coefficients of $\psi(\tau)$ are given by the *Dirichlet series* encoding the strong coupling Stokes constants evaluated at odd integer points.

Theorem 5.2.9. *The Stokes constants R_m, $m \in \mathbb{Z}_{>0}$, satisfy the exact relations*

$$\sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} = 3\frac{\zeta(2n)}{3^{2n-1}} \left(\zeta\left(2n-1, \frac{1}{3}\right) - \zeta\left(2n-1, \frac{2}{3}\right) \right) \,, \quad n \in \mathbb{Z}_{>0} \,, \tag{5.149}$$

where $\zeta(z)$ is the Riemann zeta function and $\zeta(z, a)$ is the Hurwitz zeta function.

[7] Disregarding constant prefactors, the $q, \tilde{q}$-series in Eqs. (5.59) and (5.147) differ from the logarithms of the holomorphic/anti-holomorphic blocks in Eq. (3.33) by powers of two acting on the numerator/denominator, respectively.

Proof. Substituting the original expression for the perturbative coefficients b_{2n}, $n \in n \in \mathbb{Z}_{>0}$, in Eq. (5.94) into the exact large-order relation in Eq. (5.148), we have that

$$\sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} = \pi\sqrt{3}(2\pi)^{4n-2} \frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)!} \,. \tag{5.150}$$

Using the known identities

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \,, \tag{5.151a}$$

$$B_{2n-1}(2/3) = -B_{2n-1}(1/3) \,, \quad B_{2n-1}(z) = -(2n-1)\zeta(2-2n, z) \,, \tag{5.151b}$$

the formula in Eq. (5.150) becomes

$$\sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} = \pi\sqrt{3}(-1)^n (2\pi)^{2n-2} \frac{\zeta(2n)}{(2n-2)!} \left(\zeta\left(2-2n, \frac{2}{3}\right) - \zeta\left(2-2n, \frac{1}{3}\right)\right) \,. \tag{5.152}$$

The functional identity for the Hurwitz zeta function in Eq. (5.66) yields

$$\begin{aligned} \zeta\left(2-2n, \frac{2}{3}\right) - \zeta\left(2-2n, \frac{1}{3}\right) =& \frac{2\sqrt{3}(2n-2)!}{(-1)^{n-1}(6\pi)^{2n-1}} \left(\zeta\left(2n-1, \frac{2}{3}\right) \right. \\ & \left. -\zeta\left(2n-1, \frac{1}{3}\right)\right) \,, \end{aligned} \tag{5.153}$$

and substituting this into Eq. (5.152), we obtain the desired statement. □

Note that the exact expression in Eq. (5.149) can be written equivalently in terms of the integer constants β_m, $m \in \mathbb{Z}_{\neq 0}$. Namely,

$$\sum_{m=1}^{\infty} \frac{\beta_m}{m^{2n}} = \frac{\zeta(2n)}{3^{2n-1}} \left(\zeta\left(2n-1, \frac{1}{3}\right) - \zeta\left(2n-1, \frac{2}{3}\right)\right) \,, \quad n \in \mathbb{Z}_{>0} \,, \tag{5.154}$$

which is dual to the formula in Eq. (5.68).

Remark 5.2.3. *As before, let us point out that the series in the LHS of Eq. (5.149) belongs to the family of* Dirichlet series *[158]. As a consequence of Corollary 5.2.6, the sequence of complex numbers $\{R_m\}_{m\in\mathbb{Z}_{>0}}$ defines a bounded multiplicative arithmetic function, and the corresponding Dirichlet series satisfies an expansion as an* Euler product *indexed by the set of prime numbers $\mathbb{P}$, that is,*

$$\sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} = \prod_{p\in\mathbb{P}} \sum_{e=0}^{\infty} \frac{R_{p^e}}{p^{e(2n-1)}} \,, \quad n \in \mathbb{Z}_{>0} \,, \tag{5.155}$$

*which proves that the given Dirichlet series is, indeed, an L-*series*. Similarly, the same statements apply to the Dirichlet series associated with the sequence of integers $\{\beta_m\}_{m\in\mathbb{Z}_{>0}}$ that appears in the LHS of Eq. (5.154). We will further explore this direction in Section 5.3.*

5.2.5 Exponentiating with alien calculus

Let us now translate our analytic solution to the resurgent structure of the asymptotic series $\psi(\tau)$ in Eq. (5.93) into results on the original, exponentiated perturbative series in Eq. (5.90), which we denote by

$$\varrho(\tau) = \mathrm{e}^{\psi(\tau)} = \exp\left(\sqrt{3}\sum_{n=1}^{\infty}(-1)^n \frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)}(6\pi\tau)^{2n-1}\right) \in \mathbb{Q}[\pi, \sqrt{3}][\![\tau]\!] \,, \tag{5.156}$$

which is also a Gevrey-1 asymptotic series. Its Borel transform $\hat{\varrho}(\zeta)$ inherits from $\hat{\psi}(\zeta)$ the same pattern of singularities in Eq. (5.113). Namely, there are infinitely many and discrete logarithmic branch points located along the imaginary axis of the complex ζ-plane at $\eta_n = \mathcal{A}_\infty in$, $n \in \mathbb{Z}_{\neq 0}$. Let us denote by $s_\pm(\varrho)(\tau)$ the lateral Borel resummations at the angles $\pi/2 \pm \epsilon$ with $\epsilon \ll 1$, which lie slightly above and below the Stokes line along the positive imaginary axis, respectively. The same arguments developed in Section 5.1.4 using *alien derivation* also apply here. In particular, we find that

$$s_+(\varrho)(\tau) = s_-(\varrho)(\tau) + s_-(\varrho)(\tau)\sum_{k=1}^{\infty}\mathrm{e}^{-\eta_k/\tau}\bar{R}_k \,, \tag{5.157}$$

where the asymptotic series $\varrho_k(\tau)$, which resurges from $\varrho(\tau)$ at the singularity η_k, is simply

$$\varrho_k(\tau) = \varrho(\tau) \,, \quad k \in \mathbb{Z}_{>0} \,, \tag{5.158}$$

and the Stokes constant $\bar{R}_k \in \mathbb{C}$ of $\varrho(\tau)$ at the singularity η_k is fully determined by the Stokes constants of $\psi(\tau)$ via the closed combinatorial formula

$$\bar{R}_k = \sum_{p\in\mathcal{P}(k)} \frac{1}{r!}\binom{r}{N_1, \ldots, N_k} R_{n_1}\cdots R_{n_r} \,, \quad k \in \mathbb{Z}_{>0} \,, \tag{5.159}$$

where $\mathcal{P}(k)$ is the set of all partitions $p = (n_1, \ldots, n_r)$ of the positive integer k, $r = |p|$ denotes the length of the partition, and $N_i \in \mathbb{N}$ is the number of times that the positive integer $i \in \mathbb{N}_{\neq 0}$ is repeated in the partition p. Note that $\sum_{i=1}^{k} N_i = r$. We stress that the sum over partitions in Eq. (5.159) is finite, and thus all the Stokes constants of the original perturbative series $\varrho(\tau)$ are known exactly. More precisely, the discontinuity formula in Eq. (5.157) solves the resurgent structure of $\varrho(\tau)$ analytically. Applying the same computations above to the discontinuity of $\varrho(\tau)$ across the angle $3\pi/2$, and recalling that $R_n = -R_{-n}$ for all $n \in \mathbb{Z}_{\neq 0}$, we find straightforwardly that

$$\bar{R}_k = \sum_{p\in\mathcal{P}(-k)} \frac{(-1)^r}{r!}\binom{r}{N_1, \ldots, N_k} R_{n_1}\cdots R_{n_r} \,, \quad k \in \mathbb{Z}_{<0} \,, \tag{5.160}$$

which implies that $\bar{R}_k \neq \pm\bar{R}_{-k}$ in general.

Let us point out that the formulae in Eqs. (5.159) and (5.160) immediately prove that $\bar{R}_k \in \mathbb{Q}$, $k \in \mathbb{Z}_{\neq 0}$. However, we can say more. The discontinuity formula in Eq. (5.147) can be directly exponentiated to give an exact generating function in terms of known q-series for the Stokes constants $\bar{R}_k$. Namely, we find that

$$\sum_{k=1}^{\infty}\bar{R}_k q^{k/3} = \frac{(q^{2/3};\, q)_\infty^3}{(q^{1/3};\, q)_\infty^3} \,, \quad \sum_{k=1}^{\infty}\bar{R}_{-k} q^{k/3} = \frac{(q^{1/3};\, q)_\infty^3}{(q^{2/3};\, q)_\infty^3} \,. \tag{5.161}$$

As a consequence of the q-binomial theorem, the quotients of q-series in Eq. (5.161) can be expanded in powers of $q^{1/3}$, and the resulting numerical coefficients possess a natural interpretation in terms of counting partitions. In particular, they are integer numbers. Explicitly, the first several Stokes constants $\bar{R}_k$, $k > 0$, are

$$3, 3, 1, 3, 6, 0, -3, 9, 9, -9, 0, 19, -6, -15, 27, 12, -33, 17, 33, \dots , \tag{5.162}$$

while the first several Stokes constants $\bar{R}_k$, $k < 0$, are

$$-3, 6, -10, 12, -9, 1, 9, -15, 8, 15, -42, 54, -36, -15, 73, -90, 39, 62, -153, \dots . \tag{5.163}$$

Our exact solution fully agrees with the numerical investigation of Gu and Mariño [10].

5.3 A new number-theoretic duality

We will now further develop the simple arithmetic symmetry observed in the closed formulae in Sections 5.1.2 and 5.2.3 and reformulate it into a full-fledged *analytic number-theoretic duality*—shedding new light on the statements of Sections 5.1.3 and 5.2.4.

Let us recall that the Stokes constants associated with the limits $\hbar \to 0$ and $\hbar \to \infty$ are given by the explicit divisor sum functions in Eqs. (5.32) and (5.123), respectively. We can write these formulae equivalently as

$$\frac{S_n}{S_1} = \sum_{d|n} \frac{1}{d} \chi_{3,2}(d) \, , \quad \frac{R_n}{R_1} = \sum_{d|n} \frac{d}{n} \chi_{3,2}(d) \, , \quad n \in \mathbb{Z}_{\neq 0} \, , \tag{5.164}$$

where $\chi_{3,2}(n)$ is the unique non-principal *Dirichlet character* modulo 3, that is,

$$\chi_{3,2}(n) = \begin{cases} 0 & \text{if} \quad n \equiv 0 \mod 3 \\ 1 & \text{if} \quad n \equiv 1 \mod 3 \\ -1 & \text{if} \quad n \equiv 2 \mod 3 \end{cases} , \quad n \in \mathbb{Z} \, . \tag{5.165}$$

The divisor sum functions in Eq. (5.164) can then be decomposed as

$$\frac{S_n}{S_1} = \sum_{d|n} \chi_{3,2}\left(d\right) F_{-1}\left(d\right) F_0 \left(\frac{n}{d}\right) = \left(\chi_{3,2} F_{-1} * F_0\right)(n) \, , \quad n \in \mathbb{Z}_{\neq 0} \, , \tag{5.166a}$$

$$\frac{R_n}{R_1} = \sum_{d|n} \chi_{3,2}\left(d\right) F_0\left(d\right) F_{-1} \left(\frac{n}{d}\right) = \left(\chi_{3,2} F_0 * F_{-1}\right)(n) \, , \quad n \in \mathbb{Z}_{\neq 0} \, , \tag{5.166b}$$

where the product $*$ denotes the *Dirichlet convolution* of arithmetic functions and

$$F_\alpha(n) = n^\alpha \, , \quad \alpha \in \mathbb{R} \, . \tag{5.167}$$

Note that, despite the arithmetic functions $\chi_{3,2}$, F_0, F_{-1} being totally multiplicative, the Dirichlet convolutions S_n/S_1, R_n/R_1 are multiplicative only.

We have shown that the Dirichlet series associated with the Stokes constants naturally appear in the exact large-order formulae for the perturbative coefficients in Eqs. (5.61) and (5.148), and we have proved in Remarks 5.1.1 and 5.2.3 that they are, in particular,

L-series. We can say something more. Since the multiplication of Dirichlet series is compatible with the Dirichlet convolution [158], it follows directly from the decomposition in Eq. (5.166) that we have the *formal factorization*

$$\sum_{m=1}^{\infty} \frac{S_m/S_1}{m^s} = \sum_{m=1}^{\infty} \frac{\chi_{3,2}(m)F_{-1}(m)}{m^s} \sum_{m=1}^{\infty} \frac{F_0(m)}{m^s} = L(s+1,\chi_{3,2})\zeta(s)\,, \tag{5.168a}$$

$$\sum_{m=1}^{\infty} \frac{R_m/R_1}{m^s} = \sum_{m=1}^{\infty} \frac{\chi_{3,2}(m)F_0(m)}{m^s} \sum_{m=1}^{\infty} \frac{F_{-1}(m)}{m^s} = L(s,\chi_{3,2})\zeta(s+1)\,, \tag{5.168b}$$

where $s \in \mathbb{C}$ such that $\Re(s) > 1$, $\zeta(s)$ is the Riemann zeta function, and $L(s,\chi_{3,2})$ is the Dirichlet L-series of the primitive character $\chi_{3,2}$. We have found, in this way, that the arithmetic duality which relates the weak- and strong-coupling Stokes constants S_n, R_n, $n \in \mathbb{N}_{\neq 0}$, in Eq. (5.166) is translated at the level of the L-series encoded in the perturbative coefficients into a simple unitary shift of the arguments of the factors in the RHS of Eq. (5.168). Furthermore, $L(s,\chi_{3,2})$ is absolutely convergent for $\Re(s) > 1$ and can be analytically continued to a meromorphic function on the whole complex s-plane, called a *Dirichlet L-function*. Each of the two L-series in the LHS of Eq. (5.168) is thus the product of two well-known L-functions, and such a remarkable factorization explicitly proves the convergence in the right half-plane of the complex numbers $\Re(s) > 1$ and the existence of a meromorphic continuation throughout the complex s-plane. Namely, our L-series are, themselves, *L-functions*.

In the rest of this thesis, we will refer to the meromorphic continuation of the L-series with coefficients given by the sequences of Stokes constants $\{S_m\}$ and $\{R_m\}$ as weak and strong coupling *resurgent L-functions*, and we will denote them as $L_0(s)$ and $L_\infty(s)$, $s \in \mathbb{C}$, respectively. Explicitly,

$$L_0(s) = \sum_{m=1}^{\infty} \frac{S_m}{m^s} = S_1 L(s+1,\chi_{3,2})\zeta(s)\,, \tag{5.169a}$$

$$L_\infty(s) = \sum_{m=1}^{\infty} \frac{R_m}{m^s} = R_1 L(s,\chi_{3,2})\zeta(s+1)\,. \tag{5.169b}$$

Remark 5.3.1. *If we extend the discrete index $n \in \mathbb{Z}_{>0}$ of the sequences of perturbative coefficients $\{a_{2n}\}$, $\{b_{2n}\}$ to a continuous variable $s \in \mathbb{C}$, then the exact large-order relations in Eqs. (5.61) and (5.148) allow us, in principle, to analytically continue the perturbative coefficients to meromorphic functions throughout the complex s-plane as*

$$a_s = \frac{\Gamma(s)}{\pi \mathrm{i}\,(\mathcal{A}_0 \mathrm{i})^s} L_0(s)\,, \quad b_s = \frac{\Gamma(s-1)}{\pi \mathrm{i}\,(\mathcal{A}_\infty \mathrm{i})^{s-1}} L_\infty(s-1)\,. \tag{5.170}$$

A similar observation was made in the recent work of [160] on the exact large-order relations for general resurgent trans-series.

Finally, note that the formulae in Eqs. (5.62) and (5.149) follow from Eq. (5.168) through the known relation between Dirichlet L-functions and Hurwitz zeta functions at rational values. Specifically,

$$L(s,\chi_{3,2}) = \frac{1}{3^s}\left(\zeta\left(s,\frac{1}{3}\right) - \zeta\left(s,\frac{2}{3}\right)\right)\,, \quad \Re(s) > 1\,. \tag{5.171}$$

Moreover, the Dirichlet L-function satisfies the Euler product expansion

$$L(s,\chi_{3,2}) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi_{3,2}(p)}{p^s}\right)^{-1}, \quad \Re(s) > 1\,, \tag{5.172}$$

where $\mathbb{P}$ is the set of prime numbers.

Chapter 6

Resurgent L-functions

The Stokes constants are analytic invariants that capture information about the non-perturbative corrections to a resurgent asymptotic series. Yet, in some remarkable cases, they are rational or integer numbers and possess an interpretation as enumerative invariants related to BPS counting [161–163]. Examples come from various quantum theories, including $4d$ $\mathcal{N} = 2$ supersymmetric gauge theory in the NS limit of the Omega-background [135], complex CS theory on Seifert fibered homology spheres [14] and on complements of hyperbolic knots [75, 76], the standard and NS topological strings on (toric) CY threefolds [1, 10, 38–40, 42, 44, 46–49] and their refinement [43, 45], and Walcher's real topological string [41].

In this chapter, building on the fundamental results of Chapter 5, we complete our understanding of the unique set of exact relations connecting the resurgent structures of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ for $\hbar \to 0$ and $\hbar \to \infty$. In particular, we uncover a direct way of obtaining the perturbative expansion of the generating function of the Stokes constants in one regime from the generating function in the other, which amounts to performing an operation that is the formal inverse of taking the discontinuity of the asymptotic series. Thus, we argue that a full-fledged strong-weak resurgent symmetry is at the heart of the analytic number-theoretic duality of Section 5.3. Through this newly discovered exact symmetry, the two-way exchange of perturbative/non-perturbative information between the holomorphic and anti-holomorphic blocks in the factorization of the spectral trace takes a mathematically precise form. This is a realization of underlying physical mechanisms that can be intuitively traced back to the S-type duality between the worldsheet and WKB contributions to the total grand potential of the topological string on local $\mathbb{P}^2$. Moreover, the weak and strong coupling Stokes constants are related by an arithmetic twist, while the corresponding resurgent L-functions analytically continue each other through a functional equation. We reproduce [2, Section 3].

6.1 Generating functions of the Stokes constants

In this chapter and in the rest of this thesis, we apply the conventions of Remark 1.2.1, which differ slightly from those adopted in Chapter 5 due to a change in the definitions of the Borel and Laplace transforms. In particular, with the conventions of Remark 1.2.1, the singularities of the Borel transforms of the series $\phi(\hbar)$ and $\psi(\tau)$ in Eqs. (5.1) and (5.93) are not logarithmic branch points but simple poles. Note that all the results of Chapter 5 that we use here remain unaltered.

As it will be useful in the rest of this thesis, let us now write the generating series of the weak coupling Stokes constants S_n, $n \in \mathbb{Z}_{\neq 0}$, in Eq. (5.32) in the form

$$f_0(y) := \begin{cases} \sum_{n>0} S_n \, \mathrm{e}^{2\pi i n y} & \text{if } \Im(y) > 0 \\ \\ -\sum_{n<0} S_n \, \mathrm{e}^{2\pi i n y} & \text{if } \Im(y) < 0 \end{cases} , \tag{6.1}$$

which defines a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ with the periodicity and parity properties

$$f_0(y+1) = f_0(y) , \quad f_0(-y) = -f_0(y) . \tag{6.2}$$

It follows that $f_0(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, need only be specified in the upper half of the complex y-plane, where it is known in closed form by means of the exact discontinuity formula in Eq. (5.59). Namely, we have that

$$f_0(y) = -3 \log \frac{(w;\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(w^{-1};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty} - \mathrm{i}\pi , \quad y \in \mathbb{H} , \tag{6.3}$$

where $w = \mathrm{e}^{2\pi \mathrm{i}/3}$ as before and therefore also

$$f_0\left(-\frac{2\pi}{3\hbar}\right) = \mathrm{disc}_{\frac{\pi}{2}} \, \phi(\hbar) , \quad \hbar \in \mathbb{H} . \tag{6.4}$$

Analogously, we write the generating series of the strong coupling Stokes constants R_n, $n \in \mathbb{Z}_{\neq 0}$, in Eq. (5.123) in the form

$$f_\infty(y) := \begin{cases} \sum_{n>0} R_n \, \mathrm{e}^{2\pi i n y} & \text{if } \Im(y) > 0 \\ \\ -\sum_{n<0} R_n \, \mathrm{e}^{2\pi i n y} & \text{if } \Im(y) < 0 \end{cases} , \tag{6.5}$$

which defines a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ with the periodicity and parity properties[1]

$$f_\infty(y+1) = f_\infty(y) , \quad f_\infty(-y) = f_\infty(y) . \tag{6.6}$$

Again, it follows that $f_\infty(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, need only be specified in the upper half of the complex y-plane, where it is known in closed form through the exact discontinuity formula in Eq. (5.147). Explicitly, we have that

$$f_\infty(y/3) = 3 \log \frac{(\mathrm{e}^{4\pi \mathrm{i} y/3};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{2\pi \mathrm{i} y/3};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty} , \quad y \in \mathbb{H} , \tag{6.7}$$

and therefore also

$$f_\infty\left(-\frac{1}{3\tau}\right) = \mathrm{disc}_{\frac{\pi}{2}} \, \psi(\tau) , \quad \tau \in \mathbb{H} . \tag{6.8}$$

[1] Note the exchange of even/odd parity between the Stokes constants and the corresponding generating functions in both weak and strong coupling limits.

Remark 6.1.1. *Let us now consider the discontinuities* $\mathrm{disc}_{\frac{\pi}{2}}\,\phi(\hbar)$ *and* $\mathrm{disc}_{\frac{\pi}{2}}\,\psi(\tau)$ *as functions of the variable* $\tau \in \mathbb{H}$ *using Eqs. (5.59), (5.147), and (5.92). Namely, we take the functions*

$$\left(\mathrm{disc}_{\frac{\pi}{2}}\,\phi\right)(\tau) = -3\log\frac{(w;\,\tilde{q})_\infty}{(w^{-1};\,\tilde{q})_\infty} - \mathrm{i}\pi\,, \quad \left(\mathrm{disc}_{\frac{\pi}{2}}\,\psi\right)(\tau) = 3\log\frac{(q^{2/3};\,q)_\infty}{(q^{1/3};\,q)_\infty}\,, \tag{6.9}$$

where $\tilde{q} = \mathrm{e}^{2\pi\mathrm{i}\tau}$ *and* $q = \mathrm{e}^{-2\pi\mathrm{i}/\tau}$ *as before. It follows that*

$$\left(\mathrm{disc}_{\frac{\pi}{2}}\,\phi\right)(\tau+1) = \left(\mathrm{disc}_{\frac{\pi}{2}}\,\phi\right)(\tau)\,, \quad \left(\mathrm{disc}_{\frac{\pi}{2}}\,\psi\right)\left(\frac{\tau}{3\tau+1}\right) = \left(\mathrm{disc}_{\frac{\pi}{2}}\,\psi\right)(\tau)\,, \tag{6.10}$$

that is, the discontinuities are invariant under the action of the generators

$$T : \tau \mapsto \tau + 1\,, \quad \gamma_3 : \tau \mapsto \frac{\tau}{3\tau+1} \tag{6.11}$$

of the congruence subgroup $\Gamma_1(3)$*, respectively. Recall that we have introduced* $\Gamma_1(3)$ *in Eq. (3.16) as it dictates the symmetries of the moduli space of complex structures of the mirror of local* $\mathbb{P}^2$*. The same* $\Gamma_1(3)$*-structure crucially occurs in Section 7.3, where the modular considerations above are upgraded to full quantum modularity statements on the generating functions of the Stokes constants* $f_0(y)$, $f_\infty(y)$, $y \in \mathbb{H}$.

Asymptotic series for the generating functions

Let us compute the all-orders perturbative expansions in the limit $y \to 0$ of the weak and strong coupling generating functions $f_0(y)$, $f_\infty(y)$, $y \in \mathbb{H}$, in Eqs. (6.3) and (6.7), which we denote by $\tilde{f}_0(y)$, $\tilde{f}_\infty(y)$, respectively. We apply the known asymptotic expansion formula for the quantum dilogarithm in Eq. (B.19a) to the closed expression for the generating function of the weak coupling Stokes constants in Eq. (6.3) and explicitly evaluate the special functions that appear. We find that

$$\tilde{f}_0(y) = -\frac{\pi\mathrm{i}}{2} - \frac{3}{2\pi\mathrm{i}y}\left(\mathrm{Li}_2(w) - \mathrm{Li}_2(w^{-1})\right) - 2\sqrt{3}\mathrm{i}\sum_{n=1}^{\infty}(6\pi\mathrm{i}y)^{2n-1}\frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)}\,, \tag{6.12}$$

which simplifies into

$$\tilde{f}_0(y) = -\frac{\pi\mathrm{i}}{2} - \frac{3\mathcal{V}}{2\pi\mathrm{i}y} - 2\psi(y)\,, \tag{6.13}$$

where[2] $\mathcal{V} = 2\Im\left(\mathrm{Li}_2(w)\right)$, $w = \mathrm{e}^{2\pi\mathrm{i}/3}$, and $\psi(y)$ is the asymptotic series occurring in the strong coupling perturbative expansion of $\log\mathrm{Tr}(\rho_{\mathbb{P}^2})$ in Eq. (5.93).

