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# Compactification flavors

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## Ruifeng LENG

## Composition du Jury


**Giacomo CACCIAPAGLIA**
Directeur de recherche, CNRS, Université Claude Bernard Lyon 1, IP2I Lyon — Rapporteur

**Emilian DUDAS**
Directeur de recherche, CNRS, École Polytechnique — Rapporteur

**Yann MAMBRINI**
Directeur de recherche, CNRS, Université Paris-Saclay, IJCLab — Examinateur

**Grégory MOREAU**
Maître de conférences, Université Paris-Saclay, IJCLab — Directeur de thèse

# List of Publications

During the preparation of this Ph.D. thesis, two papers were published:

- Ref. [1]: A. Angelescu, R. Leng, G. Moreau and F. Nortier, *Beyond brane-Higgs regularisation: clarifying the method and model*, Phys. Rev. D 101 (Apr, 2020) 075048.
- Ref. [2]: R. Leng, G. Moreau and F. Nortier, *Rigorous treatment of the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold model with brane-Higgs couplings*, Phys. Rev. D 103 (Apr, 2021) 075010.
- R. Leng, G. Moreau and F. Nortier, *An origin for flavors: the compact space partition.* At the level of paper writing.
- R. Leng, G. Moreau and F. Nortier, *A distribution approach for extra dimensions.* Work in progress.

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# Abstract

Since around 2000, the Standard Model extensions on extra spatial dimensions have emerged as attractive alternatives to supersymmetric scenarios. In particular, the class of warped dimension models with a brane-localized Higgs boson coupled to bulk fermions (dual to composite Higgs models) can address both the flavor puzzle and the gauge hierarchy problem. Nevertheless, a key question arises due to the possibility of fermion wave function discontinuities at the Higgs boundary: how to rigorously build the Lagrangian and calculate the fermion mass spectrum as well as the effective 4D Yukawa couplings?

In this thesis, we show that the proper treatment does not rely on any Higgs peak regularization, as usually done in the literature, but may require the presence of specific bilinear brane terms instead. This result is welcome as the Higgs regularization suffers from mathematical discrepancies reflected in the amplitude paradox of non-commutativity in some calculation steps, as debated in the literature. The bilinear brane terms could allow elaborating an ultra-violet origin of the chiral nature of the Standard Model and its chirality distribution among quarks/leptons. The introduction of bilinear brane terms can be replaced by vanishing conditions for probability currents at the interval boundaries. The current conditions are then implemented through essential boundary conditions to be contrasted with the natural boundary conditions derived from the action variation. All these theoretical conclusions are confirmed in particular by the exact converging results of the 4D versus 5D approaches. The new calculation methods presented, implying the independence of excited fermion masses and 4D Yukawa couplings on the 'wrong-chirality' Yukawa terms, have impacts on phenomenological results like the relaxing of previously obtained strong bounds on Kaluza-Klein masses induced by flavor-changing reactions generated by the tree-level exchanges of the Higgs field.

Then we extend those rigorous approaches from the interval configuration to the dual $S^1/\mathbb{Z}_2$ orbifold, which allows, in particular, a strict treatment of the fermion profile discontinuities across the characteristic branes (fixed points and Yukawa coupling brane). We also show that the $\mathbb{Z}_2$ parity transformations in the bulk do not affect the fermion chiralities, masses, and couplings, in contrast with the EBC and the BBT, but when extended to the fixed points, they can generate the chiral nature of the theory and even select the Standard Model chirality setup while fixing as well the fermion masses and couplings.

We have realized that the bilinear brane terms, located at intermediate positions along the interval, provide an opportunity to explain the existence of flavors (replicas of elementary particles with identical quantum numbers): the three families in this context correspond to three different quantum states, of a unique 5D field, localized respectively between such brane terms. This new generation partition mechanism, along the extra dimension, further generates fermion mass hierarchies automatically (from different wave function overlaps) when the Higgs boson profile is exponentially localized towards the so-called TeV-brane to address the gauge hierarchy problem. The two hierarchy problems are then solved through the same exponential scalar profile. The partition mechanism also offers a new field theory method to localize all fermions on a (thick) brane, alternatively to the standard soliton coupling approach.

# Introduction

The Standard Model of particle physics [4–7], based on the Quantum Field Theory [8–10], is undoubtedly today the most successful scenario describing the known elementary particles and fundamental forces of nature. In July 2012, the historical discovery of a 125.5 GeV resonance at CERN-Geneva's Large Hadron Collider by the ATLAS [11] and CMS [12] Collaborations, most likely constituting the Higgs boson [13–15], brought the last missing cornerstone of the SM [1] by confirming the standard Higgs mechanism of ElectroWeak Symmetry Breaking. Thanks to an undeniable experimental success, its complete elementary particle content has been discovered (see Figure 1), and the theoretical predictions of the SM have been experimentally confirmed with a good accuracy at the Large Electron-Positron collider, Tevatron and LHC [16].

