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Integrating out heavy fields in the path integral using the background-field method: general formalism

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Abstract Building on an older method used to derive non-decoupling effects of a heavy Higgs boson in the Standard Model, we describe a general procedure to integrate out heavy fields in the path integral. The derivation of the corresponding effective Lagrangian including the one-loop contributions of the heavy particle(s) is particularly transparent, flexible, and algorithmic. The background-field formalism allows for a clear separation of tree-level and one-loop effects involving the heavy fields. Using expansion by regions the one-loop effects are further split into contributions from large and small momentum modes. The former are contained in Wilson coefficients of effective operators, the latter are reproduced by one-loop diagrams involving effective tree-level couplings. The method is illustrated by calculating potential non-decoupling effects of a heavy Higgs boson in a singlet Higgs extension of the Standard Model. In particular, we work in a field basis corresponding to mass eigenstates and properly take into account non-vanishing mixing between the two Higgs fields of the model. We also show that a proper choice of renormalization scheme for the non-standard sector of the underlying full theory is crucial for the construction of a consistent effective field theory.

Contents

1 Introduction

After roughly a decade of operation, the Large Hadron Collider (LHC) at CERN has confirmed the validity of the Standard Model (SM) of particle physics generically up to energies in the TeV range, without any significant and convincing deviation from SM predictions. On the other hand, we know that the SM is incomplete, because it does not include neutrino masses nor explain phenomena like Dark Matter or the matter–antimatter asymmetry in the universe. To fully exploit the potential of the LHC on its mission to identify the limitations of the SM and to unravel the structure of potential deviations of experimental results from SM predictions, a strategy is required that is as model independent as possible and can be pushed to sufficiently high precision. Of course, precise SM predictions are the major prerequisite in this task. However, in order to establish at which accuracy the various sectors of the SM are tested or to shape observed deviations,

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it is necessary to include non-standard effects in analyses. Besides dedicated analyses in specific models for physics beyond the SM (BSM), it is desirable to provide, as far as possible, model-independent analyses that quantify the compatibility of data with the SM before confronting the results with specific models.

Standard model effective theory (SMEFT) [1,2] (see also Refs. [3–6] and references therein) is such an approach, in which it is assumed that the SM is the valid theory up to an energy scale Λ much larger than the electroweak (EW) scale $v \approx 246\,\mathrm{GeV}$ and that new particles have masses of the order of Λ. Under this assumption the leading BSM effects are generically suppressed by powers of Λ and can be parametrized in terms of Wilson coefficients of (local) dimension-5 and dimension-6 effective operators which are added to the SM Lagrangian. Although there is still a long way towards fully global fits of these Wilson coefficients to data, larger and larger subsets of operators are being considered (for up-to-date analyses see e.g. Refs. [7,8]), and SMEFT predictions are being dressed with QCD and EW corrections (for recent calculations see e.g. Refs. [9–13]). So far the results of these fits are such that all Wilson coefficients are still compatible with zero. Once some coefficients show significant deviations from zero, the question arises which BSM effects and which new particles cause them.

To answer this question, one evaluates BSM models at energies well below the mass scale Λ of the non-standard particles. In the effective field theory (EFT) describing this limit the heavy particles are *integrated out*, i.e. their fields are no longer dynamical degrees of freedom, so that the particle content is the same as in the SM. The effects of the BSM particles are reproduced by non-vanishing Wilson coefficients of the higher-dimensional effective operators mentioned above. There are roughly speaking two types of approaches to compute the relevant Wilson coefficients:

- For the *diagrammatic matching* a sufficiently large set of Green functions is evaluated both in the underlying BSM model (full theory) and the EFT in terms of Feynman diagrams to a given order in the perturbative (loop) expansion. On the EFT side this requires to first construct a generic basis of operators (composed of SM fields and respecting the symmetries of the BSM model) to the order of interest in the EFT ($1/\Lambda$) expansion. The Wilson coefficients are then fixed by demanding that corresponding EFT and full-theory amplitudes match up to higher orders in $1/\Lambda$.
- *Functional matching* is based on the path integral defining the generating functional for Green functions of the full theory. The functional integration over heavy field modes related to the BSM effects is performed and directly results in a $1/\Lambda$ expanded effective action for the low-energy (SM) degrees of freedom representing the EFT. No input on the structure of the EFT operators is required.[1] At one-loop order the relevant functional integrals are of Gaussian type and therefore straightforward to carry out. Beyond one loop the feasibility of this method seems unclear.

In this paper we describe a generic functional approach to integrate out heavy particles, which is a further development of the method introduced in Refs. [14,15]. Functional methods for EFT matching have a long history, see e.g. Refs. [16–18] for early works. Following this way, the non-decoupling effects of a heavy SM Higgs boson were computed in Refs. [14,15]. In particular, it was demonstrated there that using the background-field method (BFM) Refs. [19–25] leads to a transparent separation of tree-level and one-loop contributions in the functional derivation of the effective Lagrangian. The method of [14,15] was further refined and generalized in Ref. [26] by employing the expansion by regions [27,28] (see also Ref. [29] for a concise review).

In recent years the interest in functional matching has been revived in the context of SMEFT by Ref. [30]. This work initiated the "Universal One-Loop Effective Action" (UOLEA) program [31–37] which ultimately aims at deriving a master formula for the one-loop matching of a fully generic BSM model to SMEFT using functional methods. The basic idea is that the matching in principle only has to be performed once and for all. The relevant Wilson coefficients of the SMEFT operators could then be determined for any specific BSM model via the master formula by replacing the generic with the specific full theory parameters. So far, however, this goal has not been reached. While the UOLEA is not yet available for the most general case, for example because couplings of the heavy fields (to be integrated out) involving derivatives are not accounted for, its complexity already suggests that it will be limited to SMEFT operators with dimension ≤ 6 in the foreseeable future. For a review on the current status of the UOLEA program see Ref. [37]. In order to overcome some of the limitations of the UOLEA approach, there is a trend towards automation of the matching procedure [38–41]. Due to its algorithmic nature, functional matching turns out to be well suited for this purpose, especially at the one-loop level.

In the UOLEA and automation literature quoted above the EFT expansion is essentially based on a power counting of dimensionful quantities, i.e. masses and loop momenta of $\mathcal{O}(\Lambda)$. However, as noticed already in Ref. [42], some realistic BSM models feature dimensionless parameters like couplings or mixing angles with definite Λ scaling, i.e. they

[1] In this respect the term "matching" is actually misleading, because the Wilson coefficients of a generic EFT Lagrangian are not determined by matching EFT and full theory predictions.

must be counted as powers of v/Λ, where v represents a typical SM scale. For example, the mixing of a BSM-type with as SM-type field to form a heavy and a light mass eigenstate, as it appears in the context of spontaneous symmetry breaking, requires a mixing angle that is suppressed by powers of $1/\Lambda$ if the heavy field is supposed to decouple. We will explicitly address this aspect in the present paper using a functional method based on Refs. [14,15,26]. We will also show that combining background-field gauge invariance with a non-linear representation of the SM Higgs doublet enables further technical simplifications, because intermediate manipulations can be carried out in the unitary gauge, while full gauge invariance is restored at the end of the calculation. In this respect we generalize the matching procedure of Refs. [14,15], where light modes of the heavy field did not contribute in loops. To account for such contributions we perform the large-mass ($\sim \Lambda$) expansion according to the method of regions [27,28], which separates heavy and light modes in loop integrals as also proposed in Ref. [26]. The loop effects of the heavy modes are encoded in the Wilson coefficients of the effective Lagrangian, while the loop effects of the light modes result from insertions of tree-level effective couplings in EFT loop diagrams. The whole procedure is fully algorithmic and flexible in the sense that the underlying low-energy need not be specified in advance, i.e. the method is also applicable beyond the framework of SMEFT, which assumes the SM as the leading-order (LO) low-energy theory.

In this article we describe the general framework of our functional matching method and apply it to integrate out a non-standard Higgs boson with large mass $M_{\mathrm{H}} \gg v$ in a singlet (Higgs) extension of the SM (SESM), which is defined in different variants in Refs. [43–50]. To keep the presentation transparent, we restrict the calculation here to the level of non-decoupling effects in the bosonic sector, i.e. to terms of $\mathcal{O}(M_{\mathrm{H}}^0)$ in the effective Lagrangian, which are non-trivial in the presence of Higgs mixing. We will deal with the decoupling effects at $\mathcal{O}(M_{\mathrm{H}}^{-2})$ in a follow-up paper. A main focus of the present paper will be the issue of renormalization of the BSM sector of the underlying full theory and its consequences for the EFT. We will explain how the choice of renormalization and tadpole schemes affects the derivation of the effective Lagrangian, already at the $\mathcal{O}(M_{\mathrm{H}}^0)$ level. This aspect, which has mostly been ignored in the existing literature, generally arises in renormalization schemes where the loop contribution to a renormalization constant of a BSM parameter and the parameter itself scale differently in the large-mass limit. In models with extended Higgs sectors, such effects potentially occur in the interplay of tadpole renormalization and $\overline{\mathrm{MS}}$ renormalization conditions (see e.g. Refs. [47,49–54]).

The low-energy limit of different SESM variants has been studied repeatedly in the past, see e.g. Refs. [30,34,39,42,55–63].[2] In fact, it has become a kind of test model for different matching techniques as well as to analyze the EFT validity. In the following we give a brief overview of the most elaborate literature on SESM to SMEFT matching at $\mathcal{O}(1/\Lambda^2)$ and one-loop level. We focus on matching calculations that take into account contributions to the Wilson coefficients from both types of loops: loops that only involve heavy (virtual) particles as well as mixed heavy–light particle loops. The latter were omitted in earlier publications (cf. Refs. [30,31]). In Ref. [34] the matching was performed using functional (UOLEA) methods. While contributions from loops involving fermions were still neglected in Ref. [34], they were included later by a calculation based on Feynman diagrams [61]. The results were confirmed by purely diagrammatic matching in Ref. [62] and finally reproduced with the partly automated functional procedure of Refs. [39,40]. For a recent fit of experimental data to the effective Lagrangian of Refs. [39,61,62] see Ref. [63]. The two different functional approaches [34,39] both make use of the BFM in combination with the expansion by regions in order to streamline and simplify the calculations following Refs. [14,15,26] (and so does our method). On the other hand, none of the quoted one-loop matching references [34,39,61,62] works in a field basis corresponding to mass eigenstates, which is the safest way to consistently take into account the possibility of mixing between the SM-type Higgs doublet and the additional scalar field in the (broken phase of the) SESM. This issue was also addressed in Ref. [42], albeit using an old-fashioned functional method without the virtues of the BFM and the expansion by regions. Complications related to SESM renormalization and the treatment of tadpoles in the presence of mixing were avoided there by choosing a specific on-shell renormalization and tadpole scheme, while most renormalization procedures in BSM sectors involve $\overline{\mathrm{MS}}$ conditions to some extent. In the present article we explore the subtleties arising in different standard renormalization (e.g. $\overline{\mathrm{MS}}$) and tadpole schemes.

Our paper is organized as follows: In Sect. 2 we outline the salient steps and ingredients of the method and highlight the new features added in this paper. Section 3 describes the Singlet Higgs Extension of the SM used as test model, the relevant large-mass/low-energy scenario, the formulation of the model within the BFM, and the non-linear realization of the Higgs sector. In Sect. 4 we elaborate on the individual steps of the calculation of the effective Lagrangian: the separation of heavy and light field modes, the solution of the functional integral over the heavy quantum field, and the elimination of the light modes of the heavy Higgs field via its equation of motion. In Sect. 5 we discuss the renormalization of the full and the effective theory in detail. Our conclusions are given

[2] See Refs. [29,64,65] for the matching of a SM extension with a charged singlet scalar onto SMEFT.

in Sect. 6, and Appendix A provides further (pedagogical) details about the functional integration.

2 Outline of the general method

The method described in the following is a further development of the method introduced in Refs. [14,15], where a heavy Higgs field was integrated out in an SU(2) gauge theory and the SM, respectively, directly in the path integral. As already mentioned, some of the generalizations presented here have already been proposed in Refs. [26] (see also Refs. [29,39]). Unlike for several other approaches in the literature, no matching of free parameters between an ansatz for the effective Lagrangian and explicitly calculated Green functions or amplitudes is involved. Furthermore, the use of the BFM yields additional benefits. Particular strengths of the method are:

- (i) a clear separation of tree-level and loop effects of the heavy fields;
- (ii) the possibility to fix the (background) gauge in intermediate steps of the calculation and to restore gauge invariance of the effective Lagrangian at the end;
- (iii) transparency in the sense that at each stage of the calculation it is possible to identify the origin of all contributions to the effective Lagrangian in terms of Feynman diagrams;
- (iv) flexibility due to the fact that no ansatz is made for the effective Lagrangian. Actually not even the low-energy theory has to be specified in advance, it directly emerges as part of the result;
- (v) An automation of the method is possible, since it is fully algorithmic. In principle, given a Lagrangian, a large-mass scenario with a corresponding power-counting scheme, and some details on the renormalization of the large-mass sector, the actual determination of the effective Lagrangian at the one-loop level can be carried out by computer algebra.

Since the individual steps in the whole procedure are quite non-trivial and involve various tricks, we first sketch the different steps and ingredients before applying the method to a concrete example in the subsequent sections. This preparatory section will also motivate the splitting of a generic heavy particle field H into four conceptually different parts, $H \to \hat{H}_h + \hat{H}_l + H_h + H_l$, which is at the heart of the proposed method. In the course of this brief outline we also explain which generalizations and optimizations have been made in Ref. [26] and are made in this paper with respect to the original approach of Refs. [14,15]:

1. *Background-field formalism and non-linear Higgs realization.*
   Formulating the theory within the BFM splits all fields into background (i.e. in some sense semi-classical) and quantum parts. For a generic heavy field H, this separation reads $H \to \tilde{H} = \hat{H} + H$, with $\hat{H}$ being the background and H the quantum field. Diagrammatically this step distinguishes between fields occurring on tree and loop lines in Feynman graphs. For tree-level effects, quantum fields are not relevant. For one-loop corrections, only terms in the Lagrangian that are bilinear in quantum fields are relevant. Higher powers of quantum fields only contribute beyond the one-loop level. Thus, this step determines the terms in the full-theory Lagrangian that are needed in the subsequent derivation of the EFT Lagrangian.
   Employing a non-linear representation of the scalar sector, it is possible to absorb all background Goldstone-boson fields into the background gauge fields by a straightforward Stueckelberg transformation [66–69], which reduces the algebraic amount of work in the subsequent steps considerably. At the same time this framework remains appropriate also for cases in which heavy Higgs bosons may not decouple completely.
2. *Separation of hard and soft field modes.*
   Considering all fields consistently in momentum space, it is possible to additively split the quantum parts H of the heavy fields into field modes with small or large momenta, which we dub "light (soft) modes" H_l and "heavy (hard) modes" H_h, respectively, i.e. $H = H_l + H_h$. Diagrammatically this splitting expresses the large-mass expansion of Feynman graphs using the method of regions [27,28] in the framework of dimensional regularization. Each one-loop diagram with at least one internal heavy particle line is decomposed into a part with small and a part with large loop momentum q (carried by H_l and H_h, respectively). In the large-mass expansion the former contribution arises from a Taylor expansion of the loop integrand in $q/M, p_i/M, m_i/M \to 0$, where $\{p_i\}$ are the external momenta, m_i the small masses in the theory, and M represents the heavy particle mass. In the region of large loop momenta one expands the integrand only in $p_i/M, m_i/M \to 0$ (but not in q/M) and is thus left with vacuum-type integrals. In the EFT the small-momentum regions are reproduced by loop diagrams with insertions of (higher-dimensional) effective operators, while the large-momentum contributions are contained in the loop corrections to the Wilson coefficients of these operators.
   At one loop, the splitting of loop diagrams into two integration domains of small and large momenta can be interpreted as a splitting of the path integral into two functional

integrals extending over light and heavy field modes. The consistent mode separation according to the method of regions is a conceptual generalization of the procedure of Refs. [14,15], where only heavy field modes appeared in the calculation of the non-decoupling effects in the leading-order Lagrangian at one loop. This mode separation has also been suggested in Ref. [26] (and applied in Refs. [29,39]) within a procedure to calculate the different parts in the effective Lagrangian, but we consider our formulation in terms of heavy and light modes of background and quantum fields and their different treatments in the path integral conceptually more transparent.

3. *Integrating out the hard modes of the heavy quantum fields in the path integral.*
   Since the part of the Lagrangian that is relevant at one-loop order is only quadratic in the quantum fields, the path integral over the heavy field modes H_h of the heavy quantum field is of Gaussian type and can be done analytically. The major complication in this step is the fact that there are also terms that are linear in the heavy quantum fields H_h. As we show below, these terms can be removed by a field redefinition of the hard quantum field modes (of the light particles) in a fully algorithmic manner. This means that the resulting part of the Lagrangian quadratic in the heavy quantum field can be directly identified based again on a simple power-counting argument. This algorithmic handling, which has also been realized in Refs. [26] (see also Ref. [29] and, for a slightly different approach, Ref. [39]), establishes an important technical improvement over the procedure described in Refs. [14,15], where the "diagonalization" of the Lagrangian was performed via a non-trivial series of individual field shifts.
   The result of the straightforward (Gaussian) path integration is a functional determinant that is expanded for $M \to \infty$. The terms emerging from this expansion are exactly the vacuum-type integrals from the large-momentum regions in the large-mass expansion of the Feynman graphs described above and produce the one-loop contributions to the Wilson coefficients of the local effective operators.

4. *Equations of motion for the soft modes of the heavy fields and renormalization.*
   After the heavy modes H_h of the heavy field have been integrated out, the effective Lagrangian still involves the light modes H_l and $\hat{H}_l$ of the quantum and background fields of the heavy particle. As their momenta are much smaller than their mass M, they do not represent dynamical degrees of freedom of the EFT. In fact, they can conveniently be removed from the effective Lagrangian by applying their equations of motions (EOMs) in the large-mass expansion. This procedure can be viewed as a saddle-point approximation in the path integral over the light modes of the heavy quantum field combined with a large-mass expansion. It expresses the light modes of the heavy field in terms of all other light fields. The actual effect of these modes is revealed at a later stage in the perturbative evaluation of Green functions when effective tree-level couplings are inserted into EFT loop diagrams.
   Like the previous one, this step requires a proper power-counting of all parameters and fields in the limit $M \to \infty$. We emphasize that in order to obtain a consistent effective Lagrangian the large-mass expansion must be carefully performed taking into account that the full-theory renormalization constants may have a different scaling behaviour for $M \to \infty$ than the corresponding renormalized quantities.
   As mentioned above, in Refs. [14,15] the light mode H_l of the heavy SM Higgs field was irrelevant and ignored, i.e. the insertion of effective tree-level vertices into loops did not occur at the considered order in the heavy-mass expansion.

5. *Final form of the effective Lagrangian.*
   The effective Lagrangian resulting from the previous steps only involves light background and quantum fields, but none of the modes of the heavy fields H. The Lagrangian consists of four different types of contributions:

   (i) a tree-level part that depends only on light background fields;
   (ii) a tree-level part that depends both on light background and quantum fields;
   (iii) a part involving renormalization constants and light background fields;
   (iv) a part involving the one-loop corrections (from heavy loops) to the effective operators built from light background fields.

   Parts (i) and (ii) combine to a single effective Lagrangian at lowest order in the coupling constants, which can be used to evaluate tree-level amplitudes at different orders in the large-mass expansion and one-loop contributions resulting from insertions of effective vertices in loop diagrams (reproducing the soft momentum regions of loops in the full theory). Parts (iii) and (iv) combine to the one-loop correction to the effective Lagrangian, i.e. all NLO contributions to the Wilson coefficients of the effective operators (reproducing the hard momentum regions of loops in the full theory).
   To obtain a more transparent and compact form of the final effective Lagrangian two further steps are useful. Firstly, the EOMs of the light fields might be used to eliminate redundant effective operators that only influence off-shell Green functions, but no physical scattering

amplitudes. This step is, in particular, necessary to bring the effective Lagrangian into standard SMEFT form. Secondly, corrections to operators already present in the (light-particle sector of the) underlying full theory can be eliminated by absorbing their effect into renormalization constants of the low-energy theory as far as possible. In a decoupling scenario this means that the final effective Lagrangian differs from the SM only by operators with dimensions higher than four.