Similarly, we apply the known asymptotic expansion formula for the quantum dilogarithm in Eq. (B.19b), with the choice of $\alpha = 1/3, 2/3$, to the closed expression for the generating function of the strong coupling Stokes constants in Eq. (6.7) and explicitly evaluate the special functions that appear. We derive in this way that

$$\tilde{f}_\infty(y) = -3\log\frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(-6\pi\mathrm{i}y) - 6\sum_{n=1}^{\infty}(6\pi\mathrm{i}y)^{2n}\frac{B_{2n}B_{2n+1}(2/3)}{2n(2n+1)!}\,, \tag{6.14}$$

[2] Observe that $\frac{27}{8\pi^2}\mathcal{V} \simeq 0.462758$ is the quantum volume of local $\mathbb{P}^2$, which is given by the value of its Kähler parameter at the conifold [87]. This is related to the constant V in Eq. (5.90) by $\mathcal{V} = -2\pi^2/3 + 2\mathrm{i}V/3$.

which equals to

$$\tilde{f}_\infty(y) = -3\log\frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(-6\pi \mathrm{i} y) + 2\phi(2\pi y)\,, \tag{6.15}$$

where $\phi(y)$ is the asymptotic series occurring in the weak coupling perturbative expansion of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ in Eq. (5.1). As a consequence of Eqs. (6.13) and (6.15), the exact resurgent structures of the asymptotic series $\tilde{f}_0(y)$ and $\tilde{f}_\infty(y)$ follow straightforwardly from the work of [1] on the perturbative expansions $\psi(\tau)$ and $\phi(\hbar)$, respectively, which we have reproduced in Chapter 5.

Finally, putting together Eqs. (6.4) and (6.8) with Eqs. (6.13) and (6.15), we obtain the asymptotic expansions of the discontinuities of the original perturbative series $\phi(\hbar)$ and $\psi(\tau)$ in the large limits of the variables $\hbar$ and τ, respectively. Namely,

$$\mathrm{disc}_{\frac{\pi}{2}}\,\phi(\hbar) \sim -\frac{\pi \mathrm{i}}{2} - \frac{9\mathcal{V}}{4\pi^2 \mathrm{i}}\hbar - 2\psi\left(-\frac{2\pi}{3\hbar}\right)\,, \quad \hbar \to \infty\,, \tag{6.16a}$$

$$\mathrm{disc}_{\frac{\pi}{2}}\,\psi(\tau) \sim -3\log\frac{\Gamma(2/3)}{\Gamma(1/3)} - \log\left(\frac{2\pi \mathrm{i}}{\tau}\right) + 2\phi\left(-\frac{2\pi}{3\tau}\right)\,, \quad \tau \to \infty\,. \tag{6.16b}$$

6.2 Strong-weak resurgent symmetry

As we have described in Chapter 5, the resurgent structures of the logarithm of the spectral trace of local $\mathbb{P}^2$ in both weak and strong coupling limits display a rich analytic number-theoretic fabric centered around the key role played by the Stokes constants and their generating functions. The interaction of q-series and L-functions governs a network of relations encompassing and intertwining the regimes of $\hbar \to 0$ and $\hbar \to \infty$. Let us give a concise overview.

- The discontinuity $\mathrm{disc}_{\frac{\pi}{2}}\psi(\tau)$, which is equal to the generating function $f_\infty\left(-\frac{1}{3\tau}\right)$ n Eq. (6.7) of the strong coupling Stokes constants R_n, $n \in \mathbb{Z}_{\neq 0}$, in Eq. (5.123), reproduces the weak coupling perturbative series $\phi(\hbar)$ in Eq. (5.1) when expanded in the limit $\tau \to \infty$ (or equivalently $\hbar \to 0$), as we have seen in Eq. (6.16b).

- Resurgence associates $\phi(\hbar)$ with the collection $\{\zeta_n, S_n\}$, $n \in \mathbb{Z}_{\neq 0}$, of simple poles and Stokes constants, which satisfy $\zeta_n = \mathcal{A}_0 \mathrm{i} n$ with $\mathcal{A}_0 = 4\pi^2/3$ and $S_{-n} = S_n$ with $S_n = \sum_{d|n} \chi_{3,2}(d)/d$.

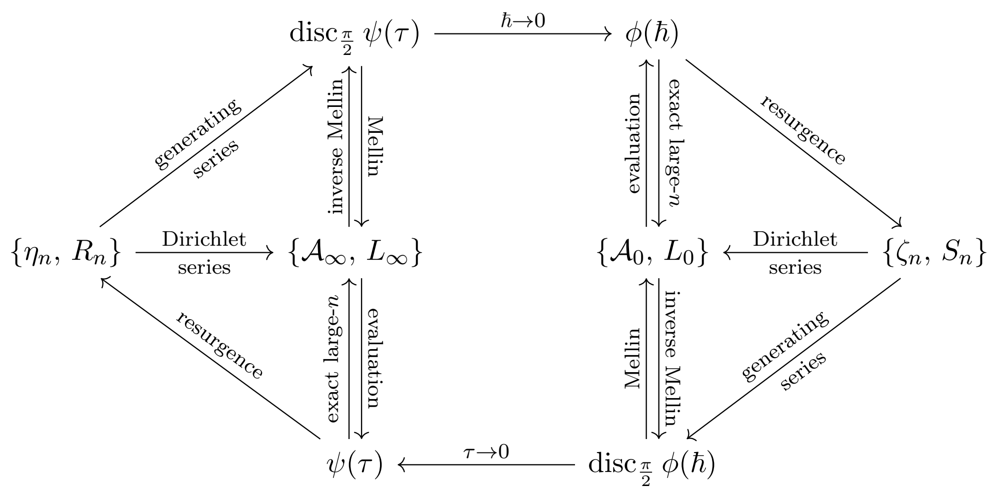
- The exact large-n relations in Eq. (5.61) for the perturbative coefficients a_{2n}, $n \in \mathbb{Z}_{>0}$, of $\phi(\hbar)$ uniquely determine the weak coupling resurgent L-function $L_0(s)$ in Eq. (5.169a). The latter reproduces, in turn, the perturbative coefficients when evaluated at $s = 2n$.

- The discontinuity $\mathrm{disc}_{\frac{\pi}{2}}\phi(\hbar)$, which is equal to the generating function $f_0\left(-\frac{2\pi}{3\hbar}\right)$ in Eq. (6.3) of the weak coupling Stokes constants S_n, $n \in \mathbb{Z}_{\neq 0}$, in Eq. (5.32), reproduces the strong coupling perturbative series $\psi(\tau)$ in Eq. (5.93) when expanded in the limit $\hbar \to \infty$ (or equivalently $\tau \to 0$), as we have seen in Eq. (6.16a).

- Resurgence associates $\psi(\tau)$ with the collection $\{\eta_n, R_n\}$, $n \in \mathbb{Z}_{\neq 0}$, of simple poles and Stokes constants, which satisfy $\eta_n = \mathcal{A}_\infty \mathrm{i} n$ with $\mathcal{A}_\infty = 2\pi/3$ and $R_{-n} = -R_n$ with $R_n = \sum_{d|n} \chi_{3,2}(d) n/d$.

- The exact large-n relations in Eq. (5.148) for the perturbative coefficients b_{2n}, $n \in \mathbb{Z}_{>0}$, of $\psi(\tau)$ uniquely determine the strong coupling resurgent L-function $L_\infty(s)$ in Eq. (5.169b). The latter reproduces, in turn, the perturbative coefficients when evaluated at $s = 2n-1$.

Assembling the above relations together, we find that the resurgent structures of the dual perturbative expansions of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ embed into a unique global construction as they compose the symmetric diagram below.



(6.17)

Notably, starting from any fixed vertex, all other vertices of this *commutative diagram* are spanned by following its directed arrows—that is, the information contained in each vertex reconstructs the whole diagram and is therefore equivalent to the information content of every other vertex. Moreover, two additional arrows are drawn in the diagram in Eq. (6.17) that were not previously stated. They originate from the well-established relation between power series and Dirichlet series sharing the same coefficients, which makes use of the *Mellin transform*.

Lemma 6.2.1. *The weak and strong coupling resurgent L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, in Eqs. (5.169a) and (5.169b) are given by the Mellin transform of the generating functions $f_0(y)$, $f_\infty(y)$, $y \in \mathbb{H}$, of the corresponding Stokes constants, that is,*

$$L_0(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty t^{s-1} f_0(\mathrm{i}t)\, dt\,, \tag{6.18a}$$

$$L_\infty(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty t^{s-1} f_\infty(\mathrm{i}t)\, dt\,. \tag{6.18b}$$

Proof. Let us start with the weak coupling regime of $\hbar \to 0$ and explicitly compute the Mellin transform of the generating function of the Stokes constants in Eq. (6.1). We find that

$$\int_0^\infty t^{s-1} f_0(\mathrm{i}t)\, dt = \int_0^\infty t^{s-1} \left(\sum_{k=1}^\infty S_k \mathrm{e}^{-k2\pi t} \right) dt = \sum_{k=1}^\infty S_k \int_0^\infty t^{s-1} \mathrm{e}^{-k2\pi t}\, dt\,, \tag{6.19}$$

where we have permuted sum and integration due to absolute convergence. The simplified integral in the RHS is then easily computed via the well-known formula

$$\int_0^\infty x^{s-1} \mathrm{e}^{-kx}\, dx = \frac{\Gamma(s)}{k^s}\,, \quad k \in \mathbb{Z}_{>0}\,, \quad s \in \mathbb{C}\,, \tag{6.20}$$

which gives the desired expression in Eq. (6.18a). The same procedure above is straightforwardly applied in the strong coupling regime of $\hbar \to \infty$ to explicitly compute the Mellin transform of the generating function of the Stokes constants in Eq. (6.5), yielding the expression in Eq. (6.18b). □

By the Mellin inversion theorem, we recover the generating functions of the Stokes constants via the *inverse Mellin transform* of the corresponding L-functions. Composing the statement of Lemma 6.2.1 with the exact large-order relations in Eqs. (5.61) and (5.148), we find a direct way of obtaining the perturbative expansion of the generating function in one asymptotic limit from the generating function in the dual limit, which amounts to performing an operation that is the formal inverse of taking the discontinuity of the perturbative series.

Proposition 6.2.2. *The perturbative coefficients a_{2n}, b_{2n}, $n \in \mathbb{Z}_{>0}$, defined in Eqs.* (5.2) *and* (5.94) *are given by the Mellin transform of the discontinuities* $\mathrm{disc}_{\frac{\pi}{2}}\,\phi(\hbar)$, $\mathrm{disc}_{\frac{\pi}{2}}\,\psi(\tau)$, *respectively, that is,*

$$a_{2n} = \frac{(-1)^n}{\pi \mathrm{i}} \int_0^\infty \hbar^{-2n-1} \mathrm{disc}_{\frac{\pi}{2}} \phi(-\mathrm{i}\hbar)\, d\hbar\,, \tag{6.21a}$$

$$b_{2n} = \frac{(-1)^n}{\pi} \int_0^\infty \tau^{-2n} \mathrm{disc}_{\frac{\pi}{2}} \psi(-\mathrm{i}\tau)\, d\tau\,. \tag{6.21b}$$

Proof. Using Eq. (6.4) and applying the change of variable $2\pi t = \mathcal{A}_0/\hbar$, the integral in the RHS of Eq. (6.21a) becomes

$$\int_0^\infty \hbar^{-2n-1} f_0\left(\frac{\mathcal{A}_0 \mathrm{i}}{2\pi\hbar}\right) d\hbar = \frac{(2\pi)^{2n}}{\mathcal{A}_0^{2n}} \int_0^\infty t^{2n-1} f_0(\mathrm{i}t)\, dt\,. \tag{6.22}$$

Using Eq. (6.18a) for $s = 2n$ and the exact large-n relations for the weak coupling perturbative coefficients in Eq. (5.61), we conclude. Similarly, using Eq. (6.8) and applying the change of variable $2\pi t = \mathcal{A}_\infty/\tau$, the integral in the RHS of Eq. (6.21b) becomes

$$\int_0^\infty \tau^{-2n} f_\infty\left(\frac{\mathcal{A}_\infty \mathrm{i}}{2\pi\tau}\right) d\tau = \frac{(2\pi)^{2n-1}}{\mathcal{A}_\infty^{2n-1}} \int_0^\infty t^{2n-2} f_\infty(\mathrm{i}t)\, dt\,. \tag{6.23}$$

Using Eq. (6.18b) for $s = 2n-1$ and the exact large-n relations for the strong coupling perturbative coefficients in Eq. (5.148) yields the desired expression. □

Consider the commutative diagram in Eq. (6.17) once more. The internal vertices and arrows composing the left-side sub-diagram that connects the q-series $\mathrm{disc}_{\frac{\pi}{2}}\psi(\tau)$ with the strong coupling perturbative series $\psi(\tau)$ reduce to one vertical two-headed arrow according to the formulae in Eqs. (5.147) and (6.21b). Analogously, the set of internal vertices and arrows composing the right-side sub-diagram that connects the $\tilde{q}$-series $\mathrm{disc}_{\frac{\pi}{2}}\phi(\hbar)$ with the weak coupling perturbative series $\phi(\hbar)$ is equivalent to one vertical two-headed arrow according to the dual formulae in Eqs. (5.59) and (6.21a). Thus, the two-way exchange of perturbative/non-perturbative information between the dual regimes in $\hbar$ takes the form of a mathematically precise mechanism, which we refer to as *strong-weak resurgent symmetry.* We represent it schematically in the box diagram below, where we stress the contribution of the Stokes constants to all quantities of interest.

$$
\begin{array}{ccc}
\operatorname{disc}_{\frac{\pi}{2}}\psi(\tau) = \sum_{n=1}^{\infty} R_n q^{n/3} & \xrightarrow{\hbar\to 0} & \phi(\hbar) = \sum_{n=1}^{\infty} \left(\frac{\Gamma(2n)}{\pi \mathrm{i}(\mathcal{A}_0 \mathrm{i})^{2n}} \sum_{m=1}^{\infty} \frac{S_m}{m^{2n}} \right) \hbar^{2n} \\
\Big\updownarrow {\scriptstyle \text{strong-weak symmetry}} & & \Big\updownarrow {\scriptstyle \text{strong-weak symmetry}} \\
\psi(\tau) = \sum_{n=1}^{\infty} \left(\frac{\Gamma(2n-1)}{\pi \mathrm{i}(\mathcal{A}_\infty \mathrm{i})^{2n-1}} \sum_{m=1}^{\infty} \frac{R_m}{m^{2n-1}} \right) \tau^{2n-1} & \xleftarrow{\tau\to 0} & \operatorname{disc}_{\frac{\pi}{2}}\phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n
\end{array}
\tag{6.24}
$$

Remark 6.2.1. *The strong-weak resurgent symmetry of the spectral trace of local $\mathbb{P}^2$, which we have presented here, fundamentally arises from the interplay of the Stokes constants, their generating functions in the form of q, $\tilde{q}$-series, and their L-functions. This is the starting point in a wider mathematical program that aims at linking the resurgent properties of q-series with the analytic number-theoretic properties of L-functions and makes contact with the notion of quantum modularity. We will present this new paradigm of resurgence in detail and full generality in Chapter 8, which reproduces [3].*

As pointed out in [5, 10], the TS/ST correspondence for a toric del Pezzo CY threefold X shares many formal similarities with complex CS theory on the complement of a hyperbolic knot $\mathcal{K}$ in the three-sphere. In particular, the spectral determinant $\Xi(\kappa, \hbar)$, which is an entire function of the complex deformation parameter κ of the mirror $\tilde{X}$, corresponds to the state integral, or Andersen–Kashaev invariant, of the knot $Z_{\mathcal{K}}(u, \mathsf{b}^2)$ [164, 165], which is an entire function of the holonomy u around the knot. Here, b^2 is a complex coupling parameter. Thus, the spectral traces $Z(N, \hbar)$ are dual to the coefficients of $Z_{\mathcal{K}}(u, \mathsf{b}^2)$ in a Taylor expansion around $u = 0$. Notably, the state integral can be expressed as a sum of products of holomorphic and anti-holomorphic blocks given by $q, \tilde{q}$-series [166, 167], where q and $\tilde{q}$ are related to b^2 by the same formulae in Eq. (3.34), and the blocks are exchanged under the transformation $\mathsf{b}^2 \mapsto -\mathsf{b}^{-2}$, which swaps q and $\tilde{q}$. In the TS/ST correspondence, a similar factorization property is observed in closed expressions for the fermionic spectral traces $Z(N, \hbar)$, although the holomorphic and anti-holomorphic blocks are generally different functions [10]. Yet, the resurgent study of the spectral trace of local $\mathbb{P}^2$ shows how such different functions are profoundly connected. Indeed, as we have already remarked in Chapter 5 following [1], the holomorphic block resurges from the perturbative expansion of the anti-holomorphic block, and vice versa. This statement assumes a concrete meaning in light of the proven, exact results on the generating functions of the weak and strong coupling Stokes constants assembled into the strong-weak resurgent symmetry. To sum up, we display a minimal scheme below.

$$
\operatorname{disc}_{\frac{\pi}{2}}\psi(\tau) = \sum_{n=1}^{\infty} R_n q^{n/3} \quad \rightleftarrows \quad \operatorname{disc}_{\frac{\pi}{2}}\phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n \tag{6.25}
$$

Here, the red arrows represent the diagonals joining the top-left and bottom-right vertices of the diagram in Eq. (6.24).

On the physics of the strong-weak resurgent symmetry

The strong-weak coupling duality in Eq. (2.52), which is at the heart of the TS/ST correspondence, allows us to describe the strongly coupled conventional topological string compactified on local $\mathbb{P}^2$ in terms of the semiclassical regime of the quantum-mechanical spectral problem defined by the quantization of its mirror curve. Analogously, the semiclassical limit in the topological string coupling constant determines the strong dynamics of the spectral theory. We point out that our strong-weak symmetry relating the exact resurgent structures of the spectral trace of local $\mathbb{P}^2$ in the limits of $\hbar \to 0$ and $\hbar \to \infty$ is a mathematically precise realization of this intuition on the exchange of perturbative and non-perturbative contributions.

The discussion on the physical mechanism underlying the strong-weak resurgent symmetry can be pushed further. Indeed, recall the definition of the total grand potential $J(\mu, \hbar)$ in Eq. (2.50). As we have briefly described in Section 2.3 and further discussed in Chapter 4, the worldsheet generating functional $J^{\mathrm{WS}}(\mu, \hbar)$, which determines the perturbative expansion of $J(\mu, \hbar)$ in the weakly interacting limit $g_s \to 0$, encodes the non-perturbative contributions in $\hbar$ from the conventional topological string partition function. At the same time, the WKB generating functional $J^{\mathrm{WKB}}(\mu, \hbar)$, which determines the perturbative expansion of $J(\mu, \hbar)$ in the semiclassical limit $\hbar \to 0$, contains the non-perturbative, exponentially small effects in g_s from the NS limit of the refined topological string partition function. Employing the TS/ST statement in Eq. (2.56) and the strong-weak coupling duality in Eq. (2.52), we can translate the above argument into the following. The non-perturbative $\hbar$-corrections to the semiclassical perturbative expansion of the fermionic spectral trace $Z(N, \hbar)$, $N \in \mathbb{Z}_{>0}$, in Eq. (3.1a), determine its perturbative expansion for $\hbar \to \infty$. Conversely, the non-perturbative $\hbar^{-1}$-corrections to the strongly coupled perturbative expansion of the fermionic spectral trace $Z(N, \hbar)$, $N \in \mathbb{Z}_{>0}$, in Eq. (3.1b), determine its perturbative expansion for $\hbar \to 0$.

Take now $N = 1$ and consider the spectral trace $Z(1, \hbar)$, as we do in this chapter. We know that the non-analytic, exponential-type contributions at strong and weak coupling are governed by two dual sequences of Stokes constants S_n, R_n, $n \in \mathbb{Z}_{\neq 0}$. We can repackage the same information in the corresponding generating functions $f_0(y)$, $f_\infty(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, or the corresponding L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$. Thus, the argument above implies that the Stokes constants/generating functions/L-functions in one regime must dictate the perturbative coefficients in the other, which we have proven true. Yet, substantially more information is discovered by analytically determining the complete resurgent structures of the spectral trace of local $\mathbb{P}^2$ in the strong and weak limits, as we have summarized in the commutative diagram in Eq. (6.17) and will further detail in the rest of this chapter. Finally, we note that our results recall the traditional notion of *S-duality* in string theory [168][3].

Remark 6.2.2. *As the ideas above apply in full generality, our conceptual argument underlying the strong-weak resurgent symmetry of the spectral trace of local $\mathbb{P}^2$ supports the existence of an appropriate generalization of our results to other toric CY threefolds and higher-order fermionic spectral traces. The detailed exploration of the physical principles of the strong-weak resurgent symmetry will be the subject of future work.*

[3] In the work of [168], it was shown that the S-duality of type IIB superstrings in ten dimensions implies the existence of an S-duality relating the A-model and B-model topological string theories on the same CY background. In particular, the D-instantons of one model correspond to the perturbative amplitudes of the other.

6.3 Functional equation from a unified perspective

Let us go back to the L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, which are defined by the meromorphic continuation to the complex s-plane of the Dirichlet series with coefficients given by the weak and strong coupling Stokes constants S_n, R_n, $n \in \mathbb{Z}_{>0}$, respectively, as we have discussed in Section 5.3. These weak and strong coupling resurgent L-functions satisfy the factorizations in Eqs. (5.169a) and (5.169b) in terms of the Dirichlet L-function $L(s, \chi_{3,2})$ and the Riemann zeta function $\zeta(s)$, whose meromorphic completions

$$\Lambda_{3,2}(s) = \frac{3^{\frac{s}{2}}}{\pi^{\frac{s+1}{2}}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_{3,2}) \,, \quad \Lambda_\zeta(s) = \frac{1}{\pi^{\frac{s}{2}}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \,, \quad s \in \mathbb{C} \,, \tag{6.26}$$

satisfy the well-known functional equations

$$\Lambda_{3,2}(s) = \Lambda_{3,2}(1-s) \,, \quad \Lambda_\zeta(s) = \Lambda_\zeta(1-s) \,, \tag{6.27}$$

which are centered at $s = 1/2$ and relate s with $1-s$. Following the factorization formulae in Eqs. (5.169a) and (5.169b) and using the completed L-functions in Eq. (6.26), we define

$$\Lambda_0(s) = \Lambda_{3,2}(s+1)\Lambda_\zeta(s) = \frac{3^{\frac{s+1}{2}}}{S_1 \pi^{s+1}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}+1\right) L_0(s) \,, \tag{6.28a}$$

$$\Lambda_\infty(s) = \Lambda_{3,2}(s)\Lambda_\zeta(s+1) = \frac{3^{\frac{s}{2}}}{R_1 \pi^{s+1}} \Gamma\left(\frac{s+1}{2}\right)^2 L_\infty(s) \,, \tag{6.28b}$$

which are meromorphic functions for $s \in \mathbb{C}$. Observe, however, that they do not individually satisfy a functional equation of the form in Eq. (6.27). Indeed, these completions of the weak and strong coupling L-functions are analytically continued to the whole complex s-plane through each other. The following statement is a consequence of the remarkable symmetry that relates the factorizations of $L_0(s)$ and $L_\infty(s)$ via a unitary cross-shift in the arguments of the factors.

Theorem 6.3.1. *The weak and strong coupling* completed L-functions $\Lambda_0(s)$ *and* $\Lambda_\infty(s)$, $s \in \mathbb{C}$, *in Eqs.* (6.28a) *and* (6.28b) *satisfy the functional equation*

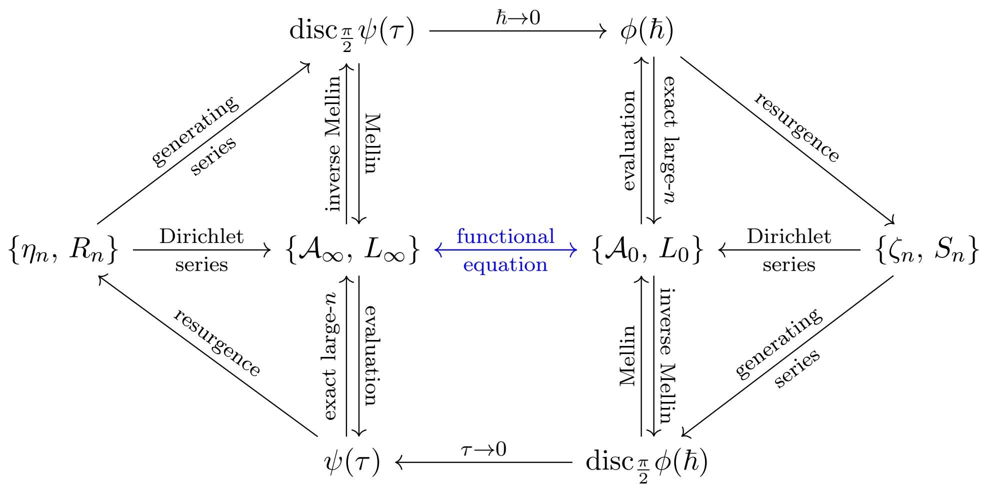
$$\Lambda_0(s) = \Lambda_\infty(-s) \,. \tag{6.29}$$

Proof. Using the definitions in Eqs. (6.28a) and (6.28b) and the functional equations in Eq. (6.27), the statement follows from the explicit computation

$$\Lambda_0(1-s) = \Lambda_{3,2}(2-s)\Lambda_\zeta(1-s) = \Lambda_{3,2}(s-1)\Lambda_\zeta(s) = \Lambda_\infty(s-1) \,, \tag{6.30}$$

after the change of variable $s \to 1-s$. □

As the weak and strong coupling L-functions are naturally paired, their combined *functional equation* in Eq. (6.29) provides an additional two-headed arrow joining them directly in the global commutative diagram in Eq. (6.17), which we explicitly reproduce below.[4]



(6.31)

6.4 Arithmetic twist of the Stokes constants

Recall that the weak and strong coupling Stokes constants S_n, R_n, $n \in \mathbb{Z}_{\neq 0}$, are explicitly given by the divisor sum functions in Eq. (5.164). As we have already observed in Section 5.2.3, an enticingly simple arithmetic symmetry relates them. Namely, the n-th Stokes constants in the two asymptotic limits are obtained from one another by exchanging the positive integer divisors d and n/d that multiply the Dirichlet character inside the sums. We can say something more. Let us start by introducing the notation

$$n = \mathrm{sign}(n) \prod_{p \in \mathbb{P}} p^{n_p} \in \mathbb{Z}_{\neq 0}\,, \quad n_p \in \mathbb{N}\,, \tag{6.32}$$

where $\mathbb{P}$ is the set of prime numbers. We then define the arithmetic function[5]

$$\mathfrak{f}(n) = 3^{n_3}\, \chi_{3,2}(n/3^{n_3})\,, \quad n \in \mathbb{Z}_{\neq 0}\,, \tag{6.33}$$

where $\chi_{3,2}(n)$ is the non-principal Dirichlet character modulo 3 in Eq. (5.165). In particular, $\mathfrak{f}(n) = \chi_{3,2}(n)$ when $3 \nmid n$. Note that $\mathfrak{f}(n)$ is completely multiplicative and satisfies $\mathfrak{f}(-n) = -\mathfrak{f}(n)$.

