Bootstrapping from a boundary point of view

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Abstract

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In this work, we study two problems in quantum field theory from a boundary point of view. Our perspective is motivated by the bootstrap philosophy, which aims to understand how principles such as kinematics, unitarity, and symmetry constrain physical observables. Regarding kinematics, we actually first relax the unitarity constraint and investigate the non-unitary representations of the boundary superconformal algebra for AdS_4 with $\mathcal{N} = 2$ supercharges. In particular, we identify multiplets containing partially massless (PM) fields, as well as other exotic shortening conditions and structures exclusive to the non-unitary regime. Then, turning on interactions, we study a problem centered in dynamics: we investigate the structure of the flat space wavefunctional in scalar field theories with nonlinearly realized symmetries. In particular, we highlight the so-called exceptional scalar field theories, which are the nonlinear sigma model, Dirac-Born-Infeld, and (special) galileon theories. We find that nonlinearly realized symmetries imply soft theorems which must be obeyed by the wavefunction. Moreover, we develop bootstrap techniques utilizing this information along with the singularity structure of the wavefunction to fix its form. In addition, we systematize this construction into a novel set of recursion relations.

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Chapter 1: Introduction and Background

This thesis is devoted to the investigation of two selected problems in quantum field theory from a boundary perspective. In Sections 1 and 2 of this Chapter, we will introduce and highlight important foundational concepts for quantum field theory in curved backgrounds through an S-matrix bootstrap lens. Overall, we may divide the main contents of this thesis into two parts: kinematics and dynamics. In Section 3 of this Chapter, we will introduce the unitary representations of the de Sitter isometry group (kinematics) and identify an exotic member thereof—the partially massless (PM) field. We will give a general overview of the current state of interacting PM field theories, which frames the content of Chapter 2. In Section 4 of this Chapter, we will give an overview of some specific aspects of the cosmological bootstrap (dynamics), which serves as the background for Chapter 3.

Chapter 2 of this thesis is based primarily on [1], which was completed in collaboration with Rachel Rosen, Kurt Hinterbichler, and Sebastian Garcia-Saenz. In this Chapter we study the non-unitary representations of the superconformal algebra for theories living on the boundary of AdS_4, and identify multiplets containing partially massless fields.

Chapter 3 of this thesis is based upon [2], which was completed in collaboration with Austin Joyce. In this Chapter we investigate the perturbative wavefunction of scalar field theories possessing nonlinearly realized symmetries, and demonstrate how they may be constructed through various bootstrap procedures.

Finally, in Chapter 4 we will conclude and provide a brief outline of future research directions.

1.1 Why the S-matrix bootstrap?

Since the 1930's, particle accelerators have been the workhorse of high energy physics. In such experiments, the S-matrix is the natural observable. Accordingly, theorists have dedicated much time not only to inventing techniques for increasingly intricate and high-precision phenomenological computations,[1] but also to developing a robust and abstracted understanding of the S-matrix itself. The latter in particular famously took off in the 1960's with the advent of S-matrix theory [4]. The aim of this ambitious program was to supplant quantum field theory by seeking a set of axioms obeyed by the non-perturbative S-matrix, along with a new scheme for performing computations. In effect, this would have done away with fields and the Lagrangian altogether. The foundational principles of the program were Lorentz invariance (which we will call kinematics), unitarity, and analyticity. Analyticity in particular was intended to encode causality, crossing, and dispersion relations. Unfortunately, it proved difficult to systematize these rules and perform computations in the resulting framework at the fully non-perturbative level, and the program was subsequently abandoned.

Despite this apparent setback, more recently there has been a modern resurgence in applying some tenets of the S-matrix philosophy to great success at the level of perturbation theory. This is the modern on-shell amplitude bootstrap program (see [5, 6, 7] for reviews), which primarily operates at the perturbative level where the analytic structure of the S-matrix is better understood. In particular, the imprint of unitarity and locality is completely understood in terms of the analytic structure of tree-level scattering amplitudes. This information about amplitudes at tree level has been organized into powerful recursion relations allowing for the efficient computation of amplitudes involving multiple exchanges [8]. The Parke-Taylor amplitude for maximal helicity violating n-point gluon amplitudes is the canonical example of this control [9]. Using modern on-shell techniques, it is possible to

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[1]See [3] for a particularly impressive example which marries a variety of perturbative and non-perturbative techniques.

immediately write down the final amplitude in half-a-line using spinor-helicty variables:

$$A_n\left(1^+...i^-...j^-...n^+\right) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle n1\rangle}\,. \tag{1.1}$$

On the other hand, a brute force perturbative computation requires managing a number of terms in the Feynman diagram expansion that grows exponentially with n.[2] Beyond tree level, there is also growing control over the analytic structure of loop amplitudes, which may be constructed from tree and lower-loop diagrams in a fashion which is simpler than standard Feynman rules. In this vein, the successes of the modern bootstrap program validate some of the noble goals of its S-matrix theory ancestor: avoiding the Lagrangian and Feynman diagrams altogether, which are plagued with redundancies from field redefinitions and gauge invariance, is a powerful computational perspective. In addition to streamlined computational techniques, there is an ever-growing industry of deriving positivity bounds on EFT parameters, which are informed by causality and dispersion relations [10, 11, 12, 13]. Moreover as our understanding of the S-matrix is abstracted, it has revealed deeper hidden and unexpected structures. A striking example of this is color-kinematics duality and the double copy, through which one may view Einstein gravity as the square of Yang-Mills [14] (among many other relations, see [15] for an exhaustive web). There is reason to believe we have only scratched the surface in our exploration of these structures, and there have already been observations of extended notions of duality, such as the recently discovered geometry-kinematics duality pointed out in [16]. Finally, there has also been the recent realization that a scattering amplitude may be thought of as the canonical volume of an abstract geometric object known as a polytope. From this perspective, the seemingly fundamental notions of locality and unitarity arise as outputs rather than inputs [17, 18, 19]. This object also contains each of the "compact" representations of the amplitude, such as those generated by BCFW relations, as different triangulations. Clearly there is much to

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[2]As a concrete example, a brute force computation of the 5-point gluon scattering amplitude requires keeping track of on the order of 10,000 terms.

be explored regarding the structure of scattering amplitudes, even at the perturbative level.

The primary aim of this thesis is to extend some aspects of the S-matrix bootstrap to the maximally symmetric cousins of Minkowski spacetime: Anti de Sitter space and de Sitter space. In each case there is a compelling reason for doing so. Regarding AdS, the original goals of the S-matrix program have more or less been solved in this spacetime for unitary theories due to the AdS/CFT correspondence. Schematically, the CFT axioms are sufficiently constraining as to non-perturbatively define what is meant by a "boundary correlator" in AdS. These are the natural observables replacing the S-matrix (see Section 1.2), which does not exist in AdS. In addition, non-perturbative CFT computations are facilitated by the existence of the operator product expansion (OPE). These rules have been systematized into the so-called conformal bootstrap program (see [20, 21] for reviews): the OPE can be combined with CFT data and crossing symmetry to constrain correlation functions on the boundary of Euclidean AdS, which may then be Wick-rotated back to Lorentzian AdS. In light of the success of the bootstrap program in AdS, there is some optimism that holographic ideas may be imported to flat space [22, 23, 24], giving new insight into the S-matrix from a boundary point of view, with the relevant boundary surface being the celestial sphere at null infinity.

The reason for extending the S-matrix bootstrap to de Sitter space is because like it or not this is the universe we live in. We find ourselves as observers living effectively beyond the future boundary of an approximately de Sitter spacetime (see Figure 1.4), and this region is all we have access to for making inferences about the universe during the inflationary epoch.[3] As in the AdS case, there is no satisfactory S-matrix in de Sitter (though for different reasons), and the natural observable replacing it is the wavefunction of the universe itself, measured on the future boundary.[4] The bootstrap point of view in de Sitter is quite young,

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[3]There is an additional de Sitter phase which is relevant to the evolution of our universe. The current accelerated expansion of our universe has been known of for 15 years [25]. As such we are static patch observers in an asymptotically de Sitter universe.

[4]Strictly speaking, the true observables are equal time correlation functions, or expectation values, on the future boundary. However once the wavefunction is known one may use the standard rules of quantum

and was only recently started in [26, 27] before taking off in [28, 29, 30] and subsequently exploding in popularity [31, 32, 33, 34, 35, 36, 37, 38, 39]. After many initial successes, there is now a steadily growing understanding of how conformal symmetry, locality, and unitarity constrain the wavefunction at tree level, which has allowed for the computation of cosmological quantities that otherwise would have been out of reach [36, 40, 30], along with the verification of some now classic results through a streamlined procedure [35, 30].

Ironically, all of the content of this thesis is inspired by de Sitter physics, but almost none of the novel computations herein will take place there. At a high level, this work may be broken into two parts: kinematics and dynamics. Regarding the first, In Chapter 2 we will essentially solve what is a very nontrivial kinematics problem (knowing what particles may exist in principle is a natural first step in any bootstrap): We derive and classify non-unitary representations of the boundary $\mathcal{N} = 2$ superconformal algebra for AdS_4 and identify multiplets which contain PM fields. As we will describe in more detail later on in Section 1.3, partially massless fields are exotic representations of the (A)dS isometry group which propagate fewer degrees of freedom than is naively expected based on their mass and spin. Essentially, at a particular mass value tied to the cosmological constant, these theories develop additional gauge symmetries which reduce the number of propagating modes. Though we study the problem in AdS, in some sense this is more appropriately thought of as a de Sitter question. This is because partially massless representations are non-unitary in AdS and propagate ghostly degrees of freedom, but are unitary in de Sitter.[5] In Chapter 3, we will turn on interactions and study dynamics. Focusing on flat space, we will derive the Ward identities satisfied by theories exhibiting nonlinearly realized symmetries, which take the form of soft theorems. From there we will demonstrate how this information may be used to systematically reconstruct the flat space wavefunction from various input data

mechanics to compute equal time correlators. At least in perturbation theory, this procedure reduces to computing Gaussian path integrals, see equation (1.22). Because of this mapping we will often abuse language and refer to the wavefunction as an observable.

[5]Despite this fact, there is some reason to believe that certain higher spin PM theories in AdS may not suffer all of the pathologies inherent to ghostly non-unitary theories [41].

through a brute force bootstrap approach, and also in a more elegant fashion by developing an analogue of BCFW recursion relations. Ultimately, the aim is to better understand the exceptional scalar field theories in de Sitter, which have mass values occurring in the exotic discrete series representation of the de Sitter isometry algebra (see Section 1.3). As pointed out in [42, 43], these theories also possess enhanced nonlinearly realized symmetries, making them analogous to the exceptional scalar field theories in flat space which we explicitly study. There is hope that many of the nontrivial lessons and techniques developed there may be directly imported to the de Sitter context.

1.2 Why not the S-matrix?

Before diving into representation theory and computations of boundary correlators, it is worth taking a moment to ask and answer a simple question: if the S-matrix possesses such a rich mathematical structure in flat space, why bother with anything else when we transition to (A)dS? In short, the answer to this question is because the S-matrix (mostly) does not exist in these spacetimes, and instead the natural quantity to compute are boundary correlators. Therefore in this section, we will give a high-level exposition of why this is the case, and how boundary correlators are defined.

Let us start with Minkowski space, since this is the background in which the S-matrix is the most familiar and well-defined. Intuitively, the S-matrix represents the transition amplitude between two Heisenberg kets, which we will refer to as $|\alpha\rangle_{\text{IN}}$ and $|\beta\rangle_{\text{OUT}}$, where α and β are stand-ins for some collection of quantum numbers. To be clear, these kets are time-independent and describe the entire history of a physical system: $|\alpha\rangle_{\text{IN}}$ describes a system evolving in time, which at $t = -\infty$ has the content α. $|\beta\rangle_{\text{OUT}}$ should be thought of analogously.[6] Both are eigenstates of the full Hamiltonian. The S-matrix then defines an

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[6]Said another way, in the Heisenberg picture a ket describing a physical system is labelled by the "boundary value" of the system at a particular time, in the ODE sense. Effectively, we are labeling entire solutions to the Schrodinger equation by their initial or final values at $t = \pm\infty$.

isomorphism from the set of IN states to the set of OUT states:

$$|\beta\rangle_{\text{OUT}} = \sum_{\alpha} S_{\beta\alpha} |\alpha\rangle_{\text{IN}} .\tag{1.2}$$

For orthonormal bases of the IN/OUT spaces, we may alternatively view the S-matrix as the overlap of an IN state with an OUT state:[7]

$$S_{\beta\alpha} = {}_{\text{OUT}}\langle\beta|\alpha\rangle_{\text{IN}} .\tag{1.3}$$

We will not repeat all of the manipulations, but it is relatively straightforward to write this overlap in terms of free theory states:

$$S_{\beta\alpha} = {}_{\text{free}}\langle\beta| T e^{i \int_{-\infty}^{\infty} dt\, H_I(t)} |\alpha\rangle_{\text{free}} .\tag{1.4}$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture, so that it evolves in time according to the free Hamiltonian. Moreover, $|\alpha, \beta\rangle_{\text{free}}$ are eigenstates of the free theory with the same quantum numbers and energy as $|\alpha, \beta\rangle_{\text{IN/OUT}}$. There are two basic requirements for the S-matrix to exist as a meaningful observable. First, there must be a notion of *asymptotic states* for us to scatter. In particular this means that there must be some sense in which wave packet superpositions of IN/OUT states become well separated for $t \to \pm\infty$ so that their interaction energy vanishes. Accordingly, they may be approximated by corresponding superpositions of eigenstates of the Hamiltonian for the free theory, which likewise have no interaction energy. The precise statement of this condition is

$$\lim_{t\to-\infty} \int d\alpha\, g(\alpha)\, \langle x; t|\alpha\rangle_{\text{IN}} = \lim_{t\to-\infty} \int d\alpha\, g(\alpha)\, {}_{\text{free}}\langle x; t|\alpha\rangle_{\text{free}} ,\tag{1.5}$$

where $g(\alpha)$ is a smooth wave packet centered around the quantum numbers of the particles

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[7]These definitions are equivalent in flat space, but not in de Sitter, where nothing may be taken for granted.
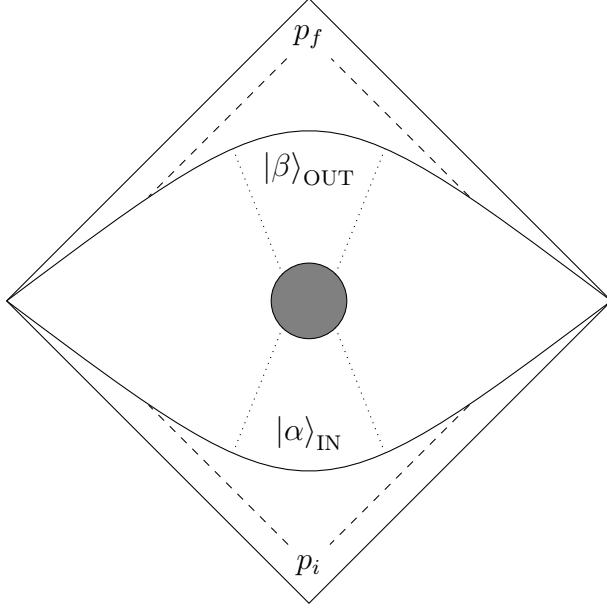
Figure 1.1: Penrose Diagram for Minkowski spacetime. The solid black lines represent the Cauchy surfaces on which $|\alpha, \beta\rangle_{\mathrm{IN,OUT}}$ are defined, and the thick dashed lines represent the causal future/past of experimentalists situated at the points p_i and p_f. The grey blob represents some interaction. In the limit that p_i and p_f are taken to the infinite past/future, their future/past light cones contain the entire IN/OUT Cauchy surfaces. Thus an experimentalist is able to both set up and measure the results of a scattering experiment.

we wish to scatter. There is an analogous condition for OUT states. The second condition for the S-matrix to exist as an observable is related to the causal structure of the underlying spacetime. In particular, it must be possible for an experimentalist to access an entire Cauchy surface of the spacetime to both set up and measure the initial and final states. This is related to the counter-intuitive fact that "particle" is a globally defined concept, and accordingly we need to define the wavefunctions of our scattering states across an entire Cauchy surface. Indeed, this is possible for an experimentalist in flat space, in the limiting sense depicted in Figure 1.1.

Another perspective on the S-matrix which will be conceptually useful later on is provided by the LSZ reduction formula. We will not got through all of the details, but one typically proceeds by defining the following creation and annihilation operators in the interacting

theory, mimicking the analogous construction in the free theory:

$$a^\dagger_{\vec{k}}(t) = -i \int \mathrm{d}^3 x\, \psi_{\vec{k}}(x) \overleftrightarrow{\partial_t} \phi(x), \qquad a_{\vec{k}}(t) = i \int \mathrm{d}^3 x\, \psi^*_{\vec{k}}(x) \overleftrightarrow{\partial_t} \phi(x)\,, \qquad (1.6)$$

where $\psi_{\vec{k}}(x)$ is a one-particle wavefunction. These operators interact with the vacuum of the interacting theory $|\Omega\rangle$ in the following way:

$$\lim_{t\to\pm\infty} a_{\vec{k}}(t)\,|\Omega\rangle = 0, \qquad \lim_{t\to\pm\infty} a^\dagger_{\vec{k}}(t)\,|\Omega\rangle = |\vec{k}\rangle_{\mathrm{OUT/IN}}\,, \qquad (1.7)$$

and it is possible to show that the IN/OUT states as defined possess the same orthogonality conditions as their free counterparts (this in turn justifies the transition from (1.2) and (1.3)). With this, one may derive the following expression for the S-matrix:

$$_{\mathrm{OUT}}\langle \vec{p}_1, ...\vec{p}_n | \vec{k}_1, ...\vec{k}_m\rangle_{\mathrm{IN}} = \int \mathrm{d}^4 x_1...\mathrm{d}^4 x_n \mathrm{d}^4 y_1...\mathrm{d}^4 y_m \psi^*_{\vec{p}_1}(x_1)...\psi^*_{\vec{p}_n}(x_n)\psi_{\vec{k}_1}(y_1)...\psi_{\vec{k}_m}(y_m)$$

$$\times \left(\Box_{x_1} - m^2\right)...\left(\Box_{x_n} - m^2\right)\left(\Box_{y_1} - m^2\right)\left(\Box_{y_m} - m^2\right) \langle\Omega|\, T\phi(x_1)...\phi(x_n)\phi(y_1)...\phi(y_m)\,|\Omega\rangle\,.$$

$$(1.8)$$

Thus the LSZ formula relates the S-matrix to time-ordered correlators, which may be algorithmically computed in perturbative field theory through the use of Feynman diagrams.

Having introduced the S-matrix and stated under which conditions it exists, let us study its status in the other maximally symmetric spacetimes. We will begin with Anti de Sitter, which passes the causality issue, evident by the Penrose diagram detailing its causal structure in Figure 1.2 (left panel). On the other hand, it fails the asymptotic state condition: it is not possible to sensibly define such states in AdS. This can be understood through a purely classical lens.[8] Point particles in AdS behave as though they are trapped in a gravitational potential well akin to a harmonic oscillator potential. In particular, AdS possesses the odd quirk that if an observer fires a photon (or any other particle following a null geodesic)

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[8]A full discussion at the quantum level may be found in [44].
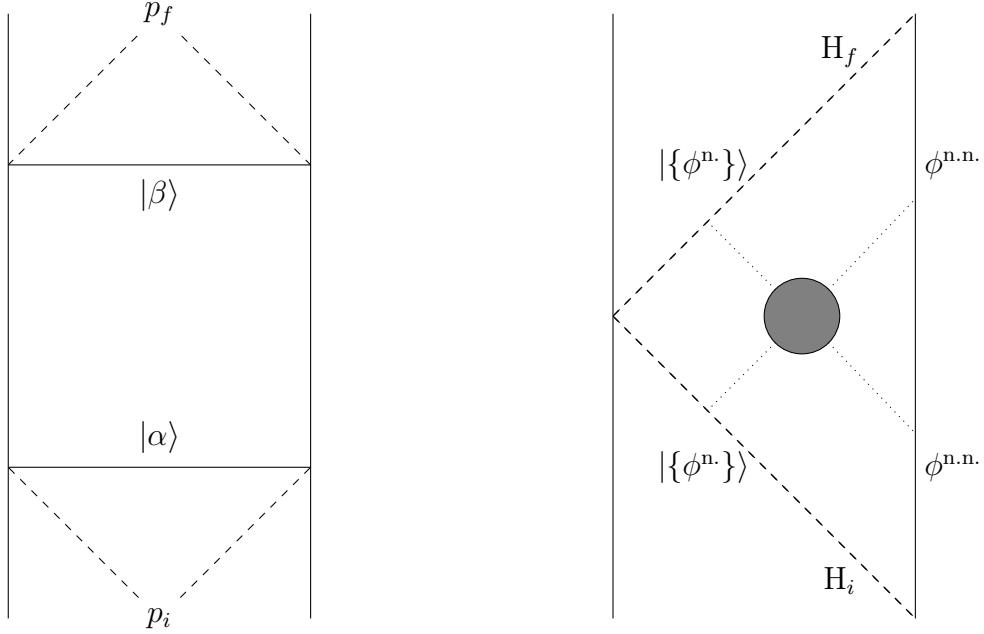
Figure 1.2: Penrose Diagram for Anti de Sitter space. Left: The Cauchy surfaces on which $|\alpha, \beta\rangle$ are defined lie in the entire future/past light cones of experimentalists situated at p_i and p_f. Right: Schematic of a general transition amplitude in the Poincare Patch of AdS, which is bounded by the horizons (thick dashed lines) $H_{i/f}$. States comprised of normalizable modes are defined on these surfaces, whereas non-normalizable modes are sourced from the boundary.

at a mirror situated at spatial infinity, it will reach the mirror and bounce back in finite observer time. Even worse, massive particles traversing time-like geodesics never make it to spatial infinity at all, and follow periodic trajectories. Accordingly, wave packets that are "well separated" will always see each other again some other day, and there is no sense in which they may be permanently approximated by superpositions of free particle states in the distant past and future.

From this discussion on the non-existence of asymptotic states, it should be clear that the bulk and spatial boundary of AdS are intimately connected. This can be made more vivid by examining solutions to the equations of motion in AdS for a scalar field. We will focus on the Poincare Patch, which is the region of AdS bounded by the horizons in Figure

1.2 (right panel). The metric in these coordinates takes the form

$$ds^2 = \frac{L^2}{z^2}\Big(-dt^2 + dx^2 + dy^2 + dz^2 \Big) , \tag{1.9}$$

with L being the AdS radius, $z = 0$ corresponding the boundary, and the $z = \infty$ surface coinciding with the horizon. Then the Klein-Gordon equation is solved as follows:[9]

$$(\Box - m^2)\phi = 0 , \tag{1.10}$$

$$\phi_{\vec{p}}^{\text{n.n.}}(z, t, \vec{x}) = \phi_{\vec{p}}^{0}\, z^{3/2} H_{\Delta_+ - \frac{3}{2}}^{(1)}(kz) e^{-i\omega_p t + i\vec{p}\cdot\vec{x}} \tag{1.11}$$

$$\phi_{\vec{p}}^{\text{n.}}(z, t, \vec{x}) = \phi_{\vec{p}}^{1}\, z^{3/2} J_{\Delta_+ - \frac{3}{2}}(kz) e^{-i\omega_p t + i\vec{p}\cdot\vec{x}} , \tag{1.12}$$

where we have defined $\vec{x} = (x, y)$, and $k^2 = \omega_{\vec{p}}^2 - \vec{p}^{\,2}$, along with the conformal weights

$$\Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2 L^2} . \tag{1.13}$$

There are two fundamental classes of positive frequency modes, described respectively by Hankel and Bessel functions. The Hankel function describes the so called non-normalizable mode,[10] which blows up near the boundary like $\phi_{\vec{p}}^{\text{n.n.}} \sim z^{3-\Delta_+}\phi_{\vec{p}}^0$. On the other hand, the Bessel function mode decays like $\phi_{\vec{p}}^{\text{n.}} \sim z^{\Delta_+}\phi_{\vec{p}}^1$.[11] If our aim were to construct an S-matrix describing the scattering of bulk particles, we would only require the normalizable mode, as only this mode can describe a state in the bulk Hilbert space. However as pointed out previously, we are forced to take a different perspective. Instead we will take the point of view of observers living on the spatial boundary, who accordingly may create localized

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[9]For the sake of a streamlined discussion we will only consider positive mass values and time-like momenta $k^2 > 0$. Results for space-like momenta may be obtained through the analytic continuation $q \to iq$.

[10]This splitting is ambiguous because one can always add multiples of the normalizable mode to the non-normalizable mode to get another non-normalizable mode. To fix the ambiguity, we have chosen the non-normalizable mode to be the analytic continuation of a mode in Euclidean AdS ($t \to -it$) which is regular in the interior. This is equivalent to requiring that the mode is purely incoming/outgoing at the edge of the Poincare Patch.

[11]There are edge cases for $m^2 < 0$ (but still above the Breitenlohner-Freedman bound [45]) where both modes are normalizable. We will not consider such cases here.

boundary sources $\mathcal{O}(t, \vec{x})$ generating non-normalizable modes which propagate into the bulk and interact. This is reflected in the following LSZ-like statement:

$$\langle \mathcal{O}(p_1)...\mathcal{O}(p_n) \rangle = \int \frac{dz_1}{z_1^4} d^3 x_1 ... \frac{dz_n}{z_n^4} d^3 x_n \, \phi_{\vec{p}_1}^{\text{n.n.}}(z_1, x_1)...\phi_{\vec{p}_1}^{\text{n.n.}}(z_n, x_n)$$
$$\times \left(\Box_1 - m_1^2 \right)...\left(\Box_n - m_n^2 \right) \langle T\phi(z_1, x_1)...\phi(z_n, x_n) \rangle , \quad (1.14)$$

where we have defined $x = (t, \vec{x})$. The object on the LHS is a boundary correlator. The bracketed object on the RHS is a bulk correlator, otherwise known as a bulk Green function, computed with normalizable boundary conditions. Perturbatively it is computed by stitching together bulk-to-bulk propagators, which will be defined later on. Again, the LHS is not an S-matrix, as we are not convolving the bulk Green function with single particle wavefunctions.[12] It can be shown that an AdS isometry acting on the bulk field $\phi(z, x)$ induces a conformal transformation on the corresponding local operator $\mathcal{O}(t, \vec{x})$, which accordingly transforms like a primary operator of dimension Δ_+. Therefore, it is reasonable to repackage (1.14) in the form

$$\langle T e^{i \int d^3 x \, \phi^0(x) \mathcal{O}(x)} \rangle_{\text{"CFT"}} = Z_{\text{bulk}} \left[\phi[\phi^0] \right] \quad (1.15)$$

$$Z_{\text{bulk}} \left[\phi[\phi^0] \right] \equiv \int^{\phi(z=0) \sim \phi^0} \mathcal{D}\phi \, e^{iS[\phi[\phi^0]]} . \quad (1.16)$$

Then (1.14) may be reproduced by taking functional derivatives with respect to ϕ^0. On the RHS, Z_{bulk} is the partition function in the bulk for field configurations behaving like $\phi(z = 0) \sim z^{3-\Delta_+} \phi^0$ near the boundary, that is, are non-normalizable. This reproduces the RHS of (1.14) because the equation of motion operators truncate the normalizable external legs of the bulk Green function and replaces them with non-normalizable legs. On the LHS of (1.15), "CFT" indicates that the boundary operators $\mathcal{O}(x)$ transform as conformal

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[12]It is possible to give AdS a finite cutoff and smoothly interpolate it with a spacetime that does allow for asymptotic states. In such cases, (1.14) *is* precisely an S-matrix, so long as the integration region is suitably adjusted [44].

primaries of weight Δ_+ under conformal transformations. The statement of AdS/CFT is then essentially that the "CFT" is a CFT, in the sense that it is unitary[13] and possesses a local stress tensor. This is only possible for bulk theories of quantum gravity, as the requisite stress tensor is dual to the bulk gravitational field (in the same sense that $\mathcal{O}$ is dual to ϕ).[14] We will not need anywhere near the full power of the AdS/CFT correspondence in this work. However, the representation theory of supersymmetric CFT's will provide a useful framework for studying the boundary imprint of exotic bulk fields in Chapter 2.

Finally, it is interesting to interpret the meaning of (1.14) if some or all of the non-normalizable modes are replaced with the normalizable modes, that is, AdS wavefunctions. This object is known as a transition amplitude[15] (see Figure 1.2 (right panel)) [48, 49]. It computes the overlap of two states specifying data on the past and future horizons of the Poincare Patch, with non-normalizable modes corresponding to operator insertions on the boundary. However we emphasize that even if all modes are made to be normalizable, this object is not an S-matrix, because the particles described by the states interact. Overall, the LSZ perspective in generic curved spacetimes (where the LHS of (1.14) are sometimes called on shell correlators) has recently gained appreciation from amplitude-minded individuals. For instance, transition amplitudes play a central role as building blocks in the AdS recursion relations developed in [48] (see Section 1.4), and their wavefunction analog (sometimes called shifted wavefunction coefficients) will also appear in the recursion relations derived in Chapter 3. They also play an essential role in the AdS Cutkosky rules developed in [49]. In addition, it has been pointed out in [50, 51] that for generic spacetimes on shell correlators exhibit some aspects of color-kinematics duality, with specific examples given in AdS. There-

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[13]The boundary notion of unitarity is inherited from the bulk; in Chapter 2 well will invoke radial quantization on the boundary, which is naturally inherited from the global AdS coordinates, in which the metric is

$$\mathrm{d}s^2 = \frac{1}{\cos\frac{\rho}{L}}\Big(-dt^2 + d\rho^2 + \sin^2\frac{\rho}{L}\,d\Omega^2 \Big), \tag{1.17}$$

where $\mathrm{d}\Omega^2$ is the metric on the 2-sphere and $\rho \sim \frac{\pi}{2}L$ is the boundary.

[14]See [46, 47] for the full dictionary.

[15]As we are restricting to the Poincare Patch, we have only given a perturbative definition. See [44] for a non-perturbative definition in global coordinates.

fore it is likely that this perspective will ultimately play some role in the generalization of the double-copy to curved spacetimes, should it exist.

Now we will address the existence of the S-matrix in de Sitter space. de Sitter fails the causality criterion and only questionably possesses asymptotic states for theories with generic mass values. As demonstrated in Figure 1.3, it is not possible for a single experimentalist to set up or measure a state on an entire Cauchy surface of global de Sitter. For cosmological applications this is a moot point anyway because there were no physicists around to build particle accelerators in the early universe. As post-inflationary observers, we may only look back and make observations of the universe at the future boundary of its quasi-de Sitter inflationary phase. Moreover, this is only possible because inflation ended and the de Sitter phase transitioned into a hot Big Bang universe. It is from this vantage point that we view correlations on the post-inflationary boundary, see Figure 1.4.

Before discussing the correlation functions inhabiting the future inflationary boundary, it is interesting to ask whether or not it is possible to define an S-matrix for global de Sitter in principle, even if it cannot be observed. This question was investigated in [52] at the perturbative level, ignoring string and gravitational back-reaction effects which call into question the stability and longevity of a de Sitter background altogether [53]. The proposed S-matrix may be computed from an LSZ-like procedure. However, since IN/OUT correlators possess IR divergences in de Sitter (recall that the IN/OUT vacua do not coincide), the LSZ formula must be applied to a time-ordered correlator computed using a more exotic time integration contour. Specifically, one defines the vacuum state $|\Omega\rangle$ along a Cauchy surface coincident with the de Sitter horizon, and then chooses a contour which extends backwards in time to the past conformal boundary, forwards in time to the future conformal boundary, and then backwards in time again to the horizon. The reason for this construction is that defining the vacuum state on the horizon Cauchy surface gives a natural way to impose the Bunch-Davies condition (see equation (1.27)).

Now that we have specified our choice of vacuum, we need to define a set of states to scatter. A reasonable looking definition is the following:

$$|\alpha\rangle_{\text{"IN"}} \equiv a_\alpha^\dagger(-\infty) |\Omega\rangle \, , \qquad |\beta\rangle_{\text{"OUT"}} \equiv a_\beta^\dagger(\infty) |\Omega\rangle \, , \qquad a_{\alpha,\beta}(t \sim \text{horizon}) |\Omega\rangle \equiv 0 \, ,$$

(1.18)

where the creation and annihilation operators are defined in analogy to (1.6).[16] We use quotes to remind the reader that these states are not constructed from the IN and OUT vacua of de Sitter. A very annoying technical point is that the "IN"/"OUT" states we have defined *do not* possess the same orthogonality properties as their free theory counterparts, and in fact, they are not orthogonal at all! To be clear this means that, for example, two "IN" states with different particle numbers may not have zero overlap. This is markedly different from the situation in flat spacetime. Physically, this subtlety arises because for interacting theories in de Sitter there is no conserved energy enforcing such a particle number selection rule. Instead, there are angular momentum selection rules on S^3, which are far less restrictive. The bottom line for the de Sitter S-matrix is that (1.3) *does not hold* in de Sitter for our definition of asymptotic states, and one must work directly with the definition in (1.2), viewing the S-matrix as an isomorphism mapping "IN" states to "OUT" states.[17] For the sake of being concrete, consider a finite dimensional toy example: take the unit vectors v_1, v_2 to span the "IN" space and the unit vectors u_1, u_2 to span the "OUT" space. Then the S-matrix element $S_{\beta\alpha} = S_{11}$ is given by

$$S_{11} = \frac{u_1 \cdot v_1 - u_1 \cdot v_2 v_1 \cdot v_2}{1 - (v_1 \cdot v_2)^2} \, .$$

(1.19)

Thus, the S-matrix receives corrections from the non-orthogonality of the "IN" basis, $v_1 \cdot v_2 \neq 0$.

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[16]Of course, the single-particle wavefunctions must obey the Bunch-Davies condition.

[17]One could always perform a Gram-Schmidt procedure to construct an orthonormal set of states spanning the "IN"/"OUT" spaces. However, these will not be multi-particle states.
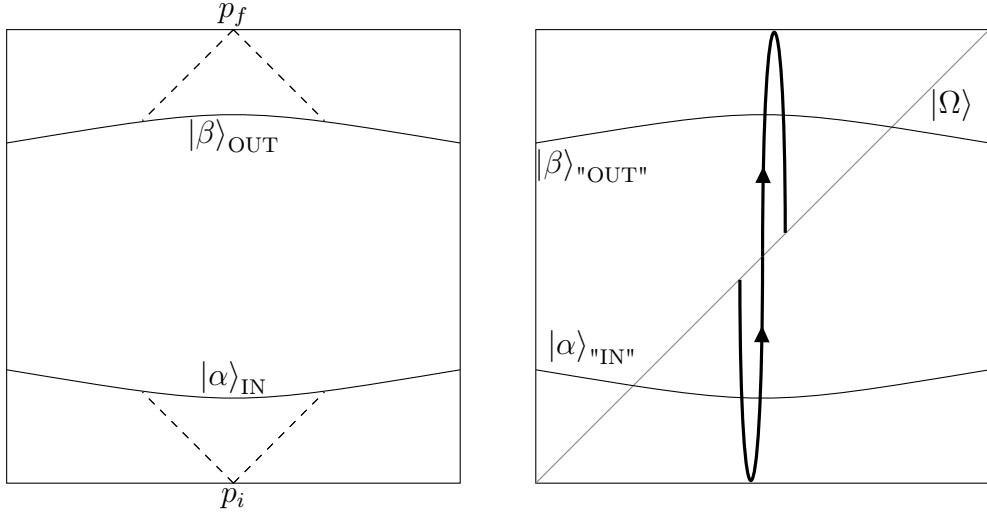
Figure 1.3: Penrose Diagram for de Sitter spacetime. Left: A single experimentalist situated at $p_{i,f}$ (which lie on the past/future conformal boundary) cannot create or observe a state on an entire Cauchy surface. Right: "Meta-observable" de Sitter S-matrix. States are created/annihilated by acting on the Euclidean vacuum. Perturbation theory is performed with respect to the time integration contour depicted by the thick black line. Field insertions on the first leg, middle leg, and final leg of the contour compute "IN"/"IN", "IN"/"OUT", and "OUT"/"OUT" overlap contributions to the S-matrix, see equation (1.19).

Turning back to field theory, for massive scalars with $m^2 > \frac{9}{4}H^2$ (that is, in the principal series, where the mass values are on the order of the Hubble parameter H–see Section 1.3), this is a computable annoyance. However for light scalars with $0 < m^2 < \frac{9}{4}H^2$ (in the complementary series), the overlaps diverge, and the procedure we have outlined is not defined. Despite this, it is possible to modify this algorithm with a suitable regularization procedure, and the remaining divergences are thought to only shift the particles' self-energy, which is familiar to flat space scattering processes.

Finally, let us turn to correlation functions on the future boundary of de Sitter, which are the main observable of interest in inflationary cosmology. Unlike any of the other objects we have considered, these are equal time correlators, and they measure how quantum fluctuations in the bulk de Sitter universe are imprinted on the boundary. We will exclusively
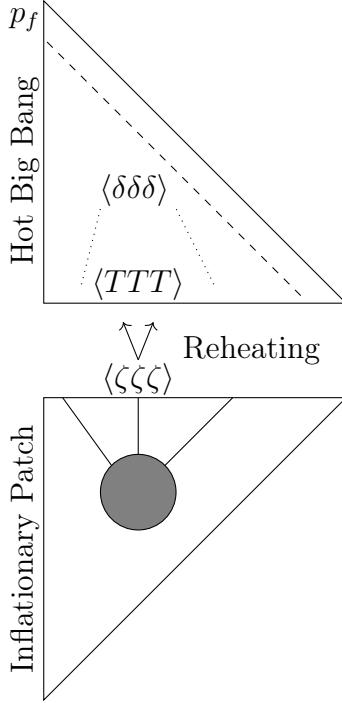
Figure 1.4: Penrose diagram for the inflationary patch of (quasi-) de Sitter and the transition to the hot Big Bang universe during Reheating. Correlations $\langle \zeta\zeta\zeta \rangle$ on the future inflationary boundary present themselves as correlations $\langle TTT \rangle$ on the past boundary of the hot Big Bang universe, or in density fluctuations $\langle \delta\delta\delta \rangle$ of the Large Scale Structure, which may also ultimately be traced back to this surface. Each of these is in the past light cone of an observer situated at p_f.

work in the inflationary patch (see Figure 1.4), using the conformal coordinates

$$ds^2 = \frac{1}{H^2\eta^2}\Big(-\,d\eta^2 + dx^2 + dy^2 + dz^2 \Big), \tag{1.20}$$

in which $\eta = 0$ is the future boundary and $\eta = -\infty$ is the past horizon. From here on we will use the notation $\vec{x} = (x, y, z)$. We will write a correlation function as

$$\langle \phi(\vec{k}_1)...\phi(\vec{k}_n) \rangle \equiv \langle \Omega | \, \phi(\vec{k}_1)...\phi(\vec{k}_n) \, | \Omega \rangle, \tag{1.21}$$

where we have defined $\phi(\vec{k}) \equiv \lim_{\eta \to 0} \phi(\eta, \vec{k})$, and again $|\Omega\rangle$ is the Bunch-Davies state. This object may be computed via so-called IN-IN perturbation theory[18] (a nice review may be

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[18]Somewhat confusingly, this has nothing to do with the IN de Sitter vacuum.

found in [54]). However, this procedure is quite cumbersome, and alternatively one may also compute the vacuum wavefunction $|\Omega\rangle$ itself. Once this is known one may derive the equal time correlation function using the standard rules of quantum mechanics for computing expectation values:

$$\langle \phi(\vec{k}_1)...\phi(\vec{k}_n)\rangle = \frac{\int \mathcal{D}\varphi \, \varphi(\vec{k}_1)...\varphi(\vec{k}_n)|\Psi[\varphi,\eta_f]|^2}{\int \mathcal{D}\varphi \, |\Psi[\varphi,\eta_f]|^2},$$ (1.22)

in which the late time wavefunction $\Psi[\phi,\eta_f] = \langle \varphi; \eta_f |\Omega\rangle$ is the projection of the vacuum onto a Heisenberg eigenstate $\langle \varphi; \eta_f|$ of the field operator $\phi(\eta,\vec{x})$ at late times $\eta_f \sim 0$. Part of the motivation for shifting focus from the final correlation function to the late time wavefunction is that the wavefunction is analogous to the bulk partition function in AdS, defined in (1.16), and as such it is possible to import intuition and in some cases entire computations from the AdS context [55, 56, 57]. We can see the connection if we study solutions to the equation of motion in dS for a scalar field:[19]

$$(\Box - m^2)\phi = 0 \,,$$ (1.23)

$$\phi_{\vec{p}}^{\text{BD}}(\eta,\vec{x}) \sim \varphi_{\vec{p}}^{\text{BD}} \, \eta^{\frac{3}{2}} H_{\Delta_+ - \frac{3}{2}}^{(2)}(p\eta)e^{i\vec{p}\cdot\vec{x}} \,,$$ (1.24)

$$\phi_{\vec{p}}^{\alpha}(\eta,\vec{x}) \sim \varphi_{\vec{p}}^{\alpha} \, \eta^{\frac{3}{2}} J_{\Delta_+ - \frac{3}{2}}(p\eta)e^{i\vec{p}\cdot\vec{x}} \,,$$ (1.25)

where we have defined $p = |\vec{p}|$ along with the conformal weight

$$\Delta \pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \,.$$ (1.26)

These solutions are structurally very similar to the AdS solutions. At late times near the future conformal boundary, the BD mode is dominant $\phi_{\vec{p}}^{\text{BD}} \sim \varphi_{\vec{p}}^{\text{BD}} \eta^{3-\Delta_+}$, whereas the α mode is subleading $\phi_{\vec{p}}^{\alpha} \sim \varphi_{\vec{p}}^{\alpha} \eta^{\Delta_+}$. Therefore, at late times we may simply identify the argument of the wavefunction with the dominant mode profile, $\varphi_{\vec{p}} \sim \varphi_{\vec{p}}^{\text{BD}}$. As in the AdS case, there

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[19]Specifically for this exposition we are considering a scalar in the complementary series.

is an ambiguity in defining the dominant mode, as we can always add on a multiple of $\phi_{\vec{p}}^{\alpha}$. However the interpretation of this ambiguity is different in de Sitter, and it captures the non-uniqueness of the de Sitter vacuum state. We will fix this ambiguity by imposing the so-called Bunch-Davies condition, which forces the mode function to be positive frequency in the far past $\eta \to -\infty$ (or in the UV). We have conveniently already defined the growing mode function so that this is the case

$$\lim_{\eta \to -\infty} \phi_{\vec{p}}^{\mathrm{BD}}(\eta, \vec{x}) \sim ip\eta e^{ip\eta} \,. \tag{1.27}$$

On the other hand, adding on the α-mode destroys this condition, as in the far past it contains an ad-mixture of positive and negative frequency modes. Beyond having this early time behavior, the Bunch-Davies condition is a natural choice because the resulting early time wavefunction is a Gaussian:

$$\Psi[\varphi, -\infty] \propto \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^2} \mathcal{E}(k) \varphi_{\vec{k}} \varphi_{-\vec{k}}\right), \tag{1.28}$$

where $\mathcal{E}(k)$ is a rotationally invariant kernel related to the power spectrum by $P(k) = 1/2 \operatorname{Re} \mathcal{E}(k)$. In addition, the mode functions defining the Bunch-Davies state are the only choice which is regular upon analytic continuation to Euclidean space. Moroever, such Gaussian initial conditions naturally give rise to the approximately scale invariant power spectrum observed in the CMB. Finally, in the case of Gaussian initial conditions, it is also straightforward to show that the wavefunction has the natural path integral representation:[20]

$$\Psi[\varphi, \eta_f] = \int^{\phi(\eta=0) \sim \varphi} \mathcal{D}\phi \, e^{iS[\phi[\varphi]]} \,, \tag{1.29}$$

which is entirely analogous to the path integral representation of the AdS bulk partition function in (1.16). As such, it is reasonable to think of $\Psi[\varphi, \eta_f]$ as a generating functional

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[20]See chapter 9 of [58].

for boundary "CFT" correlators of operators $\mathcal{O}(\vec{k})$ which are dual to $\phi(\eta, \vec{k})$:

$$\log \Psi[\varphi, \eta_f] = \sum_{n=1} \frac{1}{n!} \int \frac{\mathrm{d}^3 k_1 \cdots \mathrm{d}^3 k_n}{(2\pi)^{3n}} \varphi_{\vec{k}_1} \cdots \varphi_{\vec{k}_n} (2\pi)^3 \delta(\vec{k}_1 + \ldots + \vec{k}_n) \langle \mathcal{O}(\vec{k}_1) \cdots \mathcal{O}(\vec{k}_n) \rangle_{\text{"CFT"}},$$

(1.30)

where the "CFT" correlators (otherwise known as wavefunction coefficients, or in an abuse of language, "wavefunctions") may be extracted by taking functional derivatives of $\Psi[\varphi, \eta_f]$, just as in the AdS case. Here we will not assume that the "CFT" is any known standard CFT, and we will not invoke any dS/CFT correspondence [59].

Finally, it will be useful to describe how to compute the late time wavefunction using bulk perturbation theory. This may be done using Feynman-Witten diagrams, see Figure 1.5. The procedure is similar to the compuatation of S-matrix elements in flat space, with two essential differences. First, energy is not conserved (but 3-momentum still is), even in flat space, as our choice of the future time slice on which to compute the wavefunction breaks time translation invariance. Second, there now two types of propagators. First there is the bulk-to-boundary propagator $\mathcal{K}(k; \eta)$, which describes lines connecting a bulk point to the boundary. In addition, there is the bulk-to-bulk propagator $\mathcal{G}(k; \eta, \eta')$, which describes lines connecting two bulk points. The two propagators respectively obey the equations

$$\left(\square - m^2\right)\mathcal{K}(\eta, \vec{x}, \vec{x}') = 0$$
$$\left(\square - m^2\right)\mathcal{G}(\eta, \vec{x}, \eta', \vec{x}') = \frac{i}{\sqrt{-g}}\delta(\eta - \eta')\delta(\vec{x} - \vec{x}').$$

(1.31)

After assigning propagators to each line, one also multiplies each vertex by the corresponding vertex factor $i\mathcal{V}$, and integrates over all bulk time points. As an example, the exchange diagram for a massless $\frac{\lambda}{3!}\phi^3$ theory is given in Figure 1.5. Applying the Feynman rules gives
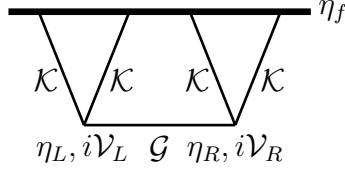
Figure 1.5: Witten-Feynman diagram computing the s-channel exchange contribution to a 4-point wavefunction coefficient.

the following integral expression for the s-channel of the four point wavefunction coefficient:

$$\psi_4^s = \int_{-\infty}^{\eta_f} \mathrm{d}\eta_L \mathrm{d}\eta_R \, i\mathcal{V}_L \mathcal{K}(p_1;\eta_L)\mathcal{K}(p_2;\eta_R)\mathcal{G}(s;\eta_L,\eta_R)\mathcal{K}(p_3;\eta_R)\mathcal{K}(p_4;\eta_R)i\mathcal{V}_R \, . \tag{1.32}$$

As a simple example, we may evaluate this expression for the flat space wavefunction (see Section 3.2 for the relevant propagators):

$$\psi_4^s = \frac{\lambda_3^2}{EE_LE_R} \, , \tag{1.33}$$

where we have defined the left and right partial energies $E_L = p_1+p_2+s$ and $E_R = p_3+p_4+s$, for $s = |\vec{p}_1 + \vec{p}_2|$.

1.3   Kinematics: Representations of the de Sitter isometry group

In this Section, we will give a brief overview of the unitary representation theory of the de Sitter isometry group $SO(1,4)$ (see [60, 61] for a full exposition). This will be useful in framing the analysis of PM fields in Chapter 2, and contextualizing future analyses to be done along the lines of Chapter 3 for the exceptional scalar theories.

Scalar Fields: For fields without spin, representations are labelled by the conformal weight Δ. The corresponding mass is given through the equation $m^2 = -\Delta(\Delta - 3)$. There are three distinct series to consider:

- Principal Series: $\Delta = \frac{3}{2} + i\rho$ for $\rho \in \mathbb{R}$. These are *heavy fields* with $m^2 \geq \frac{9}{4}H^2$

- Complementary Series: $0 < \Delta < 3$. These are *light fields* with $0 < m^2 < \frac{9}{4}H^2$

- Discrete Series: $\Delta = -n$ with $n \in \mathbf{N}$. These fields are *tachyonic* with $m^2 = -n(n+3)$

Spinning Fields: For spinning fields, representations are labelled by the conformal weight Δ and the spin s. The mass is given through the equation $m^2 = -(\Delta + s - 2)(\Delta - s - 1)$. There are again 3 distinct series:

- Principal Series: $\Delta = \frac{3}{2} + i\rho$ for $\rho \in \mathbf{R}$. These are *heavy fields* with $m^2 \geq (s - \frac{1}{2})^2 H^2$

- Complimentary Series: $1 < \Delta < 2$. These are *light fields* with
$$s(s-1)H^2 < m^2 < (s - \tfrac{1}{2})^2 H^2$$

- Discrete Series: $\Delta = 2 + t$ (or its shadow $\Delta = 1 - t$) for $t \in \{0, 1, ...s - 1\}$. These have mass values $\frac{m^2}{H^2} = s(s-1) - (t+1)$, and are often referred to as *partially massless fields* of depth t

For scalar fields, the discrete series is perhaps the most bizarre, as these states have tachyonic masses. In general they are poorly understood, but their interactions have been studied extensively in [43]. Interestingly, interacting theories of these scalars enjoy enhanced shift symmetries analogous to the family of exceptional scalar field theories in flat space, whose wavefunction we study in Chapter 3.

For spinning fields, there is a lower mass bound in the complementary series, which is required for the representation to be unitary. This is known as the Higuchi bound [62]. Note that there exist unitary representations for mass values below the Higuchi bound, analogous to the scalar discrete series which possess tachyonic masses. As mentioned previously, these exitations are allowed in the spinning case because at the specified mass values the theory develops gauge symmetries which project out the ghostly modes.

In Chapter 2 we will study partially massless fields in great detail. The field theory of these exotic representations of the de Sitter group was first studied in [63, 64, 65, 66]. At the level of the free theory, a PM field may be thought of as a massive spin-s field with the

mass parameter tuned to one of the partially massless points. In 4 spacetime dimensions, a massive spin-s field typically has $2s + 1$ degrees of freedom. However at one of the PM mass points $\frac{m_t^2}{H^2} = s(s-1) - (t+1)$, the theory develops a gauge symmetry, projecting out the modes with helicities $t = \{0, \pm 1, ..., \pm t\}$ and leaving behind a field which propagates $2(s-t)$ degrees of freedom. The simplest example of a partially massless field is that of a massive spin-2 particle with depth $t = 0$, corresponding to the mass value $m^2 = 2H^2$. This field propagates 4 degrees of freedom. It is described by the Fierz-Pauli action

$$
S = \int d^4x \sqrt{-g} \Big[-\frac{1}{2} \nabla_\lambda h_{\mu\nu} \nabla^\lambda h^{\mu\nu} + \nabla_\lambda h_{\mu\nu} \nabla^\nu h^{\mu\lambda} - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla_\mu h \nabla^\mu h
$$
$$
+ 3H^2 \Big(h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2 \Big) - \frac{1}{2} m^2 \Big(h_{\mu\nu} h^{\mu\nu} - h^2 \Big) \Big], \tag{1.34}
$$

and possesses the abelian gauge symmetry

$$
\delta h_{\mu\nu} = \nabla_\mu \nabla_\nu \alpha + H^2 g_{\mu\nu} \alpha \tag{1.35}
$$

for a scalar function α. The gauge symmetry may be made manifest by writing the action in terms of the gauge invariant PM field strength tensor $F_{\mu\nu\lambda}$:

$$
S = -\frac{1}{4} \int d^4x \sqrt{-g} \Big(F^{\lambda\mu\nu} F_{\lambda\mu\nu} - 2 F^{\lambda\mu}{}_\mu F_{\lambda\nu}{}^\nu \Big)
$$
$$
F_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} - \partial_\nu h_{\mu\lambda}. \tag{1.36}
$$

The various writings of the free PM theory hint that it is a sort of hybrid theory between gravity and traditional gauge theories like electromagnetism. Even at the level of the free theory, PM fields already possess a number of interesting properties. For instance, like Maxwell theory, the PM theory possesses an "electric-magnetic duality", in addition to electric and magnetic monopole states. In addition, there appears to be a connection between PM fields and the scalar discrete series representations, whose field theories possess nonlinearly realized shift symmetries $\delta\phi$ protecting their mass values. Somewhat surprisingly, these symmetry

transformations are precisely the reducibilities of PM gauge transformations (the functions α for which the gauge transformation vanishes). To make this connection sharper, if one begins with the Fierz-Pauli action for a generic mass parameter (so that it propagates 5 degrees of freedom), upon taking the PM mass limit the theory splits into 4 PM degrees of freedom enjoying the gauge symmetry (1.35), and also one additional scalar degree of freedom enjoying a nonlinearly realized symmetry given by a reducibility of (1.35).

The key challenge in the studying partially massless fields is to uncover interacting theories which possess PM degrees of freedom. Chief among these is the goal of writing a theory propagating a single PM field, dubbed "Partially Massless Gravity". Compared to other theories of massive gravity (which PM theories need not necssarily be a special case of), PM gravity has a number of appealing features. First, as a gauge theory, it should be possible to fix its nonlinear structure at low energies, just as diffeomorphism invariance uniquely fixes General Relativity. This is not the case for massive gravity theories in general, whose form is not fixed by symmetry. Second, it is likely that an interacting PM theory would not suffer from all of the potential pathologies and oddities of generic massive gravity theories, such as superluminalities or the vDVZ discontinuity [67]. Such features of massive gravity theories are caused by the existence of the longitudinal mode, which is happily absent from PM gravity due to the gauge symmetry. As it currently stands there is a growing body of no-go theorems carving out the space of permissible PM theories [68, 69, 68, 70, 71, 72, 73, 74]. The space of single-field PM candidate theories most similar to General Relativity has already been explored, and indeed there is no Lagrangian with at most two derivatives realizing the PM symmetry or a nonlinear extension thereof. Though there is still the possibility of single-field PM theories requiring higher derivatives, it is also interesting to study theories which include PM fields in addition to other degrees of freedom. Conformal Gravity is a known example, though it is not unitary in (A)dS. There are also constructions of Vasiliev-like higher spin theories of PM fields [41]. In addition, one may also ask if PM fields are actually more similar to traditional gauge fields than gravity, and thus search for non-abelian

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"Yang-Mills" extensions. Similar to Yang-Mills, there is a "multi-color" extension of the PM symmetry with at most two derivatives per field. However, the corresponding algebra of gauge transformations is actually abelian,[21] very much unlike Yang-Mills, and in this sense the Yang-Mills extension is a no-go. Finally in 4 spacetime dimensions, one may construct cubic vertices realizing a special case of this abelian "multi-color" PM algebra [70], though one of the kinetic terms will always have the wrong sign so the theory propagates a ghost mode.