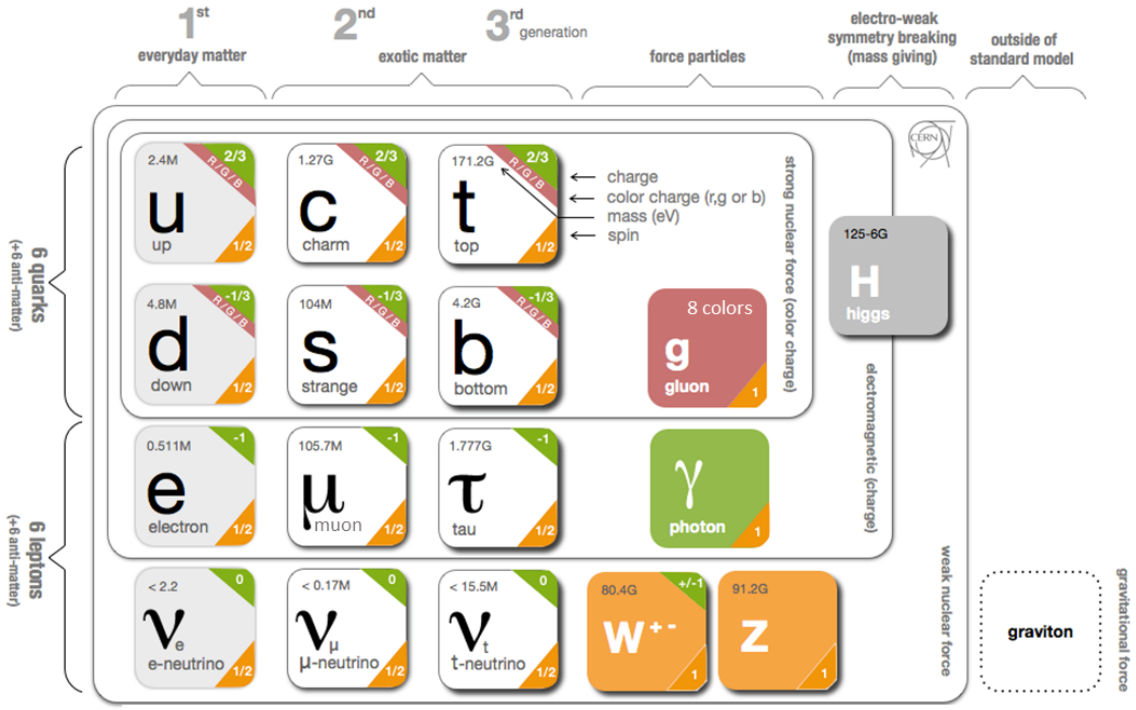



Figure 1 – Schematic view of the Standard Model of particle physics after the EW symmetry breaking [3].

Although the SM is a very successful theory that has offered us some striking agreements between experimental data and theoretical predictions, a variety of statements, both theoretical and experimental, lead to the undoubted conclusion that the SM should

1. All the abbreviations are presented explicitly in Appendix K.2.

be only an effective theory of a more fundamental one, leaving several important questions unanswered.

On the observational side, there is no confirmed interpretation of the *dark matter* of the universe [17–19]. Cosmological measurements carried out by the Planck satellite [20] indicate that approximately 27% of the total energy budget of the universe is made out of dark matter, which could be constituted by the presence of one or several species of massive particles permeating the cosmos at nonrelativistic speeds, if those neither carry electrical charge nor participate in the strong interaction and, most importantly, are stable on cosmological time scales. Among the rest of $\sim 73\%$, only 5% is represented by the ordinary, luminous matter. In contrast, the remaining 68% of the energy density appears to be made out of the even more mysterious *dark energy* (Cosmological Constant) and further raises deep questions about quantum gravity.

Another shortcoming of the SM regarding observations, in particle physics, concerns the now well measured *neutrino flavor oscillations*, which imply that at least two of them must be massive [21]. Nevertheless, in the SM, neutrinos are entirely massless, so that they should not oscillate from one flavor to another. Thus, one needs to add new physics to give them a mass. In the fermion sector of the SM, one observes a large mass hierarchy between the neutrino mass scale and the top quark mass (see Figure 1) to be explained as well.

The other experimental puzzle has to do with the flavor space: the mysterious mass hierarchies among most SM fermions [from $\sim 0.5$ MeV for the electron to $\sim 170$ GeV for the top quark], the unknown ultra-violet completion fixing the quark/lepton (CKM/PMNS matrix) mixing angle values, the origin of the three observed fermion generations and possibly the recent set of deviations from lepton flavour universality measured through neutral/charged-current semi-leptonic $B$ meson decays. Notice that the mass hierarchies are protected by the approximate chiral symmetry of the SM and remain therefore quite stable against radiative corrections.

The SM also suffers from several drawbacks of theoretical origin. First, while strong (QCD) and electroweak interactions are included in the SM, the SM does not give a description of *gravity* which consists of a back-reaction of matter and energy on the space-time geometry within the Einstein's theory of General Relativity [22]. Einstein's theory is classical and cannot describe the quantum fluctuations of spacetime, so our today's understanding of the physical world lacks a comprehensive global theoretical description. The inability to consistently incorporate a quantum theory of gravity – with *gravitons* coupled to the SM sector – lies in the fact that gravity is then a *non-renormalizable* theory: at each order in perturbation theory, new divergences appear, which mandate the introduction of an infinity of counter-terms in the renormalized Lagrangian (describing gravity). Or, each counter-term needs to be fixed by experimental measurements, meaning that one has to perform an infinite number of measurements to give sense to gravity at the quantum level, which is not consistent.

Another deep question is the origin of the ElectroWeak symmetry breaking and precise Higgs potential in the SM. The ElectroWeak sector is described by the Glashow-Salam-Weinberg (GSW) model [5, 6, 23], based on the spontaneous symmetry breaking of gauge theory. In this framework, one applies the Brout-Englert-Guralnik-Hagen-Higgs-Kibble mechanism [24–29] where the SM Higgs field $H(x^\mu)$ is a complex scalar field with a 'hat' form potential:

$$V(H) = m_H^2\, H^2 + \lambda_H\, |H|^4 \;, \quad \text{with} \;\; m_H^2 < 0\,, \lambda_H > 0\,, \tag{1}$$

which leads to a non-vanishing Vacuum Expectation Value $v \simeq 246$ GeV for $H(x^\mu)$ triggering the spontaneous EW symmetry breaking, while the fluctuations of the field around its VEV describe a spin-0 particle, the Higgs boson $h(x^\mu)$,