Proposition 6.4.1. *The normalized weak and strong coupling Stokes constants S_n/S_1, R_n/R_1, $n \in \mathbb{Z}_{\neq 0}$, in Eqs. (5.32) and (5.123) are related by the* arithmetic twist

$$\frac{S_n}{S_1} = \mathfrak{f}(n) \frac{R_n}{R_1}\,, \tag{6.34}$$

where $\mathfrak{f}(n)$ is the arithmetic function in Eq. (6.33).

Proof. Let us fix $n \in \mathbb{Z}_{\neq 0}$ and write $n = 3^{n_3} m$, where $n_3 \in \mathbb{N}$ and $m \in \mathbb{Z}_{\neq 0}$ with $3 \nmid m$. Using the closed formulae in Eqs. (5.164) and (6.33), we find that

$$\mathfrak{f}(n) \frac{R_n}{R_1} = \sum_{d|n} \frac{d\, 3^{n_3}}{n} \chi_{3,2}(dm) = \sum_{d|m} \frac{d}{m} \chi_{3,2}(m/d) = \sum_{d|m} \frac{1}{d} \chi_{3,2}(d) = \frac{S_m}{S_1}\,, \tag{6.35}$$

---

[4] For completeness, we remind that the fundamental constants $\mathcal{A}_0$, $\mathcal{A}_\infty$ are related by $\mathcal{A}_0 = 2\pi \mathcal{A}_\infty$.

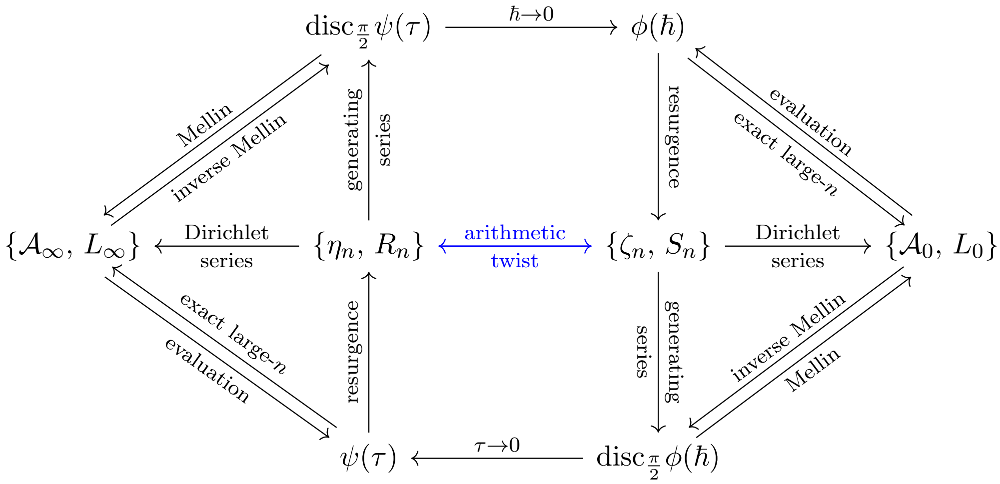
[5] The function in Eq. (6.33) can be interpreted as the Dirichlet convolution $\mathfrak{f}(n) = \sum_{d|n} f_3(d)\, \chi_{3,2}(n/d)$, where $f_3(n) = 3^{n_3}$, $n \in \mathbb{Z}_{\neq 0}$.

where we have applied the property

$$\chi_{3,2}(dm) = \chi_{3,2}(d)^2 \chi_{3,2}(m/d) = \chi_{3,2}(m/d)\,, \tag{6.36}$$

which uses that $\chi_{3,2}(d)^2 = 1$ for $3 \nmid d \mid m$. We conclude by substituting $S_m = S_n$. □

The arithmetic twist above plays a role in the global net of relations among the dual resurgent structures of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ that is analogous to the one played by the functional equation linking the two L-functions. Indeed, Proposition 6.4.1 supplies a two-headed arrow joining the weak and strong coupling Stokes constants directly in the commutative diagram in Eq. (6.17). We reproduce it below in an equivalent form that allows us to visualize the contribution from Eq. (6.34).[6]



(6.37)

L-functions and character twists

The arithmetic twist of the Stokes constants described in Eq. (6.34) appropriately translates in the language of the weak and strong coupling resurgent L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, in Eqs. (5.169a) and (5.169b). In particular, Eq. (6.34) straightforwardly implies that

$$L_0(s) = \sum_{n=1}^{\infty} \frac{S_n}{n^s} = \frac{S_1}{R_1} \sum_{n=1}^{\infty} \frac{R_n}{n^s} \mathfrak{f}(n)\,. \tag{6.38}$$

We can reformulate this arithmetic twist in a way that makes its meaning and implications more explicit. Before doing so, we introduce the principal Dirichlet character modulo 3, that is,

$$\chi_{3,1}(n) = \begin{cases} 0 & \text{if} \quad n \equiv 0 \mod 3 \\ 1 & \text{else} \end{cases}\,, \quad n \in \mathbb{Z}\,, \tag{6.39}$$

which leads us to the following corollary to Proposition 6.4.1.

Corollary 6.4.2. *The normalized weak and strong coupling Stokes constants S_n/S_1 and R_n/R_1, $n \in \mathbb{Z}_{\neq 0}$, in Eq. (5.164) satisfy*

$$\chi_{3,1}(n) \frac{S_n}{S_1} = \chi_{3,2}(n) \frac{R_n}{R_1}\,, \tag{6.40}$$

[6] For completeness, we remind that the locations of the singularities ζ_n, η_n, $n \in \mathbb{Z}_{\neq 0}$, are related by $\zeta_n = 2\pi\eta_n$.

where $\chi_{3,1}(n)$, $\chi_{3,2}(n)$ are the Dirichlet characters modulo 3 in Eqs. (5.165) and (6.39), respectively.

Proof. Observe that Eq. (6.34) implies

$$\frac{S_m}{S_1} = \chi_{3,2}(m)\frac{R_m}{R_1}\,, \quad m \in \mathbb{Z}_{\neq 0}\,, \quad 3 \nmid m\,. \tag{6.41}$$

Multiplying both sides of Eq. (6.41) by the Dirichlet character $\chi_{3,1}(m)$ in Eq. (6.39) and applying

$$\chi_{3,1}(m)\chi_{3,2}(m) = \chi_{3,2}(m)\,, \tag{6.42}$$

we obtain the desired statement. □

Note that Corollary 6.4.2 can be equivalently stated as

$$\chi_{3,2}(n)\frac{S_n}{S_1} = \chi_{3,1}(n)\frac{R_n}{R_1}\,. \tag{6.43}$$

Indeed, multiplying both sides of Eq. (6.40) by $\chi_{3,2}(n)$ and using that $\chi_{3,2}(n)^2 = \chi_{3,1}(n)$, we find the expression in Eq. (6.43).

Finally, let us introduce the following notation. We denote by $L(s,\chi)$ the L-function that results from twisting an arbitrary L-function $L(s)$, $s \in \mathbb{C}$, by an arbitrary Dirichlet character $\chi(n)$, $n \in \mathbb{Z}$. Namely,[7]

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{A_n}{n^s}\chi(n)\,, \tag{6.44}$$

where $A_n \in \mathbb{C}$ is the n-th coefficient in the L-series representation of $L(s)$.

Theorem 6.4.3. *The weak and strong coupling resurgent L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, in Eqs. (5.169a) and (5.169b) are obtained from each other by applying a* character twist. *Specifically, they satisfy*

$$L_0(s) = \frac{S_1}{R_1}\frac{1}{1-3^{-s}}L_\infty(s,\chi_{3,2})\,, \tag{6.45a}$$

$$L_\infty(s) = \frac{R_1}{S_1}\frac{1}{1-3^{-(s+1)}}L_0(s,\chi_{3,2})\,, \tag{6.45b}$$

where $\chi_{3,2}(m)$ is the unique non-principal Dirichlet character modulo 3 in Eq. (5.165).

Proof. We write an arbitrary integer $n \in \mathbb{Z}_{>0}$ in the form $n = 3^{n_3}m$, where $n_3 \in \mathbb{N}$ and $m \in \mathbb{Z}_{>0}$ with $3 \nmid m$, as before. As a straightforward consequence of their closed formulae as divisor sum functions, the weak and strong coupling Stokes constants satisfy the simple properties

$$S_n = S_m\,, \quad R_n = 3^{-n_3}R_m\,. \tag{6.46}$$

It follows from the definition of the weak coupling L-function in Eq. (5.169a) that it factorizes as

$$L_0(s) = \sum_{n_3=0}^{\infty} 3^{-sn_3} \sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{S_m}{m^s} = \frac{1}{1-3^{-s}} \sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{S_m}{m^s}\,, \tag{6.47}$$

[7]Note that the Dirichlet L-function $L(s,\chi_{3,2})$, which appears in the factorization of the weak and strong coupling L-functions in Eqs. (5.169a) and (5.169b), is obtained by twisting the Riemann zeta function $\zeta(s)$ by the Dirichlet character $\chi_{3,2}$ in Eq. (5.165).

where we have used the first formula in Eq. (6.46) and resummed the geometric series over the index n_3. As a consequence of Corollary 6.4.2, the series in the RHS of Eq. (6.47) can then be written as

$$\sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{S_m}{m^s} = \sum_{n=1}^{\infty} \frac{S_n}{n^s} \chi_{3,1}(n) = \frac{S_1}{R_1} \sum_{n=1}^{\infty} \frac{R_n}{n^s} \chi_{3,2}(n) \,. \tag{6.48}$$

Substituting Eq. (6.48) into Eq. (6.47) and using the notation introduced in Eq. (6.44) for $L_\infty(s)$ yields the statement in Eq. (6.45a). Analogously, the definition of the strong coupling L-function in Eq. (5.169b) implies the factorization

$$L_\infty(s) = \sum_{n_3=0}^{\infty} 3^{-(s+1)n_3} \sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{R_m}{m^s} = \frac{1}{1 - 3^{-(s+1)}} \sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{R_m}{m^s} \,, \tag{6.49}$$

where we have used the second formula in Eq. (6.46) and resummed the geometric series over the index n_3. As a consequence of Corollary 6.4.2 in the equivalent form of Eq. (6.43), the series in the RHS of Eq. (6.49) can then be written as

$$\sum_{\substack{m=1 \\ 3\nmid m}}^{\infty} \frac{R_m}{m^s} = \sum_{n=1}^{\infty} \frac{R_n}{n^s} \chi_{3,1}(n) = \frac{R_1}{S_1} \sum_{n=1}^{\infty} \frac{S_n}{n^s} \chi_{3,2}(n) \,. \tag{6.50}$$

Substituting Eq. (6.50) into Eq. (6.49) and using the notation introduced in Eq. (6.44) for $L_0(s)$ yields the statement in Eq. (6.45b). □

Chapter 7

Summability and quantum modularity

Resurgence, summability, and modularity have often interacted in the literature on quantum CS theory and the quantum invariants of knots and 3-manifolds [13–29], but there is currently no comprehensive framework encompassing these three promising, yet distinct research domains. Perhaps unexpectedly, a hint toward developing a unified understanding of them comes from studying the non-perturbative aspects of topological string theory compactified on toric CY threefolds.

In this chapter, we present our results on the summability and quantum modularity properties of the generating functions $f_0(y)$ and $f_\infty(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, of the Stokes constants of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ at weak and strong coupling, defined in Eqs. (6.1) and (6.5), respectively. In particular, we show how the median resummation allows us to effectively reconstruct $f_0(y)$ from its asymptotic expansions in the limit $y \to 0$, while the analogous statement at strong coupling is only conjectured in light of numerical evidence. Moreover, we prove that both generating functions and their images under Fricke involution are holomorphic quantum modular forms of weight zero under the congruence subgroup $\Gamma_1(3) \subset \mathsf{SL}_2(\mathbb{Z})$, which we have previously defined in Eq. (3.16) and is intimately related to the geometry of local $\mathbb{P}^2$. We reproduce [2, Section 4].

7.1 Borel–Laplace sums

As a preliminary step toward the summability study we perform in Section 7.2, we explicitly compute the Borel–Laplace sums of the asymptotic series $\phi(\hbar)$ and $\psi(\tau)$ in Eqs. (5.1) and (5.93) capturing the perturbative expansions of the spectral trace of local $\mathbb{P}^2$ in the limits $\hbar \to 0$ and $\hbar \to \infty$, respectively. We start by introducing the following notation. Let $\mathbf{e}_1 \colon \mathbb{C} \setminus \mathrm{i}\mathbb{R}_{\geq 0} \to \mathbb{C}$ be the function defined by[1]

$$\mathbf{e}_1(y) := \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-2\pi t} \frac{dt}{t + \mathrm{i}y} \,. \tag{7.1}$$

[1]The notation in Eq. (7.1) was suggested by M. Kontsevich as the function $\mathbf{e}_1$ plays a similar role in other examples—*e.g.*, the case of σ, σ^* [56] and the case of even/odd Maass cusp forms [3].

Note that, applying the simple change of coordinates $t \mapsto t/(2\pi) - \mathrm{i}y$, the *exponential integral* $\mathbf{e}_1(y)$ can be written in the equivalent form

$$\mathbf{e}_1(y) = \frac{1}{2\pi\mathrm{i}} \mathrm{e}^{2\pi\mathrm{i}y}\, \Gamma(0, 2\pi\mathrm{i}y)\,, \tag{7.2}$$

where $\Gamma(s, 2\pi\mathrm{i}y)$ denotes the upper incomplete gamma function. It follows from the analyticity properties of $\Gamma(0, 2\pi\mathrm{i}y)$ that the function $\mathbf{e}_1(y)$ is analytic for $y \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}_{\geq 0}$ and its jump across the branch cut along the positive imaginary axis is given by

$$\lim_{\delta \to 0^+} \mathbf{e}_1(\mathrm{i}x - \delta) - \mathbf{e}_1(\mathrm{i}x + \delta) = \mathrm{e}^{-2\pi x}\,, \tag{7.3}$$

where we have fixed $x \in \mathbb{R}_{\geq 0}$. Let us then go back to the Gevrey-1 asymptotic series $\phi(\hbar)$, $\psi(\tau)$ in Eqs. (5.1) and (5.93). As proved in Chapter 5, their perturbative coefficients a_{2n}, b_{2n}, $n \in \mathbb{Z}_{>0}$, satisfy the exact large-order relations in Eqs. (5.61) and (5.148), which can be equivalently written as

$$a_{2n} = \frac{\Gamma(2n)}{\pi\mathrm{i}} \sum_{m=1}^{\infty} \frac{S_m}{\zeta_m^{2n}}\,, \tag{7.4a}$$

$$b_{2n} = \frac{\Gamma(2n-1)}{\pi\mathrm{i}} \sum_{m=1}^{\infty} \frac{R_m}{\eta_m^{2n-1}}\,, \tag{7.4b}$$

where the Stokes constants S_m, R_m are given explicitly in Eqs. (5.32) and (5.123), while the locations ζ_m, η_m of the singularities in the complex ζ-plane are in Eqs. (5.22) and (5.113). We now prove that both $\phi(\hbar)$ and $\psi(\tau)$ are *Borel–Laplace summable* at the angles 0 and π and that their Borel–Laplace sums can be expressed in terms of the exponential integral in Eq. (7.1).

Proposition 7.1.1. *The Borel–Laplace sum of the weak coupling perturbative series $\phi(\hbar)$ in Eq.* (5.1) *along the positive real axis is*

$$s_0(\phi)(\hbar) = -\sum_{m \in \mathbb{Z}_{\neq 0}} S_m \mathbf{e}_1\left(-\frac{2\pi m}{3\hbar}\right)\,, \tag{7.5}$$

which is analytic for $\Re(\hbar) > 0$.

Proof. As summarized in Chapter 1, the Borel–Laplace resummation requires two successive steps. First, we compute the Borel transform $\hat{\phi}(\zeta)$, which gives[2]

$$\begin{aligned}\hat{\phi}(\zeta) &= \sum_{n=1}^{\infty} \frac{a_{2n}}{\Gamma(2n)} \zeta^{2n-1} \\ &= \frac{1}{\pi\mathrm{i}} \sum_{m=1}^{\infty} S_m \zeta^{-1} \sum_{n=1}^{\infty} \left(\frac{\zeta}{\zeta_m}\right)^{2n} = \frac{1}{\pi\mathrm{i}} \sum_{m=1}^{\infty} S_m \frac{\zeta}{\zeta_m^2 - \zeta^2} \\ &= -\frac{1}{2\pi\mathrm{i}} \sum_{m=1}^{\infty} \frac{S_m}{\zeta_m + \zeta} + \frac{1}{2\pi\mathrm{i}} \sum_{m=1}^{\infty} \frac{S_m}{\zeta_m - \zeta} = \frac{1}{2\pi\mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{S_m}{\zeta_m - \zeta}\,,\end{aligned} \tag{7.6}$$

[2] Note that we adopt the definitions of the Borel and Laplace transforms in Eqs. (1.23a) and (1.23b).

where we have substituted Eq. (7.4a), applied the identity

$$\frac{1}{(x-y)(x+y)} = \frac{1}{2x(x-y)} + \frac{1}{2x(x+y)}\,, \tag{7.7}$$

and used the properties $\zeta_{-m} = -\zeta_m$ and $S_{-m} = S_m$, $m \in \mathbb{Z}_{\neq 0}$. Then, assuming $\hbar \in \mathbb{R}_{\geq 0}$, we perform the Laplace transform along the positive real axis, that is,

$$\begin{aligned} s_0(\phi)(\hbar) &= \int_0^\infty d\zeta \mathrm{e}^{-\zeta/\hbar}\hat{\phi}(\zeta) \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty d\zeta \sum_{m\in\mathbb{Z}_{\neq 0}} S_m \frac{\mathrm{e}^{-\zeta/\hbar}}{\zeta_m - \zeta} \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty dt \sum_{m\in\mathbb{Z}_{\neq 0}} S_m \frac{\mathrm{e}^{-2\pi t}}{\frac{\zeta_m}{2\pi\hbar} - t} = -\sum_{m\in\mathbb{Z}_{\neq 0}} S_m \mathbf{e}_1\left(\frac{\mathrm{i}\zeta_m}{2\pi\hbar}\right)\,, \end{aligned} \tag{7.8}$$

where we have substituted Eq. (7.6), applied the change of variable $2\pi t = \zeta/\hbar$, permuted sum and integral due to absolute convergence, and used the notation introduced in Eq. (7.1). Finally, substituting Eq. (5.22) in Eq. (7.8), we find the desired expression for $s_0(\phi)(\hbar)$, which can be analytically continued to $\Re(\hbar) > 0$ thanks to the properties of the function $\mathbf{e}_1$. □

Proposition 7.1.2. *The Borel–Laplace sum of the strong coupling perturbative series $\psi(\tau)$ in Eq. (5.93) along the positive real axis is*

$$s_0(\psi)(\tau) = -\sum_{m\in\mathbb{Z}_{\neq 0}} R_m \mathbf{e}_1\left(-\frac{m}{3\tau}\right)\,, \tag{7.9}$$

which is analytic for $\Re(\tau) > 0$.

Proof. We follow the same steps of the proof of Proposition 7.1.1. First, we compute the Borel transform $\hat{\psi}(\tau)$ and obtain

$$\begin{aligned} \hat{\psi}(\tau) &= \sum_{n=1}^\infty \frac{b_{2n}}{\Gamma(2n-1)}\zeta^{2n-2} \\ &= \frac{1}{\pi\mathrm{i}}\sum_{m=1}^\infty R_m \eta_m^{-1} \sum_{n=1}^\infty \left(\frac{\zeta}{\eta_m}\right)^{2n-2} = \frac{1}{\pi\mathrm{i}}\sum_{m=1}^\infty R_m \frac{\eta_m}{\eta_m^2 - \zeta^2} \\ &= \frac{1}{2\pi\mathrm{i}}\sum_{m=1}^\infty \frac{R_m}{\eta_m + \zeta} + \frac{1}{2\pi\mathrm{i}}\sum_{m=1}^\infty \frac{R_m}{\eta_m - \zeta} = \frac{1}{2\pi\mathrm{i}}\sum_{m\in\mathbb{Z}_{\neq 0}} \frac{R_m}{\eta_m - \zeta}\,, \end{aligned} \tag{7.10}$$

where we have substituted Eq. (7.4b), applied the fractional identity in Eq. (7.7), and used the properties $\eta_{-m} = -\eta_m$ and $R_{-m} = -R_m$, $m \in \mathbb{Z}_{\neq 0}$. Then, assuming $\tau \in \mathbb{R}_{\geq 0}$, we compute the Laplace transform along the positive real axis, which gives

$$\begin{aligned} s_0(\psi)(\tau) &= \int_0^\infty d\zeta \mathrm{e}^{-\zeta/\tau}\hat{\psi}(\zeta) \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty d\zeta \sum_{m\in\mathbb{Z}_{\neq 0}} R_m \mathrm{e}^{-\zeta/\tau}\frac{1}{\eta_m - \zeta} \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty dt \sum_{m\in\mathbb{Z}_{\neq 0}} R_m \mathrm{e}^{-2\pi t}\frac{1}{\frac{\eta_m}{2\pi\tau} - t} = -\sum_{m\in\mathbb{Z}_{\neq 0}} R_m \mathbf{e}_1\left(\frac{\mathrm{i}\eta_m}{2\pi\tau}\right)\,, \end{aligned} \tag{7.11}$$

where we have substituted Eq. (7.10), performed the change of variable $2\pi t = \zeta/\tau$, permuted sum and integral due to absolute convergence, and again applied the notation in Eq. (7.1). We conclude by substituting Eq. (5.113) into Eq. (7.11). Notice that $s_0(\psi)(\tau)$ can be analytically continued to $\Re(\tau) > 0$ thanks to the properties of the function $\mathbf{e}_1$. □

In addition, following the same steps in the proofs of Propositions 7.1.1 and 7.1.2, we obtain the analogous expressions for the Borel–Laplace sums of $\phi(\hbar)$ and $\psi(\tau)$ at angle π. Namely,

$$s_\pi(\phi)(\hbar) = -\sum_{m\in\mathbb{Z}_{\neq 0}} S_m \mathbf{e}_1\left(-\frac{2\pi m}{3\hbar}\right), \tag{7.12a}$$

$$s_\pi(\psi)(\tau) = -\sum_{m\in\mathbb{Z}_{\neq 0}} R_m \mathbf{e}_1\left(-\frac{m}{3\tau}\right), \tag{7.12b}$$

which are analytic for $\Re(\hbar) < 0$ and $\Re(\tau) < 0$, respectively.

Recall that the asymptotic behavior in the limit $y \to 0$ of the generating functions $f_0(y)$, $f_\infty(y)$ of the weak and strong coupling Stokes constants in Eqs. (6.1) and (6.5) is dictated by the original perturbative series $\psi(\tau)$ and $\phi(\hbar)$, respectively, according to the formulae in Eqs. (6.13) and (6.15). Therefore, the Borel–Laplace sums of the asymptotic series $\tilde{f}_0(y)$ and $\tilde{f}_\infty(y)$ easily follow from Propositions 7.1.1 and 7.1.2. For completeness, we write their explicit formulae in the two corollaries below.

Corollary 7.1.3. *The Borel–Laplace sums of the perturbative expansion $\tilde{f}_0(y)$ defined in Eq. (6.12) along the positive and negative real axes are written in terms of the function $\mathbf{e}_1$ in Eq. (7.1) as*

$$s_0(\tilde{f}_0)(y) = -\frac{\pi \mathrm{i}}{2} - \frac{3\mathcal{V}}{2\pi \mathrm{i} y} + 2\sum_{m\in\mathbb{Z}_{\neq 0}} R_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right), \quad \Re(y) > 0, \tag{7.13a}$$

$$s_\pi(\tilde{f}_0)(y) = -\frac{\pi \mathrm{i}}{2} - \frac{3\mathcal{V}}{2\pi \mathrm{i} y} + 2\sum_{m\in\mathbb{Z}_{\neq 0}} R_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right), \quad \Re(y) < 0, \tag{7.13b}$$

where $\mathcal{V} = 2\Im\left(\mathrm{Li}_2(\mathrm{e}^{2\pi \mathrm{i}/3})\right)$ as before.