Clearly, all of the low-hanging partially massless fruit has already been picked. In Chapter 2, we will extend the survey of PM fields to include theories involving supersymmetry. This is a purely kinematics problem, and we achieve this by studying the non-unitary representations of the superconformal algebra and identifying SUSY multiplets which contain PM fields. Along the way, we will uncover novel structures such as exotic shortening conditions for multiplets and extended modules [75].

1.4 Dynamics: The cosmological bootstrap

In this Section, we will given an overview of some aspects of the cosmological bootstrap, which contextualizes the work conducted in Chapter 3. The bootstrap perspective has grown as a complementary point of view to the cosmological collider program, which was begun in [27] and continued in [76, 77, 27, 28, 29, 78, 79, 80, 81, 82]. In a sense, the cosmological collider program is an attempt to treat inflationary cosmology like particle physics. In the latter, experimentalists infer the existence of new massive particles through studying resonances and the angular distributions of scattering experiments. Likewise, the cosmological collider aims to understand how intermediate massive particles which may have been present during inflation leave their imprint on inflaton and graviton correlators. Part of the goal of the cosmological bootstrap is to efficiently generate new theoretical data

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[21]This is true at least to first order in fields, which is the order at which the non-abelian character of the Yang-Mills algebra of gauge transformations appears.

which may be compared with observational data. Beyond that, there is hope that a deeper understanding of the structure of the wavefunction and cosmological correlators will unveil novel physical and mathematical structures, in the same vein as the perturbative amplitude bootstrap.

The cosmological bootstrap is essentially a toolbox for constraining wavefunction coefficients (hopefully to the point of uniqueness) while eschewing the difficulties inherent to bulk perturbation theory, which even at tree level involves a proliferation of complicated bulk time integrals. The toolbox at this point has grown quite large, and in this section we will only highlight certain items, in particular singularities and factorization, conformal symmetry, unitarity, transmutation and lifting, and finally the constraints of additional (non-conformal) symmetries through Ward-Takahashi identities. Before concluding this Section, we will also give a brief overview of the phenomenology of the theoretical data generated by the bootstrap.

Singularities and Factorization: The singularity and factorization properties of the wavefunction will play an essential role in Chapter 3, and are thus described in more detail therein. Here we give a brief overview. In general, a tree-level wavefunction coefficient is singular when the energy flowing into to a subgraph adds to zero [28, 26]. See for instance (1.33), which is singular at $E_L \equiv p_1 + p_2 + s = 0$, among other places. At such locations, the wavefunction factorizes into the product of a lower-point amplitude and a shifted wavefunction:

$$\lim_{E_L \to 0} \psi_{n+m} = \frac{A_L \times \tilde{\psi}_R}{E_L} \,, \tag{1.37}$$

where A_L is the amplitude corresponding to the process of the left subgraph in Figure 1.5, ψ_R is the wavefunction describing the right subgraph, and $\tilde{\psi}_R$ is its shifted version, defined by

$$\tilde{\psi}_R \equiv \frac{1}{2s} \Big(\psi_R(-s) - \psi_R(s) \Big) \,. \tag{1.38}$$

The quantity s is the energy associated with the internal line connected to the singular point of the graph. In our 4-point example (1.33), the lower point objects are given by

$$A_L = \lambda_3$$
$$\tilde{\psi}_R = \frac{\lambda_3}{2s}\left(\frac{1}{p_1 + p_2 - s} - \frac{1}{p_1 + p_2 + s}\right) = \frac{\lambda_3^2}{(p_1 + p_2)^2 - s^2}, \tag{1.39}$$

and one may simply check that the factorization property holds as claimed. In addition, wavefunctions also possess a singularity when the total energy $E = p_1 + \cdots + p_n$ of a graph adds to zero. The universality of this singularity is analogous to the energy conserving δ-function that appears in the context of scattering amplitudes. However, since the time integration is cutoff at the future boundary, the corresponding signature is a singularity. Accordingly, the residue of this singularity is itself the corresponding amplitude for the processes:

$$\lim_{E \to 0} \psi = \frac{A}{E}. \tag{1.40}$$

The precise nature of these singularities may change depending on the mass values of the fields whose wavefunction we are considering in conjunction with the underlying spacetime (flat or de Sitter). For instance, it is not uncommon for the "singularities" to actually be branch points or higher order poles. However, the location of these structures is robust, and accordingly these are the only places where the tree-level wavefunction is singular.

Conformal Symmetry: Conformal symmetry is imposed on wavefunction coefficients through the conformal Ward Identities. These take the form of a set of differential equations

which the wavefunction must satisfy [28]. For 4-point wavefunctions, define[22]

$$\psi_4(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = s^{-1}\hat{F}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4), \qquad (1.41)$$

where $s = |\vec{p}_1 + \vec{p}_2|$. Then the conformal Ward identity implies that $\hat{F}$ must obey the following differential equation:

$$(\Delta_u - \Delta_v)\hat{F} = 0, \qquad \Delta_u = u^2(1 - u^2)\partial_u^2 - 2u^3\partial_u, \qquad (1.42)$$

where $u = s/(p_1 + p_2)$, $v = s/(p_3 + p_4)$. To specify a solution to this differential equation, one must impose boundary conditions. In particular, we require that ψ_4 factorizes correctly (see above) near $u, v \to -1$ (which are exactly the partial energy singularities) and also does not possess "folded singularities", namely, is regular for $u, v \to -1$. The latter condition is required due to the vacuum degeneracy of de Sitter, and this choice picks out the Bunch-Davies vacuum [83, 84, 85]. The simplest solutions to this equation are contact solutions, which do not possess partial energy singularities:

$$\hat{C}_0 = \frac{uv}{u + v} = \frac{s}{p_1 + p_2 + p_3 + p_4}, \qquad \hat{C}_n = \Delta_u^n\hat{C}_0, \qquad (1.43)$$

where $\hat{C}_0$ captures interactions with no derivatives and $\hat{C}_n$ captures derivative interactions. On the other hand, exchange interactions correspond to solutions of (1.42) which also obey

$$\left[\Delta_{u,v} - (\Delta - 1)(\Delta - 2)\right]\hat{F} = \hat{C}, \qquad (1.44)$$

where Δ is the conformal weight of the exchanged field and $\hat{C}$ is a contact solution. Roughly speaking, this is a consequence of bulk locality, as the differential operator Δ_u schematically acts on the bulk-to-bulk propagator and collapses the internal line of an exchange diagram,

---

[22]We will solely focus on wavefunctions for conformal scalars $\Delta = 2, m^2 = 2H^2$ in this exposition. Correlators for other mass values may be generated by acting the conformal scalar result with weight shifting operators, see [29].

mapping a 4-point exchange diagram to a contact diagram.

Unitarity: The constraints of unitarity will also play a major role in Chapter 3, and will be describe in more detail there. Here we will emphasize a related perspective [86]. Similar to scattering amplitudes, unitarity is encoded in an optical theorem and corresponding set of cutting rules. The derivation of the so-called cosmological optical theorem proceeds in exactly the same way as in the scattering case. One starts with the time evolution operator

$$U(\eta_0) = T \exp\left(-i \int_{-\infty}^{\eta_0} d\eta \, H_{\mathrm{I}}(\eta)\right), \tag{1.45}$$

and subtracts off the identity contribution, defining $\delta U \equiv U - 1$. Because U is unitary, δU obeys the operator equation $\delta U + \delta U^\dagger = -\delta U \delta U^\dagger$. Then we sandwich this expression between a vacuum state and a free multi-particle state $|\{\vec{k}\}_n\rangle$, and insert a complete set of free states on the RHS, leading to the following non-perturbative statement:

$$\langle\{\vec{k}\}_n| \, \delta U(\eta_0) \, |0\rangle + \langle 0| \, \delta U(\eta_0) \, |\{\vec{k}\}_n\rangle^* = - \int dX \, \langle\{\vec{k}\}_n| \, \delta U(\eta_0) \, |X\rangle \, \langle X| \, \delta U(\eta_0)^\dagger \, |0\rangle, \tag{1.46}$$

where the integral over X is a formal sum over free multi-particle states. It is not immediately obvious how this connects to the wavefunction. In fact at the non-perturbative level it is not clear that there even is a direct connection, and there are alternative avatars of non-perturbative unitary which may play a more prominent role [57]. However, we can connect this expression to the perturbative wavefunction if we expand both sides in terms of H_{I}. To linear order, we arrive at the equation

$$\int_{-\infty}^{\eta_0} d\eta \, \langle\{\vec{k}\}_n| \, H_{\mathrm{I}}(\eta) \, |0\rangle - \left(\int_{-\infty}^{\eta_0} d\eta \, \langle\{\vec{k}\}_n| \, H_{\mathrm{I}}(\eta) \, |0\rangle\right)^* = 0. \tag{1.47}$$

At the level of perturbation theory, one would compute this quantity by expanding the field variables within H_{I} in terms of free field creation and annihilation operators. Because H_{I} acts on the vacuum state, only the positive frequency mode functions survive. These are

essentially bulk-to-boundary propagators, and as a result the matrix element is proportional to the wavefunction coefficient itself:

$$\int_{-\infty}^{\eta_0} \frac{d\eta}{\eta^4} \, \langle \{\vec{k}\}_n | \, H_{\mathrm{I}}(\eta) \, |0\rangle \propto \psi_n(\{\vec{k}\}_n) \,. \tag{1.48}$$

The complex conjugate in the second term essentially flips the sign of the external energies (though the precise analytic continuation must be done with care), leading to the simple expression

$$\psi_n(k_1, ..., k_n) + \psi_n^*(-k_1, ..., -k_n) = 0 \,. \tag{1.49}$$

It is possible to derive additional relations for more complicated diagrams involving multiple vertices and exchanges by expanding the non-perturbative optical theorem to higher order in H_{I}. In effect one is deriving cosmological cutting rules, and the constraints may be organized into the following form

$$\psi_n(\{k\}_n) + \psi_n^*(\{-k\}_n) = -\sum_{\text{cuts}} \psi_n \,, \tag{1.50}$$

where $\{k\}_n$ are the energies of the external legs. The RHS is a sum over all partitions of a Witten diagram into a product of two pieces divided across an internal line, computed with this bulk-to-bulk propagator replaced by the *cut propagator*

$$\tilde{\mathcal{G}}(s; t, t') = \mathcal{G}(s, t, t') + \mathcal{G}^*(s, t, t') \,. \tag{1.51}$$

See Section 3.2 for additional details and examples.

Transmutation and Lifting: Essentially, the main idea of transmutation and lifting is the observation that for special mass values, wavefunction coefficients of scalar or spinning fields in de Sitter may be related to their flat space counterparts through a simple differential or

integral operation [31, 35]. It relies on the fact that for these special mass values, the de Sitter bulk-to-boundary propagators may be related to the analogous flat space quantities by acting with differential operators. Will not perform any heavy lifting in this thesis, but it is worth mentioning the procedure because it partially justifies our emphasis on the flat space wavefunction in Chapter 3: It should be possible to at directly lift at least a portion of our computations to de Sitter along these lines.

(Non-)Conformal Symmetries and Ward-Takahashi Identities: Additional powerful data for constraining the form of the wavefunction comes through non-conformal symmetry and the corresponding Ward-Takahashi (WT) identities. This is especially true for particles with spin, in which case WT identities are the manifestation of bulk gauge invariance on the late time wavefunction. These then represent an additional set of constraints to the conformal Ward identities that must be solved in order to derive a consistent wavefunction. Unlike their amplitude counterparts, for spinning wavefunctions the Ward-Takahasi identities are not vanishing statements. Instead, the Ward-Takahashi identity fixes the longitudinal component of the wavefunction in terms of lower point wavefunctions:

$$k_1^i \langle J^i_{\vec{k}_1} \mathcal{O}_{\vec{k}_2} ... \mathcal{O}_{\vec{k}_n} \rangle = -\sum_{a=2}^{n} \langle \mathcal{O}_{\vec{k}_2} ... \mathcal{O}_{\vec{k}_a + \vec{k}_1} ... \mathcal{O}_{\vec{k}_n} \rangle . \tag{1.52}$$

A large portion of Chapter 3 will be dedicated to deriving similar statements for the so-called exceptional scalar field theories. Like for spinning fields, their amplitudes posses WT identities which are simple vanishing statements, otherwise known as (enhanced) Adler zeros. As in the spinning case, the RHS of the Ward-Takahashi identity will not generically vanish at the level of the wavefunction, and will be given in terms of lower point wavefunctions. We will study in detail how this information may be used to bootstrap the wavefunctions of these theories.

Connection to Inflation and Phenomenology: Throughout this introduction, we have focused on quantum field theory in (Anti) de Sitter space. However for applications to

inflation, we need to account for the fact that the underlying inflationary background is only approximately de Sitter, evident from the fact that inflation has ended. Phenomenology is not the focus of this thesis, but for completeness here we will give an outline of how to do this and how some bootstrap data is imprinted on cosmological correlators in an observable way. We will only consider inflation driven by a single scalar field $\phi(x)$, which fluctuates around a spatially uniform background $\bar{\phi}(t)$. For potential energy $V(\bar{\phi}(t)) >> \frac{1}{2}\dot{\bar{\phi}}(t)$, this leads to an effective cosmological constant term in Einstein's equations, corresponding to accelerated expansion. The difference between inflationary evolution and pure de Sitter evolution is captured by the slow roll parameter

$$\epsilon \equiv \frac{1}{2}\frac{\dot{\bar{\phi}}(t)}{M_{\text{pl}}^2 H^2}, \qquad H \equiv \frac{\dot{a}(t)}{a(t)}, \tag{1.53}$$

where $a(t)$ is the scale factor. For slow roll parameters $\epsilon < 1$ is accelerated expansion, and $\epsilon \sim 0$ is pure de Sitter time evolution. Inflationary scalar fluctuations are typically characterized by the comoving curvature perturbation ζ, which in slow-roll inflation may be written as

$$\zeta = -\frac{H}{\dot{\bar{\phi}}}\phi\,, \tag{1.54}$$

where ϕ is a nearly massless field propagating in a pure de Sitter background. To be more specific, its Bunch-Davies mode function is given by (1.24), with $\Delta = 3-\epsilon$ (recall that $\Delta = 3$ corresponds to a massless scalar). Therefore, to leading order in the slowroll parameter one may compute inflationary correlators of ζ by first computing de Sitter correlators for a nearly massless scalar field, and using (1.54) to transform from ϕ to ζ.

One of the first successes of the cosmological bootstrap was the computation of the 4-point wavefunction for conformally coupled scalars and generic massive exchange. This was constructed as a power series solution to the conformal Ward identity (1.42), and possesses a

tell-tale oscillatory behavior in the collapsed limit, with $u, v \to 0$ and $u << v$ (equivalently, $s/(p_1 + p_2) \to 0$ and $s/(p_3 + p_4) \to 0$, with $p_3 + p_4 << p_1 + p_2$):

$$\lim_{u \to 0} \hat{F} = \text{Analytic} + \frac{\pi}{2 \cosh \pi\mu} \left(\frac{u}{v}\right)^{\frac{1}{2}} \frac{\sin(\mu \log u/v)}{\mu} , \qquad (1.55)$$

where the oscillation frequency $\mu = \sqrt{M^2/H^2 - 9/4}$ is set by the mass of the exchanged particle M. This characteristic oscillation is the analogue of a resonance peak for scattering experiments (see Figure 1.6). Away from this limit, the analytic contribution begins to dominate, and the 4-point function approaches that of a contact interaction, where the massive particle has been integrated out. In this way, one may search for characteristic oscillations of cosmological correlators in order to identify new massive particles present during inflation.
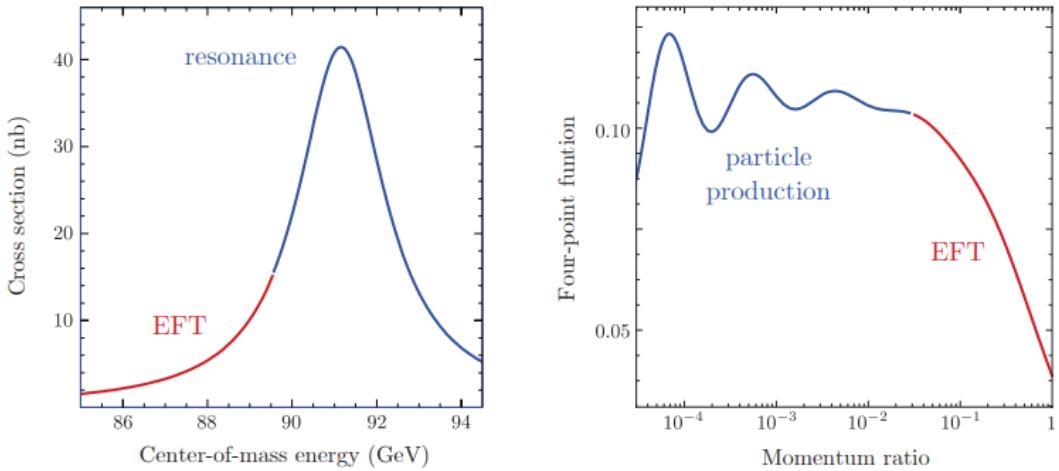
Figure 1.6: Image taken from *The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities* [28]. Left: Resonance peak in a scattering experiment. For center-of-mass energy near the mass of some intermediate particle, the cross-section is sharply peaked. Away from this energy value, the shape is governed by analytic (EFT) contributions to the amplitude. Right: Characteristic oscillation of the wavefunction in the collapsed limit. For $u/v << 1$, the 4-point function has a characteristic oscillation whose frequency is set by the mass of the intermediate particle. Away from this limit, the shape is governed by EFT contributions to the 4-point function.

Chapter 2: Partially Massless Multiplets and Non-Unitary Superconformal Representations for $\mathcal{N} = 2$

2.1 Introduction

Partially massless (PM) particles are exotic representations of the (anti) de Sitter ((A)dS) isometry algebras [87, 63, 62, 88, 64, 65, 89, 66, 90, 91, 92]. These representations occur at discrete mass values relative to the (A)dS curvature and possess a gauge symmetry, despite their mass terms. Partially massless fields exist for all spins $s \geq 1$ and are labeled by their depth t: for bosons $t \in \{0, 1, ...s-1\}$ and for fermions $t \in \{1/2, 3/2, ..., s-1\}$. For a depth-$t$ PM field, the gauge symmetry removes the helicity components with helicities $\leq t$ from the particle. The depth $t = s - 1$ field corresponds to the usual massless representation. $\mathcal{N} = 1$ supersymmetric (SUSY) extensions of partially massless representations of AdS$_4$ were studied in [93, 94]. In this work, we follow up those results by finding $\mathcal{N} = 2$ SUSY multiplets on AdS$_4$ which contain partially massless fields. We will also comment briefly on some general properties of PM representations expected for $\mathcal{N} > 2$

There are several reasons to extend the previous studies to $\mathcal{N} > 1$. Given the importance of the AdS group and its supersymmetric extensions – holography and higher-spin theory being two examples [95, 96, 97, 98, 41] – a complete classification of the SUSY AdS representations is desirable. Moreover, a deeper understanding of partially massless representations could also shed light on the difficulties encountered when constructing interacting field theories for PM particles [72, 99, 100, 73, 101, 71, 68, 102, 74, 103, 104, 105, 106, 107, 108, 109, 110, 69, 111, 112, 113, 114, 115, 116, 117]. Going beyond the restrictions of unitarity is also desirable. From the boundary conformal field theory (CFT) point of view, PM fields

in AdS are dual to CFT currents satisfying higher derivative conservation conditions [118], which occur only in non-unitary CFTs. Non-unitary CFTs and the representations they realize have found applications in condensed matter systems [119], and in understanding the analytic structure of the conformal blocks in ordinary unitary CFTs [120, 121, 122]. Unitary superconformal representations have been extensively studied and classified [123, 124, 125, 126, 127, 128, 129, 130], however the non-unitary cases remain relatively unexplored, though some results are known [131, 132, 122, 93, 133]. Here we will uncover new phenomena for $\mathcal{N} = 2$, such as exotic shortenings and extended modules, that occur in non-unitary regions.

Another reason to consider $\mathcal{N} = 2$ is that when considering dS space, the supersymmetric extensions of the dS group require an even number of supercharges [134, 135]. The essential reason is that the supercharges of SUSY-dS$_4$ must satisfy a symplectic Majorana condition, and for this $\mathcal{N}$ must be even. The equivalent statement at the Lagrangian level is that the generator of SUSY transformations cannot be a standard Majorana Killing spinor, since such spinors do not exist in dS$_4$ (see [136] for a related discussion). Thus, if one is interested in constructing supersymmetric theories with partially massless particles on dS (because the bosonic PM fields are themselves unitary on dS, but not on AdS), one needs to consider even $\mathcal{N}$.

As we will see, extended SUSY allows for a rich structure of multiplets containing PM fields. While for $\mathcal{N} = 1$, PM fields always sit in long supermultiplets [93], there is no reason to expect that this will be the case for $\mathcal{N} \geq 2$. By "long" here, we mean having as many conformal primary states as a generic long supermultiplet; the supermultiplet itself can contain conformal primaries which are themselves short states due to the presence of the PM gauge symmetries, but this is unrelated to SUSY. Indeed, for $\mathcal{N} = 2$ we find that supermultiplets with PM states can be either long or short, in addition to the exotic possibility of featuring a so-called "extended module" phenomenon. In particular, we find that for $\mathcal{N} = 2$ there are short supermultiplets where the highest-spin state in the multiplet

is a spin-s partially massless state of depth $t = s - 2$. Thus, unlike the $\mathcal{N} = 1$ case, the simplest $\mathcal{N} = 2$ supermultiplet containing a partially massless spin-2 particle is short, and it also contains several exotic fields.

Conventions: This paper relies heavily on the notation and concepts introduced in [93], and the conventions used here are detailed there.

2.2 The superconformal algebra

As in [93], we study supersymmetric extensions of the PM representations via the AdS/CFT correspondence. We are interested in AdS$_4$ SUSY, thus we study $d = 3$ superconformal symmetry on the boundary.

2.2.1 $\mathcal{N}$ extended $d = 3$ superconformal algebra

The generators of the euclidean $\mathcal{N}$ extended superconformal algebra are

$$P^i, \ J^{ij}, \ D, \ \ K^i, \ Q^{aI}, \ S^I_a, \ R^{IJ}. \tag{2.1}$$

The P^i are the translations and the anti-symmetric J^{ij} are the rotations, which together generate the Poincare transformations of $d = 3$ Euclidean space. The dilation is D and the special conformal generators are K^i, which together with the Poincare generators generate the conformal symmetries. Q^{aI} are the spinor-valued supersymmetries, labelled by the index $I = 1, \ldots, \mathcal{N}$. Together with the Poincare generators they generate $\mathcal{N}$ extended SUSY. The S^I_a are the special superconformal generators, and the anti-symmetric R^{IJ} are $so(\mathcal{N})$ R-symmetries, which together complete the SUSY generators and conformal generators into to the $\mathcal{N}$ extended superconformal algebra.

The non-vanishing (anti)commutators are as follows [127]: First there are the usual

commutators of the Poincare algebra,

$$\left[J^{ij}, P^k\right] = i\left(-\delta^{ki}P^j + \delta^{kj}P^i\right),$$

$$\left[J^{ij}, J^{kl}\right] = i\left(-\delta^{ik}J^{jl} + \delta^{jk}J^{il} - \delta^{jl}J^{ik} + \delta^{il}J^{jk}\right). \tag{2.2}$$

The commutators which when taken together with (2.2) fill out the conformal algebra are

$$\left[D, P^i\right] = P^i,$$

$$\left[D, K^i\right] = -K^i,$$

$$\left[K^i, P^j\right] = 2(\delta^{ij}D + iJ^{ij}),$$

$$\left[J^{ij}, K^k\right] = i\left(-\delta^{ki}K^j + \delta^{kj}K^i\right). \tag{2.3}$$

The commutators which when taken together with (2.2) form the $\mathcal{N}$ extended SUSY algebra are

$$\left\{Q^{aI}, Q^{bJ}\right\} = 2\sigma_i^{ab}P^i\,\delta^{IJ}, \tag{2.4}$$

$$\left[J_{ij}, Q^{aI}\right] = -\frac{i}{2}\left(\sigma_{ij}\right)^a{}_b Q^{bI}, \tag{2.5}$$

$$\left[R^{IJ}, R^{KL}\right] = i\left(-\delta^{IK}R^{JL} + \delta^{JK}R^{IL} - \delta^{JL}R^{IK} + \delta^{IL}R^{JK}\right), \tag{2.6}$$

$$\left[R^{IJ}, Q^{aK}\right] = i\left(-\delta^{IK}Q^{aJ} + \delta^{JK}Q^{aI}\right). \tag{2.7}$$

The first line (2.4) is the main anti-commutator indicative of SUSY, (2.5) shows that Q^{aI} transforms as a spinor under rotations, (2.6) is the statement that R^{IJ} forms an $so(\mathcal{N})$, and (2.7) shows that Q^{aI} transforms as a vector under this $so(\mathcal{N})$.

The remaining non-trivial commutators, which when taken with the above fill out the

superconformal algebra, are

$$
\begin{aligned}
& \left[J_{ij}, S^{aI}\right] = -\frac{i}{2}\left(\sigma_{ij}\right)^{a}{}_{b} S^{bI}, \quad \left[R^{IJ}, S^{aK}\right] = i\left(-\delta^{IK} S^{aJ} + \delta^{JK} S^{aI}\right), \\
& \left[D, Q^{aI}\right] = \frac{1}{2}Q^{aI}, \quad \left[D, S^{aI}\right] = -\frac{1}{2}S^{aI} \\
& \left\{S^{aI}, S^{bJ}\right\} = -2\sigma_{i}^{ab} K^{i}\,\delta^{IJ}, \\
& \left\{Q^{aI}, S^{bJ}\right\} = 2\delta^{IJ}\epsilon^{ab} D - i\delta^{IJ}\sigma_{ij}^{ab} J^{ij} + 2i\epsilon^{ab} R^{IJ}, \\
& \left[K_{i}, Q^{aI}\right] = -\left(\sigma_{i}\right)^{a}{}_{b} S^{bI}, \\
& \left[P_{i}, S^{aI}\right] = \left(\sigma_{i}\right)^{a}{}_{b} Q^{bI}.
\end{aligned}
\tag{2.8}
$$

The first line of (2.8) indicates that S^{aI} transforms as a spinor under rotations and a vector under $so(\mathcal{N})$ R-symmetry, the second line indicates that Q^{aI} carries scaling dimension $1/2$ and S_{a}^{I} carries scaling dimension $-1/2$.

In radial quantization, the generators satisfy the conjugation relations[1]

$$
P^{i\dagger} = K^{i}, \quad D^{\dagger} = D, \quad J^{ij\dagger} = J^{ij}, \quad Q^{aI\dagger} = S_{a}^{I}, \quad R^{IJ\dagger} = R^{IJ}. \tag{2.9}
$$

The (anti)commutation relations above are all consistent with the reality conditions (2.9).

2.2.2 Algebra in spinor form

It will be convenient to put the algebra into pure spinor form by contracting the various $3d$ vectors with sigma matrices. The translations and special conformal generators become symmetric 2-index spinors,

$$
P^{ab} = \sigma_{i}^{ab} P^{i}, \quad P^{i} = -\frac{1}{2}\sigma_{ab}^{i} P^{ab}, \tag{2.10}
$$

---

[1]Recall that with our conventions, outlined in [93], there is a subtlety with the indices; the condition $Q_{a}^{\dagger} = S_{a}$ implies $S_{a}^{\dagger} = Q^{a}$, but when both raising and lowering indices we get a sign: $Q_{a}^{\dagger} = -S^{a}$, $S^{a\dagger} = -Q_{a}$.

$$K^{ab} = \sigma_i^{ab} K^i, \qquad K^i = -\frac{1}{2}\sigma^i_{ab} K^{ab}, \tag{2.11}$$

and the rotations are dualized into a vector and then converted into a symmetric 2-index spinor,

$$J^i = -\frac{1}{2}\epsilon^{ijk} J_{jk}, \quad J^{ij} = -\epsilon^{ijk} J_k, \quad J^{ab} = \sigma_i^{ab} J^i, \quad J^i = -\frac{1}{2}\sigma^i_{ab} J^{ab}. \tag{2.12}$$

The commutators now take the form

$$
\begin{aligned}
\left[J^{ab}, P^{cd}\right] &= \frac{1}{2}\left(\epsilon^{ac} P^{bd} + \epsilon^{bc} P^{ad} + \epsilon^{ad} P^{cb} + \epsilon^{bd} P^{ca}\right), \\
\left[J^{ab}, J^{cd}\right] &= \frac{1}{2}\left(\epsilon^{ac} J^{bd} + \epsilon^{bc} J^{ad} + \epsilon^{ad} J^{cb} + \epsilon^{bd} J^{ca}\right),
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\left[D, P^{ab}\right] &= P^{ab}, \quad \left[D, K^{ab}\right] = -K^{ab}, \\
\left[K^{ab}, P^{cd}\right] &= -2\left(\epsilon^{ac}\epsilon^{bd} + \epsilon^{bc}\epsilon^{ad}\right) D - \left(\epsilon^{ac} J^{bd} + \epsilon^{bc} J^{ad} + \epsilon^{ad} J^{cb} + \epsilon^{bd} J^{ca}\right), \\
\left[J^{ab}, K^{cd}\right] &= \frac{1}{2}\left(\epsilon^{ac} K^{bd} + \epsilon^{bc} K^{ad} + \epsilon^{ad} K^{cb} + \epsilon^{bd} K^{ca}\right),
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\left[J^{ab}, Q^{cI}\right] &= \frac{1}{2}\left(\epsilon^{ac} Q^{bI} + \epsilon^{bc} Q^{aI}\right), \quad \left[R^{IJ}, Q^{aK}\right] = i\left(-\delta^{IK} Q^{aJ} + \delta^{JK} Q^{aI}\right), \\
\left[R^{IJ}, R^{KL}\right] &= i\left(-\delta^{IK} R^{JL} + \delta^{JK} R^{IL} - \delta^{JL} R^{IK} + \delta^{IL} R^{JK}\right), \\
\left\{Q^{aI}, Q^{bJ}\right\} &= 2P^{ab}\,\delta^{IJ},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\left[J^{ab}, S^{cI}\right] &= \frac{1}{2}\left(\epsilon^{ac} S^{bI} + \epsilon^{bc} S^{aI}\right), \quad \left[R^{IJ}, S^{aK}\right] = i\left(-\delta^{IK} S^{aJ} + \delta^{JK} S^{aI}\right), \\
\left[D, Q^{aI}\right] &= \frac{1}{2}Q^{aI}, \quad \left[D, S^{aI}\right] = -\frac{1}{2}S^{aI}, \\
\left\{S^{aI}, S^{bJ}\right\} &= -2K^{ab}\,\delta^{IJ}, \\
\left\{Q^{aI}, S^{bJ}\right\} &= 2\delta^{IJ}\epsilon^{ab} D - 2\delta^{IJ} J^{ab} + 2i\epsilon^{ab} R^{IJ}, \\
\left[K^{ab}, Q^{cI}\right] &= \epsilon^{ac} S^{bI} + \epsilon^{bc} S^{aI}, \quad \left[P^{ab}, S^{cI}\right] = -\left(\epsilon^{ac} Q^{bI} + \epsilon^{bc} Q^{aI}\right).
\end{aligned}
\tag{2.16}
$$

The conjugation relations (2.9) now read

$$P^{ab\dagger} = -K_{ab}, \quad D^\dagger = D, \quad J^{ab\dagger} = -J_{ab}, \quad Q^{aI\dagger} = S_a^I, \quad R^{IJ\dagger} = R^{IJ}. \tag{2.17}$$

2.2.3 $\mathcal{N} = 2$

For $\mathcal{N} = 2$, the case of primary interest for us, we have only one R symmetry generator, $R \equiv R^{12}$, generating $u(1)_R$, and two sets of supercharges Q^{a1}, Q^{a2}, S_a^1, S_a^2, which transform under the $U(1)_R$ as

$$\left[R, Q^{a1}\right] = -iQ^{a2}, \quad \left[R, Q^{a2}\right] = iQ^{a1}, \quad \left[R, S^{a1}\right] = -iS^{a2}, \quad \left[R, S^{a2}\right] = iS^{a1}. \tag{2.18}$$

We can diagonalize this $u(1)_R$ action by defining the linear combinations

$$
\begin{aligned}
Q^a &= \frac{1}{\sqrt{2}}\left(Q^{a1} - iQ^{a2}\right), \quad \bar{Q}^a = \frac{1}{\sqrt{2}}\left(Q^{a1} + iQ^{a2}\right), \\
S^a &= \frac{1}{\sqrt{2}}\left(S^{a1} - iS^{a2}\right), \quad \bar{S}^a = \frac{1}{\sqrt{2}}\left(S^{a1} + iS^{a2}\right),
\end{aligned}
\tag{2.19}
$$

in terms of which (2.18) becomes

$$[R, Q^a] = Q^a, \quad \left[R, \bar{Q}^a\right] = -\bar{Q}^a, \quad [R, S^a] = S^a, \quad \left[R, \bar{S}^a\right] = -\bar{S}^a. \tag{2.20}$$

The conjugation rules are now

$$Q^{a\dagger} = \bar{S}_a, \quad \bar{Q}^{a\dagger} = S_a, \quad S^{a\dagger} = -\bar{Q}_a, \quad \bar{S}^{a\dagger} = -Q_a. \tag{2.21}$$

The non-vanishing commutators involving more than one Q and/or S are now

$$\{Q^a, \bar{Q}^b\} = 2P^{ab}, \quad \{S^a, \bar{S}^b\} = -2K^{ab}, \tag{2.22}$$

$$\{Q^a, \bar{S}^b\} = 2\epsilon^{ab}(D - R) - 2J^{ab}, \quad \{\bar{Q}^a, S^b\} = 2\epsilon^{ab}(D + R) - 2J^{ab}. \tag{2.23}$$

2.3 $\mathcal{N} = 2$ Superconformal representations

We will be interested in finding representations of $\mathcal{N} = 2$ SUSY which contain partially massless fields. The various bosonic and fermionic PM fields on AdS_4 and their dual operators in CFT_3 are reviewed in Section 2 of [93], whose notation we follow. Let us recall here the AdS/CFT mass formula and PM mass values. A spin-s fields in AdS_4 with mass m correspond to spin-s primary operators on the boundary with scaling dimension Δ related by

$$m^2 L^2 = \begin{cases} \Delta(\Delta - 2) & s = 0 \\ (\Delta + s - 2)(\Delta - s - 1) & s \geq \frac{1}{2} \end{cases} \tag{2.24}$$

where L is the AdS radius. There are two choices of boundary conditions for the AdS fields, for the "standard quantization" boundary conditions one takes the greater root with $\Delta > \frac{3}{2}$, whereas for the "alternate quantization" [137] boundary conditions one takes the lesser root with $\Delta < \frac{3}{2}$. For bosons and fermions, the PM mass values are

$$m^2_{s,t} = \frac{1}{L^2}(t - s + 1)(s + t) \tag{2.25}$$

In the standard quantization, this means that both PM bosons and fermions of spin s and depth t have a scaling dimension of $\Delta_{s,t} = t + 2$.

Superconformal representations can be constructed by joining together conformal representations. Conformal representations are built from a conformal primary. A spin-s conformal primary of weight Δ is indicated by $|\Delta\rangle^{a_1 \dots a_{2s}}$. It has $2s$ fully symmetric spinor indices

and transforms under rotations as an irreducible representation of spin s,

$$J_{ij} |\Delta\rangle^{a_1...a_{2s}} = -\frac{i}{2}(\sigma_{ij})^{a_1}{}_b |\Delta, r\rangle^{ba_2...a_{2s}} - ... - \frac{i}{2}(\sigma_{ij})^{a_{2s}}{}_b |\Delta\rangle^{a_1...a_{2s-1}b} , \qquad (2.26)$$

$$J_{ij} J^{ij} |\Delta\rangle^{a_1...a_{2s}} = 2s(s+1) |\Delta\rangle^{a_1...a_{2s}} . \qquad (2.27)$$

In terms of the rotation generators in spinor form this becomes

$$J^{ab}|\Delta\rangle^{c_1\cdots c_{2s}} = \frac{1}{2}\epsilon^{ac_1}|\Delta\rangle^{bc_2\cdots c_{2s}} + \frac{1}{2}\epsilon^{ac_2}|\Delta\rangle^{c_1 bc_3\cdots c_{2s}} + \cdots + \frac{1}{2}\epsilon^{ac_{2s}}|\Delta\rangle^{c_1\cdots c_{2s-1}b} + (a \leftrightarrow b) \quad (2.28)$$

$$J^{ab} J_{ab}|\Delta\rangle^{c_1\cdots c_{2s}} = -J^{ij} J_{ij}|\Delta\rangle^{c_1\cdots c_{2s}} = -2J^i J_i|\Delta\rangle^{c_1\cdots c_{2s}} = -2s(s+1)|\Delta\rangle^{c_1\cdots c_{2s}}. \qquad (2.29)$$

Under dilations it has eigenvalue Δ,

$$D |\Delta\rangle^{a_1...a_{2s}} = \Delta |\Delta\rangle^{a_1...a_{2s}} . \qquad (2.30)$$

A conformal primary obeys

$$K_i |\Delta\rangle^{a_1...a_{2s}} = 0 \qquad (2.31)$$

and the rest of the representation is built by repeatedly acting with P^i. A state with l actions of P^i has dimension $\Delta + l$, the states of this dimension are

$$P^{i_1} \cdots P^{i_l} |\Delta\rangle^{a_1...a_{2s}} . \qquad (2.32)$$

These states are then further decomposed into irreducible representations under J_{ij} by symmetrizing, removing traces, etc. We will label a conformal multiplet whose primary has scaling dimension Δ and spin s by $[s]_\Delta$.

Superconformal representations are constructed by joining together conformal representations via the action of the Q's and the S's. We begin with a superconformal primary $|\Delta, r\rangle^{a_1...a_{2s}}$ which is a spin-s conformal primary with conformal weight Δ, which is also an

eigenstate of the R-symmetry generator,

$$R \left| \Delta, r \right\rangle^{a_1 \ldots a_{2s}} = r \left| \Delta, r \right\rangle^{a_1 \ldots a_{2s}}, \tag{2.33}$$

and which is also annihilated by the S's,

$$S_a \left| \Delta, r \right\rangle^{a_1 \ldots a_{2s}} = 0, \quad \bar{S}_a \left| \Delta, r \right\rangle^{a_1 \ldots a_{2s}} = 0. \tag{2.34}$$

We find the other conformal primaries within the superconformal multiplet by computing the Q descendants of the superconformal primary, which come from acting with Q (or $\bar{Q}$). This raises Δ by $1/2$ and raises (lowers) r by 1. This can be repeated until we get up to four total factors of Q, $\bar{Q}$. After this the process terminates, because adding an additional Q or $\bar{Q}$ reduces, after using the commutation relations, to acting with P^i or causes the state to vanish. In many instances, to actually construct a conformal primary from a given Q descendent, we must also add on linear combinations of P^i descendants so that the total state is annihilated by K^i. These P descendants must have the same quantum numbers as the Q-descendants we are adding them to. In this way, we construct a superconformal multiplet, which we label by $\{s\}_{\Delta, r}$.

In what follows we will list the conformal primaries present at each level for a given superconformal multiplet. The $\{0\}_{\Delta, r}$ and $\{\frac{1}{2}\}_{\Delta, r}$ multiplets are qualitatively different from the multiplets for generic spin $s > 1/2$, so we will present those first. This is because acting with certain combinations of Q lowers the spin; at most, the spin is lowered by 1, and such states do not exist in the spin-0 and spin-$\frac{1}{2}$ multiplets. The results for the $\{0\}_{\Delta, r}$ multiplet, the $\{\frac{1}{2}\}_{\Delta, r}$ multiplet, and the generic $\{s\}_{\Delta; r}$ multiplet are recorded in Table 2.1, Table 2.2, and Table 2.3 respectively. Note that there is an ambiguity in the ordering of the Q's and $\bar{Q}$'s when acting on the superconformal primary, and different choices will lead to differed P terms in the expressions. Our choice is that in states with $\delta r \leq 0$, where δr is the

difference between the state's r charge and the superconformal primary's r charge, we have the Q's acting first, and states with $\delta r > 0$ are reversed from those with $\delta r < 0$ by taking $Q \to \bar{Q}$, $\bar{Q} \to Q$ and $r \to -r$.

For each conformal primary, there is a CPT-reversed state which can be constructed by carrying out $Q \to \bar{Q}$, $\bar{Q} \to Q$ and $r \to -r$. When constructing Lagrangians or CPT invariant theories, all multiplets must appear with their CPT conjugates. In all the multiplets that follow, the masses of the various bulk fields on the AdS side can be found from Δ through the AdS/CFT mass formula (2.24).

2.3.1 $s = 0$

The conformal primaries of the $\{0\}_{\Delta,r}$ multiplet are shown in Table 2.1. The multiplet can be visualized as in Figure 2.1. From the bulk point of view, for generic values of Δ this multiplet contains 5 massive scalars (1 degree of freedom each) and a massive vector (3 degrees of freedom), for a total of 8 bosonic propagating degrees of freedom, and 4 massive spin 1/2 fermions (2 degrees of freedom each), for a total of 8 fermionic propagating degrees of freedom.

Table 2.1: Conformal primaries in the $\{0\}_{\Delta,r}$ superconformal multiplet.

Conformal Primaries of $\{0\}_{\Delta,r}$	
Level 1	
$\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2},r-1}$	$\bar{Q}^{a_1}\left|\Delta,r\right\rangle$
$\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2},r+1}$	$Q^{a_1}\left|\Delta,r\right\rangle$
Level 2	
$[0]_{\Delta+1,r}$	$\bar{Q}^c Q_c\left|\Delta,r\right\rangle$
$[0]_{\Delta+1,r-2}$	$\bar{Q}^c \bar{Q}_c\left|\Delta,r\right\rangle$

$[0]_{\Delta+1,r+2}$	$Q^c Q_c \left| \Delta, r \right\rangle$
$[1]_{\Delta+1,r}$	$\bar{Q}^{(a_1} Q^{a_2)} \left| \Delta, r \right\rangle - \left(\frac{\Delta-r}{\Delta} \right) P^{a_1 a_2} \left| \Delta, r \right\rangle$
Level 3	
$\left[\frac{1}{2} \right]_{\Delta+\frac{3}{2},r-1}$	$\bar{Q}^c \bar{Q}_c Q^{a_1} \left| \Delta, r \right\rangle - 2 \left(\frac{\Delta-r}{\Delta-\frac{1}{2}} \right) P^{a_1}{}_c \bar{Q}^c \left| \Delta, r \right\rangle$
$\left[\frac{1}{2} \right]_{\Delta+\frac{3}{2},r+1}$	$Q^c Q_c \bar{Q}^{a_1} \left| \Delta, r \right\rangle - 2 \left(\frac{\Delta+r}{\Delta-\frac{1}{2}} \right) P^{a_1}{}_c Q^c \left| \Delta, r \right\rangle$
Level 4	
$[0]_{\Delta+2,r}$	$\bar{Q}^c \bar{Q}_c Q^d Q_d \left| \Delta, r \right\rangle + 4 \left(\frac{\Delta-r-1}{\Delta-1} \right) P_{cd} \bar{Q}^{(c} Q^{d)} \left| \Delta, r \right\rangle$ $- 2 \frac{(\Delta-r)(\Delta-r-1)}{(\Delta-1)(\Delta-\frac{1}{2})} P_{cd} P^{cd} \left| \Delta, r \right\rangle$

level 4 $[0]_{\Delta+2,r}$

level 3 $\left[\tfrac{1}{2}\right]_{\Delta+\frac{3}{2},r+1}$ $\left[\tfrac{1}{2}\right]_{\Delta+\frac{3}{2},r-1}$

level 2 $[0]_{\Delta+1,r+2}$ $[0]_{\Delta+1,r}$ $[0]_{\Delta+1,r-2}$ $[1]_{\Delta+1,r}$

level 1 $\left[\tfrac{1}{2}\right]_{\Delta+\frac{1}{2},r+1}$ $\left[\tfrac{1}{2}\right]_{\Delta+\frac{1}{2},r-1}$

Δ r s

level 0 $[0]_{\Delta,r}$
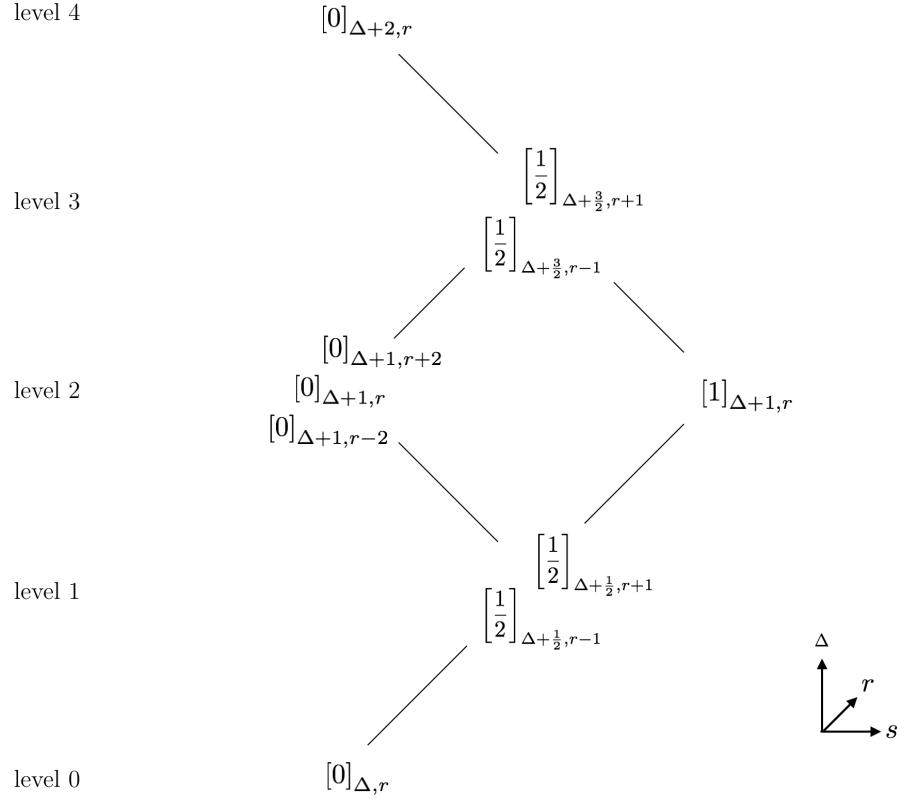
Figure 2.1: Generic scalar multiplet $\{0\}_{\Delta,r}$. The notation $[s]_{\Delta,k}$ denotes each conformal primary within the superconformal multiplet. The $[0]_{\Delta,k}$ conformal primary at level zero is also the superconformal primary. There are four levels, which corresponds to the fact that we can act on the conformal primary with at most four supercharges $Q_a, \bar{Q}_a$.

2.3.2 $s = 1/2$

The conformal primaries of the $\{\frac{1}{2}\}_{\Delta,r}$ multiplet are shown in Table 2.2. Note that at level 2 there are two different states with spin 1/2, weight $\Delta + 1$, and charge r, these are denoted by $[\frac{1}{2}]^{(1)}_{\Delta+1,r}$ and $[\frac{1}{2}]^{(2)}_{\Delta+1,r}$. This multiplet can be visualized as in Figure 2.2. From the bulk point of view, for generic values of Δ this multiplet contains 4 massive scalars (1 degree of freedom each) and 4 massive vectors (3 degrees of freedom each), for a total of 16 bosonic propagating degrees of freedom, and 6 massive spin 1/2 fermions (2 degrees of freedom each) and one massive spin 3/2 fermion (4 degrees of freedom), for a total of 16 fermionic propagating degrees of freedom.

Table 2.2: Conformal primaries in the $\{\frac{1}{2}\}_{\Delta,r}$ superconformal multiplet.

Conformal Primaries of $\{\frac{1}{2}\}_{\Delta,r}$	
Level 1	
$[0]_{\Delta+\frac{1}{2},r-1}$	$\bar{Q}_c \lvert \Delta, r \rangle^c$
$[0]_{\Delta+\frac{1}{2},r+1}$	$Q_c \lvert \Delta, r \rangle^c$
$[1]_{\Delta+\frac{1}{2},r-1}$	$\bar{Q}^{(a_1} \lvert \Delta, r \rangle^{a_2)}$
$[1]_{\Delta+\frac{1}{2},r+1}$	$Q^{(a_1} \lvert \Delta, r \rangle^{a_2)}$
Level 2	
$[\frac{1}{2}]^{(1)}_{\Delta+1,r}$	$\bar{Q}^c Q_c \lvert \Delta, r \rangle^{a_1} + \left(\frac{1}{\Delta-1} \right) P^{a_1}{}_c \lvert \Delta, r \rangle^c$
$[\frac{1}{2}]^{(2)}_{\Delta+1,r}$	$\bar{Q}_c Q^{a_1} \lvert \Delta, r \rangle^c - \left(\frac{\Delta-r-\frac{1}{2}}{\Delta-1} \right) P^{a_1}{}_c \lvert \Delta, r \rangle^c$
$[\frac{1}{2}]_{\Delta+1,r-2}$	$\bar{Q}^c \bar{Q}_c \lvert \Delta, r \rangle^{a_1}$
$[\frac{1}{2}]_{\Delta+1,r+2}$	$Q^c Q_c \lvert \Delta, r \rangle^{a_1}$
$[\frac{3}{2}]_{\Delta+1,r}$	$\bar{Q}^{(a_1} Q^{a_2} \lvert \Delta, r \rangle^{a_3)} - \left(\frac{\Delta-r+\frac{1}{2}}{\Delta+\frac{1}{2}} \right) P^{(a_1 a_2} \lvert \Delta, r \rangle^{a_3)}$

Level 3	
$[0]_{\Delta+\frac{3}{2},r-1}$	$\bar{Q}^c\bar{Q}_cQ_d\left|\Delta,r\right\rangle^d - 2\left(\frac{\Delta-r-\frac{3}{2}}{\Delta-\frac{3}{2}}\right)P_{cd}\bar{Q}^{(c}\left|\Delta,r\right\rangle^{d)}$
$[0]_{\Delta+\frac{3}{2},r+1}$	$Q^cQ_c\bar{Q}_d\left|\Delta,r\right\rangle^d - 2\left(\frac{\Delta+r-\frac{3}{2}}{\Delta-\frac{3}{2}}\right)P_{cd}Q^{(c}\left|\Delta,r\right\rangle^{d)}$
$[1]_{\Delta+\frac{3}{2},r-1}$	$\bar{Q}^c\bar{Q}_cQ^{(a_1}\left|\Delta,r\right\rangle^{a_2)} + 2\left(\frac{\Delta-r+\frac{1}{2}}{\Delta-\frac{1}{2}}\right)P^{(a_1}{}_c\bar{Q}^{(a_2)}\left|\Delta,r\right\rangle^{c)}$ $+ \left(\frac{\Delta-r+\frac{1}{2}}{\Delta+\frac{1}{2}}\right)P^{a_1a_2}\bar{Q}_c\left|\Delta,r\right\rangle^c$
$[1]_{\Delta+\frac{3}{2},r+1}$	$Q^cQ_c\bar{Q}^{(a_1}\left|\Delta,r\right\rangle^{a_2)} - 2\left(\frac{\Delta+r+\frac{1}{2}}{\Delta-\frac{1}{2}}\right)P^{(a_1}{}_cQ^{(a_2)}\left|\Delta,r\right\rangle^{c)}$ $+ \left(\frac{\Delta+r+\frac{1}{2}}{\Delta+\frac{1}{2}}\right)P^{a_1a_2}Q_c\left|\Delta,r\right\rangle^c$
Level 4 [2]	
$[\frac{1}{2}]_{\Delta+2,r}$	$\bar{Q}^c\bar{Q}_cQ^dQ_d\left|\Delta,r\right\rangle^{a_1} - 2\frac{(\Delta-r-\frac{3}{2})(\Delta-r+\frac{1}{2})}{\Delta(\Delta-\frac{3}{2})}P^{cd}P_{cd}\left|\Delta,r\right\rangle^{a_1}$ $+ 4\left(\frac{\Delta-r-\frac{3}{2}}{\Delta-\frac{3}{2}}\right)P_{cd}\bar{Q}^{(c}Q^d\left|\Delta,r\right\rangle^{a_1)} - \frac{4}{3}\left(\frac{\Delta-r+\frac{3}{2}}{\Delta}\right)P^{a_1}{}_c\bar{Q}^dQ_d\left|\Delta,r\right\rangle^c$ $- \frac{8}{3}\frac{(\Delta-r)}{\Delta}P^{a_1}{}_c\bar{Q}_dQ^c\left|\Delta,r\right\rangle^d$

---

[2]Note that at level 4, naively there are two possible P^2 terms, namely $P^{bc}P_{bc}\left|\Delta\right\rangle^{a_1}$ and $P^{a_1}{}_bP^b{}_c\left|\Delta\right\rangle^c$. However we have

$$P^{a_1}{}_bP^b{}_c\left|\Delta\right\rangle^c = P^iP^j\left(\sigma_i\sigma_j\right)^{a_1}{}_c\left|\Delta\right\rangle^c = P^iP^j\left(\frac{1}{2}\{\sigma_i,\sigma_j\}\right)^{a_1}{}_c\left|\Delta\right\rangle^c$$
$$= P^iP^j\delta_{ij}\left(\mathbb{I}\right)^{a_1}{}_c\left|\Delta\right\rangle^c = P^2\left|\Delta\right\rangle^{a_1} = -\frac{1}{2}P^{bc}P_{bc}\left|\Delta\right\rangle^{a_1},$$

thus there is really only one spin-1/2 P^2 descendent, see e.g. [138].
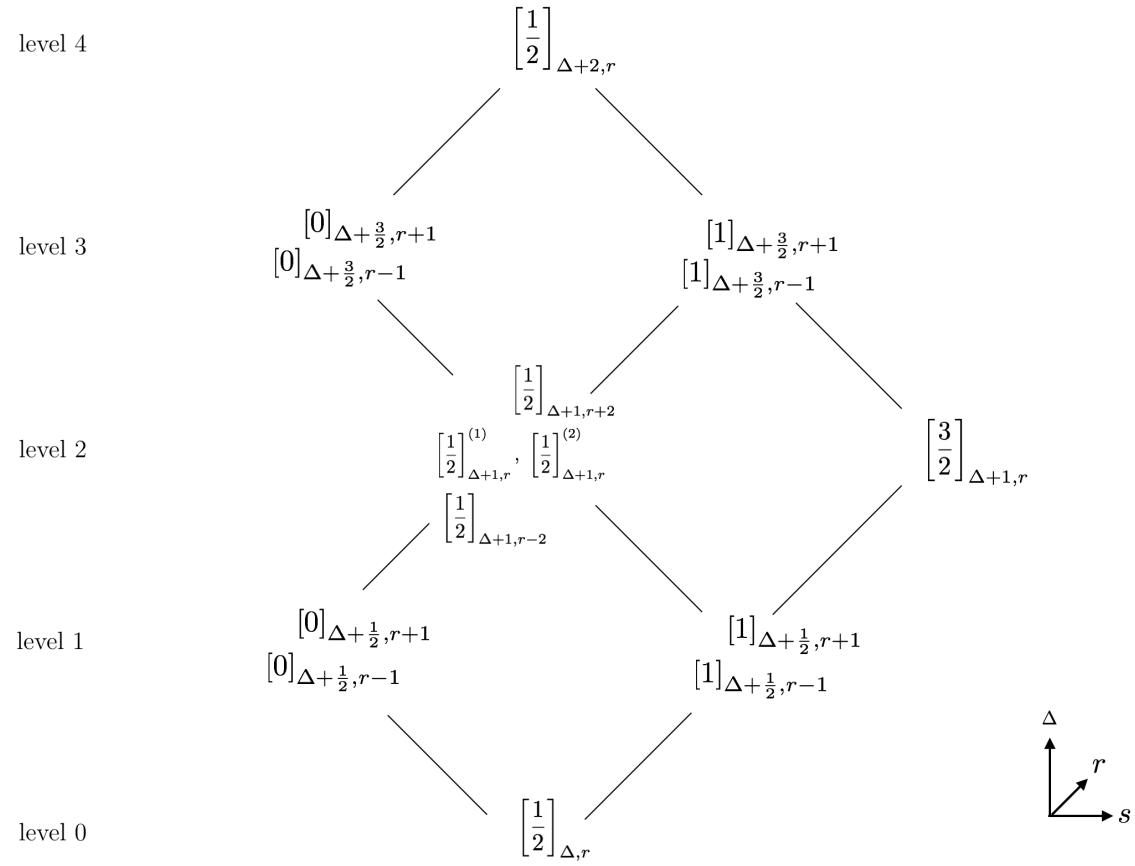
Figure 2.2: Conformal primaries of the generic multiplet $\left\{\frac{1}{2}\right\}_{\Delta,r}$. The $[\frac{1}{2}]_{\Delta,k}$ conformal primary at level zero is also the superconformal primary.