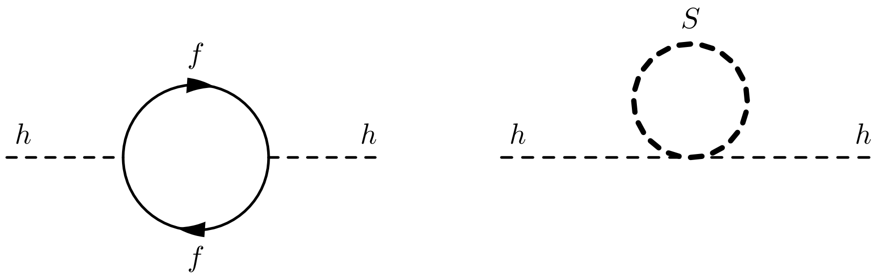
$$H(x^\mu) = \frac{1}{\sqrt{2}}\left[v + h(x^\mu)\right] \,, \quad \text{with} \quad v = \sqrt{\frac{-m_H^2}{\lambda_H}} \,, \tag{2}$$

where the mass of 125.5 GeV was measured in 2012 [11, 12]. Thus, the free Lagrangian terms of $h$ reads as,

$$\mathcal{L} \ni \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2} m_h^2 \, h^2 \,, \quad \text{with} \quad m_h^2 = -2 \times m_H^2 \,. \tag{3}$$

Next, a deep *hierarchy* problem remains unexplained in the SM: the vast discrepancy between the EW scale $\Lambda_{EW} \simeq 100$ GeV and the Planck scale $\Lambda_P \simeq 10^{19}$ GeV [2], constituting the so-called *gauge hierarchy* problem of the SM. It's not a problem to the SM itself but an uncomfortable high sensitivity of the Higgs potential to the UV completion from New Physics. More precisely, the hierarchy problem appears in the fact that $m_h^2$ is very sensitive to the high-energy behavior of quantum corrections, induced by particles directly (or indirectly) interacting with the Higgs boson. The hierarchy problem is thus related to the radiative corrections $\Delta m_h^2$ that the physical Higgs boson mass receives at the loop-level,

$$m_{h,\text{phys}}^2 \;=\; m_{h,(\text{bare})}^2 + \Delta m_h^2 \,. \tag{4}$$




(a) $h\bar{f}f$ (b) $h^2\bar{S}S$

Figure 2 – One-loop corrections to the Higgs squared mass parameter $m_{h,\text{phys}}^2$, due to (a) a Dirac fermion $f$ [antiparticle $\bar{f}$], and (b) a complex scalar $S$ [antiparticle $\bar{S}$].

For example, in Figure 2 (a), one has a correction to $m_h^2$ coming from a loop exchanging a Dirac fermion $f$ with mass $m_f$ [antiparticle $\bar{f}$]. If the Higgs field $H$ couples to $f$ via the Lagrangian term,

$$-\lambda_f H \bar{f} f \;\xrightarrow{SSB}\; -\frac{\lambda_f}{\sqrt{2}} h \bar{f} f \,, \tag{5}$$

then the Feynman diagram in the Figure 2 (a) yields a correction

$$\Delta m_h^2 \;\sim\; -\frac{\lambda_f^2}{8\pi^2}\Lambda_{UV}^2 \,, \tag{6}$$

2. Gravity should become important at this scale.

where $\Lambda_{UV} \simeq \Lambda_P$ [3] is an ultra-violet cutoff for New Physics regulating the loop integral. Since the radiative corrections are quadratic in the cutoff, the (squared) Higgs mass is quadratically sensitive to every new mass scale above $\Lambda_{EW} \simeq 100$ GeV [30–32]. Besides, for a cutoff at the Planck scale, these corrections are roughly 36 orders of magnitude larger than the value of $m^2_{h,\text{phys}} \simeq (100\text{GeV})^2$ required by the EW symmetry breaking. Of course the bare Higgs mass $m_{h,(\text{bare})}$ can be adjusted to just cancel the radiative corrections, but such extreme *fine-tuning* seems technically unnatural as further different at each order of the calculation perturbation.
The radiative corrections can also receive contributions from any heavy complex scalar particle $S$ with mass $m_S$ [antiparticle $\bar{S}$] coupled to the Higgs boson via the Lagrangian term,

$$-\lambda_S \bar{H} H \bar{S} S \xrightarrow{SSB} -\frac{\lambda_S}{2} h^2 \bar{S} S \,. \tag{7}$$

Then the Feynman diagram in Figure 2 (b) yields a one-loop correction,

$$\Delta m_h^2 \sim +\frac{\lambda_S}{16\pi^2} \Lambda_{UV}^2 \,. \tag{8}$$

Thus, the fine-tuning still arises here to reproduce the much smaller measured Higgs mass. Note that the quadratic divergences in Eq. (6)-(8) occur only for scalar particles as the masses of fermions and vector bosons are protected by chiral and gauge symmetries, respectively. In the SM indeed, the fermion and gauge boson physical masses receive corrections proportional to their bare masses, i.e. $m_{\text{phys}} - m_{\text{bare}} \propto m_{\text{bare}} \ln(\Lambda_{UV}/m)$, which is a consequence of chiral (gauge) symmetry in the case of fermions (gauge bosons).
The gauge hierarchy problem has received a lot of attention in the last decades, and in particular some generic field theory (model-independent) ways out of this problem were suggested, like absence of heavy particles interacting with the Higgs scalar field up to the Planck scale,. . . This is because so far no new state has been found around the TeV corner at the LHC, which could have provide a signature for some SM extension addressing the gauge hierarchy problem.

Finally, the absence of CP violation effects in strong interactions, described by Quantum ChromoDynamics [33–37], constitutes also an open question since a CP violating topological term is authorized by the symmetries of the SM.

Many attempts have been undertaken to provide solutions to the main *gauge hierarchy* problem. One of the most famous approaches to address the gauge hierarchy problem is SUperSYmmetry, which relates fermions to bosons, in the sense that for every known fermion, there is a bosonic partner (so-called superpartner), which retains the same quantum numbers as the original particle but whose spin differs by 1/2, and vice versa. The SUSY provides a protection mechanism for the (Higgs) scalar mass. Basically, if each of the quarks and leptons of the Standard Model is accompanied by a complex scalar with coupling $\lambda_S = 2 \times \lambda_f^2$, then the $\Lambda_{UV}^2$ quadratic divergence contributions in Figure 2 (a) and (b) would neatly cancel each other by the extra supersymmetry [38–43]. In this context, the Higgs boson gets paired with its spin-1/2 superpartner, dubbed the Higgsino, of which the SM scalar 'inherits'. The Higgsino receives protection from the chiral symmetry. One particularly popular version is the Minimal Supersymmetric Standard Model or MSSM [44]. So far, unfortunately, no superparticle has been discovered, which means that SUSY is very badly broken.