Proof. The proof of the statement follows directly from Eq. (6.13) and Proposition 7.1.2. □

Corollary 7.1.4. *The Borel–Laplace sums of the perturbative expansion $\tilde{f}_\infty(y)$ defined in Eq. (6.14) along the positive and negative real axes are written in terms of the function $\mathbf{e}_1$ in Eq. (7.1) as*

$$s_0(\tilde{f}_\infty)(y) = -3\log\frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(-6\pi \mathrm{i} y) - 2\sum_{m\in\mathbb{Z}_{\neq 0}} S_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right), \quad \Re(y) > 0, \tag{7.14a}$$

$$s_\pi(\tilde{f}_\infty)(y) = -3\log\frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(-6\pi \mathrm{i} y) - 2\sum_{m\in\mathbb{Z}_{\neq 0}} S_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right), \quad \Re(y) < 0. \tag{7.14b}$$

Proof. The proof of the statement follows directly from Eq. (6.15) and Proposition 7.1.1. □

7.2 Median resummation

We will now show that the *median resummation* of the asymptotic expansion $\tilde{f}_0(y)$ of the generating function of the weak coupling Stokes constants S_n, $n \in \mathbb{Z}_{\neq 0}$, appropriately reconstructs the generating function $f_0(y)$ itself. Consequently, our results showcase the effectiveness of the median resummation as a summability tool. We begin with the preliminary Lemma 7.2.1, where we prove that the Borel–Laplace sums at angles 0 and π of the asymptotic expansion $\tilde{f}_0(y)$ in Eq. (6.12) reproduce the original function $f_0(y)$ with a correction that is suitably encoded in the dual function $f_\infty(y)$.

Lemma 7.2.1. *Let $\tilde{f}_0(y)$ be the asymptotic expansion in the limit $y \to 0$ of the generating function $f_0(y)$ in Eq. (6.1) with $\Im(y) > 0$, which we have written explicitly in Eq. (6.12). Its Borel–Laplace sums along the positive and negative real axes can be expressed as*

$$s_0(\tilde{f}_0)(y) = f_0(y) + f_\infty\big(-\tfrac{1}{3y}\big)\,, \quad \Re(y) > 0\,, \tag{7.15a}$$

$$s_\pi(\tilde{f}_0)(y) = f_0(y) - f_\infty\big(-\tfrac{1}{3y}\big)\,, \quad \Re(y) < 0\,. \tag{7.15b}$$

Proof. Applying Eq. (B.19a) to expand the two quantum dilogarithms in Eq. (6.3) separately, $\tilde{f}_0(y)$ can be written as

$$\begin{aligned}\frac{1}{3}\tilde{f}_0(y) = &-\frac{\pi \mathrm{i}}{3} - \frac{1}{2}\left(\log(1-w) - \log(1-w^{-1})\right) \\ &- \frac{1}{2\pi \mathrm{i} y}\left(\mathrm{Li}_2(w) - \mathrm{Li}_2(w^{-1})\right) - \left(\tilde{\varphi}_+(y) - \tilde{\varphi}_-(y)\right)\,,\end{aligned} \tag{7.16}$$

where $w = \mathrm{e}^{2\pi \mathrm{i}/3}$ as before, and we have introduced the two formal power series

$$\tilde{\varphi}_\pm(y) := \sum_{k=1}^{\infty} (2\pi \mathrm{i} y)^{2k-1} \frac{B_{2k}}{(2k)!} \mathrm{Li}_{2-2k}(w^{\pm 1})\,. \tag{7.17}$$

Let us compute the Borel transform of $\tilde{\varphi}_\pm(y)$ in the variable ζ conjugate to y. We find that

$$\begin{aligned}\mathcal{B}\left[\tilde{\varphi}_\pm\right](\zeta) &= \sum_{k=1}^{\infty} (2\pi \mathrm{i})^{2k-1} \frac{B_{2k}}{(2k)!} \mathrm{Li}_{2-2k}(w^\pm) \frac{\zeta^{2k-2}}{(2k-2)!} \\ &= \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \zeta^{2k-2}\right) \diamond \left(\sum_{k=1}^{\infty} (2\pi \mathrm{i})^{2k-1} \frac{\mathrm{Li}_{2-2k}(w^\pm)}{(2k-2)!} \zeta^{2k-2}\right) \\ &=: \tilde{g}(\zeta) \diamond \tilde{f}_\pm(\zeta)\,,\end{aligned} \tag{7.18}$$

where $\diamond$ denotes the *Hadamard product*. The formal power series $\tilde{g}(\zeta)$, $\tilde{f}_\pm(\zeta)$ have a finite radius of convergence at $\zeta = 0$ and can be resummed explicitly into the functions[3]

$$g(\zeta) = -\frac{1}{\zeta^2} + \frac{1}{2\zeta} \coth\left(\frac{\zeta}{2}\right)\,, \quad |\zeta| < 2\pi\,, \tag{7.19a}$$

$$f_\pm(\zeta) = -\pi \mathrm{i} \left(\frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - 2\pi \mathrm{i}\zeta}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + 2\pi \mathrm{i}\zeta}}\right)\,, \quad |\zeta| < \frac{1}{3}\,, \tag{7.19b}$$

[3] We impose that $g(0) = 1/12$ to eliminate the removable singularity of $g(\zeta)$ at the origin.

respectively. After being analytically continued to the whole complex ζ-plane, the function $g(\zeta)$ has poles of order one along the imaginary axis at

$$\mu_m = 2\pi \mathrm{i} m\,, \quad m \in \mathbb{Z}_{\neq 0}\,, \tag{7.20}$$

while the functions $f_\pm(\zeta)$ have poles of order one along the real axis at

$$\nu_k^{(+,1)} = \frac{1}{3} + k\,, \quad \nu_k^{(+,2)} = -\frac{1}{3} + k\,, \tag{7.21a}$$

$$\nu_k^{(-,1)} = \frac{2}{3} + k\,, \quad \nu_k^{(-,2)} = -\frac{2}{3} + k\,, \tag{7.21b}$$

for $k \in \mathbb{Z}$, respectively. We consider a circle γ in the complex s-plane with center $s = 0$ and radius $0 < r < 2\pi$ and apply Hadamard's multiplication theorem [155, 156]. We refer to Appendix C for a short introduction. The Borel transform of $\tilde{\varphi}_\pm(y)$ can be written as the integral

$$\begin{aligned} \mathcal{B}\left[\tilde{\varphi}_\pm\right](\zeta) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma g(s)\, f_\pm\left(\frac{\zeta}{s}\right) \frac{ds}{s} \\ &= \frac{1}{2} \int_\gamma \left(\frac{1}{s} - \frac{1}{2}\coth\left(\frac{s}{2}\right)\right) \left(\frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - 2\pi \mathrm{i}\frac{\zeta}{s}}} + \frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + 2\pi \mathrm{i}\frac{\zeta}{s}}}\right) \frac{ds}{s^2}\,, \end{aligned} \tag{7.22}$$

for $|\zeta| < r/3$. For such values of ζ, the function $s \mapsto f_\pm(\zeta/s)$ has singular points at $s = \zeta/\nu_k^{(\pm,i)}$, $k \in \mathbb{Z}$, which sit inside the contour of integration γ and accumulate at the origin, and no singularities for $|s| > r$. The function $g(s)$ has simple poles at the points $s = \mu_m$ with residues

$$\operatorname*{Res}_{s=\pm 2\pi \mathrm{i} m} g(s) = \pm\frac{1}{2\pi \mathrm{i}}\,, \quad m \in \mathbb{Z}_{>0}\,. \tag{7.23}$$

By Cauchy's residue theorem, the integral in Eq. (7.22) can be evaluated by summing the residues at the poles of the integrand which lie outside γ, allowing us to express the Borel transform as an exact function of ζ. More precisely, we find that

$$\begin{aligned} \mathcal{B}\left[\tilde{\varphi}_\pm\right](\zeta) &= -\sum_{m \in \mathbb{Z}_{\neq 0}} \operatorname*{Res}_{s=2\pi \mathrm{i} m} \left[g(s) f_\pm\left(\frac{\zeta}{s}\right)\frac{1}{s}\right] \\ &= \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{1}{4\pi \mathrm{i} m^2} \left(\frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - \frac{\zeta}{m}}} + \frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + \frac{\zeta}{m}}}\right) \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{m=1}^{\infty} \frac{1}{m^2} \left(\frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - \frac{\zeta}{m}}} + \frac{1}{1 + \mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + \frac{\zeta}{m}}}\right)\,, \end{aligned} \tag{7.24}$$

which is a well-defined, exact function of ζ for $|\zeta| < 2\pi/3$. In fact, the Borel transform in Eq. (7.24) converges when $\frac{\zeta}{2\pi \mathrm{i}} \neq \frac{m}{3} + nm$ and $\frac{\zeta}{2\pi \mathrm{i}} \neq \frac{2m}{3} + nm$ with $m \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{Z}$, respectively.[4]

[4] The convergence of the infinite sum in the RHS of Eq. (7.24) can be easily verified by, *e.g.*, the limit comparison test. Indeed, where defined, the generic term of the series is dominated by $1/m^2$.

We can now compute the Laplace transform of $\mathcal{B}[\tilde{\varphi}_\pm](\zeta)$ along the positive and negative real axes. When $\Re(y), \Im(y) > 0$, we have that

$$
\begin{aligned}
s_0(\tilde{\varphi}_\pm)(y) &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-\zeta/y} \left[\sum_{m=1}^\infty \frac{1}{m^2} \left(\frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - \frac{\zeta}{m}}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + \frac{\zeta}{m}}} \right) \right] d\zeta \\
&= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\sum_{m=1}^\infty \frac{\mathrm{e}^{-mt/y}}{m} \right) \left(\frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - t}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + t}} \right) dt \,,
\end{aligned}
\tag{7.25}
$$

where we have applied the change of variable $\zeta = mt$. Resumming the series in m, we find that

$$
\begin{aligned}
s_0(\tilde{\varphi}_\pm)(y) &= -\frac{1}{2\pi \mathrm{i}} \int_0^\infty \log(1-\mathrm{e}^{-t/y}) \left(\frac{1}{1+e^{\pm\frac{\pi i}{3} - t}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} + t}} \right) dt \\
&= -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{\log(1-\mathrm{e}^{-t/y})}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - t}} dt + \frac{1}{2\pi \mathrm{i}} \int_0^{-\infty} \frac{\log(-\mathrm{e}^{t/y})}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - t}} dt \,,
\end{aligned}
\tag{7.26}
$$

where we have divided the integral into the sum of two contributions for simplicity. Indeed, we observe that the second term in the RHS of Eq. (7.26) gives

$$
\frac{1}{2\pi \mathrm{i}} \int_0^{-\infty} \frac{\log(-\mathrm{e}^{t/y})}{1+\mathrm{e}^{\pm\frac{\pi \mathrm{i}}{3} - t}} dt = -\frac{\mathrm{Li}_2(w^{\pm 1})}{2\pi \mathrm{i} y} - \frac{1}{2} \log(1-w^{\pm 1}) \,,
\tag{7.27}
$$

while the first term gives

$$
-\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{\log(1-\mathrm{e}^{-t/y})}{1+\mathrm{e}^{\frac{\pi \mathrm{i}}{3} - t}} dt = \log \frac{(w;\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-4\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} \,,
\tag{7.28a}
$$

$$
-\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{\log(1-\mathrm{e}^{-t/y})}{1+\mathrm{e}^{-\frac{\pi \mathrm{i}}{3} - t}} dt = \log \frac{(w^{-1};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-2\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} \,.
\tag{7.28b}
$$

The last two formulae follow from [169, Theorem A-29] as shown in [170, Eq. (5.108)].[5] Therefore, substituting Eqs. (7.27), (7.28a), and (7.28b) into the formula for the Borel–Laplace sum in Eq. (7.26), we find that

$$
s_0(\tilde{\varphi}_+)(y) = -\frac{\mathrm{Li}_2(w)}{2\pi \mathrm{i} y} - \frac{1}{2} \log(1-w) + \log \frac{(w;\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-4\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} \,,
\tag{7.29a}
$$

$$
s_0(\tilde{\varphi}_-)(y) = -\frac{\mathrm{Li}_2(w^{-1})}{2\pi \mathrm{i} y} - \frac{1}{2} \log(1-w^{-1}) + \log \frac{(w^{-1};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-2\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} \,.
\tag{7.29b}
$$

Now, it follows directly from Eqs. (7.16), (7.29a), and (7.29a) that

$$
\begin{aligned}
s_0(\tilde{f}_0)(y) &= -\pi \mathrm{i} - 3 \log \frac{(w;\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-4\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} + 3 \log \frac{(w^{-1};\, \mathrm{e}^{2\pi \mathrm{i} y})_\infty}{(\mathrm{e}^{-2\pi \mathrm{i}/(3y)};\, \mathrm{e}^{-2\pi \mathrm{i}/y})_\infty} \\
&= f_0(y) + f_\infty(-1/3y) \,,
\end{aligned}
\tag{7.30}
$$

where we have applied the closed formulae for the generating functions in Eqs. (6.3) and (6.7).

[5] We thank C. Wheeler for sharing the preliminary version of his book in preparation.

Similarly, when $\Re(y) < 0$ and $\Im(y) > 0$, the Borel–Laplace sum of the formal power series $\tilde{\varphi}_\pm(y)$ along the negative real axis is given by

$$\begin{aligned} s_\pi(\tilde{\varphi}_\pm)(y) &= \frac{1}{2\pi\mathrm{i}} \int_0^{-\infty} \mathrm{e}^{-\zeta/y} \left[\sum_{m=1}^\infty \frac{1}{m^2} \left(\frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-\frac{\zeta}{m}}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}+\frac{\zeta}{m}}} \right) \right] d\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \int_0^{-\infty} \left(\sum_{m=1}^\infty \frac{\mathrm{e}^{-mt/y}}{m} \right) \left(\frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-t}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}+t}} \right) dt\,, \end{aligned} \tag{7.31}$$

where we have applied the change of variable $\zeta = mt$. Resumming the series in m, we find that

$$\begin{aligned} s_\pi(\tilde{\varphi}_\pm)(y) &= -\frac{1}{2\pi\mathrm{i}} \int_0^{-\infty} \log(1-\mathrm{e}^{-t/y}) \left(\frac{1}{1+e^{\pm\frac{\pi i}{3}-t}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}+t}} \right) dt \\ &= \frac{1}{2\pi\mathrm{i}} \int_0^{\infty} \log(1-\mathrm{e}^{t/y}) \left(\frac{1}{1+e^{\pm\frac{\pi i}{3}+t}} + \frac{1}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-t}} \right) dt \\ &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log(1-\mathrm{e}^{t/y})}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-t}} dt - \frac{1}{2\pi\mathrm{i}} \int_0^{-\infty} \frac{\log(-\mathrm{e}^{-t/y})}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-t}} dt\,, \end{aligned} \tag{7.32}$$

where we have divided the integral into the sum of two contributions for simplicity. Let us now set $x = -1/y$, so that $\Re(x) > 0$ and $\Im(x) > 0$, then the second term in the RHS of Eq. (7.32) becomes

$$-\frac{1}{2\pi\mathrm{i}} \int_0^{-\infty} \frac{\log(-\mathrm{e}^{tx})}{1+\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{3}-t}} dt = \frac{x}{2\pi\mathrm{i}} \mathrm{Li}_2(w^{\pm1}) + \frac{1}{2}\log(1-w^{\pm1})\,, \tag{7.33}$$

while the first term gives

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log(1-\mathrm{e}^{-tx})}{1+\mathrm{e}^{\frac{\pi\mathrm{i}}{3}-t}} dt = -\log \frac{(w;\,\mathrm{e}^{2\pi\mathrm{i}/x})_\infty}{(\mathrm{e}^{-4\pi\mathrm{i}x/3};\,\mathrm{e}^{-2\pi\mathrm{i}x})_\infty}\,, \tag{7.34a}$$

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log(1-\mathrm{e}^{-tx})}{1+\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}-t}} dt = -\log \frac{(w^{-1};\,\mathrm{e}^{2\pi\mathrm{i}/x})_\infty}{(\mathrm{e}^{-2\pi\mathrm{i}x/3};\,\mathrm{e}^{-2\pi\mathrm{i}x})_\infty}\,, \tag{7.34b}$$

following [170, Eq. (5.108)]. Therefore, substituting Eqs. (7.33), (7.34a), and (7.34b), after we change back x to $-1/y$, into the formula for the Borel–Laplace sum in Eq. (7.32), we find that

$$s_\pi(\tilde{\varphi}_+)(y) = -\frac{\mathrm{Li}_2(w)}{2\pi\mathrm{i}y} + \frac{1}{2}\log(1-w) - \log \frac{(w;\,\mathrm{e}^{-2\pi\mathrm{i}y})_\infty}{(\mathrm{e}^{4\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty}\,, \tag{7.35a}$$

$$s_\pi(\tilde{\varphi}_-)(y) = -\frac{\mathrm{Li}_2(w^{-1})}{2\pi\mathrm{i}y} + \frac{1}{2}\log(1-w^{-1}) - \log \frac{(w^{-1};\,\mathrm{e}^{-2\pi\mathrm{i}y})_\infty}{(\mathrm{e}^{2\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty}\,. \tag{7.35b}$$

Putting together Eqs. (7.16), (7.35a), and (7.35b) yields

$$\begin{aligned} s_\pi(\tilde{f}_0)(y) &= 3\log \frac{(w;\,\mathrm{e}^{-2\pi\mathrm{i}y})_\infty}{(\mathrm{e}^{4\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty} - 3\log \frac{(w^{-1};\,\mathrm{e}^{-2\pi\mathrm{i}y})_\infty}{(\mathrm{e}^{2\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty} \\ &= -3\log \frac{(w\,\mathrm{e}^{2\pi\mathrm{i}y};\,\mathrm{e}^{2\pi\mathrm{i}y})_\infty}{(w^{-1}\,\mathrm{e}^{2\pi\mathrm{i}y};\,\mathrm{e}^{2\pi\mathrm{i}y})_\infty} - 3\log \frac{(\mathrm{e}^{4\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty}{(\mathrm{e}^{2\pi\mathrm{i}/(3y)};\,\mathrm{e}^{2\pi\mathrm{i}/y})_\infty}\,, \end{aligned} \tag{7.36}$$

where we have applied the relation in Eq. (B.2). Let us now recall that

$$(w^{\pm 1}\, e^{2\pi i y};\, e^{2\pi i y})_\infty = (w^{\pm 1};\, e^{2\pi i y})_\infty - \left(1 - w^{\pm 1}\right)\,, \tag{7.37}$$

which we substitute into Eq. (7.36) to get

$$\begin{aligned} s_\pi(\tilde{f}_0)(y) &= -\pi i - 3\log\frac{(w;\, e^{2\pi i y})_\infty}{(w^{-1};\, e^{2\pi i y})_\infty} - 3\log\frac{(e^{4\pi i/(3y)};\, e^{2\pi i/y})_\infty}{(e^{2\pi i/(3y)};\, e^{2\pi i/y})_\infty} \\ &= f_0(y) - f_\infty(1/3y) = f_0(y) - f_\infty(-1/3y)\,, \end{aligned} \tag{7.38}$$

where we have first employed the closed formulae for the generating functions in Eqs. (6.3) and (6.7) and then used the parity property in Eq. (6.6).[6] □

We can now prove that the median resummation of the asymptotic expansion $\tilde{f}_0(y)$ reproduces the generating function $f_0(y)$ itself.

Theorem 7.2.2. *Let $\tilde{f}_0(y)$ be the asymptotic expansion in the limit $y \to 0$ of the generating function $f_0(y)$ in Eq. (6.1) with $\Im(y) > 0$, which we have written explicitly in Eq. (6.12). Then, the median resummation of $\tilde{f}_0(y)$ reconstructs the original function $f_0(y)$, that is,*

$$\mathcal{S}^{\text{med}}_{\frac{\pi}{2}} \tilde{f}_0(y) = f_0(y)\,, \quad y \in \mathbb{H}\,. \tag{7.39}$$

Proof. Following the definition in Eq. (1.29), the median resummation of $\tilde{f}_0(y)$ along the positive imaginary axis is given by

$$\mathcal{S}^{\text{med}}_{\frac{\pi}{2}} \tilde{f}_0(y) = \begin{cases} s_0(\tilde{f}_0)(y) + \frac{1}{2}\,\text{disc}_{\frac{\pi}{2}} \tilde{f}_0(y)\,, & \Re(y) > 0\,, \\ \\ s_\pi(\tilde{f}_0)(y) - \frac{1}{2}\,\text{disc}_{\frac{\pi}{2}} \tilde{f}_0(y)\,, & \Re(y) < 0\,, \end{cases} \tag{7.40}$$

where we take $\Im(y) > 0$. As a consequence of Eqs. (6.8) and (6.13), we have that

$$f_\infty\left(-\tfrac{1}{3y}\right) = -\frac{1}{2}\text{disc}_{\frac{\pi}{2}} \tilde{f}_0(y)\,, \tag{7.41}$$

and substituting this in Eq. (7.40) yields

$$\mathcal{S}^{\text{med}}_{\frac{\pi}{2}} \tilde{f}_0(y) = \begin{cases} s_0(\tilde{f}_0)(y) - f_\infty\left(-\tfrac{1}{3y}\right)\,, & \Re(y) > 0\,, \\ \\ s_\pi(\tilde{f}_0)(y) + f_\infty\left(-\tfrac{1}{3y}\right)\,, & \Re(y) < 0\,, \end{cases} \tag{7.42}$$

where again $\Im(y) > 0$. Then, the conclusion follows from Lemma 7.2.1. □

Let us now consider the strong coupling asymptotic expansion $\tilde{f}_\infty(y)$ in Eq. (6.14). Despite its similarities with the dual expansion $\tilde{f}_0(y)$ in Eq. (6.12), the analog to the analytic argument of Lemma 7.2.1 is still missing. Therefore, relying on the support of numerical tests, we limit ourselves to conjecture the effectiveness of the median resummation for the generating function of the strong coupling Stokes constants.

[6] We note that the proof in the case of $\Re(y) > 0$ can be adapted to allow for an arbitrary choice of $w = \exp(z)$, $z \in \mathbb{C}^*$. However, the proof in the case of $\Re(y) < 0$ explicitly uses the symmetry properties of the Stokes constants and their generating functions, which do not straightforwardly generalize.

Conjecture 1. *Let $\tilde{f}_\infty(y)$ be the asymptotic expansion in the limit $y \to 0$ of the generating function $f_\infty(y)$ in Eq. (6.5) with $\Im(y) > 0$, which we have written explicitly in Eq. (6.14). Then, the median resummation of $\tilde{f}_\infty(y)$ reconstructs the original function $f_\infty(y)$, that is,*

$$\mathcal{S}^{\text{med}}_{\frac{\pi}{2}} \tilde{f}_\infty(y) = f_\infty(y) \,, \quad y \in \mathbb{H} \,. \tag{7.43}$$

Remark 7.2.1. *As a consequence Eqs. (6.4) and (6.15), we have that*

$$f_0\left(-\tfrac{1}{3y}\right) = \frac{1}{2} \text{disc}_{\frac{\pi}{2}} \tilde{f}_\infty(y) \,, \quad y \in \mathbb{H} \,. \tag{7.44}$$

Substituting this into the definition of the median resummation in Eq. (1.29) for the asymptotic expansion $\tilde{f}_\infty(y)$ at the angle $\pi\mathrm{i}/2$, Conjecture 1 straightforwardly implies that the Borel–Laplace sums of $\tilde{f}_\infty(y)$ along the positive and negative real axes reproduce the original function $f_\infty(y)$ with a correction that is suitably encoded in the dual function $f_0(y)$, and vice versa. More precisely, the formulae

$$s_0(\tilde{f}_\infty)(y) = f_\infty(y) - f_0\big(-\tfrac{1}{3y}\big) \,, \quad \Re(y) > 0 \,, \tag{7.45a}$$

$$s_\pi(\tilde{f}_\infty)(y) = f_\infty(y) + f_0\big(-\tfrac{1}{3y}\big) \,, \quad \Re(y) < 0 \,, \tag{7.45b}$$

which represent the dual statement to Lemma 7.2.1, are equivalent to Eq. (7.43).

Theorem 7.2.2 and Conjecture 1 can be placed within the context of the strong-weak resurgent symmetry discussed in Chapter 6 as they provide additional arrows in the commutative diagram in Eq. (6.24). In particular, taking the median resummation of the asymptotic expansions is the formal inverse of perturbatively expanding the generating functions. We show below the resulting completed diagram in terms of f_0, f_∞ and their asymptotic expansions.

$$\begin{array}{ccc} f_\infty(y) & \underset{\text{median resummation}}{\overset{y\to 0}{\rightleftarrows}} & \tilde{f}_\infty(y) \\ {\scriptstyle\text{strong-weak symmetry}}\updownarrow & & \updownarrow{\scriptstyle\text{strong-weak symmetry}} \\ \tilde{f}_0(y) & \underset{y\to 0}{\overset{\text{median resummation}}{\rightleftarrows}} & f_0(y) \end{array} \tag{7.46}$$

Note that all four arrows tracing the edges of the box are now invertible. We display Conjecture 1 with a dashed arrow to distinguish it from the proven results that compose the rest of the diagram.

