Link Polynomial, Crossing Multiplier and Surgery Formula*

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ABSTRACT

Relations between link polynomials constructed from exactly solvable lattice models and topological field theory are reviewed. It is found that the surgery formula for a three-sphere S^3 with Wilson lines corresponds to the Markov trace constructed from the exactly solvable models. This indicates that knot theory intimately relates various important subjects such as exactly solvable models, conformal field theories and topological quantum field theories.

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1. Introduction

A general theory has been established [1-6] to construct link polynomials, topological invariants for knots and links, from exactly solvable models in statistical mechanics. The theory, which is applicable to both vertex models and IRF models, consists of two steps. First, one makes a representation of the braid group from the Boltzmann weights of a solvable model. Second, one constructs the Markov trace on the representation. The Markov trace is essentially a trace of the braid group representation multiplied by the crossing multiplier of the model.

Recently, E. Witten has shown remarkable relations among three dimensional topological quantum field theory, two dimensional conformal field theory and link polynomials [7]. This topological field theory is the Chern-Simons gauge theory with non abelian gauge group G, whose abelian version is related to the fermi-bose transmutation and self-linking number [8].

Let us recall the formulation of the topological quantum field theory [7]. In a closed oriented 3-manifold M, we take a link L which consists of r oriented and non-intersecting knots $C_i, i = 1, 2, \cdots, r$. We assign a representation R of group G to each C_i, and consider the following path integral over all gauge orbits

$$Z(M; L) = Z(M; C_i, R_i) = \int DA \exp(ikS[A]) \prod_{i=1}^{r} W_{R_i}(C_i). \tag{1.1}$$

Here, $S[A]$ is the Chern-Simons action,

$$W_R(C) = \mathrm{Tr}_R[P\exp \oint_C A_i dx^i] \tag{1.2}$$

is the Wilson line which represents the holonomy around C and $k \in \mathbf{Z}$ is (topologically quantized) coupling.

By the method of geometric quantization, Witten has shown that the physical Hilbert space of this theory is isomorphic to the space of conformal blocks of associated Wess-Zumino-Witten model [9] at level k, and hence the Hilbert space is of finite dimensions.

We can give another argument for this important result. Note that the Gauss law constraint of the cannonical quantization is essentially equivalent to the (anomalous) Ward identity for the currents. We can show that the current Ward identity and integrability condition characterize the space of conformal blocks, and this space is shown to be of finite dimensions [10].

From the knowledge of eigenvalues of braid matrices, Witten has derived the skein relation for the expectation value of Wilson lines. For $G = SU(N)$ and Wilson lines in fundamental representation, he has explicitly given the skein relation defining the Jones polynomial [11-13]. For other gauge groups $G = SO(N)$ and $Sp(2n)$, this method can be extended straightforwardly, and yields the Kauffman polynomial [14].

Using "surgery" technique, Witten has presented the following formula:

$$Z(S^3; L) = \sum_j Z(S^2 \times S^1; R_j, B) S_0^j, \tag{1.3}$$

where $Z(S^2 \times S^1; R_j, B)$ is the partition function on $S^2 \times S^1$ of both the braid B and a parallel Wilson line in the R_j representation. The quantity S_i^j is the elements of the modular transformation matrix.

The main aim of this report is to compare Witten's results, in particular, the surgery formula (1.3) with the knot theory based on the exactly solvable models [15]. In §2, we shall briefly explain the knot theory based on exactly solvable models in statistical mechanics. In §3, the significances of the crossing symmetry will be exhibited. In §4, the relation between the surgery formula and the Markov trace constructed from solvable models is explained. In §5, graph-state IRF models are introduced to show the ubiquity of the crossing multiplier. The last section is devoted to discussions.

2. Exactly Solvable Models and Link Polynomials

We introduce braids to describe knots and links [16]. It is known that any oriented link can be expressed by a closed braid [17]. The equivalent braids expressing the same link are mutually transformed by successive applications of Markov moves, I and II (Fig.1) [18]. Markov trace is a linear functional on the representation of the braid group B_n which has the following properties (the Markov properties):

$$\begin{aligned} &\text{I.} \quad \phi(AB) = \phi(BA), \qquad A, B \in B_n, \\ &\text{II.} \quad \phi(Ab_n) = \tau\phi(A), \\ &\qquad \phi(Ab_n^{-1}) = \bar{\tau}\phi(A), \qquad A \in B_n, \quad Ab_n^{\pm 1} \in B_{n+1}, \end{aligned}$$

where

$$\tau = \phi(b_i), \qquad \bar{\tau} = \phi(b_i^{-1}), \quad i = 1, \cdots, n+1. \tag{2.1}$$

From the Markov trace we obtain a link polynomial.

The Boltzmann weight $w(a, b, c, d; u)$ of IRF model is defined for the configuration of "spins" around a face (Fig.2), where u is the spectral parameter. The Yang-Baxter relation reads as (Fig.3) [19]

$$\begin{aligned} &\sum_g w(a, b, g, f; u) w(f, g, d, e; u+v) w(g, b, c, d; v) \\ &= \sum_g w(g, c, d, e; u) w(a, b, d, g; u+v) w(f, a, g, e; v), \end{aligned} \tag{2.2}$$

In addition to the Yang-Baxter relation, the Boltzmann weights of most of exactly solvable IRF models satisfy the following basic relations [19,6].

1) standard initial condition

$$w(a, b, c, d; u = 0) = \delta(a, c), \tag{2.3}$$

where $\delta(a, b)$ is the Kronecker delta.

2) inversion relation

$$\sum_e w(e,c,d,a;u)w(b,c,e,a;-u) = \delta(b,d). \tag{2.4}$$

3) crossing symmetry (Fig.4)

$$w(a,b,c,d;u) = w(b,c,d,a;\lambda-u)\left(\frac{\psi(a)\psi(c)}{\psi(b)\psi(d)}\right)^{1/2}, \tag{2.5}$$

where $\{\psi(a)\}$ are the crossing multipliers and λ is the crossing parameter.