2.3.3 $s \geq 1$

The conformal primaries of the generic $\{s\}_{\Delta,r}$ multiplet with $s \geq 1$ are shown in Table 2.3. Note that at level 2 there are two different states with spin s, weight $\Delta+1$, and charge r, these are denoted by $[s]^{(1)}_{\Delta+1,r}$ and $[s]^{(2)}_{\Delta+1,r}$. This multiplet can be visualized as in Figure 2.3. From the bulk point of view, for generic values of Δ this multiplet contains one massive spin $s-1$ field, 4 massive spin $s-\frac{1}{2}$ fields, 6 massive spin s fields, 4 massive spin $s+\frac{1}{2}$ fields, and one massive spin $s+1$ field. Using the fact that a massive spin s field has $2s+1$ propagating degrees of freedom, the multiplet is seen to contain $8(2s+1)$ bosonic degrees of freedom and an equal number of fermionic degrees of freedom.

Table 2.3: Conformal primaries in the generic $\{s\}_{\Delta,r}$ superconformal multiplet.

Conformal Primaries of $\{s\}_{\Delta,r}$	
Level 1	
$[s-\frac{1}{2}]_{\Delta+\frac{1}{2},r-1}$	$\bar{Q}_b\,\lvert\Delta,r\rangle^{a_1...a_{2s-1}b}$
$[s-\frac{1}{2}]_{\Delta+\frac{1}{2},r+1}$	$Q_b\,\lvert\Delta,r\rangle^{a_1...a_{2s-1}b}$
$[s+\frac{1}{2}]_{\Delta+\frac{1}{2},r-1}$	$\bar{Q}^{(a}\,\lvert\Delta,r\rangle^{a_1...a_{2s})}$
$[s+\frac{1}{2}]_{\Delta+\frac{1}{2},r+1}$	$Q^{(a}\,\lvert\Delta,r\rangle^{a_1...a_{2s})}$
Level 2	
$[s-1]_{\Delta+1,r}$	$\bar{Q}_a Q_b\,\lvert\Delta,r\rangle^{a_1...a_{2s-2}ab} - \left(\frac{\Delta-r-s-1}{\Delta-s-1}\right)P^{(ab}\,\lvert\Delta,r\rangle^{a_1...a_{2s})}$
$[s]^{(1)}_{\Delta+1,r}$	$\bar{Q}^c Q_c\,\lvert\Delta,r\rangle^{a_1...a_{2s}} + \left(\frac{2s}{\Delta-1}\right)P^{(a_1}{}_b\,\lvert\Delta,r\rangle^{a_2...a_{2s})b}$
$[s]^{(2)}_{\Delta+1,r}$	$\bar{Q}_b Q^{(a_1}\,\lvert\Delta,r\rangle^{a_2...a_{2s})b} - \left(\frac{\Delta+s-r-1}{\Delta-1}\right)P^{(a_1}{}_b\,\lvert\Delta,r\rangle^{a_2...a_{2s})b}$
$[s]_{\Delta+1,r-2}$	$\bar{Q}^c \bar{Q}_c\,\lvert\Delta,r\rangle^{a_1...a_{2s}}$
$[s]_{\Delta+1,r+2}$	$Q^c Q_c\,\lvert\Delta,r\rangle^{a_1...a_{2s}}$
$[s+1]_{\Delta+1,r}$	$\bar{Q}^{(a}Q^{b}\,\lvert\Delta,r\rangle^{a_1...a_{2s})} - \left(\frac{\Delta-r+s}{\Delta+s}\right)P^{(ab}\,\lvert\Delta,r\rangle^{a_1...a_{2s})}$

	Level 3
$[s-\frac{1}{2}]_{\Delta+\frac{3}{2},r-1}$	$\bar{Q}^c\bar{Q}_c Q_b \left|\Delta,r\right\rangle^{a_1...a_{2s-1}b}$ $-2\left(\frac{1-2s}{1+2s}\right)\left(\frac{\Delta-r-s-1}{\Delta-\frac{1}{2}}\right)P^{(a_1}_{b}\bar{Q}_d\left|\Delta,r\right\rangle^{a_2...a_{2s-1})bd}$ $-2\left(\frac{\Delta-r-s-1}{\Delta-s-1}\right)P_{bd}\bar{Q}^{(a_1}\left|\Delta,r\right\rangle^{a_1...a_{2s-1}bd)}$
$[s-\frac{1}{2}]_{\Delta+\frac{3}{2},r+1}$	$Q^c Q_c \bar{Q}_b \left|\Delta,r\right\rangle^{a_1...a_{2s-1}b}$ $-2\left(\frac{1-2s}{1+2s}\right)\left(\frac{\Delta+r-s-1}{\Delta-\frac{1}{2}}\right)P^{(a_1}_{b}Q_d\left|\Delta,r\right\rangle^{a_2...a_{2s-1})bd}$ $-2\left(\frac{\Delta+r-s-1}{\Delta-s-1}\right)P_{bd}Q^{(a_1}\left|\Delta,r\right\rangle^{a_1...a_{2s-1}bd)}$
$[s+\frac{1}{2}]_{\Delta+\frac{3}{2},r-1}$	$\bar{Q}^c\bar{Q}_c Q^{(a}\left|\Delta,r\right\rangle^{a_1...a_{2s})} - 2\left(\frac{\Delta-r+s}{\Delta-\frac{1}{2}}\right)P^{(a}_{b}\bar{Q}^{(a_1}\left|\Delta,r\right\rangle^{a_2...a_{2s})b)}$ $+\frac{4s(\Delta-r+s)}{(2s+1)(\Delta+s)}P^{(aa_1}\bar{Q}_b\left|\Delta,r\right\rangle^{a_2...a_{2s})b}$
$[s+\frac{1}{2}]_{\Delta+\frac{3}{2},r+1}$	$Q^c Q_c \bar{Q}^{(a}\left|\Delta,r\right\rangle^{a_1...a_{2s})} - 2\left(\frac{\Delta+r+s}{\Delta-\frac{1}{2}}\right)P^{(a}_{b}Q^{(a_1}\left|\Delta,r\right\rangle^{a_2...a_{2s})b)}$ $+\frac{4s(\Delta+r+s)}{(2s+1)(\Delta+s)}P^{(aa_1}Q_b\left|\Delta,r\right\rangle^{a_2...a_{2s})b}$
	Level 4

$[s]_{\Delta+2,r}$	$\bar{Q}^c \bar{Q}_c Q^d Q_d \left\| \Delta, r \right\rangle^{a_1 \ldots a_{2s}}$
	$- 2 \left(\frac{1+2s}{1+s} \right) \frac{\Delta(\Delta-r-s-1)(\Delta-r+s)}{(\Delta-s-1)(\Delta+s)(\Delta-\frac{1}{2})} P_{ab} P^{(ab} \left\| \Delta, r \right\rangle^{a_1 a_2 \ldots a_{2s})}$
	$+ 4 \frac{s^2}{(1+s)^2} \frac{(\Delta-r-s-1)(\Delta+s+1)(\Delta-r+s)}{\Delta(\Delta+s)(\Delta-\frac{1}{2})} P^{(a_1}{}_c P^{(a_2}{}_b \left\| \Delta, r \right\rangle^{a_3 \ldots a_{2s})c)b}$
	$+ 4 \left(\frac{2s-1}{2s+1} \right) \left(\frac{\Delta-r+s}{\Delta+s} \right) P^{(a_1 a_2} \bar{Q}_c Q_d \left\| \Delta, r \right\rangle^{a_3 \ldots a_{2s})cd}$
	$+ 4 \left(\frac{\Delta-r-s-1}{\Delta-s-1} \right) P_{ab} \bar{Q}^{(a} Q^b \left\| \Delta, r \right\rangle^{a_1 a_2 \ldots a_{2s})}$
	$- 4 \left(\frac{s}{s+1} \right) \left(\frac{\Delta-r+s+1}{\Delta} \right) \epsilon^{dc} P^{(a_1}{}_b \bar{Q}_d Q_c \left\| \Delta, r \right\rangle^{a_2 \ldots a_{2s})b}$
	$- 8 \left(\frac{s}{s+1} \right) \frac{(\Delta-r)}{\Delta} P^{(a_1}{}_c \bar{Q}_b Q^{(a_2} \left\| \Delta, r \right\rangle^{a_3 \ldots a_{2s})c)b}$

level 4 $\qquad$ $[s]_{\Delta+2,r}$

level 3 $\qquad$ $\left[s-\dfrac{1}{2}\right]_{\Delta+\frac{3}{2},r+1}$ $\qquad$ $\left[s+\dfrac{1}{2}\right]_{\Delta+\frac{3}{2},r+1}$

$\left[s-\dfrac{1}{2}\right]_{\Delta+\frac{3}{2},r-1}$ $\qquad$ $\left[s+\dfrac{1}{2}\right]_{\Delta+\frac{3}{2},r-1}$

level 2 $\qquad$ $[s-1]_{\Delta+1,r}$ $\qquad$ $[s]_{\Delta+1,r+2}$

$[s]^{(1)}_{\Delta+1,r},\ [s]^{(2)}_{\Delta+1,r}$ $\qquad$ $[s+1]_{\Delta+1,r}$

$[s]_{\Delta+1,r-2}$

level 1 $\qquad$ $\left[s-\dfrac{1}{2}\right]_{\Delta+\frac{1}{2},r+1}$ $\qquad$ $\left[s+\dfrac{1}{2}\right]_{\Delta+\frac{1}{2},r+1}$

$\left[s-\dfrac{1}{2}\right]_{\Delta+\frac{1}{2},r-1}$ $\qquad$ $\left[s+\dfrac{1}{2}\right]_{\Delta+\frac{1}{2},r-1}$

level 0 $\qquad$ $[s]_{\Delta,r}$

Figure 2.3: Conformal primaries of the generic multiplet $\{s\}_{\Delta,r}$. The $[s]_{\Delta,k}$ conformal primary at level zero is also the superconformal primary.

2.4 Shortening conditions

The superconformal primary is taken to be normalized to unity according to

$$
{}_{b_1...b_{2s}} \langle \Delta, r | \Delta, r \rangle^{a_1...a_{2s}} = \delta^{(a_1}{}_{b_1}...\delta^{a_{2s})}{}_{b_{2s}}, \tag{2.35}
$$

where we have introduced the conjugate $|\Delta, r\rangle^{a_1...a_{2s}\dagger} = {}_{a_1...a_{2s}} \langle \Delta, r|$. Given this, we can then compute the norms of the remaining conformal primaries by repeatedly applying the superconformal algebra (anti)commutation rules in Section (2.2). At a given s, r, for large enough Δ all the norms will be positive, and the representation is unitary. As we lower Δ, some norms may pass through zero and then become negative.

The superconformal multiplet shortens for certain values of Δ where a conformal primary becomes null, meaning it has zero norm and zero overlap with all other states in the superconformal multiplet. At these values, the null states can be factored out, leaving a shorter multiplet. These short multiplets are unitary only if the remaining norms are all positive.

In this section, we compute the norms of the conformal primaries and the values of Δ at which these shortenings occur, and study the structure of the some of the resulting short representations. $s = 0$ and $s = \frac{1}{2}$ are exceptional cases, since certain conformal primaries do not exist within the superconformal multiplet, so we treat these separately in the sections that follow.

2.4.1 $s = 0$

For the $\{0\}_{\Delta,r}$ superconformal multiplet, the norm of each conformal primary in Table 2.1 is tabulated in Table 2.4. Shortening conditions occur when these norms vanish. The norms can also become singular for certain values of Δ. When this happens, one of the denominators of the P terms in the conformal primary is becoming singular. When the

denominator of one of the P terms is singular, we have instead the phenomena of extended modules, as discussed in Section 2.5. The shortening and extended module values as they occur in the r, Δ plane are visualized in Figure 2.4, and as they occur in the various states of the multiplet in Figure 2.5.

Table 2.4: Norms of the conformal primaries in the $\{0\}_{\Delta,r}$ superconformal multiplet, as they are shown in Table 2.1. The norms also include totally symmetric products of δ-functions in spinor indices which we have omitted for brevity.

Conformal Primary Norms: $\{0\}_{\Delta,r}$	
Conformal Primary	**Norm**
Level 1	
$\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2},r-1}$	$2(\Delta + r)$
$\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2},r+1}$	$2(\Delta - r)$
Level 2	
$[0]_{\Delta+1,r}$	$8(\Delta + r)(\Delta - r)$
$[0]_{\Delta+1,r-2}$	$16(\Delta + r)(\Delta + r - 1)$
$[0]_{\Delta+1,r+2}$	$16(\Delta - r)(\Delta - r - 1)$
$[1]_{\Delta+1,r}$	$4\dfrac{(\Delta+r)(\Delta-r)(\Delta+1)}{\Delta}$
Level 3	
$\left[\frac{1}{2}\right]_{\Delta+\frac{3}{2},r-1}$	$32\dfrac{(\Delta+r)(\Delta-r)(\Delta+r-1)(\Delta+1)}{\Delta-\frac{1}{2}}$
$\left[\frac{1}{2}\right]_{\Delta+\frac{3}{2},r+1}$	$32\dfrac{(\Delta+r)(\Delta-r)(\Delta-r-1)(\Delta+1)}{\Delta-\frac{1}{2}}$
Level 4	
$[0]_{\Delta+2,r}$	$256\dfrac{(\Delta+r)(\Delta-r)(\Delta+r-1)(\Delta-r-1)(\Delta+1)(\Delta+\frac{1}{2})}{(\Delta-1)(\Delta-\frac{1}{2})}$
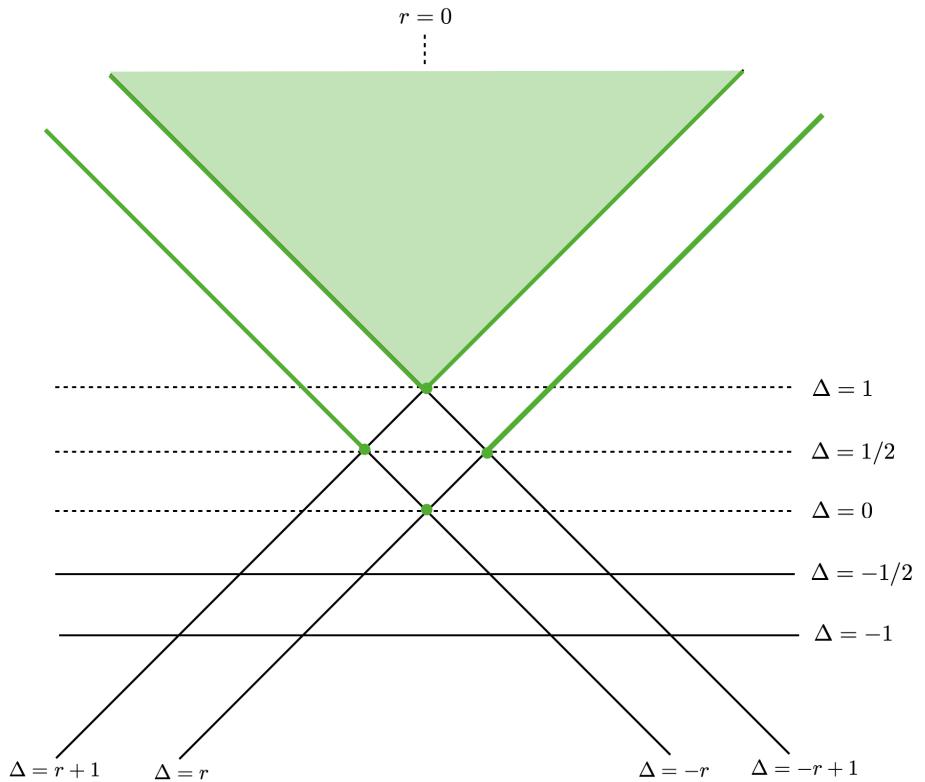
Figure 2.4: Shortening conditions for the $\{0\}_{\Delta,r}$ superconformal multiplet as they occur in the r, Δ plane. Solid lines are places where states become null and the multiplet shortens. Dotted lines are places where the extended modules discussed in Section 2.5 occur. Regions in green are unitary, where all non-null norms are positive. All other regions are non-unitary, meaning at least one non-null norm is negative.

level 4 $(\Delta+r)(\Delta-r)(\Delta+r-1)(\Delta-r-1)(\Delta+1)(\Delta+\frac{1}{2})$
                                    $\Delta-\frac{1}{2}$ $\Delta-1$

level 3 $(\Delta+r)(\Delta-r)(\Delta-r-1)(\Delta+1)$
                                            $\Delta-\frac{1}{2}$
                    $(\Delta+r)(\Delta-r)(\Delta+r-1)(\Delta+1)$
                                    $\Delta-\frac{1}{2}$

level 2 $(\Delta-r)(\Delta-r-1)$
            $(\Delta+r)(\Delta-r)$ $(\Delta+r)(\Delta-r)(\Delta+1)$
        $(\Delta+r)(\Delta+r-1)$ Δ

level 1 $\Delta-r$
                                $\Delta+r$

level 0   1
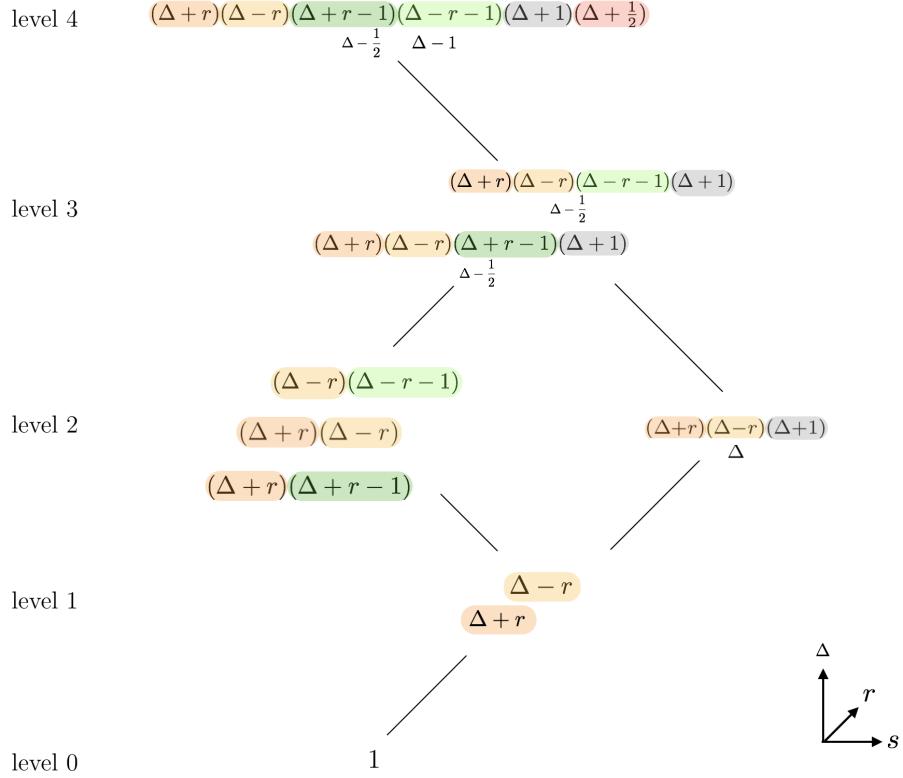
Figure 2.5: Shortening conditions at each level for the $\{0\}_{\Delta,r}$ superconformal multiplet. The placement of the states corresponds to the states in Figure 2.1. The colored factors are factors in the numerators of the norms in Table 2.4. Those with the same color vanish at the same value of Δ, where the states go null and shortening occurs. The uncolored factors underneath are values where primaries in Table 2.1 become singular. At these values the extended modules discussed in Section 2.5 occur.

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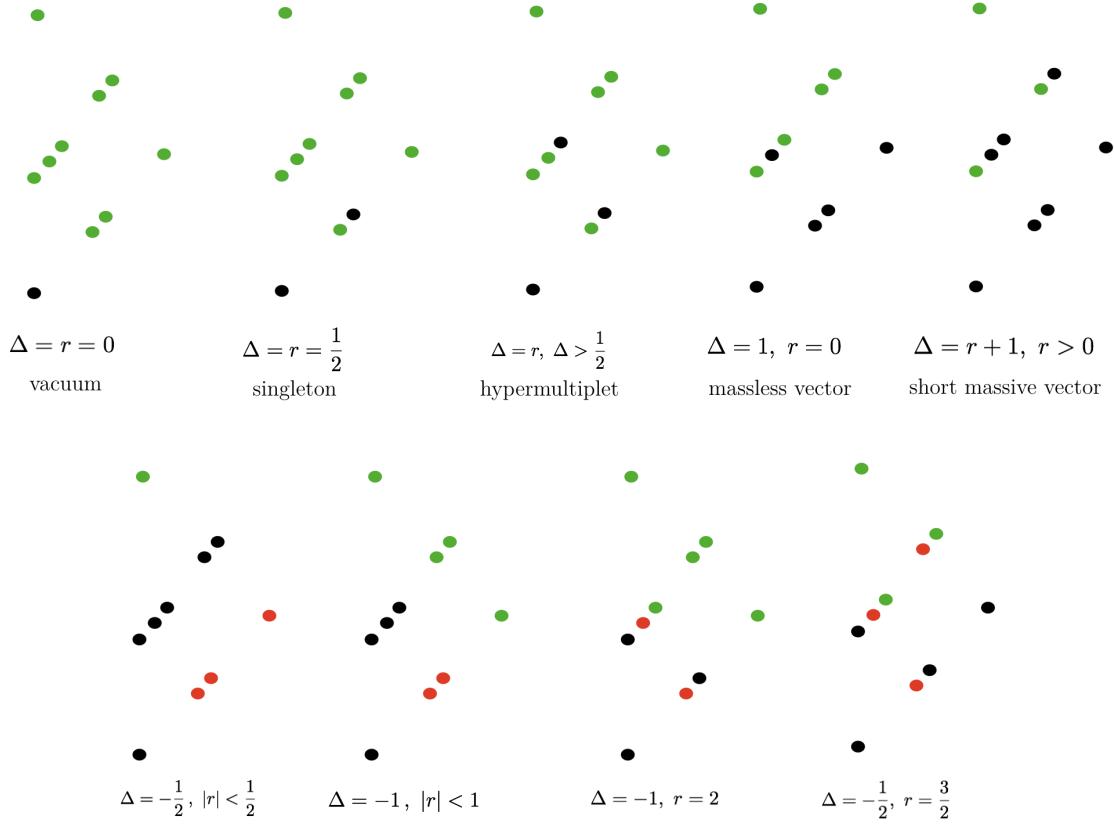
Figure 2.6: Short $\{0\}_{\Delta,r}$ multiplets. Black states have positive norm, red states have negative norm, and green states are the zero norm null states that decouple from the multiplet. The placement of the states corresponds to the states in Figure 2.1. The first line contains all the different unitary short multiplets. The second line contains some of the new non-unitary short multiplets.

In Figure 2.6 we show some of the shortened multiplets. The known unitary short multiplets are as follows: there is the vacuum multiplet at $\Delta = r = 0$, where all the higher states are null. This is an isolated unitary multiplet [130], as can be seen from Figure 2.4. Along the green lines at $\Delta = \pm r$, we have the scalar hypermultiplets which consist of two massive scalars are a massive spin 1/2 (these are the standard "matter" multiplets in $\mathcal{N} = 2$ SUSY field theories in $D = 4$). At the endpoints of these lines, at $\Delta = \pm r = 1/2$, we have a further shortening that removes a scalar and leaves the supersymmetric singleton [139, 140, 141, 142]. These fields have no bulk propagating degrees of freedom. From the point of view of CFT it is a free scalar and free fermion[3]. Along the lines $\Delta = \pm r + 1$, we have a short massive spin-1 multiplet, consisting of a massive spin-1, 3 massive spin 1/2's, and 3 scalars, for a total of 6 bosonic and 6 fermionic bulk degrees of freedom. (As discussed in Section 2.6.1, this multiplet appears in the branching rules of a massive spin 3/2 multiplet as it approaches its massless value.) Where these lines meet, at $\Delta = 1$, $r = 0$ we have a further shortening to the massless photon multiplet consisting of a massless spin 1, two scalars, and two spin 1/2's, for a total of 4 bosonic and 4 fermionic degrees of freedom (this is the standard "gauge" multiplet of $\mathcal{N} = 2$ SUSY field theories in $D = 4$). These unitary multiplets are all shown in the first line of Figure 2.6. In addition to these known unitary short multiplets, we find new non-unitary short multiplets along the solid black lines in Figure 2.4. Some of these are shown in the second line of Figure 2.6.

2.4.2 $s = \frac{1}{2}$

For the $\{\frac{1}{2}\}_{\Delta,r}$ superconformal multiplet, the norm of each conformal primary in Table 2.2 is tabulated in Table 2.5. Given that there are two different states with spin 1/2, weight $\Delta + 1$, and charge r, (the states $[\frac{1}{2}]^{(1)}_{\Delta+1,r}$ and $[\frac{1}{2}]^{(2)}_{\Delta+1,r}$ in Table 2.2) there is a 2×2 Graham

---

[3]Note that the singleton does not split upon reduction to $\mathcal{N} = 1$, it also appears as an $\mathcal{N} = 1$ multiplet [93],

$$\{0\}^{\mathcal{N}=2}_{\frac{1}{2}} = \{0\}^{\mathcal{N}=1}_{\frac{1}{2}} . \tag{2.36}$$

matrix of inner products of these states. This Graham matrix and its determinant are also shown.

Shortenings and extended modules as they occur in the r, Δ plane are visualized in Figure 2.7, and as they occur in the various levels of the multiplet in Figure 2.8.

Table 2.5: Norms of the conformal primaries in the $\{\frac{1}{2}\}_{\Delta,r}$ superconformal multiplet, as they are shown in Table 2.2. We have omitted totally symmetric products of δ-functions in spinor indices for notational brevity.

Conformal Primary Norms: $\{\frac{1}{2}\}_{\Delta,r}$	
Conformal Primary	**Norm**
Level 1	
$[0]_{\Delta+\frac{1}{2},r-1}$	$4(\Delta + r - \frac{3}{2})$
$[0]_{\Delta+\frac{1}{2},r+1}$	$4(\Delta - r - \frac{3}{2})$
$[1]_{\Delta+\frac{1}{2},r-1}$	$2(\Delta + r + \frac{1}{2})$
$[1]_{\Delta+\frac{1}{2},r+1}$	$2(\Delta - r + \frac{1}{2})$
Level 2	
$[\frac{1}{2}]^{(1,2)}_{\Delta+1,r}$ Graham Matrix	

$$\begin{pmatrix} \frac{8(\Delta+r)(\Delta-r)(\Delta-1)-6\Delta}{\Delta-1} & \frac{6r(1-2\Delta)+4r^2(\Delta-1)+\Delta(3+4\Delta-4\Delta^2)}{\Delta-1} \\ \frac{6r(1-2\Delta)+4r^2(\Delta-1)+\Delta(3+4\Delta-4\Delta^2)}{\Delta-1} & \frac{9+4r^2(1-4\Delta)-28\Delta^2+16\Delta^3+12r(2\Delta-1)}{2(\Delta-1)} \end{pmatrix}$$

$[\frac{1}{2}]^{(1,2)}_{\Delta+1,r}$ Graham Det.	$48\frac{\Delta(\Delta-r+\frac{1}{2})(\Delta+r+\frac{1}{2})(\Delta+r-\frac{3}{2})(\Delta-r-\frac{3}{2})}{\Delta-1}$
$[\frac{1}{2}]_{\Delta+1,r-2}$	$16(\Delta + r + \frac{1}{2})(\Delta + r - \frac{3}{2})$
$[\frac{1}{2}]_{\Delta+1,r+2}$	$16(\Delta - r + \frac{1}{2})(\Delta - r - \frac{3}{2})$
$[\frac{3}{2}]_{\Delta+1,r}$	$4\frac{(\Delta+r+\frac{1}{2})(\Delta-r+\frac{1}{2})(\Delta+\frac{3}{2})}{\Delta+\frac{1}{2}}$
Level 3	

|	
$[0]_{\Delta+\frac{3}{2},r-1}$	$64\dfrac{(\Delta+r-\frac{3}{2})(\Delta-r-\frac{3}{2})(\Delta+r+\frac{1}{2})\Delta}{\Delta-\frac{3}{2}}$
$[0]_{\Delta+\frac{3}{2},r+1}$	$64\dfrac{(\Delta+r-\frac{3}{2})(\Delta-r-\frac{3}{2})(\Delta-r+\frac{1}{2})\Delta}{\Delta-\frac{3}{2}}$
$[1]_{\Delta+\frac{3}{2},r-1}$	$32\dfrac{(\Delta+r+\frac{1}{2})(\Delta-r+\frac{1}{2})(\Delta+r-\frac{3}{2})(\Delta+\frac{3}{2})\Delta}{(\Delta-\frac{1}{2})(\Delta+\frac{1}{2})}$
$[1]_{\Delta+\frac{3}{2},r+1}$	$32\dfrac{(\Delta+r+\frac{1}{2})(\Delta-r+\frac{1}{2})(\Delta-r-\frac{3}{2})(\Delta+\frac{3}{2})\Delta}{(\Delta-\frac{1}{2})(\Delta+\frac{1}{2})}$
Level 4	
$[\frac{1}{2}]_{\Delta+2,r}$	$256\dfrac{(\Delta+r-\frac{3}{2})(\Delta-r-\frac{3}{2})(\Delta+r+\frac{1}{2})(\Delta-r+\frac{1}{2})(\Delta+\frac{3}{2})}{(\Delta-\frac{3}{2})}$
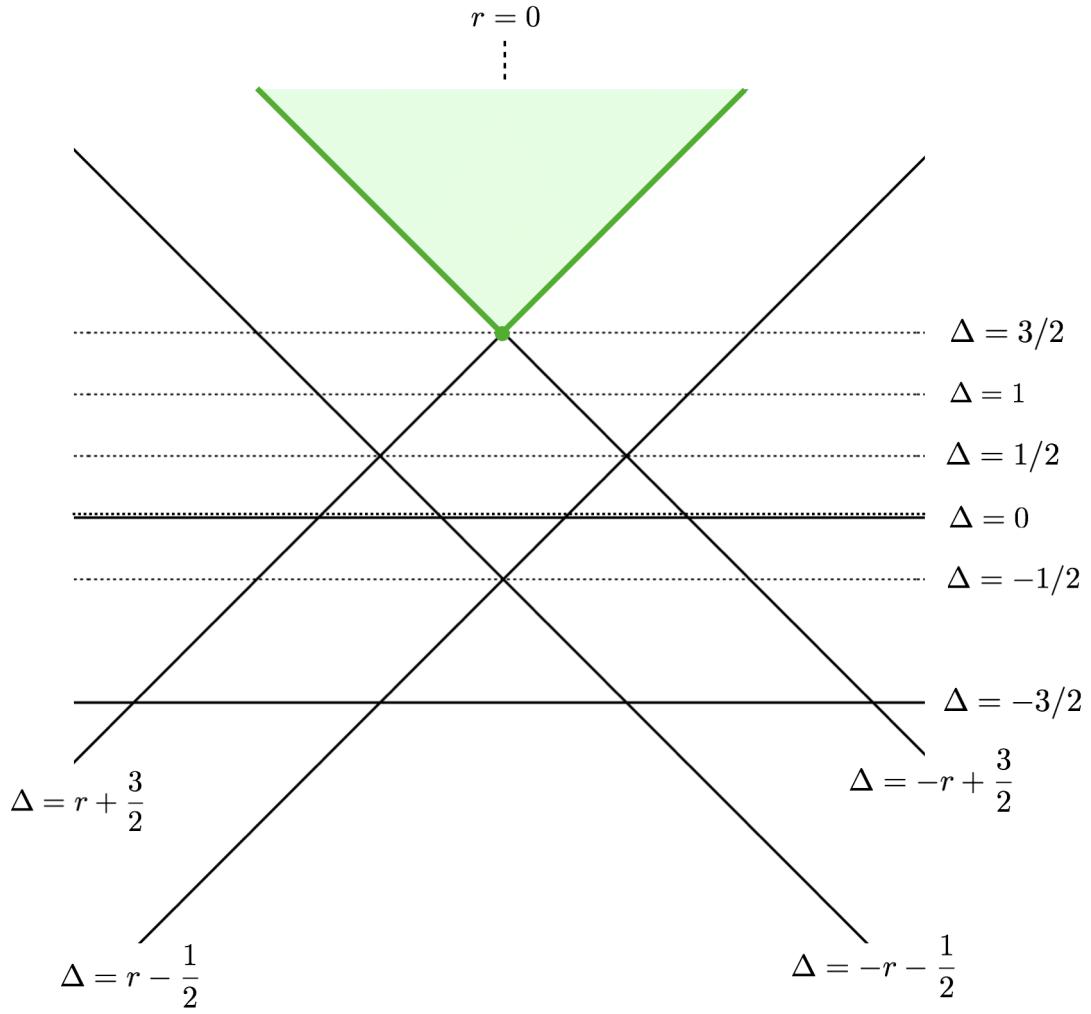
Figure 2.7: Shortening conditions for the $\{\frac{1}{2}\}_{\Delta,r}$ superconformal multiplet as they occur in the r, Δ plane. Solid lines are places where states become null and the multiplet shortens. Dotted lines are places where the extended modules discussed in Section 2.5 occur. Regions in green are unitary, where all non-null norms are positive. All other regions are non-unitary, meaning at least one non-null norm is negative.
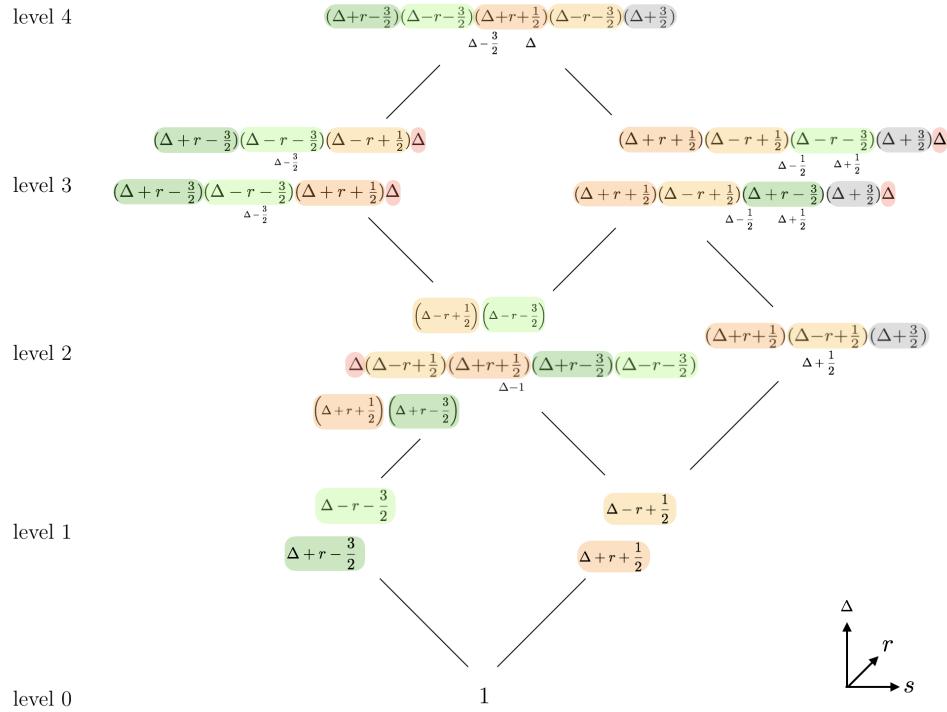
Figure 2.8: Shortening conditions at each level for the $\{\frac{1}{2}\}_{\Delta,r}$ superconformal multiplet. The placement of the states corresponds to the states in Figure 2.2. The colored factors are factors in the numerators of the norms in Table 2.5. Those with the same color vanish at the same value of Δ, where the states go null and shortening occurs. The uncolored factors underneath are values where primaries in Table 2.2 become singular. At these values the extended modules discussed in Section 2.5 occur.

The only unitary short multiplets are the short massive multiplets that occur along the lines $\Delta = 3/2 \pm r$ at the boundary of the green unitary region in Figure 2.7. They describe short multiplets for a massive spin $3/2$ particle whose structure is shown here. In this picture and those like it below, black states have positive norm, red states have negative norm, and green states are zero norm null states that decouple from the multiplet. For the degenerate state at level 2, the colors correspond to the two eigenvalues of the Graham matrix. The placement of the states is in accord with Figure 2.2:
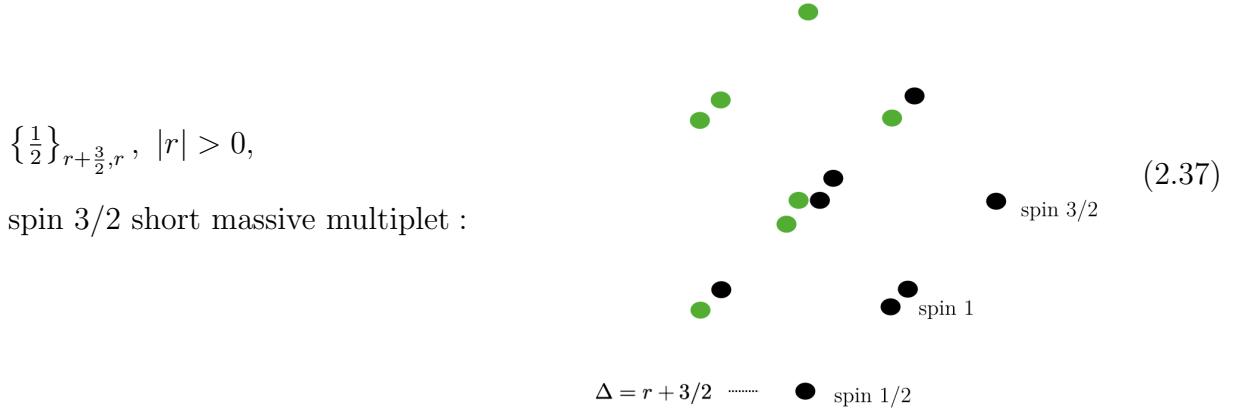
$$\left\{ \tfrac{1}{2} \right\}_{r+\frac{3}{2},r}, \quad |r| > 0,$$

spin $3/2$ short massive multiplet :

spin $3/2$

spin 1

$\Delta = r + 3/2$ .... spin $1/2$

$$(2.37)$$

This contains half as many degrees of freedom as the generic long massive multiplet. As discussed in Section 2.6.1, this multiplet appears in the branching rules of a massive spin 2 multiplet as it approaches its massless value. When $r = 0$, at the intersection of the two lines $\Delta = 3/2 \pm r$ at the apex of the triangular green unitary region in Figure 2.7, there is a further degeneration and we have the massless spin $3/2$ multiplet, shown in (2.58). In the non-unitary region there are novel shortening conditions. For example, at the intersection of the $\Delta = -\frac{3}{2}$ and $\Delta = -r - \frac{1}{2}$ shortening lines of Figure 2.7, we get the following short

multiplet which only has spins ≤ 1,

$$\left\{\tfrac{1}{2}\right\}_{-\frac{3}{2},1}, \tag{2.38}$$
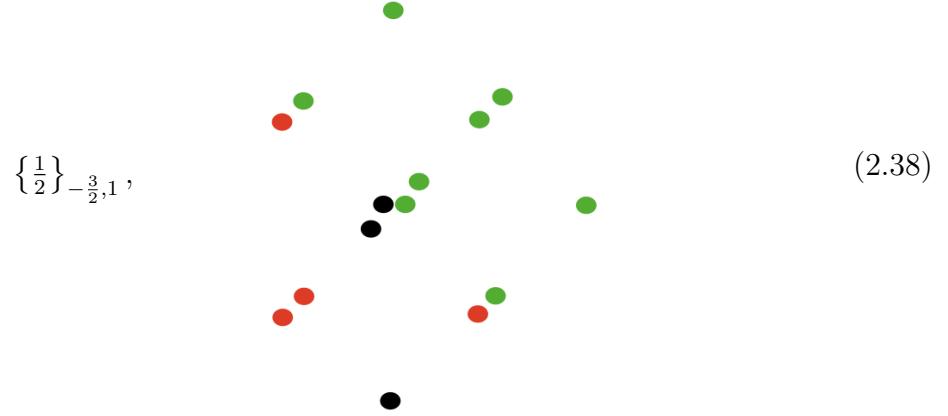
2.4.3 $\quad s \geq 1$

For the $\{s\}_{\Delta,r}$ superconformal multiplet with $s \geq 1$, the norm of each conformal primary in Table 2.3, up to an overall positive constant, is tabulated in Table 2.6. Here we have used an extrapolation process to arrive at these norms for arbitrary s, which only gives us the values up to an overall positive constant. Details about this extrapolation are presented in Appendix A. Given that there are two different states with spin s, weight $\Delta + 1$, and charge r, (the states $[s]^{(1)}_{\Delta+1,r}$ and $[s]^{(2)}_{\Delta+1,r}$ in Table 2.3) there is a 2×2 Graham matrix of inner products of these states. This Graham matrix and its determinant are also shown.

Shortenings as they occur in the r, Δ plane are visualized in Figure 2.9, and as they occur in the various levels of the multiplet in 2.10.

Table 2.6: Norms of the conformal primaries in the $\{s\}_{\Delta,r}$ superconformal multiplet, up to an overall positive constant, as they appear in Table 2.3. We have omitted totally symmetric products of δ-functions in spinor indices for notational brevity.

Conformal Primary Norms: $\{s\}_{\Delta,r}$	
Conformal Primary	**Norm**
Level 1	
$[s - \frac{1}{2}]_{\Delta+\frac{1}{2},r-1}$	$(\Delta + r - s - 1)$
$[s - \frac{1}{2}]_{\Delta+\frac{1}{2},r+1}$	$(\Delta - r - s - 1)$
$[s + \frac{1}{2}]_{\Delta+\frac{1}{2},r-1}$	$(\Delta + r + s)$
$[s + \frac{1}{2}]_{\Delta+\frac{1}{2},r+1}$	$(\Delta - r + s)$
Level 2	
$[s - 1]_{\Delta+1,r}$	$\frac{(\Delta+r-s-1)(\Delta-r-s-1)(\Delta-s)}{(\Delta-s-1)}$
$[s]^{(1,2)}_{\Delta+1,r}$ Graham Matrix	
$\begin{pmatrix} 8\left(\Delta^2 - r^2 - s(s+1)\frac{\Delta}{\Delta-1}\right) & -4\frac{(\Delta^2+s\Delta-r\Delta+r)(\Delta-s+r-1)}{(\Delta-1)} \\ -4\frac{(\Delta^2+s\Delta-r\Delta+r)(\Delta+r-s-1)}{(\Delta-1)} & 2\frac{(\Delta+r-s-1)(\Delta(\Delta-r-1)+s^2(2\Delta-1)+s(2\Delta^2-2r\Delta+r-1))}{s(\Delta-1)} \end{pmatrix}$	
$[s]^{(1,2)}_{\Delta+1,r}$ Graham Det.	$\frac{\Delta(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)}{\Delta-1}$
$[s]_{\Delta+1,r-2}$	$(\Delta + r - s - 1)(\Delta + r + s)$
$[s]_{\Delta+1,r+2}$	$(\Delta - r - s - 1)(\Delta - r + s)$
$[s + 1]_{\Delta+1,r}$	$\frac{(\Delta+s+1)(\Delta+r+s)(\Delta-r+s)}{\Delta+s}$
Level 3	
$[s - \frac{1}{2}]_{\Delta+\frac{3}{2},r-1}$	$\frac{\Delta(\Delta-s)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)}{(\Delta-\frac{1}{2})(\Delta-s-1)}$
$[s - \frac{1}{2}]_{\Delta+\frac{3}{2},r+1}$	$\frac{\Delta(\Delta-s)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta-r+s)}{(\Delta-\frac{1}{2})(\Delta-s-1)}$

$[s+\frac{1}{2}]_{\Delta+\frac{3}{2},r-1}$	$\frac{\Delta(\Delta+s+1)(\Delta+r-s-1)(\Delta+r+s)(\Delta-r+s)}{(\Delta-\frac{1}{2})(\Delta+s)}$
$[s+\frac{1}{2}]_{\Delta+\frac{3}{2},r+1}$	$\frac{\Delta(\Delta+s+1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)}{(\Delta-\frac{1}{2})(\Delta+s)}$
Level 4	
$[s]_{\Delta+2,r}$	$\frac{(\Delta+\frac{1}{2})(\Delta-s)(\Delta+s+1)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)}{(\Delta-s-1)(\Delta+s)(\Delta-\frac{1}{2})}$
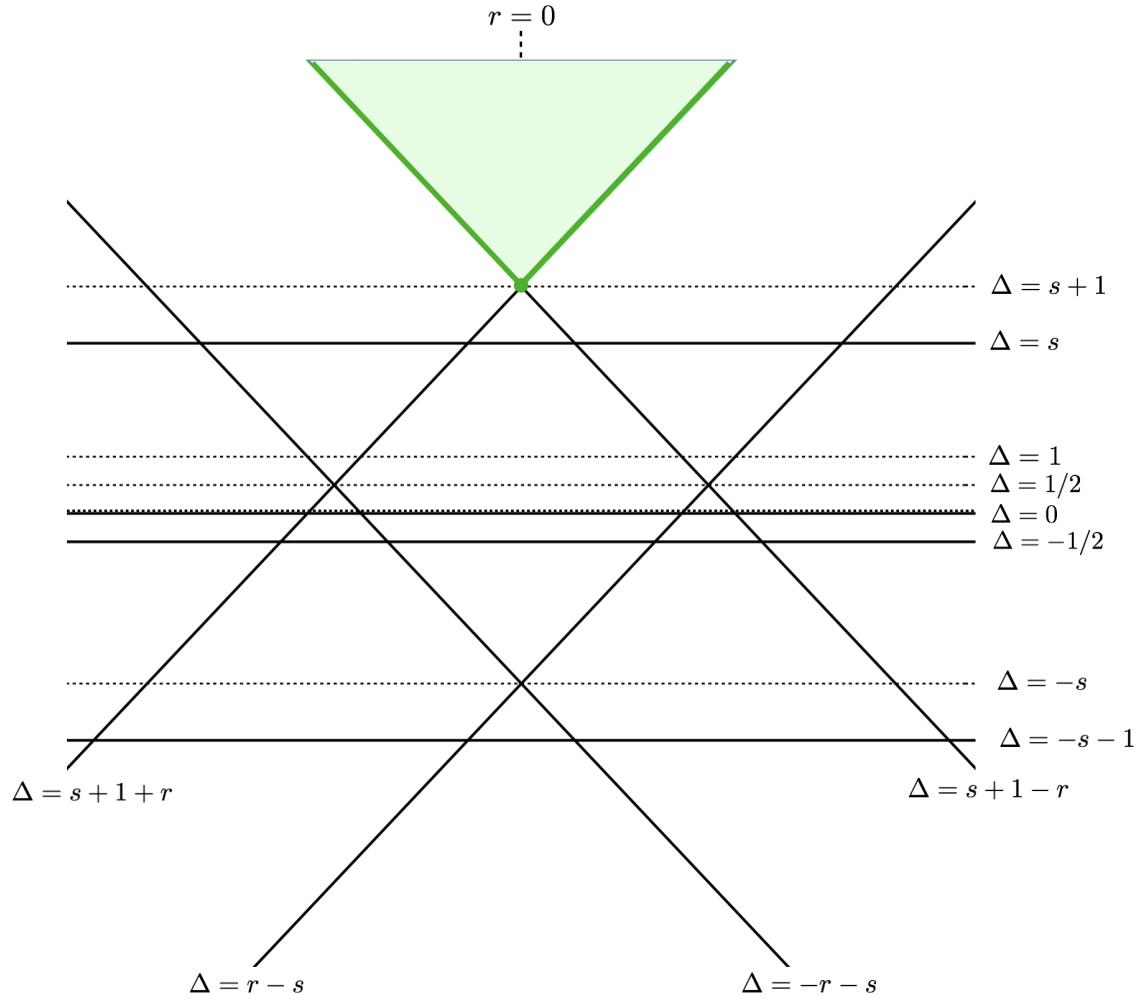
Figure 2.9: Shortening conditions for the $\{s\}_{\Delta,r}$, $s \geq 1$ superconformal multiplet as they occur in the r, Δ plane. Solid lines are places where states become null and the multiplet shortens. Dotted lines are places where the extended modules discussed in Section 2.5 occur. Regions in green are unitary, where all non-null norms are positive. All other regions are non-unitary, meaning at least one non-null norm is negative.

level 4 $(\Delta+\frac{1}{2})(\Delta-s)(\Delta+s+1)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)$
$\Delta-s-1 \quad \Delta-\frac{1}{2} \quad \Delta \quad \Delta+s$

$\Delta(\Delta-s)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta-r+s)$ $\quad$ $\Delta(\Delta+s+1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)$
$\Delta-s-1 \quad \Delta-\frac{1}{2}$ $\qquad\qquad$ $\Delta-\frac{1}{2} \quad \Delta+s$

level 3 $\Delta(\Delta-s)(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)$ $\quad$ $\Delta(\Delta+s+1)(\Delta+r-s-1)(\Delta+r+s)(\Delta-r+s)$
$\Delta-s-1 \quad \Delta-\frac{1}{2}$ $\qquad\qquad$ $\Delta-\frac{1}{2} \quad \Delta+s$

$(\Delta-r-s-1)(\Delta-r+s)$

level 2 $(\Delta+r-s-1)(\Delta-r-s-1)(\Delta-s)$ $\quad$ $\Delta(\Delta+r-s-1)(\Delta-r-s-1)(\Delta+r+s)(\Delta-r+s)$ $\quad$ $(\Delta+s+1)(\Delta+r+s)(\Delta-r+s)$
$\Delta-s-1$ $\qquad\qquad$ $\Delta-1$ $\qquad\qquad$ $\Delta+s$

$(\Delta+r-s-1)(\Delta+r+s)$

level 1 $(\Delta-r-s-1)$ $\qquad$ $(\Delta-r+s)$
$(\Delta+r-s-1)$ $\qquad$ $(\Delta+r+s)$

Δ
r
s

level 0 $\qquad\qquad\qquad$ 1
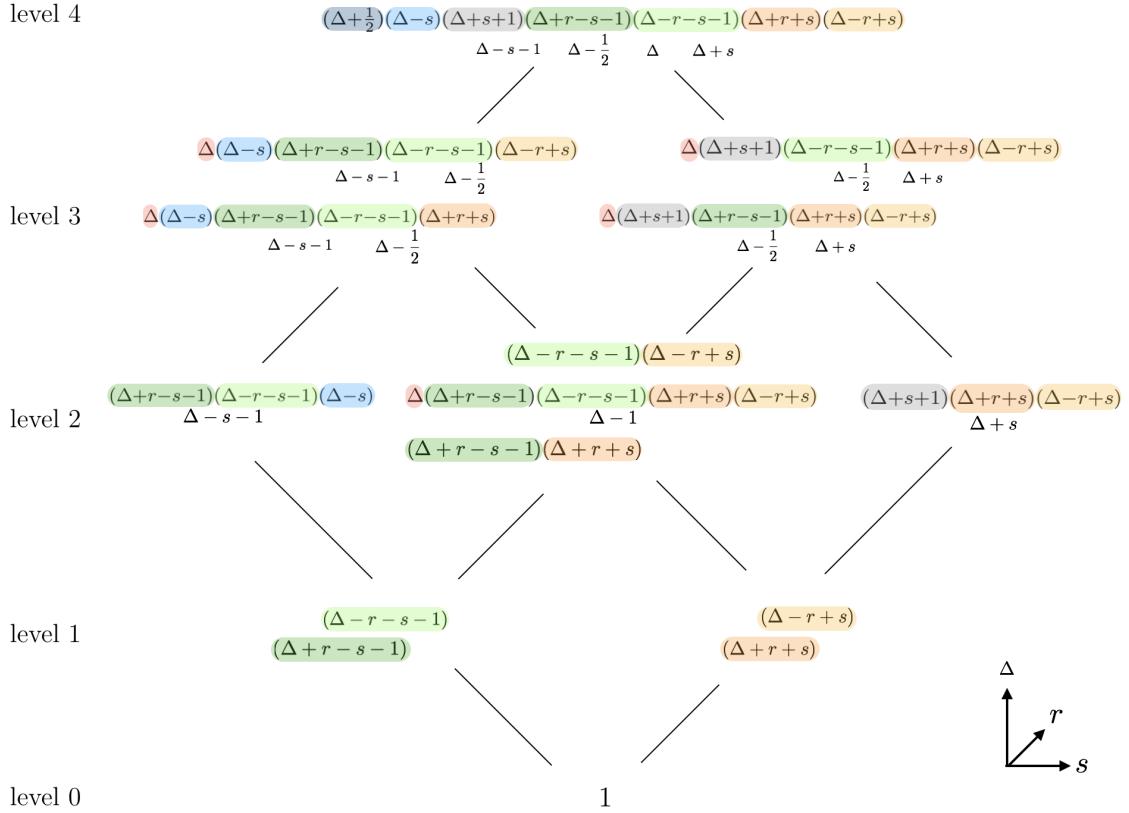
Figure 2.10: Shortening conditions at each level for the $\{s\}_{\Delta,r}$, $s \geq 1$ superconformal multiplet. The placement of the states corresponds to the states in Figure 2.3. The colored factors are factors in the numerators of the norms in Table 2.6. Those with the same color vanish at the same value of Δ, where the states go null and shortening occurs. The uncolored factors underneath are values where primaries in Table 2.3 become singular. At these values the extended modules discussed in Section 2.5 occur.