Remark 7.2.2. *The full content of the $q, \tilde{q}$-series in the block factorization of the spectral trace of local $\mathbb{P}^2$ cannot be reconstructed by a Borel–Laplace resummation. Yet, and remarkably, the missing information is not lost. Instead, it is collected by the discontinuities that are then recovered by the median resummation. The effectiveness of the median resummation as a summability tool is well-known in quantum CS theory—as it was conjectured for the quantum invariants of knots and 3-manifolds [13, Conjecture 1.1] and proven for the special cases of the trefoil knot [13] and the Seifert fibered homology sphere [14]. We expect more evidence to be discovered. Furthermore, as we discuss in Chapter 8 following [3], the effectiveness of the median resummation appears to be closely related to quantum modularity.*

7.3 Quantum modularity of the generating functions

We will now present our results on the quantum modularity properties of the generating functions of the Stokes constants in both $\hbar$-regimes, that is, the periodic holomorphic functions $f_0(y)$ and $f_\infty(y)$ defined in Eqs. (6.1) and (6.5). We refer to Appendix D for a brief introduction to quantum modular forms. Let us start by defining the holomorphic functions $F_R, F_S \colon \mathbb{H} \to \mathbb{C}$ as

$$F_R(y) := f_\infty\big(-\tfrac{1}{3y}\big) + f_0(y)\,, \tag{7.47a}$$

$$F_S(y) := f_\infty(y) - f_0\big(-\tfrac{1}{3y}\big)\,. \tag{7.47b}$$

The function $F_R(y)$ is equal to the Borel–Laplace sum $s_0(\tilde{f}_0)(y)$ in Eq. (7.15a) for $\Re(y) > 0$, while the function $-F_S\big(-\frac{1}{3y}\big)$ is equal to the Borel–Laplace sum $s_\pi(\tilde{f}_0)(y)$ in Eq. (7.15b) for $\Re(y) < 0$. They can, therefore, be expressed explicitly as functions of the strong coupling Stokes constants R_m, $m \in \mathbb{Z}_{\neq 0}$, according to Eqs. (7.13a) and (7.13b), respectively. Specifically, we find that

$$F_R(y) = -\frac{\pi \mathrm{i}}{2} - \frac{3\mathcal{V}}{2\pi \mathrm{i} y} + 2 \sum_{m \in \mathbb{Z}_{\neq 0}} R_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right)\,, \quad \Re(y) > 0\,, \tag{7.48a}$$

$$F_S(y) = \frac{\pi \mathrm{i}}{2} - \frac{9\mathcal{V} y}{2\pi \mathrm{i}} - 2 \sum_{m \in \mathbb{Z}_{\neq 0}} R_m\, \mathbf{e}_1\left(my\right)\,, \quad \Re(y) > 0\,, \tag{7.48b}$$

where $\mathcal{V} = 2\Im\left(\mathrm{Li}_2(\mathrm{e}^{2\pi \mathrm{i}/3})\right)$ and $\mathbf{e}_1$ is the exponential integral in Eq. (7.1) as before. Indeed, Theorem 7.2.2 can be restated in terms of the functions above as

$$\frac{F_R(y) - F_S\big(-\frac{1}{3y}\big)}{2} = f_0(y)\,, \quad y \in \mathbb{H}\,. \tag{7.49}$$

Remark 7.3.1. *Assuming the validity of Conjecture 1, we can complete the above discussion as follows. We observe that the function $F_R\big(-\frac{1}{3y}\big)$ is equal to the Borel–Laplace sum $s_\pi(\tilde{f}_\infty)(y)$ in Eq. (7.45b) for $\Re(y) < 0$, while the function $F_S(y)$ is equal to the Borel–Laplace sum $s_0(\tilde{f}_\infty)(y)$ in Eq. (7.45a) for $\Re(y) > 0$. They can, therefore, be expressed explicitly as functions of the weak coupling Stokes constants S_m, $m \in \mathbb{Z}_{\neq 0}$, according to Eqs. (7.14b) and (7.14a), respectively. Namely,*

$$F_R(y) = -3 \log \frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(2\pi \mathrm{i}/y) - 2 \sum_{m \in \mathbb{Z}_{\neq 0}} S_m\, \mathbf{e}_1\left(my\right)\,, \quad \Re(y) > 0\,, \tag{7.50a}$$

$$F_S(y) = -3 \log \frac{\Gamma(2/3)}{\Gamma(1/3)} - \log(-6\pi \mathrm{i} y) - 2 \sum_{m \in \mathbb{Z}_{\neq 0}} S_m\, \mathbf{e}_1\left(-\frac{m}{3y}\right)\,, \quad \Re(y) > 0\,, \tag{7.50b}$$

where $\mathbf{e}_1$ is again the exponential integral in Eq. (7.1). Indeed, Conjecture 1 is equivalent to

$$\frac{F_S(y) + F_R\big(-\frac{1}{3y}\big)}{2} = f_\infty(y)\,, \quad y \in \mathbb{H}\,. \tag{7.51}$$

In addition, the functions $F_R(y)$ and $F_S(y)$ in Eqs. (7.47a) and (7.47b) can be simply written as the logarithm of a ratio of *Faddeev's quantum dilogarithms*. Indeed, setting

$\mathsf{b}^2 = y$, it follows from the infinite product representations in Eqs. (6.3), (6.7), and (B.9) that

$$F_R(y) = 3 \log \frac{\Phi_{\mathsf{b}}\big(\frac{2\mathrm{i}}{3\mathsf{b}} - c_{\mathsf{b}}\big)}{\Phi_{\mathsf{b}}\big(\frac{\mathrm{i}}{3\mathsf{b}} - c_{\mathsf{b}}\big)} - \mathrm{i}\pi \,, \tag{7.52a}$$

$$F_S(y/3) = 3 \log \frac{\Phi_{\mathsf{b}}\big(\frac{2\mathrm{i}\mathsf{b}}{3} - c_{\mathsf{b}}\big)}{\Phi_{\mathsf{b}}\big(\frac{\mathrm{i}\mathsf{b}}{3} - c_{\mathsf{b}}\big)} \,, \tag{7.52b}$$

where c_{b} is defined in Eq. (B.6). Thus, thanks to the analyticity properties of the Faddeev's quantum dilogarithm $\Phi_{\mathsf{b}}(x)$ reviewed in Appendix B, $F_R(y)$ and $F_S(y)$ can be analytically continued to the whole cut complex plane $\mathbb{C}'$. We can now prove our statements on the *quantum modularity* of the generating functions.

Theorem 7.3.1. *The generating functions $f_0, f_\infty \colon \mathbb{H} \to \mathbb{C}$ in Eqs. (6.1) and (6.5), respectively, are holomorphic quantum modular functions for the congruence subgroup $\Gamma_1(3) \subset \mathsf{SL}_2(\mathbb{Z})$, defined in Eq. (3.16).*

Proof. Recall that the generators of $\Gamma_1(3)$ are the matrices T and γ_3 in Eq. (3.17). It is sufficient to prove that the corresponding cocycles, that is,

$$h_T[f](y) = f(y+1) - f(y) \,, \tag{7.53a}$$

$$h_{\gamma_3}[f](y) = f\big(\tfrac{y}{3y+1}\big) - f(y) \,, \tag{7.53b}$$

for $f = f_0, f_\infty$, are analytic in $\mathbb{C}'$. It follows from the periodicity properties in Eqs. (6.2) and (6.6) that the cocycles for T are trivial, *i.e.*,

$$h_T[f_0](y) = 0 \,, \quad h_T[f_\infty](y) = 0 \,. \tag{7.54}$$

Applying the definition of $F_R(y)$ in Eq. (7.47a) and the first formula in Eq. (6.6), we find that

$$\begin{aligned} f_0\big(\tfrac{y}{3y+1}\big) &= F_R\big(\tfrac{y}{3y+1}\big) - f_\infty\big(-\tfrac{1}{3y}\big) \\ &= F_R\big(\tfrac{y}{3y+1}\big) - F_R(y) + f_0(y) \,. \end{aligned} \tag{7.55}$$

Therefore, the cocycle of f_0 for γ_3 is

$$h_{\gamma_3}[f_0](y) = F_R\big(\tfrac{y}{3y+1}\big) - F_R(y) \,, \tag{7.56}$$

which is analytic in $\mathbb{C}'$ due to Eq. (7.52a) and the analyticity properties of Faddeev's quantum dilogarithm. Analogously, applying the definition of $F_S(y)$ in Eq. (7.47b) and the first formula in Eq. (6.2), we find that

$$\begin{aligned} f_\infty\big(\tfrac{y}{3y+1}\big) &= F_S\big(\tfrac{y}{3y+1}\big) + f_0\big(-\tfrac{1}{3y}\big) \\ &= F_S\big(\tfrac{y}{3y+1}\big) - F_S(y) + f_\infty(y) \,. \end{aligned} \tag{7.57}$$

Therefore, the cocycle of f_∞ for γ_3 is

$$h_{\gamma_3}[f_\infty](y) = F_S\big(\tfrac{y}{3y+1}\big) - F_S(y) \,, \tag{7.58}$$

which is analytic on $\mathbb{C}'$ due to Eq. (7.52b) and the analyticity properties of Faddeev's quantum dilogarithm. □

While the generating functions $f_0(y)$ and $f_\infty(y)$, $y \in \mathbb{H}$, have trivial cocycles for the generator T, their cocycles for the generator γ_3 in Eqs. (7.56) and (7.58) are given by the functions $F_R(y)$ and $F_S(y)$ in Eq. (7.47a) and (7.47b), respectively. Specifically,

$$h_{\gamma_3}[f_0](y) = h_{\gamma_3}[F_R](y)\,, \quad h_{\gamma_3}[f_\infty](y) = h_{\gamma_3}[F_S](y)\,, \tag{7.59}$$

which are determined by the strong and weak coupling Stokes constants R_n, S_n, $n \in \mathbb{Z}_{\neq 0}$, as a consequence of Eqs. (7.48a) and (7.50b). Once more, note the exchange of information between the two regimes in $\hbar$. The non-trivial cocycle of the generating function of the Stokes constants in one regime is controlled by the Stokes constants in the dual regime.

Remark 7.3.2. *As we learn from the proof of Theorem 7.3.1, the holomorphic quantum modular functions f_0, f_∞ are deeply related to the Faddeev's quantum dilogarithm Φ_b in Eq.* (B.9)*, which enters the formulae for the cocyles. This is indeed a general feature of the q-Pochhammer symbols, as discussed in [169, Section 8.11].*

Quantum modularity and Fricke involution

As we will show, a pair of "dual" holomorphic quantum modular functions for $\Gamma_1(3)$ is constructed by acting on the generating functions $f_0(y)$ and $f_\infty(y)$ with the transformation $y \mapsto -\frac{1}{3y}$, which is known as the *Fricke involution* of the quotient $\mathbb{H}/\Gamma_1(3)$. In particular, we define the holomorphic functions $f_0^\star, f_\infty^\star \colon \mathbb{H} \to \mathbb{C}$ as

$$f_0^\star(y) := f_0\big(-\tfrac{1}{3y}\big)\,, \quad f_\infty^\star(y) := f_\infty\big(-\tfrac{1}{3y}\big)\,, \tag{7.60}$$

where we use the symbol $\star$ to denote the action of the Fricke involution.

Theorem 7.3.2. *The functions $f_0^\star, f_\infty^\star \colon \mathbb{H} \to \mathbb{C}$, defined in Eq.* (7.60) *as the images of the generating functions under Fricke involution, are holomorphic quantum modular functions for the congruence subgroup $\Gamma_1(3) \subset \mathsf{SL}_2(\mathbb{Z})$, defined in Eq.* (3.16).

Proof. Recall that the generators of $\Gamma_1(3)$ are the matrices T and γ_3 in Eq. (3.17). It is sufficient to prove that the corresponding cocycles, that is,

$$h_T[f^\star](y) = f^\star(y+1) - f^\star(y) = f\big(-\tfrac{1}{3y+3}\big) - f\big(-\tfrac{1}{3y}\big)\,, \tag{7.61a}$$

$$h_{\gamma_3}[f^\star](y) = f^\star\big(\tfrac{y}{3y+1}\big) - f^\star(y) = f\big(-\tfrac{1}{3y} - 1\big) - f\big(-\tfrac{1}{3y}\big)\,, \tag{7.61b}$$

for $f = f_0, f_\infty$, are analytic in $\mathbb{C}'$. It follows from the periodicity properties in Eqs. (6.2) and (6.6) that the cocycles for γ_3 are trivial, *i.e.*,

$$h_{\gamma_3}[f_0^\star](y) = 0\,, \quad h_{\gamma_3}[f_\infty^\star](y) = 0\,. \tag{7.62}$$

Applying the definition of $F_S(y)$ in Eq. (7.47b) and the first formula in Eq. (6.6), we find that

$$\begin{aligned} f_0^\star(y+1) = f_0\big(-\tfrac{1}{3y+3}\big) &= f_\infty(y+1) - F_S(y+1) \\ &= F_S(y) - F_S(y+1) + f_0\big(-\tfrac{1}{3y}\big) \\ &= F_S(y) - F_S(y+1) + f_0^\star(y)\,. \end{aligned} \tag{7.63}$$

Therefore, the cocycle of $f_0^\star$ for T is

$$h_T[f_0^\star](y) = F_S(y) - F_S(y+1)\,, \tag{7.64}$$

which is analytic on $\mathbb{C}'$ due to Eq. (7.52b) and the analyticity properties of the Faddeev's quantum dilogarithm. Analogously, applying the definition of $F_R(y)$ in Eq. (7.47a) and the first formula in Eq. (6.2), we find that

$$\begin{aligned} f_\infty^\star(y+1) = f_\infty\big(-\tfrac{1}{3y+3}\big) &= F_R(y+1) - f_0(y+1) \\ &= F_R(y+1) - F_R(y) + f_\infty\big(-\tfrac{1}{3y}\big) \\ &= F_R(y+1) - F_R(y) + f_\infty^\star(y)\,. \end{aligned} \tag{7.65}$$

Therefore, the cocycle of $f_\infty^\star$ for T is

$$h_T[f_\infty^\star](y) = F_R(y+1) - F_R(y)\,, \tag{7.66}$$

which is analytic on $\mathbb{C}'$ following Eq. (7.52a) and the analyticity properties of the Faddeev's quantum dilogarithm. □

The Fricke involution has exchanged the roles played by the generators of the modular group $\Gamma_1(3)$ and, simultaneously, by the weak and strong coupling Stokes constants. Indeed, the dual functions $f_0^\star(y)$ and $f_\infty^\star(y)$, $y \in \mathbb{H}$, have trivial cocycles for the generator γ_3, while their cocycles for the generator T in Eqs. (7.64) and (7.66) are given by the functions $F_S(y)$ and $F_R(y)$ in Eqs. (7.47b) and (7.47a), respectively. Namely, we have that

$$h_T[f_0^\star](y) = -h_T[F_S](y)\,, \quad h_T[f_\infty^\star](y) = h_T[F_R](y)\,, \tag{7.67}$$

which are determined by the weak and strong coupling Stokes constants S_n, R_n, $n \in \mathbb{Z}_{\neq 0}$, as a consequence of Eqs. (7.50b) and (7.48a).

Remark 7.3.3. *Our results on the quantum modularity of the generating functions and the role of the Fricke involution pave the way for a geometric interpretation of the Stokes constants. Indeed, as briefly discussed in Section 3.2, the group $\Gamma_1(3)$ is deeply related to the geometry of local $\mathbb{P}^2$. The moduli space of complex structures of the mirror of local $\mathbb{P}^2$ is the compactification of the quotient $\mathbb{H}/\Gamma_1(3)$. Furthermore, the generating functions of the* Gromov–Witten invariants *are known to be quasi-modular functions in the modular coordinate $\tau = -\frac{1}{2} - \frac{3}{2\pi \mathrm{i}}\partial_t^2 F_0(t)$ at the conifold point in moduli space [51]. This enticing connection opens a new direction of investigation aimed at studying an explicit relationship between the Stokes constants appearing in the TS/ST correspondence and the Gromov–Witten invariants of toric CY threefolds, which we plan to address in upcoming work.*

Finally, a large class of examples of quantum modular forms comes from the study of quantum invariants of 3-*manifolds and knots* [16, 18, 22–24, 28, 53, 171]. Thus, our results on the quantum modularity of the generating functions of the Stokes constants in Theorems 7.3.1 and 7.3.2 provide new evidence in addition to the many formal similarities that are shared by topological string theory and complex CS theory [5, 10], which we have briefly discussed in Section 6.2.

Chapter 8

A theory of modular resurgence

The key example of the spectral trace of local $\mathbb{P}^2$, whose features we have extensively studied in Chapters 5, 6, and 7, propels a novel understanding of the resurgence of divergent formal power series and quantum modularity that rests on the interplay of q-series and L-functions and revolves around the role played by the Stokes constants.

In this chapter, we consider a class of resurgent Gevrey-1 asymptotic series that replicate the central aspects of the resurgence of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$, gathering numerous examples from the study of quantum invariants of knots and 3-manifolds and combinatorics. We then introduce the notion of a modular resurgent series, whose Borel plane displays a single infinite tower of singularities, the secondary resurgent series are trivial, and the Stokes constants are coefficients of an L-function, and propose a new perspective on their resurgent structures. Generalizing the strong-weak resurgent symmetry of Chapter 6, we describe the broader framework of modular resurgence. Pairs of modular resurgent structures are connected through a global exact symmetry centered around the functional equation satisfied by the resurgent L-functions. We then conjecture that the median resummation of modular resurgent series is effective and produces quantum modular forms. Finally, we illustrate the workings of modular resurgence via explicit computations in the case of local $\mathbb{P}^2$ and prove that a large class of examples of modular resurgent series originating from the theory of Maass cusp forms satisfies our conjectures. We reproduce [3, Sections 3 and 4].

8.1 New perspectives on resurgence via L-functions

Recall that one of the main features of Écalle's resurgent series is that one can compute the non-perturbative information hidden in the large-order growth of the coefficients by studying the singular structure in the Borel plane. Namely, starting from the germ at the origin, that is, the Borel transform of the divergent formal power series, and analytically continuing it to the whole complex plane, it is possible to uncover the germs at the other singularities. We refer to Chapter 1 for an introduction to the theory of resurgence. In the example of the spectral trace of local $\mathbb{P}^2$, the resurgent structure appears trivial at first glance, as the germs at the other singularities are all constants. Yet, as discovered in Chapter 5 following [1] and fully outlined in Chapters 6 and 7 following [2], when regarded as a sequence, these constants possess rich analytic number-theoretic properties that make contact with the notion of quantum modularity. Furthermore, when considering both weak and strong coupling limits, a unique strong-weak resurgent symmetry holds. As they can be generalized to a whole class of asymptotic series, our results on local $\mathbb{P}^2$ encourage us to adopt a new perspective and motivate the following discussion.

We will consider a particular type of resurgent asymptotic series and present a new *paradigm of resurgence* built upon the Stokes constants, their generating functions, and their Dirichlet series. We refer to this new paradigm as *modular resurgence* for reasons that will become clear in Section 8.2. Let us start by formally presenting and proving all relevant statements. The following result is a generalization of the exact large-order relations of Chapter 5.

Proposition 8.1.1. *Let*

$$\tilde{f}(y) = \sum_{n=1}^{\infty} c_n y^n \in \mathbb{C}[\![y]\!] \tag{8.1}$$

be a Gevrey-1 asymptotic series. The following two statements are equivalent.

(1) Its coefficients are given explicitly by

$$c_n = \frac{\Gamma(n)}{2\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta_m^n}, \quad n \in \mathbb{Z}_{>0}, \tag{8.2}$$

where A_m, ζ_m, $m \in \mathbb{Z}_{\neq 0}$, are complex numbers.

(2) Its Borel transform[1] is a simple resurgent function with simple poles at the locations ζ_m and corresponding Stokes constants A_m.

Proof. By definition, the Borel transform of $\tilde{f}$ is the function

$$\mathcal{B}[\tilde{f}](\zeta) = \sum_{n=1}^{\infty} \frac{c_n}{\Gamma(n)} \zeta^{n-1} = \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta_m^n} \zeta^{n-1} \in \mathbb{C}\{\zeta\}. \tag{8.3}$$

Since $\tilde{f}$ is Gevrey-1, the coefficients c_n have the factorially divergent growth

$$|c_n| \leq n! |\mathcal{E}|^{-n} \quad n \gg 1, \quad \mathcal{E} \in \mathbb{C}. \tag{8.4}$$

Hence, we can change the order of summation in Eq. (8.3), obtaining

$$\mathcal{B}[\tilde{f}](\zeta) = \frac{1}{2\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta_m} \sum_{n=1}^{\infty} \left(\frac{\zeta}{\zeta_m} \right)^{n-1} = -\frac{1}{2\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta - \zeta_m}, \tag{8.5}$$

which has simple poles at $\zeta = \zeta_m$ with corresponding Stokes constants A_m. The inverse statement is proven by following the same steps above in reversed order. □

Note that Eq. (8.2) can be equivalently written as

$$c_n = \frac{\Gamma(n)}{(2\pi \mathrm{i})\mathcal{E}^n} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{m^n}, \quad n \in \mathbb{Z}_{>0}, \tag{8.6}$$

when $\zeta_m = \mathcal{E} m$, $m \in \mathbb{Z}_{\neq 0}$, for some constant $\mathcal{E} \in \mathbb{C}$. Up to the prefactors, the RHS in Eq. (8.6) is the *Dirichlet series* with coefficients given by the Stokes constants A_m evaluated at the positive integer n. This does not a priori define an *L-series*. However, when it does, we can say something more.

[1] Note that we adopt the definitions of the Borel and Laplace transforms in Eqs. (1.23a) and (1.23b).

Proposition 8.1.2. *Let $\{A_m\}$, $m \in \mathbb{Z}_{\neq 0}$, be a sequence of complex numbers defining the L-series*

$$L_+(s) = \sum_{m>0} \frac{A_m}{m^s}\,, \quad L_-(s) = -\sum_{m>0} \frac{A_{-m}}{m^s} \tag{8.7}$$

that are convergent in some right half-plane $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$. Let

$$f(y) = \begin{cases} \displaystyle\sum_{m>0} A_m \mathrm{e}^{2\pi \mathrm{i} m y} & \text{if} \quad \Im(y) > 0 \\ \displaystyle -\sum_{m<0} A_m \mathrm{e}^{2\pi \mathrm{i} m y} & \text{if} \quad \Im(y) < 0 \end{cases} \tag{8.8}$$

be the generating series of the complex numbers A_m, $m \in \mathbb{Z}_{\neq 0}$. Then, $f(y)$ converges for $y \in \mathbb{C}\backslash\mathbb{R}$, and there is a canonical correspondence between the pair of L-series $L_+(s)$, $L_-(s)$ and the generating function $f(y)$.

Proof. The statement follows from the properties of the Mellin transform. Indeed, a simple computation shows that

$$(2\pi)^{-s}\Gamma(s) L_\pm(s) = \int_0^\infty t^{s-1} f(\pm \mathrm{i} t)\, dt\,, \tag{8.9}$$

that is, the L-series $L_\pm(s)$ are given by the Mellin transform of the generating function $f(y)$. Conversely, by the Mellin inversion theorem, $f(y)$ can be computed via the inverse Mellin transform of $L_\pm(s)$. Indeed, for $C > \alpha$ fixed, we have that

$$f(\mathrm{i}t) = \begin{cases} \displaystyle\frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} (2\pi t)^{-s}\Gamma(s)\, L_+(s)\, ds & \text{if} \quad t > 0 \\ \displaystyle\frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} (-2\pi t)^{-s}\Gamma(s)\, L_-(s)\, ds & \text{if} \quad t < 0 \end{cases}\,, \tag{8.10}$$

which can be analytically continued to all arguments $y \in \mathbb{C} \setminus \mathbb{R}$. □

Let us stress that the generating function $f(y)$ in Eq. (8.8) has the form of a q-series for $q = \mathrm{e}^{2\pi \mathrm{i} y}$ when restricted to either the upper or the lower halves of the complex plane. Besides, we could equivalently consider another pair of L-series, that is,

$$L_0(s) = \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{|m|^s}\,, \quad L_1(s) = \sum_{m \in \mathbb{Z}_{\neq 0}} \mathrm{sgn}(m) \frac{A_m}{|m|^s}\,, \tag{8.11}$$

which are again in a one-to-one correspondence with the generating function $f(y)$ in Eq. (8.8) as a consequence of Proposition 8.1.2. Indeed, $L_0(s)$ and $L_1(s)$ can be written as the linear combinations

$$\begin{aligned} L_0(s) &= \sum_{m>0} \frac{A_m}{m^s} + \sum_{m<0} \frac{A_m}{(-m)^s} = L_+(s) - L_-(s)\,, \\ L_1(s) &= \sum_{m>0} \frac{A_m}{m^s} - \sum_{m<0} \frac{A_m}{(-m)^s} = L_+(s) + L_-(s)\,. \end{aligned} \tag{8.12}$$

We can then consider the meromorphic extension of these L-series, or the L-series in Eq. (8.7), to the whole complex s-plane, that is, the corresponding L*-functions*. The following result states that knowing in closed form the perturbative coefficients of the asymptotic series obtained by expanding the generating function $f(y)$ at zero for $\Im(y) > 0$ is equivalent to computing the analytic continuation of the L-series $L_+(s)$. An analogous statement holds for $\Im(y) < 0$.