3 Heavy Higgs boson in a Higgs singlet extension of the standard model

3.1 The singlet Higgs extension

For the formal description of the singlet Higgs extension of the SM (SESM), which was formulated in slightly different versions in Refs. [43–50], we follow the notation and conventions of Refs. [49,50] and employ a matrix-valued non-linear representation of the Higgs doublet as defined in Refs. [14,15],

$$\Phi = \frac{1}{\sqrt{2}}\left(v_2+h_2\right)U,\;\; U=\exp\left(2\mathrm{i}\frac{\varphi}{v_2}\right),\;\; \varphi=\frac{1}{2}\varphi_a\tau_a, \tag{1}$$

where τ_a are the Pauli matrices and the usual convention for the summation over repeated indices is used throughout the paper. Here, h_2 denotes the field of the physical Higgs boson and v_2 the corresponding vacuum expectation value (vev). The real Goldstone fields φ_a are related to their counterparts $(\phi^\pm,\chi)$ in the linear representation (as used in Refs. [24,25,70]) by

$$\phi^\pm = \frac{1}{\sqrt{2}}\left(\varphi_2 \pm \mathrm{i}\varphi_1\right), \quad \chi=-\varphi_3. \tag{2}$$

The covariant derivative of Φ (and analogously of U) reads

$$D_\mu\Phi = \partial_\mu\Phi - \mathrm{i}g_2W_\mu\Phi - \mathrm{i}g_1\Phi\frac{\tau_3}{2}B_\mu, \tag{3}$$

with $W_\mu = W^a_\mu\tau_a/2$ and g_2 denoting the SU(2) gauge field and coupling, respectively, and B^μ, g_1 the U(1) gauge field and coupling. The conventions in the SM part of the SESM follow Refs. [15,24,25,70]. The Higgs sector of the SESM Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{Higgs}} = {}& \frac{1}{2}\,\mathrm{tr}\left[\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right)\right] + \frac{1}{2}\mu_2^2\,\mathrm{tr}\left[\Phi^\dagger\Phi\right] \\ & - \frac{1}{16}\lambda_2\,\mathrm{tr}\left[\Phi^\dagger\Phi\right]^2 + \frac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right) \\ & + \mu_1^2\sigma^2 - \lambda_1\sigma^4 - \frac{1}{2}\lambda_{12}\sigma^2\,\mathrm{tr}\left[\Phi^\dagger\Phi\right],\end{aligned} \tag{4}$$

where a new real scalar field σ is introduced which transforms as a singlet under the SM gauge groups. The field σ is split into its vev v_1 and its field excitation h_1 according to

$$\sigma = v_1 + h_1. \tag{5}$$

A $\mathbb{Z}_2$ symmetry under the transformation $\sigma \to -\sigma$ is assumed, so that only three new parameters, namely the mass parameter μ_1^2, the self-coupling parameter λ_1, and the mixed coupling parameter λ_{12} occur. In analogy to the Higgs sector of the SM the mass parameters fulfill $\mu_{1,2}^2 > 0$, and the coupling parameters are constrained by the vacuum stability conditions

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1\lambda_2 - \lambda_{12}^2 > 0. \tag{6}$$

The Higgs fields h, H corresponding to mass eigenstates are obtained by a rotation with the mixing angle α,

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} c_\alpha & s_\alpha \\ -s_\alpha & c_\alpha \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{7}$$

where $s_\alpha \equiv \sin(\alpha)$ and $c_\alpha \equiv \cos(\alpha)$. The Higgs-boson masses expressed in terms of the original parameters are

$$\begin{aligned} M_{\mathrm{h}}^2 &= \frac{1}{2}v_2^2\lambda_2 - 2v_1v_2\lambda_{12}\frac{s_\alpha}{c_\alpha}, \\ M_{\mathrm{H}}^2 &= \frac{1}{2}v_2^2\lambda_2 + 2v_1v_2\lambda_{12}\frac{c_\alpha}{s_\alpha}, \end{aligned} \tag{8}$$

where we enforce the mass hierarchy $M_{\mathrm{H}} > M_{\mathrm{h}}$ without loss of generality by choosing the range for the mixing angle according to

$$0 \le \alpha < \frac{\pi}{2} \text{ for } \lambda_{12} \ge 0, \text{ and } -\frac{\pi}{2} < \alpha < 0 \text{ for } \lambda_{12} < 0. \tag{9}$$

This leaves us with a SM-like Higgs field h with mass M_{h} and an additional heavier Higgs field H with mass $M_{\mathrm{H}} > M_{\mathrm{h}}$.[3] The reparametrization of the doublet and singlet fields in Eqs. (1) and (5) leads to the tadpole terms $t_{\mathrm{h}}h$ and $t_{\mathrm{H}}H$ with

$$t_{\mathrm{h}} = c_\alpha t_2 - s_\alpha t_1, \quad t_{\mathrm{H}} = s_\alpha t_2 + c_\alpha t_1, \tag{10}$$

in the Lagrangian, where

$$\begin{aligned} t_1 &= v_1\left(2\mu_1^2 - v_2^2\lambda_{12} - 4v_1^2\lambda_1\right), \\ t_2 &= \frac{v_2}{4}\left(4\mu_2^2 - 4v_1^2\lambda_{12} - v_2^2\lambda_2\right). \end{aligned} \tag{11}$$

At the bare (tree) level two parameters of the theory (here $v_{1,2}$) are fixed by requiring $t_{\mathrm{h}} = t_{\mathrm{H}} = 0$. At loop level the

[3] In principle, it is also possible to identify the heavier state H with the observed Higgs particle of mass 125 GeV, but we do not consider this (experimentally disfavoured) possibility here, because we want to analyze the heavy-mass limit of the second Higgs boson of the SESM.

tadpole terms play an important role in the course of renormalization as described in Sect. 5.

In the definition of a specific scenario for the limit $M_\mathrm{H} \to \infty$, i.e. in defining the scaling behaviour of the BSM parameters, it is useful to introduce a (dimensionless) power counting parameter $\zeta \sim M_\mathrm{H}/M_\mathrm{h} \to \infty$ that keeps track of the scaling with the heavy mass M_H. Also, it will be more transparent from now on to work with phenomenologically motivated input parameters rather than the fundamental parameters of the Lagrangian, i.e. we express the BSM parameters $\{\mu_1^2, \lambda_1, \lambda_{12}\}$ in terms of $\{M_\mathrm{H}, s_\alpha, \lambda_{12}\}$. Note that s_α is most directly related to the measured *signal strengths* of Higgs production cross sections and decay widths, which are defined by ratios of measured quantities and SM predictions. Before we define a specific large-M_H scenario of the SESM, we introduce the scaling powers a, l for s_α and λ_{12} as follows,

$$s_\alpha \sim \zeta^{-a}, \quad \lambda_{12} \sim \zeta^{-l}. \tag{12}$$

The effect of this rescaling on the fundamental parameters of the theory can be calculated from their relations to $\{M_\mathrm{H}, s_\alpha, \lambda_{12}\}$, as given in Eq. (2.15) of Ref. [49], leading to

$$\begin{aligned} &v_1 \sim \zeta^{2-a+l}, \quad \mu_1^2 \sim \zeta^{\max\{2,-2a,-l\}},\\ &\lambda_1 \sim \zeta^{\max\{2a-2l-2,-2l-4\}},\\ &v_2 \sim \zeta^0, \quad \mu_2^2 \sim \zeta^{\max\{4-2a+l,2-2a,0\}},\\ &\lambda_2 \sim \zeta^{\max\{2-2a,0\}}. \end{aligned} \tag{13}$$

In the following, we consider the scenario $a = 1,\ l = 0$, i.e.

$$v_2, \lambda_1, \lambda_2, \lambda_{12} \sim \zeta^0, \quad v_1^2, \mu_1^2, \mu_2^2 \sim \zeta^2, \tag{14}$$

in which all mass parameters of the scalar sector are considered to be large, with the exception of the vev v_2, which is tied to the known W-boson mass. Self-consistency of the scaling can be checked by applying Eq. (14) to the relation

$$\frac{s_{2\alpha}}{c_{2\alpha}} = \frac{8 v_1 v_2 \lambda_{12}}{16 v_1^2 \lambda_1 - v_2^2 \lambda_2}, \tag{15}$$

following from the diagonalization of the Higgs mass matrix [49]. This shows that s_α is naturally suppressed according to $s_\alpha \sim \zeta^{-1}$ in agreement with Eq. (12) for $a = 1$, see also Ref. [42]. This is a weakly coupled scenario, providing the minimal suppression that is required to still deliver a viable description of Higgs data, which show that the above-mentioned signal strengths are close to one, i.e. s_α has to be small. In particular, due to the ζ^{-1} suppression of the mixing angle, h equals the SM-type Higgs field h_2 at leading order in the large-mass expansion.

Other physically interesting limits are conceivable, such as the strong-coupling scenario

$$v_1, v_2 \sim \zeta^0, \quad \mu_1^2, \mu_2^2, \lambda_1, \lambda_2, \lambda_{12} \sim \zeta^2, \tag{16}$$

in which $s_\alpha \sim \zeta^0$, so that (for $s_\alpha \neq 0$) the low-energy theory does not coincide with the SM in this case. We will not consider such scenarios in this paper, although the proposed method would be capable of handling also such scenarios as long as perturbativity is guaranteed. For a tree-level study of the low-energy limit of the SESM in a non-decoupling scenario see e.g. Ref. [60].

3.2 Background-field formulation and non-linear realization

Applying the background-field transformation splits each field ϕ into a classical background field $\hat{\phi}$ and a quantum field ϕ. Gauge and physical Higgs fields are split additively,

$$\phi \to \tilde{\phi} = \hat{\phi} + \phi, \tag{17}$$

but the non-linearly parametrized matrix of the Goldstone-boson fields splits multiplicatively as [14,15]

$$U \to \tilde{U} = \hat{U} U. \tag{18}$$

Owing to the unitarity of $\tilde{U}$, the combined Lagrangian of the gauge and Higgs sectors of the SESM can be written as

$$\begin{aligned} \mathcal{L}_\text{gauge+Higgs} = &-\frac{1}{2}\,\mathrm{tr}\left[\tilde{W}_{\mu\nu}\tilde{W}^{\mu\nu}\right] - \frac{1}{4}\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu}\\ &+\frac{1}{4}\left(v_2+\tilde{h}_2\right)^2 \mathrm{tr}\left[\left(\tilde{D}_\mu\tilde{U}\right)^\dagger\left(\tilde{D}^\mu\tilde{U}\right)\right]\\ &+\frac{1}{2}\left(\partial_\mu\tilde{h}_2\right)\left(\partial^\mu\tilde{h}_2\right) + \frac{1}{2}\mu_2^2\left(v_2+\tilde{h}_2\right)^2\\ &-\frac{1}{16}\lambda_2\left(v_2+\tilde{h}_2\right)^4\\ &+\frac{1}{2}\left(\partial_\mu\tilde{h}_1\right)\left(\partial^\mu\tilde{h}_1\right) + \mu_1^2\left(v_1+\tilde{h}_1\right)^2\\ &-\lambda_1\left(v_1+\tilde{h}_1\right)^4\\ &-\frac{1}{2}\lambda_{12}\left(v_2+\tilde{h}_2\right)^2\left(v_1+\tilde{h}_1\right)^2, \end{aligned} \tag{19}$$

where the Goldstone fields occur only in their kinetic term, but not in the Higgs potential. In the BFM, separate gauge choices can be made for background and quantum fields. To eliminate the background Goldstone fields, the unitary gauge is chosen for the background fields which can be achieved by a generalized Stueckelberg transformation [66–69]

$$\begin{aligned} &\hat{W}_\mu \to \hat{U}\hat{W}_\mu\hat{U}^\dagger + \frac{\mathrm{i}}{g_2}\hat{U}\partial_\mu\hat{U}^\dagger, \quad W_\mu \to \hat{U} W_\mu \hat{U}^\dagger,\\ &\hat{B}_\mu \to \hat{B}_\mu, \quad B_\mu \to B_\mu. \end{aligned} \tag{20}$$

This transforms the covariant derivative and the field-strength tensor according to

$$\left(\tilde{D}_\mu\tilde{U}\right) \to \hat{U}\left(\tilde{D}_\mu U\right), \quad \tilde{F}_{\mu\nu} \to \hat{U}\tilde{F}_{\mu\nu}\hat{U}^\dagger. \tag{21}$$

The inversion of this Stueckelberg transformation, which restores the background Goldstone fields, is straightforward (see e.g. Sec. 5 of Ref. [15]). The gauge of the quantum fields is fixed as in the SM by an explicit gauge-fixing term. We choose [14,15]

$$\mathcal{L}_{\text{fix}} = -\frac{1}{\xi_W}\,\text{tr}\left[\left(\hat{D}_W^{\mu} W_{\mu} + \frac{1}{2}\xi_W g_2 v_2 \hat{U}\varphi\hat{U}^{\dagger}\right)^2\right] - \frac{1}{2\xi_B}\left(\partial^{\mu} B_{\mu} + \frac{1}{2}\xi_B g_1 v_2 \varphi_3\right)^2, \tag{22}$$

where

$$\hat{D}_W^{\mu}\phi = \partial^{\mu}\phi - \mathrm{i}g_2\left[\hat{W}^{\mu}, \phi\right], \tag{23}$$

for any field ϕ in the adjoint representation of SU(2). The two gauge parameters for the fields W^{μ} and B^{μ} are set equal, $\xi_W = \xi_B = \xi$, to avoid mixing in the tree-level propagators.

4 Integrating out the heavy Higgs boson

4.1 Separation of hard and soft modes of the heavy Higgs field

Although usually formulated in terms of momentum domains in Feynman integrals, the expansion by regions [27,28] provides the ideal framework for separating light and heavy field modes. At the level of a one-loop Feynman integral I with loop momentum p, the idea is to divide the integral domain into two disjunct parts containing small ($p \sim M_{\text{h}}$) or large ($p \sim M_{\text{H}} \gg M_{\text{h}}$) momenta,

$$I = \int \mathrm{d}^D p\, f(p) = \int\limits_{p\sim M_{\text{h}}} \mathrm{d}^D p\, f(p) + \int\limits_{p\sim M_{\text{H}}} \mathrm{d}^D p\, f(p), \tag{24}$$

where f denotes an arbitrary integrand. In four dimensions ($D = 4$), a momentum cutoff $M_{\text{H}} \gg \Lambda \gg M_{\text{h}}$ has to be introduced to sharply separate the two integration domains. In dimensional regularization ($D \neq 4$), however, following the method of regions the separation is effectively implemented by a strict Taylor expansion of the loop integrand in $1/\zeta \sim M_{\text{h}}/M_{\text{H}}$ before integration, where $p \sim \zeta^0$ and $p \sim \zeta^1$ in the domains of small and large momenta, respectively. To formalize this, we introduce the Taylor operators $\mathcal{T}_l(p)$ and $\mathcal{T}_h(p)$ by

$$\begin{aligned}
&\mathcal{T}_l(p)\, f(p, p_i, M_{\text{H}}, m_i, c_i)\\
&\quad = \left[\exp\left(\frac{\partial}{\partial\xi_l}\right) f(\xi_l p, \xi_l p_i, M_{\text{H}}, \xi_l m_i, \xi_l c_i)\right]_{\xi_l\to 0},\\
&\mathcal{T}_h(p)\, f(p, p_i, M_{\text{H}}, m_i, c_i)\\
&\quad = \left[\exp\left(\frac{\partial}{\partial\xi_h}\right) f(p, \xi_h p_i, M_{\text{H}}, \xi_h m_i, \xi_h c_i)\right]_{\xi_h\to 0},
\end{aligned} \tag{25}$$

where p_i^{μ} generically represents any small external momenta and $m_i \ll M_{\text{H}}$ stands for any small masses. In view of our functional approach, c.f. Eq. (42), we have generalized the integrand here to include additional quantities $c_i \sim \zeta^n$ with $n \leq 0$ like light background fields and their derivatives. The integral I of Eq. (24), thus, reads

$$I = \int \mathrm{d}^D p\, \mathcal{T}_l(p)\, f(p) + \int \mathrm{d}^D p\, \mathcal{T}_h(p)\, f(p). \tag{26}$$

Formally, the operators can be interpreted as orthogonal projectors, since they obey the relations

$$\begin{aligned}
&[\mathcal{T}_l(p)]^k = \mathcal{T}_l(p), \quad [\mathcal{T}_h(p)]^k = \mathcal{T}_h(p), \quad k \in \mathbb{N},\\
&\mathcal{T}_l(p) + \mathcal{T}_h(p) = 1,\\
&\mathcal{T}_l(p)\,\mathcal{T}_h(p) = \mathcal{T}_h(p)\,\mathcal{T}_l(p) = 0,
\end{aligned} \tag{27}$$

where the last relation holds, because the successive application of $\mathcal{T}_l(p)$ and $\mathcal{T}_h(p)$ (or vice versa) produces scaleless integrals which vanish in dimensional regularization.

Carrying the concept over to a generic quantum field variable $\phi(p)$ in momentum space, we define

$$\phi_l(p) = \mathcal{T}_l(p)\,\phi(p), \quad \phi_h(p) = \mathcal{T}_h(p)\,\phi(p), \tag{28}$$

so that $\phi(p) = \phi_l(p) + \phi_h(p)$. We stress that this definition only makes sense if the momentum p is eventually integrated over in D dimensions (in the course of a loop calculation). The separation into light and heavy modes in momentum space can be translated to the field $\phi(x)$ in position space via Fourier transformation (with unit Jacobian determinant in the path integral),

$$\begin{aligned}
\phi_l(x) &= \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}px}\, \phi_l(p),\\
\phi_h(x) &= \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}px}\, \phi_h(p),
\end{aligned} \tag{29}$$

so that $\phi(x) = \phi_l(x) + \phi_h(x)$. The parameter μ denotes the arbitrary reference scale of dimensional regularization, which is introduced to keep the mass dimensions of quantities at the same values as for $D = 4$. This additive separation implies a factorization of the path-integral measure of ϕ into factors corresponding to light and heavy modes,

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi_l \int \mathcal{D}\phi_h. \tag{30}$$

Finally, we apply this mode separation to the heavy Higgs field $\tilde{H} = H + \hat{H}$ in the BFM,

$$\begin{aligned}\tilde{H}(x) &= H(x) + \hat{H}(x)\\ &= H_l(x) + H_h(x) + \hat{H}_l(x) + \hat{H}_h(x),\end{aligned} \tag{31}$$

which gets decomposed into four contributions. Since we apply the EFT only for energies well below $M_{\rm H}$, the tree lines in Feynman diagrams only carry small momenta and we effectively have $\hat{H}_h = 0$.

For transparency, we will split the the effective Lagrangian $\mathcal{L}_{\rm eff}$ into the *tree-level effective Lagrangian* $\mathcal{L}_{\rm eff}^{\rm tree}$, which contains all tree-level effects of the heavy field (resulting from $\hat{H}_l$) and provides the effective couplings to be inserted in loops (via H_l), and the *one-loop effective Lagrangian* $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$, which contains all the (local) one-loop effects of the heavy field at large momentum transfer (via H_h). Another, third part of the Lagrangian, $\delta\mathcal{L}_{\rm eff}^{\rm ct}$ emerges in the course of renormalization, see Sect. 5.

4.2 Path integral over the hard modes of the heavy quantum Higgs field

4.2.1 Relevant terms in the Lagrangian

The goal of this section is to carry out the path integral over the quantum field $H_h(x)$ at the one-loop level. To this end, we first isolate all terms in the full-theory Lagrangian that are bilinear in the quantum fields and call the resulting part of the Lagrangian $\mathcal{L}^{\text{1-loop}}$,

$$\begin{aligned}\mathcal{L}^{\text{1-loop}} = &-\frac{1}{2}h_2\Delta_{h_2}h_2 - \frac{1}{2}h_1\Delta_{h_1}h_1 + \mathrm{tr}\left[\overline{W}_\mu\Delta_{\overline{W}}^{\mu\nu}\overline{W}_\nu\right]\\ &+\frac{1}{2}A_\mu\Delta_A^{\mu\nu}A_\nu - \mathrm{tr}\left[\varphi\Delta_\varphi\varphi\right]\\ &+h_1X_{h_1h_2}h_2 + h_2\,\mathrm{tr}\left[X^\mu_{h_2\overline{W}}\overline{W}_\mu\right] + h_2\,\mathrm{tr}\left[X_{h_2\varphi}\varphi\right]\\ &+\mathrm{tr}\left[A_\mu X^{\mu\nu}_{A\overline{W}}\overline{W}_\nu\right] + \mathrm{tr}\left[\overline{W}_\mu X^\mu_{\overline{W}\varphi}\varphi\right]\\ &+\mathrm{tr}\left[A_\mu X^\mu_{A\varphi}\varphi\right] + \mathcal{L}^{\text{1-loop}}_{\rm ghost},\end{aligned} \tag{32}$$

where

$$\overline{W}^\mu = \frac{1}{2}\left(W_1^\mu\tau_1 + W_2^\mu\tau_2 + Z^\mu\tau_3\right) \tag{33}$$

and $\mathcal{L}^{\text{1-loop}}_{\rm ghost}$ comprises all relevant terms containing Faddeev–Popov ghost fields. Since we are not interested in Green functions with external ghost fields, $\mathcal{L}^{\text{1-loop}}_{\rm ghost}$ consists of monomials with exactly two quantum ghost fields and any additional background fields. The Lagrangian $\mathcal{L}^{\text{1-loop}}_{\rm ghost}$ will play no role when integrating out the heavy quantum field $H(x)$, because ghost fields and H fields can never appear in the loop part of the same one-loop diagram.