The only unitary short multiplets are the short massive multiplets that occur along the lines $\Delta = s + 1 \pm r$ at the boundary of the green unitary region in Figure 2.9. They describe short multiplets for a massive spin $S = s + 1$ particle whose structure is shown here. In this picture and those like it below, black states have positive norm, red states have negative norm, and green states are zero norm null states that decouple from the multiplet. For the degenerate state at level 2, the colors correspond to the two eigenvalues of the Graham matrix. The placement of the states is in accord with Figure 2.3:
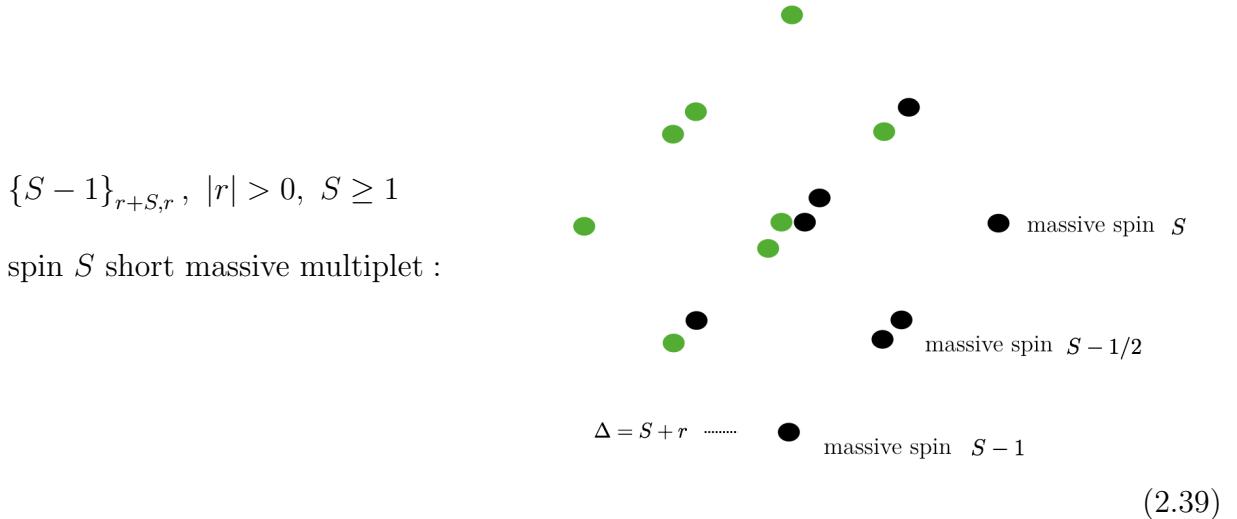
$$\{S - 1\}_{r+S,r}, \ |r| > 0, \ S \geq 1$$

spin S short massive multiplet :

massive spin S

massive spin $S - 1/2$

$\Delta = S + r$ .... massive spin $S - 1$

$$(2.39)$$

This contains half as many degrees of freedom as the generic long massive multiplet. As discussed in Section 2.6.1, this multiplet appears in the branching rules of a massive spin $S + 1/2$ multiplet as it approaches its massless value. When $r = 0$, the intersection of the two lines $\Delta = s + 1 \pm r$ at the apex of the triangular green unitary region in Figure 2.9, there is a further degeneration and we have the massless spin S multiplet, shown in (2.59). In the non-unitary region there are novel shortening conditions. For example, along the $\Delta = -\frac{1}{2}$ shortening line of Figure 2.9, we get a short multiplet where only the top state at level 4

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decouples,

$$\{s\}_{-\frac{1}{2},r}\,,\quad s\geq 1,\quad |r|<s-\tfrac{1}{2}, \tag{2.40}$$
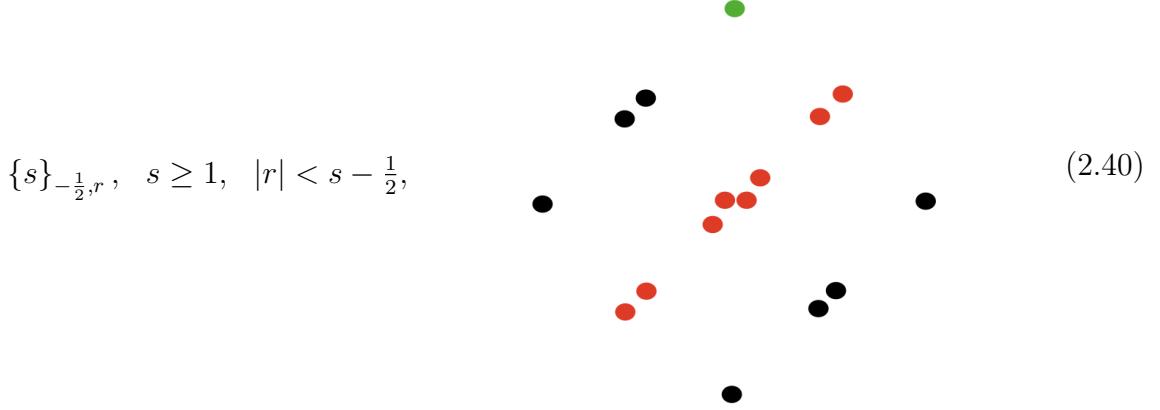
2.5 Extended modules

In the various expressions for the conformal primaries in Section 2.3, the P correction terms become singular for some values of Δ. At these values of Δ, we have the phenomena of extended modules, which also occur in the $\mathcal{N}=1$ case [93] and in non-supersymmetric non-unitary CFT's [75].

The Hilbert space of a unitary CFT is spanned by primary operators and their descendants. Often it is the case that the spectrum contains states having zero norm which are both primary and descendant. For a unitary CFT, such states and their descendants are always orthogonal to every other state. That is, they host their own null module, which may be consistently factored out, and in such cases we say that the multiplet shortens. Once such null states are factored out, the Hilbert space may be written as the direct sum

$$\operatorname{Ker} K \bigoplus_{n} \operatorname{Im} P^{n}\,. \tag{2.41}$$

Moreover, each factor may be further graded by the quantum numbers Δ, s, r.

For non-unitary CFT's, the story is more complicated. For particular values of Δ a state

may become both primary and descendant and thus have zero norm. However unlike in unitary theories, such states are generally not orthogonal to every other state in the theory, and thus do not host their own null module, in which case they cannot be factored out. This is the hallmark of an extended module [75], and typically occurs when a conformal primary and a descendant (of a different primary) which are linearly independent for generic values of Δ degenerate into a single state for a particular value of Δ. In this case there exist other states which are inaccessible by taking linear combinations of primaries and descendants. These inaccessible states must be neither primary nor descendant, because for these particular values of Δ, s the spaces $\mathrm{Ker}\,K$ and $\mathrm{Im}\,P$ coincide, and are spanned by the degenerate state. This signifies a breakdown of the decomposition in (2.41). When this happens, another zero-norm (non-null) basis state arises in the theory for these particular quantum numbers, spanning the missing direction (though not orthogonal to the degenerate states). This state hosts its own module, which we call an extended module. The module may be filled out by acting with P's, and which cannot be factored out.

Below we will work out some examples of specific cases where the extended modules occur. As we will state later on, these extended modules appear in some of the superconformal multiplets which contain partially massless fields. There are partially massless fields of spin S with depth $t = S - 2$ which exist in superconformal multiplets whose primary has $\Delta = s$, $s = S - 1$. The simplest of these, in the sense that there are no higher-spin fields in the multiplet, occurs for $\Delta = s = 1$, which contains a partially massless spin-2 field with depth $t = 0$. This superconformal multiplet contains an extended module which branches off at level 2 via the $[1]_{2,r}$ states.

2.5.1 Extended module for $\{0\}_{0,r}$

In the $\{0\}_{\Delta,r}$ superconformal multiplet, consider the space of states at level 2 with dimension $\Delta + 1$, R-charge r, and spin 1. This space is two dimensional and is spanned by

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the states

$$|1\rangle^{ab} \equiv \bar{Q}^{(a}Q^{b)} \, |\Delta, r\rangle \,, \quad |2\rangle^{ab} \equiv P^{ab} \, |\Delta, r\rangle \,. \tag{2.42}$$

For $\Delta \neq 0$, we can find an orthogonal basis consisting of the conformal primary at level 2 and the first descendant of the conformal primary at level 0,

$$|P\rangle^{ab} \equiv \bar{Q}^{(a}Q^{b)} \, |\Delta, r\rangle - \left(\frac{\Delta - r}{\Delta}\right) P^{ab} \, |\Delta, r\rangle \,, \quad |D\rangle^{ab} \equiv P^{ab} \, |\Delta, r\rangle \,. \tag{2.43}$$

When $\Delta = 0$ on the other hand, there is a degeneration: for $r = 0$, both states (2.42) are primary, while for $r \neq 0$ only $|2\rangle^{ab}$ is primary and is also descendent. The decomposition (2.41) fails. The Graham matrix of the basis (2.42) for $\Delta = 0$ reads

$$\begin{pmatrix} {}_{cd}\langle 1|1\rangle^{ab} & {}_{cd}\langle 1|2\rangle^{ab} \\ {}_{cd}\langle 2|1\rangle^{ab} & {}_{cd}\langle 2|2\rangle^{ab} \end{pmatrix} = \begin{pmatrix} -4r(r+2) & -4r \\ -4r & 0 \end{pmatrix} \delta^{(a}{}_c \delta^{b)}{}_d \,. \tag{2.44}$$

When $r = 0$, the Graham matrix vanishes so both states are null and decouple. This is the vacuum multiplet. This phenomena of extended states occurring within the null states often happens even in the unitary representations. For example in the hypermultiplet $\{0\}_{1,1}$ and in the massless gauge multiplet $\{0\}_{0,1}$ there is an extended module occurring at level 2 where everything is null.

For $r \neq 0$, the eigenvalues of (2.44) are both non-zero and of opposite sign to each other, so in diagonalizing (2.44) we would find one positive norm state and one negative norm state. Thus the Graham matrix is Lorentzian, and we can find a "light-like" basis consisting of two zero-norm states whose overlap with each other is non-zero:

$$|N_1\rangle^{ab} \equiv \bar{Q}^{(a}Q^{b)} \, |0, r\rangle - \frac{r+2}{2} P^{ab} \, |0, r\rangle \,, \quad |N_2\rangle^{ab} \equiv P^{ab} \, |0, r\rangle \,. \tag{2.45}$$

$$\begin{pmatrix} {}_{cd}\langle N_1|N_1\rangle^{ab} & {}_{cd}\langle N_1|N_2\rangle^{ab} \\ {}_{cd}\langle N_2|N_1\rangle^{ab} & {}_{cd}\langle N_2|N_2\rangle^{ab} \end{pmatrix} = \begin{pmatrix} 0 & -4r \\ -4r & 0 \end{pmatrix} \delta^{(a}{}_{c}\delta^{b)}{}_{d} \,. \tag{2.46}$$

$|N_2\rangle^{ab}$ spans the subspace of primary and descendent, whereas $|N_1\rangle^{ab}$, termed an "extension state" in [75], is neither primary nor descendant, and takes the place of the would-be primary at this level. Though both states are zero-norm, they are not null because their overlap is non-zero and so there is no decoupling.

Note that from the point of view of the conformal algebra, a scalar primary with $\Delta = 0$ is just the vacuum module with a single state because all the conformal descendants are null, leaving a short module with only a single state. Here in the superconformal case with $r \neq 0$, because of non-trivial mixing with $|1\rangle^{ab}$ and its descendants, the descendants, while still zero-norm, are no longer null and do not decouple, and so the module is no longer shortened.

One can also ask what happens if one acts on $|N_1\rangle^{ab}$ with Q. For $\Delta = 0$, none of the basis vectors at level three degenerate, so the action of Q simply moves one back into the original module. There can however be instances for different combinations of Δ and s where level 2 basis vectors degenerate, along with the basis vectors at higher levels. In such cases, the extended module spans multiple levels.

2.5.2 Extended module for $\{1\}_{1,r}$

As another example we look at the extended module that occurs in the $\{1\}_{1,r}$ representation. For $r = 0$ this is the extended module that appears in the PM spin-2 multiplet discussed in Section 2.6. In this superconformal multiplet, consider the space of states at level 2 with dimension 2, R-charge r, and spin 1. This space is three dimensional and is spanned by the states

$$|1\rangle^{ab} \equiv \bar{Q}^c Q_c |\Delta, r\rangle^{ab}, \quad |2\rangle^{ab} \equiv \bar{Q}_c Q^{(a} |\Delta, r\rangle^{b)c}, \quad |3\rangle^{ab} \equiv P^{(a}{}_c |\Delta, r\rangle^{b)c} \,. \tag{2.47}$$

For $\Delta \neq 1$, there is a basis consisting of two conformal primaries at level 2 (the states $[s]^{(1)}_{\Delta+1,r}$ and $[s]^{(2)}_{\Delta+1,r}$ in Table 2.3) and the first spin-1 descendant of the conformal primary at level 0,

$$|P_1\rangle^{ab} \equiv \bar{Q}^c Q_c |\Delta, r\rangle^{ab} + \left(\frac{2}{\Delta - 1}\right) P^{(a}_{\ c} |\Delta, r\rangle^{b)c} ,$$

$$|P_2\rangle^{ab} \equiv \bar{Q}_c Q^{(a} |\Delta, r\rangle^{b)c} - \left(\frac{\Delta - r}{\Delta - 1}\right) P^{(a}_{\ c} |\Delta, r\rangle^{b)c} ,$$

$$|D\rangle^{ab} \equiv P^{(a}_{\ c} |\Delta, r\rangle^{b)c} . \tag{2.48}$$

On the other hand, for $\Delta = 1$, there is degeneration, and we cannot construct such a basis. For $r \neq \pm 2, \pm 1$, $|3\rangle^{ab}$ is a primary, and it is also a descendant. The Graham matrix for the basis (2.47) for $\Delta = 1$ reads

$$\begin{pmatrix} {}_{cd}\langle 1|1\rangle^{ab} & {}_{cd}\langle 1|2\rangle^{ab} & {}_{cd}\langle 1|3\rangle^{ab} \\ {}_{cd}\langle 2|1\rangle^{ab} & {}_{cd}\langle 2|2\rangle^{ab} & {}_{cd}\langle 2|3\rangle^{ab} \\ {}_{cd}\langle 3|1\rangle^{ab} & {}_{cd}\langle 3|2\rangle^{ab} & {}_{cd}\langle 3|3\rangle^{ab} \end{pmatrix} = \begin{pmatrix} -8(r^2 + 1) & 4(r^2 - 4r - 1) & -8 \\ 4(r^2 - 4r - 1) & -2(r-1)(3r-1) & -4(r-1) \\ -8 & -4(r-1) & 0 \end{pmatrix} \delta^{(a}_{\ c} \delta^{b)}_{\ d} .$$

$$\tag{2.49}$$

The determinant of this matrix is $128(r+1)(r-1)(r+2)(r-2)$, so for the special cases of $r = \pm 1, \pm 2$, this Graham matrix has a vanishing eigenvalue. The remaining two eigenvalues are nonvanishing and of opposite sign. For these particular values of r, we have an extended module and a null module. Looking at Figure 2.9, we see that $r = \pm 1$ are the cases where the line $\Delta = 1$ intersects the $\Delta = s + 1 \mp r$ shortening line, and $r = \pm 2$ are the cases where the line $\Delta = 1$ intersects the $\Delta = \pm r - sr$ shortening line, so we are seeing these shortenings impact the extended module. We will return to these cases at the end of this section.

On the other hand, for generic values of r (in particular, this applies to the PM spin-2 multiplet discussed in Section 2.6 which has $r = 0$), the Graham matrix always has two

positive and one negative eigenvalue, or two negative and one positive eigenvalue. We can find a basis of zero-norm states whose overlap with each other is non-zero:

$$
\begin{aligned}
|N_1\rangle^{ab} &= \bar{Q}^c Q_c \, |\Delta, r\rangle^{ab} - \frac{1+r^2}{2} P^{(a}{}_c \, |\Delta, r\rangle^{b)c} \, , \\
|N_2\rangle^{ab} &= \bar{Q}_c Q^{(a} \, |\Delta, r\rangle^{b)c} + \frac{1-3r}{4} P^{(a}{}_c \, |\Delta, r\rangle^{b)c} \, , \\
|N_3\rangle^{ab} &= P^{(a}{}_c \, |\Delta, r\rangle^{b)c} \, ,
\end{aligned}
\tag{2.50}
$$

$$
\begin{pmatrix}
{}_{cd}\langle N_1|N_1\rangle^{ab} & {}_{cd}\langle N_1|N_2\rangle^{ab} & {}_{cd}\langle N_1|N_3\rangle^{ab} \\[6pt]
{}_{cd}\langle N_2|N_1\rangle^{ab} & {}_{cd}\langle N_2|N_2\rangle^{ab} & {}_{cd}\langle N_2|N_3\rangle^{ab} \\[6pt]
{}_{cd}\langle N_3|N_1\rangle^{ab} & {}_{cd}\langle N_3|N_2\rangle^{ab} & {}_{cd}\langle N_3|N_3\rangle^{ab}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 & 2(r+1)(r+2)(r-2) & -8 \\[6pt]
2(r+1)(r+2)(r-2) & 0 & -4(r-1) \\[6pt]
-8 & -4(r-1) & 0
\end{pmatrix} \delta^{(a}{}_c \delta^{b)}{}_d .
\tag{2.51}
$$

$|N_3\rangle^{ab}$ spans the subspace of primaries and descendants. On the other hand, $|N_1\rangle^{ab}$ and $|N_2\rangle^{ab}$ are extension states, and are neither primary nor descendant. They take the place of the would-be pair of primaries at this level, and host the extended module, which can be filled out by the action of P's on linear combinations of these states.

Now we will return to the cases $r = \pm 1, \pm 2$. In each of these cases, the Graham matrix (2.49) has a zero eigenvalue, and two additional eigenvalues of opposite sign. This indicates that in each case there is a basis of zero-norm vectors, one of which will have zero overlap with the others. The other two will have non-zero overlap with each other. The basis vector with zero-overlap is a true null state, and may be consistently factored out. On the other hand, one of the remaining two will be an extension state, and host an extended module.

The remaining state will span the space of states which are simultaneously primary and descendant.

Cases $r = \pm 2, -1$: We can handle these cases together because for each of these values of r, the $(1,2), (2,1)$ entries of the Graham matrix (2.51) vanish. Thus, any linear combination of $|N_1\rangle$ and $|N_2\rangle$ has zero norm by default, and we need only find a linear combination which is both a conformal primary and orthogonal to $|N_3\rangle$. A particular choice of basis is

$$
\begin{aligned}
|\mathcal{P}_1\rangle^{ab} &= |N_1\rangle^{ab} + \frac{2}{1-r}|N_2\rangle^{ab} \\
&= \bar{Q}^c Q_c |\Delta, r\rangle^{ab} + \frac{2}{1-r}\bar{Q}_c Q^{(a}|\Delta, r\rangle^{b)c} - \frac{r(r+1)(r-2)}{2(r-1)}P^{(a}{}_c|\Delta, r\rangle^{b)c} \; , \\
|N_2\rangle^{ab} &= \bar{Q}_c Q^{(a}|\Delta, r\rangle^{b)c} + \frac{1-3r}{4}P^{(a}{}_c|\Delta, r\rangle^{b)c} \; , \\
|N_3\rangle^{ab} &= P^{(a}{}_c|\Delta, r\rangle^{b)c} \; ,
\end{aligned}
\tag{2.52}
$$

$$
\begin{pmatrix}
{}_{cd}\langle \mathcal{P}_1|\mathcal{P}_1\rangle^{ab} & {}_{cd}\langle \mathcal{P}_1|N_2\rangle^{ab} & {}_{cd}\langle \mathcal{P}_1|N_3\rangle^{ab} \\
{}_{cd}\langle N_2|\mathcal{P}_1\rangle^{ab} & {}_{cd}\langle N_2|N_2\rangle^{ab} & {}_{cd}\langle N_2|N_3\rangle^{ab} \\
{}_{cd}\langle N_3|\mathcal{P}_1\rangle^{ab} & {}_{cd}\langle N_3|N_2\rangle^{ab} & {}_{cd}\langle N_3|N_3\rangle^{ab}
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -4(r-1) \\
0 & -4(r-1) & 0
\end{pmatrix}
\delta^{(a}{}_c \delta^{b)}{}_d, \quad r = \pm 2, -1.
\tag{2.53}
$$

The state $|\mathcal{P}_1\rangle$ is null and a conformal primary, and hosts its own null module which may be factored out. On the other hand, the extension state $|N_2\rangle$ is zero-norm but not null, and hosts an extension module, taking the place of the second conformal primary. Finally, $|N_3\rangle$ spans the space of states which are simultaneously primary and descendant.

Case $r = 1$: In this case, the $(2,3), (3,2)$ entries of the Graham matrix (2.51) vanish, which suggests that we should construct the conformal primary which hosts the null module by taking a linear combination of $|N_2\rangle$ and $|N_3\rangle$. Moreover, for $r = 1$ both $|N_2\rangle$ and $|N_3\rangle$ are primary, so we need only worry about constructing a state which is null. A particular

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choice of basis is

$$|N_1\rangle^{ab} = \bar{Q}^c Q_c |\Delta, 1\rangle^{ab} - P^{(a}{}_c |\Delta, 1\rangle^{b)c} \, ,$$

$$|\mathcal{P}_2\rangle^{ab} = |N_2\rangle^{ab} - \frac{3}{2} |N_3\rangle^{ab} = \bar{Q}_c Q^{(a} |\Delta, 1\rangle^{b)c} - 2P^{(a}{}_c |\Delta, 1\rangle^{b)c} \, ,$$

$$|N_3\rangle^{ab} = P^{(a}{}_c |\Delta, 1\rangle^{b)c} \, , \tag{2.54}$$

$$\begin{pmatrix} {}_{cd}\langle N_1 | N_1 \rangle^{ab} & {}_{cd}\langle N_1 | \mathcal{P}_2 \rangle^{ab} & {}_{cd}\langle N_1 | N_3 \rangle^{ab} \\ {}_{cd}\langle \mathcal{P}_2 | N_2 \rangle^{ab} & {}_{cd}\langle \mathcal{P}_2 | \mathcal{P}_2 \rangle^{ab} & {}_{cd}\langle \mathcal{P}_2 | N_3 \rangle^{ab} \\ {}_{cd}\langle N_3 | N_1 \rangle^{ab} & {}_{cd}\langle N_3 | \mathcal{P}_2 \rangle^{ab} & {}_{cd}\langle N_3 | N_3 \rangle^{ab} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 0 \end{pmatrix} \delta^{(a}{}_c \delta^{b)}{}_d. \tag{2.55}$$

Similar to the prior case, the state $|\mathcal{P}_2\rangle$ is null and a conformal primary, a hosts its own null module which may be factored out. The extension state is $|N_1\rangle$, and $|N_3\rangle$ spans the space of states which are simultaneously primary and descendant.

Finally, note that at level 3 in this superconformal multiplet, there is no degeneration for $\Delta = s = 1$. Therefore, acting with Q on the extension states and their P-descendants moves one back into the original module. This applies for all values of r.

2.6 Partially massless multiplets

In the boundary CFT, partially massless particles of spin S and depth t correspond to short multiplets with conformal dimension $\Delta = t + 2$, which have a null descendent at level $S - t$.

The simplest partially massless SUSY multiplets are those whose highest spin component is a PM field. Thus the superconformal primary should have spin $S-1$ and weight $\Delta = t+1$. If we don't want the PM field to be charged under R-symmetry then we should take $r = 0$.

Thus the $\mathcal{N} = 2$ SUSY multiplet of interest for a PM field of spin S depth t is

$$\{S - 1\}_{t+1,0} . \tag{2.56}$$

The PM field of interest occurs at level 2, with weight $\Delta + 1 = t + 2$. If we compare this with the shortening conditions found above, we see that there are short supermultiplets where the highest-spin state in the multiplet is a partially massless state of depth $t = S - 2$.

The various partially massless multiplets described in the paragraph above are as follows. In all the pictures, black states have positive norm, red states have negative norm, and green states are zero norm null states that decouple from the multiplet. For the degenerate state at level 2, the colors correspond to the two eigenvalues of the Graham matrix. Blue states are where the extended modules described in Section 2.5 occur. The placement of the states is in accord with Figures 2.1, 2.2, 2.3. The partially massless fields of spin S and depth t are labelled by the notation (S, t).

Spin 1 massless: The only partially massless point for spin 1 is the standard massless value at $t = 0$, where we hit the unitary shortening condition at $\Delta = 1$. This multiplet is $\{0\}_{1,0}$, shown here:
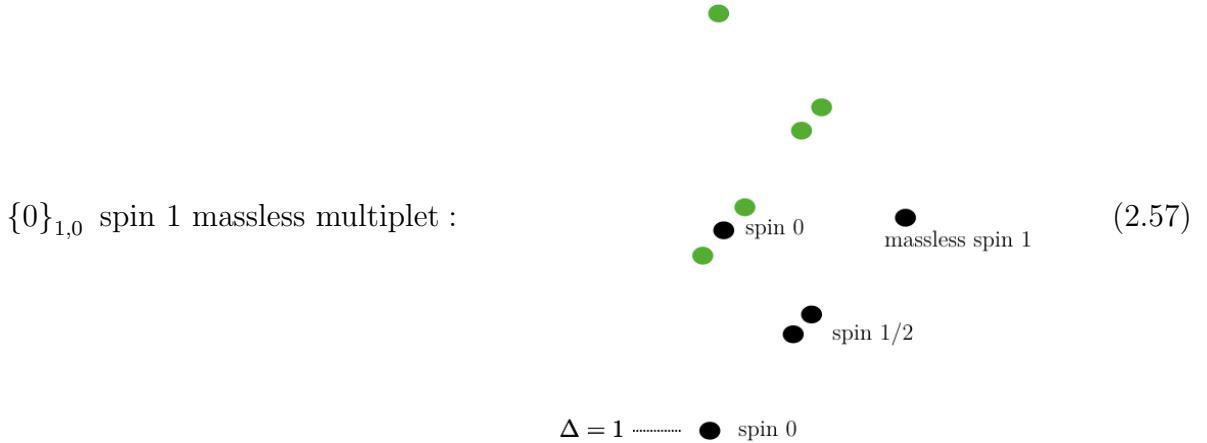
$\{0\}_{1,0}$ spin 1 massless multiplet :

$$\tag{2.57}$$

The bulk has 4 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin 3/2 massless: The only partially massless point for spin $3/2$ is the standard massless value at $t = 1/2$, where we hit the unitary shortening condition at $\Delta = 3/2$. This multiplet is $\{\frac{1}{2}\}_{\frac{3}{2},0}$, shown here:
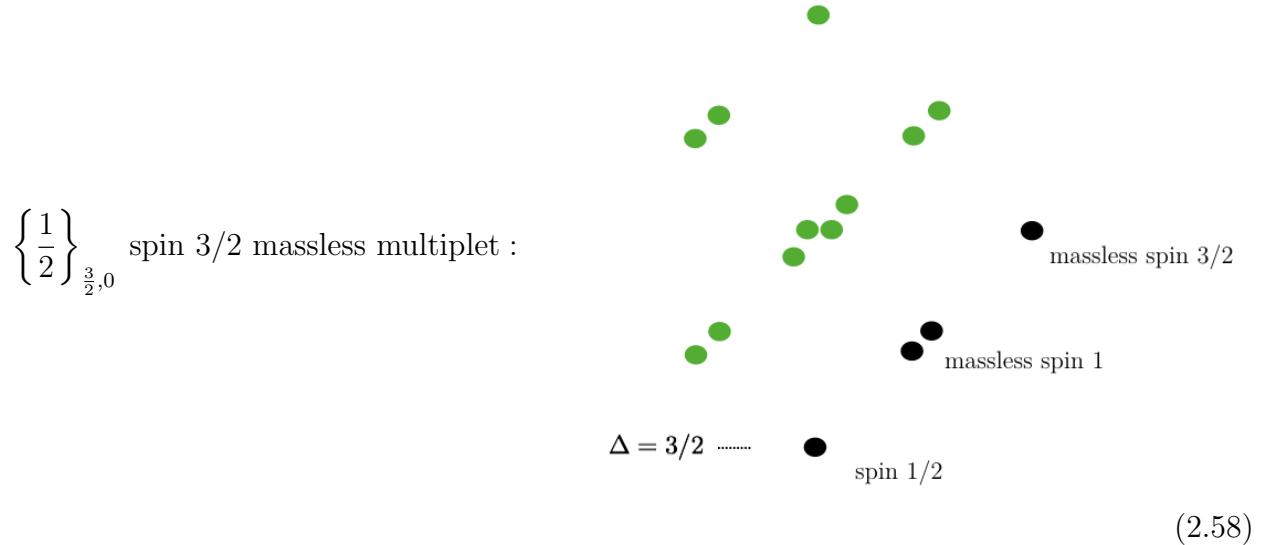
$$\left\{\frac{1}{2}\right\}_{\frac{3}{2},0} \quad \text{spin 3/2 massless multiplet}:$$

$\Delta = 3/2$ ⋯⋯ ● spin 1/2

massless spin 3/2

massless spin 1

$$(2.58)$$

The bulk has 4 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin $S \geq 2$ massless: For spin $S \geq 2$ the massless case is $t = S - 1$ where we hit the

unitary shortening condition at $\Delta = s + 1$. This multiplet is $\{S-1\}_{S,0}$, shown here:

$\{S-1\}_{S,0}$ spin S massless multiplet :

massless spin S

massless spin $S - \dfrac{1}{2}$

$\Delta = S$ .... massless spin $S - 1$

$$(2.59)$$
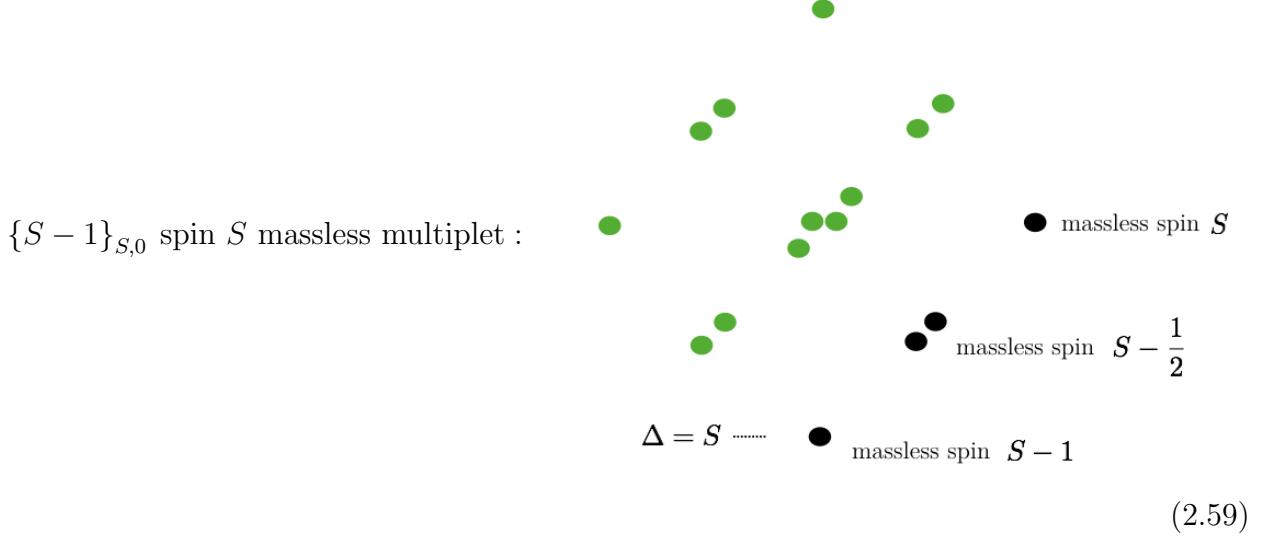
It contains only massless fields. The bulk has 4 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin 2, $t = 0$ PM short multiplet: Here we hit the non-unitary level 2 shortening condition given by $\Delta = s$ which also coincides with the extended module condition. This multiplet is $\{1\}_{1,0}$, shown here:

$\{1\}_{1,0}$ spin 2 PM :

massless spin 3/2

massless spin 1

PM spin 2

spin 1/2

massive spin 3/2

$$(2.60)$$

$\Delta = 1$ .... alternate
massless spin 1
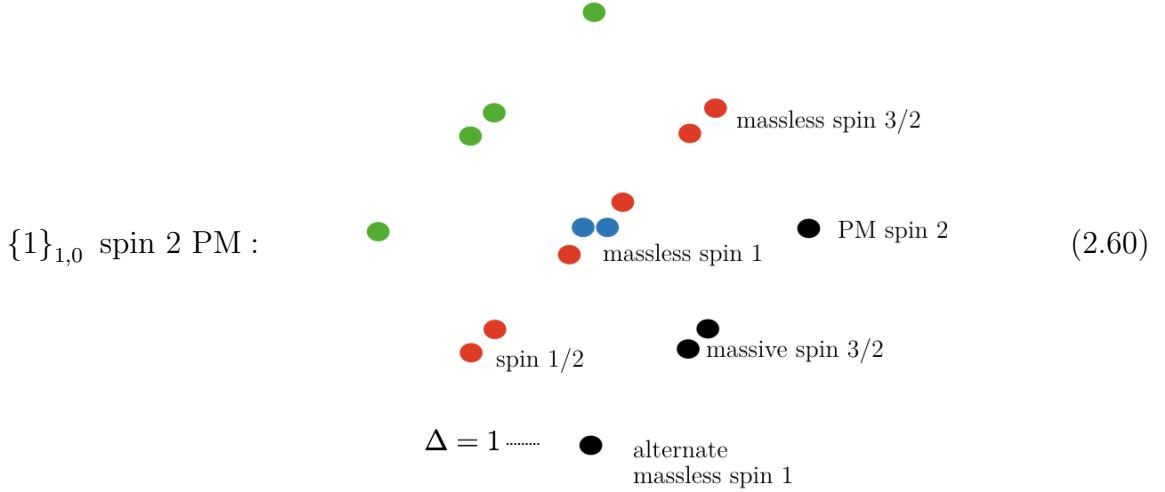
This short multiplet includes massless and massive fields in addition to the partially massless spin-2. It also contains the field $[1]_0$, obtained by performing an alternate quantization on

a bulk massless spin-1. Some of the other massless spin-1's live in an extended module. If we demand that the multiplet has an equal number of bosonic and fermionic degrees of freedom, then the alternately quantized photon must carry 4 propagating degrees of freedom. Given this, the bulk has 16 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin $5/2$, $t = 1/2$ **PM short multiplet:** Here we hit the non-unitary level 2 shortening condition given by $\Delta = s$. This multiplet is $\{\frac{3}{2}\}_{\frac{3}{2},0}$, shown here:

$$\left\{\frac{3}{2}\right\}_{\frac{3}{2},0} \quad \text{spin } \frac{5}{2}, \ t = \frac{1}{2} \text{ PM}:$$

massless spin 2

PM spin 5/2

massless spin 3/2

PM spin 2

massless spin 1

$\Delta = 3/2$ ----- massive spin 3/2

$$(2.61)$$
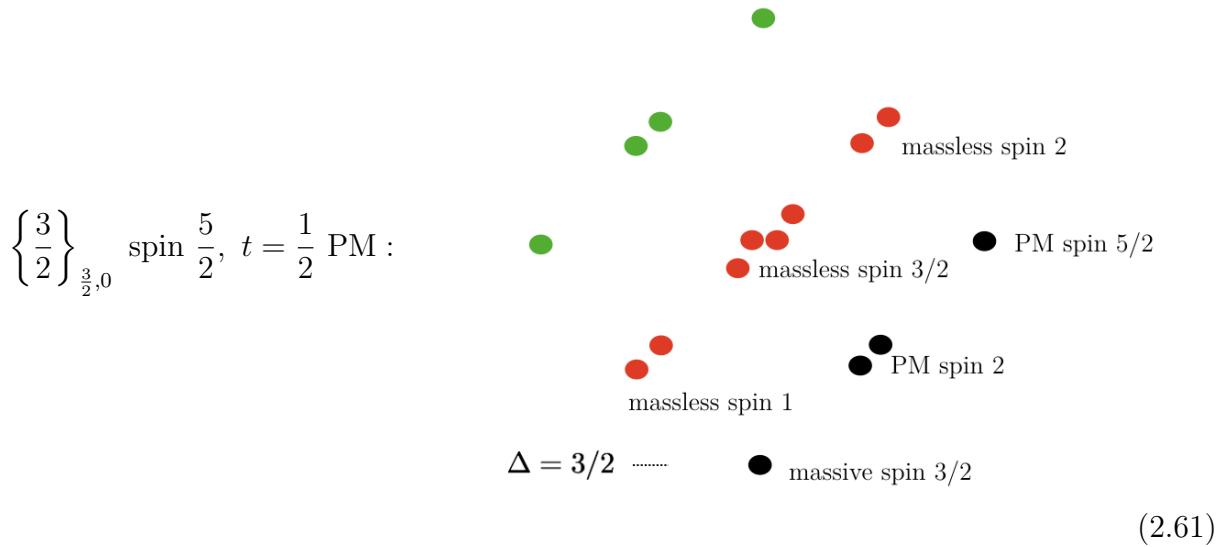
It contains PM fields and massless fields, and a single massive field of spin $3/2$. The bulk has 16 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin $S \geq 3$, $t = S - 2$ **PM short multiplet:** At the next PM point beyond the massless point we hit the non-unitary level 2 shortening condition given by $\Delta = s$. This

multiplet is $\{S-1\}_{S-1,0}$, shown here:

$\{S-1\}_{S-1,0}$ Spin $S \geq 3$, $t = S-2$ PM short

massless
$S - 1/2$

$(S, S-2)$

massless
$S - 1$

massless
$S - 3/2$

$(S - 1/2, S - 5/2)$

$\Delta = S - 1$ ⸱⸱⸱⸱⸱⸱ $(S - 1, S - 3)$

$$(2.62)$$
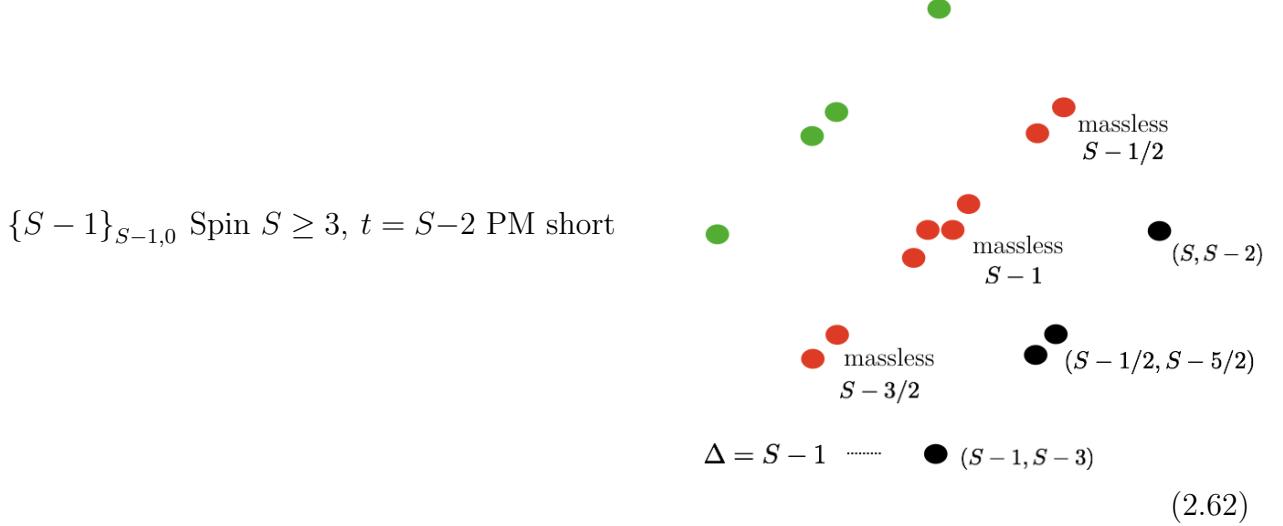
It contains massless and depth $t = S-2$ partially massless fields. The bulk has 16 propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin $S \geq 4$, $1 \leq t \leq S - 3$ PM long multiplet: For these PM values we have no shortening condition and the PM multiplet is a generic long multiplet. This multiplet is $\{S-1\}_{t+1,0}$, shown here:

$\{S-1\}_{t+1,0}$ spin $S \geq 4$, $1 \leq t \leq S-3$ PM long :

$(S - 1, t + 1)$

$(S - 3/2, t + 1/2)$ $(S - 1/2, t + 1/2)$

$(S - 2, t)$ $(S - 1, t)$ (S, t)

$(S - 3/2, t - 1/2)$ $(S - 1/2, t - 1/2)$

$\Delta = t + 1$ ⸱⸱⸱⸱⸱⸱ $(S - 1, t - 1)$

$$(2.63)$$
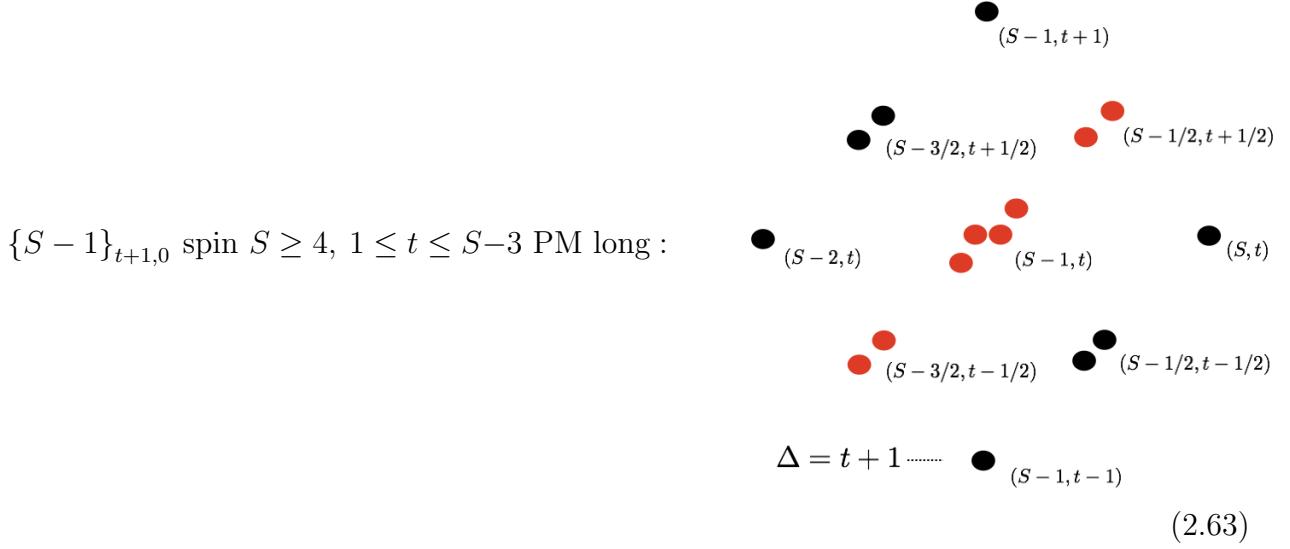
It contains only PM fields, and massless fields only if $t = S - 3$. The bulk has $16(S - t - 1)$ propagating bosonic degrees of freedom and the same number of propagating fermionic

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degrees of freedom.

Spin $S \geq 3$, $t = 0$ **PM multiplet:** This is the minimal depth bosonic multiplet. There is no shortening condition and it is a generic long multiplet. This multiplet is $\{S - 1\}_{1,0}$, shown here:

$$\{S - 1\}_{1,0} \text{ spin } S \geq 3, \, t = 0 \text{ PM}: \qquad\qquad\qquad\qquad (2.64)$$
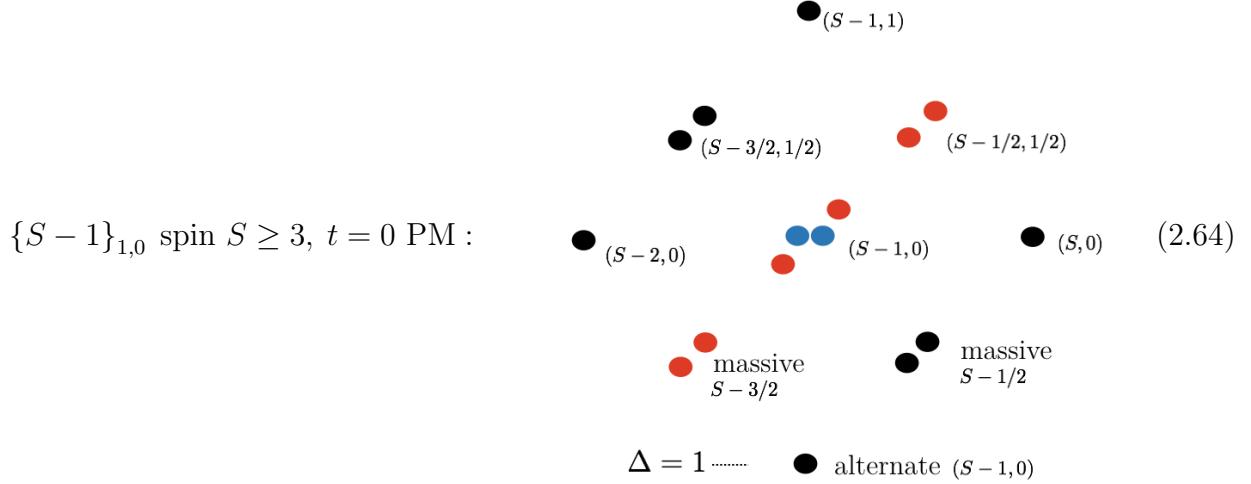
It contains PM fields (which are massless only for $S = 3$) and massive fields. In addition, it contains the field $[S - 1]_0$. This is obtained by performing an alternate quantization on a bulk PM field of spin $S - 1$ with $t = 0$. The bulk degree of freedom counting of this alternately quantized field is unclear, but if we demand that the multiplet has an equal number of bosonic and fermionic degrees of freedom, we can infer that it must propagate $2s$ degrees of freedom. Given this, the bulk has $16(S - 1)$ propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

Spin $S \geq 7/2$, $t = 1/2$ PM multiplet: This is the minimal depth fermionic multiplet. There is no shortening condition and it is a generic long multiplet. This multiplet is $\{S - 1\}_{\frac{3}{2},0}$, shown here:

$$\{S - 1\}_{\frac{3}{2},0} \text{ spin } S \geq 7/2, \ t = 1/2 \text{ PM}:$$

$$(2.65)$$
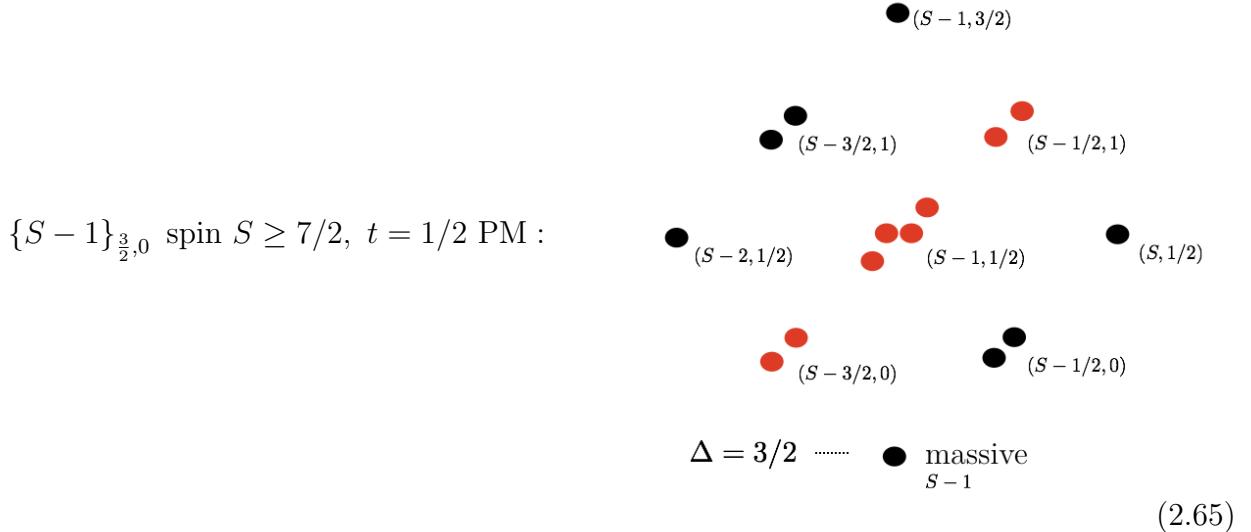
It contains PM fields and a single massive field . The bulk has $16(S - t - 1)$ propagating bosonic degrees of freedom and the same number of propagating fermionic degrees of freedom.

The various PM fields and how they fit into the multiplets described in this section are visualized in Figure 2.11.

2.6.1 Branching rules

As a generic massive multiplet approaches the values of a partially massless or other multiplet containing null states, the null states decouple into their own multiplet and we have a branching rule. Here we determine the branching rules for the PM multiplets described in Section 2.6.

In the partially massless case, the null states corresponding to the partially conserved operators can be thought of as the gauge modes of the corresponding bulk partially massless field. As a conformal primary approaches a partially massless value, we have the branching
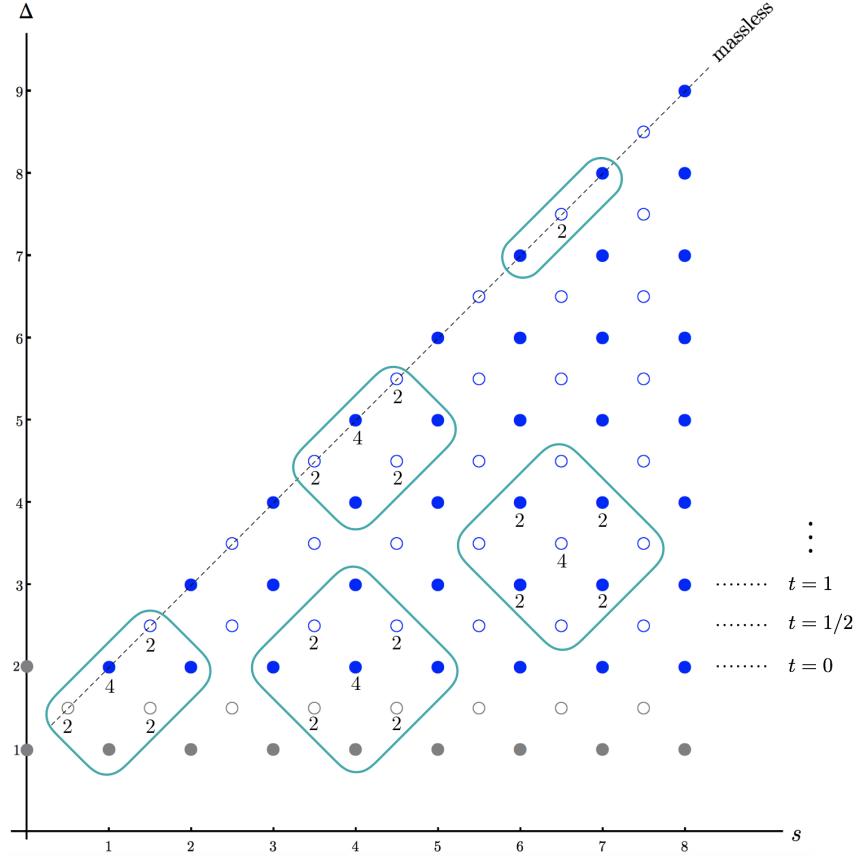
Figure 2.11: $\mathcal{N} = 2$ partially massless multiplets. Filled circles are bosons, hollow circles are fermions. The blue circles are all the partially massless points. The grey circles, such as those on the bottom two rows, are massive states that participate in some of the multiplets. The generic PM multiplets are 2×2 diamonds, like the example showing the $\{6\frac{1}{2}\}_{2\frac{1}{2},0}$ spin $7\frac{1}{2}$ depth $t = 3/2$ PM multiplet (the numbers inside the diamond show the multiplicities of the fields which occur more than once in the representation). As the diamonds approach the massless line $\Delta = s + 1$ at the top, we get the shortened PM multiplets, like the $\{4\}_{4,0}$ spin 5 depth $t = 3$ example shown, and the massless multiplets, like the $\{6\}_{7,0}$ spin 7 example shown. When the diamond reaches the bottom line $\Delta = 1$, we get the PM multiplets containing extended modules, like the $\{4\}_{1,0}$ spin 5 depth $t = 0$ example shown. These are the PM multiplets which cross the $\Delta = 3/2$ divide between ordinary and alternate quantization of bulk AdS_4 fields. The case where we get both a shortening and extended modules is the PM spin 2 multiplet $\{1\}_{1,0}$, also shown.

rule [43]

$$[s]_\Delta \underset{\Delta \to t+2}{\to} [s]_{t+2} \oplus [t]_{s+2} \,, \tag{2.66}$$

where the first summand is the PM field and the second summand is the gauge mode. From this we can deduce the following branching rules of the PM multiplets:

Spin 1 massless: The longitudinal modes form two short hypermultiplets of opposite r charge,

$$\{0\}_{\Delta,0} \underset{\Delta\to 1}{\to} \{0\}_{1,0} \oplus \{0\}_{2,2} \oplus \{0\}_{2,-2} \tag{2.67}$$

Spin 3/2 massless: The longitudinal modes form two short massive spin 1 reps of opposite r charge,

$$\left\{\frac{1}{2}\right\}_{\Delta,0} \underset{\Delta\to\frac{3}{2}}{\to} \left\{\frac{1}{2}\right\}_{\frac{3}{2},0} \oplus \{0\}_{2,1} \oplus \{0\}_{2,-1} \tag{2.68}$$

Spin $S \geq 2$ massless: The longitudinal modes form two short massive reps of opposite r charge,

$$\{S-1\}_{\Delta,0} \underset{\Delta\to S}{\to} \{S-1\}_{S,0} \oplus \left\{S-\frac{3}{2}\right\}_{S+\frac{1}{2},1} \oplus \left\{S-\frac{3}{2}\right\}_{S+\frac{1}{2},-1} \tag{2.69}$$

Spin 2, $t=0$ PM short multiplet: The longitudinal modes form a long massive scalar multiplet,

$$\{1\}_{\Delta,0} \underset{\Delta\to 1}{\to} \{1\}_{1,0} \oplus \{0\}_{2,0} \tag{2.70}$$

Note that the state $[1]_1$ does not spin off any gauge modes.