4) second inversion relation

$$\sum_e w(c,e,a,b;\lambda-u)w(a,e,c,d;\lambda+u)\frac{\psi(e)\psi(b)}{\psi(a)\psi(c)} = \delta(b,d). \tag{2.6}$$

The crossing multipliers of the model are introduced in the crossing symmetry and the second inversion relation. They are important in a general theory to construct link polynomials from the exactly solvable models. As will be shown in §3, the basic relations correspond to local moves, known as the Reidemeister moves, on the link diagrams.

Let us explain the constraint of the model. Nearest neighboring spins of IRF model satisfy the constraint. When spin b is allowed (or admissible) to be nearest neighbor of spin a, then we write it as $b \sim a$. Of course, the Boltzmann weight is equal to 0 if the configuration is not admissible. A sequence of spins $\ell = (\ell_0, \ell_1, \cdots, \ell_n)$ is also called admissible if $\ell_i \sim \ell_{i-1} (i = 1, \cdots, n)$.

Let us introduce the Yang-Baxter operators and construct representations of the braid group. The Yang-Baxter operator $X_i(u)$ for IRF models are defined by [2] (Fig.5)

$$[X_i(u)]^{p_0p_1\cdots p_n}_{\ell_0\ell_1\cdots\ell_n} = \left(\prod_{j=0}^{i-1}\delta^{p_j}_{\ell_j}\right) w(\ell_i, \ell_{i+1}, p_i, \ell_{i-1}; u) \left(\prod_{j=i+1}^{n}\delta^{p_j}_{\ell_j}\right), \tag{2.7}$$

Here and hereafter multi-indicies $(p_0, p_1, \cdots, p_n)$ are assumed to be admissible. The Yang-Baxter operator $X_i(u)$ is a constituent of the transfer matrix $T(u)$

in a 2-dimensional lattice system. In terms of the Yang-Baxter operator the Yang-Baxter relation is written as [19,6]

$$\begin{aligned} X_i(u)X_{i+1}(u+v)X_i(v) &= X_{i+1}(v)X_i(u+v)X_{i+1}(u), \\ X_i(u)X_j(v) &= X_j(v)X_i(u), \qquad |i-j| \geq 2. \end{aligned} \tag{2.8}$$

We call this algebra the Yang-Baxter algebra. We see that the defining relations of the Yang-Baxter algebra (2.8) are analogous to the braid relations. From the Yang-Baxter operators $\{X_i(u)\}$ we can construct the representation of the braid group $\{G_i\}$ by the following formula [1]

$$G_i^{\pm 1} = \lim_{u \to \infty} X_i(\pm u). \tag{2.9}$$

The well-definedness of the limit requires that the model is critical, that is, the Boltzmann weight is parametrized by trigonometric (or hyperbolic) function.

Let us construct Markov trace $\phi(\cdot)$ on the braid group representation. Using the crossing multipliers of the model, we introduce a "constrained trace" $\tilde{\mathrm{Tr}}(A)$ [2]:

$$\tilde{\mathrm{Tr}}(A) = \tilde{\sum_{\ell_1 \cdots \ell_n}} A_{\ell_0 \ell_1 \cdots \ell_n}^{\ell_0 \ell_1 \cdots \ell_n} \frac{\psi(\ell_n)}{\psi(\ell_0)} \qquad (\ell_0 : \text{fixed}), \tag{2.10a}$$

then the Markov trace $\phi(\cdot)$ is written as

$$\phi(A) = \frac{\tilde{\mathrm{Tr}}(A)}{\tilde{\mathrm{Tr}}(I^{(n)})}, \qquad A, I^{(n)} \in B_n, \tag{2.10b}$$

where $I^{(n)}$ is the identity. The symbol $\tilde{\Sigma}$ represents the summataion over admissible multi-indices with ℓ_0 being fixed. We also have a formula of the Markov trace for vertex models [1,6], which is related to (2.10) by the Wu-Kadanoff-Wegner transformation [6,20].

We can show that the trace $\phi(\cdot)$ defined in (2.10) satisfies the Markov properties (2.1) by proving the extended Markov property:

$$\sum_{b\sim a} w(a,b,a,c;u)\frac{\psi(b)}{\psi(a)} = H(u;\lambda) \qquad \text{(independent of a, c)}, \tag{2.11}$$

where the function $H(u;\lambda)$ is called the characteristic function [6].

3. Crossing Symmetry, Temperley-Lieb Algebra and Graphical formulation

Let us discuss the meanings and consequences of the crossing symmetry: It has a remarkable significance to algebraic and graphical approaches in knot theory [6]. We use the notation of the factorized S-matrices. It is noted that the factorized S-matrices and solvable vertex models are equivalent. The discussion also holds for solvable IRF models [6].

We denote the amplitude of the scattering process $(\alpha,\beta) \to (\mu,\nu)$ by $S^{\alpha\nu}_{\beta\mu}(u)$ (Fig.6). The Yang-Baxter relation for the factorized S-matrices reads as (Fig.7)

$$\sum_{\alpha\beta\gamma} S^{\alpha\nu}_{\gamma\mu}(u) S^{\beta\lambda}_{\tau\gamma}(u+v) S^{\rho\beta}_{\sigma\alpha}(v) = \sum_{\alpha\beta\gamma} S^{\beta\lambda}_{\alpha\nu}(v) S^{\rho\beta}_{\gamma\mu}(u+v) S^{\sigma\alpha}_{\tau\gamma}(u). \tag{3.1}$$

The relation (3.1), often referred to as the factorization equation, was introduced as the consistency condition for the Bethe ansatz wavefunction [21]. The factorized S-matrices represent the elastic scattering of particles in that only the exchanges of momenta and the phase shifts occur.