Supersymmetry benefits from other strong motivations of theoretical nature, like representing the first non-trivial extension of the Poincaré group or relying on gravity in

3. Under the assumption that new physics enters at $\Lambda_P$.

its local form (supergravity), as well as of phenomenological nature, like allowing gauge coupling and group unification within a Grand Unified Theory context – the extended SU(5) gauge group being probably the most famous one – or providing possible Weakly Interacting Massive Particles being realistic candidates for the dark matter of the universe: Lightest Supersymmetric Particles (LSP) being stable thanks to the conserved R-parity.

Over the last two decades, an appealing alternative solution to the gauge hierarchy problem has been developed. In QFT, spacetime is a simple background where fields propagate. In general relativity, spacetime is dynamic and the theory describes how classical sources backreact on the geometry. In both theories, the number of spacetime dimensions is not determined by the first principles. On the one hand, compactified timelike extra dimensions seem to lead to physically inconsistent theories because the KK excitations of the fields propagating in the extra dimensions are tachyons, which imply violation of physically reasonable conditions like causality and unitarity [45, 46]. On the other hand, spacelike extra dimensions have a long history in fundamental physics since the pioneer works by G. Nordström [47], T. Kaluza [48] and O. Klein [49, 50] who has built the first consistent EFTs. Possibly, in a complete theory of gravity that describes Planckian physics, the dimensionality of spacetime can be determined by the dynamics or the consistency of the theory like string theories [51]. Interestingly, both SUSY and extra dimensions appear as necessary ingredients of superstring theories. In the absence of a particular UV completion of gravity, one is free to build models adding an arbitrary number of spacetime dimensions compactified on some geometries with a given topology. The only criteria are that the higher-dimensional model should be consistent with all the observations indicating that our Universe appears four-dimensional in current low-energy experiments. These extra spatial dimensions are thus in general constrained to be typically microscopic (from several collider physics and gravitational tests) – except in some specific gravity-localized scenarios (RS2).

The scenario proposed by Arkani-Hamed, Dimopoulos and Dvali [52–54], with compactified Large Extra Dimensions between *two branes*, beyond the ordinary 4D Minkowski space, and the SM fields localized on a 3D wall (3-brane), allows to have a higher-dimensional Planck scale at a few TeV. Consequently, gravity gets diluted in the extra-dimensional volume, which is why, in such models, gravity appears so weak in the effective 4D description. The 4D Planck scale $\Lambda_P$ is just an effective scale. This model also allows to generate small Dirac masses for the neutrinos if the right-handed neutrinos are KK modes of a gauge singlet field propagating into the whole spacetime (the bulk) [55–57]. In the simplest version with toroidal compactification and with less than seven LEDs motivated by superstring/M theory, the compactification radii are large compared to the higher-dimensional Planck length: the gauge hierarchy disappears at the price of introducing a geometrical hierarchy to stabilize. The ADD proposition is thus a kind of reformulation of the gauge hierarchy problem.

A popular way to overcome the ADD geometrical hierarchy question is to use a single warped extra dimension as proposed concretely by Randall and Sundrum [58]: the RS1 model. The SM fields are localized at the boundary of a slice of an $AdS_5$ (5D Anti-de Sitter) spacetime where the warp factor redshifts the scale at which gravity becomes strongly coupled (from the Planck scale $\Lambda_P \simeq 10^{19}$ GeV to the TeV scale) thanks to an exponentially suppressed scale [4]. Quickly, it was realized that only the Higgs field has to

4. Strongly coupled gravity at the TeV scale may generate dangerous brane-localized higher-dimensional operators inducing proton decay, large Majorana neutrino masses and Flavor Changing Neutral Currents [53]. The value of these operator Wilson coefficients is suppressed only by the TeV scale, and one has to add new ingredients to the scenario to forbid them, like gauging some global symmetries of the SM as the baryon and lepton numbers [59] and other flavor symmetries [60–62], or spreading the brane-localized SM

be localized (or highly peaked) at the boundary where the effective cut-off is of the order of the TeV (if the EW scale is to be stabilized by such a geometrical structure), while gauge bosons and fermions can propagate into the bulk [64–69].

Finally, this RS version provides a totally new physical interpretation [70, 71] for the origin of the large mass hierarchy prevailing among different flavors and types of SM fermions [5]. The zero modes of the fields are identified with the SM particles: the fermion zero modes can be localized near one of the two boundaries thanks to 5D Dirac masses. The wave function of a heavy (light) SM fermion can then have a significant (small) overlap with the boundary-localized Higgs field. Therefore, without *ad hoc* hierarchies in these fundamental 5D masses and Yukawa couplings, one can generate the flavor mass hierarchy observed in Nature (see for instance Ref. [68, 76]). Such an interpretation of the whole SM fermion mass hierarchy is attractive, as it does not rely on the presence of any new symmetry in the short-distance theory, in contrast with the usual Froggatt-Nielsen mechanism [77] where a flavor symmetry is crucial. This interpretation is purely geometrical: it is based on the possibility of different localizations for SM fermions along an extra dimension, depending on their flavor and type [6]. In such a scenario, the quark masses and CKM mixing angles can be effectively accommodated [80–82], as well as the lepton masses and MNS mixing angles in both cases where neutrinos acquire Majorana masses (via either dimension five operators [83] or the specific see-saw mechanism [84]) and Dirac masses (see Ref. [85] and [86] for order unity Yukawa couplings leading to mass hierarchies essentially generated by the higher-dimensional mechanism).