Proposition 8.1.3. *Let* $\{A_m\}$, $m \in \mathbb{Z}_{>0}$, *be a sequence of complex numbers defining an* L*-series*

$$L(s) = \sum_{m=1}^{\infty} \frac{A_m}{m^s} \tag{8.13}$$

that is convergent in some right half-plane $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$ *and can be analytically continued to* $\{\Re(s) < 0\} \subset \mathbb{C}$ *through a functional equation. Let*

$$\tilde{f}(y) = \sum_{n=0}^{\infty} c_n y^n \in \mathbb{C}[\![y]\!] \tag{8.14}$$

be the asymptotic series of the generating function $f(y)$*, defined in Eq. (8.8), as* $y \to 0$ *with* $\Im(y) > 0$*. Then, the perturbative coefficients* c_n*,* $n \in \mathbb{Z}_{\geq 0}$*, are given by*

$$c_n = L(-n)\frac{(2\pi\mathrm{i})^n}{n!}\,, \tag{8.15}$$

where $L(-n)$ *is defined by the functional equation.*

Proof. Taking the formal power series expansion of the exponential $\mathrm{e}^{2\pi\mathrm{i}my}$ at $y = 0$, the generating function in Eq. (8.8) gives

$$\begin{aligned} f(y) = \sum_{m=1}^{\infty} A_m \mathrm{e}^{2\pi\mathrm{i}my} &\sim \sum_{m=1}^{\infty} A_m \sum_{n=0}^{\infty} \frac{(2\pi\mathrm{i})^n}{n!} m^n y^n \\ \text{“}=\text{”} &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_m}{m^{-n}} \frac{(2\pi\mathrm{i})^n}{n!} y^n \\ &= \sum_{n=0}^{\infty} L(-n) \frac{(2\pi\mathrm{i})^n}{n!} y^n\,, \end{aligned} \tag{8.16}$$

where we have changed the order of summation.[2] Finally, we make sense of $L(-n)$ through the analytic continuation of $L(s)$. □

Remark 8.1.1. *Depending on the explicit form of the functional equation satisfied by the* L*-function* $L(s)$ *in Eq. (8.13), it might happen that* $L(-n)$ *is not well-defined for some* $n \in \mathbb{Z}_{\geq 0}$*. In this case, the computation of the corresponding perturbative coefficients* c_n *through the formula in Eq. (8.15) requires a separate ad-hoc analysis. A complete first-principles understanding of the domain of validity of Eq. (8.15) necessitates a specific investigation of the role played by the parameters entering the functional equation, its central point, and the motivic weight of the* L*-function. We leave this endeavor for the future.*

[2]The change of summation order is a formal operation here as the coefficients c_n might be factorially divergent.

We can now apply the three propositions above to build the new modular resurgence paradigm. To do so, we assemble the key ingredients in this construction and the net of relations among them along the same lines of Chapter 6. Let us give a step-by-step overview below.

- Let $\tilde{f}_\infty(y) \in \mathbb{C}[\![y]\!]$ be a Gevrey-1 formal power series whose perturbative coefficients can be written in the form of Eq. (8.2) with $\zeta_m = \mathcal{E}_0 m$, $m \in \mathbb{Z}_{\neq 0}$, for some constant $\mathcal{E}_0 \in \mathbb{C}$, and $S_m \in \mathbb{C}$. By Proposition 8.1.1, $\tilde{f}_\infty(y)$ is a simple resurgent series whose Borel transform has a single tower of simple poles at the locations ζ_m with Stokes constants S_m.

- Assume that the complex numbers S_m, $m \in \mathbb{Z}_{\neq 0}$, are the coefficients of two L-series $L_{0,+}(s)$ and $L_{0,-}(s)$, where $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$, defined as in Eq. (8.7). Following Proposition 8.1.2, we can build their generating function $f_0(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, defined as in Eq. (8.8), by taking the inverse Mellin transform of $L_{0,\pm}(s)$ as in Eq. (8.10).

- Assume that $L_{0,\pm}(s)$ can be meromorphically continued to $\{\Re(s) < 0\} \subset \mathbb{C}$. Let $\tilde{f}_{0,\pm}(y) \in \mathbb{C}[\![y]\!]$ be the asymptotic series of $f_0(y)$ as $y \to 0$ with $\Im(y) > 0$ and $\Im(y) < 0$, respectively. Following Proposition 8.1.3, the perturbative coefficients of $\tilde{f}_{0,\pm}(y)$ are explicitly given by the values of the L-functions $L_{0,\pm}(s)$ at negative integers as in Eq. (8.15), respectively. Note that $L_{0,\pm}(-n)$ for $n \in \mathbb{Z}_{>0}$ are computed through the functional equations that give the analytic continuation of the corresponding L-series. Sometimes, the analytic continuation of an L-function is described by another L-function with different coefficients. Assume this is the case and denote the new coefficients by $R_m \in \mathbb{C}$, $m \in \mathbb{Z}_{\neq 0}$.

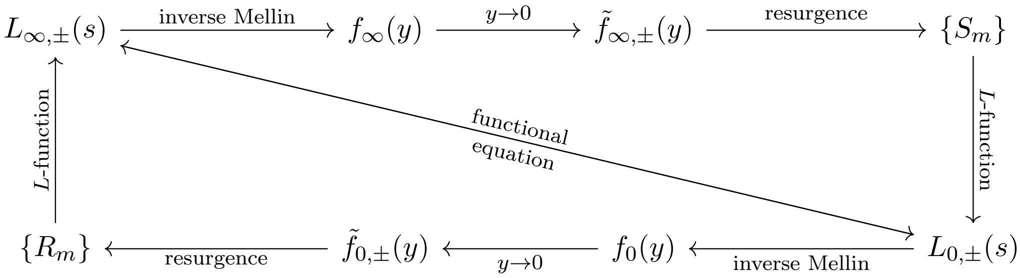
- Crucially, the resurgence structures of $\tilde{f}_{0,\pm}(y)$ are completely determined by the functional equations satisfied by $L_{0,\pm}(s)$, respectively. Assume that the perturbative coefficients of $\tilde{f}_0(y) := \tilde{f}_{0,+}(y) - \tilde{f}_{0,-}(y)$ can be written in terms of η_m and R_m in the form of Eq. (8.2), where $\eta_m = \mathcal{E}_\infty m$, $m \in \mathbb{Z}_{\neq 0}$, for some constant $\mathcal{E}_\infty \in \mathbb{C}$. By Proposition 8.1.1, $\tilde{f}_0(y)$ is a simple resurgent series whose Borel transform has a single tower of simple poles at the locations η_m with Stokes constants R_m.

- Let $L_{\infty,\pm}(s)$, where $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$, be the two L-series, defined as in Eq. (8.7), with coefficients R_m, $m \in \mathbb{Z}_{\neq 0}$. Following Proposition 8.1.2, we can build their generating function $f_\infty(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, defined as in Eq. (8.8), by taking the inverse Mellin transform of $L_{\infty,\pm}(s)$ as in Eq. (8.10).

- Finally, let $\tilde{f}_{\infty,\pm}(y) \in \mathbb{C}[\![y]\!]$ be the asymptotic series of $f_\infty(y)$ as $y \to 0$ with $\Im(y) > 0$ and $\Im(y) < 0$, respectively. Again following Proposition 8.1.3, the perturbative coefficients of $\tilde{f}_{\infty,\pm}(y)$ are explicitly given by the values of the L-functions $L_{\infty,\pm}(s)$ at negative integers as in Eq. (8.15), respectively. Once more, $L_{\infty,\pm}(-n)$ for $n \in \mathbb{Z}_{>0}$ are computed through the functional equations that give the analytic continuation of the corresponding L-series $L_{\infty,\pm}(s)$. By construction, these are determined by $L_{0,\pm}(s)$. We recover $\tilde{f}_\infty(y)$ as the combination $\tilde{f}_\infty(y) = \tilde{f}_{\infty,+}(y) - \tilde{f}_{\infty,-}(y)$.

Notice that the above relations among the dual resurgent structures of the pair of asymptotic series $\tilde{f}_0$, $\tilde{f}_\infty$ assemble into a unique global construction that generalizes the strong-weak symmetry of Chapter 6. For instance, the L-functions $L_{0,\pm}(s)$ and $L_{\infty,\pm}(s)$

are what we previously called *resurgent L-functions*. We summarize the previous circular argument in the *commutative diagram* below.



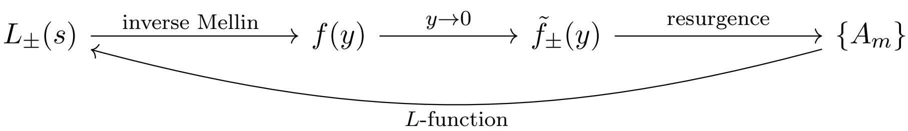
(8.17)

Notably, starting from any vertex, all vertices of this symmetric diagram are spanned by following its directed arrows—that is, the information contained in each vertex reconstructs the whole diagram and is therefore equivalent to the information content of every other vertex. Moreover, a special place is occupied by the two-headed diagonal arrow representing the *functional equation* connecting the two pairs of L-functions $L_{0,\pm}(s)$ and $L_{\infty,\pm}(s)$.

The role of L-functions

When it applies, Proposition 8.1.3 implies that the asymptotic series of the generating function of the Stokes constants is governed by the functional equation satisfied by the corresponding L-function. This is the core of the modular resurgence paradigm and explains why, starting from the above asymptotic series, one might find a new divergent formal power series with a new set of Stokes constants whose resurgent structure is, however, determined by the analytic continuation of the original L-function. In some examples, this is dictated by the same L-function, in which case modular resurgence simplifies.

It might happen that the functional equations for the L-functions $L_\pm(s)$ in Eq. (8.7) are written in terms of the same L-functions—as in the case of the Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s, \chi_{3,2})$ in Eq. (6.27). Equivalently, it might happen that the asymptotic expansion in the limit $y \to 0$ with $\Im(y) > 0$ of the generating function $f(y)$ of a set of Stokes constants $\{A_m\}$, $m \in \mathbb{Z}_{\neq 0}$, which are obtained from the resurgence of the Gevrey-1 formal power series $\tilde{f}_+(y)$ and $\tilde{f}_-(y)$, is equal to $\tilde{f}_+(y)$—and similarly for $\Im(y) < 0$. In this case, the commutative diagram in Eq. (8.17) reduces to the one below.



(8.18)

This simplified version of the modular resurgence construction is observed, among others, in the examples of the series σ and σ^*, whose L-function was first studied by Cohen in [172].[3] More generally, a similar behavior can be observed when $f(y)$ is the function associated with a Maass cusp form of the kind considered in [173]. See Proposition 8.3.1.

Remark 8.1.2. *From the conventional viewpoint in resurgence, looking at the generating series of the Stokes constants is unusual, as one would not generally expect them to define*

[3] The study of the resurgent structure of σ and σ^* was presented in [56]. From a number-theoretic point of view, the relation of these examples with quantum modular forms was studied by Zagier in [18].

a function—except in those examples where there are finitely many of them, in which case their generating series is a polynomial. However, we have shown that when the resurgent structure of a given asymptotic series is simple enough, and the Stokes constants have unique analytic and number-theoretic properties, studying their generating series allows us to construct a dual formal power series whose resurgent structure is deeply related to the original one. In particular, when the Stokes constants are coefficients of an L-function, the modular resurgence construction applies and is controlled by the corresponding functional equation. Therefore, we advocate a new perspective on resurgence that aims to investigate the properties of the Stokes constants.

8.2 Modular resurgent structures

Our previous discussion highlights the leading role of the L-functions and their functional equation in describing new features of a particular type of resurgent series. In this section, we address the relationship between quantum modularity and resurgent structures, which will clarify our choice of name for the paradigm of modular resurgence. We start by proposing the following definitions.

Definition 8.2.1. *A Gevrey-1 asymptotic series $\tilde{f}(y) \in \mathbb{C}[\![y]\!]$ has a* modular resurgent structure *if the following conditions hold.*

1. *The Borel transform $\mathcal{B}[\tilde{f}](\zeta) \in \mathbb{C}\{\zeta\}$ has a tower of singularities at the locations $\zeta_m = \mathcal{E}m$, $m \in \mathbb{Z}_{\neq 0}$, for some constant $\mathcal{E} \in \mathbb{C}$, in the complex ζ-plane.*
2. *For every $m \in \mathbb{Z}_{\neq 0}$, the resurgent series at the singularity ζ_m is the constant function $A_m \in \mathbb{C}$,* i.e., *the Stokes constant.*
3. *The Stokes constants A_m, $m \in \mathbb{Z}_{\neq 0}$, are the coefficients of two L-series*

$$L_+(s) = \sum_{m>0} \frac{A_m}{m^s} \,, \quad L_-(s) = -\sum_{m>0} \frac{A_{-m}}{m^s} \tag{8.19}$$

   that are convergent in some right half-plane $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$ and can be analytically continued to $\{\Re(s) < 0\} \subset \mathbb{C}$.

Definition 8.2.2. *A Gevrey-1 asymptotic series with a modular resurgent structure is a* modular resurgent series. *The Borel transform of a modular resurgent series is a* modular resurgent function.

It follows from Proposition 8.1.2 that a modular resurgent series is equivalently characterized by the generating function $f(y)$ in Eq. (8.8). Moreover, the discontinuities across the Stokes rays identified by the tower of singularities are given by

$$\mathrm{disc}_\theta \, \tilde{f}_+(y) = \sum_{m=1}^{\infty} A_m \, \mathrm{e}^{-\mathcal{E}m/y} = f\left(-\tfrac{\mathcal{E}}{2\pi \mathrm{i} y}\right) \,, \quad y \in \mathbb{H} \cap \{\Re(\mathrm{e}^{-\mathrm{i}\theta} y) > 0\} \,, \tag{8.20a}$$

$$\mathrm{disc}_{2\pi-\theta} \, \tilde{f}_-(y) = \sum_{m=1}^{\infty} A_{-m} \, \mathrm{e}^{\mathcal{E}m/y} = -f\left(-\tfrac{\mathcal{E}}{2\pi \mathrm{i} y}\right) \,, \quad y \in \mathbb{H}_- \cap \{\Re(\mathrm{e}^{-\mathrm{i}\theta} y) > 0\} \,, \tag{8.20b}$$

where $0 \leq \theta < \pi$ is the argument of the singularities in the upper half of the Borel plane, and $\tilde{f}_\pm(y)$ are the asymptotic series of $f(y)$ as $y \to 0$ with $\Im(y) > 0$ and $\Im(y) < 0$, respectively.

Let us now state our main conjectures. Specifically, a q-series $f\colon \mathbb{H} \to \mathbb{C}$ whose asymptotic series at zero has a modular resurgent structure is expected to agree with the median resummation of its asymptotic expansion and to be a holomorphic quantum modular form.

Conjecture 2. *Let $f\colon \mathbb{H} \to \mathbb{C}$ be a q-series where $q = e^{2\pi i y}$. If its asymptotic expansion $\tilde{f}(y)$ as $y \to 0$ with $\Im(y) > 0$ has a modular resurgent structure, then the* median resummation *of $\tilde{f}(y)$ reconstructs the original function $f(y)$, that is,*

$$\mathcal{S}_\theta^{\text{med}} \tilde{f}(y) = f(y)\,, \quad y \in \mathbb{H} \cap \{\Re(e^{-i\theta} y) > 0\}\,, \tag{8.21}$$

where θ is the argument of the singularities in the Borel plane.

Conjecture 3. *Let $f\colon \mathbb{H} \to \mathbb{C}$ be a q-series where $q = e^{2\pi i y}$. If its asymptotic expansion $\tilde{f}(y)$ as $y \to 0$ with $\Im(y) > 0$ has a modular resurgent structure, then the function $f(y)$ is a* holomorphic quantum modular form *for a subgroup $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$.*

We can relax the assumptions in the conjectures above as in Zagier's strange identity [53]. Namely, let $f'\colon \mathbb{Q} \to \mathbb{C}$ be such that there exists a q-series $f\colon \mathbb{H} \to \mathbb{C}$ whose radial limit agrees with f' to all orders. If the asymptotic expansion $\tilde{f}(y)$ as $y \to 0$ with $\Im(y) > 0$ has a modular resurgent structure, then we expect that the median resummation of $\tilde{f}(y)$ reconstructs the original function $f'(y)$, that is,

$$\mathcal{S}_\theta^{\text{med}} \tilde{f}(y) = f'(y)\,, \quad y \in \mathbb{Q} \cap \{\Re(e^{-i\theta} y) > 0\}\,, \tag{8.22}$$

where θ is again the argument of the singularities in the Borel plane. At the same time, we expect that $f'(y)$ is a quantum modular form for a subgroup $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$.

Remark 8.2.1. *Note that the generating function of the Stokes constants in Eq. (8.8) is supported on $\mathbb{C} \setminus \mathbb{R}$. Indeed, all statements above can be recast for a q-series $f\colon \mathbb{H}_- \to \mathbb{C}$, where $\mathbb{H}_-$ denotes the lower half of the complex plane. In this case, the relevant asymptotic expansion is obtained in the limit $y \to 0$ with $\Im(y) > 0$.*

Finally, we consider functions that are defined only over the rationals. This was, in fact, the setting where quantum modular forms were originally introduced [18], as briefly discussed in Appendix D. Conjectures 2 and 3 then become the following.

Conjecture 4. *Let $f\colon \mathbb{Q} \to \mathbb{C}$ be a q-series where $q = e^{2\pi i y}$. If its asymptotic expansion $\tilde{f}(y)$ as $y \to 0$ extends to $y \in \mathbb{C}$ and has a modular resurgent structure, then the* median resummation *of $\tilde{f}(y)$ reconstructs the original function $f(y)$, that is,*

$$\mathcal{S}_\theta^{\text{med}} \tilde{f}(y) = f(y)\,, \quad y \in \mathbb{Q} \cap \{\Re(e^{-i\theta} y) > 0\}\,, \tag{8.23}$$

where θ is the argument of the singularities in the Borel plane.

Conjecture 5. *Let $f\colon \mathbb{Q} \to \mathbb{C}$ be a q-series where $q = e^{2\pi i y}$. If its asymptotic expansion $\tilde{f}(y)$ as $y \to 0$ extends to $y \in \mathbb{C}$ and has a modular resurgent structure, then the function $f(y)$ is a* quantum modular form *for a subgroup $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$.*

We stress that, to perform a resurgent analysis of $\tilde{f}(y)$ in Conjectures 4 and 5, it is necessary to complexify the variable y. Namely, $\tilde{f}(y)$ must be analytically continued to $y \in \mathbb{C}$. In Conjecture 4, the median resummation $\mathcal{S}_\theta^{\rm med}\tilde{f}(y)$ is then restricted to the original domain of the function $f(y)$. We conclude by listing the currently available evidence in support of our conjectures.

- In support of Conjectures 2 and 3, we have the generating functions of the weak and strong coupling Stokes constants for $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ [2] (see also Section 8.3.1) and a collection of examples built from the theory of Maass cusp forms and their periods (see Section 8.3.2).

- In support of Conjecture 4, we have the Kontsevich–Zagier series[4] [13] and the WRT invariants for a family of Seifert fibered homology spheres [14, Theorem 3].

- In support of Conjecture 5, we cite again the Kontsevich–Zagier series, whose quantum modularity was proven in [53], the LMO ($\hat{Z}$) invariants for a family of plumbed 3-manifolds [25, 26], and the q-series σ and σ^*, which are quantum modular forms according to [18, Example 1].

According to [13, Conjecture 1.4], other examples satisfying Conjecture 4 might originate from the quantum invariants of knots and 3-manifolds. For instance, the Kashaev invariants of hyperbolic knots are conjecturally vector-valued quantum modular forms [19] whose resurgent structure displays a full-fledged peacock pattern [75, 76]. Peacock patterns are also conjectured to appear in the resurgence of the fermionic spectral traces of toric CY threefolds according to [1, 10], as we have described in Section 3.1. It would be interesting to investigate examples exhibiting multiple towers of singularities in the Borel plane and possibly extend our definition of modular resurgent structures accordingly. Furthermore, at this stage, other assumptions may be needed to fully characterize the L-functions associated with the generating function of the Stokes constants and enable us to prove the conjectures above within the paradigm of modular resurgence. In particular, a complete characterization of the admissible L-functions would require assumptions on the gamma factors appearing in their completions—or equivalently, in their functional equation. We plan to probe these ideas in the future.

8.3 Examples of modular resurgence

In this section, we describe two complete examples of the proposed paradigm of modular resurgence. In particular, we show how the global exact symmetry of Section 8.1 reproduces the strong-weak resurgent symmetry of Chapter 6 once specialized to the logarithm of the spectral trace of local $\mathbb{P}^2$. Then, we prove our conjectural statements in Section 8.2 for a large class of modular resurgent series emerging from the theory of Maass cusp forms.

8.3.1 From topological string theory

Let us compare the paradigm of *modular resurgence* with the *strong-weak resurgent symmetry* of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$, showing that the construction presented here and the one of Section 6.2 are coincident and complementary. We refer the reader to Section 3.2 and Chapters 5

[4] In the example of the Kontsevich–Zagier series, the resurgent structure has a half-tower of fractional power singularities, and not simple poles, contrary to what occurs in the examples discussed in this thesis.

and 6 for the necessary background. We start by observing that the asymptotic series $\phi(\hbar)$ and $\psi(\tau)$ in Eqs. (5.1) and (5.93) dictating the perturbative expansion of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ in the limits $\hbar \to 0$ and $\tau \propto \hbar^{-1} \to 0$, respectively, are modular resurgent series according to Definition 8.2.1—not only, they are *paired* modular resurgent structures. In fact, the corresponding Stokes constants S_m, R_m, $m \in \mathbb{Z}_{\neq 0}$, their generating functions $f_0(y)$, $f_\infty(y)$, $y \in \mathbb{C} \setminus \mathbb{R}$, and their resurgent L-functions $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, perfectly fit the symmetric diagram in Eq. (8.17).

Let us repeat the main steps in the modular resurgence construction for this key example. This offers a supplementary understanding of the strong-weak resurgent symmetry of local $\mathbb{P}^2$, particularly highlighting the central role played by the functional equation in Eq. (6.29). In what follows, we neglect the terms that do not enter the perturbative series in the RHS of Eqs. (6.13) and (6.15).

- The resurgence of the asymptotic series $\tilde{f}_\infty(y)$ is dictated by the formal power series $2\phi(2\pi y)$ according to the formula in Eq. (6.15). Thus, its resurgent structure is modular and can be written in the coordinate conjugate to y as $\{\eta_m, S_m\}$, $m \in \mathbb{Z}_{\neq 0}$, where η_m is defined in Eq. (5.113) and the Stokes constants satisfy $S_{-m} = S_m$.

- The weak coupling L-series $L_0(s)$, $s \in \mathbb{C}$ with $\Re(s) > 1$, has coefficients given by the Stokes constants S_m for $m \in \mathbb{Z}_{>0}$. Note that $L_0(s) = \pm L_{0,\pm}(s)$ in the notation of Eq. (8.7). Because of the parity of the Stokes constants, we need only consider one L-series.

- Taking the inverse Mellin transform of $L_0(s)$ as in Eq. (8.10), we obtain the generating function $f_0(y)$. Let

$$\tilde{f}_0(y) = \sum_{n=1}^{\infty} c_n^0 y^n \in \mathbb{C}[\![y]\!] \tag{8.24}$$

be its asymptotic series as $y \to 0$ with $\Im(y) > 0$. By Proposition 8.1.3, particularly Eq. (8.15), the perturbative coefficients c_n^0, $n \in \mathbb{Z}_{>0}$, are given by

$$c_n^0 = L_0(-n) \frac{(2\pi \mathrm{i})^n}{n!} \,, \tag{8.25}$$

where we make sense of the RHS via the functional equation for the completed L-function $\Lambda_0(s)$. In particular, substituting Eqs. (6.28a) and (6.28b) into Eq. (6.29), we find that

$$L_0(-s) = -\frac{2\mathrm{i} 3^s}{\pi^{2s} s} \frac{\Gamma\left(\frac{s+1}{2}\right)^2}{\Gamma\left(-\frac{s}{2}\right)^2} L_\infty(s) \,. \tag{8.26}$$

Notice that $L_0(-s)$ tends to zero for $s = n \in \mathbb{Z}_{>0}$ even as a consequence of the divergence of the gamma function in the denominator. Thus, we set

$$c_{2n}^0 = 0 \,, \quad n \in \mathbb{Z}_{>0} \,. \tag{8.27}$$

Then, taking $s = n \in \mathbb{Z}_{>0}$ odd, applying the well-known properties of the gamma function, and using the series representation of $L_\infty(s)$ in Eq. (5.169b), Eq. (8.26) becomes

$$L_0(-n) = -\frac{2\mathrm{i}}{\pi} n!(n-1)!(-2\pi \mathrm{i})^{-n} \sum_{m=1}^{\infty} \frac{R_m}{\eta_m^n} \,. \tag{8.28}$$

Using Eq. (8.25), the non-trivial perturbative coefficients are

$$c^0_{2n+1} = -2\frac{\Gamma(2n+1)}{\pi \mathrm{i}} \sum_{m=1}^{\infty} \frac{R_m}{\eta_m^{2n+1}} = -2b_{2n+2}\,, \quad n \in \mathbb{Z}_{\geq 0}\,, \tag{8.29}$$

where we have applied the exact large-order identity in Eq. (8.2) for the perturbative coefficients $\{b_{2k}\}$, $k \in \mathbb{Z}_{>0}$, of the formal power series $\psi(\tau)$ in Eq. (5.93)—that is, Eq. (5.148) from Chapter 5. This is what we expect from the formula in Eq. (6.13).