For the factorized S-matrices, the Yang-Baxter operator $X_i(u)$ is defined by [1,6] (Fig.8)

$$[X_i(u)]^{\kappa_1\cdots\kappa_n}_{\mu_1\cdots\mu_n} = \left(\prod_{j=1}^{i-1} \delta^{\kappa_j}_{\mu_j}\right) S^{\mu_i\kappa_{i+1}}_{\mu_{i+1}\kappa_i}(u) \left(\prod_{j=i+2}^{n} \delta^{\kappa_j}_{\mu_j}\right), \tag{3.2}$$

In terms of the Yang-Baxter operator the Yang-Baxter relation (3.1) is again expressed as (2.8).

Generally, the factorized S-matrices satisfy the following crossing symmetry and the standard initial condition.

1) crossing symmetry (Fig.9)

$$S^{\alpha\nu}_{\beta\mu}(u) = S^{\beta\mu}_{\bar{\nu}\bar{\alpha}}(\lambda - u)\left(\frac{r(\alpha)r(\mu)}{r(\beta)r(\nu)}\right)^{1/2}, \tag{3.3}$$

where we have introduced the notation $\bar{\alpha} = -\alpha$ for "charge conjugation" and $r(\alpha)$ are the crossing multipliers with a relation $r(-\alpha) = 1/r(\alpha)$.

2) standard initial condition (Fig.10)

$$S^{\alpha\nu}_{\beta\mu}(u=0) = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}. \tag{3.4}$$

The above relations have the following physical meanings. We can interepret u as the rapidity difference of the scattering particles. Also it can be considered as the scattering angle (Fig.6). The standard initial condition indicates that there is no scattering between two particles with zero relative velocity. The crossing symmetry describes the invariance of the system under 90 degree rotation in a 2-dimensional space. Note that from the standard initial condition and the crossing symmetry, the inversion relation and the second inversion relation for the factorized S-matrices (solvable vertex models) are derived.

It is important in the critical (vertex and IRF) models with the crossing symmetry that the Yang-Baxter operator becomes the Temperley-Lieb operator at the point $u = \lambda$ [6]. Setting

$$E_i = X_i(\lambda), \tag{3.5}$$

we find that the operators $\{E_i\}$ satisfy the following relations

$$E_i E_{i\pm1} E_i = E_i,$$

$$E_i^2 = q^{1/2} E_i,$$
$$E_i E_j = E_j E_i, \qquad |i-j| \geq 2, \tag{3.6}$$

where the quantity $q^{1/2}$ is related to the crossing multipliers $\psi(i)$ (or $\{r(\alpha)\}$) by

$$q^{1/2} = \sum_{\alpha} r^2(\alpha), \qquad \text{for S matrix (vertex model)}, \tag{3.7a}$$
$$= \sum_{b \sim a} \frac{\psi(b)}{\psi(a)}, \qquad \text{for IRF model}. \tag{3.7b}$$

The relations (3.6) are the defining relations of the Temperley-Lieb algebra [21].

Furthermore, the relation (3.5) is of importance in an algebraic formulation of the knot theory [6]. We only point out two key observations. First, the operators $\{E_i, G_i\}$ form an braid-monoid algebra. Second, using the Temperley-Lieb operator E_i, we can show that the extended Markov property is equivalent to the relation (projection relation) [6]

$$X_i(u) E_i = \beta(u) E_i. \tag{3.8}$$

where $\beta(u)$ is a function which is related to the characteristic function $H(u;\lambda)$ by $\beta(u) = H(\lambda - u; \lambda)$.

Let us consider the graphical meanings of the relation (3.6) [6]. From the crossing symmetry and the standard initial condition we have (Fig.11)

$$S_{\beta\mu}^{\alpha\nu}(\lambda) = \left(\frac{r(\alpha) r(\mu)}{r(\beta) r(\nu)} \right)^{1/2} S_{\bar{\nu}\bar{\alpha}}^{\beta\mu}(0) = r(\alpha)\delta_{\beta}^{\bar{\alpha}} \cdot r(\mu)\delta_{\bar{\nu}}^{\mu}. \tag{3.9}$$

We can regard the elements $r(\alpha)\delta_{\beta}^{\bar{\alpha}}$ and $r(\mu)\delta_{\bar{\nu}}^{\mu}$ as the weights for the pair-annihilation and the pair-creation diagrams, respectively (Fig.12). Then, the Yang-Baxter operator at $u = \lambda$ is depicted as the monoid diagram, by which the Temperley-Lieb algebra is explained. This way of thinking is consistent with a fact that the energy at the point λ is related to the pair-creation energy. For IRF models, the weights $(\psi(a)/\psi(b))^{1/2}$ and $(\psi(c)/\psi(b))^{1/2}$ correspond to the pair-annihilation and pair-creation diagrams, respectively (Fig.13) [6].

Let us introduce link diagram $\hat{L}$, which is 2-dimensional projection of a link L (Fig.14). The writhe $w(\hat{L})$ is the sum of signs for all crossings $\{C_i\}$ in the link diagram (Fig.15):

$$w(\hat{L}) = \sum_i \epsilon(C_i), \tag{3.10}$$

We can formulate link polynomial directly on link diagrams. First we calculate statistical sum $Tr(\hat{L})$ on the diagram $\hat{L}$ by the rules given in Fig.12 (Fig.13). The link polynomial for the link L is calculated as

$$\alpha(L) = c^{-w(\hat{L})} \frac{Tr(\hat{L})}{Tr(\hat{K}_0)}, \tag{3.11}$$

where $\hat{K}_0$ is the link diagram for the trivial knot (a loop) and the constant c is defined by

$$c = \lim_{u \to \infty} \beta(u). \tag{3.12}$$

or by a relation

$$G_i E_i = c E_i. \tag{3.13}$$

We can confirm $\alpha(L)$ is invariant under the Reidemeister moves of link diagrams (Fig.16), and hence $\alpha(L)$ is a topological invariant of the link L. Thus we have shown that a model with the crossing symmetry gives a graphical construction of the link polynomial.