In addition, this RS version with bulk matter turns out to constitute a suitable framework with respect to model building in general and various phenomenological aspects. For instance, unification of the gauge couplings at a high scale is possible within a Grand Unified Theory framework [87–91]. Secondly, from the cosmological point of view, there exists a viable new Weakly Interacting Massive Particle candidate for dark matter – the so-called Lightest KK Particle (LKP) being stable thanks e.g. to a residual KK-parity [92–94]. The scenarios with an extra warped dimension also appear to constitute a really new paradigm in the sense that those are approximately dual, through the Anti-de Sitter / Conformal Field Theory correspondence, to composite Higgs models which shed a new light on the origin of the ElectroWeak symmetry breaking mechanism as well as the little hierarchy problem.

Therefore, in order to develop a clear understanding and study the phenomenology of higher-dimensional models, it is crucial to have in particular a rigorous field theoretical treatment of 5D fermions that can accommodate couplings to a brane-localized Higgs field – a standard geometrical configuration arising in attractive models as discussed above. This is the first topics we propose to study in this Ph.D. thesis. A few words here on our general methodology. An higher-dimensional field theory with compactified extra dimensions can be rewritten as an effective 4D theory by a procedure called Kaluza-Klein dimensional reduction. An higher-dimensional field gives then rise to an infinite tower of 4D fields: the KK modes. Most of the authors use a perturbative approach [95, 96], which we call the 4D approach (due to a KK mixing at the 4D field level), where the KK spectrum and wave functions of the 5D fermion fields are worked out without including immediately the brane-localized mass terms. An alternative equivalent method is to treat the brane-localized terms directly when one solves the 5D equations and boundary conditions for the

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fermion fields via a finite brane thickness [63].

5. Within the RS context, other higher-dimensional mechanisms [72–75] apply specifically to neutrinos to explain their lightness compared to the rest of SM fermions.

6. This possibility of fermion localizations along extra dimension(s) was also considered in the context of large flat extra dimension models, in order to generate quark [78] and lepton [79] masses/mixings.

fermion wave functions: the called 5D method. The exact matching between these 4D and 5D approaches will constitute a solid validation of several obtained analytical results, one of which being in contrast with the literature on this topics: many authors were puzzled by an apparent discontinuity in the KK wave functions at the Higgs field brane, and they have thus introduced a kind of *Higgs profile regularization* smoothing or shifting away from the boundary the brane-localized Higgs boson [96–108]. We will present what we claim to be the appropriate method to treat the bulk fermion couplings with a brane-localized Higgs boson. There the non-trivial regularization procedure appears to be useless. Then we will apply the new approaches developed in the simple interval model to the case of the dual orbifold model. In particular, we will build a strict mathematical method to treat properly possible profile jumps along an extra dimension. Those studies will point out the necessity to introduce some new kinds of fermionic terms that we call the bilinear brane terms. Based on this resulting statement, we will propose a surprisingly convenient formalism for higher-dimensional models based on fields defined as distributions rather than simple functions. The bilinear terms will also turn out to allow us to build scenarios explaining the existence of SM fermion replicas, namely the flavors. Indeed, in those scenarios, a replication of 4D fermions, with identical quantum numbers, arises. The original reason being that the similar 4D fermions – only differing by their quantum position states along a fifth dimension and hence their masses – originate from a common higher-dimensional field. These scenarios realize the mixings among quarks/leptons via different fermion partition mechanisms and further allow to reproduce the SM fermion mass hierarchies for a standard exponential Higgs profile along the extra dimension, which is interestingly yet required by the gauge hierarchy problem within a warped version of the present type of model. Our partition mechanism also allows to build models strictly localising fermions on thick branes with a controlled width as small as wanted (independently of flavor considerations), representing thus an original alternative to the usual mechanism based on the fermion interaction with a solitonic background.

The manuscript of this Ph.D. thesis is organized as follows:

- Part I gives the scheme of holography beyond the Standard Model via extra spatial dimensions whose purpose is to present regular treatments of 5D model building in the literature:
  - Chapter 1 is a short review of the SM of particle physics and the motivations for BSM model buildings, insisting on the gauge hierarchy problem.
  - Chapter 2 is an introduction to the models with the SM Higgs field localized at the boundary of a slice of an $\mathrm{AdS}_5$ spacetime with bulk fermion and gauge fields.
- Part II contains the main original research work made during this Ph.D. thesis:
  - Chapter 3 describes the method to treat 5D fermions coupled to a boundary localized Higgs field with a compactification on an interval.
  - Chapter 4 contains a generalization of the method of Chapter 3 towards a compactification on 5D orbifolds, including a Higgs field localized on a brane away from a boundary and some other brane-localized terms.
  - Chapter 5 contains a distribution formalism on the the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold, reformulating Chapter 4.
  - Chapter 6 describes our intermediate brane model to reveal a new mechanism to split fermion generations and realize mass hierarchy simultaneously.
- Our notations & conventions are given in Appendix A.
- The acronyms used in this manuscript are listed on a Glossary page 204.

# Part I

# A Holography Beyond the Standard Model

# Chapter 1

# The Kaluza-Klein Theory of Bulk Fields

This chapter is to present a general treatment in a simplified model with a flat extra dimension, which already possesses all the key ingredients to study the delicate brane-Higgs aspects. Hence, our conclusions can be directly extended to the warped models.

## 1.1 A Simple Spacetime Geometry

We consider a 5D toy model on the product spacetime geometry, $\mathcal{E}^5 = \mathcal{M}^4 \times \mathcal{I}$:

(i) $\mathcal{M}^4$ represents the usual 4D Minkowski spacetime whose coordinates are denoted by $x^\mu$ where $\mu \in [\![0,3]\!]$ is the Lorentz index of the covariant formalism. The metric conventions are given in Appendix A.

(ii) $\mathcal{I}$ is a compact 1D flat interval of the extra spatial dimension, which is denoted by $y \in [0, L]$, with a length, $L \in \mathbb{R}^*$, and bounded by two flat 3-branes at $y = 0$ and $y = L$.