- The resurgence of the asymptotic series $\tilde{f}_0(y)$ is dictated by the formal power series $-2\psi(y)$ according to the formula in Eq. (6.13). Thus, its resurgent structure is modular and can be written in the coordinate conjugate to y as $\{\eta_m, R_m\}$, $m \in \mathbb{Z}_{\neq 0}$, where η_m is defined in Eq. (5.113) and the Stokes constants satisfy $R_{-m} = -R_m$.
- The strong coupling L-series $L_\infty(s)$, $s \in \mathbb{C}$ with $\Re(s) > 1$, has coefficients given by the Stokes constants R_m for $m \in \mathbb{Z}_{>0}$. Note that $L_\infty(s) = L_{\infty,\pm}(s)$ in the notation of Eq. (8.7). Again, because of the parity of the Stokes constants, we need only consider one L-series.
- Taking the inverse Mellin transform of $L_\infty(s)$ as in Eq. (8.10), we obtain the generating function $f_\infty(y)$. Let

$$\tilde{f}_\infty(y) = \sum_{n=1}^{\infty} c_n^\infty y^n \in \mathbb{C}[\![y]\!] \tag{8.30}$$

be its asymptotic series as $y \to 0$ with $\Im(y) > 0$. By Proposition 8.1.3, particularly Eq. (8.15), its perturbative coefficients c_n^∞, $n \in \mathbb{Z}_{>0}$, are given by

$$c_n^\infty = L_\infty(-n)\frac{(2\pi \mathrm{i})^n}{n!}\,, \tag{8.31}$$

where we make sense of the RHS via the functional equation for the completed L-function $\Lambda_\infty(s)$ as before. In particular, Eq. (8.26) gives

$$L_\infty(-s) = \frac{3^s s}{2\mathrm{i}\pi^{2s}} \frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma\left(\frac{1-s}{2}\right)^2} L_0(s)\,. \tag{8.32}$$

Notice that $L_\infty(-s)$ tends to zero for $s = n \in \mathbb{Z}_{\geq 0}$ odd due to the divergence of the gamma function in the denominator. Hence, we set

$$c^\infty_{2n+1} = 0\,, \quad n \in \mathbb{Z}_{\geq 0}\,. \tag{8.33}$$

Then, taking $s = n \in \mathbb{Z}_{>0}$ even, applying the well-known properties of the gamma function, and using the series representation of $L_0(s)$ in Eq. (5.169a), Eq. (8.32) becomes

$$L_\infty(-n) = \frac{2}{\mathrm{i}\pi} n!(n-1)!(-2\pi)^n \sum_{m=1}^{\infty} \frac{S_m}{\zeta_m^n}\,. \tag{8.34}$$

Using Eq. (8.31), the non-trivial perturbative coefficients are

$$c^\infty_{2n} = 2(2\pi)^{2n}\frac{\Gamma(2n)}{\pi \mathrm{i}} \sum_{m=1}^{\infty} \frac{S_m}{\zeta_m^{2n}} = 2(2\pi)^{2n} a_{2n}\,, \quad n \in \mathbb{Z}_{>0}\,, \tag{8.35}$$

where we have applied the exact large-order identity in Eq. (8.2) for the perturbative coefficients $\{a_{2k}\}$, $k \in \mathbb{Z}_{>0}$, of the formal power series $\phi(\hbar)$ in Eq. (5.1)—that is, Eq. (5.61) from Chapter 5. Note again that this is what we expect from the formula in Eq. (6.15).

Remark 8.3.1. *The term of order y^0 does not enter our definitions of $\tilde{f}_0(y)$ and $\tilde{f}_\infty(y)$. We observe, however, that direct evaluation of the weak coupling L-function $L_0(s)$ at $s = 0$ using the closed formula in Eq. (5.169a), together with Eq. (8.25), gives the perturbative coefficient $c_0^0 = L_0(0) = -\pi\mathrm{i}/2$—in agreement with the asymptotic expansion in Eq. (6.13). At the same time, the strong coupling L-function $L_\infty(s)$ in Eq. (5.169b) is not well-defined at $s = 0$, as it can also be observed from the functional equation in Eq. (8.32). In particular, following Eq. (8.31), the perturbative coefficient c_0^∞ would appear to diverge to infinity. We conclude that the constant term in the asymptotic expansion in Eq. (6.15) cannot be consistently derived from the analytic continuation of the L-function $L_\infty(s)$.*

Finally, we stress that both Conjectures 2 and 3 are explicitly proven for the generating function f_0 in Theorems 7.2.2 and 7.3.1. Namely, f_0 is a holomorphic quantum modular function for the congruence subgroup $\Gamma_1(3)$ and is reconstructed by the median resummation of its asymptotic expansion. As for the generating function f_∞, Conjecture 3 is proven for the same modular group $\Gamma_1(3)$ in Theorem 7.3.1, while Conjecture 2 is only supported by numerical evidence and previously stated in Conjecture 1.

8.3.2 From the theory of Maass cusp forms

Given a pair of L-series satisfying a special type of functional equations, a large class of examples of modular resurgent series can be derived from this data. In particular, building on the results of [173], we consider L-functions related to *Maass cusp forms* with spectral parameter 1/2.[5]

Proposition 8.3.1. *Let $\{A_m\}$, $m \in \mathbb{Z}_{\neq 0}$, be a sequence of complex numbers such that the two L-series*

$$L_\epsilon(s) = \sum_{m=1}^{\infty} \frac{A_m + (-1)^\epsilon A_{-m}}{m^s}\,, \quad \epsilon = 0, 1\,, \tag{8.36}$$

are convergent in some right half-plane $\{\Re(s) > \alpha \geq 0\} \subset \mathbb{C}$ and can be meromorphically continued to $\{\Re(s) < 0\} \subset \mathbb{C}$. Assume that the corresponding completed L-functions

$$\Lambda_\epsilon(s) = \gamma(s+\epsilon) L_\epsilon(s)\,, \quad \textit{where} \quad \gamma(s) = \frac{1}{4\pi^s}\Gamma\left(\frac{s}{2}\right)^2\,, \tag{8.37}$$

are entire functions of finite order and satisfy the functional equations

$$\Lambda_\epsilon(1-s) = (-1)^\epsilon \Lambda_\epsilon(s)\,, \quad \epsilon = 0, 1\,. \tag{8.38}$$

Let $f\colon \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ be the generating function

$$f(y) = \begin{cases} \displaystyle\sum_{m>0} A_m \mathrm{e}^{2\pi\mathrm{i} m y} & \textit{if} \quad \Im(y) > 0 \\ \displaystyle -\sum_{m<0} A_m \mathrm{e}^{2\pi\mathrm{i} m y} & \textit{if} \quad \Im(y) < 0 \end{cases}\,. \tag{8.39}$$

[5] A Maass cusp form with spectral parameter $s \in \mathbb{C}$ is a smooth $\mathsf{PSL}_2(\mathbb{Z})$-invariant function $u\colon \mathbb{H} \to \mathbb{C}$ that is an eigenfunction of the hyperbolic Laplacian Δ with eigenvalue $s(1-s)$. Namely, $\Delta u = s(1-s)u$. These functions give a basis for L^2 on $\mathbb{H}/\mathsf{PSL}_2(\mathbb{Z})$.

Then, the asymptotic series $\tilde{f}_+(y)$ and $\tilde{f}_-(y)$ of $f(y)$ in the limit $y \to 0$ with $\Im(y) > 0$ and $\Im(y) < 0$, respectively, are modular resurgent series.

Proof. We introduce the auxiliary L-functions

$$L_+(s) = \sum_{m=1}^{\infty} \frac{A_m}{m^s} = \frac{L_0(s) + L_1(s)}{2}\,, \tag{8.40a}$$

$$L_-(s) = -\sum_{m=1}^{\infty} \frac{A_{-m}}{m^s} = \frac{L_1(s) - L_0(s)}{2}\,, \tag{8.40b}$$

as in Eq. (8.7). It follows from Proposition 8.1.3, particularly from Eq. (8.15), that the asymptotic series of $f(y)$ as $y \to 0$ with $\Im(y) > 0$ can be written as

$$\tilde{f}_+(y) = \sum_{n=0}^{\infty} L_+(-n) \frac{(2\pi \mathrm{i})^n}{n!} y^n\,. \tag{8.41}$$

Similarly, the asymptotic series of $f(y)$ as $y \to 0$ with $\Im(y) < 0$ is

$$\tilde{f}_-(y) = \sum_{n=0}^{\infty} L_-(-n) \frac{(2\pi \mathrm{i})^n}{n!} y^n\,. \tag{8.42}$$

We can now apply the functional equation for $\Lambda_\epsilon(s)$ in Eq. (8.38) to make sense of $L_\pm(-n)$ for $n \in \mathbb{Z}_{\geq 0}$. To do so, let us consider separately the case of $n \in \mathbb{Z}_{\geq 0}$ even and odd for $L_1(-n)$ and $L_0(-n)$, respectively. For $n \in \mathbb{Z}_{\geq 0}$ even, we have that

$$L_1(-n) = -\frac{\gamma(n+2)}{\gamma(1-n)} L_1(n+1) = -\frac{2^{-2n}}{\pi^{2n+2}} \Gamma(n+1)^2 L_1(n+1)\,, \tag{8.43}$$

where we have used the expression for the gamma factor in Eq. (8.37) and applied the well-known identities for the gamma function. Similarly, for $n \in \mathbb{Z}_{>0}$ odd, we have that

$$L_0(-n) = \frac{\gamma(n+1)}{\gamma(-n)} L_0(n+1) = \frac{2^{-2n}}{\pi^{2n+2}} \Gamma(n+1)^2 L_0(n+1)\,. \tag{8.44}$$

Starting again from Eq. (8.38) and recalling that the gamma function has simple poles at the non-positive integers, we set $L_1(-n) = 0$ for $n \in \mathbb{Z}_{>0}$ odd and $L_0(-n) = 0$ for $n \in \mathbb{Z}_{\geq 0}$ even. Therefore, putting together Eqs. (8.40a), (8.43), and (8.44), we find that

$$\begin{aligned}
\frac{1}{n!} L_+(-n) &= \begin{cases} -\dfrac{2^{-2n-1}}{\pi^{2n+2}} n!\, L_1(n+1) = -\dfrac{2^{-2n-1}}{\pi^{2n+2}} n! \displaystyle\sum_{m=1}^{\infty} \frac{A_m - A_{-m}}{m^{n+1}} & \text{if} \quad n \in \mathbb{Z}_{\geq 0} \text{ even} \\[2ex] \dfrac{2^{-2n-1}}{\pi^{2n+2}} n!\, L_0(n+1) = \dfrac{2^{-2n-1}}{\pi^{2n+2}} n! \displaystyle\sum_{m=1}^{\infty} \frac{A_m + A_{-m}}{m^{n+1}} & \text{if} \quad n \in \mathbb{Z}_{>0} \text{ odd} \end{cases} \\
&= \frac{2^{-2n-1}}{\pi^{2n+2}} n! \left(\sum_{m=1}^{\infty} \frac{A_m}{(-m)^{n+1}} + \sum_{m=1}^{\infty} \frac{A_{-m}}{m^{n+1}} \right) \\
&= \frac{2^{-2n-1}}{\pi^{2n+2}} n! \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{(-m)^{n+1}}\,.
\end{aligned} \tag{8.45}$$

Analogously, putting together Eqs. (8.40b), (8.43), and (8.44), we find that

$$\begin{aligned}\frac{1}{n!}L_-(-n) &= \begin{cases} -\frac{2^{-2n-1}}{\pi^{2n+2}} n!\, L_1(n+1) = -\frac{2^{-2n-1}}{\pi^{2n+2}} n! \sum_{m=1}^{\infty} \frac{A_m - A_{-m}}{m^{n+1}} & \text{if} \quad n \in \mathbb{Z}_{\geq 0} \text{ even} \\ -\frac{2^{-2n-1}}{\pi^{2n+2}} n!\, L_0(n+1) = -\frac{2^{-2n-1}}{\pi^{2n+2}} n! \sum_{m=1}^{\infty} \frac{A_m + A_{-m}}{m^{n+1}} & \text{if} \quad n \in \mathbb{Z}_{>0} \text{ odd} \end{cases} \\ &= -\frac{2^{-2n-1}}{\pi^{2n+2}} n! \left(\sum_{m=1}^{\infty} \frac{A_m}{m^{n+1}} + \sum_{m=1}^{\infty} \frac{A_{-m}}{(-m)^{n+1}} \right) \\ &= -\frac{2^{-2n-1}}{\pi^{2n+2}} n! \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{m^{n+1}} \,. \end{aligned} \tag{8.46}$$

To sum up, substituting Eq. (8.45) into Eq. (8.41) yields

$$\begin{aligned} \tilde{f}_+(y) &= 2 \sum_{n=0}^{\infty} n! \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{(-4\pi^2 m)^{n+1}} (2\pi \mathrm{i})^n y^n \\ &= \sum_{n=0}^{\infty} \left(\frac{n!}{\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{(2\pi \mathrm{i} m)^{n+1}} \right) y^n \,, \end{aligned} \tag{8.47}$$

which is a Gevrey-1 asymptotic series. Using Proposition 8.1.1, we conclude that the Borel transform of $y\tilde{f}_+(y)$ has a modular resurgent structure with simple poles at $\rho_m = 2\pi \mathrm{i} m$, $m \in \mathbb{Z}_{\neq 0}$, and Stokes constants $2A_m$. Analogously, substituting Eq. (8.46) into Eq. (8.42) gives

$$\begin{aligned} \tilde{f}_-(y) &= -2 \sum_{n=0}^{\infty} n! \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{(4\pi^2 m)^{n+1}} (2\pi \mathrm{i})^n y^n \\ &= -\sum_{n=0}^{\infty} \left(\frac{n!}{\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_{-m}}{(2\pi \mathrm{i} m)^{n+1}} \right) y^n \,, \end{aligned} \tag{8.48}$$

which is a Gevrey-1 asymptotic series. Once more, using Proposition 8.1.1, we conclude that the Borel transform of $y\tilde{f}_-(y)$ has a modular resurgent structure with simple poles at $\rho_m = 2\pi \mathrm{i} m$, $m \in \mathbb{Z}_{\neq 0}$, and Stokes constants $-2A_{-m}$. □

Observe that the class of examples presented in Proposition 8.3.1 fits into the simplified diagram in Eq. 8.18. Additionally, the generating series of the complex numbers A_m, $m \in \mathbb{Z}_{\neq 0}$, satisfies Conjecture 3.

Theorem 8.3.2. *Within the assumptions of Proposition 8.3.1, the generating function $f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined in Eq. (8.39) restricted to either the upper or lower half of the complex plane is a weight-1 holomorphic quantum modular form for the group $\mathsf{SL}_2(\mathbb{Z})$.*

Proof. Recall from part (*d*) of Theorem 1 in [173, Chapter 1] that there exists an analytic function $\psi \colon \mathbb{C}' \to \mathbb{C}$ such that[6]

$$c\,\psi(y) = f(y) - \frac{1}{y} f\left(-\frac{1}{y}\right) \,, \tag{8.49}$$

[6] The function ψ in Eq. (8.49) is the so-called *period* of the function f in Eq. (8.39) and satisfies a three-terms functional equation. See [173] for details.

for some constant $c \in \mathbb{C}^*$. Moreover, recall that the generators of the modular group $\mathsf{SL}_2(\mathbb{Z})$ are the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8.50}$$

Hence, if we restrict the generating function to the upper half-plane, *i.e.*, we take $f\colon \mathbb{H} \to \mathbb{C}$, the cocycle for T is trivial due to the periodicity property $f(y+1) = f(y)$, that is,

$$h_T[f](y) = 0\,, \tag{8.51}$$

while the cocycle for S is given by

$$h_S[f](y) = -c\,\psi(y)\,. \tag{8.52}$$

The same arguments apply to the restriction of the generating function to the lower half-plane, *i.e.*, for $f\colon \mathbb{H}_- \to \mathbb{C}$. □

With an extra assumption on the parity of the coefficients A_m, $m \in \mathbb{Z}_{\neq 0}$, appearing in the definition of the L-functions $L_\epsilon(s)$, $\epsilon = 0, 1$, in Eq. (8.36), we prove the effectiveness of the median resummation for the generating function in Eq. (8.39), which therefore satisfies Conjecture 2. In particular, assuming that $A_m = A_{-m}$, we have that $f(-y) = -f(y)$ for $y \in \mathbb{C} \setminus \mathbb{R}$, so that $f(y)$ need only be specified in the upper half of the complex y-plane.

Theorem 8.3.3. *Within the assumptions of Proposition 8.3.1, if $A_m = A_{-m}$ for every $m \in \mathbb{Z}_{\neq 0}$, the generating function $f\colon \mathbb{H} \to \mathbb{C}$ defined in Eq. (8.39) is recovered from its asymptotic expansion $\tilde{f}_+(y)$ as $y \to 0$ with $\Im(y) > 0$ through the median resummation. Precisely,*

$$y f(y) = \mathcal{S}^{\mathbf{med}}_{\frac{\pi}{2}}\big[y\tilde{f}_+(y)\big]\,, \quad y \in \mathbb{H}\,. \tag{8.53}$$

Proof. Due to the parity of the coefficients, $L_1(s)$ is identically zero, while

$$L_0(s) = 2\sum_{m=1}^{\infty} \frac{A_m}{m^s}\,, \tag{8.54}$$

so that the two L-functions in Eqs. (8.40a) and (8.40b) reduce to

$$L_\pm(s) = \frac{L_0(s)}{2}\,. \tag{8.55}$$

As a consequence of Proposition 8.3.1, and particularly Eqs. (8.47) and (8.48), the formal power series $\tilde{f}_\pm(y)$ can be written as

$$\tilde{f}_\pm(y) = \pm\frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} (2n+1)! \sum_{m=1}^{\infty} \frac{A_m}{\rho_m^{2n+2}} y^{2n+1}\,, \tag{8.56}$$

where $\rho_m = 2\pi \mathrm{i} m$ as before. Recall that the median resummation along the Stokes ray at angle $\pi/2$ can be expressed in terms of the Borel–Laplace sums at angles 0 and π using Eq. (1.29). Specifically,

$$\mathcal{S}^{\mathbf{med}}_{\frac{\pi}{2}}\big[y\tilde{f}_+(y)\big] = \begin{cases} s_0\big(y\tilde{f}_+(y)\big) + \frac{1}{2}\mathrm{disc}_{\frac{\pi}{2}}\big[y\tilde{f}_+(y)\big]\,, & \Re(y) > 0\,, \\ s_\pi\big(y\tilde{f}_+(y)\big) - \frac{1}{2}\mathrm{disc}_{\frac{\pi}{2}}\big[y\tilde{f}_+(y)\big]\,, & \Re(y) < 0\,, \end{cases} \tag{8.57}$$

for $\Im(y) > 0$. To compute the Borel–Laplace sums of $\tilde{f}_+(y)$, let us first evaluate its Borel transform as in the proof of Proposition 8.1.1, that is,

$$\mathcal{B}[y\tilde{f}_+(y)](\zeta) = \frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{A_m}{\rho_m^{2n+2}} \zeta^{2n+1} = -\frac{1}{\pi \mathrm{i}} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta - \rho_m}\,, \tag{8.58}$$

where we have exchanged the order of summation, resummed the geometric series over the index n, and applied the identity in Eq. (7.7) in the second step. Thus, the Borel–Laplace sum at angle 0 is

$$\begin{aligned} s_0(y\tilde{f}_+(y)) &= -\frac{1}{\pi \mathrm{i}} \int_0^{\infty} \mathrm{e}^{-\zeta/y} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta - \rho_m} d\zeta \\ &= -2 \sum_{m \in \mathbb{Z}_{\neq 0}} A_m \mathbf{e}_1\left(-\frac{m}{y}\right), \quad y > 0\,, \end{aligned} \tag{8.59}$$

while the Borel–Laplace sum at angle π is

$$\begin{aligned} s_\pi(y\tilde{f}_+(y)) &= -\frac{1}{\pi \mathrm{i}} \int_0^{-\infty} \mathrm{e}^{-\zeta/y} \sum_{m \in \mathbb{Z}_{\neq 0}} \frac{A_m}{\zeta - \rho_m} d\zeta \\ &= -2 \sum_{m \in \mathbb{Z}_{\neq 0}} A_m \mathbf{e}_1\left(-\frac{m}{y}\right), \quad y < 0\,, \end{aligned} \tag{8.60}$$

where the exponential integral $\mathbf{e}_1 \colon \mathbb{C}\backslash \mathrm{i}\mathbb{R}_{\geq 0} \to \mathbb{C}$ is defined in Eq. (7.1). Note that the Borel–Laplace sums above can be analytically continued to $\Re(y) > 0$ and $\Re(y) < 0$, respectively, as a consequence of the properties of the function $\mathbf{e}_1$. Besides, by the standard residue argument, the discontinuity across the Stokes ray at angle $\pi/2$ is

$$\mathrm{disc}_{\frac{\pi}{2}}\left[y\tilde{f}_+(y)\right] = 2 \sum_{m=1}^{\infty} A_m \mathrm{e}^{-2\pi \mathrm{i} m/y} = 2f\left(-\frac{1}{y}\right), \quad \Im(y) > 0\,. \tag{8.61}$$

We can then apply the results of Section 3, Chapter II of [173], which uses results from the theory of even Maass cusp forms. In particular, adopting the notation of [173, Eqs. (2.11) and (2.13)], we define the function $\psi_1 \colon \mathbb{C} \setminus \mathrm{i}\mathbb{R}_{\geq 0} \to \mathbb{C}$ as

$$\psi_1(y) := 2\pi \mathrm{i} \sum_{m \in \mathbb{Z}_{\neq 0}} A_m \mathbf{e}_1(my)\,, \tag{8.62}$$

which satisfies the properties [173, Eq. (2.10)]

$$y\psi_1(y) = \psi_1\left(\frac{1}{y}\right), \quad \psi_1(-y) = \psi_1(y)\,, \tag{8.63}$$

where the first formula holds for $\Re(y) > 0$. Furthermore, we have that [173, Eq. (2.20)][7]

$$c_*\, f(y) = \psi_1(y) + \frac{1}{y}\psi_1\left(-\frac{1}{y}\right), \tag{8.64}$$

[7] The same formula in Eq. (8.64) holds for $y \in \mathbb{H}_-$ with a change of sign in the constant c'.

for some constant $c_* \in \mathbb{C}^*$. Let us set the constant $c_* = -2\pi\mathrm{i}$. Using Eqs. (8.64) and (8.63), we deduce that

$$yf(y) = \begin{cases} -\frac{1}{\pi\mathrm{i}}\psi_1\left(-\frac{1}{y}\right) + f\left(-\frac{1}{y}\right), & \Re(y) > 0\,, \\ -\frac{1}{\pi\mathrm{i}}\psi_1\left(-\frac{1}{y}\right) - f\left(-\frac{1}{y}\right), & \Re(y) < 0\,. \end{cases} \tag{8.65}$$

It follows from Eqs. (8.62), (8.59), and (8.60) that we can identify $\psi_1(-1/y)$ with the Borel–Laplace sum of $y\tilde{f}_+(y)$ along the positive real axis for $\Re(y) > 0$, that is,

$$-\frac{1}{\pi\mathrm{i}}\psi_1\left(-\frac{1}{y}\right) = s_0\big(y\tilde{f}_+(y)\big)\,, \quad \Re(y) > 0\,, \tag{8.66}$$

and with the Borel–Laplace sum of $y\tilde{f}_+(y)$ along the negative real axis for $\Re(y) < 0$, that is,

$$-\frac{1}{\pi\mathrm{i}}\psi_1\left(-\frac{1}{y}\right) = s_\pi\big(y\tilde{f}_+(y)\big)\,, \quad \Re(y) < 0\,. \tag{8.67}$$

Substituting the discontinuity in Eq. (8.61) and the Borel–Laplace sums in Eqs. (8.66) and (8.67) into the formula for the median resummation in Eq. (8.57), we find the same expression in Eq. (8.65), thus proving the desired statement. □

A result analogous to Theorem 8.3.3 applies for an odd sequence of complex numbers $A_m = -A_{-m}$, $m \in \mathbb{Z}_{\neq 0}$. In this case, the generating function in Eq. (8.39) satisfies $f(-y) = f(y)$ for $y \in \mathbb{C} \setminus \mathbb{R}$, while $L_1(s) = 2\sum_{m=1}^{\infty} \frac{A_m}{m^s}$ is the only non-trivial L-function. Moreover, we expect a similar result for Maass cusps forms that are invariant under a discrete subgroup of $\mathsf{PSL}_2(\mathbb{Z})$.