To conclude this section, we again emphasize that the crossing symmetry has algebraic and graphical significances. Algebraically, it leads to the Temperley-Lieb algebra and the braid-monoid algebra. Graphically, the pair-creation and pair-annihilation diagrams are introduced through the crossing symmetry.

4. Modular Transformation, Fusion Rule and Crossing Multiplier

In this section we study the relation between the surgery formula and the Markov trace. In the surgery formula (1.3), modular transformation matrix S_i^j plays an essential role. For WZW model, conformal blocks on the torus are given by the characters of the Kac-Moody algebra. The modular transformation matrix of the characters is given by [9,23]

$$\frac{S_{j0}}{S_{00}} = \mathrm{Tr}_{V_j}(\exp\frac{2\pi i\rho}{k+g}), \tag{4.1}$$

where ρ is the half sum of all positive roots, g is the dual Coxeter number, and k is the level of the integrable representation whose ground state is irreducible representation V_j of group G.

The quantity defined by (4.1) appears in several places in mathematics and physics. In the context of exactly solvable models, it is the crossing multiplier of critical 8VSOS model [24,19], and its generalizations, A, B, C and D IRF models [25]. Explicitly we find that the crossing multiplier and the matrix elements S_{ij} are related by

$$\frac{\psi(j)}{\psi(0)} = \frac{S_{0j}}{S_{00}}. \tag{4.2}$$

For the vertex models [26,1,28,29], the same correspondence also holds. It is known that the partition function $Z(S^2 \times S^1; R_j, B)$ on $S^2 \times S^1$ corresponds to the trace of the braid matrix

$$Z(S^2 \times S^1; R_j, B) = \sum_{\ell_1\ell_2\cdots\ell_{n-1}} B^{\ell_0\ell_1\cdots\ell_{n-1}\ell_n}_{\ell_0\ell_1\cdots\ell_{n-1}\ell_n} \qquad (\ell_0 = 0, \ell_n = j). \tag{4.3}$$

This braid matrix B is the monodromy matrix on the conformal blocks on S^2. For the case of $SU(2)$, the monodromy matrix has been explicitly obtained [29] by solving the Knizhnik-Zamolodchikov equation [30], and the monodromy matrix is equivalent to the braid matrix appearing from 8VSOS model or IRF model associated to $SU(2)$ [2,25]. Recently it has been shown [31] that the above equivalence between monodromy matrices in WZW models and the Boltzmann weights in IRF models (or the R−matrices in quantum groups) also holds for other gauge groups of type A, B, C and D. These facts indicate that the surgery formula (1.3) is expressed as [15]

$$\begin{aligned} Z(S^3;L) &= \sum_j Z(S^2\times S^1;R_j,B)S_0^j \\ &= \widetilde{\sum_{\ell_1\ell_2\cdots\ell_n}} B^{\ell_1\ell_2\cdots\ell_n}_{\ell_1\ell_2\cdots\ell_n}\frac{\psi(\ell_n)}{\psi(\ell_0)} \qquad (\ell_0=0), \end{aligned} \tag{4.4}$$

where the symbol $\widetilde{\Sigma}$ denotes the constrained trace. Thus, the surgery formula of a link L in S^3 has the same form with the Markov trace given by the constrained trace (2.10) for the IRF models [15].

The skein relations of the link polynomials constructed from 8VSOS and A,B,C,D, IRF models can be explained by the conformal weights of the WZW models [15]. This fact also agrees with the correspondence between the surgery formula and the Markov trace for the IRF model [15].

The relation (4.2) can be also explained as folows. Recall the fusion algebra of the rational conformal field theory [32,33]:

$$\phi_i\phi_j=\sum_k N_{ij}^{\ k}\phi_k, \tag{4.5}$$

where generators $\{\phi_i\}$ correspond to primary fields, and $N_{ij}^{\ k}$ counts the multiplicity of ϕ_k appearing in OPE of ϕ_i and ϕ_j. Let us consider one dimensional representations $\{\lambda_i^{(n)}\in \mathbf{C}\}$ of the fusion algebra

$$\lambda_i^{(n)}\lambda_j^{(n)}=\sum_k N_{ij}^{\ k}\lambda_k^{(n)}. \tag{4.6}$$

By the Verlinde's formula [32], the $\lambda_i^{(n)}$'s are given by modular transformation matrix S_{ij} as follows

$$\lambda_i^{(n)} = \frac{S_{in}}{S_{0n}}. \tag{4.7}$$

The value $\lambda_i^{(0)} = S_{0i}/S_{00}$ can be seen as the relative dimension [33] of the representation space of the chiral algebra. On the other hand, the crossing multipliers $\{\psi(j)\}$ of the solvable IRF models are determined by the eigenvalue problem [6]

$$\sum_{\ell\sim j} \psi(\ell) = q^{1/2}\psi(j), \tag{4.8}$$

where the summation is over all states admissible to j. The admissibility condition can be described by the coefficients $N_{ij}^{\ k}$ for corresponding conformal field theory. This fact explains the relation (4.2) between the matrix elements of the modular transformation and the crossing multipliers.

For the case of restricted 8VSOS model, the condition (4.8) is

$$\sum_{\ell} N_{1j}^{\ \ell}\psi(\ell) = q^{1/2}\psi(j), \tag{4.9}$$

with

$$N_{1j}^{\ \ell} = \delta_{j+1}^{\ell} + \delta_{j-1}^{\ell}. \tag{4.10}$$

Here $N_{1j}^{\ \ell}$ is nothing but the fusion coefficient for $SU(2)$ WZW model.