(iii) A point of the 5D spacetime $\mathcal{E}^5$ is labeled by the coordinates, $x^M \triangleq (x^\mu, y)$, $M \in [\![0,4]\!]$ with the 5D metric is given by,

$$ds^2 = \eta_{MN} dx^M dx^N \, ,$$

where $\eta_{MN}$ with $M$, $N \in [\![0,4]\!]$ is the 5D Minkowski metric in Eq. (A.2).

## 1.2 Bulk Scalar Fields

We recall the common approach of performing Hamilton's principle in a holographic context of 5D scalar fields of mass dimension 1 following Ref. [109]. We want to describe a 5D real scalar field $H\,(x^\mu, y)$ with quadratic mass terms in a 5D bulk with the extra dimension compactified on an interval $[0, L]$.

The physical mass spectrum arises when the bulk fermions couple to a Higgs-like scalar field. In a realistic extension of the SM, this scalar field should be a $SU(2)_W$ doublet. In our toy model, it is enough to take a real scalar field $H$, with a $\mathbb{Z}_2$ symmetry $H \mapsto -H$, such as only powers of $|H|^2$ appear in the action, like in a realistic extension of the SM. In contrast to the brane-localized field in Ref. [1], here we consider a bulk real scalar field $H$ (mass dimension 3/2) [110] such as the Yukawa interactions exist into the whole extra dimension $\mathcal{I}$. The 5D action of the field $H$ is

$$S_H = \int d^4x \left[ \left( \int_0^L dy \; \mathcal{L}_H \right) - V_0(H) - V_L(H) \right] , \tag{1.1}$$

where $\mathcal{L}_H$ is the bulk scalar Lagrangian, and $V_{0/L}$ is the scalar potential on the brane at $y = 0/L$. Motivated by models where the EWSB occurs only at the boundary of a warped extra dimension (as in RS models with bulk Higgs in Ref. [110]), we will consider only a non-quadratic potential on the brane at $y = L$, and we will comment after what we expect by relaxing this hypothesis. For our purpose, it is enough to keep only the operator quartic in $H$ for the self-interaction term, since it is the dominant one. Therefore, we consider a bulk scalar Lagrangian with no bulk self-interaction of $H$,

$$\mathcal{L}_H = \frac{1}{2} \partial_M H \partial^M H - \frac{M_H^2}{2} \, H^2 \, , \tag{1.2}$$

and the boundary-localized potentials:

$$V_0(H) = \frac{M_0}{2} \, H^2 \Big|_0 \, , \quad V_L(H) = \left( -\frac{M_L}{2} \, H^2 + \frac{\lambda_H}{4!} \, H^4 \right) \Big|_L \, , \tag{1.3}$$

where $M_H^2$, $M_{0/L}$, $\lambda_H > 0$. $M_H$ and $M_{0/L}$ have mass dimension 1, and $\lambda_H$ has mass dimension $-2$.

We perform Hamilton's principle by varying the field $H$, and we get

$$\begin{aligned} \delta S_H = & \int d^4x \int_0^L dy \; \delta H \left[ \frac{\partial \mathcal{L}_H}{\partial H} - \partial_M \left( \frac{\partial \mathcal{L}_H}{\partial(\partial_M H)} \right) \right] + \left[ \delta H \, \frac{\partial \mathcal{L}_H}{\partial(\partial_4 H)} \right] \Bigg|_0^L \\ & - \int d^4x \left[ \left( \delta H \, \frac{\partial V_0}{\partial H} \right) \Bigg|_0 + \left( \delta H \, \frac{\partial V_L}{\partial H} \right) \Bigg|_L \right] . \end{aligned} \tag{1.4}$$

With generic field variations $\delta H$ at every point of the 5D spacetime, the variations of the action in the bulk and on the boundaries vanish separately. We get the Euler-Lagrange equation of $H$ in the bulk:

$$\forall \, x^\mu, \; \forall \, y \in \mathcal{I} = [0, L] \, , \quad \left( \partial_M \partial^M + M_H^2 \right) H = 0 \, , \tag{1.5}$$

and the natural BC on the branes:

$$\left( \partial_4 - M_0 \right) H|_0 = 0 \, , \quad \left( \partial_4 - M_L \right) H|_L = -\frac{\lambda_H}{3!} \, H^3 \Big|_L \, . \tag{1.6}$$

In analogy to the Higgs mechanism, the Mexican hat potential $V_L(H)$ (1.3) at the brane $y = L$ makes the scalar field $H$ having a non-vanishing vacuum expectation value $v(y)$ of mass dimension $3/2$ with respect to the spontaneous $\mathbb{Z}_2$ symmetry breaking, such as

$$H(x^\mu, y) = \frac{v(y) + h(x^\mu, y)}{\sqrt{2}} \, . \tag{1.7}$$

From the Euler-Lagrange equation (1.5) of $H$ in the bulk, and its BC (1.6), we get the equation for the VEV in the bulk:

$$\left( \partial_4^2 - M_H^2 \right) v(y) = 0 \, , \tag{1.8}$$

and its BC:

$$\left( \partial_4 - M_0 \right) v|_0 = 0, \quad \left( \partial_4 - M_L \right) v|_L = -\frac{\lambda_H}{12} \, v^3 \Big|_L \, , \tag{1.9}$$

where we have imposed the $x^\mu$ independence of $v(y)$ and the 4D asymptotic condition of $h(x^\mu, y)$, which is explicitly presented in Appendix B. From the equation (1.8) of the VEV in the bulk, and the BC for the VEV (1.9) at $y = 0$, we obtain:

$$v(y) = \mathcal{N}_v \left( e^{M_H\, y} + \frac{M_H - M_0}{M_H + M_0}\, e^{-M_H\, y} \right), \tag{1.10}$$

where $\mathcal{N}_v$ is a constant of mass dimension 3/2 fixed by the BC at $y = L$. It is natural to take $M_H \sim M_0 \sim M_L$. In this toy model, we choose $M_0 = M_H$ to simplify the profile of the VEV (1.10), such as

$$v(y) = \mathcal{N}_v\, e^{M_H\, y}\,. \tag{1.11}$$

In this case, the BC (1.9) for the VEV at $y = L$ gives

$$\mathcal{N}_v = \sqrt{\frac{12}{\lambda_H}\,(M_L - M_H)}\, e^{-M_H\, L}. \tag{1.12}$$

The VEV is thus peaked at $y = L$, as in the scenario of Ref. [111]. Note that in this reference, the authors consider a more general flat extra-dimensional model where the scalar field has a quartic self-interaction in the bulk and on both branes. The general profile of the VEV is then more involved. However, as argued in the same article [111], one can choose a natural regime of parameters where the VEV profile is well approximated by an exponential function peaked on the brane at $y = L$. One may notice that the quartic potential terms (1.3) only exist on the IR brane, the motivation of which would be discussed in Chapter 2.2.