The relevant Δ- and X-operators in Eq. (32) are given by

$$\begin{aligned}\Delta_{h_1} &= \Box - 2\mu_1^2 + 12\lambda_1\big(v_1+\hat{h}_1\big)^2 + \lambda_{12}\big(v_2+\hat{h}_2\big)^2,\\ \Delta_{h_2} &= \Box - \mu_2^2 + \frac{3}{4}\lambda_2\big(v_2+\hat{h}_2\big)^2 + \lambda_{12}\big(v_1+\hat{h}_1\big)^2 - \frac{1}{2}g_2^2\,\mathrm{tr}\big[\hat{C}^2\big],\\ \Delta_\varphi &= \hat{D}_\mu\left(1+\frac{\hat{h}_2}{v_2}\right)^2\hat{D}^\mu + g_2^2\left(1+\frac{\hat{h}_2}{v_2}\right)^2\hat{C}^2\\ &\quad+\xi M_W^2\left(1+\frac{s_{\rm w}^2}{c_{\rm w}^2}P_3\right),\\ \Delta^{\mu\nu}_{\overline{W},0} &= g^{\mu\nu}\Box + \frac{1-\xi}{\xi}\partial^\mu\partial^\nu + g^{\mu\nu}M_W^2\left(1+\frac{s_{\rm w}^2}{c_{\rm w}^2}P_3\right),\\ X_{h_1h_2} &= -2\lambda_{12}\big(v_1+\hat{h}_1\big)\big(v_2+\hat{h}_2\big),\\ X^\mu_{h_2\overline{W}} &= 2g_2\left(1+\frac{\hat{h}_2}{v_2}\right)M_W\hat{C}^\mu\left(1+\frac{1-c_{\rm w}}{c_{\rm w}}P_3\right),\\ X_{h_2\varphi} &= 2g_2\left(1+\frac{\hat{h}_2}{v_2}\right)\big(\mathrm{i}g_1\hat{B}_\mu\tau_3\hat{W}^\mu - \hat{C}_\mu\partial^\mu\big),\\ X_{\overline{W}\varphi,0} &= -g_2\hat{h}_2\left(2+\frac{\hat{h}_2}{v_2}\right)\left(1+\frac{1-c_{\rm w}}{c_{\rm w}}P_3\right)\partial^\mu,\end{aligned} \tag{34}$$

where (we suppressed 2×2 unit matrices for compactness and) $\hat{C}^2 = \hat{C}^\mu\hat{C}_\mu$ with

$$\hat{C}^\mu = \hat{W}^\mu + \frac{s_{\rm w}}{c_{\rm w}}\hat{B}^\mu\frac{\tau_3}{2} = \frac{1}{2}\left(\hat{W}_1^\mu\tau_1 + \hat{W}_2^\mu\tau_2 + \frac{1}{c_{\rm w}}\hat{Z}^\mu\tau_3\right). \tag{35}$$

Moreover, we have introduced the operators P_a projecting any 2×2 matrix M onto the Pauli matrix τ_a,

$$P_aM = \frac{\tau_a}{2}\,\mathrm{tr}\left[\tau_aM\right]. \tag{36}$$

Note that we have not given Δ_A, $X^{\mu\nu}_{A\overline{W}}$, and $X^\mu_{A\varphi}$ explicitly, because they will not be needed for our purposes as will become clear below. Likewise, for $\Delta^{\mu\nu}_{\overline{W}}$ and $X_{\overline{W}\varphi}$ we show only the leading-order terms indicated by the subscript "0", because the rest will not be needed in the derivation of the effective Lagrangian to $\mathcal{O}(\zeta^{-2})$. In fact, for the purpose of the present paper, where we only aim for the $\mathcal{O}(\zeta^0)$ effective Lagrangian, $\Delta^{\mu\nu}_{\overline{W},0}$ and $X_{\overline{W}\varphi,0}$ are not required either.

Since we want to integrate out the heavy field H, we have to express $\mathcal{L}^{\text{1-loop}}$ in terms of the Higgs fields corresponding to mass eigenstates as defined in Eq. (7),

$$\begin{aligned}\mathcal{L}^{\text{1-loop}} = &-\frac{1}{2}H\Delta_HH - \frac{1}{2}h\Delta_hh + \mathrm{tr}\left[\overline{W}_\mu\Delta^{\mu\nu}_{\overline{W}}\overline{W}_\nu\right]\\ &+\frac{1}{2}A_\mu\Delta_A^{\mu\nu}A_\nu - \mathrm{tr}\left[\varphi\Delta_\varphi\varphi\right]\\ &+HX_{Hh}h + H\,\mathrm{tr}\left[X^\mu_{H\overline{W}}\overline{W}_\mu\right] + h\,\mathrm{tr}\left[X^\mu_{h\overline{W}}\overline{W}_\mu\right]\\ &+H\,\mathrm{tr}\left[X_{H\varphi}\varphi\right] + h\,\mathrm{tr}\left[X_{h\varphi}\varphi\right]\\ &+\mathrm{tr}\left[A_\mu X^{\mu\nu}_{A\overline{W}}\overline{W}_\nu\right] + \mathrm{tr}\left[\overline{W}_\mu X^\mu_{\overline{W}\varphi}\varphi\right]\\ &+\mathrm{tr}\left[A_\mu X^\mu_{A\varphi}\varphi\right] + \mathcal{L}^{\text{1-loop}}_{\rm ghost},\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Delta_H &= s_\alpha^2 \Delta_{h_2} + c_\alpha^2 \Delta_{h_1} - 2c_\alpha s_\alpha X_{h_1h_2},\\ \Delta_h &= c_\alpha^2 \Delta_{h_2} + s_\alpha^2 \Delta_{h_1} + 2c_\alpha s_\alpha X_{h_1h_2},\\ X_{Hh} &= c_\alpha s_\alpha \left(\Delta_{h_1} - \Delta_{h_2}\right) + \left(c_\alpha^2 - s_\alpha^2\right) X_{h_1h_2},\\ X^\mu_{H\overline{W}} &= s_\alpha X^\mu_{h_2\overline{W}}, \quad X^\mu_{h\overline{W}} = c_\alpha X^\mu_{h_2\overline{W}},\\ X_{H\varphi} &= s_\alpha X_{h_2\varphi}, \quad X_{h\varphi} = c_\alpha X_{h_2\varphi}. \end{aligned} \tag{38}$$

Up to this point we have not yet split the quantum fields into light and heavy modes in the Lagrangian $\mathcal{L}^{\text{1-loop}} = \mathcal{L}^{\text{1-loop}}(H, \phi_i)$, where ϕ_i denotes any quantum field other than H. At the one-loop level, we can simply write

$$\mathcal{L}^{\text{1-loop}}(H, \phi_i) = \mathcal{L}^{\text{1-loop}}(H_h, \phi_{i,h}) + \mathcal{L}^{\text{1-loop}}(H_l, \phi_{i,l}), \tag{39}$$

because one-loop diagrams split into two contributions corresponding to large and small loop momenta. This can also be understood from Eq. (27). At the same time recall that all background fields should be interpreted as light modes, because momenta on external and on tree lines in diagrams are assumed to be small.

4.2.2 Diagonalization and functional integration

Our next step is to express the Lagrangian $\mathcal{L}^{\text{1-loop}}(H_h, \phi_{i,h})$ for the heavy modes of the quantum fields in the ("diagonal") form

$$\mathcal{L}^{\text{1-loop}}(H_h, \phi_{i,h}) = -\frac{1}{2} H_h \tilde{\Delta}_H H_h + \mathcal{L}^{\text{1-loop}}_{\text{rem}}(\phi_{i,h}), \tag{40}$$

where $\mathcal{L}^{\text{1-loop}}_{\text{rem}}(\phi_{i,h})$ does not depend on $H_h(x)$, but only on the other quantum fields generically denoted $\phi_{i,h}$. This can be achieved by a suitable linear field redefinition of the $\phi_{i,h}$ as demonstrated below. Of course, both $\tilde{\Delta}_H$ and $\mathcal{L}^{\text{1-loop}}_{\text{rem}}$ also depend on all background fields including $\hat{H}_l(x)$. The diagonalization of $\mathcal{L}^{\text{1-loop}}(H_h, \phi_{i,h})$ w.r.t. H_h transforms the functional integral over $H_h(x)$ into an integral of Gaussian type, which can be evaluated as

$$\begin{aligned}&\int \mathcal{D}H_h \exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, H_h \tilde{\Delta}_H H_h\right\}\\ &\quad\propto \left\{\mathcal{D}et_h\Big[\delta(x-y)\tilde{\Delta}_H(x,\partial_x)\Big]\right\}^{-\frac{1}{2}}\\ &\quad= \exp\left\{-\frac{1}{2}\mathcal{T}r_h\Big[\ln\Big(\delta(x-y)\tilde{\Delta}_H(x,\partial_x)\Big)\Big]\right\}\\ &\quad= \exp\left\{\mathrm{i}\,\mu^{D-4}\int \mathrm{d}^Dx\, \delta\mathcal{L}^{\text{1-loop}}_{\text{eff}}\right\}. \end{aligned} \tag{41}$$

Dropping an irrelevant constant contribution the one-loop effective Lagrangian $\delta\mathcal{L}^{\text{1-loop}}_{\text{eff}}$ describing the hard loop contributions of $H_h(x)$ can thus be obtained by translating the functional determinant $\mathcal{D}et_h$ of the differential operator $\tilde{\Delta}_H(x, \partial_x)$ into a functional trace $\mathcal{T}r_h$, which in turn can be evaluated in terms of a hard momentum-space integral. The subscript h of $\mathcal{D}et_h$ and $\mathcal{T}r_h$ indicates the restriction to the subspace of large-momentum modes. We explain the details of the corresponding functional manipulations in Appendix A and proceed with the well-known result

$$\delta\mathcal{L}^{\text{1-loop}}_{\text{eff}} = \frac{\mathrm{i}}{2}\mu^{4-D}\int \frac{\mathrm{d}^Dp}{(2\pi)^D}\,\mathcal{T}_h(p)\,\ln\left(\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p)\right), \tag{42}$$

where $\mathcal{T}_h(p)$ is defined in Eq. (25). Unlike $\tilde{\Delta}_H$ the term $\mathcal{L}^{\text{1-loop}}_{\text{rem}}$ in Eq. (40) is (by construction) independent of any hard scale $\sim M_{\mathrm{H}}$. Diagonalizing it w.r.t. the $\phi_{i,h}$ and performing the corresponding functional integrations in analogy to Eq. (42) therefore yields scaleless large-momentum integrals which vanish in dimensional regularization. The part $\mathcal{L}^{\text{1-loop}}_{\text{rem}}(\phi_{i,h})$ is therefore irrelevant for the derivation of $\delta\mathcal{L}^{\text{1-loop}}_{\text{eff}}$. Note that the x dependence of $\tilde{\Delta}_H$ is only due to background fields. Thus, ∂_x only acts on background fields, which all carry small momenta, and therefore scales like ζ^{-1} relative to the large momentum p.

At this point, we need the explicit form of the differential operator $\tilde{\Delta}_H(x, \partial_x)$ which results from the diagonalization of the Lagrangian $\mathcal{L}^{\text{1-loop}}(H_h, \phi_{i,h})$ in Eq. (40). We first formulate this diagonalization in a generic way and subsequently specialize the result to our model Lagrangian. Considering Eq. (37) and suppressing the subscripts "h" indicating heavy modes in the following, $\mathcal{L}^{\text{1-loop}}(H, \phi_i)$ has the generic form

$$\mathcal{L}^{\text{1-loop}}(H, \phi_i) = -\frac{1}{2} H \Delta_H H + H \mathcal{X}_{Hi}\phi_i - \frac{1}{2}\phi_i \mathcal{A}_{ij}\phi_j \tag{43}$$

with implicit summations over the labels i, j of the light fields ϕ_i, ϕ_j which are assumed to be real ($\phi_k = \phi_k^\dagger$). Taking into account the hermiticity of $\mathcal{L}^{\text{1-loop}}$, the generic operators Δ_H, $\mathcal{A}_{ij}$, and $\mathcal{X}_{Hi}$ can be assumed to obey the relations

$$\Delta_H = \Delta_H^\dagger, \quad \mathcal{X}_{Hi} = \mathcal{X}_{iH}^\dagger, \quad \mathcal{A}_{ij} = \mathcal{A}_{ji}^\dagger. \tag{44}$$

The following shifts of the light quantum fields,

$$\phi_i \;\rightarrow\; \phi_i + \left(\mathcal{A}^{-1}\right)_{ij} \mathcal{X}_{jH}\, H, \tag{45}$$

which are inspired by the field transformations described in Refs. [14,15], have unit Jacobian in the functional integral and change the Lagrangian $\mathcal{L}^{\text{1-loop}}(H, \phi_i)$ only by terms containing H,

$$\mathcal{L}^{\text{1-loop}}(H, \phi_i) \;\rightarrow\; -\frac{1}{2} H \tilde{\Delta}_H H - \frac{1}{2}\phi_i \mathcal{A}_{ij}\phi_j \tag{46}$$

with

$$\tilde{\Delta}_H = \Delta_H - \mathcal{X}_{Hi}\left(\mathcal{A}^{-1}\right)_{ij}\mathcal{X}_{jH}. \tag{47}$$

To evaluate the inverse $\left(\mathcal{A}^{-1}\right)_{ij}$, we split the operators $\mathcal{A}_{ij}$ into large and small contributions $\mathcal{D}_{ij}$ and $\mathcal{X}_{ij}$, respectively, in the sense that all $\mathcal{X}_{ij}$ are suppressed w.r.t. all diagonal parts $\mathcal{D}_{ii}$ at least by one power of $1/\zeta$,

$$\mathcal{A}_{ij} = \mathcal{D}_{ij} - \mathcal{X}_{ij}, \qquad \mathcal{X}_{ii} = 0, \tag{48}$$

so that $\mathcal{D}_{ij}$ is invertible (but not necessarily fully diagonal).[4] Without loss of generality, we take the diagonal parts of the $\mathcal{X}_{ij}$ to vanish. Below we will relate the $\mathcal{D}_{ij}$ to the Δ_u and the $\mathcal{X}_{ij}$ to the X_{uv} of Eq. (37) (with $u, v = \overline{W}, \varphi, H, h, \ldots$). The scaling assumption holds because, upon the replacement $\partial_x \to \partial_x + \mathrm{i}p$ according to Eq. (42), the kinetic terms in the Δ_u contain at least one power of the large momentum $p \sim M_{\mathrm{H}}$ more than the interaction terms X_{ij}. As realized also in Ref. [26], the inverse of the operator of $\mathcal{A}_{ij}$, can then be expressed as ($\mathcal{D}^{-1}$ times) a Neumann series,

$$\begin{aligned}\left(\mathcal{A}^{-1}\right)_{ij} &= \left((\mathcal{D}-\mathcal{X})^{-1}\right)_{ij} = \left(\mathcal{D}^{-1}+\mathcal{D}^{-1}\mathcal{X}\mathcal{D}^{-1}\right.\\ &\qquad \left.+\mathcal{D}^{-1}\mathcal{X}\mathcal{D}^{-1}\mathcal{X}\mathcal{D}^{-1}+\ldots\right)_{ij}\\ &= \left(\mathcal{D}^{-1}\right)_{ij} + \left(\mathcal{D}^{-1}\right)_{ik}\mathcal{X}_{kl}\left(\mathcal{D}^{-1}\right)_{lj}\\ &\qquad + \left(\mathcal{D}^{-1}\right)_{ik}\mathcal{X}_{kl}\left(\mathcal{D}^{-1}\right)_{lm}\mathcal{X}_{mn}\left(\mathcal{D}^{-1}\right)_{nj} + \cdots .\end{aligned} \tag{49}$$

Here the intermediate field indices such as $k, l, \ldots$ (but not the external indices i, j) are summed over. This implies that the Lorentz and internal symmetry group indices of adjacent $\left(\mathcal{D}^{-1}\right)_{ij}$ and $\mathcal{X}_{ij}$ factors are properly contracted (except for the left- and rightmost indices).

The application of this generic diagonalization procedure to our model requires a careful identification of the operators $\mathcal{D}_{ij}$, and $\mathcal{X}_{ij}$ with their concrete realizations in Eq. (37). Explicitly writing out also the adjoint SU(2) and Lorentz indices we have the following assignments for the non-vanishing $\mathcal{D}_{ij}$, which are diagonal in the field type (but not in the Lorentz and SU(2) indices),

$$\begin{aligned}&\mathcal{D}_{\overline{W}^a_\mu \overline{W}^b_\nu} = -2\,\mathrm{tr}\Big[\tfrac{\tau_a}{2}\Delta^{\mu\nu}_{\overline{W}}\tfrac{\tau_b}{2}\Big], \quad \mathcal{D}_{A_\mu A_\nu} = -\Delta^{\mu\nu}_A,\\ &\mathcal{D}_{\varphi_a\varphi_b} = 2\,\mathrm{tr}\Big[\tfrac{\tau_a}{2}\Delta_\varphi\tfrac{\tau_b}{2}\Big], \quad \mathcal{D}_{hh} = \Delta_h.\end{aligned} \tag{50}$$

For our model the relevant Δ_u expressions on the r.h.s. are given in Eqs. (34) and (38). The inverse $\left(\mathcal{D}(x, \partial_x + \mathrm{i}p)^{-1}\right)_{ij}$ required in Eq. (49) can now be easily obtained, again in terms of a Neumann series, by realizing that its leading-order term in the ζ expansion is the usual momentum-space propagator of the respective light field with momentum p. Accordingly, for $u = \overline{W}, \varphi$ the Δ_u are, at leading order in $1/\zeta$, proportional to the unit matrix with fundamental SU(2) indices. Hence, we can also compute[5]

$$\begin{aligned}\left(\mathcal{D}^{-1}\right)_{\varphi_a\varphi_b} &= 2\,\mathrm{tr}\Big[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\Big],\\ \left(\mathcal{D}^{-1}\right)_{\overline{W}^a_\mu \overline{W}^b_\nu} &= -2\,\mathrm{tr}\Big[\tfrac{\tau_a}{2}\big(\Delta_{\overline{W}}^{-1}\big)^{\mu\nu}\tfrac{\tau_b}{2}\Big].\end{aligned} \tag{51}$$

Corresponding expansions to the order required in this work are given below. For the non-vanishing non-diagonal parts $\mathcal{X}_{ij}$ $(= \mathcal{X}^\dagger_{ji})$ we have

$$\begin{aligned}\mathcal{X}_{A_\mu \overline{W}_{a,\nu}} &= \mathrm{tr}\Big[\tfrac{\tau_a}{2}X^{\mu\nu}_{A\overline{W}}\Big], \quad \mathcal{X}_{\overline{W}_{a,\mu}\varphi_b} = \mathrm{tr}\Big[\tfrac{\tau_a}{2}X^\mu_{\overline{W}\varphi}\tfrac{\tau_b}{2}\Big],\\ \mathcal{X}_{A_\mu\varphi_a} &= \mathrm{tr}\Big[X^\mu_{A\varphi}\tfrac{\tau_a}{2}\Big], \quad \mathcal{X}_{Hh} = X_{Hh},\\ \mathcal{X}_{H\overline{W}_{a,\mu}} &= \mathrm{tr}\Big[X^\mu_{H\overline{W}}\tfrac{\tau_a}{2}\Big], \quad \mathcal{X}_{h\overline{W}_{a,\mu}} = \mathrm{tr}\Big[X^\mu_{h\overline{W}}\tfrac{\tau_a}{2}\Big],\\ \mathcal{X}_{h\varphi_a} &= \mathrm{tr}\Big[X_{h\varphi}\tfrac{\tau_a}{2}\Big], \quad \mathcal{X}_{H\varphi_a} = \mathrm{tr}\Big[X_{H\varphi}\tfrac{\tau_a}{2}\Big].\end{aligned} \tag{52}$$

4.2.3 *Large-mass expansion*

Aiming at a final effective Lagrangian $\delta\mathcal{L}^{\text{1-loop}}_{\text{eff}}$ that includes all (non-decoupling) effects of $\mathcal{O}(M^0_{\mathrm{H}})$, we need for the calculation of $\ln\left(\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p)\right)$ in Eq. (42), where $p \sim M_{\mathrm{H}} \sim \zeta$, all terms of order ζ^{-4}. Since

$$\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p) = -\left(p^2 - M^2_{\mathrm{H}}\right) + \Pi(x, p, \partial_x) \tag{53}$$

with $\Pi(x, p, \partial_x)$ at most of $\mathcal{O}(\zeta^1)$, the operator $\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p)$ is required to $\mathcal{O}(\zeta^{-2})$. The scaling behaviour of the individual operators $X_{uv}(x, \partial_x + \mathrm{i}p)$ and $\Delta_u(x, \partial_x + \mathrm{i}p)$ (and hence $\mathcal{X}_{ij}$ and $\mathcal{D}_{ij}$) can be easily determined from Eqs. (34)

[4] This is always possible, because the leading terms of $\mathcal{A}_{ij}$ for large p correspond to the inverse propagators of the light fields.

[5] Note that defining $\mathcal{A}_{ij}$, $\mathcal{D}_{ij}$, $\mathcal{X}_{ij}$ in such a way that their indices are individual SU(2) components rather than complete SU(2) multiplets ($\overline{W}$, φ) makes it unnecessary to project onto the subspace of SU(2) generators when inverting Δ_u as was done in (Sections 3 of) Refs. [14,15].