Spin 5/2, $t=1/2$ PM short multiplet: The longitudinal mode is a long massive spin 3/2 multiplet,

$$\left\{\frac{3}{2}\right\}_{\Delta,0} \underset{\Delta\to\frac{3}{2}}{\to} \left\{\frac{3}{2}\right\}_{\frac{3}{2},0} \oplus \left\{\frac{1}{2}\right\}_{\frac{5}{2},0} \tag{2.71}$$

Spin $S \geq 3$, $t = s - 2$ PM short multiplet: The longitudinal modes form a generic long massive multiplet,

$$\{S-1\}_{\Delta,0} \underset{\Delta\to S-1}{\to} \{S-1\}_{S-1,0} \oplus \{S-2\}_{S,0} \tag{2.72}$$

Spin $S \geq 4$, $1 \leq t \leq S-3$ PM long multiplet: The longitudinal modes form a generic

long massive multiplet, all of whose members are PM gauge modes,

$$\{S-1\}_{\Delta,0} \underset{\Delta \to t+1}{\to} \{S-1\}_{t+1,0} \oplus \{t\}_{S,0} \tag{2.73}$$

Spin $S \geq 3$, $t = 0$ **PM multiplet:** The longitudinal modes form a generic long massive scalar multiplet, all of whose members are PM gauge modes,

$$\{S-1\}_{\Delta,0} \underset{\Delta \to 1}{\to} \{S-1\}_{1,0} \oplus \{0\}_{S,0} \tag{2.74}$$

Note that the state $[S-1]_1$ does not spin off any gauge modes, even though it develops a null descendent from the point of view of the conformal algebra. This is because that null state is not null with respect to the full superconformal algebra, because it is part of the extended module that develops.

Spin $S \geq 7/2$, $t = 1/2$ **PM multiplet:** The longitudinal mode is a long massive spin $3/2$ multiplet, all of whose members are PM gauge modes,

$$\{S-1\}_{\Delta,0} \underset{\Delta \to \frac{3}{2}}{\to} \{S-1\}_{\frac{3}{2},0} \oplus \left\{\frac{1}{2}\right\}_{S,0} \tag{2.75}$$

2.6.2 Reduction to $\mathcal{N} = 1$

All of the $\mathcal{N} = 2$ multiplets can be decomposed into the $\mathcal{N} = 1$ multiplets classified in [93] by simply finding the unique $\mathcal{N} = 1$ multiplets which combine into the desired $\mathcal{N} = 2$ multiplet. The generic massive spin $s \geq 1$ long multiplet splits as

$$\{s\}_{\Delta}^{\mathcal{N}=2} = \{s\}_{\Delta}^{\mathcal{N}=1} \oplus \left\{s+\frac{1}{2}\right\}_{\Delta+\frac{1}{2}}^{\mathcal{N}=1} \oplus \left\{s-\frac{1}{2}\right\}_{\Delta+\frac{1}{2}}^{\mathcal{N}=1} \oplus \{s\}_{\Delta+1}^{\mathcal{N}=1}. \tag{2.76}$$

The short PM multiplets are also easily reduced:

Spin 1 massless: The massless spin 1 multiplet splits into two $\mathcal{N} = 1$ multiplets, a massless spin 1 multiplet and a massive scalar multiplet,

$$\{0\}_{1,0}^{\mathcal{N}=2} = \left\{\frac{1}{2}\right\}_{\frac{3}{2}}^{\mathcal{N}=1} \oplus \{0\}_1^{\mathcal{N}=1} \tag{2.77}$$

Spin 3/2 massless: The massless spin 3/2 multiplet splits into two massless $\mathcal{N} = 1$ multiplets, a massless spin 3/2 multiplet and a massless spin 1 multiplet,

$$\left\{\frac{1}{2}\right\}_{\frac{3}{2},0}^{\mathcal{N}=2} = \{1\}_2^{\mathcal{N}=1} \oplus \left\{\frac{1}{2}\right\}_{\frac{3}{2}}^{\mathcal{N}=1} \tag{2.78}$$

Spin $S \geq 2$ massless: The massless spin S multiplet splits into two massless $\mathcal{N} = 1$ multiplets, a massless spin S multiplet and a massless spin $S - \frac{1}{2}$ multiplet,

$$\{S-1\}_{S,0}^{\mathcal{N}=2} = \left\{S - \frac{1}{2}\right\}_{S+\frac{1}{2}}^{\mathcal{N}=1} \oplus \{S-1\}_S^{\mathcal{N}=1} \tag{2.79}$$

Spin 2, $t = 0$ PM short multiplet: The PM spin-2 multiplet splits as,

$$\{1\}_{1,0}^{\mathcal{N}=2} = \{1\}_1^{\mathcal{N}=1} \oplus \left\{\frac{3}{2}\right\}_{\frac{3}{2}}^{\mathcal{N}=1} \oplus \left\{\frac{1}{2}\right\}_{\frac{3}{2}}^{\mathcal{N}=1} \oplus \{1\}_2^{\mathcal{N}=1} . \tag{2.80}$$

The extended module and alternate quantized photon go into $\{1\}_1^{\mathcal{N}=1}$. Apart from this we have a PM spin-2 multiplet a massless spin 3/2 and a spin 1 multiplet on the $\mathcal{N} = 1$ side.

Spin 5/2, $t = 1/2$ PM short multiplet:

$$\left\{\frac{3}{2}\right\}_{\frac{3}{2},0}^{\mathcal{N}=2} = \left\{\frac{3}{2}\right\}_{\frac{3}{2}}^{\mathcal{N}=1} \oplus \{2\}_2^{\mathcal{N}=1} \oplus \{1\}_2^{\mathcal{N}=1} \oplus \left\{\frac{3}{2}\right\}_{\frac{5}{2}}^{\mathcal{N}=1} . \tag{2.81}$$

PM spin-2 multiplet, PM spin 5/2 multiplet, massless spin 3/2 and massless spin 2.

Spin $S \geq 3$, $t = S - 2$ PM short multiplet: The short PM multiplets each split into

two partially massless $\mathcal{N} = 1$ multiplets and two massless $\mathcal{N} = 1$ multiplets,

$$\{S-1\}^{\mathcal{N}=2}_{S-1,0} = \{S-1\}^{\mathcal{N}=1}_{S-1} \oplus \left\{S-\frac{1}{2}\right\}^{\mathcal{N}=1}_{S-\frac{1}{2}} \oplus \left\{S-\frac{3}{2}\right\}^{\mathcal{N}=1}_{S-\frac{1}{2}} \oplus \{S-1\}^{\mathcal{N}=1}_{S} . \qquad (2.82)$$

Spin $S \geq 4$, $1 \leq t \leq S-3$ PM long multiplet: The long PM multiplets each split into four different partially massless $\mathcal{N} = 1$ multiplets,

$$\{S-1\}^{\mathcal{N}=2}_{t+1,0} = \{S-1\}^{\mathcal{N}=1}_{t+1} \oplus \left\{S-\frac{1}{2}\right\}^{\mathcal{N}=1}_{t+\frac{3}{2}} \oplus \left\{S-\frac{3}{2}\right\}^{\mathcal{N}=1}_{t+\frac{3}{2}} \oplus \{S-1\}^{\mathcal{N}=1}_{t+2} . \qquad (2.83)$$

Spin $S \geq 3$, $t = 0$ PM multiplet:

$$\{S-1\}^{\mathcal{N}=2}_{1,0} = \{S-1\}^{\mathcal{N}=1}_{1} \oplus \left\{S-\frac{1}{2}\right\}^{\mathcal{N}=1}_{\frac{3}{2}} \oplus \left\{S-\frac{3}{2}\right\}^{\mathcal{N}=1}_{\frac{3}{2}} \oplus \{S-1\}^{\mathcal{N}=1}_{2} . \qquad (2.84)$$

The extended module and alternate quantized spin $S-1$ go into $\{S-1\}^{\mathcal{N}=1}_{1}$. Apart from this we have only PM multiplets on the $\mathcal{N} = 1$ side.

Spin $S \geq 7/2$, $t = 1/2$ PM multiplet: Splits into four different partially massless $\mathcal{N} = 1$ multiplets,

$$\{S-1\}^{\mathcal{N}=2}_{\frac{3}{2},0} = \{S-1\}^{\mathcal{N}=1}_{\frac{3}{2}} \oplus \left\{S-\frac{1}{2}\right\}^{\mathcal{N}=1}_{2} \oplus \left\{S-\frac{3}{2}\right\}^{\mathcal{N}=1}_{2} \oplus \{S-1\}^{\mathcal{N}=1}_{\frac{5}{2}} . \qquad (2.85)$$

2.7 $\mathcal{N} > 2$ Supersymmetry

We now present a heuristic argument that higher $\mathcal{N}$ supersymmetries should admit short, non-unitary multiplets containing partially massless fields of depth $t = S - \mathcal{N}$ where, as before, S is the spin of the partially massless field (in contrast to the spin of the superconformal primary). We follow the same general argument as is presented in Appendix A.

Let us consider the quadratic Casimir operator of the superconformal algebra as it acts on a superconformal primary with zero r-charge. Extending the $\mathcal{N} = 2$ case [143], we find

$$\mathcal{C}_2^{super} = D^2 + J_i J_i - \frac{1}{2}\{P_i, K_i\} + \frac{1}{4}[S^{aI}, Q_a^{\ I}] + \dots \, , \qquad (2.86)$$

where $I = 1, \dots, \mathcal{N}$ and the $\dots$ in the above expression denote operators that vanish on a primary state of zero r-charge. Acting on an uncharged superconformal primary, we thus have

$$\mathcal{C}_2^{super} |\Delta, 0\rangle^{a_1 \dots a_{2s}} = [\Delta(\Delta - 3 + \mathcal{N}) + s(s + 1)] |\Delta, 0\rangle^{a_1 \dots a_{2s}} \, . \qquad (2.87)$$

We expect multiplet shortenings to occur when a descendent state is itself a superconformal primary, i.e., when $S^{aI} |\Delta', r'\rangle^{a_1 \dots a_{2s'}} = 0$ for the descendent state. Let us consider the lowest spin state in the supermultiplet: for a superconformal primary given by $|\Delta, 0\rangle^{a_1 \dots a_{2s}}$, this state will have quantum numbers $\Delta' = \Delta + \frac{\mathcal{N}}{2}$, $s' = s - \frac{\mathcal{N}}{2}$ and $r = 0$ (see, e.g., [130]). If this state is also a superconformal primary then its quantum numbers must obey

$$\Delta(\Delta - 3 + \mathcal{N}) + s(s + 1) = \Delta'(\Delta' - 3 + \mathcal{N}) + s'(s' + 1) \, . \qquad (2.88)$$

Solving gives $\Delta = s - \mathcal{N} + 2$. We now consider the highest spin conformal primary in the supermultiplet with quantum numbers $\Delta'' = \Delta + \frac{\mathcal{N}}{2}$, $S \equiv s'' = s + \frac{\mathcal{N}}{2}$ and $r = 0$. This gives the condition $\Delta'' = S - \mathcal{N} + 2 = t + 2$. We see that the highest spin state in the candidate short supermultiplet is partially massless particle of depth $t = S - \mathcal{N}$. For $\mathcal{N} = 1$ this is simply the usual unitary massless representation. For $\mathcal{N} = 2$ we have the depth $t = s - 2$ representations found above which include the partially massless spin-2 particle.

Allowing for non-zero r-charge and based on the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, we might extrapolate to the case of general $\mathcal{N}$. We would predict that in fact, higher $\mathcal{N}$ supersymmetries should admit short, non-unitary multiplets containing partially massless fields of depths $t = S - \mathcal{N}, \dots, S - 1$. We can anticipate that the generic $\mathcal{N}$-extended partially massless

representations will be $\mathcal{N} \times \mathcal{N}$ diamonds in Figure 2.11, with the $\mathcal{N}$ types of shortening (including the massless case) happening as the diamond approaches the massless line from below.

2.8   Conclusions

In this paper we have extended earlier work [93] on supersymmetric versions of partially massless fields by analyzing the case of extended SUSY with $\mathcal{N} = 2$ supercharges. Our results go beyond partially massless particles: they provide new non-unitary representations of the 3-dimensional $\mathcal{N} = 2$ superconformal algebra, and hence of the equivalent super-AdS$_4$ algebra, of which partially massless SUSY multiplets are a special case. We have found a very rich set of possibilities for the structure of non-unitary representations that have no analogues in the unitary domain. These include a range of short multiplets and extended multiplets that differ qualitatively from the known unitary ones. All shortening and extended multiplet conditions are summarized in Figures. 2.4, 2.7 and 2.9.

Concerning the $\mathcal{N} = 2$ representations that include PM fields, we have shown that the corresponding multiplets can be either long or short, unlike what occurs for $\mathcal{N} = 1$ SUSY where PM particles always live in multiplets with no null states (with the exception of the exactly massless case). The generic long PM multiplet is given in (2.63). This multiplet is interesting in that it includes only gauge fields and yet no shortening occurs, something that doesn't exist in the unitary region. For lower spins and for special values of the PM depth we find several short multiplets, which can be fully PM or also include massive particles. Of special interest is the multiplet shown in (2.60) which contains a single PM spin-2 field and no higher-spin states, and also involves an extended module of spin-1 states.

Chapter 3: Soft limits of the wavefunction in exceptional scalar theories

3.1 Introduction

Some of the deepest insights arising from the study of scattering amplitudes have been the discovery of structural motifs that appear in seemingly unrelated theories. In many cases this is a reflection of the underlying rigidity of consistent quantum field theories. These recurring patterns in field theories take on various forms. A famous manifestation of this underlying structure is Weinberg's soft theorem [144], which shows that gauge theory amplitudes in the soft limit have a universal form. Another important example is provided by BCFW recursion relations, which make it possible to systematically construct amplitudes for complicated processes from simpler building blocks [8]. Similarly, the double copy makes precise the notion that some theories can be thought of as squares of others [14]. Taken together, these examples are hints of deeper structures that relate different field theories. In some cases these relations can be made more explicit via the double copy and other transmutation operations that transform theories into each other [145, 146, 147, 148]. Not only are these relations conceptually illuminating, but they are practically useful in enabling computations that would otherwise be out of reach.

These rich structures also appear in certain scalar field theories. Consistent theories of massless spinning particles are highly constrained—with Yang–Mills and Einstein gravity being essentially unique at low energies [149, 150, 151, 152, 153]—and this rigid structure partially explains the recurrence of various features. Massless scalar theories are somewhat

less constrained, in the absence of any additional assumptions. However, requiring that the scalar field behave as a Nambu–Goldstone boson—nonlinearly realizing certain symmetries—is a sufficiently strong demand to make interesting structures reemerge. For example, scalar field theories can then be classified by their amplitudes' behavior in the soft limit [147, 154, 155], with the nonlinear sigma model (NLSM), Dirac–Born–Infeld (DBI), and special galileon theories appearing as distinguished points in theory space. These exceptional scalar theories exhibit interesting relations to each other, and also to Yang–Mills and Gravity [147, 156, 148], and their amplitudes can be constructed by means of recursion relations, similar to BCFW [157, 158, 159, 160]. The interesting features and relative simplicity of these exceptional scalar field theories, along with their connections to gauge theory amplitudes, make them ideal places to explore the hidden structures in scattering amplitudes.

In this paper, we explore the analogues of these on-shell structures in the wavefunctional of Nambu–Goldstone scalar theories. This is motivated by recent progress in the study of cosmological correlation functions. Compared to our understanding of scattering amplitudes, our knowledge of correlators (or the wavefunction) is considerably less sophisticated, even at tree level. Nevertheless, much is now known about the singularity structure of cosmological correlators [161, 162, 26, 28, 30], including how they encode locality. Other properties of bulk time evolution are captured by the way that correlators behave as we vary the kinematic parameters to move away from singular configurations [28, 31]. For example, bulk unitarity has been studied both perturbatively [86, 163, 164, 49] and non-perturbatively [55, 165, 57, 56] in the cosmological context. Aside from providing insights into the structure of quantum field theory, these formal developments have also enabled the calculation of inflationary signatures that would be otherwise intractable (see, e.g., [27, 28, 29, 166, 167, 168, 169, 170, 171, 37, 36, 40, 35]).[1]

Despite recent progress, many deep mysteries remain in the study of cosmological corre-

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[1]An important phenomenological motivation for these studies is the possibility of using the inflationary background as a sort of cosmological collider [76, 77, 27, 28, 29, 78, 79, 80, 81, 82, 172, 173, 174, 175, 176].

lators, and our investigation is aimed at shedding light on some of these issues. Much in the same way that exceptional scalar field theories have been useful in the study of flat space scattering, it is natural to expect that there will be hidden structures in the wavefunction of these exceptional scalar theories. Though our ultimate interest is in cosmology, in this paper we specialize to the study of the wavefunction in flat space. The study of the flat space wavefunction has already proven to be useful for the understanding of de Sitter correlators, leading to many insights that can be imported into the cosmological setting [26, 177, 178, 179]. Even more concretely, in many cases of interest the cosmological wavefunction can be obtained from these flat space expressions by acting with appropriate transmutation operations [26, 171, 30, 35]. We therefore anticipate that the lessons learned for exceptional scalar theories in the flat space context can similarly be abstracted into the cosmological setting.

Summary: For convenience, we summarize our main results:

- We derive the soft theorems that wavefunction coefficients satisfy as a consequence of nonlinearly realized shift symmetries. The general soft theorem is given by equation (3.33), which we apply to a number of theories exhibiting these symmetries. Evidently, the Ward identity is far more complicated than the (enhanced) Adler zero condition enjoyed by the analogous scattering amplitudes. Ultimately this is because, compared to scattering amplitudes, wavefunction coefficients depend on a single additional variable—the total energy involved in the relevant process. Though this difference might seem somewhat innocuous, it leads to important structural differences between these two objects. In particular wavefunction coefficients obey soft theorems relating n-point wavefunction coefficients to lower point wavefunction coefficients when one of their external momenta is taken to be soft.[2]

- We organize soft information into a bootstrap-like construction and use this to fix the

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[2]Scattering amplitudes can also satisfy soft theorems when there are cubic vertices present that are compatible with the nonlinearly realized symmetries, for example in the conformal dilaton [180], or in non-relativistic cases [181] (see also [182] for another relativistic example).

wavefunction coefficients of theories with shift symmetries. A conceptually interesting question is whether the wavefunction contains more, less, or the same information as the flat space S-matrix. In some cases it is known that these two objects can be constructed from each other [178]. In these soft scalar theories, in addition to the scattering information, one requires information about the soft theorems that the wavefunction satisfies in order to reconstruct wavefunction coefficients uniquely.[3]

- We systematize the bootstrap by deriving recursion relations for wavefunction coefficients. The general recursion formula is given in equation (3.115), and is obtained by deforming the kinematics that the wavefunction depends on into the complex plane. The recursion relations are conceptually different from those in [26, 34, 31], which deform the energy variables. In order to input information about the soft structure of the wavefunction, it is important to deform the momentum variables directly. An interesting feature of the complexified wavefunction in this case is that it has branch cuts. In a sense, these branch cuts are avatars of particle exchange, and it turns out that the wavefunction factorizes into a product of lower point shifted wavefunctions along these branch cuts. This is analogous to how scattering amplitudes factorize into lower point amplitudes on their poles. We recursively construct the wavefunction with two different sets of inputs. First, we recurse wavefunction coefficients from their respective scattering amplitudes and soft theorems. Then, we demonstrate that for theories exhibiting higher order soft theorems (NLSM, DBI, special galileon), it is possible to construct the wavefunction without knowledge of any scattering information at all, though at the expense of complicating the procedure. An interesting feature of these constructions it that we are able to effectively give a definition of these exceptional scalar theories directly at the level of the wavefunction without referring to the underlying action.

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[3]This is perhaps unsurprising because there are ambiguities related to (position space) contact terms and field redefinitions that have to be fixed in order to uniquely specify a wavefunction coefficient. See [183] for a similar discussion in the context of inflation.

- There are a number of technical intermediate results that may be of independent interest. Many of these details are given in the appendices. In particular, it is interesting to note that the classical canonical momentum is a generating functional for tree level wavefunction coefficients, which is expressed in (D.6). Perhaps more surprisingly, we also show that the classical field profile at early times is a generating functional for tree-level shifted wavefunction coefficients. This is expressed in (D.20), and indicates that shifted wavefunction coefficients possess information about the system at early times. These statements are analogues of the fact that the classical field profile in the presence of a source with a Feynman pole prescription is a generating functional of in/out correlators. We expect that both of these formulae hold at loop order as well, and also have de Sitter analogues.

Outline: In Section 3.2 we first review the definition and perturbative calculation of the quantum field theory wavefunctional, which is the object of interest. We then describe how wavefunction coefficients in scalar theories with nonlinearly realized symmetries obey soft theorems. In Section 3.3 we first derive the relevant soft theorems for the NLSM, DBI, and the special galileon. We then show how information about the wavefunction's singularities (including the fact that the residue of one of its singularities is the corresponding scattering amplitude) along with partial information about the soft limit is sufficient to uniquely reconstruct the wavefunction. In Section 3.4, we systematize the construction of the wavefunction in these theories by deriving recursion relations that input information about singularities and soft limits in two ways. The first is a systematic implementation of the arguments of Section 3.3, which relies on scattering information as one of the inputs. It is reasonable to ask if it is possible to replace the amplitude information with knowledge of the full soft theorems that a given theory obeys, and indeed we show that this is the case by explicitly constructing recursion relations relying only on soft information, but restricting our discussion to the NLSM and DBI cases for simplicity. We conclude in Section 3.5. A

number of appendices collect technical information that is somewhat outside the main line of development. In Appendix B we provide a brief review of the exceptional scalar theories that we study in this paper. In Appendix C, we discuss many of the important technical subtleties that must be addressed in order to give a boundary definition of the wavefunction of a higher-derivative bulk theory. In Appendix D we show how the early time classical field profile can be viewed as the generating functional of shifted wavefunction coefficients. In Appendix E we discuss some details of the analytic structure of the wavefunction.

Conventions & Notation: We work with the mostly plus metric signature in four space-time dimensions, use Greek letters, e.g., $\mu, \nu, \rho, \cdots$ to indicate spacetime indices, and use Roman letters from the middle of the alphabet, e.g., $i, j, k, \cdots$ to indicate spatial indices. We index various particles/lines/operators by Roman letters from the beginning of the alphabet, e.g., $a, b, c, \cdots$. We label spatial momenta by $\vec{k}_a$ (or $\vec{p}_a$) with magnitude $k_a \equiv \sqrt{\vec{k}_a^2}$ (or $p_a \equiv \sqrt{\vec{p}_a^2}$), which we will refer to as "energies". We Fourier transform with the convention

$$f(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} f_{\vec{k}} \,. \tag{3.1}$$

We denote sums of energies by $k_{12\cdots n} = k_1 + k_2 + \cdots k_n$. For a given process, we denote the total energy involved in the process by E (irrespective of the number of external lines). In many cases we use the following partial energies: $E_{a_1 a_2 \cdots a_n} \equiv k_{a_1} + k_{a_2} + \cdots + k_{a_n} + |\vec{k}_{a_1} + \vec{k}_{a_2} + \cdots + \vec{k}_{a_n}|$, which denote the energy flowing into a vertex of an exchange diagram. We also define the exchanged momenta $\vec{s}_{a_1 \cdots a_n} \equiv \vec{k}_{a_1} + \vec{k}_{a_2} + \cdots + \vec{k}_{a_n}$ and its corresponding energy $s_{a_1 \cdots a_n} \equiv |\vec{k}_{a_1} + \vec{k}_{a_2} + \cdots + \vec{k}_{a_n}|$. Combined, we see that for example, $E_{12} = k_{12} + s_{12} \equiv E_L^{(s)}$ flows into the left vertex of a four-point tree level exchange diagram in the s-channel. When describing scattering amplitudes, we will also often make use of generalized Mandelstam variables, which we define to be $S_{i_1 \dots i_n} = -(P_{i_1} + \dots + P_{i_n})^2$. The variable ϕ denotes a field propagating in the four-dimensional bulk spacetime, while we use φ to denote its profile on the time slice where we compute the wavefunction. Other notational conventions are

introduced as they arise.

3.2 Wavefunction soft theorems

Our goal is to explore the features of the soft limit, where one of the external momenta is taken to zero, in the wavefunction of scalar theories. We are particularly interested in understanding the extent to which the wavefunction is fixed by its singularity structure and soft limit. In the context of scattering amplitudes, enhanced Adler zeroes—where amplitudes vanish faster than expected in the soft limit—correspond to nonlinearly realized shift symmetries from the field theory perspective [154, 155]. We therefore begin by exploring the consequences of these symmetries for the wavefunction. As we will show, an important difference between the wavefunction and the S-matrix is that wavefunction coefficients arising from exceptional scalar theories typically obey soft theorems rather than having Adler zeroes.[4] Therefore one must understand how to translate the symmetries enjoyed by these theories into the relevant soft theorems. We will later utilize these soft theorems as a bootstrap input to generate wavefunction coefficients.

3.2.1 Review of the wavefunction

We begin by briefly reviewing the definition of the quantum field theory wavefunctional and its associated wavefunction coefficients, which will be the objects of central interest. (For more details see, e.g., [184, 26, 185, 30].)

The wavefunctional of interest is a representation of the ground state of an interacting field theory given by projecting onto the Heisenberg-picture eigenstates of the fields, ϕ, as $\Psi[\varphi(\vec{x}), t_*] \equiv \langle \varphi | 0 \rangle$.[5] The wavefunctional is therefore naturally a function of the field profile $\varphi(\vec{x}) \equiv \phi(\vec{x}, t_*)$ at the time t_*. Given the wavefunctional, we can recover correlation functions

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[4]A simple way to understand the difference between the amplitude and wavefunction case is to note that the highest-order enhanced Adler zeroes of amplitudes are a result of a cancellation between exchange and contact contributions. In the wavefunction context, these two contributions have a different analytic structure with respect to internal energies, s_I, so they cannot cancel.

[5]The field eigenstates $|\varphi\rangle$ satisfy $\phi(\vec{x}, t)|\varphi\rangle = \varphi(\vec{x})|\varphi\rangle$.

of the field ϕ at time t_* by employing the usual quantum mechanics formula:

$$\langle \varphi(\vec{x}_1) \cdots \varphi(\vec{x}_n) \rangle = \frac{\int \mathcal{D}\varphi \, \varphi(\vec{x}_1) \cdots \varphi(\vec{x}_n) \, |\Psi[\varphi]|^2}{\int \mathcal{D}\varphi \, |\Psi[\varphi]|^2} \, . \tag{3.2}$$

It is convenient to organize the late-time wavefunctional in a series of connected *wavefunction coefficients*, $\psi_n(\vec{k}_N)$, in Fourier space as:[6]

$$\log \Psi[\varphi, t_f] = \sum_{n=2} \frac{1}{n!} \int \frac{\mathrm{d}^3 k_1 \cdots \mathrm{d}^3 k_n}{(2\pi)^{3n}} \, \varphi_{\vec{k}_1} \cdots \varphi_{\vec{k}_n} \, (2\pi)^3 \delta(\vec{k}_1 + \cdots + \vec{k}_n) \, \psi_n(\vec{k}_N) \, , \tag{3.3}$$

where the wavefunction coefficients are functions of the set of momenta $\vec{k}_N \equiv \{\vec{k}_1, \ldots, \vec{k}_n\}$. We will also often make use of the canonical commutation relation

$$[\phi(\vec{x}, t), \Pi^{(\phi)}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}) \, , \tag{3.4}$$

which given our Fourier convention reads as follows in momentum space

$$[\phi_{\vec{k}_1}(t), \Pi^{(\phi)}_{\vec{k}_2}(t)] = i(2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \, . \tag{3.5}$$

In this representation, the canonical momentum is realized as a functional derivative $\Pi^{(\phi)}_{\vec{k}} = (2\pi)^3 \frac{\delta}{i\delta\phi_{-\vec{k}}}$.

Perturbation theory

In many cases it is convenient to express the wavefunction at some late time t_f as a path integral that evolves a wavefunction from an initial time t_i, which we denote by Ψ_i, in the

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[6] We will often refer to these wavefunction coefficients—in a slight abuse of terminology—as wavefunctions.

following way

$$\Psi[\varphi_f, t_f] = \langle \varphi_f | 0 \rangle = \int \mathcal{D}\varphi_i \, \langle \varphi_f | \varphi_i \rangle \, \langle \varphi_i | 0 \rangle = \int \mathcal{D}\varphi_i \int_{\substack{\phi(t_f) = \varphi_f \\ \phi(t_i) = \varphi_i}} \mathcal{D}\phi \, e^{iS[\phi]} \, \Psi_i[\varphi_i, t_i] \,. \tag{3.6}$$

The path integration is done over all field configurations connecting the profile φ_i at the initial time to φ_f at the final time. In the cases of interest, we will take the initial time to be in the infinite past ($t_i \to -\infty$) with an initial wavefunctional that is a gaussian

$$\Psi_i[\varphi_i, t_i] = \langle \varphi_i | 0 \rangle \propto \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, \mathcal{E}(E_k) \, \varphi_{i\vec{k}} \, \varphi_{i-\vec{k}} \right) \,, \tag{3.7}$$

where $\mathcal{E}(E_k)$ is a kernel capturing the statistics of the initial fluctuations, which depends on the energy of a mode with momentum $\vec{k}$, denoted as $E_k = \sqrt{\vec{k}^2 + m^2}$. In what follows, we will always set $m^2 = 0$ and consider massless fields, so that $E_k = k$.[7] We will primarily be interested in wavefunction coefficients in Minkowski spacetime, where $\mathcal{E}(E_k) = E_k$. Sometimes when performing the path integral (3.6) it is convenient to trade the dependence on the initial state for an unconstrained path integral, at the cost of introducing $i\epsilon$ terms into the action. However, since the initial state is typically not invariant under the symmetries of interest—and thus contributes to the Ward identities we will derive—we will often keep it explicit.

Similar to scattering amplitudes, the computation of the wavefunction coefficients appearing in (3.3) can be organized into a diagrammatic perturbative expansion. There are two essential differences from the computation of S-matrix elements. The first is that energy is not necessarily conserved, because we have broken time translation invariance by choosing a surface on which to compute the wavefunction. Relatedly, the second difference is that

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[7]In the massless case, the vacuum wavefunction is actually highly degenerate due to the free action having an infinite tower of nonlinearly realized symmetries which are all spontaneously broken. We will always take (3.7) to be the initial state in our computations, and routinely abuse language by referring to this state as *the* vacuum state.

there are now two different kinds of propagators that appear in diagrams. First, there are those that connect bulk vertices to the boundary $t = t_f$ surface:

$$\mathcal{K}(k, t) = e^{iE_k t}, \tag{3.8}$$

which we refer to as the *bulk-to-boundary propagator* in analogy to AdS/CFT. In the expression (3.8) we have taken $t_f = 0$, without loss of generality. In contrast, lines that connect bulk vertices to each other represent the *bulk-to-bulk propagator*[8]

$$\mathcal{G}(k; t, t') = \frac{1}{2E_k} \left(e^{iE_k(t'-t)}\theta(t - t') + e^{iE_k(t-t')}\theta(t' - t) - e^{iE_k(t+t')} \right), \tag{3.9}$$

which differs from the usual Feynman propagator by an un-time-ordered piece that enforces the boundary condition $\mathcal{G}(k; 0, t') = 0$.

In order to compute wavefunction coefficients in perturbation theory, we follow a recipe that is quite similar to the computation of scattering amplitudes. We derive Feynman rules from the vertices in the action in the same way that we would do for the S-matrix (only Fourier transforming in the spatial directions), and call the corresponding vertex factors iV. Then, we draw all possible Feynman–Witten diagrams with the desired number of lines ending on the $t = 0$ surface (for concreteness, n), associate to the bulk vertices factors of iV, use $\mathcal{G}$ to connect bulk vertices to each other, and use $\mathcal{K}$ to connect bulk vertices to the boundary at $t = 0$. We then integrate over all the bulk vertex insertion times to produce the wavefunction coefficient ψ_n on the $t = 0$ surface.[9]

Effectively this Feynman diagram expansion is computing the saddle-point approximation of the path integral (3.6) by first constructing the classical solution to the nonlinear equations of motion with vacuum initial conditions and a given field profile $\varphi_{\vec{k}} = \phi_{\vec{k}}(0)$ at

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[8]This Green's function satisfies $(\partial_t^2 + E_k^2)\mathcal{G}(k; t, t') = -i\delta(t - t')$.

[9]In cases that involve loops of internal lines, one should also integrate over the undetermined loop momenta. However, we will restrict ourselves to tree level computations in the following.

time $t = 0$, and then evaluating the action on-shell. The wavefunctional is then a functional of the boundary field profile $\varphi_{\vec{k}}$. From the bulk perspective the computation of wavefunction coefficients is completely algorithmic, but becomes quite complicated even in flat space as the multiplicity of external lines increases, which is part of the motivation to search for more efficient computational methods.

Singularities and cuts

An important lesson about the structure of wavefunction coefficients is that their singularities largely control their behavior [26, 28, 30, 31]. Indeed, in some cases the singularity structure completely specifies the wavefunction [178, 30]. Information about the singularities of the wavefunction and their residues therefore serves as useful input from the boundary perspective. The characteristics of the singularities can be thought of as a boundary manifestation of bulk locality. There are also boundary manifestations of bulk unitarity that provide important constraints on the wavefunction [86, 163, 164, 49]. Specifically, bulk unitarity implies that the wavefunction satisfies an analogue of Cutkosky rules [49, 33, 32, 31], that also serve as a useful input to reconstruct the wavefunction [34, 31, 40]. We will utilize both of these pieces of information in the following, so we briefly review both the singularities and cuts of the wavefunction.

Singularities: We first review the possible singularities of the wavefunction and their residues. Interestingly, this information can be specified in a general way, without specializing to a specific model. Essentially, from the bulk perspective wavefunction coefficients can become singular when the energy flowing into a subgraph happens to add up to zero [26, 30]. An important special case is when the total energy involved in a process adds up to zero. At this kinematic location, the wavefunction has a singularity whose residue is the

corresponding flat space scattering amplitude [161, 162]:

$$\lim_{E\to 0} \psi_n = \frac{A_n}{E} \, , \tag{3.10}$$

where we have denoted the total energy as $E \equiv k_1 + k_2 + \cdots k_n$.[10] This *total energy singularity* provides a beautiful connection between the wavefunction and the S-matrix: wavefunction coefficients are in a precise sense deformations of scattering amplitudes. Physically, the $E \to 0$ divergence arises from integrating the bulk vertices all the way into the infinite past, which typically is suppressed by an oscillatory factor $\sim e^{iEt}$. When the total energy vanishes, this integration is unsuppressed and diverges.

The coefficients of singularities where the energy flowing into a subgraph vanishes—so-called *partial energy singularities*—can also be understood in generality. At these locations, the wavefunction factorizes into a product of a lower-point amplitude and a shifted wavefunction. As a concrete example, consider an $(n + m)$-point wavefunction coefficient in the limit that the energies flowing into some particular n-point subgraph, $E_{1\cdots n}$, add up to zero. We will assume for simplicity that there is a single internal line connected to the subgraph. Pictorially, we can write this as

$$= \lim_{E_{1\cdots n}\to 0} \psi_{n+m} = \frac{A_n \times \tilde{\psi}_m}{E_{1\cdots n}} \, . \tag{3.11}$$

The residue of this singularity is a product of the scattering amplitude, A_n, corresponding to the subgraph whose energy is conserved, multiplied by a shifted version of the wavefunction coefficient corresponding to the rest of the graph. Here the shifting is with respect to the

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[10]The intuition for this singularity is that the diagrammatics involved in computing the wavefunction and the S-matrix are very similar. However, the different integration region for time—in the amplitude case we go to frequency space which involves integrating over all times—leads to a pole in the total energy rather than an energy-conserving delta function $\delta(E)$.

internal energy, s_I, and is defined by

$$\tilde{\psi}_m(k_{n+1}, \cdots, k_{n+m}, s_I) \equiv \frac{1}{2s_I}\Big(\psi_m(k_{n+1}, \cdots, k_{n+m}, -s_I) - \psi_m(k_{n+1}, \cdots, k_{n+m}, s_I)\Big). \quad (3.12)$$

The expression (3.11) can straightforwardly be understood from the form of the bulk-to-bulk propagator with the terms corresponding to the left subgraph taken to the infinite past, where the divergence is localized. These partial energy subgraph singularities are signatures of exchange—wavefunctions arising from contact interactions have only a total energy singularity.[11]

Importantly, the total energy and partial energy singularities are the *only* tree level singularities of the wavefunction, which places strong constraints on its analytic structure. In flat space, all of these singularities are simple poles, so that wavefunction coefficients are rational functions in the total and partial energies.[12]

Cuts: We can get some further insight into the structure of the wavefunction from the form of the bulk-to-bulk propagator (3.9). If we add to it its complex conjugate, all the time-ordering disappears:

$$\widetilde{\mathcal{G}}(k; t, t') \equiv \mathcal{G}(k; t, t') + \mathcal{G}^*(k; t, t') = -\frac{1}{2E_k}\Big(e^{-iE_k t} - e^{iE_k t}\Big)\Big(e^{-iE_k t'} - e^{iE_k t'}\Big). \quad (3.13)$$

This suggests that certain combinations of wavefunction coefficients should simplify and be writeable in terms of shifted lower-point wavefunctions. This is indeed the case, and such relations can be systematized as a set of cutting rules satisfied by the wavefunction [49, 33, 32, 31], which are consequences of unitarity [86, 163, 164, 49]. Schematically, the statement

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[11]Though we have focused for simplicity on a graph with a single internal line, the fact that the wavefunction has a pole when the energy flowing into any subgraph vanishes is true for an arbitrary graph and at arbitrary order in perturbation theory, and the corresponding residues can be characterized.

[12]In de Sitter space, or other cosmological backgrounds, the nature of the singularities can change (for example there are sometimes branch cuts), but the presence of singularities at these—and only these—locations is robust (at least at tree level).

is that

$$\psi_n(X) + \psi_n^*(-X) = -\sum_{\text{cuts}} \psi_n \,, \tag{3.14}$$

where X is a multi-index standing for all the external energies of a given wavefunction.[13] On the right hand side, the sum runs over all partitions of the graph in two, flipping the signs of the external energies of all the vertices to the right of the cut and replacing any internal lines that the cut crosses with the cut propagator (3.13). This transforms the original graph into a pair of graphs, each of which computes a shifted wavefunction coefficient.

In practice we will only require the simplest of these cutting rules. For contact diagrams, the right hand side of (3.14) is zero, indicating that the wavefunction added to itself with its external energies flipped will vanish. We will also make use of the case of single exchange, which takes the pictorial form (for an $(n+m)$-point function)
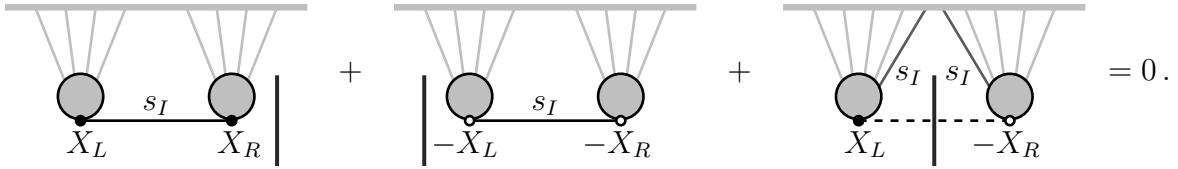
Here $X_{L,R}$ stand schematically for all the energies in the left (right) subgraph, while s_I is the energy of the internal line, and we denote the vertices with their energies flipped by white dots. Translating this into an equation we obtain

$$\psi_{n+m}(X_L, X_R) + \psi_{n+m}^*(-X_L, -X_R) = -2s_I \, \tilde{\psi}_{n+1}(X_L, \mp s) \, \tilde{\psi}_{m+1}(-X_R, \mp s) \,. \tag{3.15}$$

Analogues of this formula can be found for more complicated graph topologies, but we will not need them.

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[13]In general in a cosmological spacetime one has to be careful about the precise analytic continuation to negative energies, but this subtlety is unimportant in flat space.

(Non)uniqueness of wavefunction coefficients

An important issue that we have to face is that wavefunction coefficients are not completely uniquely defined. In particular, field redefinitions and boundary terms can change the wavefunction, and we must deal with these ambiguities.

The field redefinition ambiguity is relatively straightforward to resolve. Essentially it is fixed by demanding that the wavefunction coefficients satisfy soft theorems in a particular form. Intuitively, the soft theorems follow from certain symmetry transformations. If we were to perform a field redefinition, this would change the form of the symmetry transformation, and the wavefunction would correspondingly satisfy a different soft theorem.[14] There is, of course, the residual ambiguity that there could exist field redefinitions that preserve the form of the symmetry transformation but nevertheless change the wavefunction coefficients. However, we will see that this possibility does not arise because the wavefunctions of interest are fixed uniquely by the soft information.

The question of boundary terms is more subtle. Given a wavefunction coefficient bootstrapped via some set of criteria, it must correspond to some action, with some particular choice of boundary terms. How are we to know which one? It turns out that the relevant action is the one that has a well-posed variational principle. That is, the boundary terms are such that the interactions have only a single time derivative per field. This is a necessary condition for the on-shell action to actually be computing the transition amplitude of interest. Any other choice of boundary terms will change the states involved or, equivalently, correspond to matrix elements with operator insertions. This is a somewhat technical point that we elaborate on in Appendix C, but the takeaway is fairly simple to state: there is a distinguished choice of boundary terms—those that make the variational principle well-posed—and interestingly it is the wavefunction coefficients in this presentation that are most naturally generated by the soft bootstrap.

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[14]A similar point was made in the context of the inflationary curvature perturbation ζ in [183].

3.2.2 Derivation of soft theorems

We now derive the various soft theorems that control the wavefunction coefficients of derivatively coupled scalar field theories. Here we present the general formalism before specializing to the relevant theories of interest in the following sections.

The philosophy is to start from the algebra of symmetries and their representation on fields in the theory, $\delta\phi$. We avoid as much as possible directly using the action that is invariant under the relevant symmetries (though we do assume that one exists). Instead, we want to extract the consequences of these symmetries for wavefunction coefficients directly, without passing through some intermediate Lagrangian (see Appendix C.2). The most essential fact that we use is that there is a conserved charge that generates the symmetry $\delta\phi$, which we denote $Q(t)$. Conservation of this charge implies the following equality of matrix elements

$$\langle\varphi_f|\,Q(t_f)\,|0\rangle = \langle\varphi_f|\,Q(t_i)\,|0\rangle \ . \tag{3.16}$$

Our goal in this Section is to express this as a relation between wavefunction coefficients.

In order to simplify the left hand side of this equation, we take advantage of the fact that the charge at time t_f can be expressed in terms of the field φ_f and its conjugate momentum, which in the field basis takes the form $\Pi^{(\phi)}(t_f) = -i\delta/\delta\varphi_f$. We can then write the charge acting on the late-time wavefunctional abstractly as

$$\langle\varphi_f|\,Q(t_f)\,|0\rangle = Q\Big[\varphi, \frac{\delta}{i\delta\varphi}, t_f\Big]\Psi[\varphi, t_f]\,. \tag{3.17}$$

In explicit examples it will be useful to perform further manipulations to simplify this expression, but for now we leave it abstract.

To simplify the right hand side of the expression (3.16) it is convenient to split the charge into a piece that generates the nonlinear part of the relevant symmetry—denoted by

Q^{NL}—and the rest, denoted by Q^{L}:

$$Q = Q^{\mathrm{NL}} + Q^{\mathrm{L}}. \tag{3.18}$$

The nonlinear part of the charge can always be written in terms of the canonical momentum as

$$Q^{\mathrm{NL}} = \int \mathrm{d}^3 x\, \delta^{\mathrm{NL}} \phi\, \Pi^{(\phi)} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \delta^{\mathrm{NL}} \phi_{-\vec{k}} \Pi^{(\phi)}_{\vec{k}}, \tag{3.19}$$

where $\delta^{\mathrm{NL}} \phi$ denotes the nonlinear part of the symmetry transformation on the field variable.[15] In momentum space, we may write this as

$$\delta^{\mathrm{NL}} \phi_{\vec{k}}(t) = (2\pi)^3 D^n_{\vec{k}}(t) \delta^3(\vec{k}) , \tag{3.20}$$

where $D^n_{\vec{k}}(t)$ is some (possibly time dependent) order n differential operator. It is then convenient to evaluate $\langle \varphi_f | Q^{\mathrm{NL}}(t_i) | 0 \rangle$ by introducing a complete set of early-time field eigenstates

$$\langle \varphi_f | Q^{\mathrm{NL}}(t_i) | 0 \rangle = \int \mathcal{D}\varphi_i\, \langle \varphi_f | \varphi_i \rangle\, \langle \varphi_i | Q^{\mathrm{NL}}(t_i) | 0 \rangle = \int \mathcal{D}\varphi_i \int_{\substack{\phi(t_f)=\varphi_f \\ \phi(t_i)=\varphi_i}} \mathcal{D}\phi\, e^{iS[\phi]}\, Q^{\mathrm{NL}}\!\left[\varphi_i, \frac{\delta}{i\delta\varphi_i}\right] \Psi[\varphi_i, t_i] .$$

$$\tag{3.21}$$

We have written the charge $Q^{\mathrm{NL}}(t_i)$ in the field basis as we did for the charge at t_f, now acting on the initial wavefunctional. A benefit is that we know the initial wavefunction—we work in the infinite past so that it is just a gaussian (as in (3.7))

$$Q^{\mathrm{NL}}\!\left[\varphi_i, \frac{\delta}{i\delta\varphi_i}, t_i\right] \Psi[\varphi_i, t_i] = i(-1)^n \Psi[\varphi_i, t_i] \int \mathrm{d}^3 k\, \delta^{(3)}(\vec{k}) D^n_{-\vec{k}}(t_i) \left[\mathcal{E}(k)\varphi_{i\vec{k}}\right] , \tag{3.22}$$

where $\mathcal{E}(k)$ is a kernel parameterizing the two-point function of φ_i, which is given by

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[15]More properly, this should be called the *sub-linear* part of the symmetry—i.e., the part independent of the fields themselves.

$\langle \varphi_{i\vec{k}} \varphi_{i-\vec{k}} \rangle = 1/2 \Re \mathcal{E}(k)$. We can then remove the $|\varphi_i\rangle$ eigenstates to obtain

$$\langle \varphi_f | Q^{\mathrm{NL}}(t_i) | 0 \rangle = i(-1)^n \int \mathrm{d}^3 k \, \delta^{(3)}(\vec{k}) D^n_{-\vec{k}}(t_i) \Big[\mathcal{E}(k) \langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle \Big] . \qquad (3.23)$$

We still have to simplify the matrix element $\langle \varphi_f | Q^{\mathrm{L}}(t_i) | 0 \rangle$. In order to do this, it is convenient to further split the linear part of the charge as

$$Q^{\mathrm{L}} = -\int \mathrm{d}^3 x \, \Delta_t \phi(\vec{x}, t) + Q^{\mathrm{L}}_{(2)} . \qquad (3.24)$$

This first piece, proportional to Δ_t, is present only when the free Lagrangian shifts by a temporal boundary term under the symmetry of interest, and $\Delta_t \phi$ is precisely this boundary term. It is worth noting that this type of contribution is special to nonlinearly realized symmetries, and this is the only part of the charge linear in the field ϕ. The remaining part of the charge $Q^{\mathrm{L}}_{(2)}$ captures all of the pieces that start at quadratic order and higher in ϕ and $\Pi^{(\phi)}$. Then, we can write

$$\langle \varphi_f | Q^{\mathrm{L}}(t_i) | 0 \rangle = -\Delta_t \int \mathrm{d}^3 k \, \delta^{(3)}(\vec{k}) \langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle + \langle \varphi_f | Q^{\mathrm{L}}_{(2)}(t_i) | 0 \rangle . \qquad (3.25)$$

Putting this together with (3.23) we obtain

$$\langle \varphi_f | Q(t_i) | 0 \rangle = \int \mathrm{d}^3 k \, \delta^{(3)}(\vec{k}) \left(i(-1)^n D^n_{-\vec{k}}(t_i) \Big[\mathcal{E}(k) \langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle \Big] - \Delta_t \langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle \right)$$
$$+ \langle \varphi_f | Q^{\mathrm{L}}_{(2)}(t_i) | 0 \rangle . \qquad (3.26)$$

Now we need to evaluate each of these pieces separately.

First, we will study $\langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle$, which will allow us to simplify the first and second terms in (3.26). In general, the full quantum object is difficult to compute. However, at tree level this object (properly normalized) solves the classical equations of motion with boundary

sources φ_f:[16]

$$\phi_{\vec{k}}^{\text{cl}}(t) \simeq \frac{\int^{\varphi_f} \mathcal{D}\phi\, e^{iS_\epsilon[\phi]} \phi_{\vec{k}}(t)}{\int^{\varphi_f} \mathcal{D}\phi\, e^{iS_\epsilon[\phi]}} = \frac{\langle \varphi_f | \phi_{\vec{k}}(t_i) | 0 \rangle}{\langle \varphi_f | 0 \rangle}, \tag{3.27}$$

where we have deformed the action $S_\epsilon[\phi]$ by $i\epsilon$ terms that project onto the vacuum in the far past, and the path integral is done subject to the boundary condition that ϕ approaches φ_f at $t = t_f$. In (3.26) we are interested more specifically in the classical field profile at early times $t \to -\infty$. It turns out that this object is a generating functional for shifted wavefunction coefficients (see Appendix D):

$$\phi_{\vec{k}}^{\text{cl}}(-\infty) = \mathcal{K}(k, -\infty) \sum_{n=2} \frac{1}{n!} \int \frac{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}} \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}}$$
$$\times \delta^{(3)}(\vec{p}_1 + ... + \vec{p}_{n-1} - \vec{k}) \tilde{\psi}_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, -\vec{k}), \tag{3.28}$$

where we have defined the shifted wavefunction coefficient

$$\tilde{\psi}_n(\vec{k}) \equiv \frac{1}{2k}\left(\psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, \vec{k}; -k) - \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, \vec{k}; k) \right), \tag{3.29}$$

which is the difference of two wavefunction coefficients with the sign of the energy corresponding to $\vec{k}$ flipped. We will only need the classical field profile in the soft limit, which reduces to

$$\lim_{\vec{k}\to 0} \phi_{\vec{k}}^{\text{cl}}(-\infty) = -\sum_{n=2} \frac{1}{(n-1)!} \int \frac{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}} \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}}$$
$$\times \delta^{(3)}(\vec{p}_1 + ... + \vec{p}_{n-1}) \partial_k \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, 0). \tag{3.30}$$

Using this expression, we can simplify the first two terms on the right hand side of (3.26). Further simplification will require specifying the charge in a particular theory.

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[16]The factor in the denominator is simply the late time wavefunction itself, $\langle \varphi_f | 0 \rangle$, and serves to cancel the disconnected contributions from computing the path integral in the numerator.

Finally we have to consider the third term in (3.26). Schematically its contribution will take the form

$$\langle\varphi_f|\, Q^{\mathrm{L}}_{(2)}(t_i)\, |0\rangle \sim \int \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_m\, \delta^{(3)}(\vec{p}_1 + \cdots + \vec{p}_m)\, \langle\varphi_f|\, \phi_{\vec{p}_1}(t_i) \cdots \phi_{\vec{p}_m}(t_i)\, |0\rangle\,, \qquad (3.31)$$

for some integer m. At tree level, we can substitute in the classical field profile, sourced by the boundary field φ_f. From (3.28), we see that at early times all of the time dependence is contained in $\mathcal{K}(p, t_i) \sim e^{ipt_i}$. Thus, we will arrive at an integral of the form

$$\langle\varphi_f|\, Q^{\mathrm{L}}_{(2)}(t_i)\, |0\rangle \sim \int \mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_m\, e^{i(p_1 + \cdots + p_m)t_i} F(\vec{p}_1, \cdots, \vec{p}_m) \delta^3(\vec{p}_1 + \cdots + \vec{p}_m)\,. \qquad (3.32)$$

For generic kinematics, the integrand is highly oscillatory as $t_i \to \infty$. Thus the integral will vanish so long as the function F is sufficiently smooth in $p_1 + \cdots + p_m$, which we will assume to be the case. Thus, the matrix element $\langle\varphi_f|\, Q^{\mathrm{L}}_{(2)}\, |0\rangle$ simply vanishes.[17]

Putting all of this together, we obtain the soft theorem

$$Q\left[\varphi, \frac{\delta}{i\delta\varphi}, t_f\right]\Psi[\varphi_f, t_f] = \lim_{\vec{k}\to 0}\left(i(-1)^n D^n_{-\vec{k}}(t_i)\left[\mathcal{E}(k)\phi^{\mathrm{cl}}_{\vec{k}}(t_i)\right] - \Delta_t \phi^{\mathrm{cl}}_{\vec{k}}(t_i)\right)\Psi[\varphi_f, t_f]\,, \qquad (3.33)$$

where we have evaluated the matrix elements involving a single field in (3.26) using (3.28). Note that this implies that the soft theorem (3.33) is only valid at tree level, but this is sufficient for our purposes.[18] This expression should be utilized by substituting in the expansion of the wavefunction into wavefunction coefficients (3.3) and replacing the classical field profile with (3.28). Then, to isolate the soft theorem for a particular wavefunction coefficient one simply acts repeatedly with $\delta/\delta\varphi$, setting φ to zero in the end. We now turn

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[17]Importantly, the same is not true of the contribution from Q^{L} with a single ϕ because it is evaluated in the soft limit, which removes the oscillatory factor.