Remark 8.3.2. *The example of even Maass cusp forms discussed here displays an interesting feature. Indeed, the cocycle $\psi(y)$ in Eq. (8.49) is proportional to the function $\psi_1(y)$ in Eq. (8.62) by means of Eq. (8.64). Consequently, it follows from Theorems 8.3.2 and 8.3.3 that the Borel–Laplace sums $s_0\big(y\tilde{f}_\pm(y)\big)$ are proportional to $\psi(-1/y)$. We expect this property to hold more generally when both Conjectures 2 and 3 (or, equivalently, Conjectures 4 and 5) are verified—potentially, under some additional assumptions on the q-series. In other words, we expect the cocycles of (a well-defined class of) q-series whose asymptotic expansions have a modular resurgent structure to occur as Borel–Laplace sums.*

Part IV

Appendices

Appendix A

Wigner transform of the inverse operator

Let O be a quantum-mechanical operator acting on $L^2(\mathbb{R})$. We describe how to obtain the WKB expansion in phase space of the inverse operator $\rho = \mathsf{O}^{-1}$ at NLO in the semiclassical limit $\hbar \to 0$. See [174] for an introduction to the phase-space formulation of quantum mechanics. The *Wigner transform* of the operator O is defined as

$$O_{\mathrm{W}}(x,y) = \int_{\mathbb{R}} \mathrm{e}^{\frac{\mathrm{i}yx'}{\hbar}} \left\langle x - \frac{x'}{2} \middle| \mathsf{O} \middle| x + \frac{x'}{2} \right\rangle dx' \,, \tag{A.1}$$

where x, $y \in \mathbb{R}$ are the phase-space coordinates, and the diagonal element of O in the coordinate representation is given by

$$\langle x | \, \mathsf{O} \, | x \rangle = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} O_{\mathrm{W}}(x,y) \, dy \,. \tag{A.2}$$

The trace of O is obtained by integrating its Wigner transform over phase space, that is,

$$\mathrm{Tr}(\mathsf{O}) = \int_{\mathbb{R}} \langle x | \, \mathsf{O} \, | x \rangle \; dx = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} O_{\mathrm{W}}(x,y) \, dx dy \,. \tag{A.3}$$

Example A.0.1. *Let us consider the quantum-mechanical operator*

$$\mathsf{O} = \mathrm{e}^{m\mathsf{x}+n\mathsf{y}} = \mathrm{e}^{-\mathrm{i}\hbar mn/2} \mathrm{e}^{m\mathsf{x}} \mathrm{e}^{n\mathsf{y}} \,, \quad m, n \in \mathbb{Z} \,, \tag{A.4}$$

where x, y *are the Heisenberg operators corresponding to the classical variables* x, y *and satisfying* $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$*, and the second equality follows from the Baker–Campbell–Hausdorff formula. The Wigner transform in Eq.* (A.1) *becomes*

$$\begin{aligned} O_{\mathrm{W}}(x,y) &= \mathrm{e}^{-\mathrm{i}\hbar mn/2} \int_{\mathbb{R}} dx' \mathrm{e}^{\frac{\mathrm{i}yx'}{\hbar}} \left\langle x - \frac{x'}{2} \middle| \mathrm{e}^{m\mathsf{x}} \mathrm{e}^{n\mathsf{y}} \middle| x + \frac{x'}{2} \right\rangle \\ &= \mathrm{e}^{-\mathrm{i}\hbar mn/2} \int_{\mathbb{R}} dx' \mathrm{e}^{\frac{\mathrm{i}yx'}{\hbar}} \mathrm{e}^{m(x-x'/2)} \left\langle x - \frac{x'}{2} \middle| \mathrm{e}^{n\mathsf{y}} \middle| x + \frac{x'}{2} \right\rangle \\ &= \mathrm{e}^{-\mathrm{i}\hbar mn/2} \mathrm{e}^{mx} \int_{\mathbb{R}^2} \frac{dx'dy'}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}(y-y')x'}{\hbar}} \mathrm{e}^{-mx'/2} \mathrm{e}^{ny'} \,. \end{aligned} \tag{A.5}$$

After performing the change of variable $u = x'/\hbar$, and Taylor expanding the exponential factor $\mathrm{e}^{-mu\hbar/2}$ around $\hbar = 0$, we get

$$\begin{aligned} O_\mathrm{W}(x,y) &= \mathrm{e}^{-\mathrm{i}\hbar mn/2}\mathrm{e}^{mx}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\mathrm{i}\hbar m}{2}\right)^k\int_{\mathbb{R}}dy'\mathrm{e}^{ny'}\int_{\mathbb{R}}\frac{du}{2\pi}\mathrm{e}^{\mathrm{i}(y-y')u}(-\mathrm{i}u)^k \\ &= \mathrm{e}^{-\mathrm{i}\hbar mn/2}\mathrm{e}^{mx}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\mathrm{i}\hbar m}{2}\right)^k\int_{\mathbb{R}}dy'\mathrm{e}^{ny'}\delta^{(k)}(y-y') \\ &= \mathrm{e}^{-\mathrm{i}\hbar mn/2}\mathrm{e}^{mx}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}\hbar mn}{2}\right)^k\mathrm{e}^{ny} = \mathrm{e}^{mx+ny}\,, \end{aligned} \tag{A.6}$$

where $\delta^{(k)}$ denotes the k-th derivative of the Dirac delta function.

We recall now the definition of the *Moyal $\star$-product* of two quantum operators A, B acting on $L^2(\mathbb{R})$, that is,

$$\mathsf{A}\star\mathsf{B} = A_\mathrm{W}(x,y)\,\exp\left[\frac{\mathrm{i}\hbar}{2}\overleftrightarrow{\Lambda}\right]B_\mathrm{W}(x,y)\,, \tag{A.7}$$

where $\overleftrightarrow{\Lambda} = \overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}$, and the arrows indicate the direction in which the derivatives act. Expanding around $\hbar = 0$, we get

$$\mathsf{A}\star\mathsf{B} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n\partial_x^m\partial_y^{n-m}A_\mathrm{W}(x,y)\,\partial_y^m\partial_x^{n-m}B_\mathrm{W}(x,y)\,. \tag{A.8}$$

Theorem A.0.2. *The Wigner transform of the inverse operator $\rho = \mathsf{O}^{-1}$ can be expressed in terms of O_W as*

$$\rho_\mathrm{W} = \sum_{r=0}^{\infty}(-1)^r\frac{\mathcal{G}_r}{O_\mathrm{W}^{r+1}}\,, \tag{A.9}$$

where the quantities

$$\mathcal{G}_r = [(\mathsf{O} - O_\mathrm{W}(x,y))^r]_\mathrm{W} \tag{A.10}$$

are evaluated at the same point $(x,\,y)$ in phase space.

We can compute the functions in Eq. (A.10) explicitly by using the Moyal $\star$-product in Eq. (A.7) and expand them in formal power series in $\hbar$. More precisely, the first four functions are $\mathcal{G}_0 = 1$, $\mathcal{G}_1 = 0$,

$$\mathcal{G}_2 = O_\mathrm{W}\star O_\mathrm{W} - O_\mathrm{W}^2 = -\frac{\hbar^2}{4}\left[\frac{\partial^2 O_\mathrm{W}}{\partial x^2}\frac{\partial^2 O_\mathrm{W}}{\partial y^2} - \left(\frac{\partial^2 O_\mathrm{W}}{\partial x\partial y}\right)^2\right] + \mathcal{O}(\hbar^4)\,, \tag{A.11a}$$

$$\begin{aligned} \mathcal{G}_3 &= O_\mathrm{W}\star O_\mathrm{W}\star O_\mathrm{W} - 3(O_\mathrm{W}\star O_\mathrm{W})O_\mathrm{W} + 2O_\mathrm{W}^3 \\ &= -\frac{\hbar^2}{4}\left[\left(\frac{\partial O_\mathrm{W}}{\partial x}\right)^2\frac{\partial^2 O_\mathrm{W}}{\partial y^2} + \frac{\partial^2 O_\mathrm{W}}{\partial x^2}\left(\frac{\partial O_\mathrm{W}}{\partial y}\right)^2 - 2\frac{\partial O_\mathrm{W}}{\partial x}\frac{\partial O_\mathrm{W}}{\partial y}\frac{\partial^2 O_\mathrm{W}}{\partial x\partial y}\right] + \mathcal{O}(\hbar^4)\,. \end{aligned} \tag{A.11b}$$

It follows then from Eq. (A.9) that the Wigner transform of ρ is obtained up to order $\hbar^2$ by substituting Eqs. (A.11a) and (A.11b) into

$$\rho_\mathrm{W} = \frac{1}{O_\mathrm{W}} + \frac{\mathcal{G}_2}{O_\mathrm{W}^3} - \frac{\mathcal{G}_3}{O_\mathrm{W}^4} + \ldots \tag{A.12}$$

We note that the same result can be obtained using the properties of the $\star$-product only. Indeed, the identity

$$\rho_{\rm W} \star O_{\rm W} = O_{\rm W} \star \rho_{\rm W} = 1\,, \tag{A.13}$$

together with the definition in Eq. (A.7), implies that

$$\rho_{\rm W} \cos\left[\frac{\hbar}{2}\overleftrightarrow{\Lambda}\right] O_{\rm W} = 1\,. \tag{A.14}$$

Expanding Eq. (A.14) in powers of $\hbar^2$, we have

$$\rho_{\rm W} O_{\rm W} - \frac{\hbar^2}{8}\rho_{\rm W}\left(\overleftrightarrow{\Lambda}\right)^2 O_{\rm W} + \mathcal{O}(\hbar^4) = 1\,, \tag{A.15}$$

where

$$\rho_{\rm W}\left(\overleftrightarrow{\Lambda}\right)^2 O_{\rm W} = \frac{\partial^2\rho_{\rm W}}{\partial x^2}\frac{\partial^2 O_{\rm W}}{\partial y^2} + \frac{\partial^2\rho_{\rm W}}{\partial y^2}\frac{\partial^2 O_{\rm W}}{\partial x^2} - 2\frac{\partial^2\rho_{\rm W}}{\partial x\partial y}\frac{\partial^2 O_{\rm W}}{\partial x\partial y}\,, \tag{A.16}$$

and solving order by order in $\hbar$, we find the Wigner transform of ρ at NLO in the limit $\hbar \to 0$. Note that the formalism described here can be used to systematically extract the expansion up to any order by extending all intermediate computations beyond order $\hbar^2$.

Appendix B

Quantum dilogarithms

We call *quantum dilogarithm* the function of two variables defined by the series [175, 176]

$$(xq^{\alpha};\, q)_{\infty} = \prod_{n=0}^{\infty} (1 - xq^{\alpha+n})\,, \quad \alpha \in \mathbb{R}\,, \tag{B.1}$$

which is analytic in $x, q \in \mathbb{C}$ with $|q| < 1$ and has asymptotic expansions around q a root of unity. It satisfies the relation

$$(x;\, q^{-1})_{\infty} = (xq;\, q)_{\infty}^{-1}\,. \tag{B.2}$$

The *q-Pochhammer symbols*, also known as q-shifted factorials, are denoted by

$$(x;\, q)_m = \prod_{n=0}^{m-1} (1 - xq^n)\,, \quad (x;\, q)_{-m} = \frac{1}{(xq^{-m};\, q)_m}\,, \quad m \in \mathbb{Z}_{>0}\,, \tag{B.3}$$

with $(x;\, q)_0 = 1$. Equivalently, we can write

$$(x;\, q)_m = \frac{(x;\, q)_{\infty}}{(xq^m;\, q)_{\infty}}\,, \quad m \in \mathbb{Z}\,. \tag{B.4}$$

Moreover, the (unilateral) *q-hypergeometric series*, also called (unilateral) basic hypergeometric series, is

$${}_{r+1}\phi_s \begin{pmatrix} a_0, & a_1, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{pmatrix}; q,\, x \Bigg) = \sum_{n=0}^{\infty} \frac{(a_0;\, q)_n (a_1;\, q)_n \dots (a_r;\, q)_n}{(q;\, q)_n (b_1;\, q)_n \dots (b_s;\, q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r} x^n\,, \tag{B.5}$$

where $a_i, b_j, x \in \mathbb{C}$, $r, s \in \mathbb{N}$.

The *Faddeev's quantum dilogarithm* [175] $\Phi_{\mathsf{b}}(x)$ is defined in the strip $|\Im(x)| < |\Im(c_{\mathsf{b}})|$, where

$$c_{\mathsf{b}} = \mathrm{i}(\mathsf{b} + \mathsf{b}^{-1})/2\,, \tag{B.6}$$

by the integral representation

$$\Phi_{\mathsf{b}}(x) = \exp\left(\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{e}^{-2\mathrm{i}xz}}{4\sinh(z\mathsf{b})\sinh(z\mathsf{b}^{-1})} \frac{dz}{z} \right)\,, \tag{B.7}$$

which implies the symmetry properties

$$\Phi_{\mathsf{b}}(x) = \Phi_{-\mathsf{b}}(x) = \Phi_{\mathsf{b}^{-1}}(x)\,. \tag{B.8}$$

We recall some of the main properties of Faddeev's quantum dilogarithm here and refer to [165, Appendix A] and [169, Section 8.10] for detailed expositions. When $\Im(\mathsf{b}^2) > 0$, the formula in Eq. (B.7) is equivalent to

$$\Phi_{\mathsf{b}}(x) = \frac{(\mathrm{e}^{2\pi\mathsf{b}(x+c_{\mathsf{b}})};\, q)_\infty}{(\mathrm{e}^{2\pi\mathsf{b}^{-1}(x-c_{\mathsf{b}})};\, \tilde{q})_\infty} = \prod_{n=0}^{\infty} \frac{1-\mathrm{e}^{2\pi\mathsf{b}(x+c_{\mathsf{b}})}q^n}{1-\mathrm{e}^{2\pi\mathsf{b}^{-1}(x-c_{\mathsf{b}})}\tilde{q}^n}\,, \tag{B.9}$$

where

$$q = \mathrm{e}^{2\pi\mathrm{i}\mathsf{b}^2}\,, \quad \tilde{q} = \mathrm{e}^{-2\pi\mathrm{i}\mathsf{b}^{-2}}\,. \tag{B.10}$$

Note that the function in Eq. (B.9) can be extended to the region $\Im(\mathsf{b}^2) < 0$ by means of Eq. (B.8) and further admits an analytic continuation to all values of b such that $\mathsf{b}^2 \notin \mathbb{R}_{\leq 0}$. Moreover, $\Phi_{\mathsf{b}}(x)$ can be extended to the whole complex x-plane as a meromorphic function with an essential singularity at infinity, poles at the points

$$x = c_{\mathsf{b}} + \mathrm{i}m\mathsf{b} + \mathrm{i}n\mathsf{b}^{-1}\,, \tag{B.11}$$

and zeros at the points

$$x = -c_{\mathsf{b}} - \mathrm{i}m\mathsf{b} - \mathrm{i}n\mathsf{b}^{-1}\,, \tag{B.12}$$

for $m, n \in \mathbb{N}$. It satisfies the inversion formula

$$\Phi_{\mathsf{b}}(x)\Phi_{\mathsf{b}}(-x) = \mathrm{e}^{\pi\mathrm{i}x^2}\Phi_{\mathsf{b}}(0)^2\,, \quad \Phi_{\mathsf{b}}(0) = \left(\frac{q}{\tilde{q}}\right)^{1/48} = \mathrm{e}^{\pi\mathrm{i}(\mathsf{b}^2+\mathsf{b}^{-2})/24}\,, \tag{B.13}$$

the complex conjugation formula

$$\overline{\Phi_{\mathsf{b}}(x)} = \frac{1}{\Phi_{\overline{\mathsf{b}}}(\overline{x})}\,, \tag{B.14}$$

and the quasi-periodicity relations

$$\Phi_{\mathsf{b}}(x \pm \mathrm{i}\mathsf{b}) = \Phi_{\mathsf{b}}(x)\left(1+\mathrm{e}^{2\pi\mathsf{b}x\pm\pi\mathrm{i}\mathsf{b}^2}\right)^{\mp 1}\,, \tag{B.15a}$$

$$\Phi_{\mathsf{b}}(x \pm \mathrm{i}\mathsf{b}^{-1}) = \Phi_{\mathsf{b}}(x)\left(1+\mathrm{e}^{2\pi\mathsf{b}^{-1}x\pm\pi\mathrm{i}\mathsf{b}^{-2}}\right)^{\mp 1}\,. \tag{B.15b}$$

In addition, when $\mathsf{b}^2 = M/N \in \mathbb{Q}$ with $M, N \in \mathbb{Z}_{>0}$ coprime, the expression for $\Phi_{\mathsf{b}}(x)$ in Eq. (B.9) simplifies into [177, Theorem 1.9]

$$\Phi_{\mathsf{b}}\left(\frac{x}{2\pi\sqrt{MN}} - c_{\mathsf{b}}\right) = \frac{\mathrm{e}^{\frac{\mathrm{i}}{2\pi MN}\mathrm{Li}_2(\mathrm{e}^x)}\,(1-\mathrm{e}^x)^{1+\frac{\mathrm{i}x}{2\pi MN}}}{D_N(\mathrm{e}^{x/N};\, \mathrm{e}^{2\pi\mathrm{i}M/N})\, D_M(\mathrm{e}^{x/M};\, \mathrm{e}^{2\pi\mathrm{i}N/M})}\,, \tag{B.16}$$

where

$$D_N(z;\, q) = \prod_{k=1}^{N-1}(1-zq^k)^{\frac{k}{N}} \tag{B.17}$$

is the cyclic quantum dilogarithm.[1]

Finally, we recall the known asymptotic behavior of Faddeev's quantum dilogarithm. We refer to Chapter 1 for an introduction to Écalle's theory of resurgence. In the limit of $\mathsf{b} \to 0$ with $\Im(\mathsf{b}^2) > 0$, $\Phi_{\mathsf{b}}(x)$ gives the asymptotic series [165]

$$\log \Phi_{\mathsf{b}}\left(\frac{x}{2\pi \mathsf{b}}\right) \sim \sum_{k=0}^{\infty} (2\pi \mathsf{i} \mathsf{b}^2)^{2k-1} \frac{B_{2k}(1/2)}{(2k)!} \mathrm{Li}_{2-2k}(-\mathrm{e}^x)\,, \tag{B.18}$$

where x is kept fixed, $\mathrm{Li}_n(z)$ is the polylogarithm of order n, and $B_n(z)$ is the n-th Bernoulli polynomial. Similarly, for $\mathsf{b} \to 0$ with $\Im(\mathsf{b}^2) > 0$ and x fixed, the following special cases of the quantum dilogarithm in Eq. (B.1) have the asymptotic expansions [178]

$$\log(x;\, q)_\infty \sim \frac{1}{2}\log(1-x) + \sum_{k=0}^{\infty} (2\pi \mathsf{i} \mathsf{b}^2)^{2k-1} \frac{B_{2k}}{(2k)!} \mathrm{Li}_{2-2k}(x)\,, \tag{B.19a}$$

$$\begin{aligned}\log(q^\alpha;\, q)_\infty \sim &-\frac{\pi \mathsf{i}}{12\mathsf{b}^2} - B_1(\alpha)\log(-2\pi \mathsf{i} \mathsf{b}^2) - \log\frac{\Gamma(\alpha)}{\sqrt{2\pi}} \\ &- B_2(\alpha)\frac{\pi \mathsf{i} \mathsf{b}^2}{2} - \sum_{k=2}^{\infty} (2\pi \mathsf{i} \mathsf{b}^2)^k \frac{B_k B_{k+1}(\alpha)}{k(k+1)!}\,, \quad \alpha > 0\,,\end{aligned} \tag{B.19b}$$

where $B_n = B_n(0)$ is the n-th Bernoulli number and $\Gamma(\alpha)$ is the gamma function. Besides, $\Phi_{\mathsf{b}}(x)$ is Borel–Laplace summable [179, Theorem 1.3].

[1] We use the simplified formula for Faddeev's quantum dilogarithm in Eq. (B.16) to perform numerical checks at rational points of the results presented in Section 7.2.

Appendix C

Hadamard's multiplication theorem

We briefly recall the content of *Hadamard's multiplication theorem* [155, 156] following the introduction by Titchmarsh [180].

Theorem C.0.1. *Consider the two formal power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \,, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \,, \tag{C.1}$$

and suppose that $f(z), g(z)$ are convergent for $|z| < R, R'$, respectively, where $R, R' \in \mathbb{R}_{>0}$, and that their singularities in the complex z-plane are known. Let us denote by $\{\alpha_i\}_{i\in I}$ the set of singularities of $f(z)$ and by $\{\beta_j\}_{j\in J}$ the set of singularities of $g(z)$, where I, J are countable sets. We introduce the formal power series

$$F(z) = (f \diamond g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \,, \tag{C.2}$$

also known as the Hadamard product *of the given series $f(z)$ and $g(z)$, which we denote with the symbol $\diamond$. Then, $F(z)$ has a finite radius of convergence $r > RR'$, and its singularities belong to the set $\{\alpha_i\beta_j\}_{i\in I,\, j\in J}$ of products of the singular points of $f(z)$ and $g(z)$. Furthermore, $F(z)$ admits the integral representation*

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(s) g(z/s) \frac{ds}{s} \,, \tag{C.3}$$

where γ is a closed contour encircling the origin $s = 0$ on which

$$|s| < R \,, \quad \left|\frac{z}{s}\right| < R' \,. \tag{C.4}$$

Let us conclude by citing two results of [181] that we use in the explicit resummation of the Hadamard factors considered in Chapters 5 and 7. Namely,

$$\sum_{k=1}^{\infty} \frac{\zeta(2k, a)}{2k} z^{2k} = \frac{1}{2} \log\left(\Gamma(a-z)\Gamma(a+z)\right) - \log\Gamma(a) \,, \quad |z| < |a| \,, \tag{C.5a}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1, a)}{2k+1} z^{2k+1} = \frac{1}{2} \log\left(\frac{\Gamma(a-z)}{\Gamma(a+z)}\right) + z\Psi(a) \,, \quad |z| < |a| \,, \tag{C.5b}$$

where $\zeta(z, a)$ denotes the Hurwitz zeta function, $\Psi(a)$ denotes the digamma function, $z \in \mathbb{C}$, and $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Appendix D

Quantum modular forms

A first description of *quantum modular forms* was given by Zagier in [18]. Namely, a function $f\colon \mathbb{Q} \to \mathbb{C}$ is called a weight-ω quantum modular form with respect to a subgroup $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$, where ω is a fixed integer or half-integer, if the cocycle

$$h_\gamma[f](y) := (cy+d)^{-\omega} f\left(\frac{ay+b}{cy+d}\right) - f(y) \tag{D.1}$$

is *better behaved* than $f(y)$ for all choices of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Conceptually, being better behaved means having better analyticity properties than the function $f(y)$ itself—*e.g.*, being a real analytic function over $\mathbb{R} \setminus \{\gamma^{-1}(\infty)\}$. Quantum modular forms of weight zero are called *quantum modular functions*.

Some examples of quantum modular forms are built from classical modular forms. For instance, Zagier proved in [53] that the function $g\colon \mathbb{Q} \to \mathbb{C}$ defined by

$$g(y) = q^{1/24}\left(1 + \sum_{n=1}^{\infty}(1-q)(1-q^2)\dots(1-q^n)\right), \quad q = \mathrm{e}^{2\pi \mathrm{i} y}, \tag{D.2}$$

which is closely related to the Kashaev invariant for the trefoil knot [13][1], is a weight-3/2 quantum modular form for $\mathsf{SL}_2(\mathbb{Z})$. Its cocycle is written explicitly as

$$h_\gamma[g](y) = \int_{\gamma^{-1}(\infty)}^{\infty} \eta(t)(y-t)^{-3/2} dt, \quad \gamma \in \mathsf{SL}_2(\mathbb{Z}), \tag{D.3}$$

where η is the Dedekind eta function. In particular, the cocycle $h_\gamma[g]$ in Eq. (D.3) is real analytic on $\mathbb{R} \setminus \{\gamma^{-1}(\infty)\}$ and is expressed in terms of the original weight-1/2 modular form η. This is a common feature of those quantum modular forms that arise from a classical counterpart and is manifest in other examples as well. See [18, Examples 1, 3, 4] and [24–26, 55].

A function f that is well-defined and analytic in the upper half of the complex plane, which we denote by $\mathbb{H}$, already has good analyticity properties. Yet, a version of quantum modularity is obtained by requiring that the cocycle $h_\gamma[f]$ in Eq. (D.1) is analytic in a domain larger than $\mathbb{H}$. The quantum modular forms that belong to this class of examples are called *holomorphic quantum modular forms*. The following definition was proposed by Zagier [182, min 21:45].

[1] Other examples arising from the quantum invariants of hyperbolic knots have been discussed in the literature, although their cocycles generally have a more complicated structure [19]. It is unknown whether they are related to classical modular forms.

Definition D.0.1. *An analytic function* $f\colon \mathbb{H} \to \mathbb{C}$ *is a weight-*ω holomorphic quantum modular form *for a subgroup* $\Gamma \subseteq \mathsf{SL}_2(\mathbb{Z})$*, where* ω *is integer or half-integer, if the cocycle* $h_\gamma[f]\colon \mathbb{H} \to \mathbb{C}$ *in Eq.* (D.1) *extends holomorphically to*[2]

$$\mathbb{C}_\gamma = \{y \in \mathbb{C}\colon cy + d \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\} \tag{D.4}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

For instance, let $\Delta\colon \mathbb{H} \to \mathbb{C}$ be the modular discriminant of the Dedekind eta function η, which is given by

$$\Delta(y) = \eta(y)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = \mathrm{e}^{2\pi \mathrm{i} y}. \tag{D.5}$$

It was shown in [18, Example 2] that its Eichler integral $\tilde{\Delta}\colon \mathbb{H} \to \mathbb{C}$, which is

$$\tilde{\Delta}(y) = -\frac{(2\pi\mathrm{i})^{11}}{10!} \int_y^\infty \Delta(t)(y-t)^{10} dt, \tag{D.6}$$

is a weight-10 holomorphic quantum modular form for $\mathsf{SL}_2(\mathbb{Z})$ whose cocycle is known as

$$h_\gamma[\tilde{\Delta}](y) = \frac{(2\pi\mathrm{i})^{11}}{10!} \int_{\gamma^{-1}(\infty)}^\infty \Delta(t)(y-t)^{10} dt, \quad \gamma \in \mathsf{SL}_2(\mathbb{Z}). \tag{D.7}$$

In particular, the cocycle $h_\gamma[\tilde{\Delta}]$ in Eq. (D.7) is a polynomial of degree less than or equal to 10 and therefore holomorphic in $\mathbb{C}$. This is indeed an exceptional example since the cocycle of a holomorphic quantum modular form does not typically extend to the whole complex plane, but rather some cut plane—*e.g.*, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. More general examples can be constructed from the theory of *Maass cusp forms* and their periods, as discussed in Section 8.3.2. Additionally, we show in Section 7.3 that the resurgence of the *spectral trace of local* $\mathbb{P}^2$ is a new source of examples of holomorphic quantum modular forms.

Note that the analog of Definition D.0.1 can be written for a function $f\colon \mathbb{H}_- \to \mathbb{C}$, where $\mathbb{H}_-$ denotes the lower half of the complex plane. Even more generally, a notion of *vector-valued quantum modular form* on $\mathbb{H} \cup \mathbb{Q} \cup \mathbb{H}_-$ was proposed in [169, Definition 22], although we will not use it in this thesis.

[2]Note that $\mathbb{C}_\gamma = \mathbb{C} \setminus (-\infty; -d/c]$ when $c > 0$ and $\mathbb{C}_\gamma = \mathbb{C} \setminus [-d/c; +\infty)$ when $c < 0$.

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