5. Graph State IRF Models

The theory to construct link polynomials from exactly solvable models takes advantage of the crossing multipliers of the models. To show a wide applicability of the theory, we shall present examples of solvable models other than A, B, C and D models. In fact, there exist various models with non-trivial crossing multipliers from which link polynomials can be constructed.

Let us introduce graph-state IRF models [6](Fig.17). The constraint on the model can be expressed by a graph [34,2,35]. Each point of the graph corresponds to a possible spin value of the model. The point for a spin b is connected to the point for a spin a, if and only if b is admisssible to a: $b \sim a$. For any graph in any dimensions we consider the following relation [2]

$$\sum_{b\sim a}\psi(b) = q^{1/2}\psi(a), \tag{5.1}$$

where the summation is over all spin values admissible to a. This relation has already appeared in (3.7) and (4.8). From the solutions of the eigenvalue problem (5.1) we have the Temperley-Lieb operators by

$$[U_i]^{p_0p_1\cdots p_n}_{\ell_0\ell_1\cdots\ell_n} = \left(\prod_{j=0}^{i-1}\delta^{p_j}_{\ell_j}\right)\delta^{\ell_{i+1}}_{\ell_{i-1}}\frac{(\psi(\ell_i)\psi(p_i))^{1/2}}{\psi(\ell_{i-1})}\left(\prod_{j=i+1}^{n}\delta^{p_j}_{\ell_j}\right). \tag{5.2}$$

It is easy to see that the operator U_i satisfies the defining relations of the Temperley-Lieb algebra. Choosing the crossing parameter λ by

$$2\cos\lambda = q^{1/2}, \tag{5.3}$$

and from the Temperley-Lieb operator U_i , we can construct Yang-Baxter operator $X_i(u)$ by [2]

$$X_i(u) = \frac{\sin(\lambda-u)}{\sin\lambda}(I + \frac{\sin u}{\sin(\lambda-u)}U_i) \tag{5.4}$$

Let us give an example. For a graph of the two-dimensional square lattice depicted in Fig.17(f), the crossing multiplier is given by

$$\psi(\overrightarrow{a}) = \sin(\overrightarrow{a} \cdot \overrightarrow{n} + \omega_0), \tag{5.5}$$

where the lattice points on the two-dimensional square lattice are expressed in terms of

$$\overrightarrow{a} = (a_1, a_2), \qquad a_1, a_2 \in \mathbf{Z}, \tag{5.6}$$

and $\overrightarrow{n} = (n_1, n_2)$. For this model the quantity $q^{1/2}$ is given by

$$q^{1/2} = 2\cos n_1 + 2\cos n_2. \tag{5.7}$$

Thus we have shown that we can construct a solvable model for any graph. These models are generalizations of the 8VSOS models. When a graph has a finite size, the graph-state IRF model is a restricted solid-on-solid (RSOS) model.

Since the extended Markov property (2.11) holds for graph-state model, we can construct the Markov trace on the braid group representation. Therefore we have link polynomial corresponding to arbitrary graph. In other words, from the crossing symmetry, solvable models and then link polynomials are constructed in a systematic manner.

6. Discussions

1) For link polynomial given by the topological quantum field theory, we have found that its expression by the surgery formula corresponds to the general construction of link polynomial based on the exactly solvable models [15]. The crossing multiplier is the keypoint of the latter construction. Crossing multiplier and modular transformation matrix is closely related by the fusion algebra. Therefore, it seems that the construction of link polynomial by the surgery formula is as general as the knot theory based on the exactly solvable models.

2) An advantage of the knot theory based on the exactly solvable models is that the crossing multiplier and the crossing symmetry are naturally introduced. In §5, we have shown that from the crossing symmetry with non-trivial crossing multipliers solvable models and link polynomials are systematically derived.

3) We can construct composite solvable models. The key of the construction is that the projectors can be made from the Yang-Baxter operators [36,6]. The method is called composition method or fusion method. The projectors are considered as generalized Young operators [6]. We can construct "3-point vertex" using the projectors (Fig.18). For graph-state models, the projectors consist of the crossing multipliers. Further, it has been shown that the projectors satisfy the pass-through condition [6] (Fig.19) and also they are compatible with the Markov properties [6]. Therefore by using the projectors we have topological invariant for linking graphs.

4) The row-to-row transfer matrix $T(u)$ of two-dimensional lattice system is constructed from the Yang-Baxter operator as

$$T(u) = U(u)V(u), \tag{6.1}$$

where

$$U(u) = X_1(u)X_3(u)\cdots X_{2n-1}(u), \tag{6.2}$$

$$V(u) = X_2(u)X_4(u)\cdots X_{2n}(u), \tag{6.3}$$

The partition function of the system is written as

$$Z_N(u) = Tr((U(u)V(u))^n), \qquad N = 2n \times 2n, \tag{6.4}$$

and then the free energy f per site is given by

$$f(u) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(u). \tag{6.5}$$

We note that if we start from the Yang-Baxter operator for the graph-state model, then the partition function is written as a summation of the crossing multiplers of the model.