[18]Of course, this expression could be corrected at higher orders in perturbation theory by evaluating $\langle\varphi_f|\, \phi(\vec{k}, t_i)\, |0\rangle$ at higher order.

to applying this general formalism to a few specific theories of interest.

3.3 The wavefunction from the S-matrix

As a first step toward constructing the wavefunction in theories with enhanced soft limits, we take as an input the S-matrix of the relevant theory and ask: *how much more information is contained in the wavefunction?* Interestingly, we find that if one is willing to input information about the singularities of the wavefunction coefficients, the remaining part of the wavefunction is fixed by a soft theorem at one lower order in the soft momentum than the final wavefunction actually satisfies. Further, this lower-order soft theorem is actually an Adler zero-like vanishing condition in general, which is simpler to implement.[19]

3.3.1 The nonlinear sigma model

We begin by considering the wavefunction of the nonlinear sigma model (NLSM). This theory is the effective description of the Nambu–Goldstone modes arising from the spontaneous breaking of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ global symmetries to its diagonal subgroup. (For more details, see Appendix B.1.)

The nonlinear sigma model is a somewhat exceptional case because one does not need a proper soft theorem to bootstrap its wavefunction. Its amplitudes exhibit an Adler zero and vanish like $\mathcal{O}(p)$ in the soft limit, and the wavefunction coefficients correspondingly satisfy a soft theorem that constrains their $\mathcal{O}(p^0)$ behavior.[20] However for a general wavefunction coefficient this soft theorem is difficult to write down without knowing the precise form of the symmetry itself to all orders. Fortunately, the wavefunction soft theorem is not necessary— the wavefunction can be reconstructed from its singularities (including the total energy

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[19]The two-point and four-point wavefunctions for the (special) galileon are exceptions to this, see (3.86) and (3.87).

[20]To be clear, we say that a theory has an $\mathcal{O}(p^n)$ soft theorem if there is a Ward identity that controls the behavior of the wavefunction or amplitude at order $\mathcal{O}(p^n)$ in the soft limit. Note that the this slightly different than the language used for Adler zeros—if an object possesses an order $\mathcal{O}(p^{n+1})$ Adler zero, then it vanishes like p^{n+1} in the soft limit.

scattering pole) along with the requirement that the U(1) mode decouples (see Appendix B.1.1 for details about flavor ordering and U(1) decoupling).

Wavefunction coefficients

In order to construct NLSM wavefunction coefficients, our strategy is to parameterize the most general wavefunction, subject to the constraints that it has the correct singularity structure and obeys the U(1) decoupling identity (see Appendix B.1.1).

Four points: To see explicitly how this works, we begin by bootstrapping the flavor ordered four-point wavefunction, which is generated by a contact interaction in the bulk. The most general ansatz for the four-point wavefunction coefficient, having only a total energy singularity and a vanishing cut, is of the form

$$\psi_4^{(\mathrm{nlsm})}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{A_4}{E} + R, \tag{3.34}$$

where A_4 is some representation of the corresponding scattering amplitude, and R is the most general polynomial with mass dimension one in the variables p_a, s_{ab} for $a, b = 1, 2, 3, 4$. For brevity, we will often suppress the argument of the wavefunction. When this is the case, the momenta are ordered as indicated above. Without loss of generality, we may choose a representation for A_4 which is manifestly Lorentz invariant. The difference between this choice and any other choice can be absorbed into R. The symmetries of the flavor decomposition imply that ψ_4 must be even under cyclic permutations. This is most easily achieved by defining A_4 and R to separately be invariant under cyclic permutations. Since the amplitude has two powers of momenta in each term, it will be constructible out of the building blocks

$$
\begin{aligned}
C_1 &= P_1 \cdot P_2 + P_2 \cdot P_3 + P_3 \cdot P_4 + P_4 \cdot P_1 \,, \\
C_2 &= P_1 \cdot P_3 + P_2 \cdot P_4 \,,
\end{aligned}
\tag{3.35}
$$

where we have defined the four-momentum-like object $P_a^\mu \equiv (p_a, \vec{p}_a)$ and the dot products are the contraction $P_a \cdot P_b = \eta_{\mu\nu} P_a^\mu P_b^\nu$ where $\eta_{\mu\nu}$ is the ordinary Minkowski space metric (mostly plus signature). The two building blocks C_1 and C_2 are not actually independent, but satisfy

$$C_1 + C_2 = \frac{1}{2}\left(P_1 + P_2 + P_3 + P_4\right)^2 = -\frac{1}{2}E^2 \,. \tag{3.36}$$

Among many, one viable representation of A_4 is (temporarily setting the coupling $1/f^2$ to 1)

$$A_4 = \frac{1}{2}C_1 \,. \tag{3.37}$$

Next we construct R. Since we are studying a contact interaction, the cut of ψ_4 must vanish. The scattering part of our ansatz already has this property, so it must be obeyed separately by R. Therefore, each term in R must have an odd number of p_a's, which by dimensional analysis means that s_{ab} cannot appear. Therefore, the only cyclic permutation-invariant combination of mass dimension one is the total energy: $R = a_1 E$ where a_1 is some constant that remains to be determined. Our ansatz for the wavefunction is then

$$\psi_4 = \frac{C_1}{2E} + a_1 E \,. \tag{3.38}$$

This ansatz manifestly has the correct total energy singularity and has a vanishing cut. We must finally impose the constraint that the U(1) mode decouples, which is enforced by the U(1) decoupling identity (see Appendix B.1.1). At four points, this is

$$\psi_4(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) + \psi_4(\vec{p}_1, \vec{p}_3, \vec{p}_4, \vec{p}_2) + \psi_4(\vec{p}_1, \vec{p}_4, \vec{p}_2, \vec{p}_3) = \left(3a_1 - \frac{1}{2}\right)E = 0 \,. \tag{3.39}$$

Requiring that this vanishes imposes $a_1 = -1/6$. Thus after restoring the coupling, we have completely determined the flavor ordered four-point wavefunction to be

$$\psi_4^{(\text{nlsm})} = \frac{1}{2f^2 E}\left(P_1 \cdot P_2 + P_2 \cdot P_3 + P_3 \cdot P_4 + P_4 \cdot P_1\right) + \frac{1}{6f^2}E. \qquad (3.40)$$

It is straightforward to check that this matches a direct perturbative computation. Using 3-momentum conservation, we may also bring the wavefunction to the form

$$\psi_4^{(\text{nlsm})} = \frac{1}{3f^2 E}\left(P_1 \cdot P_2 + P_2 \cdot P_3 - 2P_1 \cdot P_3\right) - \frac{1}{6f^2}\left(p_1 - 2p_2 + p_3\right). \qquad (3.41)$$

By writing the wavefunction in this form, we have sacrificed cyclic invariance in order to eliminate all instances of $\vec{p}_4$ and p_4 (except in the total energy singularity). As we will see momentarily, writing the wavefunction in this way is useful for bootstrapping wavefunction coefficients that involve exchange interactions.[21]

Six points: As a more nontrivial example, one can construct the six-point wavefunction coefficient from knowledge of its singularities along with the U(1) decoupling identity. Since we are constructing the flavor ordered wavefunction, the only possible singularities occur when adjacent sums of three energies add up to zero. There are three such factorization channels: one is when $E_{123} = p_1 + p_2 + p_3 + s_{123}$ or $E_{456} = p_4 + p_5 + p_6 + s_{456}$ vanish, or the analogous partial energy singularities in the other factorization channels. A natural ansatz is then of the form (again setting the coupling $1/f^2$ to 1)

$$\psi_6^{(\text{nlsm})}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) = \frac{1}{E}\left(\frac{N_{123}N_{456}}{E_{123}\,E_{456}} + \frac{N_{561}N_{234}}{E_{561}\,E_{234}} + \frac{N_{612}N_{345}}{E_{612}\,E_{345}}\right) + \frac{C}{E} + \text{regular}, \quad (3.43)$$

where the regular terms do not have any singularities. There are various ways to fix the form of the kinematic numerators. The simplest is to note that the expression (3.43) must factorize

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[21] There is also a more economical way of writing the wavefunction, which takes the form of an NLSM scattering amplitude divided by the total energy:

$$\psi_4^{(\text{nlsm})} = \frac{1}{6f^2 E}\left(P_1 \cdot P_2 + P_2 \cdot P_3 + P_3 \cdot P_4 + P_4 \cdot P_1 - 2P_1 \cdot P_3 - 2P_2 \cdot P_4\right). \qquad (3.42)$$

appropriately into products of shifted four-point functions when we take cuts according to the prescription (3.15).[22] These shifted four-point functions can be computed using (3.40):

$$\tilde{\psi}_4^{(\text{nlsm})}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_I) = \frac{P_1 \cdot P_2 + P_2 \cdot P_3 - 2P_1 \cdot P_3}{3(p_{123}^2 - s_{123}^2)}, \tag{3.44}$$

where we have shifted the internal line, I. This implies that we should take

$$N_{123} \equiv \frac{1}{3}\left(P_1 \cdot P_2 + P_2 \cdot P_3 - 2P_1 \cdot P_3\right), \tag{3.45}$$

and similarly for the other permutations. With this choice (3.43) will have the correct cuts (and by extension the correct partial energy singularities). We must then fix the C and regular terms in (3.43). The C terms have mass dimension two and can be built from the cyclic permutation-invariant building blocks[23]

$$C_1 = P_1 \cdot P_2 + P_2 \cdot P_3 + P_3 \cdot P_4 + P_4 \cdot P_5 + P_5 \cdot P_6 + P_6 \cdot P_1, \tag{3.46}$$

$$C_2 = P_1 \cdot P_3 + P_2 \cdot P_4 + P_3 \cdot P_5 + P_4 \cdot P_6 + P_5 \cdot P_1 + P_6 \cdot P_2.$$

Finally, the regular terms in (3.43) must also be cyclic permutation invariant, have mass dimension one, and have a vanishing cut, which implies they must be proportional to E. We then have

$$\psi_6^{(\text{nlsm})} = \frac{1}{E}\left(\frac{N_{123}N_{456}}{E_{123}E_{456}} + \frac{N_{561}N_{234}}{E_{561}E_{234}} + \frac{N_{612}N_{345}}{E_{612}E_{345}}\right) + \frac{a_1 C_1 + a_2 C_2}{E} + a_3 E. \tag{3.47}$$

---

[22]Equivalently, we can require that the residues of the partial energy singularities are the appropriate combination of shifted four-point wavefunction coefficients and four-point amplitudes.

[23]There is another possible cyclic-invariant building block with the right mass dimension: $C_3 = P_1 \cdot P_4 + P_2 \cdot P_5 + P_3 \cdot P_6$. As in the four-point function case, these quantities are not independent, but are related by $C_1 + C_2 + C_3 = -\frac{E^2}{2}$ so we can eliminate C_3 in terms of C_1, C_2 at the cost of shifting around the regular terms with no singularities.

The coefficients a_1 and a_2 can be fixed from the $E \to 0$ limit, whose residue must be the scattering amplitude

$$A_6 = -\left(\frac{N_{123}N_{456}}{S_{123}} + \frac{N_{561}N_{234}}{S_{561}} + \frac{N_{612}N_{345}}{S_{612}}\right) - \frac{1}{18}\left(C_1 + 2C_2\right). \tag{3.48}$$

We can then immediately read off the coefficients

$$a_1 = -\frac{1}{18}, \qquad a_2 = -\frac{1}{9}. \tag{3.49}$$

To fix the last coefficient, we use the U(1) decoupling identity, which at six points reads

$$0 = \psi_6(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) + \psi_6(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) + \psi_6(\vec{p}_2, \vec{p}_3, \vec{p}_1, \vec{p}_4, \vec{p}_5, \vec{p}_6)$$
$$+ \psi_6(\vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_1, \vec{p}_5, \vec{p}_6) + \psi_6(\vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_1, \vec{p}_6) = \left(5a_3 + \frac{1}{6}\right)E .$$

This must vanish, which implies $a_3 = -1/30$, so that the six-point wavefunction is given by

$$\psi_6^{(\mathrm{nlsm})} = \frac{1}{E}\left(\frac{N_{123}N_{456}}{E_{123}E_{456}} + \frac{N_{561}N_{234}}{E_{561}E_{234}} + \frac{N_{612}N_{345}}{E_{612}E_{345}}\right) - \frac{C_1 + 2C_2}{18E} - \frac{1}{30}E , \tag{3.50}$$

which can again be matched to a bulk perturbative calculation. At higher points, the boot-strap procedure generalizes in a straightforward fashion.[24]

Soft theorem

Though the NLSM wavefunction is fixed by the residues of its singularities along with the U(1) decoupling identity, it is nevertheless interesting to explore the soft theorem that the wavefunction satisfies at $\mathcal{O}(p^0)$. The NLSM wavefunction soft theorem is somewhat less powerful than its amplitude counterpart. Essentially this is because it is necessary to know

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[24]A natural question is to understand how this discussion changes when studying double trace theories—in particular the SO(N+1)/SO(N) NLSM—where the Goldstones transform in the fundamental representation of SO(N). In particular, there is no analogue of the U(1) decoupling identity for such theories, and it would interesting to understand what (if any) piece of data replaces it.

the precise form of the symmetry (B.5) (or, equivalently, the corresponding Noether charge) to a given order in fields in order to derive the relevant soft theorem. Further, because the symmetry transformation has infinitely many terms, there is no universal form of the soft theorem that holds for a general n-point wavefunction. Instead, one must work out the soft theorem order by order.

We begin by considering the NLSM symmetry transformation:

$$\delta\phi^c = B^c - \frac{1}{3f^2}B^{a_1}\phi^{b_1}\phi^{b_2}f^{a_1b_1b_3}f^{cb_2b_3} - \frac{1}{45f^4}B^{a_1}\phi^{b_1}\phi^{b_2}\phi^{b_3}\phi^{b_4}f^{a_1b_1\,c}f^{a_2b_2a_3}f^{b_3ca_4}f^{b_4a_3a_4} + \mathcal{O}(\phi^6)\,,$$

$$(3.51)$$

where f is the symmetry breaking scale and f^{abc} are the Lie algebra structure constants (hopefully the difference is clear from context). The corresponding symmetry charge is given by

$$Q = \int \mathrm{d}^3x\, \delta\phi^b \Pi_b^{(\phi)}\,. \tag{3.52}$$

We want to apply the formalism of Section 3.2.2 to this particular charge. First, note that Q only has a piece which is linear in Π, so computing the left hand side of (3.33) is straightforward. Moreover, the differential operator coming from the nonlinear part of the symmetry is just a constant $D_{\vec{k}}^n(t) = 1$. Finally, note that Q does not contain a term with an isolated ϕ, so $\Delta_t = 0$.[25]

With these considerations, we can write the soft theorem (3.2.2) as

$$-i\lim_{\vec{k}\to 0}\delta\varphi_{\vec{k}}^a\frac{\delta\log\Psi[\varphi_f,t_f]}{\delta\varphi_{\vec{k}}^a} = i\lim_{\vec{k}\to 0}B^a\,\mathcal{E}(k)\phi_{\vec{k}}^{\mathrm{cl}\,a}(t_i)\,, \tag{3.53}$$

where we have divided through by $\Psi[\varphi_f,t_f]$ in order to write the LHS in terms of $\log\Psi$.

---

[25]This is because the symmetry transformation $\delta\phi^a$ does not induce a temporal boundary term on the kinetic part of the action.

Next we expand both sides in terms of wavefunction coefficients, which gives

$$
-\lim_{\vec{k}\to 0}\sum_{n=2}\frac{1}{(n-1)!}\int\frac{\mathrm{d}^3 p_1\cdots\mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-1)}}\,\varphi^{b_1}_{\vec{p}_1}\cdots\varphi^{b_{n-1}}_{\vec{p}_{n-1}}\delta\varphi^a_{\vec{k}}\delta^{(3)}(\vec{p}_1+...+\vec{p}_{n-1})
$$

$$
\times\,\psi^{b_1\cdots b_{n-1}a}_n(\vec{p}_1,\cdots,\vec{p}_{n-1},\vec{k})
$$

$$
=\lim_{\vec{k}\to 0}B^a\mathcal{E}(k)\sum_{n=2}\frac{1}{(n-1)!}\int\frac{\mathrm{d}^3 p_1\cdots\mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}}\,\varphi^{b_1}_{\vec{p}_1}\cdots\varphi^{b_{n-1}}_{\vec{p}_{n-1}}\delta^{(3)}(\vec{p}_1+...+\vec{p}_{n-1})
$$

$$
\times\,\partial_k\psi^{b_1\cdots b_{n-1}a}_n(\vec{p}_1,\cdots,\vec{p}_{n-1},\vec{k})\,. \quad (3.54)
$$

Note that the RHS vanishes because $\mathcal{E}(k)$ vanishes in the soft limit, and $\partial_k\psi$ is finite. At this point, we may take functional derivatives with respect to φ to extract particular wavefunction coefficients from these sums.

Since $\delta\phi$ has infinitely many terms, there is no uniform way to write the resulting soft theorem for a general n-point wavefunction coefficient. Therefore, one must derive results coefficient by coefficient. This is straightforward to do; after stripping the flavor indices, one finds that the four-point soft theorem is

$$
\lim_{\vec{k}\to 0}\psi_4(\vec{k},\vec{p}_2,\vec{p}_3,\vec{p}_4)=-\frac{1}{6f^2}\Big(\psi_2(p_4)-2\psi_2(p_3)+\psi_2(p_2)\Big)\,. \quad (3.55)
$$

It is easy to verify that (3.40) satisfies this identity. We can similarly work out the six-point soft theorem

$$\lim_{\vec{k}\to 0} \psi_6(\vec{k}, \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) =$$
$$- \frac{1}{6f^2}\Big(\psi_4(\vec{p}_2 + \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) - 2\psi_4(\vec{p}_6 + \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5) + \psi_4(\vec{p}_5 + \vec{p}_6, \vec{p}_2, \vec{p}_3, \vec{p}_4) \Big)$$
$$+ \frac{1}{180f^4}\Big(\psi_2(p_6) - 4\psi_2(p_5) + 6\psi_2(p_4) - 4\psi_2(p_3) + \psi_2(p_2) \Big) , \tag{3.56}$$

and check that it is satisfied by the true answer (3.50). It is worth emphasizing that in contrast to the Adler zero that amplitudes satisfy, these soft theorems are comparatively less useful, because we do not know how to write them down without knowing precisely form of the full symmetry transformation. Nevertheless, given knowledge of these soft theorems, it is possible to construct the NLSM wavefunction recursively, without using any scattering information, as we show in Section 3.4.4.

3.3.2 $P(X)$ and Dirac–Born–Infeld

The next example we consider is that of a Dirac–Born–Infeld (DBI) scalar. In addition to the ordinary shift symmetry $\delta_C \phi = 1$, the theory is invariant under the symmetry

$$\delta_{B_\mu} \phi = x_\mu + \frac{1}{f^4}\phi \partial_\mu \phi . \tag{3.57}$$

In the context of scattering amplitudes, this symmetry causes the S-matrix elements to vanish as $\mathcal{O}(p^2)$ in the soft limit.

We will require input from these symmetries in the form of soft theorems in order to fix the wavefunction. Since we are utilizing scattering information, we only need the soft theorem corresponding to the shift symmetry, which in this case implies that the wavefunction also has an Adler zero.[26]

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[26]Later in Section 3.4 we will construct two types of recursion relations, one that uses the same input as this section, but also one that uses the soft theorem for the DBI symmetry (3.57) instead of the scattering amplitude.

Soft theorems

In the $P(X)$ and DBI cases, bootstrapping wavefunction coefficients requires input from the lowest-order soft theorem. This soft theorem is particularly simple—it is just a vanishing statement like the Adler zero. We therefore first derive this identity. We will also derive the corresponding statement for the DBI symmetry. This latter statement is not directly needed to bootstrap the wavefunction from the S-matrix, but can be used to recursively construct the wavefunction via a different method (as we will do in Section 3.4).

Wavefunction Adler zero: We first derive the lowest-order soft theorem satisfied by $P(X)$ wavefunctions (and in particular DBI). This is the soft theorem corresponding to the shift symmetry $\delta_C \phi = 1$. This symmetry is generated by the charge

$$Q_C = \int \mathrm{d}^3 x \, \Pi^{(\phi)} \,, \tag{3.58}$$

and the differential operator appearing in (3.33) is $D^0_{\vec{k}}(t_i) = 1$. From here we proceed exactly as in the case of the NLSM. However in this case the shift symmetry does not have quadratic and higher terms, so there is a simple expression holding for all wavefunction coefficients:

$$\lim_{\vec{k}\to 0} \psi_n(\vec{k}, \vec{p}_2, \cdots, \vec{p}_n) = 0 \,. \tag{3.59}$$

That is, the wavefunction coefficients possess an Adler zero.

DBI symmetry soft theorem: We now sketch the derivation of the higher-order soft theorem associated to the DBI symmetry (3.57). Notice that this symmetry has a full Lorentz vector of charges. Since we are breaking Lorentz symmetry by choosing a time slice on which to define the wavefunction, the spatial and temporal components of this charge (and the corresponding soft theorems) behave slightly differently. Explicitly, these charges

are given by

$$Q_B^i = \int \mathrm{d}^3x \left(x^i + \frac{1}{f^4} \phi \partial^i \phi \right) \Pi^{(\phi)} \,, \tag{3.60}$$

$$Q_B^t = \int \mathrm{d}^3x \, t\Pi^{(\phi)} - \phi \sqrt{\left(1 + \frac{1}{f^4}(\nabla\phi)^2\right)\left(1 + \frac{1}{f^4}(\Pi^{(\phi)})^2\right)} \,. \tag{3.61}$$

Knowing only the symmetry transformation $\delta\phi$, we may immediately write down the spatial charge by appealing to the logic outlined in Appendix C.2. On the other hand, the temporal charge must be computed by explicitly working out the boundary term that the Lagrangian shifts by under this symmetry. We now consider the soft theorems arising from each of these charges in turn.

Spatial soft theorem: First consider the spatial charge (3.60). The spatial DBI symmetry does not have a boundary term, so $\Delta_t = 0$ in (3.33). The differential operator corresponding to the x^i part of the symmetry is $D_{\vec{k}}^1 = -i\mathrm{d}/\mathrm{d}\vec{k}$.[27] The general soft theorem (3.33) then takes the form

$$(2\pi)^3 \lim_{\vec{k}\to 0} \frac{\mathrm{d}}{\mathrm{d}\vec{k}} \left[\frac{\delta \log \Psi[\varphi, t_f]}{\delta\varphi_{-\vec{k}}}\right] + \frac{1}{f^4} \int \frac{\mathrm{d}p_1^3 \mathrm{d}p_2^3 \mathrm{d}p_3^3}{(2\pi)^3} \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \left(\frac{\vec{p}_1 + \vec{p}_2}{2}\right) \varphi_{\vec{p}_1} \varphi_{\vec{p}_2} \frac{\delta \log \Psi[\varphi, t_f]}{\delta\varphi_{-\vec{p}_3}}$$
$$= -\lim_{\vec{k}\to 0} \frac{\mathrm{d}}{\mathrm{d}\vec{p}} \left[\mathcal{E}(k)\phi_{\vec{k}}^{\mathrm{cl}}(t_i)\right] \,, \tag{3.62}$$

where the two terms on the LHS come from the two terms in the charge (3.60). Now we may expand both sides in terms of wavefunction coefficients and take functional derivatives to obtain (after relabelling)

$$\lim_{\vec{k}\to 0} \frac{\mathrm{d}}{\mathrm{d}\vec{k}}\psi_n(-\vec{k}, \vec{p}_2, \cdots, \vec{p}_n) + \sum_{\text{pairs } a,b} \frac{1}{f^4}(\vec{p}_a + \vec{p}_b)\psi_{n-2}(\vec{p}_a + \vec{p}_b, \vec{p}_2, \cdots, \overline{\vec{p}}_a, \cdots, \overline{\vec{p}}_b, \cdots, \vec{p}_n)$$
$$= \lim_{\vec{k}\to 0} \frac{\mathrm{d}}{\mathrm{d}\vec{k}} \left[\mathcal{E}(q)\psi_n(-\vec{k}, \vec{p}_2, \cdots, \vec{p}_n)\right] \,, \tag{3.63}$$

where the bar over a particular momentum indicates that it should be removed. Using the

---

[27]A helpful fact to remember is $\vec{x} = i\int \mathrm{d}^3k \, e^{i\vec{k}\cdot\vec{x}} \vec{\partial}_k \delta^{(3)}(\vec{k})$.

fact that $\lim_{\vec{k}\to 0} \psi_n = -\partial_k \psi_n|_0$ along with the chain rule, we can simplify (3.63) to the schematic form[28]

$$\lim_{\vec{k}\to 0} \left(\vec{\partial}_k \psi_n(-\vec{k}) + \hat{k}\partial_k \psi_n(-\vec{k}) \right) + \psi_{n-2}^{ab} = \lim_{\vec{k}\to 0} \left(\hat{k}\partial_k \psi_n(-\vec{k}) - \mathcal{E}(k)\frac{\mathrm{d}}{\mathrm{d}\vec{k}}\left[\psi_n(-\vec{k}) \right] \right) \quad (3.64)$$

where ψ_{n-2}^{ab} is the sum in the first line of (3.63). Note that the terms involving $\lim_{\vec{k}\to 0} \hat{k}$—which strictly speaking are ill-defined because they are direction dependent—cancel between the two sides. The second term on the RHS vanishes in the soft limit. Thus we are left with

$$\lim_{\vec{k}\to 0} \vec{\partial}_k \psi_n(\vec{k}, \vec{p}_2, \cdots, \vec{p}_n) = \sum_{\text{pairs } a,b} \frac{1}{f^4}(\vec{p}_a + \vec{p}_b)\psi_{n-2}(\vec{p}_a + \vec{p}_b, \vec{p}_2, \cdots, \overline{p}_a, \cdots, \overline{p}_b, \cdots, \vec{p}_n) . \tag{3.65}$$

This implies that the DBI wavefunction coefficients at order $\mathcal{O}(p)$ satisfy a soft theorem rather than an Adler zero.

Temporal soft theorem: Finally, we want to derive the soft theorem associated to the temporal DBI charge (3.61). Notice that this charge does have a term linear in ϕ, which means that $\Delta_t = 1$. Moreover, the differential operator for the temporal symmetry is $D_{\vec{k}}^0(t_i) = t_i$. The soft theorem (3.33) then can be written as

$$\bar{Q}_B^t \left[\varphi, \frac{\delta}{i\delta\varphi}, t_f \right] \Psi[\varphi, t_f] = \lim_{\vec{k}\to 0} \left(it_i\mathcal{E}(k) - 1 \right) \phi^{\text{cl}}(\vec{k}, t_i)\Psi[\varphi, t_f] , \tag{3.66}$$

where $\bar{Q}_B^t$ is only the square root term in (3.61).[29] Now we expand this expression in wavefunction coefficients and take functional derivatives to extract their soft theorems. Unfortunately, there is no simple way to write the LHS in general, but for particular examples everything can be straightforwardly worked out. For example, we have at four-points

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[28]Specifically we use the relation $\frac{\mathrm{d}\psi_n}{\mathrm{d}\vec{k}} = \vec{\partial}_k \psi_n + \hat{k}\partial_k \psi_n$.

[29]At late times, we may ignore the nonlinear piece if we set $t_f = 0$.

$$\lim_{\vec{k} \to 0} \partial_k \psi_4(\vec{k}, \vec{p}_2, \vec{p}_3, \vec{p}_4) = -\frac{1}{f^4} \Big(\vec{p}_2 \cdot \vec{p}_3 + \psi_2(p_2)\psi_2(p_3) \Big) + \text{perms.}, \qquad (3.67)$$

where the overall δ-function on each side is $\delta^{(3)}(\vec{p}_2 + \vec{p}_3 + \vec{p}_4)$, and "perms." indicates a symmetrization over these momenta. At six points, the soft theorem reads

$$\lim_{\vec{k} \to 0} \partial_k \psi_6(\vec{k}, \vec{p}_2, \vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) = \frac{1}{f^4} \psi_4(-\vec{p}_{234}, \vec{p}_2, \vec{p}_3, \vec{p}_4) \psi_2(p_5)$$
$$+ \frac{1}{f^8} \Big(-\vec{p}_2 \cdot \vec{p}_3 \psi_2(p_4)\psi_2(p_5) + 3\psi_2(p_2)\psi_2(p_3)\psi_2(p_4)\psi_2(p_5) + \vec{p}_2 \cdot \vec{p}_3 \vec{p}_4 \cdot \vec{p}_5 \Big) + \text{perms.}$$
$$(3.68)$$

As in the case of the nonlinear sigma model, these soft theorems may be used to recursively construct the six-point wavefunction coefficient if the scattering amplitude is not known, see Section 3.4.4. However, it is often simpler to use the scattering amplitude plus the Adler zero that wavefunction coefficients satisfy as a consequence of the ordinary shift symmetry, as we now demonstrate.

Wavefunction coefficients

Much like the NLSM, we can construct wavefunction coefficients of DBI from their singularities, but now supplemented with the Adler zero condition. We will demonstrate this procedure for a number of simple examples.

Four-point wavefunction: To bootstrap the four-point function, we will take an ansatz of the form

$$\psi_4^{(\text{dbi})} = \frac{A_4}{E} + R, \qquad (3.69)$$

where A_4 is an arbitrary representation of the scattering amplitude and R is analytic in the total energy E. For convenience, we will choose a manifestly Lorentz invariant representation

of the scattering amplitude which is also manifestly Bose-symmetric (temporarily setting the coupling $1/f^4$ to 1):

$$A_4 = \frac{1}{2}\Big((P_1 \cdot P_2)^2 + (P_1 \cdot P_3)^2 + (P_1 \cdot P_4)^2 + (P_2 \cdot P_3)^2 + (P_2 \cdot P_4)^2 + (P_3 \cdot P_4)^2\Big). \quad (3.70)$$

To parametrize R, we write down the most general Bose-symmetric polynomial which is cubic in p_a, s_{ab}. As in the NLSM case, we are assuming that the wavefunction coefficient is purely generated by a contact interaction in the bulk, which means that the cut of the wavefunction vanishes. This implies that each term in R must have an odd number of external energies, so by dimensional analysis s_{ab} can only appear in even powers. Modulo dimension-dependent Gram identities, the most general polynomial of this form is

$$R = a_1\Big(p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4\Big) + a_2\Big(p_1^2 p_2 + \text{perms.}\Big) + a_3\Big(p_1^3 + p_2^3 + p_3^3 + p_4^3\Big). \quad (3.71)$$

All of the unfixed coefficients are completely determined by the simple Adler zero. Sending $\vec{p}_1 \to 0$ gives:

$$\lim_{\vec{p}_1 \to 0} \psi_4 = \Big(a_1 - \frac{3}{4}\Big)p_2 p_3 p_4 + \Big(a_2 + \frac{3}{8}\Big)\Big(p_2^3 + p_3^3 + p_4^3\Big) + \Big(a_3 + \frac{1}{8}\Big)\Big(p_2^2 p_3 + p_2^2 p_4 + p_3^2 p_4\Big). \quad (3.72)$$

Since all the terms on the right hand side are independent, their coefficients must each vanish, so that

$$a_1 = \frac{3}{4}, \qquad a_2 = -\frac{3}{8}, \qquad a_3 = -\frac{1}{8}. \qquad (3.73)$$

Restoring the coupling, this fixes the wavefunction to be

$$\psi_4^{(\text{dbi})} = \frac{1}{2f^4 E}\Big((P_1 \cdot P_2)^2 + (P_1 \cdot P_3)^2 + (P_1 \cdot P_4)^2 + (P_2 \cdot P_3)^2 + (P_2 \cdot P_4)^2 + (P_3 \cdot P_4)^2\Big)$$
$$+ \frac{3}{4f^4}\Big(p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4\Big) - \frac{3}{8f^4}\Big(p_1^2 p_2 + \text{perms.}\Big)$$
$$- \frac{1}{8f^4}\Big(p_1^3 + p_2^3 + p_3^3 + p_4^3\Big).$$
$$\tag{3.74}$$

Later on, when it comes to performing recursion, the following form of the wavefunction will also be useful:[30]

$$\psi_4^{(\text{dbi})} = -\frac{2}{f^4 E}\Big(P_1 \cdot P_2\, P_2 \cdot P_3 + P_2 \cdot P_3\, P_3 \cdot P_1 + P_3 \cdot P_1\, P_1 \cdot P_2\Big)$$
$$- \frac{1}{f^4}\Big(P_1 \cdot P_2 p_3 + P_1 \cdot P_3 p_2 + P_2 \cdot P_3 p_1\Big).$$
$$\tag{3.76}$$

It is possible to build the six-point wavefunction in the same systematic way, but in Section 3.4 we give a more elegant recursive construction using the same input.

3.3.3 Galileon theories

As a final example, we consider galileon field theories. These are theories that have a shift symmetry similar to the DBI symmetry (3.57), but which is field independent

$$\delta_{B^\mu}\phi = x^\mu, \tag{3.77}$$

---

[30]Even this is not the most economical representation of the wavefunction. Using 3-momentum conservation, it may be reduced to

$$\psi_4^{(\text{dbi})} = \frac{1}{f^4 E}\Big(P_1 \cdot P_2\, P_3 \cdot P_4 + P_1 \cdot P_3\, P_2 \cdot P_4 + P_1 \cdot P_4\, P_2 \cdot P_3\Big). \tag{3.75}$$

This is, of course, a particular representation of the scattering amplitude divided by the total energy. Notice that this form of the amplitude in the numerator of (3.75) manifestly has the $\mathcal{O}(p)$ Adler zero, while this is not manifest in (3.70). Noting that the different forms of the amplitude correspond to different forms of the Lagrangian related by an integration by parts, we can see that this simple way of writing the answer corresponds to the Lagrangian with one time derivative per field (see Appendix C.1).

along with the ordinary shift symmetry $\delta_C \phi = 1$. In this section we will focus solely on the quartic galileon vertex. However, our results should generalize to any galileon theory. In addition, among the class of galileon theories, there is a distinguished subset [147, 154, 186]—often called the special galileon—that has an additional symmetry of the form

$$\delta_{S_{\mu\nu}} \phi = s_{\mu\nu} \left(x^\mu x^\nu + \frac{1}{f^6} \partial^\mu \phi \partial^\nu \phi \right) . \tag{3.78}$$

As before, we first derive the soft theorems associated to these symmetries and then use them to bootstrap the wavefunction.

Soft theorems

As was mentioned before, the special galileon has three different symmetries: an ordinary shift symmetry, a symmetry linear in x^μ (galileon), and a symmetry quadratic in x^μ (special galileon). We will treat each of these in turn.

Shift symmetry: Like the DBI case, the shift symmetry, $\delta_C \phi = 1$ leads to an Adler zero—the wavefunction coefficients vanish in the soft limit as in (3.59).

Galileon symmetry: We next consider the $\delta_{B^\mu}\phi = x^\mu$ symmetry. We first need the generators of the spatial and temporal parts of this symmetry:[31]

$$Q_B^i = -i \lim_{\vec{k}\to 0} \frac{\mathrm{d}}{\mathrm{d}k^i}\Pi^{(\phi)}(\vec{k})\,, \tag{3.81}$$

$$Q_B^0 = \lim_{\vec{k}\to 0}\left(t\Pi^{(\phi)}(\vec{k}) - \phi_{\vec{k}}(t)\right)$$
$$+ \frac{N}{f^6}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2\mathrm{d}^3p_3}{(2\pi)^{3\cdot 2}}\,\delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\phi_{\vec{p}_1}(t)\phi_{\vec{p}_2}(t)\phi_{\vec{p}_3}(t)\left(p_2^2 p_3^2 - (\vec{p}_2\cdot\vec{p}_3)^2\right)\,. \tag{3.82}$$

Here, N is a constant which fixes the normalization of the interaction term in the action. It may be absorbed into the coupling parameter $1/f^6$. However we will keep it explicit for convenience later on when we bootstrap wavefunction coefficients. Interestingly, the K^0 term in Q_B^0 only has a term with one ϕ and one with three ϕs. Eventually we will see that these terms only contribute to the Ward identities satisfied by the two-point and four-point wavefunction coefficients. All other wavefunction coefficients will actually have higher-order Adler zeroes, vanishing like $\mathcal{O}(p^2)$.

Spatial soft theorem: We first consider the soft theorem associated to the symmetry (3.81). Everything proceeds essentially identically to the DBI case, except that there is no $\phi\partial\phi$ part of the symmetry. The resulting soft theorem is the same as (3.62) with the middle term removed, because it comes from the $\phi\partial\phi$ part of the symmetry. In terms of wavefunction

---

[31]The spatial charge may be written down immediately if $\delta\phi$ is known. The temporal charge is more subtle, but can be obtained by following the procedure outlined in Appendix C.2. For the charge to act correctly, it must be of the form

$$Q_B^0 = \int \mathrm{d}^3x\, t\Pi^{(\phi)}(\vec{x},t) - K^0[\phi(\vec{x},t)]\,. \tag{3.79}$$

Using the explicit expression for the charges, we can compute the commutator $[Q_B^i, Q_B^0] = \int \mathrm{d}^3x\, \delta_{B^i}K^0[\phi(\vec{x},t)]$. We know from the algebra of symmetries that this commutator should vanish, which requires that K^0 is invariant under a spatial galileon symmetry transformation. The terms with this symmetry and with the right derivative counting and number of fields are themselves (three-dimensional euclidean) galileon terms:

$$K^0[\phi] = \phi - \frac{N}{f^6}\phi\left[(\nabla^2\phi)^2 - (\partial_i\partial_j\phi)^2\right]\,. \tag{3.80}$$

Fourier transforming (3.79) then produces (3.82).

coefficients, we find

$$\lim_{\vec{k}\to 0} \vec{\partial}_k \psi_n(\vec{k}, \vec{p}_2, \cdots, \vec{p}_n) = 0\,, \tag{3.83}$$

which implies that the galileon wavefunction exhibits a spatial enhanced Adler zero.

Temporal soft theorem: We next turn to the soft theorem that is a consequence of the temporal symmetry. In this case the general Ward identity (3.33) can be written as

$$Q_B^0 \Psi[\phi, t_f] = \lim_{\vec{q}\to 0} \left[it_i \mathcal{E}(k) - 1 \right] \phi_{\vec{k}}^{\text{cl}}(t_i) \Psi[\phi, t_f]\,, \tag{3.84}$$

where Q_B^0 is given by (3.82). Now we may expand both sides in terms of wavefunction coefficients. After relabelling we have

$$- \int \mathrm{d}^3k\, \delta^{(3)}(\vec{k}) \varphi_{\vec{k}}(t_f) + \frac{N}{f^6} \int \frac{\mathrm{d}p_1^3 \mathrm{d}p_2^3 \mathrm{d}p_3^3}{(2\pi)^{3\cdot2}} \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \varphi_{\vec{p}_1}(t) \varphi_{\vec{p}_2}(t) \varphi_{\vec{p}_3}(t) \left(p_2^2 p_3^2 - (\vec{p}_2 \cdot \vec{p}_3)^2 \right)$$

$$= - \lim_{\vec{k}\to 0} \left[\frac{it_i}{(2\pi)^3} \mathcal{E}(k) - 1 \right] \sum_{n=2} \frac{1}{(n-1)!} \int \frac{\mathrm{d}^3p_1 \cdots \mathrm{d}^3p_{n-1}}{(2\pi)^{3(n-2)}} \delta^{(3)}(\vec{p}_1 + \dots + \vec{p}_{n-1})$$

$$\times \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}} \partial_k \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, \vec{k})\,. \tag{3.85}$$

In the soft limit, the $\mathcal{E}(k)$ term vanishes. We may take functional derivatives to extract the soft theorems obeyed by the wavefunction coefficients:

$$\lim_{\vec{k}\to 0} \partial_k \psi_2(\vec{k}) = -1\,, \tag{3.86}$$

$$\lim_{\vec{k}\to 0} \partial_k \psi_4(\vec{k}, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{2N}{f^6} \left(p_2^2 p_3^2 - (\vec{p}_2 \cdot \vec{p}_3)^2 \right) + \text{perms.}\,, \tag{3.87}$$

$$\lim_{\vec{k}\to 0} \partial_k \psi_n(\vec{k}, \vec{p}_2, ..., \vec{p}_n) = 0, \qquad n > 4\,. \tag{3.88}$$

Notice that $n = 2, 4$ are exceptional cases where the wavefunction's energy derivative in the

soft limit does not vanish, but rather obeys a soft theorem.

Special galileon symmetry: Finally we consider the soft theorems associated to the special galileon symmetry (3.78). This symmetry has a full symmetric traceless tensor of charges. As in the DBI case, the spatial and temporal components of the charges behave differently. The spatial (traceless) charge is given by

$$s_{ij}Q_S^{ij} = \int \mathrm{d}^3x \left(x^i x^j + \frac{1}{f^6}\partial^i\phi\partial^j\phi \right)\Pi^{(\phi)} \tag{3.89}$$

where s_{ij} is a symmetric and traceless tensor. This charge may be immediately written down from the symmetry transformation following the logic in Appendix C.2, and does not require knowledge of the action. To derive the temporal charges, one needs to compute the boundary term the action develops by brute force. Since we do not require the temporal charges for our purposes, we leave these details for future work.

Spatial Soft Theorem: To derive the special galileon spatial soft theorem, we essentially follow the same steps as in the derivation of the DBI spatial soft theorem. The end result is

$$\lim_{\vec{k}\to 0} s_{ij}\partial_k^i\partial_k^j\psi_n(\vec{k},\vec{p}_2,\cdots,\vec{p}_n) = -\sum_{\text{pairs }a,b} \frac{1}{f^6}s_{ij}p_a^i p_b^j\, \psi_{n-2}(\vec{p}_a+\vec{p}_b,\vec{p}_2,\cdots\bar{p}_a,\cdots,\bar{p}_b,\cdots,\vec{p}_n)\,. \tag{3.90}$$

Wavefunction coefficients

We now want to use the soft theorems discussed in this section to bootstrap the wavefunction of the galileon. We begin with an ansatz for the wavefunction of the form

$$\psi_4^{(\text{gal})} = \frac{A_4}{E} + R\,, \tag{3.91}$$

where A_4 is some representation of the scattering amplitude and R is a remainder term without any singularities. Concretely, we can write the amplitude as (temporarily setting the coupling $1/f^6$ to 1)

$$A_4 = -\frac{1}{2}\left(P_1 \cdot P_2 P_1 \cdot P_3 P_1 \cdot P_4 + \text{perms.}\right). \tag{3.92}$$

This way of writing the amplitude manifestly vanishes as $\mathcal{O}(p)$ in the soft limit. However when energy is conserved, this object actually vanishes as $\mathcal{O}(p^2)$, though this cannot be made manifest.[32] As in the case of NLSM and DBI, the cut of the wavefunction coefficient must vanish, which means that each term must have an odd power of external energies. Then by dimensional analysis, s_{ab} can only appear in even powers. Up to dimension-dependent Gram identities, the most efficient ansatz takes the form

$$\begin{aligned} R = \ & a_1\, p_1^2 p_2 p_3 p_4 + a_2\, p_1^3 p_2 p_3 + a_3\, p_1^4 p_2 + a_4\, p_1^5 + a_5\, p_1^2 p_2^2 p_3 + a_6 p_1^3 p_2^2 \\ & + a_7\, p_1 s_{12}^4 + a_8\, p_1^2 p_2 s_{12}^2 + \text{perms.} \end{aligned} \tag{3.93}$$

In order to restrict this ansatz, we first check the simple Adler zero coming from the ordinary shift symmetry. The amplitude part manifestly has this property, but in the soft limit, R becomes

$$\lim_{\vec{p}_1 \to 0} R = a_2\, p_2^3 p_3 p_4 + \left(a_3 + a_7\right) p_2^4 p_3 + \left(a_4 + a_7\right) p_2^5 + \left(a_5 + 2a_8\right) p_2^2 p_3^2 p_4 + a_6 p_2^3 p_3^2 + \text{perms.} \tag{3.94}$$

Since this is an independent basis of polynomials, in order for this quantity to vanish we have to set

$$a_2 = a_6 = 0, \qquad a_3 = a_4 = -a_7, \qquad a_5 + 2a_8 = 0. \tag{3.95}$$

---

[32] Actually it vanishes like $\mathcal{O}(p^3)$ due to the special galileon symmetry, but we will not utilize this property.

Next, we impose the galileon symmetry soft theorem. First, we impose the $\mathcal{O}(p)$ spatial soft theorem, which is an Adler zero condition:

$$
\begin{aligned}
\lim_{\vec{p}_1 \to 0} \vec{\partial}_{p_1} \psi_4^{(\text{gal})} &= \vec{p}_2 \frac{E}{4} (p_2 - p_4) \Big[(1 + 16a_7)(p_2 + p_4) - (1 + 8a_8)p_3 \Big] \\
&\quad + \vec{p}_3 \frac{E}{4} (p_3 - p_4) \Big[(1 + 16a_7)(p_3 + p_4) - (1 + 8a_8)p_2 \Big].
\end{aligned}
\tag{3.96}
$$

In order for this quantity to vanish we must set

$$
a_7 = -\frac{1}{16}, \qquad a_8 = -\frac{1}{8}.
\tag{3.97}
$$

Now we proceed to impose the $\mathcal{O}(p)$ temporal soft theorem, which reads:[33]

$$
\begin{aligned}
\lim_{\vec{p}_1 \to 0} \partial_{p_1} \psi_4^{(\text{gal})} &= \frac{1}{4} p_2^4 - \frac{1}{2} p_2^2 p_3^2 + \Big(a_1 + \frac{1}{2} \Big) p_2^2 p_3 p_4 + \text{perms.} \\
&= -\frac{3N}{2} E \Big(p_2 - p_3 - p_4 \Big) \Big(p_4 - p_2 - p_3 \Big) \Big(p_3 - p_2 - p_4 \Big).
\end{aligned}
\tag{3.98}
$$

Note that there is no choice of a_1 which will cause the top line to vanish. This implies that there is no theory that matches the galileon scattering amplitudes when $E = 0$, has an $\mathcal{O}(p^2)$ spatial Adler zero, and also exhibits an $\mathcal{O}(p^2)$ temporal Adler zero. The best one can do is fix a_1 so that the wavefunction coefficient satisfies the soft theorem in (3.87). The equation above is uniquely solved by

$$
a_1 = \frac{1}{2}, \qquad N = \frac{1}{6}.
\tag{3.99}
$$

Thus, we have successfully bootstrapped the wavefunction. The final result is given by

---

[33]Recall that N corresponds to the normalization of the interaction term in the action. However, we have already fixed this normalization by our choice of the overall constant in the scattering amplitude in (3.92). Thus in this way of framing our input data, N is an additional parameter we must solve for.

$$\psi_4^{(\text{gal})} = \frac{1}{2f^6 E}\Big(P_1 \cdot P_2 P_1 \cdot P_3 P_1 \cdot P_4 + \text{perms.}\Big)$$

$$+ \frac{1}{f^6}\left(\frac{1}{2}\, p_1^2 p_2 p_3 p_4 + \frac{1}{16}\, p_1^4 p_2 + \frac{1}{16}\, p_1^5 + \frac{1}{4}\, p_1^2 p_2^2 p_3 - \frac{1}{16}\, p_1 s_{12}^4 - \frac{1}{8}\, p_1^2 p_2 s_{12}^2\right) + \text{perms.}$$

$$(3.100)$$

Contrary to the NLSM and DBI cases, there does not exist a way of writing the wavefunction which takes the form of a manifestly Lorentz invariant scattering amplitude divided by the total energy. This is related to the fact that after accounting for the boundary term, the interaction term in the action is not manifestly Lorentz invariant (see Appendix B.3). This form of the wavefunction is somewhat cumbersome to work with when it comes to performing recursion. A nicer form of the four-point function, which does not have manifest Bose symmetry, is given by

$$\psi_4^{(\text{gal})} = \frac{2}{f^6 E} P_1 \cdot P_2\, P_2 \cdot P_3\, P_3 \cdot P_1 + \frac{1}{f^6}\Big(p_1\big(p_2^2 p_3^2 - (\vec{p}_2 \cdot \vec{p}_3)^2\big) + p_2\big(p_1^2 p_3^2 - (\vec{p}_1 \cdot \vec{p}_3)^2\big) + p_3\big(p_1^2 p_2^2 - (\vec{p}_1 \cdot \vec{p}_2)^2\big)\Big)\,.$$

$$(3.101)$$

This form of the wavefunction coefficient may be straightforwardly derived by using three-momentum conservation to eliminate all instances of $\vec{p}_4$ and p_4 in (3.100), except in the total energy singularity.

3.4 Recursion relations

So far, we have explored how the wavefunction in exceptional scalar theories can be fixed in terms of the corresponding scattering amplitudes, supplemented by some information about soft limits. However, the brute-force approach we have followed quickly becomes cumbersome, motivating us to search for a more efficient algorithm. In the S-matrix context, similar problems have been overcome via the construction of powerful recursion relations— the most famous of these being the BCFW relations [8]—which have been applied to scalar theories [157, 155, 158, 160]. It is therefore natural to look for a similar recursive construction of the wavefunction.

Recursion relations for the wavefunction both in flat space and in de Sitter space have been studied already by [26, 34, 31]. These relations are obtained by deforming the energy variables that the wavefunction depends on into the complex plane and writing the true wavefunction as a sum over residues of the poles of the complex function. This is natural because, as we reviewed in Section 3.2.1, all the singularities of the wavefunction occur at loci in energy space. However, the wavefunction is fundamentally a function of *momenta*, rather than only energies. We are therefore motivated to look for a construction where the 3-momenta themselves are deformed. Another motivation for this approach is that we will need additional input from soft theorems in our recursion relations. This is information about the behavior of the wavefunction and its derivatives in the limit $\vec{p} \to 0$, and shifts that only deform the energies do not allow us to access this regime.

3.4.1 Recursion generalities

Here we describe the philosophy underlying the recursion relations that we are going to construct. As orientation—and to contrast with the wavefunction case—it is useful to quickly review the construction for scattering amplitudes. At the most basic level, the idea is to deform the amplitude into the complex plane by shifting some of its kinematic variables.

This deformed amplitude, $A(z)$, now is a complex function of the parameter of this shift, z, while the undeformed (true) amplitude is $A(0)$. We can then use Cauchy's formula to write

$$A(0) = \frac{1}{2\pi i} \oint_{z=0} dz \frac{A(z)}{z}, \tag{3.102}$$

where the subscript on the integral indicates that we are integrating around a small contour encircling the point $z = 0$. Then, we can deform the contour out to infinity. If all the singularities of A are poles (which they are for amplitudes at tree level), we can write $A(0)$ as a sum of residues, plus a possible contribution from a pole at infinity:

$$A(0) = -\sum_j \text{Res}_{z=z_j} \left(\frac{A(z)}{z} \right) - B_\infty. \tag{3.103}$$

In many cases of interest, the pole at infinity vanishes and we can therefore reconstruct the amplitude from its residues, which are given by products of lower point amplitudes.

The situation for the wavefunction is conceptually similar. We imagine analytically continuing the wavefunction coefficients into the complex plane by deforming the 3-momenta as

$$\vec{p}_a(z) = \vec{p}_a + z\vec{q}_a, \qquad \vec{p}_a \cdot \vec{q}_a = p_a q_a, \qquad \sum_a \vec{q}_a(z) = 0, \tag{3.104}$$

where we have ensured that the $\vec{q}_a$ sum up to zero, so that momentum conservation continues to hold for the deformed variables. In addition, as a result of the middle constraint the shifted energies satisfy

$$\sqrt{\vec{p}_a{}^2(z)} = p_a(z) = p_a + z q_a. \tag{3.105}$$

That is, the energies get deformed by the lengths of $\vec{q}_a$.

There is a very important difference between scattering amplitudes and the wavefunction: while analytically continued scattering amplitudes have *only* pole-like singularities, the wavefunction has branch cuts in the complex z-plane due to the presence of $s_{a_1\cdots a_m}$ factors,

(where there are m momenta adding up to the exchange momentum).[34] We must therefore learn to deal with the branch cuts. This is equivalent to understanding the analytic structure of the deformed partial energies $E_{1\cdots m}(z)$. Under the shifts in (3.104), in the complex z plane $E_{1\cdots m}(z)$ has a cut which extends between the branch points

$$b_{\pm} = \frac{-\vec{p}_{1\cdots m} \cdot \vec{q}_{1\cdots m} \pm \sqrt{(\vec{p}_{1\cdots m} \cdot \vec{q}_{1\cdots m})^2 - \vec{p}_{1\cdots m}^2 \vec{q}_{1\cdots m}^2}}{\vec{q}_{1\cdots m}^2} \, . \tag{3.106}$$

In addition, because the partial energies generically appear in the denominator, one might also be concerned about the partial energy poles $E_{1\cdots m}(z) = 0$. These lie at

$$z_{\pm} = \frac{-P_{1\cdots m} \cdot Q_{1\cdots m} \pm \sqrt{(P_{1\cdots m} \cdot Q_{1\cdots m})^2 - P_{1\cdots m}^2 Q_{1\cdots m}^2}}{Q_{1\cdots m}^2} \, , \tag{3.107}$$

where $P_{1\cdots m} = (p_{1\cdots m}, \vec{p}_{1\cdots m})$ and $Q_{1\cdots m} = (q_{1\cdots m}, \vec{q}_{1\cdots m})$. However the partial energies are multi-valued functions, and it turns out that we can always define $\psi(z)$ by choosing a branch which does not have zeros (see Appendix E for more details). Because of this, we only have to worry about partial energy branch cuts.