Table 1 Scaling behaviour of the operators $\Delta_u^{-1}(x, \partial_x + \mathrm{i}p)$ and $X_{uv}(x, \partial_x + \mathrm{i}p)$ according to Eqs. (34) and (38) in the hard-momentum region where $p \sim M_\mathrm{H} \sim \zeta$, $\hat{H} \sim s_a \sim \zeta^{-1}$

Operator	$\Delta^{-1}_{u=\overline{W},A,\varphi,h,H}$	$X_{A\overline{W}}$	$X_{\overline{W}\varphi}$	$X_{A\varphi}$	X_{Hh}	$X_{H\overline{W}}$	$X_{h\overline{W}}$	$X_{H\varphi}$	$X_{h\varphi}$
Scaling	ζ^{-2}	ζ^1	ζ^1	ζ^0	ζ^1	ζ^{-1}	ζ^0	ζ^0	ζ^1

and (38) and is summarized in Table 1.[6] From Eqs. (47)–(50) we thus obtain

$$\begin{aligned}\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p) = {} & \Delta_H - X_{Hh}\,\Delta_h^{-1}\,X_{hH} - \mathrm{tr}\left[X_{H\varphi}\tfrac{\tau_a}{2}\right] \\ & \times 2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\,\mathrm{tr}\left[X_{\varphi H}\tfrac{\tau_b}{2}\right] \\ & - 2\,\mathrm{tr}\left[X_{H\varphi}\tfrac{\tau_a}{2}\right]\,2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\,\mathrm{tr}\left[X_{\varphi h}\tfrac{\tau_b}{2}\right] \\ & \times \Delta_h^{-1}\,X_{hH} - X_{Hh}\,\Delta_h^{-1}\,\mathrm{tr}\left[X_{h\varphi}\tfrac{\tau_a}{2}\right] \\ & \times 2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\mathrm{tr}\left[X_{\varphi h}\tfrac{\tau_b}{2}\right]\,\Delta_h^{-1}\,X_{hH} + \mathcal{O}(\zeta^{-3}),\end{aligned} \tag{54}$$

where the fourth term on the r.h.s. actually represents two equal contributions, corresponding to the two different orders $\mathcal{X}_{H\varphi_a}\ldots\mathcal{X}_{hH}$ and $\mathcal{X}_{Hh}\ldots\mathcal{X}_{\varphi_a H}$ of the operator chain.

The operators $\Delta_u(x, \partial_x + \mathrm{i}p)$ and $X_{uv}(x, \partial_x + \mathrm{i}p)$ appearing in Eq. (54) can be directly read from Eqs. (34) and (38) to the needed order in $1/\zeta$. With these ingredients the individual contributions $\Pi^{(\kappa)}$ of order $\zeta^{-\kappa}$ to $\Pi(x, p, \partial_x)$, as defined in Eq. (53), follow in a straightforward way, and we can evaluate $\ln\big(\Delta_H(x, \partial_x + \mathrm{i}p)\big)$ as series expansion,

$$\ln\left(\tilde{\Delta}_H\,(x, \partial_x + \mathrm{i}p)\right) = \ln(-p^2 + M_\mathrm{H}^2) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\Pi}{p^2 - M_\mathrm{H}^2}\right)^n, \tag{55}$$

where the nth term of the sum contributes at most at order ζ^{-n}. After that we can drop the $\mathcal{T}_h(p)$ operator in Eq. (42). Taking into account that odd powers of p^μ integrate to zero and dropping an irrelevant constant we arrive at[7]

$$\delta\mathcal{L}_\mathrm{eff}^\text{1-loop} = \frac{\mathrm{i}}{2}\,\mu^{4-D}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\left[-\frac{\Pi^{(0)} + \Pi^{(2)}}{p^2 - M_\mathrm{H}^2} - \frac{(\Pi^{(0)})^2}{2(p^2 - M_\mathrm{H}^2)^2}\right] + \mathcal{O}(\zeta^{-2}). \tag{56}$$

Note that the p^μ-even terms $\propto \Pi^{(-1)}\Pi^{(1)}$ and $\propto \Pi^{(-1)}\Pi^{(0)}\Pi^{(-1)}$(+perm.) with $\Pi^{(-1)} = 2\mathrm{i}p\cdot\partial_x$ vanish in Eq. (56) like total derivatives or because there is no background field for the partial derivative to act on. The relevant terms of the $\Delta_y^{-1}(x, \partial_x + \mathrm{i}p)$ read

$$\begin{aligned}\Delta_h^{-1} = {} & -\frac{1}{p^2} - \frac{1}{p^4}\Bigg[2\mathrm{i}p\cdot\partial_x + \Box_x + M_\mathrm{h}^2 + \frac{3M_\mathrm{h}^2}{v_2}\hat{h} \\ & + \frac{M_\mathrm{H}^2 s_\alpha}{v_2}\hat{H} + \frac{3(M_\mathrm{h}^2 + M_\mathrm{H}^2 s_\alpha^2)}{2v_2^2}\hat{h}^2 \\ & - \frac{g_2^2}{2}\,\mathrm{tr}\big[\hat{C}^2\big]\Bigg] + 4\frac{(p\cdot\partial_x)^2}{p^6} + \mathcal{O}(\zeta^{-5}),\end{aligned} \tag{57}$$

$$\Delta_\varphi^{-1} = -\frac{1}{p^2}\left(1 + \frac{\hat{h}}{v_2}\right)^{-2}\mathbb{1} + \mathcal{O}(\zeta^{-3}), \tag{58}$$

where $\mathbb{1}$ is the 2×2 unit matrix. Furthermore we have according to Eqs. (34) and (38)

$$\begin{aligned}\Delta_H = {} & -p^2 + M_\mathrm{H}^2 + 2\mathrm{i}p\cdot\partial_x + \Box_x - \frac{g_2^2 s_\alpha^2}{2}\,\mathrm{tr}\big[\hat{C}^2\big] \\ & + \left[\frac{3s_\alpha^2\,(M_\mathrm{h}^2 + s_\alpha^2 M_\mathrm{H}^2)}{2v_2^2} + \frac{6\lambda_{12}^2 v_2^2}{M_\mathrm{H}^2} + \lambda_{12}\left(1 - 6s_\alpha^2\right)\right]\hat{h}^2 \\ & + \left[2\lambda_{12}v_2\left(s_\alpha^2 - \frac{3M_\mathrm{h}^2}{M_\mathrm{H}^2} - 2\right) + \frac{s_\alpha^2\,(M_\mathrm{h}^2 - M_\mathrm{H}^2\,(s_\alpha^2 - 2))}{v_2}\right]\hat{h} \\ & + \frac{6\lambda_{12}^2 v_2^2}{M_\mathrm{H}^2 s_\alpha^2}\hat{H}^2 + \left[\frac{6\lambda_{12}v_2}{s_\alpha}\left(\frac{M_\mathrm{h}^2}{M_\mathrm{H}^2} - s_\alpha^2 + 1\right) + \frac{3M_\mathrm{H}^2 s_\alpha^3}{v_2}\right]\hat{H} \\ & + \left(6\lambda_{12}s_\alpha - \frac{12\lambda_{12}^2 v_2^2}{M_\mathrm{H}^2 s_\alpha}\right)\hat{h}\hat{H} + \mathcal{O}(\zeta^{-3}),\end{aligned} \tag{59}$$

which corresponds to the contribution from loops involving heavy Higgs modes only. With Eqs. (57) and (58) the remaining terms in Eq. (54) are

$$\begin{aligned}X_{Hh}\,\Delta_h^{-1}\,X_{hH} = {} & \frac{g_2^2 M_\mathrm{H}^4 s_\alpha^2}{2p^4 v_2^2}\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h}^2 \\ & + \frac{g_2^2 M_\mathrm{H}^2 s_\alpha^2}{p^2 v_2}\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h} - \frac{3M_\mathrm{H}^4 s_\alpha^2}{2p^4 v_2^4}\left(M_\mathrm{h}^2 + M_\mathrm{H}^2 s_\alpha^2\right)\hat{h}^4 \\ & - \frac{3M_\mathrm{H}^2 s_\alpha^2}{p^4 v_2^3}\Big[M_\mathrm{h}^2\left(M_\mathrm{H}^2 + p^2\right) \\ & + p^2\left(M_\mathrm{H}^2 s_\alpha^2 - 2\lambda_{12}v_2^2\right)\Big]\hat{h}^3\end{aligned}$$

[6] Here we anticipate that $\hat{H}_l \sim \zeta^{-1}$. This scaling behaviour is confirmed by the explicit result for the equation of motion of $\hat{H}_l$ in Sect. 4.3, but can also be directly understood from the scaling of the heavy Higgs propagator: $\langle H_l H_l\rangle \sim 1/M_\mathrm{H}^2$.

[7] In the corresponding diagrammatic calculation the loop integrands, which are expanded in ζ depend only quadratically on M_H and s_α (upon eliminating $\hat{H}$ using its EOMs). It is therefore intuitively clear that $\Pi^{(\kappa)}$ with odd κ is proportional to odd powers of p^μ. This can be easily verified with the explicit expressions given in Eqs. (54) and (59)–(63).

$$
\begin{aligned}
&+\frac{M_{\rm H}^2 s_\alpha^2}{p^4 v_2^2}\Big[\big(2s_\alpha^2-1\big)p^2 M_{\rm H}^2 \\
&-4\lambda_{12}p^2 v_2^2 - M_{\rm h}^2\big(M_{\rm H}^2+4p^2\big)\Big]\hat{h}^2 \\
&-\frac{M_{\rm H}^2 s_\alpha}{p^4 v_2^3}\left(M_{\rm H}^4 s_\alpha^2 + 4\lambda_{12}p^2 v_2^2\right)\hat{h}^2\hat{H} \\
&+\frac{4M_{\rm H}^2 s_\alpha}{p^2}\left(2\lambda_{12}-\frac{M_{\rm H}^2 s_\alpha^2}{v_2^2}\right)\hat{h}\hat{H} \\
&-\frac{M_{\rm H}^4 s_\alpha^2}{p^6 v_2^2}\left[p^2\hat{h}\Box\hat{h} + 2\mathrm{i}p^2\,\hat{h}\,p^\mu\partial_\mu\hat{h} - 4\hat{h}(p^\mu\partial_\mu)^2\hat{h}\right] \\
&+\mathcal{O}(\zeta^{-3}),
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
&\mathrm{tr}\left[X_{H\varphi}\tfrac{\tau_a}{2}\right]2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\mathrm{tr}\left[X_{\varphi H}\tfrac{\tau_b}{2}\right] \\
&= -\frac{2g_2^2 s_\alpha^2}{p^2}\,p^\mu p^\nu\,\mathrm{tr}\big[\hat{C}_\mu\hat{C}_\nu\big] + \mathcal{O}(\zeta^{-3}),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&2\,\mathrm{tr}\left[X_{H\varphi}\tfrac{\tau_a}{2}\right]2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\mathrm{tr}\left[X_{\varphi h}\tfrac{\tau_b}{2}\right]\Delta_h^{-1}X_{hH} \\
&= -\frac{4g_2^2 M_{\rm H}^2 s_\alpha^2}{p^4 v_2}\,p^\mu p^\nu\,\mathrm{tr}\big[\hat{C}_\mu\hat{C}_\nu\big]\hat{h} + \mathcal{O}(\zeta^{-3}),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
&X_{Hh}\,\Delta_h^{-1}\,\mathrm{tr}\left[X_{h\varphi}\tfrac{\tau_a}{2}\right]2\,\mathrm{tr}\left[\tfrac{\tau_a}{2}\Delta_\varphi^{-1}\tfrac{\tau_b}{2}\right]\mathrm{tr}\left[X_{\varphi h}\tfrac{\tau_b}{2}\right]\Delta_h^{-1}X_{hH} \\
&= -\frac{2g_2^2 M_{\rm H}^4 s_\alpha^2}{p^6 v_2^2}\,p^\mu p^\nu\,\mathrm{tr}\big[\hat{C}_\mu\hat{C}_\nu\big]\hat{h}^2 + \mathcal{O}(\zeta^{-3}).
\end{aligned}
\tag{63}
$$

At this point the correspondence between the individual terms in $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$ (56) and Feynman graphs in a diagrammatic calculation is most obvious: The external lines of the diagrams are uniquely given by the background fields contained in each monomial of $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$, the internal lines of the light fields ϕ_i originate from the factors Δ_{ϕ_i} with $\phi_i = h, \varphi, \dots$, and the heavy internal H lines correspond to the factors $1/(p^2 - M_{\rm H}^2)$. Note, however, that in general internal loop lines in diagrams lead to sequences of powers of the corresponding propagators owing to the Taylor expansion for $p_i \ll p, M_{\rm H}$, where p_i stands for external momenta represented by ∂ operators in $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$. Therefore, the terms in $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$ actually correspond to the individual terms of the Taylor-expanded Feynman diagrams in the hard momentum region.

Inserting the results of Eqs. (57)–(63) into Eq. (56) effectively leads to

$$
\begin{aligned}
\Pi^{(0)} &= \left(\lambda_{12} + \frac{M_{\rm H}^4 s_\alpha^2}{p^2 v_2^2}\right)\hat{h}^2 + \left(\frac{2M_{\rm H}^2 s_\alpha^2}{v_2} - 4\lambda_{12}v_2\right)\hat{h} \\
&\quad + \frac{6\lambda_{12}v_2}{s_\alpha}\,\hat{H},
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
\Pi^{(2)} &= -\frac{(D-4)g_2^2 M_{\rm H}^4 s_\alpha^2}{2D\,p^4 v_2^2}\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h}^2 \\
&\quad -\frac{(D-4)g_2^2 M_{\rm H}^2 s_\alpha^2}{D\,p^2 v_2}\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h} \\
&\quad -\frac{(D-4)g_2^2 s_\alpha^2}{2D}\,\mathrm{tr}\big[\hat{C}^2\big] \\
&\quad +\frac{3M_{\rm H}^4 s_\alpha^2\left(M_{\rm h}^2 + M_{\rm H}^2 s_\alpha^2\right)}{2p^4 v_2^4}\,\hat{h}^4 + 3s_\alpha^2\Bigg[\frac{M_{\rm h}^2 M_{\rm H}^4}{p^4 v_2^3} \\
&\quad +\frac{M_{\rm H}^2\left(M_{\rm h}^2 + M_{\rm H}^2 s_\alpha^2\right)}{p^2 v_2^3} - \frac{2\lambda_{12}M_{\rm H}^2}{p^2 v_2}\Bigg]\hat{h}^3 \\
&\quad +s_\alpha^2\Bigg[\frac{M_{\rm h}^2 M_{\rm H}^4}{p^4 v_2^2} + \frac{4M_{\rm h}^2 M_{\rm H}^2 - 2M_{\rm H}^4 s_\alpha^2}{p^2 v_2^2} \\
&\quad +3\frac{M_{\rm h}^2 + M_{\rm H}^2 s_\alpha^2}{2v_2^2} + \lambda_{12}\left(\frac{4M_{\rm H}^2}{p^2} - 6\right) + \frac{6\lambda_{12}^2 v_2^2}{M_{\rm H}^2 s_\alpha^2}\Bigg]\hat{h}^2 \\
&\quad +\left[2\lambda_{12}v_2\left(s_\alpha^2 - \frac{3M_{\rm h}^2}{M_{\rm H}^2}\right) + \frac{s_\alpha^2\left(M_{\rm h}^2 - M_{\rm H}^2 s_\alpha^2\right)}{v_2}\right]\hat{h} \\
&\quad +\frac{6\lambda_{12}^2 v_2^2}{M_{\rm H}^2 s_\alpha^2}\hat{H}^2 + \left[6\lambda_{12}v_2\left(\frac{M_{\rm h}^2}{M_{\rm H}^2 s_\alpha} - s_\alpha\right) + \frac{3M_{\rm H}^2 s_\alpha^3}{v_2}\right]\hat{H} \\
&\quad +\left(\frac{M_{\rm H}^6 s_\alpha^3}{p^4 v_2^3} + \frac{4\lambda_{12}M_{\rm H}^2 s_\alpha}{p^2 v_2}\right)\hat{h}^2\hat{H} \\
&\quad +\Bigg[\frac{4M_{\rm H}^4 s_\alpha^3}{p^2 v_2^2} + \lambda_{12}\left(6s_\alpha - \frac{8M_{\rm H}^2 s_\alpha}{p^2}\right) \\
&\quad -\frac{12\lambda_{12}^2 v_2^2}{M_{\rm H}^2 s_\alpha}\Bigg]\hat{h}\hat{H} + \frac{(D-4)M_{\rm H}^4 s_\alpha^2}{D\,p^4 v_2^2}\,\hat{h}\Box\hat{h}\,,
\end{aligned}
\tag{65}
$$

under the integral over p in Eq. (56), where we have already performed the tensor reduction of the $p^\mu p^\nu$ terms, which for rank-2 vacuum integrals is achieved by the replacement

$$
p^\mu p^\nu \;\to\; \frac{p^2}{D}\,g^{\mu\nu}. \tag{66}
$$

The loop integration over p involves only the very simple vacuum integrals

$$
\begin{aligned}
I_{ab} &= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D p\,\frac{1}{(p^2 - M_{\rm H}^2 + \mathrm{i}0)^a(p^2 + \mathrm{i}0)^b} \\
&= (4\pi\mu^2)^{(4-D)/2}\,(-1)^{a+b}\,\frac{\Gamma\big(\frac{D}{2}-b\big)\Gamma\big(a+b-\frac{D}{2}\big)}{\Gamma(a)\Gamma\big(\frac{D}{2}\big)} \\
&\quad\times M_{\rm H}^{D-2a-2b},
\end{aligned}
\tag{67}
$$

which obey the useful relations

$$
\begin{aligned}
&I_{0b} = 0, \quad I_{11} = I_{10}/M_{\rm H}^2, \quad I_{12} = I_{10}/M_{\rm H}^4, \\
&I_{21} = I_{20}/M_{\rm H}^2 - I_{10}/M_{\rm H}^4, \quad I_{22} = I_{20}/M_{\rm H}^4 - 2I_{10}/M_{\rm H}^6.
\end{aligned}
\tag{68}
$$

The integrals I_{0b} vanish, because they are scaleless; the other relations follow from partial fractioning. We can thus express $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$ solely in terms of I_{10} and I_{20} and obtain

$$
\delta\mathcal{L}_{\rm eff}^{\text{1-loop}} = \frac{1}{32\pi^2}\Bigg\{-\frac{(D-4)g_2^2 s_\alpha^2}{2D}I_{10}\,\mathrm{tr}\big[\hat{C}^2\big]\left(1+\frac{\hat{h}}{v_2}\right)^2
$$

$$
\begin{aligned}
&+ \frac{(D-4)s_\alpha^2}{D\,v_2^2} I_{10}\,\hat{h}\Box\hat{h} \\
&+ \Bigg[\frac{1}{2}\lambda_{12}^2 I_{20} - \frac{\lambda_{12}s_\alpha^2}{v_2^2}(I_{10} - M_{\rm H}^2 I_{20}) \\
&+ \frac{s_\alpha^2}{2v_2^4}\Big(3M_{\rm h}^2 I_{10} + M_{\rm H}^2 s_\alpha^2 (I_{10} + M_{\rm H}^2 I_{20})\Big)\Bigg]\hat{h}^4 \\
&- \Bigg[4\lambda_{12}^2 v_2 I_{20} + \frac{2\lambda_{12}s_\alpha^2}{v_2}(I_{10} + M_{\rm H}^2 I_{20}) \\
&- \frac{s_\alpha^2}{v_2^3}\Big(6M_{\rm h}^2 I_{10} + M_{\rm H}^2 s_\alpha^2 (I_{10} + 2M_{\rm H}^2 I_{20})\Big)\Bigg]\hat{h}^3 \\
&+ \Bigg[\frac{2\lambda_{12}^2 v_2^2}{M_{\rm H}^2}(3I_{10} + 4M_{\rm H}^2 I_{20}) \\
&- \lambda_{12}\Big(8M_{\rm H}^2 s_\alpha^2 I_{20} - (1-2s_\alpha^2)I_{10}\Big) \\
&+ \frac{s_\alpha^2}{2v_2^2}\Big(13M_{\rm h}^2 I_{10} + 4M_{\rm H}^4 s_\alpha^2 I_{20} \\
&+ M_{\rm H}^2(2-s_\alpha^2)I_{10}\Big)\Bigg]\hat{h}^2 \\
&+ \Bigg[\frac{s_\alpha^2}{v_2}\Big(M_{\rm h}^2 + M_{\rm H}^2(2-s_\alpha^2)\Big) \\
&- \frac{2\lambda_{12}v_2}{M_{\rm H}^2}\Big(3M_{\rm h}^2 + M_{\rm H}^2(2-s_\alpha^2)\Big)\Bigg] I_{10}\hat{h} \\
&+ \frac{6\lambda_{12}^2 v_2^2}{M_{\rm H}^2 s_\alpha^2}(I_{10} + 3M_{\rm H}^2 I_{20})\hat{H}^2 + \frac{3}{v_2}\Bigg[M_{\rm H}^2 s_\alpha^3 \\
&+ \frac{2\lambda_{12}v_2^2}{M_{\rm H}^2 s_\alpha}\Big(M_{\rm h}^2 + M_{\rm H}^2(1-s_\alpha^2)\Big)\Bigg] I_{10}\hat{H} \\
&+ \frac{1}{s_\alpha v_2^3}\Big[6\lambda_{12}^2 v_2^4 I_{20} - 2\lambda_{12}s_\alpha^2 v_2^2 (I_{10} \\
&- 3M_{\rm H}^2 I_{20}) + M_{\rm H}^2 s_\alpha^4 I_{10}\Big]\hat{h}^2\hat{H} \\
&+ \frac{2}{M_{\rm H}^2 s_\alpha v_2^2}(M_{\rm H}^2 s_\alpha^2 - 2\lambda_{12}v_2^2)\Big[2M_{\rm H}^2 s_\alpha^2 I_{10} \\
&+ 3\lambda_{12}v_2^2(I_{10} + 2M_{\rm H}^2 I_{20})\Big]\hat{h}\hat{H}\Bigg\} \\
&+ \mathcal{O}\big(\zeta^{-2}\big),
\end{aligned}
\tag{69}
$$