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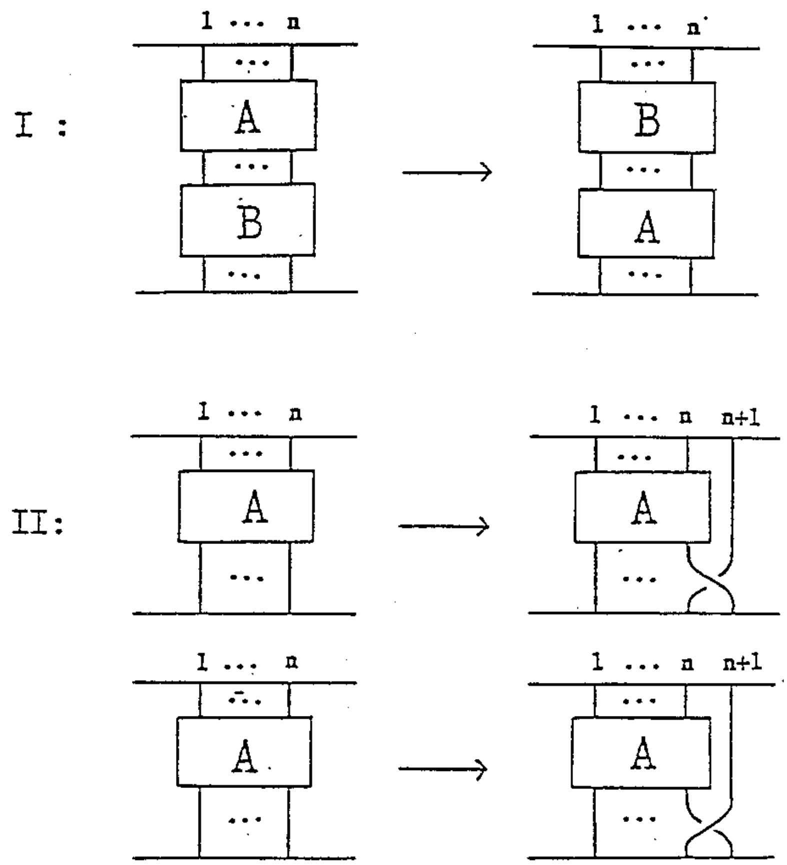

Fig.1 Markov moves I and II.

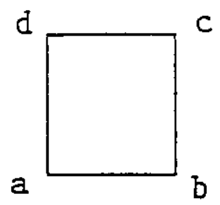

Fig.2 Boltzmann weight of IRF model.

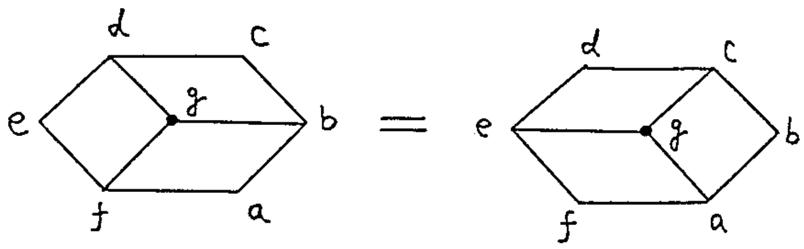

Fig.3 Yang-Baxter relation for IRF model.

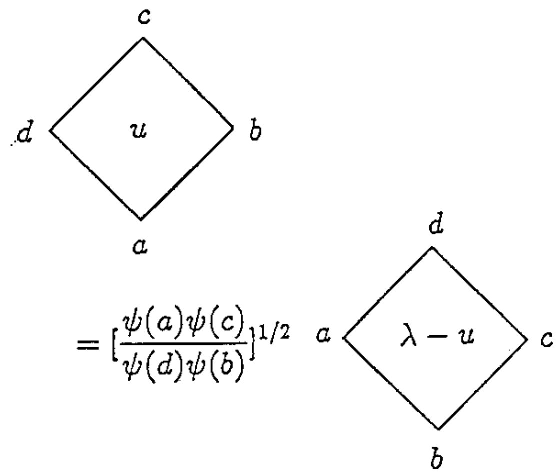

Fig.4 Crossing of IRF model.

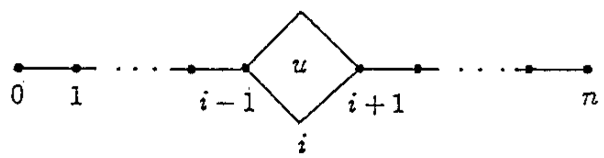

Fig.5 Yang-Baxter operator for IRF model.

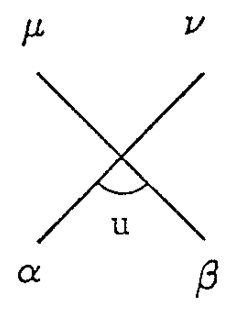

Fig.6 Scattering amplitude $S^{\alpha\nu}_{\beta\mu}(u)$.

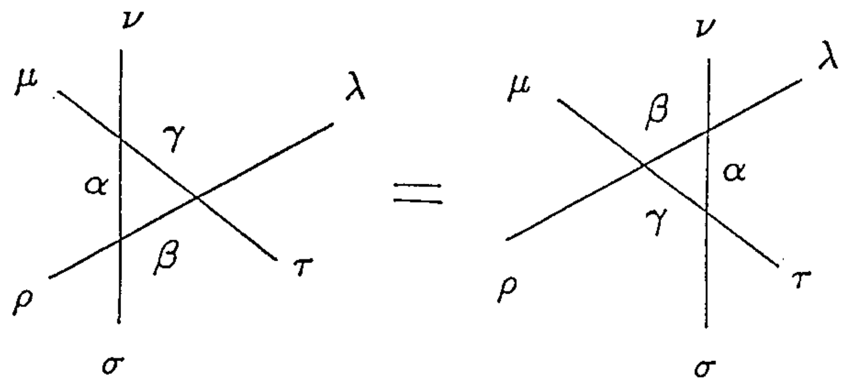

Fig.7 Yang-Baxter relation for S-matrix (vertex model).

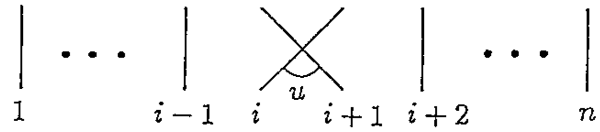

Fig.8 Yang-Baxter operator for S-matrix (vertex model).