In addition to the partial energy branch cuts, a generic wavefunction will also have a singularity when $E(z) = 0$, whose residue is the corresponding scattering amplitude:

$$\psi(z) \xrightarrow[z \to z_E]{} \frac{A(z_E)}{E(z_E)}, \quad \text{where} \quad \sum_i p_i(z_E) = 0 \, . \tag{3.108}$$

Now that we have catalogued all the relevant structures in the complex plane, we may arrange them into a recursion formula. The procedure is similar to the scattering amplitude

---

[34]An exception is if we choose a special deformation such that $\vec{q}_1 + \cdots + \vec{q}_m = 0$. In this case, the z dependence under the square root falls out, and the branch cut issue is avoided, but only for the exchange channel where the internal momentum is $\vec{p}_1 + \vec{p}_2 + ... + \vec{p}_m$. It is typically impossible to choose such set of constraints that simultaneously simplifies all of the possible channels in this way.

case. We write the true wavefunction as

$$\psi(0) = \frac{1}{2\pi i} \oint_{z=0} \frac{\mathrm{d}z}{z} \psi(z) \,, \tag{3.109}$$

and then deform the contour out to infinity, which picks up contributions from the total energy singularity, from the integration contour running along the partial energy branch cuts, and a possible contribution from infinity. All together, this means that we can write the wavefunction as

$$\psi(0) = -\sum_I \frac{1}{2\pi i} \oint_{\text{cut } I} \frac{\mathrm{d}z}{z} \psi(z) - \frac{1}{2\pi i} \oint_{E(z)=0} \mathrm{d}z \frac{A(z)}{E(z)} - B_\infty \,, \tag{3.110}$$

where "cut I" indicates that the contour encircles the branch cut associated to the partial energy singularities of factorization channel I. We have also included a possible boundary contribution, B_∞.[35] Note that the integral around $E(z) = 0$ just extracts the residue of the integrand at z_E—the point where $E(z_E) = 0$—which is $-\frac{A(z_E)}{z_E}$. However, we will later see that it is often possible to do further contour deformations to simplify the evaluation of this expression.

Equation (3.110) is true, but only useful if we know how to deal with the integrations along the branch cuts of $\psi(z)$, which appear because of the square roots in the partial energies. At tree level, we can gain some insight into this problem by examining where the branch cuts come from in the first place. In terms of bulk perturbation theory, partial energy singularities and their associated branch cuts arise from exchanges of particles. We can understand the analytic structure of these exchanges by inspecting the bulk-to-bulk propagator (3.9) in frequency space:

$$\mathcal{G}(s_I; t_1, t_2) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi i} \frac{\left(e^{i\omega t_1} - e^{-i\omega t_1}\right)\left(e^{i\omega t_2} - e^{-i\omega t_2}\right)}{\omega^2 - s_I^2 + i\epsilon} \,, \tag{3.111}$$

---

[35]The integrand must vanish strictly faster than $1/z$ for large z in order for this contribution to vanish.

where s_I is the magnitude of the exchanged momentum $\vec{s}_I$. Consider a wavefunction with an arbitrary number of exchanges, and for now focus on just a single one:

The grey blobs are any completion of the graph and may contain an arbitrary number of exchanges. For exchange I, the energy flowing into the left vertex from the boundary is K_I and the energy flowing into the right vertex is $K_{\bar{I}}$.[36] By I we therefore mean the set of external lines corresponding to the "left" subgraph and by $\bar{I}$ we mean the set of external lines associated to the "right" subgraph. In addition, $\mathcal{V}_I$ and $\mathcal{V}_{\bar{I}}$ abstractly parametrize the vertex factors appearing on the left (resp. right) side of the exchange. We can translate this into an expression as

$$
\psi_{\text{exc}} = \frac{1}{2} \int \frac{d\omega}{2\pi i} \int dt_1 dt_2 \, i\mathcal{V}_I(\vec{k}_I, \omega) e^{iK_I t_1} \frac{\left(e^{i\omega t_1} - e^{-i\omega t_1} \right) \left(e^{i\omega t_2} - e^{-i\omega t_2} \right)}{\omega^2 - s_I^2 + i\epsilon} e^{iK_{\bar{I}} t_2} \, i\mathcal{V}_{\bar{I}}(\vec{k}_{\bar{I}}, \omega) \,,
$$
(3.112)

At the level of this expression, it is somewhat obscure from where the branch cuts in the final wavefunction originate. However, this may be made manifest by performing the time integrals:

$$
\psi_{\text{exc}} = \frac{1}{2} \int \frac{d\omega}{2\pi i} i\mathcal{V}_I(\vec{k}_I, \omega) \frac{2\omega}{(K_I - i\epsilon)^2 - \omega^2} \frac{1}{\omega^2 - s_I^2 + i\epsilon} \frac{2\omega}{(K_{\bar{I}} - i\epsilon)^2 - \omega^2} \, i\mathcal{V}_{\bar{I}}(\vec{k}_{\bar{I}}, \omega) \,. \quad (3.113)
$$

Notice that at the level of this integrand, only s_I^2 appears, so $\psi(z)$ does not have a branch cut inside the integral. Instead, the cut arises when we actually perform the frequency integral. In particular, the ω-integral may be computed by residues, and upon doing so, it is only the residue at $\omega = s_I$ which has a branch cut. Therefore, it is only this residue which contributes

---

[36]For example, for a four-point function we could refer to the s channel as the $I = (12)$ channel, with $\bar{I} = (34)$. The momentum flowing through the internal line is $\vec{s}_I = \vec{p}_1 + \vec{p}_2$.

to the integral along the branch cut in (3.110). Looking back at the time integral (3.112), we see that evaluating the integrand on the $\omega = s$ pole physically corresponds to putting the internal propagator on-shell, and serves to cut the internal line as in (3.15). In particular, the time integrand is exactly the cut propagator in (3.13), and carrying out the time integrals computes a product of shifted wavefunctions, so that we have[37]

$$\frac{1}{2\pi i} \oint_{\text{cut } I} \frac{\mathrm{d}z}{z} \psi(z) = -\frac{1}{2\pi i} \oint_{\text{cut } I} \frac{\mathrm{d}z}{z} s_I(z)\, \tilde{\psi}_I(z) \times \tilde{\psi}_{\bar{I}}(z)\,. \tag{3.114}$$

We have written the integral along the branch cut of the full wavefunction as an integral along the branch cut of a product of lower point shifted wavefunction coefficients. In this way, the wavefunction "factorizes" into lower point objects around its branch cuts, much in the way that a scattering amplitude factors into lower point amplitudes on its poles. Putting everything together, we can write the following recursion relation for the wavefunction:

$$\psi(0) = \sum_I \frac{1}{2\pi i} \oint_{\text{cut } I} \frac{\mathrm{d}z}{z} s_I(z)\, \tilde{\psi}_I(z) \times \tilde{\psi}_{\bar{I}}(z) - \frac{1}{2\pi i} \oint_{E(z)=0} \frac{\mathrm{d}z}{z} \frac{A(z)}{E(z)} - B_\infty\,. \tag{3.115}$$

Given this general formula, we will now apply it to some specific examples. Typically we will want to choose our shifts so that the boundary term B_∞ is absent, allowing us to recursively construct higher point functions from simpler building blocks.

3.4.2 Simple examples

In order to demonstrate the use of (3.115), we first consider two simple examples— the wavefunction coefficients in a theory with ϕ^3 and ϕ^4 interactions, and the four-point wavefunction coefficient in scalar QED.

---

[37]Notice the factor of 1/2 difference between the integrand on the RHS and the cut diagram in e.g., (3.15). This stems from the prefactor in (3.111), indicating that on the $\omega = s$ pole, the bulk-to-bulk propagator is half the cut propagator.

Scalar theory with ϕ^3 and ϕ^4 interactions: Possibly the simplest example of a non-trivial wavefunction coefficient is the 4-point wavefunction in a scalar theory with both contact and exchange contributions. Concretely we consider a theory with interactions $\mathcal{L}_{\text{int}} \sim \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4$. We shift the momenta as

$$\hat{\vec{p}}_a(z) = \vec{p}_a + z\vec{q}_a, \quad \text{with} \quad \vec{p}_a \cdot \vec{q}_a = p_a q_a, \quad \text{and} \quad \sum_a^4 \vec{q}_a = 0. \tag{3.116}$$

Under these shifts, the energies are deformed as

$$\hat{p}_a(z) = p_a + zq_a. \tag{3.117}$$

Note that we are not requiring that the total energy is conserved by the shifts (in contrast to what is normally done for scattering amplitudes).[38] This choice of shifts causes $z^{-1}\psi_4(z)$ to scale as z^{-2} when $z \to \infty$, so that the boundary term is absent.

In order to use the formula (3.115), we will require as inputs the three-point wavefunction coefficient:

$$\psi_3 = \frac{\lambda_3}{p_1 + p_2 + p_3}, \tag{3.119}$$

along with the four-point scattering amplitude

$$A_4 = -\lambda_3^2 \left(\frac{1}{S} + \frac{1}{T} + \frac{1}{U} \right) + \lambda_4, \tag{3.120}$$

where $S = p_{12}^2 - s^2 = p_{34}^2 - s^2$ on the total energy singularity (and similarly for T, U). For

---

[38]We can write these shifts in four-vector notation more simply as

$$P_a(z) = P_a + zQ_a, \qquad P_a \cdot Q_a = 0, \qquad Q_i^2 = 0, \qquad \sum_a^4 \vec{q}_a = 0, \tag{3.118}$$

where $P_a^\mu \equiv (p_a, \vec{p}_a)$ and $Q_a^\mu \equiv (q_a, \vec{q}_a)$.

the s-channel, the shifted 3-point wavefunction coefficients are given by

$$\tilde{\psi}_{(12)} = \frac{\lambda_3}{p_{12}^2 - s^2} \qquad \tilde{\psi}_{(34)} = \frac{\lambda_3}{p_{34}^2 - s^2} \qquad (3.121)$$

and similarly for the other channels. With this information, we can write (3.115) as

$$\psi_4 = \frac{1}{2\pi i} \oint_{s \text{ cut}} \frac{\mathrm{d}z}{z} \frac{\lambda_3^2 \hat{s}}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} + \frac{1}{2\pi i} \oint_{t \text{ cut}} \frac{\mathrm{d}z}{z} \frac{\lambda_3^2 \hat{t}}{(\hat{p}_{14}^2 - \hat{t}^2)(\hat{p}_{23}^2 - \hat{t}^2)} \\ + \frac{1}{2\pi i} \oint_{u \text{ cut}} \frac{\mathrm{d}z}{z} \frac{\lambda_3^2 \hat{u}}{(\hat{p}_{13}^2 - \hat{u}^2)(\hat{p}_{24}^2 - \hat{u}^2)} - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \frac{A_4(z)}{\hat{E}} , \qquad (3.122)$$

where the hatted variables indicate that the energies are deformed by the complex param-
eter, z. One lingering point of concern might be that the answer depends on the choice of
representation of the scattering amplitude. In particular, there are many different represen-
tations which are all related by 4-momentum conservation, but which are inequivalent when
energy is not conserved. Thus a natural question is which one should be used? Fortunately
according to (3.122), one only needs information about $A_4(z)$ on the $E(z) = 0$ singularity,
that is, when energy is conserved. On this singularity, all amplitudes related by 4-momentum
conservation are equivalent, so it does not matter which representation we take.

Our goal now is to evaluate the integrals in (3.122). This is straightforward to do. For
now, focus on the s-channel integral. The most efficient strategy is to deform the contour
so that it surrounds poles corresponding to different kinematic relations, and then to use
those relations to simplify the integrand. This may be done until the only pole remaining is
the pole at $z = 0$, which is easy to evaluate. For instance, we may first deform the contour
away from the s cut and onto the poles at $z = 0, \hat{p}_{12} = \hat{s}$, and $\hat{p}_{34} = \hat{s}$. Note that the last
two are *folded singularities*. On each locus, we apply the corresponding kinematic identity

to simplify the integrand, which yields

$$
\begin{aligned}
\psi_{s\,\text{cut}} &= \frac{1}{2\pi i} \oint_{s\,\text{cut}} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{s}}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} , \\
&= -\frac{\lambda_3^2 \, s}{(p_{12}^2 - s^2)(p_{34}^2 - s^2)} - \frac{1}{2\pi i} \oint_{\hat{p}_{12}=\hat{s}} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{p}_{12}}{\hat{E}(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34} - \hat{p}_{12})} \\
&\quad - \frac{1}{2\pi i} \oint_{\hat{p}_{34}=\hat{s}} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{k}_{34}}{\hat{E}(\hat{p}_{12} - \hat{p}_{34})(\hat{p}_{34}^2 - \hat{s}^2)} .
\end{aligned}
\tag{3.123}
$$

Now we simply iterate the process by deforming the contour again to obtain:

$$
\begin{aligned}
\psi_{s\,\text{cut}} &= -\frac{\lambda_3^2 \, s}{(p_{12}^2 - s^2)(p_{34}^2 - s^2)} + \frac{\lambda_3^2 \, p_{12}}{(p_{12}^2 - s^2)E(p_{34} - p_{12})} + \frac{\lambda_3^2 \, p_{34}}{E(p_{12} - p_{34})(p_{34}^2 - s^2)} \\
&\quad + \frac{1}{2\pi i} \oint_{\hat{p}_{12}=\hat{p}_{34}} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{p}_{12}}{\hat{E}(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34} - \hat{p}_{12})} + \frac{1}{2\pi i} \oint_{\hat{p}_{34}=\hat{p}_{12}} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{p}_{34}}{\hat{E}(\hat{p}_{12} - \hat{p}_{34})(\hat{p}_{34}^2 - \hat{s}^2)} \\
&\quad + \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{p}_{12}}{\hat{E}(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34} - \hat{p}_{12})} + \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{dz}{z} \frac{\lambda_3^2 \, \hat{p}_{34}}{\hat{E}(\hat{p}_{12} - \hat{p}_{34})(\hat{p}_{34}^2 - \hat{s}^2)} .
\end{aligned}
\tag{3.124}
$$

In the second line, the $\hat{p}_{12} = \hat{p}_{34}$ residues cancel one another. The total energy residues may also be simplified, leaving us with

$$
\begin{aligned}
\psi_{s\,\text{cut}} &= -\frac{\lambda_3^2 \, s}{(p_{12}^2 - s^2)(p_{34}^2 - s^2)} + \frac{\lambda_3^2 \, p_{12}}{(p_{12}^2 - s^2)E(p_{34} - p_{12})} + \frac{\lambda_3^2 \, p_{34}}{E(p_{12} - p_{34})(p_{34}^2 - s^2)} \\
&\quad - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{dz}{z} \frac{\lambda_3}{\hat{p}_{12}^2 - \hat{s}^2} .
\end{aligned}
\tag{3.125}
$$

The second and third terms are (minus) the folded singularities of the first term, and are analytic in the s branch cut region.[39] Simplifying this expression yields

$$
\psi_{s\,\text{cut}} = \frac{\lambda_3^2}{E_{12} E_{34} E} - \frac{1}{2\pi i} \oint_{E=0} \frac{dz}{z} \frac{\lambda_3}{\hat{p}_{12}^2 - \hat{s}^2},
\tag{3.126}
$$

with a similar equation holding for the integrals along the t and u cuts.[40]

---

[39] To be clear, this means that there are no odd powers of s, which contains a branch cut, or poles at $s = 0$, which are the branch points of the cut.

[40] An alternative procedure that is faster (but less systematic) is to simplify the integrands (3.122) before

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Combining together all of the channels, one has

$$
\begin{aligned}
\psi_4 = {} & \frac{\lambda_3^2}{EE_{12}E_{34}} + \frac{\lambda_3^2}{EE_{14}E_{23}} + \frac{\lambda_3^2}{EE_{13}E_{24}} \\
& - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \left(\frac{\lambda_3^2}{\hat{p}_{12}^2 - \hat{s}^2} + \frac{\lambda_3^2}{\hat{p}_{14}^2 - \hat{t}^2} + \frac{\lambda_3^2}{\hat{p}_{13}^2 - \hat{u}^2} \right) - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \frac{A_4(z)}{\hat{E}} .
\end{aligned}
\tag{3.129}
$$

From (3.120), it is clear that all of the exchange pieces of the scattering amplitude cancel with the other term in the second line. This leaves only the contact contribution, whose contour integral is straightforward to compute. Ultimately we end up with

$$
\psi_4 = \frac{\lambda_3^2}{EE_{12}E_{34}} + \frac{\lambda_3^2}{EE_{14}E_{23}} + \frac{\lambda_3^2}{EE_{13}E_{24}} + \frac{\lambda_4}{E} .
\tag{3.130}
$$

This formula has all of the correct singularities, and it agrees with the result of a direct bulk calculation. It is worth noting that the analogous scattering amplitude is not recursively constructible, essentially because the presence or absence of the contact λ_4 contribution cannot be determined from three-point information. In the way that we have proceeded, we have used the residue of the total energy singularity—the scattering amplitude—as an input, which fixes the coefficient of this contact term, making the wavefunction recursively constructible.

We may apply precisely the same procedure to construct higher point wavefunction coefficients. For instance, to recurse the 5-point wavefunction one simply needs to compute shifted four-point and 3-point coefficients and stitch them together according to the general

evaluating the branch cut integrals. For instance, we may use the identity

$$
\frac{\hat{s}}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} = -\frac{1}{\hat{E}_{12}\hat{E}_{34}\hat{E}} + \frac{1}{\hat{E}} \frac{\hat{p}_{12}\hat{p}_{34} + \hat{s}^2}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} ,
\tag{3.127}
$$

and similarly for the t, u channels. Since the second term in (3.127) depends only on $\hat{s}^2$, it has no branch cuts in the z plane, so that inside the integral along the s branch cut we can replace

$$
\oint_{s\ \mathrm{cut}} \frac{\mathrm{d}z}{z} \frac{\lambda_3^2\,\hat{s}}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} = -\oint_{s\ \mathrm{cut}} \frac{\mathrm{d}z}{z} \frac{\lambda_3^2}{\hat{E}\hat{E}_{12}\hat{E}_{34}} = \frac{\lambda_3^2}{E_{12}E_{34}E} - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \frac{\lambda_3}{\hat{p}_{12}^2 - \hat{s}^2} ,
\tag{3.128}
$$

which is precisely the same as in (3.126).

recursion formula (3.115). The conceptual role of the integral along the branch cut continues to hold: an exchange channel I is equal to the product of lower point shifted wavefunction coefficients, plus correction pieces which are analytic in the s_I-cut region and serve to subtract off folded singularities.

Scalars with spin-1 exchange: As a more complicated example, we consider the four-point wavefunction that arises from scalars exchanging a spin-1 field. This is the wavefunction coefficient in a theory of scalar QED. This example is interesting not only because it involves the exchange of a spinning particle, but also because the three-point coupling involved is conformally invariant so the wavefunction coefficient for conformally coupled scalars is the same in flat space and in de Sitter.

As before, one of the building blocks that we need is the four-point scattering amplitude[41]

$$A_4 = 1 + \frac{2U}{S}, \tag{3.131}$$

where, for simplicity, we will consider only the s-channel contribution to the wavefunction. This truncation is consistent because the corresponding wavefunction is gauge invariant. We emphasize that any form of the amplitude related to this by 4-momentum conservation would work equally well. We also require the three-point wavefunction coefficient

$$\psi_{J\varphi\varphi} = \frac{\vec{\xi}_3 \cdot (\vec{p}_1 - \vec{p}_2)}{p_1 + p_2 + p_3}. \tag{3.132}$$

From this, we can compute the product of shifted wavefunction coefficients

$$\tilde{\psi}_{\varphi\varphi J} \times \tilde{\psi}_{J\varphi\varphi} = \frac{s^2 \Pi_{1,1}^{(s)}}{(p_{12}^2 - s^2)(p_{34}^2 - s^2)}, \tag{3.133}$$

---

[41]Like in the pure scalar example, this amplitude makes a particular choice of scalar contact interaction, whose coupling we have normalized to 1.

where we have defined the sum over internal helicities

$$s^2 \Pi_{1,1}^{(s)} \equiv (\vec{p}_1 - \vec{p}_2)^i \pi_{ij}^{(s)} (\vec{p}_3 - \vec{p}_4)^j = u^2 - t^2 + \frac{p_{12}p_{34}(p_1 - p_2)(p_3 - p_4)}{s^2} , \qquad (3.134)$$

by contracting the external momenta with the projector

$$\pi_{ij}^{(s)} \equiv \delta_{ij} - \frac{s_i s_j}{s^2} = \sum_\lambda \xi_i^\lambda \xi_j^{-\lambda} . \qquad (3.135)$$

With these components, we can write (3.115) as

$$\psi_4 = \frac{1}{2\pi i} \oint_{s\,\mathrm{cut}} \frac{\mathrm{d}z}{z} \hat{s} \frac{\hat{s}^2 \hat{\Pi}_{1,1}^{(s)}}{(\hat{p}_{12}^2 - \hat{s}^2)(\hat{p}_{34}^2 - \hat{s}^2)} - \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \frac{1}{\hat{E}} \left(1 + \frac{2\hat{U}}{\hat{S}} \right) . \qquad (3.136)$$

As in the ϕ^3 example, we will now proceed to evaluate the integral along the branch cut in s. The manipulations are exactly the same as in the scalar case, leading to the expression

$$\psi_{s\,\mathrm{cut}} = -s \frac{s^2 \Pi_{1,1}^{(s)}}{(p_{12}^2 - s^2)(p_{34}^2 - s^2)} + \frac{p_{12}(u^2 - t^2) + p_{34}(p_1 - p_2)(p_3 - p_4)}{(p_{12}^2 - s^2)E(p_{34} - p_{12})}$$
$$+ \frac{k_{34}(u^2 - t^2) + p_{12}(p_1 - p_2)(p_3 - p_4)}{E(p_{12} - p_{34})(p_{34}^2 - s^2)} + \frac{1}{2\pi i} \oint_{\hat{E}=0} \frac{\mathrm{d}z}{z} \frac{1}{\hat{E}} \frac{(\hat{p}_1 - \hat{p}_2)(\hat{p}_3 - \hat{p}_4) + \hat{t}^2 - \hat{u}^2}{\hat{p}_{12}^2 - \hat{s}^2} , \qquad (3.137)$$

where $\psi_{s\,\mathrm{cut}}$ stands for the first term in (3.136). The structure of this formula is the same as in the scalar case. The first term is a product of shifted lower-point wavefunction coefficients, and the second and third terms subtract off the folded singularities, and are analytic in the s-cut region. The final term is a total energy correction which is exactly canceled by the scattering amplitude piece in (3.136). Performing some algebra to clean up the result, we are left with

$$\psi_4^{(1)} = \frac{s^2 \Pi_{1,1}^{(s)}}{E E_{12} E_{34}} - \frac{\Pi_{1,0}^{(s)}}{E} , \qquad (3.138)$$

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where we have defined the quantity

$$\Pi_{1,0}^{(s)} \equiv \frac{(p_1 - p_2)(p_3 - p_4)}{s^2} \, .$$

<div align="right">(3.139)</div>

This matches a direct calculation [30].

3.4.3 Recursion from the S-matrix

In this subsection, we will apply the recursion procedure to compute six-point functions for the exceptional scalar field theories. We are using the same information as in the brute-force bootstrap approach, but systematized using complex analysis. We will illustrate the recursion algorithm for $P(X)$/DBI theories and galileon theories. As it turns out, NLSM is an exceptional case. The data we used in addition to scattering information was the U(1) decoupling identity, which unlike soft theorems, cannot be formulated in terms of analytic structures in an obvious way. For this reason, it is easier to bootstrap the wavefunction as in Section 3.3.1.

$P(X)$ *and DBI*

First we demonstrate how to recurse the six-point wavefunction coefficient for a general $P(X)$ theory. For an arbitrary $P(X)$ the required input is the scattering amplitude, the product of four-point wavefunctions corresponding to cutting the internal propagator, and the $\mathcal{O}(p^0)$ soft theorem, which is an Adler zero condition. For DBI, one may substitute the enhanced soft theorem for the scattering amplitude. The latter procedure (though conceptually interesting) is more cumbersome and requires the full DBI soft theorem. We have included an example computation in Section 3.4.4.

To use scattering information, we employ the following shifts, which are suitable for

probing soft behavior:

$$\vec{p}_a(z) = (1 - c_a z)\vec{p}_a, \qquad p_a(z) = (1 - c_a z)p_a, \qquad \sum_{i=1}^{6} \vec{p}_a = 0. \qquad (3.140)$$

Note that the shifted momenta do not obey energy conservation, and therefore probe the total energy singularity.

Similar to [155], it is necessary to introduce a mollifying function to improve the large z behavior of the wavefunction coefficient. We will take this to be

$$F(z) = \prod_{a=1}^{n}(1 - c_a z), \qquad (3.141)$$

so that the quantity $(zF(z))^{-1}\psi_6(z)$ scales as z^{-2} when $z \to \infty$, causing the boundary term to vanish.[42] In general, the cost of introducing $F(z)$ is that one must account for its singularities in the derivation of (3.115). For a general $n > 4$ point function, the modified recursion formula is

$$\psi_n(0) = \sum_I \frac{1}{2\pi i} \oint_{\text{cut } I} \frac{dz}{zF(z)} s_I \tilde{\psi}_I(z) \times \tilde{\psi}_{\bar{I}}(z) - \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{zF(z)} \frac{A_n(z)}{E(z)} - \frac{1}{2\pi i} \oint_{F(z)=0} \frac{dz}{zF(z)} \psi_n(z).$$
$$(3.142)$$

The contour integral in the last term is a sum over the residues of the $F(z) = 0$ poles. However, due to the $\mathcal{O}(p)$ Adler zero, the soft limit of the wavefunction vanishes like

$$\lim_{z \to 1/c_a} \psi_n(z) \sim (1 - c_a z). \qquad (3.143)$$

Thus the residue of the $F(z) = 0$ poles vanish, so that the last term in (3.142) is zero.

Now all we need to do is compute the building blocks for the recursion formula. A

---

[42]In the recursion of DBI scattering amplitudes, one actually needs $F(z) \sim \prod(1 - za_i)^2$. However, the presence of the total energy singularity in the wavefunction coefficient allows us to use one fewer power of z.

convenient form of the four-point wavefunction coefficient is (3.76).[43] From this, we can compute the product of shifted four-point wavefunction coefficients. For instance, for the (123) factorization channel, we get

$$\tilde{\psi}_{(123)} \times \tilde{\psi}_{(456)} = \frac{4}{f^8} \frac{\tilde{N}_{123}}{p_{123}^2 - s_{123}^2} \frac{\tilde{N}_{456}}{p_{456}^2 - s_{123}^2},$$

$$\text{where } \tilde{N}_{123} = P_1 \cdot P_2 \, P_2 \cdot P_3 + P_2 \cdot P_3 \, P_3 \cdot P_1 + P_3 \cdot P_1 \, P_1 \cdot P_2.$$

(3.144)

Quantities in the other channels are defined analogously. We will also need an expression for the scattering amplitude

$$A_6 = \sum_I \frac{4}{f^8} \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{P_I^2} + \frac{3}{f^8} P_1 \cdot P_2 P_3 \cdot P_4 P_5 \cdot P_6 + \text{perms.},$$

(3.145)

where we have defined $P_I^2 \equiv -p_I^2 + s_I^2$, which is the square of the sum of four-momenta associated to the exchanged particle. As an example, for the $I = (123)$ channel $P_{123}^2 = (P_1 + P_2 + P_3)^2$. Putting these pieces together in the recursion formula gives

$$\psi_6^{(\text{dbi})} = \frac{1}{2\pi i} \sum_I \oint_{\text{cut } I} \frac{dz}{zF(z)} \frac{4}{f^8} s_I(z) \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{(\hat{p}_I^2 - \hat{s}_I^2)(\hat{p}_{\bar{I}}^2 - \hat{s}_{\bar{I}}^2)}$$
$$- \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{zF(z)} \frac{1}{\hat{E}} \left(\sum_I \frac{4}{f^8} \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{\hat{P}_I^2} + \frac{3}{f^8} \hat{P}_1 \cdot \hat{P}_2 \hat{P}_3 \cdot \hat{P}_4 \hat{P}_5 \cdot \hat{P}_6 + \text{perms.} \right).$$

We can employ the analogue of the identity (3.127) to reduce this expression to

$$\psi_6^{(\text{dbi})} = \frac{1}{2\pi i} \sum_I \oint_{\text{cut } I} \frac{dz}{zF(z)} \frac{4}{f^8} \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{\hat{E}_I \hat{E} \hat{E}_{\bar{I}}}$$
$$- \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{zF(z)} \frac{1}{\hat{E}} \left(\sum_I \frac{4}{f^8} \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{\hat{P}_I^2} + \frac{3}{f^8} \hat{P}_1 \cdot \hat{P}_2 \hat{P}_3 \cdot \hat{P}_4 \hat{P}_5 \cdot \hat{P}_6 + \text{perms.} \right),$$

(3.146)

and again add a contour at infinity in the first integral and deform the contour off the branch cuts to sum over the residues of the simple poles in the complex plane. These are the poles

---

[43]This form is not manifestly Bose symmetric, but it is convenient to treat one of the lines differently because it is the one that will be shifted.

at $z = 0$, $F(z) = 0$, and $E(z) = 0$. The $E(z) = 0$ pole of the first integral exactly cancels the exchange part in the second integral. Moreover, because of our choice of representation for the shifted four-point wavefunction coefficients in the first integral, the factors of $(1 - c_a z)$ in the numerator and those in $F(z)$ cancel.[44]

Making these cancellations leaves us with

$$\psi_6^{(\text{dbi})} = \frac{1}{2\pi i} \sum_I \oint_{z=0} \frac{dz}{z} \frac{4}{f^8} \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{\hat{E}_I \hat{E} \hat{E}_{\bar{I}}} - \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{z} \frac{3}{f^8 \hat{E}} \hat{P}_1 \cdot \hat{P}_2 \hat{P}_3 \cdot \hat{P}_4 \hat{P}_5 \cdot \hat{P}_6 + \text{perms.}$$

$$(3.147)$$

Finally, we add a contour at infinity in the second integral, and deform the contour so that it only picks up the pole at $z = 0$. Evaluating the residue and combining it with the residue of the first integral, the final answer is

$$\psi_6^{(\text{dbi})} = \sum_I \frac{4}{f^8} \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{E_I E E_{\bar{I}}} + \frac{3}{f^8 E} P_1 \cdot P_2 P_3 \cdot P_4 P_5 \cdot P_6 + \text{perms.,} \qquad (3.148)$$

which matches a direct computation. The algorithm can be readily generalized to higher points.

(Special) galileon

Now we will demonstrate how to recursively construct the six-point wavefunction coefficient for the general galileon. The procedure is very similar to the $P(X)$ discussion in the prior subsection. For the general galileon, one may recurse the wavefunction coefficient using information about factorization along its unitarity cut, the scattering amplitude, and

---

[44]The same cancellation happens for the contact part of the amplitude in the second integral. Note that this is the only form of the amplitude such that this the case. If we had chosen a different form of the amplitude related to this one by energy conservation, the poles of $F(z)$ in the second integral would *not* cancel, and their residues must be computed, though the final answer would be the same. It is also interesting to point out that the natural form of the scattering amplitude which we have chosen is also the one such that the $\mathcal{O}(p)$ Adler zero is manifestly obeyed.

the $\mathcal{O}(p)$ soft theorem. For n-point wavefunctions with $n > 4$, this is an Adler zero condition. Similar to how DBI stands out as a privileged $P(X)$ theory, the special galileon is a distinguished point in the space of general galileon theories. In particular, one may replace scattering information with the enhanced, $\mathcal{O}(p^2)$ soft theorem.

In order to recurse a general galileon using scattering information, we will use the same shifts as in (3.140). To preclude the boundary term, one must introduce the mollifying function[45]

$$F(z) = \prod_{a=1}^{n}(1 - c_a z)^2 \, . \tag{3.149}$$

The cost of introducing $F(z)$ is that we have to account for its singularities in the derivation of (3.115). The modified recursion formula is the same as in (3.142), and again the final contour integral is a sum over the $F(z) = 0$ poles. However, for $n > 4$, the soft limit vanishes like

$$\lim_{z \to 1/c_a} \psi_n(z) \sim (1 - c_a z)^2 \, . \tag{3.150}$$

Thus the residue of the $F(z)$ poles vanishes, so the term again is zero.

Now all we need to do is compute the building blocks for the recursion formula. The form of the four-point wavefunction which makes the computation simple is (3.101). Happily, when it comes to computing the shifted wavefunction coefficient the long tail of terms in this expression which are analytic in the total energy does not contribute. For the (123) channel,

$$\tilde{\psi}_4^{(123)} \times \tilde{\psi}_4^{(123)} = \frac{4}{f^{12}} \frac{\tilde{N}_{123}\tilde{N}_{456}}{(p_{123}^2 - s_{123}^2)(p_{456}^2 - s_{456}^2)} \tag{3.151}$$

$$\tilde{N}_{123} = P_1 \cdot P_2 \; P_2 \cdot P_3 \; P_3 \cdot P_1 \, .$$

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[45]For special galileon scattering amplitudes, one may derive additional recursion relations by defining $F(z)$ with an additional power of $(1 - c_a z)$.

Quantities in the other channels are defined analogously. We will also need the amplitude[46]

$$A_6 = \sum_I \frac{4}{f^{12}} \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{P_I^2}, \tag{3.152}$$

Putting these pieces together in the recursion formula gives

$$\psi_6^{(\text{gal})} = \frac{1}{2\pi i} \sum_I \oint_{\text{cut } I} \frac{dz}{zF(z)} \frac{4}{f^{12}} \hat{s}_I \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{(\hat{p}_I^2 - \hat{s}_I^2)(\hat{p}_{\bar{I}}^2 - \hat{s}_{\bar{I}}^2)} - \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{zF(z)} \frac{1}{\hat{E}} \sum_I \frac{4}{f^{12}} \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{\hat{P}_I^2}. \tag{3.153}$$

We employ the same identity as in (3.127) to reduce this expression to

$$\psi_6^{(\text{gal})} = -\frac{1}{2\pi i} \sum_I \oint_{\text{cut } I} \frac{dz}{zF(z)} \frac{4}{f^{12}} \hat{s}_I \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{\hat{E}_I \hat{E} \hat{E}_{\bar{I}}} - \frac{1}{2\pi i} \oint_{E(z)=0} \frac{dz}{zF(z)} \frac{1}{\hat{E}} \sum_I \frac{4}{f^{12}} \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{\hat{P}_I^2}, \tag{3.154}$$

and again add a contour at infinity in the first integral and deform the contour so that we can evaluate it as a sum over the resides of its poles at $z = 0, F(z) = 0$, and $E(z) = 0$. The $E(z) = 0$ pole of the first integral exactly cancels the second integral. Because we have made a convenient choice for the form of our shifted four-point wavefunctions, all of the factors of $(1 - c_a z)$ contained in $F(z)$ cancel with factors in the numerator. Thus, all that remains is the $z = 0$ pole, with residue

$$\psi_6^{(\text{gal})} = \sum_I \frac{4}{f^{12}} \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{E_I E E_{\bar{I}}}. \tag{3.155}$$

This computation clearly demonstrates the advantage of the recursion procedure over doing a direct perturbative computation with the action (B.32). From the perspective of this action, it is completely non-obvious that the numerators will organize themselves into a simple Lorentz invariant form, with no analytic pieces left over.

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[46]It is actually possible to recursively construct this wavefunction coefficient without using scattering amplitude information. One may define shifts that are insensitive to the total energy singularity but still cause the deformed wavefunction to vanish sufficiently fast at infinity that there is no boundary term.

3.4.4 Recursion from soft theorems

In the previous sections, we implemented recursion relations for the wavefunction using the corresponding scattering amplitudes as input. The advantage of this approach is that the full information about the soft limits of the wavefunction is not needed. However, from a conceptual viewpoint, we might want to construct the wavefunction without inputting scattering information explicitly. This is indeed possible, but we have to use the full information about the soft limit. This is technically more involved, but may be useful in some situations, particularly in the cosmological context. In this section, we demonstrate how this works for the simple examples of the NLSM and DBI.

Nonlinear sigma model

We first consider the nonlinear sigma model. In order to input the full soft behavior of the wavefunction, we will use the following shifts:

$$\vec{p}_a = (1 - c_a z)\vec{p}_a, \qquad p_a(z) = (1 - c_a z)p_a, \qquad \sum_a c_a \vec{p}_a = \sum_a c_a p_a = 0 \,. \qquad (3.156)$$

The final condition ensures that our shifts are not sensitive to the total energy pole, so that we will not need to make use of scattering information. The cost of this modification is that $\psi_n(z)$ has more divergent large-z behavior than if we allowed the shifts to probe the total energy singularity. In fact, it has the same large-z behavior as the corresponding scattering amplitude. To improve this behavior we need to introduce the mollifying function

$$F(z) = \prod_{a=1}^{n} (1 - c_a z), \qquad (3.157)$$

which is the same as is required to recurse the amplitude. The corresponding modified recursion formula reads

$$\psi_n(0) = \sum_I \frac{1}{2\pi i} \oint_{\text{cut } I} \frac{\mathrm{d}z}{zF(z)} s_I(z)\tilde{\psi}_I(z) \times \tilde{\psi}_{\bar{I}}(z) - \frac{1}{2\pi i} \oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \psi_n(z) \,. \tag{3.158}$$

In order to evaluate this formula, we need the following building blocks. First, there is the four-point wavefunction coefficient. Its most convenient form is (3.41), so that the product of shifted wavefunction coefficients is

$$\tilde{\psi}_{(123)} \times \tilde{\psi}_{(456)} = \frac{1}{9f^4} \frac{\tilde{N}_{123}\tilde{N}_{456}}{(p_{123}^2 - s_{123}^2)(p_{456}^2 - s_{456}^2)} \,, \tag{3.159}$$

$$\tilde{N}_{123} = P_1 \cdot P_2 + P_2 \cdot P_3 - 2P_1 \cdot P_3 \,.$$

To compute the residues at $F(z) = 0$, we will need the six-point soft theorem, given by (3.56). Then we may write the residues of at $F(z) = 0$ in (3.158) as

$$\oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \psi_n(z) = \oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \lim_{\vec{p}_a \to 0} \psi_n(z) \,, \tag{3.160}$$

because $F(z) = 0$ probes the soft limit of the wavefunction. Overall, this reduces (3.158) to

$$\psi_6 = \frac{1}{2\pi i} \sum_I \frac{1}{9f^4} \oint_{\text{cut } I} \frac{\mathrm{d}z}{zF(z)} s_I(z) \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{(\hat{p}_I^2 - \hat{s}_I^2)(\hat{p}_{\bar{I}}^2 - \hat{s}_{\bar{I}}^2)} - \oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \lim_{\vec{p}_a \to 0} \psi_6(z) \,. \tag{3.161}$$

To simplify the contour integral over the cut region, we use the analogue of (3.127):

$$\psi_6 = -\frac{1}{2\pi i} \sum_I \frac{1}{9f^4} \oint_{\text{cut } I} \frac{\mathrm{d}z}{zF(z)} \frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{E\hat{E}_I\hat{E}_{\bar{I}}} - \oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \lim_{\vec{p}_a \to 0} \psi_6(z) \,. \tag{3.162}$$

To compute the first integral, we add a contour at infinity and deform the contour to pick up the residues of the simple poles of the integrand. These are at $z = 0$ and $F(z) = 0$. This

gives

$$\psi_6 = \frac{1}{9f^4}\sum_I \frac{\tilde{N}_I\tilde{N}_{\bar{I}}}{EE_IE_{\bar{I}}} + \frac{1}{9f^4}\sum_I\sum_a \mathrm{Res}_{z=1/c_a}\left[\frac{1}{zF(z)}\frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{E\hat{E}_I\hat{E}_{\bar{I}}}\right] - \sum_a \mathrm{Res}_{z=1/c_a}\left[\frac{1}{zF(z)}\lim_{\vec{p}_a\to 0}\psi_n(z)\right] .$$

$$(3.163)$$

The first sum captures the exchange pieces, in the sense that it is exactly what one finds from computing exchange diagrams from a bulk computation. Thus, the last two sums must combine to reconstruct the bulk contact piece. We will now verify that this is indeed correct. We can compute the second two terms in (3.163) at for example the $z = 1/c_1$ soft pole:

$$\sum_I \mathrm{Res}_{z=1/c_1}\left[\frac{1}{zF(z)}\frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{E\hat{E}_I\hat{E}_{\bar{I}}}\right] - \mathrm{Res}_{z=1/c_1}\left[\frac{1}{zF(z)}\lim_{\vec{p}_a\to 0}\psi_6(z)\right]$$

$$= \mathrm{Res}_{z=1/c_1}\left[\frac{1}{zF(z)}\right]\frac{1}{f^4}\left(\frac{1}{18}\frac{1}{E}\left(\tilde{N}_{456}(z) + \tilde{N}_{234}(z) - 2\tilde{N}_{345}(z)\right) - \frac{1}{45}\left(\hat{p}_{26} - 4\hat{p}_{35} + 6\hat{p}_4\right)\right)\Bigg|_{z=\frac{1}{c_1}} .$$

$$(3.164)$$

This expression can be simplified using the following identity

$$\left(\hat{C}_1 - 4\hat{C}_2 + 6\hat{C}_3\right)\Big|_{z=\frac{1}{c_1}} = -\left[5\left(\tilde{N}_{456}(z) + \tilde{N}_{234}(z) - 2\tilde{N}_{345}(z)\right) - 2\left(\hat{p}_{26} - 4\hat{p}_{35} + 6\hat{p}_4\right)\right]\Big|_{z=\frac{1}{c_1}} .$$

$$(3.165)$$

where we have used the cyclic building blocks defined in (3.46). A similar identity holds for the other residues. Now summing over all of the residues, we finally arrive at

$$\sum_I\sum_a \mathrm{Res}_{z=1/c_a}\left[\frac{1}{zF(z)}\frac{\tilde{N}_I(z)\tilde{N}_{\bar{I}}(z)}{E\hat{E}_I\hat{E}_{\bar{I}}}\right] - \sum_a \mathrm{Res}_{z=1/c_a}\left[\frac{1}{zF(z)}\lim_{\vec{p}_a\to 0}\psi_6(z)\right]$$

$$= -\sum_a \mathrm{Res}_{z=1/c_a}\left[\frac{1}{zF(z)}\frac{1}{90f^4}\frac{1}{E}\left(\hat{C}_1 - 4\hat{C}_2 + 6\hat{C}_3\right)\right] .$$

$$(3.166)$$

This set of residues can be written as a contour integral encircling the locations where $F(z) = 0$. If we add an arc at infinity, we can deform the contour to only pick up the $z = 0$ pole, whose residue is easy to evaluate. Performing this computation and substituting the result into (3.163), we obtain

$$\psi_6^{(\text{nlsm})} = \frac{1}{9f^4} \sum_I \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{E E_I E_{\bar{I}}} + \frac{1}{90f^4} \frac{1}{E} \left(C_1 - 4C_2 + 6C_3 \right), \qquad (3.167)$$

which matches the bulk perturbative computation, along with the bootstrap procedure in section 3.3.1 which utilized the scattering amplitude. As is clear from this example, the computation is a bit cumbersome without using scattering information. Apart from knowing the soft theorem, the limiting factor in this procedure's utility is knowing how to compute the sum of resides in (3.163). This approach in general requires nontrivial kinematic identities as in (3.165). Nevertheless, this construction is conceptually useful, as it shows that higher point functions can be reconstructed from soft information alone, without explicitly using scattering amplitudes.

DBI

As another example, we can construct DBI wavefunction coefficients using soft limits. The set up is largely the same as for the NLSM. We will use the same shifts as in (3.156). To improve the large z behavior, we divide by

$$F(z) = \prod_{a=1}^{n} (1 - c_a z)^2, \qquad (3.168)$$

which is the same as is required to recurse the scattering amplitude. Formally, the recursion relation is also identical to (3.158). Moreover, the shifted wavefunction coefficients are already computed in (3.144). To input soft information, consider expanding $\psi(z)$ around one of the $z = 1/c_a$ poles:

$$\psi_n(z \sim 1/c_a) = \left(\vec{p}_a(z) \cdot \vec{\partial}_{p_a} + p_a(z) \partial_{p_a} \right) \psi_n(\vec{p}_a(z), p_a(z)) \Big|_{\vec{p}_a(z)=0} + \mathcal{O}(p_a^2(z)), \qquad (3.169)$$

Then in the recursion formula, we can write the residues of $F(z) = 0$ as

$$\oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \, \psi_n(z) = \oint_{F(z)=0} \frac{\mathrm{d}z}{zF(z)} \left(\vec{p}_a(z) \cdot \vec{\partial}_{p_a} + p_a(z) \partial_{p_a} \right) \psi_6(\vec{p}_a(z), p_a(z)) \Big|_{\vec{p}_a(z)=0} , \quad (3.170)$$

where the right hand side is controlled by the spatial and temporal DBI soft theorems. Performing the same contour manipulations as in the NLSM case, we arrive at an analogous residue formula:

$$\psi_6^{(\mathrm{dbi})} = \frac{4}{f^8} \sum_I \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{E E_I E_{\bar{I}}} + \frac{4}{f^8} \sum_I \sum_a \mathrm{Res}_{z=1/c_a} \left[\frac{1}{zF(z)} \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{E \hat{E}_I \hat{E}_{\bar{I}}} \right]$$
$$- \sum_a \mathrm{Res}_{z=1/c_a} \left[\frac{1}{zF(z)} \left(\vec{p}_a(z) \cdot \vec{\partial}_{p_a} + p_a(z) \partial_{p_a} \right) \psi_6(\vec{p}_a(z), p_a(z)) \right]. $$
$$(3.171)$$

where we have defined the quantity

$$\tilde{N}_{123} = P_1 \cdot P_2 P_2 \cdot P_3 + P_2 \cdot P_3 P_3 \cdot P_1 + P_3 \cdot P_1 P_1 \cdot P_2 \quad (3.172)$$

As in the NLSM case, the first sum in (3.171) correctly produces the exchange contribution, and the challenge is to show that the last two terms conspire to give the correct contact contribution. Combining everything, we obtain[47]

$$\frac{4}{f^8} \sum_I \sum_a \mathrm{Res}_{z=\frac{1}{c_a}} \left[\frac{1}{zF(z)} \frac{\tilde{N}_I(z) \tilde{N}_{\bar{I}}(z)}{E \hat{E}_I \hat{E}_{\bar{I}}} \right] - \sum_a \mathrm{Res}_{z=\frac{1}{c_a}} \left[\frac{1}{zF(z)} \left(\vec{p}_a(z) \cdot \vec{\partial}_{p_a} + p_a(z) \partial_{p_a} \right) \psi_6(\vec{p}_a(z), p_a(z)) \right]$$
$$= -3 \sum_a \mathrm{Res}_{z=\frac{1}{c_a}} \left[\frac{1}{zF(z)} \hat{P}_1 \cdot \hat{P}_2 \hat{P}_3 \cdot \hat{P}_4 \hat{P}_5 \cdot \hat{P}_6 \right]. $$
$$(3.174)$$

---

[47]Note that this step is rather nontrivial and requires the use of the kinematic identity:

$$\left(2(\hat{P}_2^\mu + \hat{P}_3^\mu) \tilde{N}_{456}(z) + (\hat{P}_2^\mu + \hat{P}_3^\mu) \left(\hat{p}_4 \hat{P}_5 \cdot \hat{P}_6 + \hat{p}_5 \hat{P}_4 \cdot \hat{P}_6 + \hat{p}_6 \hat{P}_4 \cdot \hat{P}_5 \right) E \right.$$
$$\left. + \delta_0^\mu \left(\hat{\vec{p}}_2 \cdot \hat{\vec{p}}_3 \hat{\vec{p}}_4 \cdot \hat{\vec{p}}_5 - \hat{\vec{p}}_2 \cdot \hat{\vec{p}}_3 \hat{p}_4 \hat{p}_5 + 3 \hat{p}_2 \hat{p}_3 \hat{p}_4 \hat{p}_5 \right) E + \mathrm{perms.} \right) \Big|_{z=\frac{1}{c_1}} = -3 \hat{P}_2^\mu \hat{P}_3 \cdot \hat{P}_4 \hat{P}_5 \cdot \hat{P}_6 \Big|_{z=\frac{1}{c_1}} + \mathrm{perms.} ,$$
$$(3.173)$$

along with its analogues for the other $z = 1/c_a$ residues, and where perms. indicates we should sum over permutations of the $2, 3, 4, 5, 6$ lines.

As in the NLSM case, we may easily compute this final set of residues by writing it as a contour integral, and deforming the contour onto the $z = 0$ pole. Performing this procedure and combining with the exchange contribution, the final result for the six-point wavefunction coefficient is

$$\psi_6^{(\text{dbi})} = \sum_I \frac{4}{f^8} \frac{\tilde{N}_I \tilde{N}_{\bar{I}}}{E E_I E_{\bar{I}}} + \frac{3}{f^8 E} P_1 \cdot P_2 P_3 \cdot P_4 P_5 \cdot P_6 + \text{perms.} , \tag{3.175}$$

which matches the bulk perturbative computation, along with the recursion procedure in 3.4.3 which utilized the scattering amplitude.

3.5   Conclusions

We have studied the soft structure of the wavefunction of exceptional scalar field theories in flat space. We find that, while most of the structure survives, there are interesting and important differences with the scattering amplitude case. Foremost, wavefunction coefficients of exceptional scalar theories like the NLSM, DBI, and the special galileon satisfy nontrivial soft theorems in the limit where one of their external momenta is taken to zero, in contrast to their amplitude counterparts which display Adler zeroes.

The soft theorems obeyed by these shift-symmetric scalar theories can be used to reconstruct wavefunction coefficients in two ways. The most straightforward approach is to input information about the total energy singularity of the wavefunction, which has the corresponding scattering amplitude as a residue. In order to completely specify the wavefunction we additionally need to use part of the soft limit of the wavefunction: theories that have a shift symmetry that scales like $\mathcal{O}(x^n)$ in coordinates have a soft theorem that fixes the $\mathcal{O}(q^n)$ part of wavefunction, where $\vec{q}$ is the soft momentum. If we input scattering information, we only require the $\mathcal{O}(q^{n-1})$ soft theorem. Alternatively, we can use the highest-order soft theorem to reconstruct the wavefunction, which does not require the scattering amplitude as an

input. We have also derived recursion relations that systematize each of these constructions. These recursion relations directly deform the momentum variables of the wavefunction, and we have described how to handle the resulting subtleties involving the analytic structure.

Many rich and interesting structures have already been uncovered in the study of correlation functions in cosmological spacetimes, and there are many future discoveries to be made. In this journey, the flat space wavefunction, and in particular wavefunction coefficients of exceptional scalar theories, will serve as illuminating guideposts. We have already seen that they possess interesting structures akin to those in scattering amplitudes, and we expect that they will provide further structural insights into the nature of the wavefunction and of quantum field theory in cosmology.

Chapter 4: Conclusions

In this thesis, we have investigated how bootstrap principles may be imported to the boundary of (A)dS to great effect. We have invoked this perspective to study two disparate problems. First, we studied the non-unitary regime of the representation theory for the 3-dimensional $\mathcal{N} = 2$ superconformal algebra, and uncovered a wide variety of possibilities for the structure of non-unitary representations which have no unitary counterparts. Second, we have extensively investigated the soft structure of the flat space wavefunction for the exceptional scalar field theories. Moreover, we have classified how the wavefunction for these theories may be bootstrapped from various sets of input data. These investigations suggest a number of directions for future study:

- It would interesting to conduct a detailed extension of our work for SUSY PM fields in de Sitter space, and the $\mathcal{N} = 2$ AdS multiplets should have a counterpart there. More specifically, it would be enlightening to explicitly study the Lagrangian formulations for these multiplets in de Sitter, and specifically to understand what form the AdS extended modules take after this transmutation (though even in AdS the bulk Lagrangian description of the extended modules is unclear). A more complete understanding of supersymmetric Lagrangians for PM particles, both in AdS and dS, would also open the door to the study of interactions. In-line with the bootstrap philosophy, the exploration of interactions via non-Lagrangian methods is also especially intriguing.

- Although a detailed analysis of SUSY multiplets with $\mathcal{N} > 2$ would be messy, it would be worthwhile to have a better grasp on their generic properties, and we have taken only a first step here by showing some basic features of PM multiplets with higher

$\mathcal{N}$. Perhaps more intriguing would be to explore supersymmetric extensions of PM fields beyond $D = 4$ dimensions. It is known that some cubic vertices for PM spin-2 particles can be constructed when $D = 4$ [72, 99, 100, 73, 101, 106, 113, 116, 117, 70], which makes this value of the dimension somewhat special, and it would therefore be interesting to see if analogous constraints exist in the supersymmetric context.

- In Chapter 3, we have focused on the properties of flat space wavefunction coefficients. From a formal standpoint, this situation is already of interest, but of course our ultimate aim is to understand the properties of the wavefunction in cosmological spacetimes. The most obvious and straightforward extension is to study the analogue of our results in a fixed de Sitter background for the exceptional scalar theories constructed in [43, 42]. The fields appearing in these theories have particular masses in de Sitter space, so that the relevant time integrals are related to those in flat space in a simple fashion. Consequently, it should be possible to uplift parts of our analysis directly to de Sitter space along the lines of [171, 30, 35].

- In Section 3.4, we studied recursion relations for wavefunction coefficients that rely on complex deformations of spatial momenta. The analytic structure of the deformed wavefunction is slightly different in this case than when the energies are deformed [26, 34, 31]. It would be interesting to investigate these recursion relations in the de Sitter context (and to compare them with those in [162]). Previous constructions have mostly avoided dealing directly with the branch cuts that appear when complexifying the s_I variables, and the techniques developed here may be useful to recursively construct de Sitter wavefunctions, especially in cases where external particles have spin.

- An important reason to study the structure of the exceptional scalar field theories is that they have interesting inter-relations and relations to Yang–Mills and gravity. At the level of scattering amplitudes, aspects of these relations have been systematized using the double copy [147] and by operations that transform amplitudes of the var-

ious theories into each other [148]. It would be very interesting to understand both of these things from the perspective of the wavefunction. It is natural to suspect that the transmutation operations of [148] have some relation to the weight-shifting operators studied in [187, 29], and perhaps analogues can be found also for the flat space wavefunction.