In the end of this section, we need to give a glance at the 5D scalar field $h(x^\mu, y)$. From the Euler-Lagrange equation (1.5) of $H$ in the bulk, and its BC (1.6), one also derives the EOM for $h(x^\mu, y)$ in the bulk by 4D dependence and localization comments:

$$\forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,,\quad \left(\partial_M \partial^M + M_H^2\right) h = 0\,, \tag{1.13}$$

and the NBC on the branes:

$$(\partial_4 - M_0)\, h|_0 = 0\,,\quad \left[ (\partial_4 - M_L)\, h + \frac{\lambda_H}{4}\, v^2 h \right]\Big|_L = 0\,. \tag{1.14}$$

which is explicitly derived in Appendix B.

## 1.3 Bulk Fermion Fields

We recall the common approach of performing Hamilton's principle in a holographic context of 5D fermion fields following Ref [109], to show disappointing results with the lack of the BBT. A general bulk action of a 5D Fermion field is developed as:

$$S_{\text{bulk}} = \int d^4x \int_0^L dy\ \mathcal{L}_{\text{bulk}}\,, \tag{1.15}$$

where $\mathcal{L}_{\text{bulk}}$ includes the fermion kinetic and the mass terms of the Lagrangian density, which is integrable over the entire region, $\mathcal{I} = [0, L]$. The 5D fermion fields, $F(x^\mu, y)$, – of mass dimension 2 – have the following kinetic terms [entering Eq. (1.15)] which allow to recover canonical covariant kinetic terms for the associated fermions in the 4D effective action (as imposed by the argument of decoupling limit [1]):

$$\mathcal{L}_{\text{kin}} = \frac{i}{2}\,\bar{F}\Gamma^M \overleftrightarrow{\partial_M} F\,, \tag{1.16}$$

using the standard notations $\overleftrightarrow{\partial_M} = \overrightarrow{\partial_M} - \overleftarrow{\partial_M}$, $\partial_M = \partial/\partial x^M$, $x^M = (x^\mu, y)$ with $M \in [\![0,4]\!]$ for the coordinates, $x^M \in \mathcal{E}^5$, and $\Gamma^M$ for the 5D Dirac matrices (cf. Appendix A). In the used conventions, the 5D Dirac spinor of mass dimension 2 is an irreducible representation of the Lorentz group as,

$$F = \mathcal{F}_L + \mathcal{F}_R \quad \text{with} \quad \mathcal{F}_L = \begin{pmatrix} F_L \\ 0 \end{pmatrix} = \mathcal{P}_L F\,, \quad \mathcal{F}_R = \begin{pmatrix} 0 \\ F_R \end{pmatrix} = \mathcal{P}_R F\,, \tag{1.17}$$

in terms of the two two-component Weyl spinors $F_L$, $F_R$, $L/R$ standing for the Left/Right chirality after the chiral projection,

$$\mathcal{P}_{L/R} \mathrel{\hat{=}} \frac{1 \mp \gamma^5}{2}\,, \tag{1.18}$$

and $\bar{F} = F^\dagger \gamma^0$ as usual. Based on Eq. (1.17), we can rewrite the kinetic terms $\mathcal{L}_{\text{kin}}$ in the bulk Lagrangian of Eq. (1.16) in the following form where the Left/Right chiralities are obvious:

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \frac{1}{2}\left( iF_R^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu} F_R + iF_L^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F_L - F_R^\dagger \overleftrightarrow{\partial_4} F_L + F_L^\dagger \overleftrightarrow{\partial_4} F_R \right) \\ &= \frac{1}{2}\left( i\overline{\mathcal{F}_R} \gamma^\mu \overleftrightarrow{\partial_\mu} \mathcal{F}_R + i\overline{\mathcal{F}_L} \gamma^\mu \overleftrightarrow{\partial_\mu} \mathcal{F}_L - \overline{\mathcal{F}_R} \overleftrightarrow{\partial_4} \mathcal{F}_L + \overline{\mathcal{F}_L} \overleftrightarrow{\partial_4} \mathcal{F}_R \right)\,, \end{aligned} \tag{1.19}$$

using the matrices $\sigma^\mu, \bar{\sigma}^\mu$ defined in Appendix A and $\overline{\mathcal{F}_{L/R}} = \mathcal{F}_{L/R}^\dagger \gamma^0$.

Besides, we can add bulk mass terms to the bulk terms as additional physical ingredients,

$$\mathcal{L}_{\text{mass}} = -\widetilde{m}_F \bar{F} F\,, \tag{1.20}$$

where $\widetilde{m}_F$ is the bulk mass of the fermion, $F$, which is a constant such that $\partial_4 \widetilde{m}_F = 0$ on the whole physical domain, $y \in \mathcal{I}$. The mass terms, $\mathcal{L}_{\text{mass}}$, can also be rewritten by the chiral decomposition (1.17) as,

$$\mathcal{L}_{\text{mass}} = -\widetilde{m}_F \left( F_L^\dagger F_R + F_R^\dagger F_L \right) = -\widetilde{m}_F \left( \overline{\mathcal{F}_L} \mathcal{F}_R + \overline{\mathcal{F}_R} \mathcal{F}_L \right)\,. \tag{1.21}$$

In order to extract the equations of motion and the boundary conditions from the relevant Lagrangian for the bulk fermions, we apply the least action principle – or Hamilton's variational principle – for each of them. The least action principle leads to two relations of the kind, $\delta_{\bar{F}} S_{\text{bulk}} = 0$, for the unknown 5D field $F$, and the corresponding one, $\delta_F S_{\text{bulk}} = 0$, involving the complex conjugate fields [2], since the elementary field variations $\delta F_\alpha$, $\delta \bar{F}_\alpha$ (see Appendix C.1) are generic and hence independent from each other. Assuming, at a first level, the boundary fields $F(x^\mu, y = \{0, L\}) \mathrel{\hat{=}} F|_{0,L}$ to be initially unknown (unfixed), they should be deduced from the action minimization with respect to

1. From the theoretical consistency and phenomenological points of view, the SM must be approximately recovered at low-energies in the limit of infinitely heavy KK excitations.