Upon inserting

$$
I_{10} = M_{\rm H}^2(L_\epsilon + 1) + \mathcal{O}(\epsilon), \quad I_{20} = L_\epsilon + \mathcal{O}(\epsilon), \tag{70}
$$

with

$$
\begin{aligned}
L_\epsilon &= \Delta + \ln\bigg(\frac{\mu^2}{M_{\rm H}^2}\bigg), \\
\Delta &= \frac{1}{\epsilon} - \gamma_{\rm E} + \ln(4\pi),
\end{aligned}
\tag{71}
$$

and expanding in $\epsilon = (4-D)/2$ we have

$$
\begin{aligned}
\delta\mathcal{L}_{\rm eff}^{\text{1-loop}} = \frac{1}{32\pi^2}\Bigg\{&\frac{1}{4}g_2^2 M_{\rm H}^2 s_\alpha^2\,{\rm tr}\big[\hat{C}^2\big]\bigg(1+\frac{\hat{h}}{v_2}\bigg)^2 - \frac{M_{\rm H}^2 s_\alpha^2}{2v_2^2}\hat{h}\Box\hat{h} \\
&+ \bigg[\frac{\lambda_{12}^2}{2}L_\epsilon + \frac{3M_{\rm h}^2 M_{\rm H}^2 s_\alpha^2}{2v_2^4}(L_\epsilon+1) + \frac{M_{\rm H}^4 s_\alpha^4}{2v_2^4}(2L_\epsilon+1) \\
&- \frac{\lambda_{12}M_{\rm H}^2 s_\alpha^2}{v_2^2}\bigg]\hat{h}^4 \\
&+ \bigg[\frac{6M_{\rm h}^2 M_{\rm H}^2 s_\alpha^2}{v_2^3}(L_\epsilon+1) + \frac{M_{\rm H}^4 s_\alpha^4}{v_2^3}(3L_\epsilon+1) \\
&- \frac{2\lambda_{12}M_{\rm H}^2 s_\alpha^2}{v_2}(2L_\epsilon+1) - 4\lambda_{12}^2 v_2 L_\epsilon\bigg]\hat{h}^3 \\
&+ \bigg[\frac{13M_{\rm h}^2 M_{\rm H}^2 s_\alpha^2}{2v_2^2}(L_\epsilon+1) \\
&+ \frac{M_{\rm H}^4 s_\alpha^2}{2v_2^2}\Big(3s_\alpha^2 L_\epsilon + 2L_\epsilon - s_\alpha^2 + 2\Big) \\
&+ \lambda_{12}M_{\rm H}^2\Big(1 - 10s_\alpha^2 L_\epsilon + L_\epsilon - 2s_\alpha^2\Big) \\
&+ 2\lambda_{12}^2 v_2^2(7L_\epsilon+3)\bigg]\hat{h}^2 \\
&+ \bigg[\frac{M_{\rm H}^2 s_\alpha^2}{v_2}\Big(M_{\rm h}^2 + (2-s_\alpha^2)M_{\rm H}^2\Big)(L_\epsilon+1) \\
&- 2\lambda_{12}v_2\Big(3M_{\rm h}^2 + (2-s_\alpha^2)M_{\rm H}^2\Big)(L_\epsilon+1)\bigg]\hat{h} \\
&+ \frac{6\lambda_{12}^2 v_2^2}{s_\alpha^2}(4L_\epsilon+1)\hat{H}^2 + 3\bigg[\frac{2\lambda_{12}v_2}{s_\alpha}\Big(M_{\rm h}^2 + (1-s_\alpha^2)M_{\rm H}^2\Big) \\
&+ \frac{M_{\rm H}^4 s_\alpha^3}{v_2}\bigg](L_\epsilon+1)\hat{H} \\
&+ \bigg[\frac{M_{\rm H}^4 s_\alpha^3}{v_2^3}(L_\epsilon+1) + \frac{2\lambda_{12}M_{\rm H}^2 s_\alpha}{v_2}(2L_\epsilon-1) \\
&+ \frac{6\lambda_{12}^2 v_2}{s_\alpha}L_\epsilon\bigg]\hat{h}^2\hat{H} \\
&+ \frac{2}{s_\alpha v_2^2}\big(M_{\rm H}^2 s_\alpha^2 - 2\lambda_{12}v_2^2\big)\Big[2M_{\rm H}^2 s_\alpha^2(L_\epsilon+1) \\
&+ 3\lambda_{12}v_2^2(3L_\epsilon+1)\Big]\hat{h}\hat{H}\Bigg\} \\
&+ \mathcal{O}(\zeta^{-2},\epsilon).
\end{aligned}
\tag{72}
$$

This expression represents the bare effective Lagrangian from integrating out heavy modes at one loop in unitary (background) gauge. In order to bring $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$ into a manifestly gauge-invariant form, we can invert the Stueckelberg transformation in Eq. (20) by replacing

$$
\hat{C}_\mu \to \frac{\rm i}{g_2}\hat{U}^\dagger\big(\hat{D}_\mu\hat{U}\big). \tag{73}
$$

We emphasize that the (seemingly non-decoupling) $\delta\mathcal{L}_{\rm eff}^{\text{1-loop}}$ must be properly renormalized, taking into account full-theory as well as EFT counterterms, before it can be used to compute physical observables. We will come back to this point in Sect. 5.

4.3 Heavy Higgs equation of motion and lowest-order effective Lagrangian

At the end of Sect. 4.1, we have already outlined how the final effective Lagrangian $\mathcal{L}_{\text{eff}}$ breaks up into different parts,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}}(\hat{\phi}_i, \phi_i) + \delta\mathcal{L}_{\text{eff}}^{\text{tree}}(\hat{\phi}_i, \phi_i) + \delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}(\hat{\phi}_i) + \delta\mathcal{L}_{\text{eff}}^{\text{ct}}(\hat{\phi}_i), \tag{74}$$

where all field arguments correspond to light field modes. The arguments of the full SM Lagrangian $\mathcal{L}_{\text{SM}}(\hat{\phi}_i, \phi_i)$ comprise all background and quantum fields of the SM, since all SM particles can propagate along tree and loop lines in EFT Feynman diagrams.

The part $\delta\mathcal{L}_{\text{eff}}^{\text{tree}}(\hat{\phi}_i, \phi_i)$ of the effective Lagrangian quantifies all lowest-order couplings between SM fields that are induced by exchange of a heavy Higgs boson. The terms in $\mathcal{L}_{\text{eff}}^{\text{tree}} = \mathcal{L}_{\text{SM}} + \delta\mathcal{L}_{\text{eff}}^{\text{tree}}$ built from background fields $\hat{\phi}_i$ only are sufficient for the construction of all tree-level diagrams contributing to Green functions up to some target order ζ^{-n}. The effective couplings in $\delta\mathcal{L}_{\text{eff}}^{\text{tree}}$ involving (SM) quantum fields ϕ_i give rise to loop diagrams that are related to the small-momentum regions of full-theory loop diagrams involving the quantum field H. Note that most of the terms in $\delta\mathcal{L}_{\text{eff}}^{\text{tree}}$ depend on the background and quantum fields only via their sum $\tilde{\phi}_i = \hat{\phi}_i + \phi_i$ by construction within the BFM. [8] In this section we derive $\delta\mathcal{L}_{\text{eff}}^{\text{tree}}$. To this end, we eliminate $\hat{H}_l$ and H_l from the full SESM Lagrangian by solving the EOM for the H_l field in terms of a series in inverse powers of ζ. This is possible, since all derivatives ∂ acting on light field modes scale as ζ^0 are therefore ζ^{-1} suppressed compared to the heavy Higgs mass M_{H}. The effects of the heavy field modes in hard loops, where ∂ effectively counts as ζ^1, are contained in $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}(\hat{\phi}_i)$ constructed in the previous section. The last contribution to the effective Lagrangian, $\delta\mathcal{L}_{\text{eff}}^{\text{ct}}(\hat{\phi}_i)$, which accounts for counterterm contributions from the renormalization of the heavy-H-boson sector in the full and effective theory, is constructed in the next section.

To derive the EOM for the light modes $\hat{H}_l$ and H_l, we start from the dependence of the full-theory Lagrangian on light and SM field modes, which we summarize in a Lagrangian dubbed $\mathcal{L}^{\text{tree}}(\hat{\phi}_i, \phi)$. This part is given by (with $\tilde{H}_l \sim \zeta^{-1}$)

$$\mathcal{L}^{\text{tree}} = \mathcal{L}_{\text{SM}} - \frac{s_\alpha^2 M_{\text{H}}^2}{8v_2^2}\tilde{h}^4 - \frac{1}{2}M_{\text{H}}^2\tilde{H}_l^2 - \frac{s_\alpha M_{\text{H}}^2}{2v_2}\tilde{h}^2\tilde{H}_l + \mathcal{O}\left(\zeta^{-2}\right). \tag{75}$$

Here and in the following we suppress the subscript l of the soft modes of light (SM) particles, which represent the degrees of freedom of the EFT. Since $\mathcal{L}_{\text{SM}}$ does not depend on $\hat{H}_l$ and H_l, the EOM resulting from the variation of H_l reads

$$0 = M_{\text{H}}^2\tilde{H}_l + \frac{s_\alpha M_{\text{H}}^2}{2v_2}\tilde{h}^2 + \mathcal{O}\left(\zeta^{-1}\right) \tag{76}$$

with the straightforward solution

$$\tilde{H}_l = -\frac{s_\alpha}{2v_2}\tilde{h}^2 + \mathcal{O}\left(\zeta^{-3}\right). \tag{77}$$

Note that this result a posteriori confirms our counting $\tilde{H}_l \sim \zeta^{-1}$. Inserting this solution back into $\mathcal{L}^{\text{tree}}$ given in Eq. (75) leads to

$$\mathcal{L}_{\text{eff}}^{\text{tree}} = \mathcal{L}_{\text{SM}} + \delta\mathcal{L}_{\text{eff}}^{\text{tree}}, \qquad \delta\mathcal{L}_{\text{eff}}^{\text{tree}} = \mathcal{O}\left(\zeta^{-2}\right), \tag{78}$$

showing that there are no non-decoupling effects of the SESM with a heavy H boson at tree level in the weak-coupling scenario in Eq. (14). Note, however, that the individual Feynman rules of the full theory do not all simply turn into their SM versions in this limit. The non-standard $\tilde{h}^4$ coupling in $\mathcal{L}^{\text{tree}}$, for instance, is rather compensated by the leading contribution of the four-point interaction of $\tilde{h}$ fields induced by tree-level $\tilde{H}$ exchange, when the $\tilde{H}$ propagator shrinks to a point and $\tilde{H}$ is effectively given by Eq. (77).

In order to obtain $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}$ in Eq. (72) in terms of SM fields, we have to eliminate the light mode $\hat{H}_l$ of the heavy-Higgs background field, which proceeds along the same lines as above using the EOM (76). There are, however, two differences. Firstly, the dependence of the solution on the quantum field H_l is irrelevant and can be discarded at the one-loop level, because these terms would only contribute as part of a second loop. Secondly, the term proportional to $\hat{H}_l$ in $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}$ of Eq. (72) has a prefactor scaling like ζ^3, so that the solution for $\hat{H}_l$ is needed to order ζ^{-3}, i.e. the solution in Eq. (76) has to be supplemented by further terms. This task is straightforward and yields

$$\begin{aligned}\hat{H}_l = &-\frac{s_\alpha}{2v_2}\hat{h}^2 + \frac{s_\alpha}{2v_2 M_{\text{H}}^2}\left(\Box - 2M_{\text{h}}^2 - 2\lambda_{12}v_2^2 + s_\alpha^2 M_{\text{H}}^2\right)\hat{h}^2 \\ &- \frac{s_\alpha}{2v_2^2 M_{\text{H}}^2}\left(M_{\text{h}}^2 - s_\alpha^2 M_{\text{H}}^2 + 2\lambda_{12}v_2^2\right)\hat{h}^3 - \frac{s_\alpha\lambda_{12}}{4v_2 M_{\text{H}}^2}\hat{h}^4 \\ &+ \frac{g_2^2 s_\alpha}{2M_{\text{H}}^2}\text{tr}\left[\hat{C}^2\right](v_2 + \hat{h}) + \mathcal{O}\left(\zeta^{-5}\right).\end{aligned} \tag{79}$$

For later convenience we also derive the EOM for the light Higgs field $\tilde{h}$,

$$0 = \left(\Box + M_{\text{h}}^2\right)\tilde{h} - \frac{g_2^2}{2}\text{tr}\left[\tilde{C}^2\right](v_2 + \tilde{h})$$

[8] The only parts of the BFM quantized full-theory Lagrangian that do not depend on the sum $\hat{\phi}_i + \phi_i$ of background and quantum fields are the gauge-fixing Lagrangian of the quantum fields and the ghost Lagrangian.

$$
\begin{aligned}
&+\frac{3M_\mathrm{h}^2}{2v_2}\tilde{h}^2+\frac{M_\mathrm{h}^2+M_\mathrm{H}^2s_\alpha^2}{2v_2^2}\tilde{h}^3\\
&+\frac{M_\mathrm{H}^2s_\alpha}{v_2}\tilde{h}\tilde{H}_l+\mathcal{O}(\zeta^{-2})\\
&=\left(\Box+M_\mathrm{h}^2\right)\tilde{h}-\frac{g_2^2}{2}\mathrm{tr}\big[\tilde{C}^2\big](v_2+\tilde{h})\\
&+\frac{3M_\mathrm{h}^2}{2v_2}\tilde{h}^2+\frac{M_\mathrm{h}^2}{2v_2^2}\tilde{h}^3+\mathcal{O}(\zeta^{-2}),
\end{aligned}
\tag{80}
$$

where the solution Eq. (77) has been inserted for $\tilde{H}_l$ in the last line.

5 Renormalization

5.1 Renormalization of the SM

Of course, the one-loop renormalization of the SM is by now standard, both in the conventional quantization formalism and in the BFM (see e.g. Refs. [24,25,70] and references therein). As shown in the previous section, the SM coincides with the EFT describing the large-M_H limit of the SESM at tree level in the leading order of the large-M_H expansion. To prepare ourselves for the renormalization of the SESM and the EFT, it is therefore instructive to first recall some aspects of the SM renormalization. In the formulation below, we closely follow Ref. [25] both conceptually and concerning notation and conventions for field-theoretical quantities.

Before renormalization, the defining "bare" Lagrangian depends on parameters whose physical meaning is obscure when they are used to parametrize physical observables. Likewise, the fields occurring in the bare Lagrangian are, in general, not canonically normalized. In order to introduce parameters and fields with clear meaning and well-defined normalization, respectively, the original "bare" quantities are split into renormalized quantities and renormalization constants. Denoting all bare quantities with subscript "0", we write

$$
c_{i,0}=c_i+\delta c_i,\qquad \hat{\phi}_{i,0}=\left(\delta_{ij}+\frac{1}{2}\delta Z_{ij}\right)\hat{\phi}_j,
\tag{81}
$$

for generic parameters c_i and background fields $\hat{\phi}_i$. The renormalized parameters are denoted by c_i and the corresponding renormalization constants by δc_i. The renormalization constants δc_i are fixed by renormalization conditions in order to tie the renormalized parameters to measurable quantities, which in turn give them their precise physical meaning. The choice of the field renormalization constants δZ_{ij}, on the other hand, is only a matter of convenience. The matrix structure of the field renormalization constants δZ_{ij} is conveniently determined by demanding that (at least) at some specific momentum transfer the renormalized fields $\hat{\phi}_i$ do not mix. The renormalization of the (virtual) quantum fields ϕ_i is not necessary.

Specifically, we perform the "renormalization transformations" for the relevant physical parameters in the SM as follows:

$$
\begin{aligned}
&e_0=(1+\delta Z_e)e,\quad s_{\mathrm{w},0}=s_\mathrm{w}+\delta s_\mathrm{w},\\
&M_{\mathrm{W},0}^2=M_\mathrm{W}^2+\delta M_\mathrm{W}^2,\quad M_{\mathrm{h},0}^2=M_\mathrm{h}^2+\delta M_\mathrm{h}^2.
\end{aligned}
\tag{82}
$$

This fixes the renormalization of the gauge couplings g_1, g_2 and the parameters $c_\mathrm{w}, M_\mathrm{Z}, v_2$, which are related to the gauge-boson masses by[9]

$$
g_1=\frac{e}{c_\mathrm{w}},\quad g_2=\frac{e}{s_\mathrm{w}},\quad c_\mathrm{w}^2=1-s_\mathrm{w}^2=\frac{M_\mathrm{W}^2}{M_\mathrm{Z}^2},\quad v_2=\frac{2M_\mathrm{W}}{g_2}.
\tag{83}
$$

These relations are valid for bare and renormalized quantities. This, in particular, implies

$$
\delta v_2=v_2\left(\frac{\delta M_\mathrm{W}^2}{2M_\mathrm{W}^2}+\frac{\delta s_\mathrm{w}}{s_\mathrm{w}}-\delta Z_e\right).
\tag{84}
$$

The renormalization of the parameters μ_2^2 and λ_2 of the Higgs potential depends on the scheme that is employed to treat the SM tadpole parameter

$$
t_{\mathrm{h},0}=v_{2,0}\left(\mu_{2,0}^2-\frac{1}{4}\lambda_{2,0}v_{2,0}^2\right).
\tag{85}
$$

The SM tadpole term $t_{\mathrm{h},0}\tilde{h}_0$ in $\mathcal{L}_\mathrm{SM}$ is the term linear in the bare Higgs field $\tilde{h}_0$, while $v_{2,0}/\sqrt{2}$ is the constant contribution from the bare Higgs doublet field. The (squared) bare Higgs-boson mass is given by

$$
M_{\mathrm{h},0}^2=-\mu_{2,0}^2+\frac{3}{4}\lambda_{2,0}v_{2,0}^2,
\tag{86}
$$

and for the renormalized Higgs parameters we adopt the renormalization conditions

$$
\mu_2^2=\frac{M_\mathrm{h}^2}{2},\quad \lambda_2=\frac{2M_\mathrm{h}^2}{v_2^2}.
\tag{87}
$$

In order to determine the renormalization constants $\delta\mu_2^2$ and $\delta\lambda_2$ in

$$
\mu_{2,0}^2=\mu_2^2+\delta\mu_2^2,
$$

[9] Note that we write the parameters of the SM Higgs potential here with a subscript "2", i.e. v_2, μ_2, λ_2, whereas the Higgs field h has no subscript in order to match the notation for the SM-like part of the SESM in view of the next sections.

$$\lambda_{2,0} = \lambda_2^2 + \delta\lambda_2^2, \tag{88}$$

we still have to fix the tadpole parameter t_h. Similar to the descriptions of Ref. [25] we use two different prescriptions in parallel: [10]

- *Parameter-renormalized tadpole scheme* (PRTS) [70]: Demanding that the renormalized vev v_2 corresponds to the true (corrected) minimum of the Higgs potential implies that the renormalized tadpole parameter vanishes,

$$t_\mathrm{h} = t_{\mathrm{h},0} - \delta t_\mathrm{h} = 0. \tag{89}$$

  The tadpole renormalization constant δt_h is then simply given by the bare tadpole parameter $t_{\mathrm{h},0}$ in Eq. (85),

$$\delta t_\mathrm{h} = t_{\mathrm{h},0} = v_{2,0}\left(\mu_{2,0}^2 - \frac{1}{4}\lambda_{2,0}v_{2,0}^2\right) = v_2\left(\delta\mu_2^2 - \frac{1}{4}\delta\lambda_2 v_2^2 - \frac{1}{2}\lambda_2 v_2\delta v_2\right). \tag{90}$$

  Together with Eq. (86) this fixes $\delta\mu_2^2$ and $\delta\lambda_2$ in terms of δM_h^2, δv_2, and δt_h.
- *Fleischer–Jegerlehner tadpole scheme* (FJTS) [51]: The bare tadpole parameter is consistently set to zero, $t_{\mathrm{h},0} = 0$, so that, according to Eq. (85) $v_{2,0} = 2\sqrt{\mu_{2,0}^2/\lambda_{2,0}}$, and no renormalization of the tadpole parameter is performed. The bare Higgs-boson mass is thus given by

$$M_{\mathrm{h},0}^2 = 2\mu_{2,0}^2 = \frac{1}{2}\lambda_{2,0}v_{2,0}^2. \tag{91}$$

  This directly fixes $\delta\mu_2^2$ and $\delta\lambda_2$, in terms of δM_h^2 and δv_2. A tadpole counterterm

$$\delta t_\mathrm{h} = -M_\mathrm{h}^2\Delta v_\mathrm{h} \tag{92}$$

  is effectively generated by a field shift $\hat{h} \to \hat{h} + \Delta v_\mathrm{h}$ in the Lagrangian, which does not affect physical observables.