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Appendix A: Finding norms for $s > 1$

Here we present some details on the computations of the norms of the conformal primaries within the superconformal multiplets $\{s\}_{\Delta,r}$. For $s = 0$ in Section 2.4.1, $s = 1/2$ in Section 2.4.2 and $s = 1$ in Section 2.4.3, we have computed the norm of each conformal primary explicitly. However, this brute force calculation of the norms becomes prohibitively involved at higher spin. Instead, we make reasonable assumptions to extrapolate the norms of the multiplets at arbitrary s, up to an overall constant of determined sign.

First, we exploit the fact that each of the conformal primaries in the superconformal multiplet will have the same eigenvalue of the superconformal quadratic Casimir operator. In $d = 3$, the quadratic conformal Casimir is given by

$$\mathcal{C}_2 = D^2 + J_i J_i - \frac{1}{2}\{P_i, K_i\} = D(D-3) + J_i J_i - P_i K_i. \tag{A.1}$$

For the $\mathcal{N} = 2$ superconformal algebra, the quadratic Casimir gets extended to [143]

$$\mathcal{C}_2^{super} = \mathcal{C}_2 + \frac{1}{4}[S^{aI}, Q_a{}^I] - \frac{1}{2}R^2 + \dots. \tag{A.2}$$

The ... denotes operators that vanish on a superconformal primary $|\Delta, r\rangle^{a_1 \dots a_{2s}}$, i.e., a spin-$s$ conformal primary with conformal weight Δ, which is also an eigenstate of the R-symmetry generator as in (2.33) and which is also annihilated by S^{aI}. The quadratic Casimir acting on a superconformal primary becomes

$$\mathcal{C}_2^{super} |\Delta, r\rangle^{a_1 \dots a_{2s}} = \left[\Delta(\Delta - 1) + s(s+1) - \tfrac{1}{2}r^2\right] |\Delta, r\rangle^{a_1 \dots a_{2s}}. \tag{A.3}$$

We expect shortenings to occur when a descendent state is itself a superconformal primary, i.e., when $S^{\alpha I}\,|\Delta',r'\rangle^{a_1...a_{2s'}} = 0$ for the descendent state, and thus this state decouples from the original multiplet. In order for this to be true, given the original superconformal primary $|\Delta,r\rangle^{a_1...a_{2s}}$, there must exist a descendent state $|\Delta',r'\rangle^{a_1...a_{2s'}}$ with eigenvalues such that

$$\Delta(\Delta - 1) + s(s+1) - \tfrac{1}{2}r^2 = \Delta'(\Delta' - 1) + s'(s'+1) - \tfrac{1}{2}r'^2\,. \tag{A.4}$$

We can solve this equation for each conformal primary in the superconformal multiplet to find all possible new shortening conditions at each level.

We note that the superconformal multiplets do not necessarily take advantage of each of these shortening conditions. For the $s = 0$ multiplet there is no $\Delta = 0$ shortening condition at level 2 and no new shortening at level 3; for the $s = \tfrac{1}{2}$ multiplet there is no $\Delta = 1 \pm r$ shortening condition at level 2 and there are no new shortening conditions at either level 3 or level 4; for the $s = 1$ multiplet there is also no $\Delta = 1 \pm r$ shortening condition at level 2 and no new shortening conditions level 3. We can predict when a conformal primary exhibits new shortening by looking at the structure of the primary itself. First, we remark that if a level-N conformal primary $|P\rangle_N$ is the Q-descendent of $|P\rangle_{N-1}$ which shortens at $\Delta_\star$, then $|P\rangle_N$ must also shorten at $\Delta_\star$, unless $\Delta_\star$ appears in the denominator of a term in $|P\rangle_N$, as in the case of $\Delta_\star = 0$ at level 4 in the generic superconformal multiplet. Moreover, there is maximal number of shortenings which a conformal primary may admit. For example, the norm of the $[s+1]_{\Delta+1,r}$ conformal primary

$$\bar{Q}^{(a}Q^{b}\,|\Delta,r\rangle^{a_1...a_{2s})} - \left(\frac{\Delta - r + s}{\Delta + s}\right)P^{(ab}\,|\Delta,r\rangle^{a_1...a_{2s})} \tag{A.5}$$

is a third degree polynomial in Δ, and thus admits 3 shortenings. It is the Q-descendent of either $[s+1]_{\Delta+\frac{1}{2},r\pm1}$, which uses two of the available shortenings. This leaves room for one new shortening, which is predicted by the Casimir. Generically however, the norm of a generic conformal primary may not have enough roots to host the shortenings of the previous

levels, as well as a new shortening condition. In such cases, the predicted shortenings of the Casimir are not realized. In this manner, we can predict the numerator of the norm of conformal primary, up to an overall factor. We can predict the denominator of the norm by multiplying together all of the denominators which appear in the coefficients of the P-descendants in the conformal primary. This leaves only an overall undetermined constant. The sign of this constant can be fixed by requiring that the norms be positive for large Δ.

We emphasize that we are merely conjecturing the structure of the norm for conformal primaries with $s \geq 1$. There may be for instance, accidental cancellations, so that not all of the denominators in a conformal primary make an appearance in the norm. For example, this happens at level 4 in the $\{\frac{1}{2}\}_{\Delta,r}$ multiplet. However, we have explicitly computed all of the norms for $s = 1$ and our conjecture correctly predicts all of them. For generic Δ, the $s = 1$ case should be generic, so we take this as good evidence that our conjectured norms are valid.

Appendix B: Exceptional scalar theories

In this Appendix, we give a brief overview of the different exceptional scalar field theories discussed in the main text. These are the SU(N) nonlinear sigma model, Dirac–Born–Infeld theory, and the special galileon. These theories are exceptional in the broader class of scalar EFTs, because they have nonlinearly realized symmetries that control the structure of the theory and lead to enhanced Adler zeroes, meaning their amplitudes vanish faster in the soft limit than one would expect from derivative counting. These exceptional scalar theories have a large amount of structure and interesting inter-relations and relations to Yang–Mills and gravity.

B.1 The nonlinear sigma model

The nonlinear sigma model is the low energy effective field theory corresponding to the symmetry breaking pattern $G_L \times G_R \to G_V$, where $G_{L,R}$ are two copies of some Lie group (in our case $\mathrm{SU}(N)$ or $\mathrm{U}(N)$) and G_V is a diagonal subgroup. We will follow the conventions of [188].[1] On a group element $U \in G$, the symmetry acts like

$$U \longrightarrow V_R U V_L^{-1}, \qquad V_{L,R} \in G_{L,R}. \tag{B.1}$$

At lowest order in derivatives, the Lagrangian invariant under this symmetry is simply

$$\mathcal{L} = -\frac{f^2}{4}\mathrm{Tr}[U^{-1}\partial_\mu U U^{-1}\partial^\mu U], \qquad U = \exp\left(\sqrt{2}\frac{i}{f}\phi^a t^a\right), \tag{B.2}$$

---

[1]In particular, we take $[t^a, t^b] = i\sqrt{2}f^{abc}t^c$, where f^{abc} are the totally anti-symmetric structure constants. Moreover the generators are normalized such that $\mathrm{Tr}[t^a t^b] = \delta^{ab}$.

where t^a are the generators of G in the fundamental representation. For $G = \mathrm{SU}(N)$, we will use $a = 1, ..., N^2 - 1$. For $G = \mathrm{U}(N)$, we append an extra generator $t^0 \propto 1$, which spans the additional $\mathrm{U}(1)$ direction and commutes with all of the $\mathrm{SU}(N)$ generators. In addition, note that we have used the *exponential parametrization* to represent the Goldstone fields, see [189].

Expanding in the Goldstone fields ϕ^a, we may write the Lagrangian as

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{1}{6f^2}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\phi^{a_3}\phi^{a_4}f^{a_1a_3b_1}f^{a_2a_4b_1} \\
&+ \frac{1}{45f^4}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\phi^{a_3}\phi^{a_4}\phi^{a_5}\phi^{a_6}f^{a_1a_3b_1}f^{a_4b_1b_2}f^{a_5b_2b_3}f^{a_2a_6b_3} + \mathcal{O}(\phi^8) \, .
\end{aligned}
\tag{B.3}
$$

The benefit of this choice of field variables is that if we embed the $\mathrm{SU}(N)$ model in a $\mathrm{U}(N)$ model, the decoupling of the $\mathrm{U}(1)$ degree of freedom occurs manifestly at the level of the Lagrangian.[2]

The diagonal subgroup G_V acts linearly on the fields ϕ^a, which transform in the adjoint representation. However, the axial subgroup of $G_L \times G_R$ acts on U by

$$
U \longrightarrow VUV, \qquad V \in G, \tag{B.4}
$$

and is realized on the Goldstone modes nonlinearly as

$$
\delta\phi^c = B^c - \frac{1}{3f^2}B^{a_1}\phi^{b_1}\phi^{b_2}f^{a_1b_1b_3}f^{cb_2b_3} - \frac{1}{45f^4}B^{a_1}\phi^{b_1}\phi^{b_2}\phi^{b_3}\phi^{b_4}f^{a_1b_1\,c}f^{a_2b_2a_3}f^{b_3ca_4}f^{b_4a_3a_4} + \mathcal{O}(\phi^6) \, .
\tag{B.5}
$$

The Ward identity corresponding to this symmetry is responsible for controlling the soft behavior of the NLSM scattering amplitudes and wavefunctions. NLSM scattering amplitudes exhibit an $\mathcal{O}(p)$ Adler zero, despite having fewer than one derivative per field at the level of

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[2]To see this, note that any structure constant with a zero index vanishes $f^{0bc} = 0$ because the U(1) generator is proportional to the identity. Because all of the fields in the Lagrangian are contracted with structure constants, all interaction vertices involving the U(1) mode ϕ^0 must be zero. Thus the mode decouples.

the action.

There is an alternative representation of the Goldstone fields which is also commonly used, known as the *Cayley parametrization*. In this case the group element U is given by

$$U = \frac{1 + \frac{i}{\sqrt{2}f}\phi}{1 - \frac{i}{\sqrt{2}f}\phi}, \tag{B.6}$$

and expanding out the Lagrangian gives

$$\mathcal{L} = \text{Tr}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2f^2}\phi^2\partial_\mu\phi\partial^\mu\phi + \mathcal{O}(\phi^6) \right]. \tag{B.7}$$

The axial symmetry is realized very simply in this representation:

$$\delta\phi = B + \frac{1}{2f^2}\phi B\phi. \tag{B.8}$$

However, the disadvantage is that the U(1) mode does not decouple at the level of the Lagrangian, though it does at the level of scattering amplitudes.[3] This non-decoupling does however appear in the wavefunction. For this reason, we do not use the Cayley parametrization, but it would be interesting to study the wavefunction of the U(N) NLSM further.

B.1.1   A brief tour of flavortown

Computing scattering amplitudes and wavefunction coefficients for the nonlinear sigma model is made somewhat cumbersome by the presence of group theoretic flavor structures. For the SU(N) nonlinear sigma model, one may circumvent this difficulty by working instead with so-called *flavor ordered* objects. By prescribing a particular ordering of the legs of a Feynman diagram, one may effectively remove the group-theoretic structures and reduce the number of channels necessary to describe a process. While the full "dressed" object will

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[3]This of course must be the case, because the exponential and Cayley parametrization are related by a field redefinition.

be a sum over channels composed of every combination of external momenta, the flavor ordered version only sums channels composed of consecutive momenta, drastically reducing the number of channels that need to be accounted for. In addition, flavor ordered objects possess the same analytic structure and Ward identities (or a flavor ordered version thereof) as their dressed counterparts. This is precisely the same as how color ordered amplitudes for Yang–Mills maintain the correct singularity structure of the full amplitude and also obey the same Ward identities enforced by gauge invariance. We will begin by outlining the general procedure for computing a flavor ordered wavefunction from a dressed wavefunction. Then we will derive some useful facts about flavor ordered wavefunction which are used in the text.

We begin with a flavor dressed object (e.g., a scattering amplitude or wavefunction coefficient) $\mathcal{M}^{a_1 \cdots a_n}$. In the exponential parametrization, the group theoretic factors will always be products of structure constants.[4] For tree level diagrams, it turns out to always be possible to reduce the products of structure constants to a single trace over group generators t^a by successively applying only the following two identities

$$[t^a, t^b] = \frac{i}{\sqrt{2}} f^{abc} t^c, \qquad f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}\left[[t^a, t^b] t^c\right]. \tag{B.9}$$

In this way, we may write a flavor dressed object in terms of the following expansion (for either U(N) or SU(N)):

$$\mathcal{M}^{a_1 a_2 \cdots a_n}(p_1, p_2, \cdots, p_n) = \sum_{\sigma \in S_n / Z_n} \text{Tr}[t^{\sigma(a_1)} t^{\sigma(a_2)} \cdots t^{\sigma(a_n)}] \mathcal{M}_\sigma(p_1, p_2, \cdots, p_n) \tag{B.10}$$

where σ is a non-cyclic permutation. Due to the Bose symmetry of the dressed object, all

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[4]This remains true for amplitudes in any choice of field variables, but not for the wavefunction. Regardless, the group factors for each interaction vertex may always be reduced to a single trace for any choice of field variables.

of the $\mathcal{M}_\sigma$ are related to one another via permutations of the momenta. More explicitly,

$$\mathcal{M}_\sigma(p_1, p_2, \cdots, p_n) = \mathcal{M}\big(\sigma(p_1), \sigma(p_2), \cdots, \sigma(p_n)\big),\tag{B.11}$$

where $\mathcal{M}(p_1, p_2, \cdots, p_n)$ is defined to be the coefficient of $\text{Tr}[t^{a_1} t^{a_2} \cdots t^{a_n}]$. Each of the quantities $\mathcal{M}\big(\sigma(p_1), \sigma(p_2), \cdots, \sigma(p_n)\big)$ are called flavor ordered objects, and once just one is known, we can apply this formula to generate the full flavor dressed object. Therefore, we need only keep track of and work with one of the orderings as opposed to the fully dressed object. In effect, we may view the set of possible single trace structures as a basis of flavor structures. For $\text{SU}(N)$, this basis is complete in the sense that[5]

$$\sum_{a=1}^{N^2-1} \text{Tr}[X, t^a]\text{Tr}[t^a, Y] = \text{Tr}[XY] - \frac{1}{N}\text{Tr}[X]\text{Tr}[Y],\tag{B.13}$$

where X and Y are arbitrary matrices. Using this relation, one may also prove that the trace structures are also approximately orthogonal:

$$\sum_{a_1,\ldots a_n=1}^{N^2-1} \text{Tr}[t^{\sigma(a_1)} t^{\sigma(a_2)} \cdots t^{\sigma(a_n)}]\text{Tr}[t^{\rho(a_1)} t^{\rho(a_2)} \cdots t^{\rho(a_n)}] = N^{n-2}(N^2-1)\Big(\delta_{\sigma\rho} + \mathcal{O}\Big(\frac{1}{N^2}\Big)\Big)$$
$$\tag{B.14}$$

So long as the flavor ordered objects do not depend on N (which they cannot, because they were generated by applying (B.9), which does not have any N dependence), this is sufficient to extract the flavor ordered object from the dressed object.

The flavor ordered objects enjoy a number of properties (see [188, 190, 5] for more details)

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[5]This is actually just a rewriting of the $\text{U}(N)$ completeness relation

$$\frac{1}{N}\text{Tr}[X]\text{Tr}[Y] + \sum_{a=1}^{N^2-1} \text{Tr}[X, t^a]\text{Tr}[t^a, Y] = \text{Tr}[XY],\tag{B.12}$$

which may be derived from the fact that any complex matrix may be written as a complex linear combination of hermitian matrices.

and interrelations. In the main text, we use two in an essential way:[6]

- **Cyclicity:** The flavor ordered objects obey $\mathcal{M}(p_1, p_2, \cdots, p_n) = \mathcal{M}(\sigma(p_1), \sigma(p_2), \cdots, \sigma(p_n))$, where σ is a cyclic permutation.

- U(1) **Decoupling:** The flavor ordered objects obey the relations

$$\mathcal{M}(p_1, p_2, p_3, \cdots, p_n) + \mathcal{M}(p_2, p_1, p_3, \cdots, p_n) + \mathcal{M}(p_2, p_3, p_1, \cdots, p_n)$$
$$+ \cdots + \mathcal{M}(p_2, p_3, \cdots, p_1, p_n) = 0 \,. \tag{B.16}$$

The first identity follows from cyclicity of the trace. To prove the second identity, carry out the expansion in (B.10) for U(N). Then set $a_1 = 0$, and all of the other indices to SU(N) values. On the left hand side, a_1 always appears as the index of a structure constant. Thus the left hand side vanishes. On the right hand side, this sets $t^{a_1} \propto 1$, which simplifies the traces, allowing flavor ordered objects to be grouped together. This is best demonstrated through a simple example. At three points, we have

$$\mathcal{M}^{a_1 a_2 a_3} = \mathrm{Tr}[t^{a_1} t^{a_2} t^{a_3}] \mathcal{M}(p_1, p_2, p_3) + \mathrm{Tr}[t^{a_1} t^{a_3} t^{a_2}] \mathcal{M}(p_1, p_3, p_2) \,. \tag{B.17}$$

After setting $a_1 = 0$, this may be simplified to

$$\mathrm{Tr}[t^{a_2} t^{a_3}] \Big(\mathcal{M}(p_1, p_2, p_3) + \mathcal{M}(p_1, p_3, p_2) \Big) = 0 \,, \tag{B.18}$$

which forces the term in parentheses to vanish. After applying cyclicity, this may be brought to the form of the U(1) decoupling relation.

---

[6]Note that U(1) decoupling *does not* hold for the wavefunction in the Cayley parametrization, for which the flavor-ordered wavefunction reads

$$\psi_4^{\mathrm{Cayley}}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{1}{2f^2 E} \Big(P_1 \cdot P_2 + P_1 \cdot P_4 + P_2 \cdot P_3 + P_3 \cdot P_4 \Big) \,. \tag{B.15}$$

This is to be expected, because the U(1) mode does not decouple at the level of the Cayley Lagrangian.

For convenience, below we have included single trace representations for the products of 2 and 4 structure constants:

$$f^{a_1 a_2 e} f^{a_3 a_4 e} = \frac{1}{2}\Big(-\operatorname{Tr}[1234] + \operatorname{Tr}[1243] + \operatorname{Tr}[2134] - \operatorname{Tr}[2143] \Big), \qquad \text{(B.19)}$$

$$
\begin{aligned}
f^{a_1 a_2 e} f^{a_3 e g} f^{a_4 g h} f^{a_5 a_6 h} = \frac{1}{4}\Big(&\operatorname{Tr}[123456] - \operatorname{Tr}[123465] - \operatorname{Tr}[123564] + \operatorname{Tr}[123654] \\
& - \operatorname{Tr}[124563] + \operatorname{Tr}[124653] + \operatorname{Tr}[125643] - \operatorname{Tr}[126543] \\
& - \operatorname{Tr}[213456] + \operatorname{Tr}[213465] + \operatorname{Tr}[213564] - \operatorname{Tr}[213654] \\
& + \operatorname{Tr}[214563] - \operatorname{Tr}[214653] - \operatorname{Tr}[215643] + \operatorname{Tr}[216543] \Big).
\end{aligned}
$$

$$\text{(B.20)}$$

B.2 Dirac–Born–Infeld

Dirac–Born–Infeld is the theory of a D-dimensional brane probing a flat $(D+1)$-dimensional bulk. At leading order in derivatives, the action is given by

$$S = -f^4 \int \mathrm{d}^D x \left(\sqrt{1 + \frac{1}{f^4}(\partial\phi)^2} - 1 \right) \qquad \text{(B.21)}$$

This action nonlinearly realizes the ISO$(D+1,1)$ symmetry of the bulk. Translations in the transverse bulk direction correspond to the simple shift symmetry $\phi \mapsto \phi + c$, whereas the extra boost symmetry generates the transformation

$$\delta\phi = v_\mu x^\mu + \frac{1}{f^4} v^\mu \phi \partial_\mu \phi \,. \qquad \text{(B.22)}$$

The Ward identity for the enhanced shift symmetry implies an $\mathcal{O}(p^2)$ Adler zero for scattering amplitudes. For our application, we will need the boundary term generated by the enhanced shift symmetry in order to compute the corresponding temporal charge. From the probe

brane formalism [191], this can be easily seen to be

$$\delta\mathcal{L} = v^\mu \partial_\mu \big(\phi\mathcal{L}\big). \tag{B.23}$$

For the spatial symmetry, which has $v^t = 0$, the boundary term is purely spatial and may be dropped. On the other hand, for the temporal symmetry the boundary term has a time component and thus contributes to the charge. This is described in more detail in Appendix C.2.

B.3 (Special) galileon

Galileon theories are those which are invariant under the enhanced shift symmetry [192]

$$\delta\varphi = c + b_\mu x^\mu, \tag{B.24}$$

and which have second-order equations of motion. The Ward identity resulting from this symmetry controls the soft behavior of the galileon scattering amplitudes, which exhibit a $\mathcal{O}(p^2)$ Adler zero. In D spacetime dimensions, a generic galileon action may be written as

$$\mathcal{L} = \sum_{n=1}^{D+1} c_n \mathcal{L}_n, \qquad \mathcal{L}_n = (\partial\phi)^2 \mathcal{L}_{n-2}^{\mathrm{TD}}, \tag{B.25}$$

where c_n are arbitrary coefficients and $\mathcal{L}_n^{\mathrm{TD}}$ is the unique total derivative that may be formed with n copies of ϕ and two derivatives per field. Abstractly, these can be written

$$\mathcal{L}_n^{\mathrm{TD}} = \sum_\sigma (-1)^\sigma \eta^{\mu_1 \sigma(\nu_1)} \cdots \eta^{\mu_1 \sigma(\nu_n)} \partial_{\mu_1}\partial_{\nu_1}\phi \cdots \partial_{\mu_n}\partial_{\nu_n}\phi, \tag{B.26}$$

where the sum is over all permutations σ. The first few of these take the explicit form

$$\mathcal{L}_1^{\mathrm{TD}} = \Box\phi \tag{B.27}$$

$$\mathcal{L}_2^{\mathrm{TD}} = (\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi \tag{B.28}$$

$$\mathcal{L}_3^{\mathrm{TD}} = (\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\varphi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\alpha\phi\partial_\alpha\partial^\mu\phi\,. \tag{B.29}$$

The standard forms of the galileon action where Lorentz invariance is manifest all have more than one time derivative per field. For Dirichlet boundary conditions, the action as written therefore does not have a well-posed variational principle, and evaluating the on-shell action computes wavefunction coefficients with operator insertions present (see Appendix C.1). To cure this, we must add a boundary term to the action, which breaks manifest Lorentz invariance. We can do this explicitly for the quartic galileon, which is relevant to the discussion in the main text. The bulk and boundary contributions to the action are

$$S_{\mathrm{bulk}}^{(4)} = \int \mathrm{d}^4x\,(\partial\phi)^2\Big[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\Big]\,, \tag{B.30}$$

$$S_{\mathrm{boundary}}^{(4)} = -2\int \mathrm{d}^3x\,\nabla^2\phi\Big[\frac{1}{3}\dot\phi^3 - (\nabla\phi)^2\dot\phi\Big]\,. \tag{B.31}$$

The boundary contribution may be absorbed into the bulk contribution by writing it as a total time derivative. After doing this and performing spatial integrations by parts, we arrive at the following simple form:

$$S^{(4)} = -\frac{1}{4f^6}\int \mathrm{d}^4x\left(\dot\phi^2\left[(\nabla^2\phi)^2 - (\nabla_i\nabla_j\varphi)^2\right] - \frac{1}{3}\partial_i\phi\partial^i\phi\left[(\nabla^2\phi)^2 - (\nabla_i\nabla_j\phi)^2\right]\right)\,. \tag{B.32}$$

This way of writing the action behaves simply under the galileon shift symmetry. First, the action $S^{(4)}$ does not pick up a temporal boundary term under the spatial galileon symmetry. This is not the case for the bulk action in (B.31), which does develop a temporal boundary term. In addition, under the temporal shift symmetry $\delta\phi = b_0 t$, the action $S^{(4)}$ does pick up

a temporal boundary term, which is given by

$$\delta S = -2b_0 \int \mathrm{d}^3 x \, \phi \Big[(\nabla \phi)^2 - (\partial_i \partial_j \phi)^2 \Big] . \tag{B.33}$$

More generally, after fixing the boundary term the action for the nth galileon term obeys

$$
\begin{aligned}
\delta \phi = \vec{b} \cdot \vec{x}, && \delta S^{(n)} = 0 \,, \\
\delta \phi = b_0 t, && \delta S^{(n)} = -2 \int \mathrm{d}^3 x \, \phi \mathcal{L}^{\mathrm{TD}}_{n-2}[\partial_i \phi] \,,
\end{aligned}
\tag{B.34}
$$

where we have discarded spatial boundary terms and $\mathcal{L}^{\mathrm{TD}}_n[\partial_i \phi]$ is a galileon term comprised of spatial derivatives only. That is, the variation of galileon terms does not generate a temporal boundary term under the spatial symmetry $\delta \phi = \vec{b} \cdot \vec{x}$, but does generate a temporal boundary term for the $\delta \phi = b_0 t$ symmetry.

In the space of galileon theories, there is a subset whose actions are invariant under an enhanced field dependent shift symmetry. The simplest of these is the quartic galileon [186]

$$S_{\mathrm{sgal}} = \int \mathrm{d}^4 x \, -\frac{1}{2}(\partial \phi)^2 + \frac{1}{12 f^6} \Big[(\Box \phi)^2 - (\partial_\mu \partial_\nu) \phi)^2 \Big] \,, \tag{B.35}$$

which is often called the *special galileon*. It is invariant under the symmetry

$$\delta \phi = s_{\mu\nu} x^\mu x^\nu + \frac{1}{f^6} s^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \,, \tag{B.36}$$

for $s_{\mu\nu}$ a traceless symmetric tensor. As a result of this symmetry, the amplitudes of the special galileon have a $\mathcal{O}(p^3)$ Adler zero. Being a subset of the galileon, we have already described which boundary terms one must add to the action in order for the variational principle to be well-posed. For tensors $s^{\mu\nu}$ that have no temporal component, i.e., $s^{0\mu} = 0$, the action (with the appropriate boundary terms) does not develop a temporal boundary term under the symmetry, in accordance with the logic laid out in C.2. On the other hand

if these components are not zero, the action does pick up a temporal boundary term. This boundary term is cumbersome to specify and not essential for our main argument, so we have omitted it.[7]

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[7]There is one way of writing of the special galileon action where the boundary term is simple to deduce:

$$S_{\text{sgal}} = -\frac{f^6}{8} \int \mathrm{d}^4 x \left(x^2 + \frac{1}{f^6} \partial \phi \cdot \partial \phi \right) \left(1 - \frac{1}{2f^6} \mathcal{L}_2^{\text{TD}} + \frac{1}{24 f^{12}} \mathcal{L}_4^{\text{TD}} \right). \tag{B.37}$$

Under special galileon symmetry, the Lagrangian changes by the total derivative

$$\delta \mathcal{L} = \frac{2}{f^6} \partial_\alpha \left(s^{\alpha \beta} \partial_\beta \phi \mathcal{L} \right), \tag{B.38}$$

which is made explicit by the probe brane formalism of [193]. However, this way of writing the action does not have a well-posed variational principle, and it turns out that the resulting charge will have time derivatives of canonical momentum, so that it cannot be quantized in a straightforward fashion. It would be interesting to determine whether or not there is a way to apply the probe brane formalism to determine the temporal symmetry boundary term from the fixed action (B.32) in a simple manner.

Appendix C: A boundary view on the wavefunction

In the main text, we take a boundary point of view on the computation of the wavefunction, in the sense that we try to define and compute this object directly on some particular time slice. However, we know from the bulk perspective that there are various ambiguities and subtleties in the definition of the wavefunction associated to, for example, boundary terms. These subtleties should have an avatar from the boundary perspective. In this Appendix we explore these technical subtleties and their resolution.

C.1 Boundary terms and the variational principle

An important subtlety that we have to address is the presence of boundary terms in the action. When it comes to computing scattering amplitudes, such boundary terms are typically harmless. However, they may affect the form of wavefunction coefficients, and thus it is worth asking whether there is a natural choice at the level of the action. There are two types of boundary terms which we will consider. The first type contain derivatives which are normal to the future time slice on which we are evaluating the wavefunction. That is, they contain time derivatives. We will refer to such boundary terms as *nonlocal*. The second type of boundary term does not contain normal derivatives, and we will refer to such terms as *contact terms*.[1] We will argue that there is a natural way to fix nonlocal boundary terms, and that contact terms may be precluded by derivative counting.

Consider first nonlocal boundary terms, which contain a time derivative. Boundary terms of this type may be generated through integration by parts in time, or they may present

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[1]This nomenclature is motivated by the way that these boundary terms contribute to the wavefunction.

on their own. The natural choice of nonlocal boundary terms are those which make the variational principle of the action well-posed. Furthermore, this is actually the only choice of boundary terms for which the on-shell action is computing a vacuum wavefunctional (a transition amplitude from the vacuum to a Heisenberg picture eigenstate) as opposed to a transition amplitude with an operator insertion (or equivalently a transition amplitude between an excited state and a Heisenberg picture eigenstate).

As a simple instructional example, consider a model which describes a free particle. Traditionally, we take the action to be

$$S_1 = \frac{1}{2} \int_{t_1}^{t_2} dt\, \dot{q}^2, \tag{C.1}$$

which has one time derivative per field. This is also the action that naturally appears when computing a transition amplitude for a model described by the Hamiltonian $H = p^2/2$:

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int_{\substack{q(t_2)=x_2 \\ q(t_1)=x_1}} \mathcal{D}q\, e^{iS_1[q]} \ . \tag{C.2}$$

If we imagine that terms that differ by integrations by parts are indistinguishable, there is no reason to prefer this writing of the action over

$$S_2 = -\frac{1}{2} \int_{t_1}^{t_2} dt\, q\ddot{q} = \frac{1}{2} \int_{t_1}^{t_2} dt\, \dot{q}^2 - \frac{1}{2} q\dot{q} \Big|_{t_1}^{t_2}. \tag{C.3}$$

However, actions with more than one time derivative per field typically do not have a well-posed variational principle. Following the logic of [194], we can check this with our simple point-particle example. The variation of S_2 is given by

$$\delta S_2 = -\frac{1}{2} \int_{t_1}^{t_2} dt\, \delta q\ddot{q} - \frac{1}{2} \int_{t_1}^{t_2} dt\, q\delta\ddot{q} = -\int_{t_1}^{t_2} dt\, \delta q\ddot{q} - q\delta\dot{q} \Big|_{t_1}^{t_2} + \dot{q}\delta q \Big|_{t_2}^{t_1}. \tag{C.4}$$

The variation δq is defined to vanish on the boundary, but there is no such constraint on $\delta \dot{q}(t_{1,2})$, which may take on any value. Thus in order for the variational problem to be well-posed, one must add a boundary term to the action whose variation cancels the $\delta \dot{q}(t_{1,2})$ term in (C.4). The proper boundary term is

$$S_{\text{boundary}} = \frac{1}{2}\Big(q(t_2)\dot{q}(t_2) - q(t_1)\dot{q}(t_1) \Big), \tag{C.5}$$

and altogether the action with a well-posed variational principle is

$$S_2 + S_{\text{boundary}} \equiv S_1, \tag{C.6}$$

leading us back to the writing of the action with one time derivative per field.

We can also study what happens when we compute transition amplitudes. Traditionally, we expect transition amplitudes for a model with Hamiltonian $H = p^2/2$ to be computed by (C.2). On the other hand, if we take S_2 and plug it into the path integral, then we have

$$\int_{\substack{q(t_2)=x_2 \\ q(t_1)=x_1}} \mathcal{D}q\, e^{iS_2[q]} = \int_{\substack{q(t_2)=x_2 \\ q(t_1)=x_1}} \mathcal{D}q\, e^{iS_1[q]} e^{-\frac{i}{2}(q(t_2)\dot{q}(t_2)-q(t_1)\dot{q}(t_1))} = \langle x_2, t_2| \, e^{\frac{i}{2}(x_2 p_2 - x_1 p_1)} \, |x_1, t_1\rangle . \tag{C.7}$$

This is still a transition amplitude in the theory with the Hamiltonian $H = p^2/2$, but now with an extra operator insertion present. Alternatively, we can view the boundary term as modifying the states that we are computing a transition amplitude between. When it comes to computing wavefunction coefficients, we do not want there to be any spurious operator insertions as we time evolve the ground state. Therefore, we *only* want to study actions which have at most one time derivative per field. In the theories that we are interested in studying, this is enough to completely resolve the ambiguities of nonlocal boundary terms.

In practice, we can essentially resolve the problem of nonlocal boundary terms by taking

an action and performing integrations by parts in time until there is at most one time derivative per field. Then the variational principle is manifestly well-posed. For the theories that we study in the main text, it is usually the case that the natural and most familiar form of the action is already written so that there is no more than one time derivative per field. However, this is not the case for the (special) galileon. Most of the standard forms of the galileon are manifestly Lorentz invariant, and as it turns out there is no writing of the action which is both manifestly Lorentz invariant and has at most one time derivative per field.

Now consider the case of contact boundary terms, which do not contain derivatives normal to the boundary. For instance, for DBI we might add a term like

$$S \supset \int \mathrm{d}^3 p_1 \mathrm{d}^3 p_2 \mathrm{d}^3 p_3 \mathrm{d}^3 p_4 \, \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4) \, \vec{p}_1 \cdot \vec{p}_2 \vec{p}_3 \cdot \vec{p}_4 \, \varphi_{\vec{p}_1}(t_f) \varphi_{\vec{p}_2}(t_f) \varphi_{\vec{p}_3}(t_f) \varphi_{\vec{p}_4}(t_f) \,. \quad \text{(C.8)}$$

Such a term is called a contact term because its contribution to the wavefunction is proportional to a δ-function in position space. For the theories we are studying, such terms are forbidden by power counting. For instance, the contribution to the DBI four-point function from the bulk action contains 4 derivatives, and so boundary terms should only contain 3 derivatives at this order in fields. However, the only way to form a rotationally invariant boundary term with 3 derivatives is to use the magnitude of one of the momenta. This in effect would require the boundary term to have a normal derivative, and thus it is nonlocal, and covered by the discussion above. In this way, we may fix the contact terms in the action for all of the theories that we study in the main text.

At the level of the action, we have addressed how to fix the boundary terms in order to compute proper vacuum wavefunction coefficients. However in the bootstrap philosophy the opposite question is more natural: If someone hands us a putative wavefunction coefficient that has all of the expected singularities (or, if we bootstrap it ourselves), how do we know it came from an action with a well-posed variational principle, or equivalently, that

it was computed with no spurious operator insertions in the path integral? We may test for this by deriving the Ward identities obeyed by vacuum wavefunction coefficients, and check that whatever we have bootstrapped obeys these Ward identities. If it does not, then there must be an operator insertion. However the converse is not true; just because the object we bootstrapped *does* obey the Ward identity does not mean there is no operator insertion. The proper strategy is then to impose so many Ward identities that there is only one object satisfying them all. Then we may be sure that the object we have bootstrapped is a wavefunction coefficient.

As a simple example, consider the following two objects, computed with a DBI Hamiltonian at four points:

$$\psi_4^{(a)} = \frac{1}{f^4 E} \left(P_1 \cdot P_2 P_3 \cdot P_4 + P_1 \cdot P_3 P_2 \cdot P_4 + P_1 \cdot P_4 P_2 \cdot P_3 \right), \tag{C.9}$$

$$\psi_4^{(b)} = -\frac{1}{2 f^4 E} \left(P_1 \cdot P_2 P_2 \cdot P_3 + \text{perms.} \right). \tag{C.10}$$

Both of these have the correct singularity structure and appear to be perfectly reasonable wavefunction coefficients computed by evaluating the on-shell DBI action. However, one of these was computed from an action with a well-posed variational principle, while the other is not. To detect which is the true wavefunction coefficient, we test each for the $\mathcal{O}(p)$ Adler zero. The first object has the Adler zero, and the second does not, telling us that $\psi_4^{(a)}$ is the true vacuum wavefunction coefficient, whereas $\psi_4^{(b)}$ was computed in the presence of operator insertions. Indeed, each object comes from the respective action

$$S_{(a)} \supset \int \mathrm{d}^4 x \, \partial \phi^4, \tag{C.11}$$

$$S_{(b)} \supset \int \mathrm{d}^4 x \, \phi \partial_\mu \left(\partial^\mu \phi \partial \phi^2 \right), \tag{C.12}$$

which only differ by an integration by parts. However, only the first has a well-posed variational principle. In this way, we can use Ward identities to detect well-posedness of the

variational principle.

C.2 Symmetry generators

In the main text, we are interested in deriving Ward identities for charges that generate nonlinearly realized symmetries. If we know the action, it is a simple matter to compute the associated current and derive the charge which generates a particular symmetry. However, from the on-shell point of view, in many cases we want to be able to derive Ward identities without starting from the action. This means that we need a way to write down the symmetry charges without utilizing the action. As it turns out, we may often intuit the structure of the charges and their realization on fields just by knowing the symmetry transformation $\delta\phi$ alone.

Our starting point will be the action

$$S[\phi] = \int \mathrm{d}^4 x\, \mathcal{L}[\phi, \dot{\phi}]\,. \tag{C.13}$$

Note that we are specifically writing the action so that there is at most one time derivative per field. There may be arbitrarily many spatial derivatives acting on a single field, and we have suppressed their dependence in the Lagrangian. If $\delta\phi$ is a symmetry of the action, then the Lagrangian changes by at most a total derivative:

$$\delta S[\phi] = \int \mathrm{d}^4 x\, \partial_\mu K^\mu, \tag{C.14}$$

and the conserved current and its corresponding charge are given by

$$J^\mu = \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - K^\mu, \tag{C.15}$$

$$Q = \int \mathrm{d}^3 x\, \delta\phi \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - K^0 = \int \mathrm{d}^3 x\, \delta\phi \Pi^{(\phi)} - K^0\,. \tag{C.16}$$

We have used that by definition, $\Pi^{(\phi)} = \partial \mathcal{L}/\partial \dot{\phi}$. Note that the K^0 piece comes from the temporal boundary term generated by the symmetry. The symmetry may also generate spatial boundary terms, but they do not contribute to the charge.

Our strategy will be to use commutation relations to determine the form of K^0. The symmetry transformation $\delta \phi$ may contain powers of $\dot{\phi}$ (or equivalently Π_ϕ) and have dependence on the spacetime coordinate x. First, suppose that the symmetry $\delta \phi$ does not involve any time derivatives or the time coordinate itself. Q must act on $\delta \phi$ via $[Q, \phi] = i \delta \phi$. From (C.16), this requires that

$$\int \mathrm{d}^3 x \, [K^0(\vec{x}, t), \phi(\vec{y}, t)] = -i \int \mathrm{d}^3 x \, \frac{\delta K^0(\vec{x}, t)}{\delta \Pi^{(\phi)}(\vec{y}, t)} = 0 \implies K^0 = K^0[\phi] \,. \qquad \text{(C.17)}$$

Therefore K^0 does not have any powers of canonical momenta. Next, we will make the following assertion, without proof: If Q commutes with the Hamiltonian for a Lorentz invariant system, then the symmetry $\delta \phi$ *does not* generate a *temporal* boundary term in the action, and in turn $K^0 = 0$. Said another way, if Q commutes with the Hamiltonian for a Lorentz invariant system, the Lagrangian (not the Lagrangian density) is invariant.[2]

Now consider the case when the symmetry $\delta \phi$ does contain the time coordinate, but not time derivatives. Such charges typically do not commute with the Hamiltonian. However, requiring again that $[Q, \phi] = i \delta \phi$, we can again conclude that $K^0 = K^0[\phi]$. However, if Q and the Hamiltonian do not commute, we will *not* assume that $K^0 = 0$. Despite this, it is sometimes the case that K^0 may be determined solely by the commutation relations of Q with the other charges in the algebra, and does not need to be computed from the action at

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[2] We do not have a general proof of this assertion. However, we will take it as an input; we are only interested in studying theories where the charges are implemented on fields such that this condition is true. It is worth noting that this is usually the case, so long as one properly fixes the boundary term so that there is no more than one time derivative per field as prescribed in Appendix C.1. One can indeed check this for spacetime symmetries, as well as more exotic symmetries such as the shift symmetry, the spatial DBI symmetry, and the spatial (special) galileon symmetry, where "spatial" indicates that the symmetry generated commutes with the Hamiltonian. In fact, it would be interesting if there were a theory with a symmetry charge such that this is *not* the case, but we are not aware of any examples.

all (see (3.79) for an example involving the general galileon).

Finally, consider the case when $\delta\phi$ contains time derivatives. In such cases, it is difficult to write down the charges without starting from the action itself. Because $\delta\phi$ contains $\Pi^{(\phi)}$, even implementing $[Q, \phi] = i\delta\phi$ becomes nontrivial. From (C.16) we have

$$[Q, \phi] = \int d^3x \, [\delta\phi(\vec{x}, t), \phi(\vec{y}, t)] + i\delta^{(3)}(\vec{x} - \vec{y})\delta\phi(\vec{x}, t) - [K^0(x), \phi(\vec{y}, t)] . \qquad \text{(C.18)}$$

In order for Q to generate the symmetry properly, the first and third terms must cancel, which means that K^0 must contain factors of $\Pi^{(\phi)}$, and there is not a straightforward way to deduce the form of K^0 in such cases. This is equivalent to the fact that we cannot anticipate the boundary term which the action develops under a temporal symmetry without simply computing it. This is the fact that limits the utility of using the temporal DBI soft theorem and the temporal special galileon soft theorem in a bootstrap procedure.

Appendix D: Classical field profile and generating functionals

In this Appendix, we highlight some features of the classical field profile which are used in the text to prove tree level soft theorems. These observations essentially relate the classical field profile to combinations of wavefunction coefficients. First, we will derive relations for the late time classical canonical momentum, which are similar to those pointed out in [137] in the AdS context. These are the analogue of the statement that the classical field profile with a Feynman pole prescription in the presence of a source is a generating functional for tree level in/out correlators [195]. In the wavefunction context, we can actually derive additional relations, using the Bunch–Davies condition to relate early time classical field profiles to late time canonical momentum profiles in a simple and universal way.

To begin, we demonstrate the analogous statement for flat space in/out correlators: the classical field profile with a Feynman pole prescription is a generating functional for tree level Feynman diagrams. We start by defining $W[J]$, the generating functional for connected in/out correlators:

$$e^{iW[J]} \equiv \int \mathcal{D}\phi \, e^{iS[\phi] + i \int \mathrm{d}^4 x \, \phi J} \,. \tag{D.1}$$

Taking a functional derivative of this form of the generating functional, we have

$$\frac{\delta W[J]}{\delta J} = e^{-iW[J]} \int \mathcal{D}\phi \, e^{iS[\phi] + i \int \mathrm{d}^4 x \, \phi J} \phi \equiv \langle \phi \rangle_J \,. \tag{D.2}$$

Thus, $\langle \phi \rangle_J$ is a generating functional for in/out correlators. If we take the tree level approximation, then the result follows:

$$\phi_{\mathrm{cl}}[J] = \frac{\delta W[J]}{\delta J} \,, \tag{D.3}$$

where $\phi_{\text{cl}}[J]$ is the classical field profile in the presence of the source, J.

We want to carry out at a similar computation for the classical field profile with Dirichlet boundary conditions. We begin with the generating functional for connected Witten diagrams, which we will also call $W[\varphi]$. Of course, this is the logarithm of the wavefunction itself:

$$e^{W[\varphi]} \equiv \Psi[\varphi, t_f] = \int^{\phi(t_f)=\varphi} \mathcal{D}\phi(t) \, e^{iS[\phi]} \,. \tag{D.4}$$

Similar to the in/out computation, we can take a functional derivative of the generating functional:

$$-i\frac{\delta W[\varphi]}{\delta\varphi(\vec{x})} = \frac{1}{\langle \varphi, t_f | \Psi \rangle} \langle \varphi, t_f | \, \Pi^{(\phi)}(\vec{x}, t_f) \, |\Psi\rangle \,. \tag{D.5}$$

Therefore, the connected part of the canonical momentum, computed with Dirichlet boundary conditions, is a generating functional for connected Witten diagrams. At tree level, we have

$$\Pi_{\text{cl}}^{(\phi)}(\vec{p}, t_f) = -i(2\pi)^3 \frac{\delta W[\varphi]}{\delta\varphi_{-\vec{p}}}$$

$$= \sum_{n=2}^{\infty} \frac{-i}{(n-1)!} \int \frac{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}} \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}} \delta^{(3)}(\vec{p}_1 + \cdots - \vec{p}) \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, -\vec{p}) \,. \tag{D.6}$$

To relate this expression to the field profile itself, first write the canonical momentum as

$$\Pi^{(\phi)}(\vec{p}, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} + (2\pi)^3 \frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}} \,. \tag{D.7}$$

We will also need the following recursively defined expression for the classical field profile:

$$\phi_{\vec{p}}(t) = \mathcal{K}(p, t)\varphi_{\vec{p}} + i(2\pi)^3 \int_{-\infty}^{0} \mathrm{d}t' \, \mathcal{G}(t, t', \vec{p}) \frac{\delta S_{\text{int}}}{\delta\phi_{-\vec{p}}(t')} \,. \tag{D.8}$$

Using the following property of the bulk-to-bulk propagator

$$\partial_t \mathcal{G}(p; t, t') \Big|_{t=0} = -i\mathcal{K}(p, t'), \tag{D.9}$$

leads to an expression for the time derivative of the classical field on the future boundary:

$$\dot{\phi}_{\vec{p}}(t_f) = ip\,\varphi_{\vec{p}} + (2\pi)^3 \int_{-\infty}^{0} dt'\, \mathcal{K}(p, t') \frac{\delta S_{\text{int}}}{\delta \phi_{-\vec{p}}(t')}. \tag{D.10}$$

If the theory does not have interactions involving time derivatives, then $\Pi^{(\phi)} = \dot{\phi}$ exactly, and recursively computing the above expression provides an alternative way to compute wavefunction coefficients. If the theory *does* have time derivative interactions, then this expression is not a generating functional for wavefunction coefficients, and one must include corrections to account for the difference between $\Pi^{(\phi)}$ and $\dot{\phi}$. Using (D.10) and (D.7), we can write the following expression for $\Pi^{(\phi)}$ evaluated on the future boundary:

$$\Pi^{(\phi)}(\vec{p}, t_f) = ip\varphi_{\vec{p}} + (2\pi)^3 \frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}_{-\vec{p}}(t)} \Big|_{t=t_f} + (2\pi)^3 \int_{-\infty}^{0} dt'\, \mathcal{K}(p, t') \frac{\delta S_{\text{int}}}{\delta \phi_{-\vec{p}}(t')}. \tag{D.11}$$

Writing out explicitly the Euler–Lagrange derivative inside the integral we get

$$\Pi^{(\phi)}(\vec{p}, t_f) = ip\varphi_{\vec{p}} + (2\pi)^3 \frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}_{-\vec{p}}(t)} \Big|_{t=t_f} + (2\pi)^3 \int_{-\infty}^{0} dt'\, \mathcal{K}(p, t') \left[\frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi_{-\vec{p}}(t')} - \frac{d}{dt} \frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}_{-\vec{p}}(t')} \right], \tag{D.12}$$

and after integrating the time derivative by parts, we find

$$\Pi^{(\phi)}(\vec{p}, t_f) = ip\varphi_{\vec{p}} + (2\pi)^3 \int_{-\infty}^{0} dt'\, \mathcal{K}(p, t') \left[\frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi_{-\vec{p}}(t')} + ip \frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}_{-\vec{p}}(t')} \right]. \tag{D.13}$$

This expression may be used to compute wavefunction coefficients for theories with arbitrary scalar interactions.

As it turns out, we may use the above expression to derive statements about the classical

field profile at *early* times as well. Using the form of the bulk-to-bulk propagator in (3.9) we have that at early times

$$\lim_{t \to -\infty} \mathcal{G}(p; t, t') = \frac{1}{2p} \Big(\mathcal{K}(-p, t') - \mathcal{K}(p, t') \Big) \mathcal{K}(p, t) \,. \tag{D.14}$$

Thus, at early times, the classical field profile (D.8) reduces to

$$\lim_{t \to -\infty} \phi_{\vec{p}}(t) = \mathcal{K}(p, t) \Bigg(\varphi_{\vec{p}} + \frac{i}{2p} (2\pi)^3 \int_{-\infty}^0 \mathrm{d}t' \bigg[\Big(\mathcal{K}(-p, t') - \mathcal{K}(p, t') \Big) \frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial \phi_{-\vec{p}}(t')} $$
$$- \Big(\mathcal{K}(-p, t') + \mathcal{K}(p, t') \Big) ip \frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial \dot{\phi}_{-\vec{p}}(t')} \bigg] \Bigg) \,, \tag{D.15}$$

where we have made the same simplifications of the $\delta S/\delta\phi$ term that we performed to get from (D.11) to (D.13). Note there is no future boundary term because of the relative minus sign between the bulk-to-boundary propagators.

Interestingly (D.15) is essentially the difference of two copies of (D.13) with the sign of the external energy flipped:

$$\lim_{t \to -\infty} \phi_{\vec{p}}(t) = \lim_{t \to -\infty} \frac{i}{2p} \Big(\Pi^{(\phi)}(\vec{p}, -p, t_f) - \Pi^{(\phi)}(\vec{p}, p, t_f) \Big) \mathcal{K}(p, t) \,. \tag{D.16}$$

Therefore, the classical field profile at early times is a generating functional for shifted wavefunction coefficients:

$$\lim_{t \to -\infty} \phi_{\vec{p}}(t) = \lim_{t \to -\infty} \mathcal{K}(p, t) \sum_{n=2} \frac{1}{(n-1)!} \int \frac{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}} \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}}$$
$$\times \delta^{(3)}(\vec{p}_1 + \cdots + \vec{p}_{n-1} - \vec{p}) \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, -\vec{p}) \,, \tag{D.17}$$

where we have defined the quantity

$$\psi_n(\vec{p}) \equiv \frac{1}{2p} \Big(\psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, \vec{p}_n; -p_n) - \psi_n(\vec{p}_1, \cdots, \vec{p}_{n-1}, \vec{p}_n; p_n) \Big). \qquad \text{(D.18)}$$

We can in fact go one step further by noticing that a shifted wavefunction coefficient is in the form of a finite difference in the energy associated with the classical field. In the soft limit, we can make the replacement

$$\lim_{\vec{p} \to 0} \psi_n(\vec{p}) = -\partial_p \psi_n(\vec{p})\Big|_{\vec{p}=0}. \qquad \text{(D.19)}$$

Thus, the soft classical field profile at early times is the generating functional for partial derivatives of wavefunction coefficients with respect to energy:

$$\lim_{\substack{t \to -\infty \\ \vec{p} \to 0}} \phi_{\vec{p}}(t) = -\sum_{n=2} \frac{1}{(n-1)!} \int \frac{\mathrm{d}^3 p_1 \cdots \mathrm{d}^3 p_{n-1}}{(2\pi)^{3(n-2)}} \, \varphi_{\vec{p}_1} \cdots \varphi_{\vec{p}_{n-1}} \delta^{(3)}(\vec{p}_1 + \cdots + \vec{p}_{n-1}) \partial_p \psi_n\Big|_{\vec{p}=0}.$$

$$\text{(D.20)}$$

This final formula indicates that shifted wavefunction coefficients contain information about the system at early times. This fact is essential in understanding the role that the initial state plays in various soft theorems.

Appendix E: Navigating branch cuts

In this Appendix, we will show that it is always possible to choose a branch of the complexified partial energy functions which does not have any zeros. We want to study the zeros of the multivalued functions of the form

$$f(z) = A + Bz + \sqrt{(z - c_-)(z - c_+)}, \tag{E.1}$$

where A, B are real and $c_\pm$ are complex. The candidate zeros of the function are

$$z_\pm = \frac{-(2AB + c_- + c_+) \pm \sqrt{(2AB + c_- + c_+)^2 - 4(B^2 - 1)(A^2 - c_- c_+)}}{2(B^2 - 1)}. \tag{E.2}$$

We will prove that there is a branch of $f(z)$ with no zeros, and a branch with two zeros. Define the two branches as

$$
\begin{aligned}
f_1(z) &= A + Bz + |(z - c_-)(z - c_+)|^{\frac{1}{2}} \exp\left(\frac{i}{2}[\arg(z - c_-) + \arg(z - c_+)]\right), \\
f_2(z) &= A + Bz + |(z - c_-)(z - c_+)|^{\frac{1}{2}} \exp\left(\frac{i}{2}[\arg(z - c_-) + \arg(z - c_+) + 2\pi]\right).
\end{aligned} \tag{E.3}
$$

We define both arg functions in both branches to take values in the range $[0, 2\pi)$. This unambiguously defines two single-valued functions which are (at least) meromorphic on the space $\mathbb{C} - [c_-, c_+]$. On the putative zeros, each branch becomes

$$
\begin{aligned}
f_1(z_\pm) &= A + Bz_\pm + |A + Bz_\pm| \exp\left(\frac{i}{2}[\arg(z_\pm - c_-) + \arg(z_\pm - c_+)]\right), \\
f_2(z_\pm) &= A + Bz_\pm + |A + Bz_\pm| \exp\left(\frac{i}{2}[\arg(z_\pm - c_-) + \arg(z_\pm - c_+) + 2\pi]\right).
\end{aligned} \tag{E.4}
$$

We can simplify this further by manipulating the arg functions, which are both defined mod 2π. We have that

$$\arg(z_\pm - c_-) + \arg(z_\pm - c_+) = \arg\left((z_\pm - c_-)(z_\pm - c_+)\right)$$
$$= \arg\left((A + Bz_\pm)^2\right) = 2\arg\left(A + Bz_\pm\right).$$

(E.5)

Substituting this into the prior equation, we come to

$$f_1(z_\pm) = A + Bz_\pm + |A + Bz_\pm|\exp\left(i\arg(A + Bz_\pm)\right) = A + Bz_\pm + A + Bz_\pm \neq 0\,,$$

$$f_2(z_\pm) = A + Bz_\pm + |A + Bz_\pm|\exp\left(i\arg(A + Bz_\pm) + i\pi\right) = A + Bz_\pm - (A + Bz_\pm) = 0\,.$$

(E.6)

Thus, the branch $f_1(z)$ has no zeros while the branch $f_2(z)$ has two zeros.