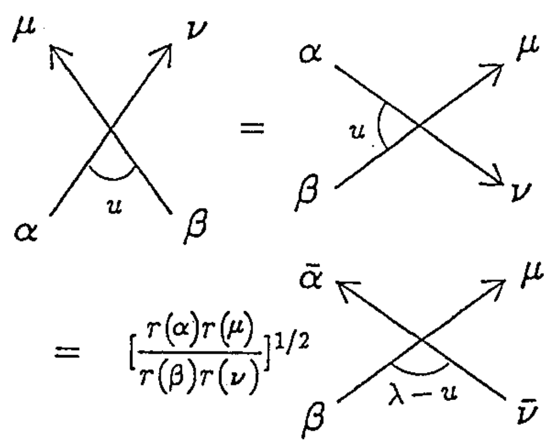

Fig.9 Crossing symmetry of S-matrix.

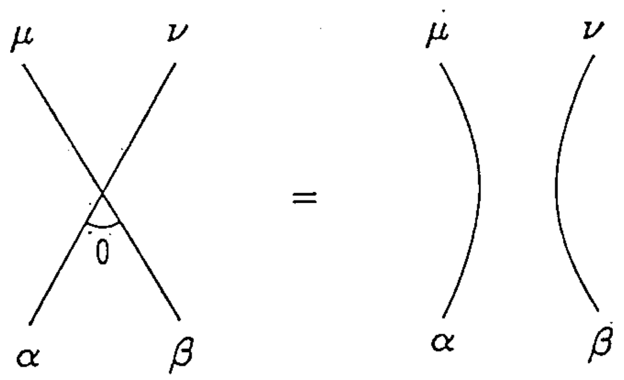

Fig.10 Standard initial condition.

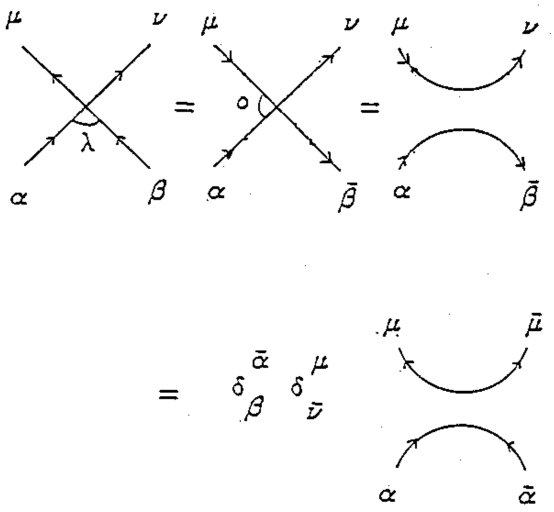

Fig.11 Scattering with $u = \lambda$ corresponds to annihilation- creation diagram (monoid diagram).

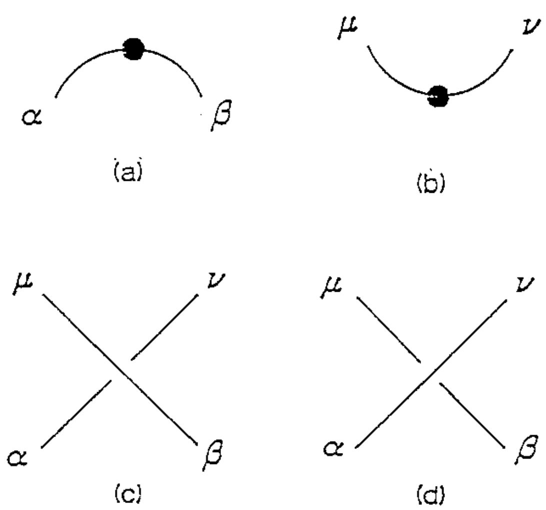

Fig.12 Weights for elements of link diagrams.

(a) pair-annihilation diagram $r(\alpha)\delta^{\bar{\alpha}}_{\beta}$.

(b) pair-creation diagram $r(\mu)\delta^{\mu}_{\bar{\nu}}$.

(c) braid diagram with $w = -1$, $G^{\alpha\nu}_{\beta\mu}(+) = S^{\alpha\nu}_{\beta\mu}(\infty)$.

(d) braid diagram with $w = 1$, $G^{\alpha\nu}_{\beta\mu}(-) = S^{\alpha\nu}_{\beta\mu}(-\infty)$.

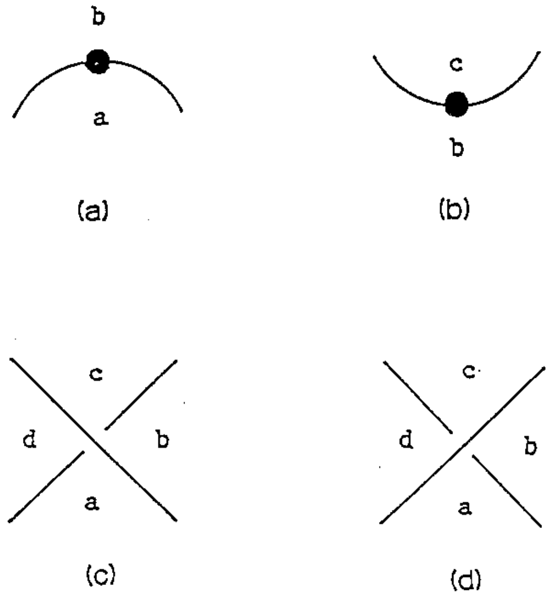

Fig.13 Weights for elements of link diagrams.

(a) pair-annihilation diagram $(\psi(a)/\psi(b))^{1/2}$.

(b) pair-creation diagram $(\psi(c)/\psi(b))^{1/2}$.

(c) braid diagram with $w = -1$, $\quad G(a,b,c,d;+) = w(a,b,c,d;\infty)$.