In both schemes there is a term $\delta t_\mathrm{h}\hat{h}$ in the counterterm Lagrangian, and δt_h is chosen to compensate explicit tadpole diagrams in Green functions, i.e.

$$\delta t_\mathrm{h} = -T^{\hat{h}}, \tag{93}$$

[10] Our description differs from the procedure described in Sect. 3.1.6 of Ref. [25] by introducing the bare vev $v_{2,0}$. In the FJTS our $v_{2,0}$ effectively plays the same role as the parameter v_0 in Ref. [25] for the FJTS; in the PRTS our $v_{2,0}$ corresponds to the PRTS parameter $\bar{v}$ of Ref. [25]. The formal treatment described here seems somewhat more generic, but the PRTS and FJTS schemes are fully equivalent to the ones of Ref. [25].

where $T^{\hat{h}}$ $(= \Gamma^{\hat{h}})$ denotes the unrenormalized one-point vertex function of the background Higgs field at one loop. The tadpole renormalization constant δt_h also enters many other contributions in the counterterm Lagrangian. These terms depend on the tadpole scheme. For the sake of compact notation we introduce the expressions $\delta t_\mathrm{h}^\mathrm{PRTS}$ and $\delta t_\mathrm{h}^\mathrm{FJTS}$, where $\delta t_\mathrm{h}^\mathrm{PRTS}$ equals δt_h only in the PRTS and is zero in the FJTS, and vice versa.

The field renormalization can either be performed in the basis of the gauge multiplets $\hat{W}_\mu$, $\hat{B}_\mu$, $\hat{\Phi}$ or in the basis spanned by the fields $\hat{W}_\mu^\pm$, $\hat{A}_\mu$, $\hat{Z}_\mu$, $\hat{h}$ that correspond to mass eigenstates. For our purposes, the gauge field renormalization will not play a role. In the following, the only relevant field renormalization transformation is the one of the Higgs field, which we formulate directly for $\hat{h}$:

$$\hat{h}_0 = \left(1 + \frac{1}{2}\delta Z_{\hat{h}\hat{h}}\right)\hat{h}. \tag{94}$$

The part of the counterterm Lagrangian $\delta\mathcal{L}_\mathrm{SM}^\mathrm{ct}$ that results from the SM Higgs sector by the renormalization transformations described above is denoted $\delta\mathcal{L}_\mathrm{SM}^\mathrm{Hct}$ and (in compact notation for both schemes) given by

$$\begin{aligned}\delta\mathcal{L}_\mathrm{SM}^\mathrm{Hct} = &-\frac{1}{2}\delta M_\mathrm{h}^2\,\hat{h}^2\left(1 + \frac{\hat{h}}{v_2} + \frac{\hat{h}^2}{4v_2^2}\right)\\ &+ \delta v_2\left[\frac{g_2^2}{2}\,\mathrm{tr}\big[\hat{C}^2\big](v_2 + \hat{h}) + \frac{M_\mathrm{h}^2}{4v_2^2}\,\hat{h}^3\left(2 + \frac{\hat{h}}{v_2}\right)\right]\\ &- \frac{1}{2}\delta Z_{\hat{h}\hat{h}}\hat{h}\left[\Box\hat{h} - \frac{g_2^2}{2}\,\mathrm{tr}\big[\hat{C}^2\big](v_2 + \hat{h})\right.\\ &\left.+ M_\mathrm{h}^2\hat{h}\left(1 + \frac{3\hat{h}}{2v_2} + \frac{\hat{h}^2}{2v_2^2}\right)\right] + \delta t_\mathrm{h}\,\hat{h}\\ &- \delta t_\mathrm{h}^\mathrm{PRTS}\,\frac{\hat{h}^3}{2v_2^2}\left(1 + \frac{\hat{h}}{4v_2}\right) + \delta t_\mathrm{h}^\mathrm{FJTS}\left[\frac{\hat{h}^2}{2v_2}\left(3 + \frac{\hat{h}}{v_2}\right)\right.\\ &\left.- \frac{g_2^2}{2M_\mathrm{h}^2}\,\mathrm{tr}\big[\hat{C}^2\big](v_2 + \hat{h})\right].\end{aligned} \tag{95}$$

In the SM, the mass parameters for the W, Z, and Higgs bosons are usually defined as on-shell (OS) masses, which determine the locations of the poles in the respective propagators. This fixes the mass renormalization constants according to

$$\delta M_\mathrm{W}^2 = \Sigma_\mathrm{T}^{\hat{W}}(M_\mathrm{W}^2), \quad \delta M_\mathrm{Z}^2 = \Sigma_\mathrm{T}^{\hat{Z}\hat{Z}}(M_\mathrm{Z}^2), \quad \delta M_\mathrm{h}^2 = \Sigma^{\hat{h}}(M_\mathrm{h}^2), \tag{96}$$

where $\Sigma_{(\mathrm{T})}^{\cdots}(p^2)$ denotes the corresponding self-energy (with "T" indicating its transverse part) for momentum transfer p. Following the conventions of Ref. [25], at the one-loop level $\Sigma^{\cdots}_{(\mathrm{T})}(p^2)$ includes the contributions from one-particle-irreducible (1PI) loop diagrams, explicit tadpole diagrams, as well as tadpole counterterms, but no contributions from other renormalization constants. Note that according to Eq. (83) fixing δM_W^2 and δM_Z^2 also fixes δs_w.

We complement these OS renormalization conditions by the OS condition for the electric charge e, where δZ_e is fixed by requiring that e does not receive any correction in the Thomson limit, where a physical charged particle interacts with a photon of vanishing momentum. The explicit form of δZ_e, which involves only loops of charged particles in the AA and AZ propagators, will not be needed in the following, because neutral Higgs bosons do not contribute to δZ_e at one loop. The explicit form of the field renormalization constants, which we assume to be fixed in the OS renormalization scheme, will not be required either. Only their scaling properties in the considered large-mass limit of the SESM will be relevant and are quoted below.

5.2 Renormalization of the SESM

Renormalization schemes for the SESM were worked out in Refs. [47–50,54] in different variants. We follow the proposals of Refs. [49,50] which employ the parameters M_H, s_α, and λ_{12} (or alternatively λ_1) as independent parameters in the BSM sector of the model. We apply the renormalization transformations

$$M_{\mathrm{H},0}^2 = M_\mathrm{H}^2 + \delta M_\mathrm{H}^2, \quad s_{\alpha,0} = s_\alpha + \delta s_\alpha,\\ \lambda_{12,0} = \lambda_{12} + \delta\lambda_{12}, \tag{97}$$

which are supplemented by the renormalization transformations of the SM-like parameters described in Sect. 5.1. In Refs. [49,50] several conceptually different renormalization schemes for the (sine of the) mixing angle (s_α) are discussed:

- $\overline{\mathrm{MS}}$ renormalization [49] with the PRTS or FJTS for treating tadpoles,
- OS renormalization [50] based on the ratio of amplitudes with external h/H bosons with the PRTS or FJTS for treating tadpoles,
- symmetry-inspired renormalization [50] based on rigid (global) and BFM gauge invariance of the model.

The benefits and drawbacks of these schemes for the renormalization of s_α are discussed in Ref. [50] in detail. In the present paper, we focus on $\overline{\mathrm{MS}}$ and OS renormalization. In the OS scheme, the renormalization constant δs_α can be calculated from the field renormalization constants of the $\hat{h}/\hat{H}$ system, which are introduced below, using Eq. (3.13) of Ref. [50]. The result for δs_α in the $\overline{\mathrm{MS}}$ scheme can be obtained from δs_α in the OS scheme upon taking only its ultraviolet (UV) divergent parts. Explicitly, we have in these schemes

$$\delta s_\alpha = \begin{cases} \mathcal{O}(\zeta^{-1}) & \text{for the OS/PRTS and } \overline{\mathrm{MS}}\text{/PRTS schemes,} \\ -\dfrac{s_\alpha}{M_\mathrm{h}^2 v_2}\, T^{\hat{h}} + \mathcal{O}(\zeta^{-1}) & \text{for the OS/FJTS scheme,} \\ -\dfrac{s_\alpha}{M_\mathrm{h}^2 v_2}\, T^{\hat{h}}\big|_\mathrm{UV} + \mathcal{O}(\zeta^{-1}) & \text{for the } \overline{\mathrm{MS}}\text{/FJTS scheme,} \end{cases} \tag{98}$$

where the "UV" label indicates that only UV-divergent parts proportional to Δ, as given in Eq. (71), are absorbed into δs_α. The $\overline{\mathrm{MS}}$ renormalization constants can be deduced from the corresponding OS counterparts upon dropping the UV-finite parts. Here and in the following we only give explicit expressions for the terms in the large-M_H expansion that will be relevant for the final effective Lagrangian to $\mathcal{O}(\zeta^0)$.

The tadpole contributions from the large-momentum region of all relevant one-loop tadpole diagrams can be directly read off the linear Higgs field terms in Eq. (69). At leading order in the large-mass expansion the explicit expressions are

$$T^{\hat{h}} = -\frac{2\lambda_{12}v_2^2 - M_\mathrm{H}^2 s_\alpha^2}{16\pi^2 v_2}\, I_{10} + \mathcal{O}(\zeta^0),\\ T^{\hat{H}} = \frac{3\lambda_{12}v_2}{16\pi^2 s_\alpha}\, I_{10} + \mathcal{O}(\zeta^1), \tag{99}$$

for the background light ($\hat{h}$) and heavy Higgs ($\hat{H}$) fields, respectively, where $I_{10} = \mathcal{O}(\zeta^2)$ is given by Eq. (70). The soft-momentum regions contribute to Eq. (99) only at $\mathcal{O}(\zeta^0)$. Like in the SM, the SESM tadpole counterterms are fixed by

$$\delta t_\mathrm{h} = -T^{\hat{h}}, \qquad \delta t_\mathrm{H} = -T^{\hat{H}}, \tag{100}$$

which applies both in the PRTS and FJTS.

In all SESM renormalization schemes considered here, the mass M_H is on-shell renormalized, and the coupling parameter λ_{12} (or λ_1) with the $\overline{\mathrm{MS}}$ prescription. Explicit results for δM_H^2 and $\delta\lambda_{12}$ can be obtained in a straightforward way (see also the explicit results in Ref. [49]), but for our purpose we actually only need their scaling behaviour as $M_\mathrm{H} \to \infty$, namely

$$\delta M_\mathrm{H}^2 = \Sigma^{\hat{H}}(M_\mathrm{H}^2) = \mathcal{O}(\zeta^2), \quad \delta\lambda_{12} = \mathcal{O}(\zeta^0), \tag{101}$$

both in the PRTS and FJTS.

The renormalization constants of the SM-like parameters are obtained in full analogy to their counterparts in the SM. The required leading terms in the large-M_H limit are

$$\delta M_\mathrm{h}^2 = \frac{M_\mathrm{H}^2 s_\alpha^2 + \lambda_{12}v_2^2}{16\pi^2 v_2^2} I_{10} + \frac{3}{v_2}\,\delta t_\mathrm{h}^\mathrm{FJTS} + \frac{s_\alpha}{v_2}\,\delta t_\mathrm{H}^\mathrm{FJTS}\\ + \mathcal{O}(\zeta^0),$$

$$\delta M_\mathrm{W}^2 = \frac{g_2^2 v_2}{2M_\mathrm{h}^2}\,\delta t_\mathrm{h}^\mathrm{FJTS} + \mathcal{O}(\zeta^0),$$
$$\delta s_\mathrm{w} = \mathcal{O}(\zeta^0), \quad \delta Z_e = \mathcal{O}(\zeta^0). \tag{102}$$

The first three terms of δM_h^2 and the first term of δM_W^2 can also be directly read off from the $\hat{h}^2$ and $\mathrm{tr}[\hat{C}^2]$ terms of Eqs. (69) and (95).

As for the field renormalization, only the two Higgs fields and their mixing are of interest in the following,

$$\begin{pmatrix}\hat{H}_0\\ \hat{h}_0\end{pmatrix} = \begin{pmatrix}1+\frac{1}{2}\delta Z_{\hat{H}\hat{H}} & \delta Z_{\hat{H}\hat{h}}\\ \delta Z_{\hat{h}\hat{H}} & 1+\frac{1}{2}\delta Z_{\hat{h}\hat{h}}\end{pmatrix}\begin{pmatrix}\hat{H}\\ \hat{h}\end{pmatrix}. \tag{103}$$

For practical calculations, these field renormalization constants are fixed by OS conditions, which guarantee that the particle residues in the diagonal propagators are equal to one and that different field types do not mix on their mass shells; the explicit prescription for calculating $\delta Z_{...}$ from the Higgs self-energies can, e.g., be found in Eqs. (4.8)–(4.11) of Ref. [49] (see also Ref. [50]). As a matter of fact, the explicit form of none of the Higgs field renormalization constants δZ_{ij} will be required for the calculation of the final effective Lagrangian (not even their scaling behaviour as $M_\mathrm{H}\to\infty$). Nevertheless, it is helpful to know some of their leading terms in the large-M_H expansion,

$$\delta Z_{\hat{H}\hat{H}} = \mathcal{O}(\zeta^0), \quad \delta Z_{\hat{H}\hat{h}} = \frac{2s_\alpha}{v_2 M_\mathrm{h}^2}\,\delta t_\mathrm{h}^\mathrm{FJTS} + \mathcal{O}(\zeta^{-1}),$$
$$\delta Z_{\hat{h}\hat{h}} = \mathcal{O}(\zeta^0),$$
$$\delta Z_{\hat{h}\hat{H}} = -\frac{3M_\mathrm{H}^2 s_\alpha}{16\pi^2 v_2^2}\,\mathrm{Re}\{B_0(M_\mathrm{H}^2,0,0)\} - \frac{2s_\alpha}{v_2 M_\mathrm{h}^2}\,\delta t_\mathrm{h}^\mathrm{FJTS} + \mathcal{O}(\zeta^{-1}). \tag{104}$$

The scalar two-point one-loop integral B_0 is defined as in Refs. [25,70] and given by

$$B_0(p_1^2 = M_\mathrm{H}^2, 0, 0) = L_\epsilon + 2 + \mathrm{i}\pi + \mathcal{O}(\epsilon). \tag{105}$$

The results in Eq. (104) can be easily derived from the one-loop Higgs-boson self-energies, as e.g. described in Refs. [49,50], and by applying our power-counting in ζ. They can, for instance, be used to derive the OS scheme renormalization constant δs_α in Eq. (98) as suggested in Ref. [50].

Before we turn to the renormalization of the EFT and the contribution of the SESM counterterm Lagrangian to the effective Lagrangian, we comment on the use of a non-linear parametrization of the Higgs doublet Φ and potential implications on the renormalization procedure. In fact, great care is mandatory when adopting renormalization schemes that have been designed for linear realizations of the Higgs doublet. Note that vertex functions even with the same external field content in general change by switching from a linear to a non-linear Higgs realization. This also concerns the structure of UV divergences of vertex functions, and the differences might be quite drastic. In the non-linear Higgs realization, for instance, the Higgs self-energy $\Sigma^{\hat{h}\hat{h}}(p^2)$, involves UV-divergent terms proportional to p^4, which cannot appear for linearly realized Higgs bosons. This, in particular, implies that the "renormalized" Higgs self-energy $\Sigma_\mathrm{R}^{\hat{h}\hat{h}}(p^2) = \Sigma^{\hat{h}\hat{h}}(p^2) - \delta M_\mathrm{h}^2 + \delta Z_{\hat{h}\hat{h}}(p^2 - M_\mathrm{h}^2)$ is not UV finite. Of course, this does not spoil the UV finiteness of S-matrix elements, since the theory with non-linearly realized Higgs doublet is still renormalizable. The compensation of UV divergences simply does not happen inside 1PI vertex functions (such as the self-energies) after renormalization, but results from a non-trivial conspiracy of the divergences between different renormalized vertex functions.

As long as the same renormalization transformations in the linearly and non-linearly realized theories are used with the same OS renormalization conditions, the resulting renormalized theories are fully equivalent, because OS conditions make use of properties of S-matrix elements that are independent of the nature of the Higgs field realizations. Thus, the OS renormalization [50] of the mixing angle α works in the SESM with linear or non-linear Higgs realizations exactly in the same way. More care is already needed for $\overline{\mathrm{MS}}$ renormalization, where the determination of the renormalization constant $\delta s_\alpha^{\overline{\mathrm{MS}}}$ has to be carried out based on S-matrix elements in the non-linear realization, while it is sufficient to consider some appropriate 1PI vertex function in the linear realization, as e.g. in Ref. [49]. A safe way to determine $\delta s_\alpha^{\overline{\mathrm{MS}}}$ is to take the UV-divergent part of $\delta s_\alpha^{\mathrm{OS}}$. On the other hand, the translation of the symmetry-inspired BFM schemes of Ref. [50] to the non-linearly realized theory is non-trivial, because these schemes are based on properties of the UV structure of specific vertex functions, which drastically differ from the ones in the non-linear realization. We, therefore, do not consider these symmetry-inspired renormalization schemes in this paper. Of course, one possibility to apply these schemes would be to integrate out the heavy Higgs field directly starting from the linearly realized SESM Lagrangian.

5.3 Renormalization of the EFT

In Sect. 4.2 we have integrated out the hard modes from the SESM at one-loop order. The result is the contribution to the effective Lagrangian given in Eq. (72), which contains $1/\epsilon$ singularities of UV and infrared (IR) origin. The UV divergences are absorbed by the (ζ-expanded) SESM renormalization constants of the previous section, while the IR divergences correspond to UV divergences of the EFT (with opposite sign) and can thus be interpreted as part of the counterterms of the EFT.

In Sect. 4.3 we have worked out the effective Lagrangian $\mathcal{L}_\mathrm{eff}^\mathrm{tree}$ describing all tree-level effects of the SESM and found that to $\mathcal{O}(\zeta^0)$ it has the same form as the SM Lagrangian, see Eq. (78). Using bare parameters and fields in the original (full-theory) Lagrangian in Eq. (75), this procedure automatically includes the SESM counterterms and yields the effective Lagrangian $\mathcal{L}_{\text{eff}}^{\text{tree},0}$. The "0" superscript indicates that the parameters and fields of $\mathcal{L}_{\text{eff}}^{\text{tree},0}$ are the bare quantities of Eq. (81), where the δc_i and δZ_i are the one-loop SESM renormalization constants expanded to sufficiently high order in $1/\zeta$. As long as the one-loop SESM renormalization constants have the same large-M_{H} scaling behaviour as originally assumed for the associated renormalized quantities, there is no further contribution from SESM counterterms to the effective Lagrangian. Note, in particular, that δM_{H}^2 in Eq. (101) is eliminated at $\mathcal{O}(\zeta^0)$ together with M_{H} in $\mathcal{L}_{\text{eff}}^{\text{tree}}$ upon using the EOM in Eq. (77).

In Eqs. (98), (102), and (104) we have observed, however, that, depending on the scheme, some of the renormalization constants are enhanced by positive powers of ζ compared to the scaling assumed for the corresponding renormalized quantities. The associated SESM counterterms thus give rise to additional contributions to the effective Lagrangian, which we dub $\delta\mathcal{L}_{\text{eff}}^{\text{SESMct}}$. They are derived in the same way as Eq. (78), i.e. employing the heavy-Higgs EOM, and also comprise the tadpole counterterms of Eq. (100) in analogy to Eq. (95). We find

$$
\begin{aligned}
\delta\mathcal{L}_{\text{eff}}^{\text{SESMct}} &= \delta M_{\text{h}}^2 \frac{3 s_\alpha^2}{4 v_2} \hat{h}^3 \left(1 + \frac{7\hat{h}}{6 v_2} + \frac{\hat{h}^2}{3 v_2^2}\right) \\
&\quad + \delta v_2 \frac{s_\alpha^2}{4 v_2^3} \hat{h} \Bigg[\hat{h}\,\Box \hat{h}^2 - g_2^2 v_2 \operatorname{tr}\big[\hat{C}^2\big](v_2^2 - \hat{h}^2) \\
&\quad - v_2 M_{\text{h}}^2 \hat{h}^2 \left(3 + \frac{7\hat{h}}{v_2} + \frac{3\hat{h}^2}{v_2^2}\right)\Bigg] \\
&\quad + \frac{s_\alpha \delta s_\alpha}{4 v_2^2} \hat{h} \Bigg[-\hat{h}\,\Box \hat{h}^2 - 2 g_2^2 v_2 \operatorname{tr}\big[\hat{C}^2\big](v_2 + \hat{h})^2 \\
&\quad + v_2 M_{\text{h}}^2 \hat{h}^2 \left(6 + \frac{7\hat{h}}{v_2} + \frac{2\hat{h}^2}{v_2^2}\right)\Bigg] \\
&\quad + \delta Z_{\hat{h}\hat{H}} \frac{s_\alpha}{4 v_2} \hat{h}^2 \Bigg[\Box \hat{h} - \frac{g_2^2}{2} \operatorname{tr}\big[\hat{C}^2\big](v_2 + \hat{h}) \\
&\quad + M_{\text{h}}^2 \hat{h} \left(1 + \frac{3\hat{h}}{2 v_2} + \frac{\hat{h}^2}{2 v_2^2}\right)\Bigg] \\
&\quad + \delta t_{\text{h}}^{\text{PRTS}} \frac{s_\alpha^2}{v_2^2} \hat{h}^3 \left(1 + \frac{17\hat{h}}{16 v_2} + \frac{\hat{h}^2}{4 v_2^2}\right) \\
&\quad + \delta t_{\text{H}}^{\text{PRTS}} \frac{s_\alpha}{M_{\text{H}}^2} \Bigg[\frac{g_2^2}{2} \operatorname{tr}\big[\hat{C}^2\big](v_2 + \hat{h}) \\
&\quad - \frac{2 M_{\text{h}}^2 + (1 - s_\alpha^2) M_{\text{H}}^2 + 2\lambda_{12} v_2^2}{2 v_2} \hat{h}^2 \\
&\quad - \frac{2 M_{\text{h}}^2 + (2 - 5 s_\alpha^2) M_{\text{H}}^2 + 4\lambda_{12} v_2^2}{4 v_2^2} \hat{h}^3 \\
&\quad - \frac{(1 - 8 s_\alpha^2) M_{\text{H}}^2 + 2\lambda_{12} v_2^2}{8 v_2^3} \hat{h}^4 + \frac{M_{\text{H}}^2 s_\alpha^2}{4 v_2^4} \hat{h}^5\Bigg] \\
&\quad + \delta t_{\text{h}}^{\text{FJTS}} \frac{s_\alpha^2}{2 v_2^2 M_{\text{h}}^2} \Bigg[\hat{h}^2\,\Box \hat{h} \\
&\quad + \frac{g_2^2 v_2}{2} \operatorname{tr}\big[\hat{C}^2\big](v_2 + \hat{h})(v_2 + 3\hat{h}) \\
&\quad - v_2 M_{\text{h}}^2 \hat{h}^2 \left(\frac{9}{2} + \frac{7\hat{h}}{v_2} + \frac{5\hat{h}^2}{2 v_2^2}\right)\Bigg] + \mathcal{O}(\zeta^{-2}),
\end{aligned} \tag{106}
$$

where δv_2 obeys Eq. (84) and, according to Eq. (102), scales like ζ^2 in the FJTS. The other renormalization constants and tadpole counterterms in Eq. (106) are given in the previous section. Note that $\delta\mathcal{L}_{\text{eff}}^{\text{SESMct}}$ is independent of $\delta Z_{\hat{H}\hat{H}}$, $\delta Z_{\hat{H}\hat{h}}$, and $\delta t_{\text{H}}^{\text{FJTS}} = -M_{\text{H}}^2 \Delta v_{\text{H}}$ in analogy to Eq. (92). This is true to any order in $1/\zeta$, because these renormalization constants are connected to field redefinitions of the heavy field $\hat{H}$, which is eliminated via its EOM. In fact, the EOM effectively eliminates the combination $\hat{H}(1+\delta Z_{\hat{H}\hat{H}}/2)+\delta Z_{\hat{H}\hat{h}}/2\hat{h}+\Delta v_{\text{H}}$. Similarly, the term in Eq. (106) that is proportional to $\delta Z_{\hat{h}\hat{H}}$ can be removed upon using the EOM (80) for the field $\hat{h}$ (again the quantum part h can be dropped here, because it would only contribute at two loops). Recall that the use of the EOM for $\hat{h}$ changes off-shell parts of Green functions, but not S-matrix elements, so that predictions for observables remain unaffected.

With Eqs. (72), (78), (106) and the SESM renormalization constants of Sect. 5.2 we can now write down the complete "bare" effective Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{eff}}^{\text{tree},0} + \delta\mathcal{L}_{\text{eff}}^{\text{1-loop}} + \delta\mathcal{L}_{\text{eff}}^{\text{SESMct}} \\
&= \mathcal{L}_{\text{eff}}^{\text{tree}} + \delta\mathcal{L}_{\text{eff}}^{\text{1-loop,ren}} + \delta\mathcal{L}_{\text{eff}}^{\text{ct}}.
\end{aligned} \tag{107}
$$

In the second line we have reshuffled the terms contributing to $\mathcal{L}_{\text{eff}}$ in such a way that $\mathcal{L}_{\text{eff}}^{\text{tree}}$ equals Eq. (78), $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop,ren}}$ is finite, and $\delta\mathcal{L}_{\text{eff}}^{\text{ct}}$ consists of the one-loop EFT counterterms. Although, the effective Lagrangian in Eq. (107) is already suitable for phenomenological studies at (fixed) NLO in the loop expansion, this Lagrangian is "bare" in the sense that it explicitly includes the UV-divergent counterterms (containing $1/\epsilon$ poles) required to render the one-loop corrections to physical observables based on $\mathcal{L}_{\text{eff}}^{\text{tree}}$ finite. The one-loop contributions from hard momentum ($\sim M_{\text{H}}$) modes are encoded in $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}$.

The exact form of $\mathcal{L}_{\text{eff}}^{\text{tree}}$, $\delta\mathcal{L}_{\text{eff}}^{\text{1-loop,ren}}$, and $\delta\mathcal{L}_{\text{eff}}^{\text{ct}}$ is only unique after fixing a renormalization scheme for the EFT. In the course of our derivation, the renormalization scheme for the parameters and fields of the $\mathcal{O}(\zeta^0)$ part of $\mathcal{L}_{\text{eff}}^{\text{tree}}$, which (here) equals $\mathcal{L}^{\rm SM}$, is initially inherited from the underlying renormalized full theory. For instance, the masses in the EFT are, according to Sect. 5.2, initially on-shell renormalized. Note, however, that the associated counterterms in $\delta\mathcal{L}_{\rm eff}^{\rm ct}$ differ in general from the respective SESM counterterms in $\delta\mathcal{L}_{\rm eff}^{\rm SESMct}$, because the contributions from large momentum modes cancel with terms in $\delta\mathcal{L}_{\rm eff}^{\rm 1\text{-}loop}$. The (bare) Wilson coefficients of the remaining (BSM-type) effective operators are initially composed of renormalized full-theory parameters. The renormalization conditions for these Wilson coefficients are in general not predetermined by the chosen full-theory renormalization scheme, because the respective operators are usually not part of the full-theory Lagrangian.

Once the (bare) $\mathcal{L}_{\rm eff}$ in Eq. (107) is derived, we can of course adopt any suitable renormalization scheme for the Wilson coefficients as well as for the SM-type parameters in the EFT by moving finite terms between $\mathcal{L}_{\rm eff}^{\rm tree}+\delta\mathcal{L}_{\rm eff}^{\rm 1\text{-}loop,ren}$, and $\delta\mathcal{L}_{\rm eff}^{\rm ct}$. In particular, to resum large logarithms ($\propto \ln M_{\rm h}/M_{\rm H}$) via renormalization group equations (RGEs) one may want to choose a (modified) minimal subtraction ($\overline{\rm MS}$) scheme for the Wilson coefficients and couplings of the EFT. In that case we write

$$C_{i,0}=C_i(\mu_{\rm R})+\delta C_i(\mu_{\rm R}), \tag{108}$$

where $C_{i,0}$ is the already determined bare coefficient of some effective operator and $\mu_{\rm R}$ denotes the renormalization scale on which the renormalized coefficient as well as its (one-loop) renormalization constant depend. The boundary (matching) condition of the corresponding one-loop RGE at a matching scale $\mu_{\rm M}$ is then given by

$$C_i(\mu_{\rm M})=C_{i,0}-\delta C_i(\mu_{\rm M}), \tag{109}$$

where in the $\overline{\rm MS}$ scheme $\delta C_i(\mu_{\rm M})$ equals the (divergent) terms proportional to Δ in $C_{i,0}$, with Δ as defined in Eq. (71). The $\overline{\rm MS}$ matching scale is identified with μ in Eq. (71) and $\mu_{\rm M}\equiv\mu\sim M_{\rm H}$ should be chosen to render the logarithms in Eq. (72) small. In this way a good convergence behaviour of the perturbative expansion of $C_i(\mu_{\rm M})$ is maintained. Solving the one-loop RGE for $C_i(\mu_{\rm R})$ then resums logarithms $\propto$ $(\ln\mu_R/\mu_{\rm M})^n$ at leading logarithmic order in renormalization-group-improved perturbation theory.[11] In EFT computations of physical observables $\mu_{\rm R}$ is fixed to a typical low-energy scale ($\sim M_{\rm h}$).

[11] The corresponding one-loop anomalous dimension matrix for all dimension-6 SMEFT operators was computed in Refs. [71–77].

5.4 Final form of the effective Lagrangian

Finally, to check the decoupling of all BSM effects, we have to investigate whether

$$\delta\mathcal{L}_{\rm eff}^{\rm BSM}\equiv\mathcal{L}_{\rm eff}-\mathcal{L}_{\rm SM}^{0}=\mathcal{O}(\zeta^{-2}), \tag{110}$$

where $\mathcal{L}_{\rm SM}^{0}$ denotes the "bare" SM Lagrangian including appropriate one-loop counterterms such that $\mathcal{L}_{\rm SM}^{0}$ cancels all SM-type operators of $\mathcal{L}_{\rm eff}$.[12] In particular, the part of the SM counterterm Lagrangian relevant here takes the form of Eq. (95). Consequently, $\delta\mathcal{L}_{\rm eff}^{\rm BSM}$ consists of non-SM-type operators only.

Using Eq. (107) we can thus write

$$\delta\mathcal{L}_{\rm eff}^{\rm BSM}=\left[\delta\mathcal{L}_{\rm eff}^{\rm 1\text{-}loop}+\delta\mathcal{L}_{\rm eff}^{\rm SESMct}\right]^{\rm BSM}, \tag{111}$$

where $[\mathcal{L}]^{\rm BSM}$ returns only the non-SM-type operators in $\mathcal{L}$. Subtracting from $\delta\mathcal{L}_{\rm eff}^{\rm SESMct}$ in Eq. (106) a SM-type term of the form of Eq. (95) with appropriately adjusted renormalization constants and tadpole counterterms we have

$$\begin{aligned}
\left[\delta\mathcal{L}_{\rm eff}^{\rm SESMct}\right]^{\rm BSM}=&\left(\delta v_2-\frac{\delta t_{\rm h}^{\rm FJTS}}{M_{\rm h}^2}\right)\frac{s_\alpha^2}{4v_2^2}\,\hat h\\
&\times\Big[-g_2^2\,{\rm tr}\big[\hat C^2\big](v_2+\hat h)(3v_2+\hat h)\\
&+M_{\rm h}^2\,\hat h^2\Big(3-\frac{\hat h^2}{v_2^2}\Big)\Big]\\
&+\big(3\delta t_{\rm h}^{\rm PRTS}+\delta t_{\rm H}^{\rm PRTS}s_\alpha+\delta M_{\rm h}^2 v_2\big)\,\frac{s_\alpha^2}{8v_2}\,\hat h^3\\
&\times\left(6+\frac{7\hat h}{v_2}+\frac{2\hat h^2}{v_2^2}\right)-\delta t_{\rm h}\,\frac{s_\alpha^2}{16v_2^2}\,\hat h^3\\
&\times\left(20+\frac{25\hat h}{v_2}+\frac{8\hat h^2}{v_2^2}\right)\\
&+\left(\frac{\delta s_\alpha}{s_\alpha}-\frac{\delta v_2}{v_2}\right)\frac{s_\alpha^2}{4v_2^2}\\
&\times\Big[-\hat h^2\,\Box\hat h^2-2g_2^2v_2\,{\rm tr}\big[\hat C^2\big]\hat h(\hat h+v_2)^2\\
&+M_{\rm h}^2v_2\,\hat h^3\left(6+\frac{7\hat h}{v_2}+\frac{2\hat h^2}{v_2^2}\right)\Big]+\mathcal{O}(\zeta^{-2}).
\end{aligned} \tag{112}$$

Here we have eliminated the terms involving the field renormalization constant $\delta Z_{\hat h\hat H}$ by using the EOM (80) for $\hat h$ as described below Eq. (106). The remaining (SESM) renormalization constants appearing on the r.h.s. of Eq. (112) are grouped in a way that makes the simultaneous use of

[12] This is, e.g., achieved by choosing the same renormalization scheme(s) for the SM parameters as for the corresponding full theory quantities and posing corresponding renormalization conditions.

the PRTS and FJTS particularly simple. The first term in Eq. (112) does not contribute at the considered order (ζ^0) at all, because in δv_2 as derived from Eqs. (84) and (102) only the term $\delta t_{\mathrm{h}}^{\mathrm{FJTS}}/M_{\mathrm{h}}^2$ contributes at $\mathcal{O}(\zeta^2)$ which is cancelled by the explicit $\delta t_{\mathrm{h}}^{\mathrm{FJTS}}$ term. In the second term on the r.h.s. of Eq. (112), the explicit PRTS tadpole terms and the FJTS tadpoles implicitly contained in δM_{h}^2 combine with the 1PI parts of δM_{h}^2 exactly in the same way, so that the overall contribution of the second line is independent of the tadpole scheme. The third term is given by the same tadpole term in the PRTS and FJTS. Only the last term on the r.h.s. of Eq. (112), which involves δs_α, depends on the tadpole scheme as well as on the renormalization scheme chosen for the (sine of the) mixing angle α in the SESM. Combining δv_2 with the scheme-dependent results for δs_α given in Eq. (98), we find that $(\delta s_\alpha/s_\alpha - \delta v_2/v_2)s_\alpha^2$ vanishes at $\mathcal{O}(\zeta^0)$ in all but the $\overline{\mathrm{MS}}$/FJTS scheme, where it is proportional to the UV-finite part of $s_\alpha^2 T^{\hat{h}}/(M_{\mathrm{h}}^2 v_2)$.

To simplify the final step towards the effective Lagrangian at $\mathcal{O}(\zeta^0)$, we now insert the explicit (ζ-expanded) expressions for the SESM renormalization constants and tadpole counterterms into Eq. (112) everywhere but in the last term to obtain

$$\left[\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{SESMct}}\right]^{\mathrm{BSM}} = -\frac{M_{\mathrm{H}}^2 s_\alpha^2}{64\pi^2 v_2^3}\left(M_{\mathrm{H}}^2 s_\alpha^2 - 2\lambda_{12}v_2^2\right)\hat{h}^3 \times\left(1+\frac{3\hat{h}}{4v_2}\right)(L_\epsilon+1) + \left(\frac{\delta s_\alpha}{s_\alpha}-\frac{\delta v_2}{v_2}\right)\frac{s_\alpha^2}{4v_2^2} \times\left[-\hat{h}^2\,\Box\hat{h}^2 - 2g_2^2 v_2\,\mathrm{tr}\left[\hat{C}^2\right]\hat{h}(\hat{h}+v_2)^2 + M_{\mathrm{h}}^2 v_2\,\hat{h}^3\left(6+\frac{7\hat{h}}{v_2}+\frac{2\hat{h}^2}{v_2^2}\right)\right] + \mathcal{O}(\zeta^{-2}). \tag{113}$$

Now, adding $\delta\mathcal{L}_{\mathrm{eff}}^{\text{1-loop}}$ of Eq. (72) and again dropping terms that can be absorbed in the SM counterterm in Eq. (95), we end up with

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{BSM}} = \left(\frac{\delta s_\alpha}{s_\alpha}-\frac{\delta v_2}{v_2}\right)\frac{s_\alpha^2}{4v_2^2}\left[-\hat{h}^2\,\Box\hat{h}^2 - 2g_2^2 v_2\,\mathrm{tr}\left[\hat{C}^2\right]\hat{h}(\hat{h}+v_2)^2 + M_{\mathrm{h}}^2 v_2\,\hat{h}^3\left(6+\frac{7\hat{h}}{v_2}+\frac{2\hat{h}^2}{v_2^2}\right)\right] + \mathcal{O}(\zeta^{-2}), \tag{114}$$

with the SESM renormalization constants δs_α as given in Eq. (98) and

$$\delta v_2 = v_2\frac{\delta M_{\mathrm{W}}^2}{2M_{\mathrm{W}}^2} + \mathcal{O}(\zeta^0) = \frac{\delta t_{\mathrm{h}}^{\mathrm{FJTS}}}{M_{\mathrm{h}}^2} + \mathcal{O}(\zeta^0) \tag{115}$$

according to Eqs. (84) and (102). Hence, we observe decoupling of the heavy Higgs boson H in the SESM for $M_{\mathrm{H}}\to\infty$, i.e.

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{BSM}} = \mathcal{O}(\zeta^{-2}),\quad \text{for the schemes OS/PRTS, OS/FJTS, and } \overline{\mathrm{MS}}\text{/PRTS}, \tag{116}$$

but non-decoupling in the $\overline{\mathrm{MS}}$/FJTS scheme,

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{BSM}}\Big|_{\overline{\mathrm{MS}}/\mathrm{FJTS}} = \frac{M_{\mathrm{H}}^2 s_\alpha^2(2\lambda_{12}v_2^2 - M_{\mathrm{H}}^2 s_\alpha^2)}{64\pi^2 M_{\mathrm{h}}^2 v_2^4} \times\left[\ln\left(\frac{\mu_{\mathrm{M}}^2}{M_{\mathrm{H}}^2}\right)+1\right] \times\left[\hat{h}^2\,\Box\hat{h}^2 + 2g_2^2 v_2\,\mathrm{tr}\left[\hat{C}^2\right]\hat{h}(\hat{h}+v_2)^2 - M_{\mathrm{h}}^2 v_2\,\hat{h}^3\left(6+\frac{7\hat{h}}{v_2}+\frac{2\hat{h}^2}{v_2^2}\right)\right] + \mathcal{O}(\zeta^{-2}). \tag{117}$$

Here we have used Eqs. (99) and (100), and identified the reference scale μ of dimensional regularization with the $\overline{\mathrm{MS}}$ renormalization scale of the full theory, which is in turn interpreted as the matching scale μ_{M} of the EFT (not to be confused with the EFT renormalization scale μ_R). Accordingly, the $\overline{\mathrm{MS}}$ renormalized mixing angle of the SESM depends on this scale, i.e. $s_\alpha \equiv \bar{s}_\alpha(\mu_{\mathrm{M}}^2)$ in Eq. (117).

At first sight, the explicit appearance of the renormalization scale μ_{M} in $\delta\mathcal{L}_{\mathrm{eff}}|_{\overline{\mathrm{MS}}/\mathrm{FJTS}}$ of Eq. (117) seems odd, because it potentially appears in NLO corrections to observables without being compensated by some implicit μ_{M} dependence in LO contributions, since the tree-level effective Lagrangian is just $\mathcal{L}_{\mathrm{SM}}$ to $\mathcal{O}(\zeta^0)$ and thus independent of s_α. To resolve this puzzle, we have to remember that our weak-coupling scenario in Eq. (14) assumes that $s_\alpha \equiv \bar{s}_\alpha(\mu_{\mathrm{M}}^2) = \mathcal{O}(\zeta^{-1})$. Here we have emphasized that the renormalized parameter s_α, in which the perturbative expansion works, is a running parameter $\bar{s}_\alpha(\mu_{\mathrm{M}}^2)$ tied to the SESM renormalization scale μ_{M}. The running of $\bar{s}_\alpha(\mu_{\mathrm{M}}^2)$ follows from the μ_{M} independence of the bare parameter $s_{\alpha,0}$ and the UV divergence of the renormalization constant in the SESM,

$$\frac{\partial\bar{s}_\alpha(\mu_{\mathrm{M}}^2)}{\partial\ln\mu_{\mathrm{M}}^2} = \beta_{s_\alpha}, \tag{118}$$

where the one-loop β-function of s_α in the $\overline{\mathrm{MS}}$/FJTS scheme can be directly read off Eq. (98):

$$\beta_{s_\alpha}\Big|_{\overline{\mathrm{MS}}/\mathrm{FJTS}} = \frac{\partial\delta s_\alpha}{\partial\Delta} = \frac{M_{\mathrm{H}}^2 s_\alpha(2\lambda_{12}v_2^2 - M_{\mathrm{H}}^2 s_\alpha^2)}{16\pi^2 M_{\mathrm{h}}^2 v_2^2} + \mathcal{O}(\zeta^{-1}). \tag{119}$$

Solving Eq. (118) iteratively to NLO in the loop ($\lambda_{12} \sim g_2^2$) expansion for $M_{\mathrm{H}} \to \infty$, we find

$$\bar{s}_\alpha(\mu_{\mathrm{M}}^2)|_{\overline{\mathrm{MS}}/\mathrm{FJTS}} = \bar{s}_\alpha(\hat{\mu}_{\mathrm{M}}^2) + \frac{M_{\mathrm{H}}^2 \bar{s}_\alpha(\hat{\mu}_{\mathrm{M}}^2)\big(2\lambda_{12} v_2^2 - M_{\mathrm{H}}^2 \bar{s}_\alpha(\hat{\mu}_{\mathrm{M}}^2)^2\big)}{16\pi^2 M_{\mathrm{h}}^2 v_2^2} \times \ln\left(\frac{\mu_{\mathrm{M}}^2}{\hat{\mu}_{\mathrm{M}}^2}\right) + \mathcal{O}(\zeta^{-1}), \qquad (120)$$

where $\hat{\mu}_{\mathrm{M}}$ can take an arbitrary value different from μ_{M}. From Eq. (120) we see that if we start from a specific renormalization scale μ_{M} for which the assumption $\bar{s}_\alpha(\mu_{\mathrm{M}}^2) = \mathcal{O}(\zeta^{-1})$ holds, this assumption is not fulfilled for $\bar{s}_\alpha(\hat{\mu}_{\mathrm{M}}^2)$ anymore if $\mu_{\mathrm{M}} \neq \hat{\mu}_{\mathrm{M}}$. Thus, if we want to change the SESM renormalization scale, we have to take into account the terms in $\mathcal{L}_{\mathrm{eff}}^{\mathrm{tree}}$ that are promoted from $\mathcal{O}(\zeta^{-2})$ to $\mathcal{O}(\zeta^{0})$, when substituting Eq. (120) for s_α and counting the one-loop correction to $s_\alpha(\mu_{\mathrm{M}}^2)$ as $\mathcal{O}(\zeta^{1})$. Note that two-loop terms have to be dropped consistently after the replacement. The relevant terms in $\mathcal{L}_{\mathrm{eff}}^{\mathrm{tree}}$ are derived in a straightforward way following Sect. 4.3 and read (before the substitution)