(d) braid diagram with $w = 1$, $\quad G(a,b,c,d;-) = w(a,b,c,d;-\infty)$.

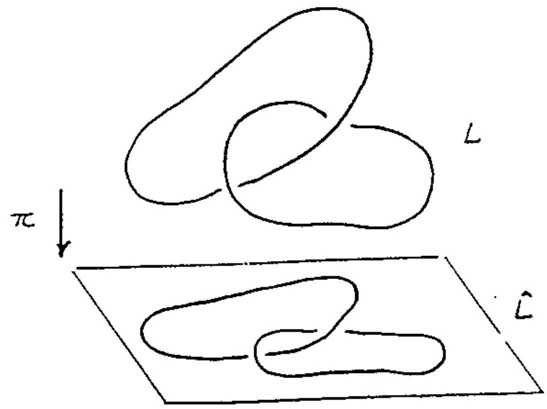

Fig.14 Link diagram $\hat{L}$.

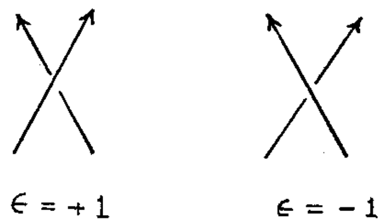

Fig.15 Sign $\epsilon(C)$ of crossings C.

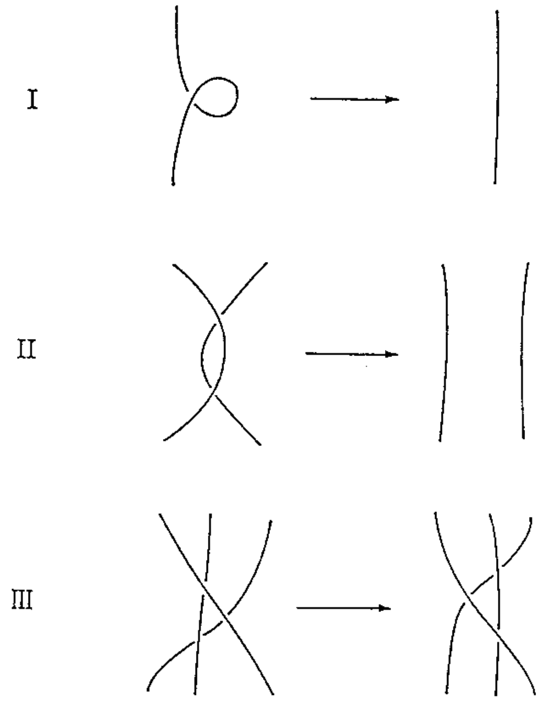

Fig.16 Reidemeister moves I, II and III.

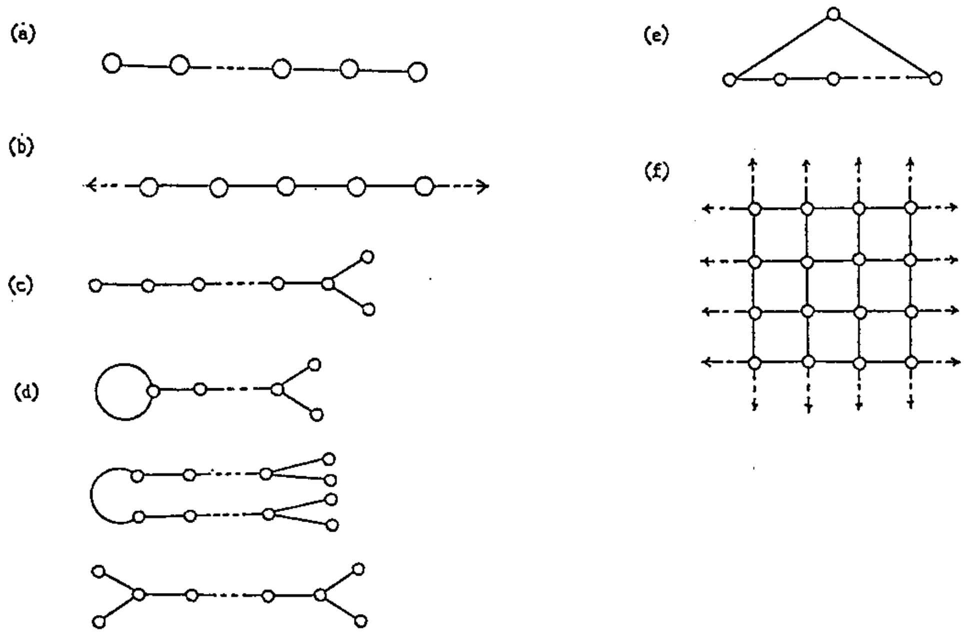

Fig.17 Graph-state models. (a) restricted 8VSOS model (A type), (b) unrestricted 8VSOS model, (c) D type model, (d) special S_2-generation ($D^{(1)}$) model, (e) periodic 8VSOS model ($A^{(1)}$ type), (f) a two-dimensional square lattice.

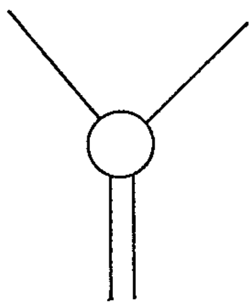

Fig.18 3-point vertex. Circle denotes the projector.

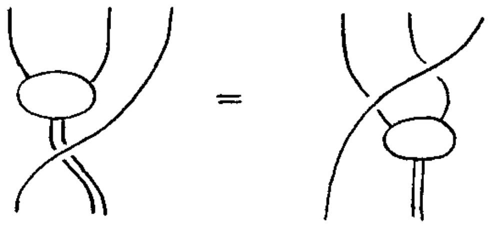

Fig.19 Pass-through condition.