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{tree}} = -\frac{\bar{s}_\alpha(\mu_{\mathrm{M}}^2)^2}{8 v_2^2}\Big[\hat{h}^2\,\Box\hat{h}^2 + 2 g_2^2 v_2\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h}(\hat{h}+v_2)^2 - M_{\mathrm{h}}^2 v_2\,\hat{h}^3\left(6 + \frac{7\hat{h}}{v_2} + \frac{2\hat{h}^2}{v_2^2}\right)\Big] + \cdots, \qquad (121)$$

where the ellipses refer to terms that are of $\mathcal{O}(\zeta^{-2})$ even after incorporating the enhanced one-loop correction of Eq. (120). Combining $\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{tree}}$ with the one-loop part of $\delta\mathcal{L}_{\mathrm{eff}}$ given in Eq. (117), we can now write the final effective Lagrangian to $\mathcal{O}(\zeta^{0})$ in the form

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{BSM}}|_{\overline{\mathrm{MS}}/\mathrm{FJTS}} = \frac{\bar{s}_\alpha(M_{\mathrm{H}}^2)^2}{8 v_2^2}\left[-1 + \frac{M_{\mathrm{H}}^2\big(2\lambda_{12} v_2^2 - M_{\mathrm{H}}^2 \bar{s}_\alpha(M_{\mathrm{H}}^2)^2\big)}{8\pi^2 M_{\mathrm{h}}^2 v_2^2}\right] \times \Big[\hat{h}^2\,\Box\hat{h}^2 + 2 g_2^2 v_2\,\mathrm{tr}\big[\hat{C}^2\big]\hat{h}(\hat{h}+v_2)^2 - M_{\mathrm{h}}^2 v_2\,\hat{h}^3\left(6 + \frac{7\hat{h}}{v_2} + \frac{2\hat{h}^2}{v_2^2}\right)\Big] + \mathcal{O}(\zeta^{-2}), \qquad (122)$$

which is renormalization/matching scale independent at one-loop order. To compactify the result, we have set $\hat{\mu}_{\mathrm{M}} = M_{\mathrm{H}}$. Note that in Eq. (122), and accordingly in Eq. (120), the loop and large-mass expansions are intertwined in the sense that one should treat $\bar{s}_\alpha(\hat{\mu}_{\mathrm{M}}^2)^n$ to be of $\mathcal{O}(\zeta^{2-n})$ or $\mathcal{O}(\zeta^{-n})$ when it appears in a tree-level or a one-loop term, respectively. This non-uniform scaling behaviour of s_α in the $\overline{\mathrm{MS}}$/FJTS scheme continues at higher loop orders and is particularly problematic when it comes to resummation in the EFT (for quantities where s_α appears in the anomalous dimension). Concerning Eq. (122), we can loosely speaking say that adding the one-loop corrections to the effective Lagrangian in the $\overline{\mathrm{MS}}$/FJTS scheme effectively changes the scale at which $\bar{s}_\alpha$ is evaluated from μ_{M}, where $\bar{s}_\alpha(\mu_{\mathrm{M}}^2)$ is strongly suppressed, to $\hat{\mu}_{\mathrm{M}} = M_{\mathrm{H}}$, where $\bar{s}_\alpha(M_{\mathrm{H}}^2)$ is enhanced by one-loop corrections of $\mathcal{O}(\zeta^{1})$.

Using the EOM of $\hat{h}$, given in Eq. (80), but with $\hat{h}$ interpreted as SM Higgs field (i.e. dropping again the quantum part of $\tilde{h}$), and absorbing some terms into the SM renormalization constants in $\mathcal{L}_{\mathrm{SM}}^{\mathrm{Hct}}$, the operator appearing in Eqs. (117) and (122) can be rewritten as

$$\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{BSM}}|_{\overline{\mathrm{MS}}/\mathrm{FJTS}} = C_{\Phi\Box}(\mu_{\mathrm{M}}^2)\, Q_{\Phi\Box} + \mathcal{O}(\zeta^{-2}), \qquad (123)$$

where $Q_{\Phi\Box}$ is one of the SMEFT operators in the Warsaw basis [2] usually written as

$$Q_{\Phi\Box} = \frac{1}{4}\big(\mathrm{tr}\big[\hat{\Phi}_{\mathrm{SM}}^\dagger \hat{\Phi}_{\mathrm{SM}}\big]\big)\Box\big(\mathrm{tr}\big[\hat{\Phi}_{\mathrm{SM}}^\dagger \hat{\Phi}_{\mathrm{SM}}\big]\big) = \big(\hat{\phi}_{\mathrm{SM}}^\dagger \hat{\phi}_{\mathrm{SM}}\big)\Box\big(\hat{\phi}_{\mathrm{SM}}^\dagger \hat{\phi}_{\mathrm{SM}}\big). \qquad (124)$$

Here $\hat{\Phi}_{\mathrm{SM}}$ denotes the matrix-valued SM background Higgs field and $\hat{\phi}_{\mathrm{SM}}$ is the corresponding two-component SM background Higgs doublet field in the linear realization, i.e.

$$\hat{\phi}_{\mathrm{SM}} = \begin{pmatrix} \hat{\phi}^+ \\ (v_2 + \hat{h} + \mathrm{i}\hat{\chi})/\sqrt{2} \end{pmatrix}. \qquad (125)$$

Note that we write $\hat{h}$ instead of $\hat{h}_2$ for the SM Higgs field. The background Goldstone-boson fields $\hat{\phi}^{\pm}$ and $\hat{\chi}$ are defined as in Eq. (2). According to Eq. (122) the Wilson coefficient $C_{\Phi\Box}(\mu_{\mathrm{M}}^2)$ in Eq. (123) is

$$C_{\Phi\Box}(\mu_{\mathrm{M}}^2) = -\frac{\bar{s}_\alpha(\mu_{\mathrm{M}}^2)^2}{2 v_2^2} + \frac{M_{\mathrm{H}}^2 \bar{s}_\alpha(\mu_{\mathrm{M}}^2)^2\,\big(2\lambda_{12} v_2^2 - M_{\mathrm{H}}^2 \bar{s}_\alpha(\mu_{\mathrm{M}}^2)^2\big)}{16\pi^2 M_{\mathrm{h}}^2 v_2^4} \times \left[\ln\left(\frac{\mu_{\mathrm{M}}^2}{M_{\mathrm{H}}^2}\right) + 1\right] + \mathcal{O}(\zeta^{-2}) = \frac{\bar{s}_\alpha(M_{\mathrm{H}}^2)^2}{2 v_2^2}\left[-1 + \frac{M_{\mathrm{H}}^2\,\big(2\lambda_{12} v_2^2 - M_{\mathrm{H}}^2 \bar{s}_\alpha(M_{\mathrm{H}}^2)^2\big)}{8\pi^2 M_{\mathrm{h}}^2 v_2^2}\right] + \mathcal{O}(\zeta^{-2}). \qquad (126)$$

In this simple example, no particular effort was needed to bring the final result into SMEFT form. In more complicated cases, we first have to translate the final effective Lagrangian into a basis of gauge-invariant operators upon inverting the Stueckelberg transformation in Eq. (20) as described in Refs. [14,15]. In a second step EOMs for the light fields, in our case the SM fields, can be used to bring all occurring operators in the effective Lagrangian into canonical form, which is the SMEFT basis in our case. We note that despite $Q_{\Phi\Box}$ being a dimension-six operator it formally contributes, because of its Wilson coefficient $C_{\Phi\Box}$, to the effective Lagrangian at $\mathcal{O}(\zeta^{0})$ in the $\overline{\mathrm{MS}}$/FJTS scheme. On the other hand, re-expressing $\bar{s}_\alpha(M_{\mathrm{H}}^2)$ in terms of the on-shell renormalized s_α and consistently re-expanding to one-loop order renders $C_{\Phi\Box}$ to be of $\mathcal{O}(\zeta^{-2})$.

Comparing the different renormalization schemes, we have to conclude that the $\overline{\text{MS}}$/FJTS scheme does not reflect the true nature of the $M_\text{H} \to \infty$ limit of the SESM in a sound way. The decoupling behaviour observed in the other schemes and at tree level is broken at the one-loop level or, more precisely, decoupling at one loop only happens at the fine-tuned scale $\mu_\text{M} = M_\text{H}/\text{e}$, with e being Euler's constant, where $C_{\Phi\Box}(\mu_\text{M}^2)$ is of $\mathcal{O}(\zeta^{-2})$. The origin of this odd behaviour is the fact that the renormalization constant δs_α and thus the one-loop contribution to $\bar{s}_\alpha(\mu_\text{M}^2)$ does not scale in the same way as initially assumed for the bare (tree-level) parameter $s_{\alpha,0}$ in the heavy-mass limit. As a result, some NLO corrections in the SESM tend to get unnaturally large in the $\overline{\text{MS}}$/FJTS scheme for large M_H, so that this scheme is not recommendable for the SESM with large M_H. In particular, this scheme does not allow to retain $\bar{s}_\alpha(\mu_\text{M}^2)$ (without loop-order dependent fine-tuning) as a parameter in the EFT describing the large-mass limit in a consistent way, because it obscures the power counting.[13]

Finally, we note that the unpleasant behaviour of the $\overline{\text{MS}}$/FJTS scheme is certainly not tied to the specific case of the SESM. Artificially large corrections in the $\overline{\text{MS}}$/FJTS scheme for mixing angles have, for instance, also been found in scenarios of the Two-Higgs-Doublet Models with large Higgs-boson masses in Refs. [52,53,78,79]. In general, this typically occurs when the full-theory loop expansion of an EFT parameter has non-uniform scaling behaviour in the heavy-mass limit. Nevertheless a case-by-case study is always recommended in order to analyse the decoupling behaviour of each renormalization scheme.

6 Conclusion

Building on earlier work, in this article we have described a general procedure to integrate out heavy fields directly in the path integral and to derive an effective Lagrangian at the one-loop level. The method is based on the *background-field formalism*, which implies a natural separation of tree-level and loop effects of the heavy fields, and on the *expansion by regions*, which further separates loop effects into contributions from large and small momentum modes. Combining these concepts, together with additional technical tricks (non-linear Higgs realization, field redefinitions, EOMs, etc.), lends the method some particular strengths:

- *Transparency:* The clear separation of tree-level and loop effects of the heavy fields and the further decomposition of field modes into light and heavy degrees of freedom render the procedure very transparent. At every stage of our calculation it is possible to identify the origin of all contributions to the effective Lagrangian in terms of (combinations of) Feynman diagrams.
- *Flexibility:* The method for integrating out heavy fields is fully flexible in the sense that no preknowledge of the low-energy effective theory is needed, i.e. no ansatz for the effective Lagrangian is made in advance. The fields in the full theory just have to be divided into sets of light fields, providing the dynamical degrees of freedom at low energies, and heavy fields, which will be integrated out and the effects of which will appear in effective operators composed of the light fields. Besides that, given a large mass scale Λ, a proper definition of the large-mass scenario has to be specified by a power-counting scheme for all model parameters in the limit $\Lambda \to \infty$.
- *Gauge invariance:* In the the background-field method the gauge of the background fields, which correspond to the fields on tree lines in Feynman diagrams, can be fixed independent of the gauge of the quantum fields, which are the fields appearing inside loops. This feature can, for instance, be exploited to simplify the explicit calculation of the effective Lagrangian, by choosing a specific background gauge in intermediate steps and restoring gauge invariance at the end. This proves particularly powerful in combination with a non-linear realization of the SM Higgs sector.
- *Algorithmic organization:* The method is fully algorithmic and suitable for automation. Given a properly defined large-mass scenario and some details on the renormalization of the large-mass sector, the actual determination of the effective Lagrangian at the one-loop level can, in principle, be carried out by computer algebra. Recently, some steps in this procedure have already been automated, see Refs. [40,41].

Compared to other related approaches described in the literature (such as the UOLEA approach) for integrating out heavy fields, our presentation might seem somewhat lengthy, but to a large extent this is due to the fact that our formulation is very close to the actual NLO machinery used in precision calculations in SM extensions. In the first place this means that we work in a field basis corresponding to mass eigenstates by diagonalizing mass matrices involving heavy fields before integrating out the heavy degrees of freedom. This procedure does not only avoid doubts on the consistent treatment of mixing effects raised in the literature w.r.t. other approaches, it also very naturally prepares an appropriate framework to include renormalization prescriptions that

[13] In Ref. [42] an on-shell renormalization scheme was adopted for the SESM (with linear Higgs realization), so that the results are independent of the chosen ("β_h") tadpole scheme. Hence, there are no subtleties connected to tadpole renormalization like the spurious $\mathcal{O}(\zeta^0)$ non-decoupling terms in $\overline{\text{MS}}$/FJTS.

are designed for phenomenological analyses (e.g. by taking mixing angles as independent parameters).

We have illustrated the method by considering a singlet Higgs extension of the Standard Model in which a heavy Higgs boson H exists in addition to the known Standard-Model-like Higgs boson h, which is experimentally investigated at the LHC. To be precise, we have calculated potential non-decoupling effects of H in the limit $M_{\mathrm{H}} \to \infty$, assuming a weak coupling scenario in which the mixing angle α between H and the singlet scalar of the model is suppressed by a factor $\sim M_{\mathrm{h}}/M_{\mathrm{H}}$. We have carried out our calculation in a field basis corresponding to mass eigenstates, in order to avoid issues in the mixing between fields of light and heavy particles. In the course of the calculation we have emphasized the issue of renormalization of the non-standard sector of the theory – an aspect that is widely ignored in the literature on the construction of effective Lagrangians for heavy-particles effects. Non-trivial contributions connected to renormalization appear whenever model parameters and the corresponding renormalization constants scale differently in the heavy-mass limit. Spontaneously broken gauge theories with extended scalar sectors are particularly prone to such issues, because heavy Higgs-boson masses often enhance scalar self-couplings. In this context, the renormalization of vacuum expectation values and corresponding tadpole contributions in the full SESM deserve particular care. In the specific model with the heavy Higgs singlet H we observe for example full decoupling for $M_{\mathrm{H}} \to \infty$ using an on-shell renormalization scheme for the Higgs mixing angle α. For commonly used $\overline{\mathrm{MS}}$ renormalization schemes for α, on the other hand, we find decoupling or non-decoupling depending on the treatment of tadpole contributions. In the latter case the construction of a consistent EFT is problematic.

Owing to its transparent, flexible, and algorithmic structure the method opens a vast field of applications. The natural next step is to extend the calculation of all heavy-Higgs effects associated with the singlet extension considered in this paper to order $1/M_{\mathrm{H}}^2$ and the determination of the corresponding dimension-six SMEFT Lagrangian for various renormalization schemes. The effects of integrating out the heavy Higgs field on the fermionic sector of the SESM, neglected here for brevity, also remain to be analyzed in detail. These tasks and some phenomenological applications will be addressed in a forthcoming publication.

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Appendix A: Evaluation of the functional determinant

In this appendix we describe the evaluation of the functional determinant in Eq. (41). Let us first introduce the Hilbert space version of the heavy-mode projection operator in Eq. (25):

$$\langle p|T_h|\phi\rangle \equiv \langle p|\phi_h\rangle \equiv \mathcal{T}_h(p)\langle p|\phi\rangle = \mathcal{T}_h(p)\,\phi(p) = \phi_h(p). \tag{A.1}$$

We now write the differential operator $\tilde{\Delta}_H(x,\partial_x)$, which operates on functions $\psi(x)$, $\phi(x)$ in Minkowski space, as matrix elements of a linear operator M acting on the elements $|\psi\rangle$, $|\phi\rangle$ of the corresponding Hilbert space. In the usual bracket notation, we thus have

$$|\phi\rangle = M\,|\psi\rangle, \quad \psi(x) = \langle x|\psi\rangle, \quad \phi(x) = \langle x|\phi\rangle, \tag{A.2}$$

$$\phi(x) = \int \mathrm{d}^D y\,\langle x|M|y\rangle\,\psi(y) = \tilde{\Delta}_H(x,\partial_x)\,\psi(x), \tag{A.3}$$

so that we can identify

$$\langle x|M|y\rangle = \tilde{\Delta}_H(x,\partial_x)\,\delta(x-y) = \delta(x-y)\,\tilde{\Delta}_H(y,\partial_y). \tag{A.4}$$

The last relation is obtained via partial integration under the y-integral and expresses the hermiticity of $\tilde{\Delta}_H$. With this notation we can replace the clumsy expression $\mathcal{D}et_h\big[\delta(x-y)\tilde{\Delta}_H(x,\partial_x)\big]$ in Eq. (41) by a more accurate one: $\mathcal{D}et_h[M]$, which represents the functional determinant of the suboperator of M that acts only on the subspace of hard-momentum states $|H_h\rangle$. We then evaluate the 1-loop part of the EFT action as

$$\begin{aligned}\mu^{D-4}\int \mathrm{d}^D x\;\delta\mathcal{L}_{\mathrm{eff}}^{\text{1-loop}} &= \frac{\mathrm{i}}{2}\ln\big(\mathcal{D}et_h[M]\big) = \frac{\mathrm{i}}{2}\,\mathcal{T}r_h\,[\ln(M)]\\ &= \frac{\mathrm{i}}{2}\,\mathcal{T}r\,[T_h\ln(M)]\\ &= \frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\,\langle p|T_h\ln(M)|p\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p)\, \langle p|\ln(M)|p\rangle \\
&= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p) \int \mathrm{d}^D x \\
&\quad \times \int \mathrm{d}^D y\, \langle p|x\rangle\, \langle x|\ln(M)\,|y\rangle\, \langle y|p\rangle \\
&= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p) \int \mathrm{d}^D x \int \mathrm{d}^D y \\
&\quad \times \mathrm{e}^{-\mathrm{i}px}\, \langle x|\ln(M)\,|y\rangle\, \mathrm{e}^{\mathrm{i}py}, \qquad \text{(A.5)}
\end{aligned}
$$

where $\langle x|p\rangle = \mathrm{e}^{ipx}$ is the eigenfunction of the derivative operator $-\mathrm{i}\partial_x^\mu$ with eigenvalue p^μ and $|p\rangle$ denotes the corresponding momentum eigenstate. The matrix element $\langle x|\ln(M)\,|y\rangle$ is evaluated via the usual power series of the logarithm of the operator M, which we express in terms of the deviation N from the unit operator $\mathbb{1}$,

$$
M \equiv \mathbb{1} - N, \quad \ln(M) = -\sum_{k=1}^{\infty} \frac{N^k}{k}. \qquad \text{(A.6)}
$$

Writing the matrix element of N according to Eq. (A.4) as

$$
\langle x|N|y\rangle = \delta(x-y)\, n(y, \partial_y), \quad \tilde{\Delta}_H = 1 - n, \qquad \text{(A.7)}
$$

we have

$$
\begin{aligned}
\langle x|\ln(M)\,|y\rangle &= -\sum_{k=1}^{\infty} \frac{1}{k} \langle x|N^k\,|y\rangle \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \int \mathrm{d}^D x_1 \cdots \int \mathrm{d}^D x_{k-1} \\
&\quad \times \langle x|N\,|x_1\rangle\, \langle x_1|N\,|x_2\rangle \cdots \langle x_{k-1}|N\,|y\rangle \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \int \mathrm{d}^D x_1 \cdots \int \mathrm{d}^D x_{k-1} \\
&\quad \times \langle x|N\,|x_1\rangle\, \delta(x_1 - x_2)\, n(x_2, \partial_{x_2}) \cdots \\
&= -\sum_{k=1}^{\infty} \frac{1}{k} \langle x|N\,|y\rangle\, n(y, \partial_y)^{k-1} \\
&= -\delta(x-y) \sum_{k=1}^{\infty} \frac{n(y, \partial_y)^k}{k} \\
&= \delta(x-y) \ln\left[1 - n(y, \partial_y)\right] = \delta(x-y) \\
&\quad \times \ln\left[\tilde{\Delta}_H(y, \partial_y)\right]. \qquad \text{(A.8)}
\end{aligned}
$$

Inserting this into Eq. (A.5) we obtain for the 1-loop effective Lagrangian

$$
\begin{aligned}
&\delta\mathcal{L}_{\text{eff}}^{\text{1-loop}}(x) \\
&= \frac{\mathrm{i}}{2} \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p) \int \mathrm{d}^D y\, \mathrm{e}^{-\mathrm{i}px}\, \delta(x-y) \\
&\quad \times \ln\left[\tilde{\Delta}_H(y, \partial_y)\right] \mathrm{e}^{\mathrm{i}py} \\
&= \frac{\mathrm{i}}{2} \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p)\, \mathrm{e}^{-\mathrm{i}px} \ln\left[\tilde{\Delta}_H(x, \partial_x)\right] \mathrm{e}^{\mathrm{i}px} \\
&= \frac{\mathrm{i}}{2} \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathcal{T}_h(p) \ln\left[\tilde{\Delta}_H(x, \partial_x + \mathrm{i}p)\right], \quad \text{(A.9)}
\end{aligned}
$$

which is the result given in Eq. (42).

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