2. The equations of motion and boundary conditions derived from the least action principle for the fields and their conjugates are trivially related through Hermitian conjugation.

them, considering thus non-vanishing generic [3] variations $\delta F|_{0,L} \neq 0$ [4]. In other words, $F|_{0,L}$ should be then obtained from the so-called Natural Boundary Conditions. Using compact notations, like for example,

$$\sum_{\alpha=1}^{4} \delta \bar{F}_\alpha \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \bar{F}_\alpha} \hat{=} \delta \bar{F} \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \bar{F}} \,, \tag{1.22}$$

we can write in particular [5],

$$\begin{aligned} \delta_{\bar{F}} S_{\text{bulk}} &= \int d^4x \int_0^L dy \, \left\{ \delta \bar{F} \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \bar{F}} + \delta \left( \partial_M \bar{F} \right) \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \, \partial_M \bar{F}} \right\} \\ &= \int d^4x \int_0^L dy \, \left\{ \delta \bar{F} \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \bar{F}} + \partial_M \left[ \delta \bar{F} \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \, \partial_M \bar{F}} \right] - \delta \bar{F} \, \partial_M \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \, \partial_M \bar{F}} \right\} \\ &= \int d^4x \int_0^L dy \, \left\{ \delta \bar{F} \left[ \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \bar{F}} - \partial_M \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \, \partial_M \bar{F}} \right] \right\} + \int d^4x \;\; \delta \bar{F} \frac{\partial \mathcal{L}_{\text{bulk}}}{\partial \, \partial_4 \bar{F}} \bigg|_0^L \,. \end{aligned} \tag{1.23}$$

Then, a generic 5D Fermion field can be decomposed by the following KK decomposition,

$$F_{L/R} \left( x^\mu, y \right) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} f_{L/R}^n(y) \, F_{L/R}^n \left( x^\mu \right) , \tag{1.24}$$

where the 4D fields $F_{L/R}^n$ represent the KK states and satisfy the Dirac-Weyl equations,

$$\forall \, n \in \mathbb{N}, \; \left\{ \begin{array}{rcl} i\bar{\sigma}^\mu \partial_\mu F_L^n \left( x^\mu \right) - m_n^F \, F_R^n \left( x^\mu \right) &=& 0 \, , \\ i\sigma^\mu \partial_\mu F_R^n \left( x^\mu \right) - m_n^F \, F_L^n \left( x^\mu \right) &=& 0 \, , \end{array} \right. \tag{1.25}$$

involving the KK mass eigenvalues $m_n^F$. Besides, the two (for L/R) following ortho-normalisation conditions over the full domain for non-vanishing profiles,

$$\forall \, n, m \in \mathbb{N}, \; \frac{1}{L} \int_0^L dy \; f_{L/R}^{n*}(y) \, f_{L/R}^m(y) = \delta_{nm} \, , \tag{1.26}$$

originating from the condition of a canonical form for the 4D effective kinetic terms. The integer $n$ is defined as being the level index of the fermion mode tower and is chosen to be non-negative, i.e. $n \in \mathbb{N}$; the meaningful feature about the general KK decomposition (1.24) is rather the infinite summation (possibly also from $-\infty$ to $+\infty$) dictated by field expressions as Fourier series on a finite interval.

3. A field variation reads as $\delta F(x^M) = \epsilon \, \eta(x^M)$ with a generic function $\eta(x^M)$ and an infinitesimal parameter $\epsilon \to 0$.

4. Then in the final step, once for instance the field $F|_L$ is found and fixed by the solution (not initially fixed as an hypothesis in this considered case), its resulting determined form does not imply $\delta F|_L = 0$ which would be incompatible with the starting non-vanishing field variation: there are sometimes confusions in articles about these chronological aspects of the variational calculus.

5. We omit the global 4-divergence which vanishes in the action integration due to vanishing fields at the boundaries at infinities. Indeed, when minimising the action, the varied terms must vanish separately at infinite boundaries, since the non-vanishing field variations at boundaries are independent from each other and from the bulk ones (see also Ref. [8]). This is realized by the local physics statement which induces vanishing fields at infinities due to the wave function normalization conditions (see also Ref. [112]).

### 1.3.1 Bulk Massless Fermion Fields

Let's start from the bulk massless case where

$$\mathcal{L}_{\text{bulk}} = \mathcal{L}_{\text{kin}} \,. \tag{1.27}$$

Based on the Lagrangian $\mathcal{L}_{\text{kin}}$ of Eq. (1.19), bulk terms and remaining brane terms can be calculated:

$$\begin{aligned} \delta_{\bar{F}} S_{\text{bulk}} &= \int d^4x \left\{ \int_0^L dy \; \left[ \delta\bar{F} \left( i\Gamma^M \partial_M F \right) \right] + \delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \Big|_0^L \right\} \\ &= \int d^4x \left\{ \int_0^L dy \; \left[ \delta\bar{F} \left( i\Gamma^M \partial_M F \right) \right] + \frac{1}{2} \left[ \delta F_R^\dagger F_L - \delta F_L^\dagger F_R \right] \Big|_0^L \right\} , \end{aligned} \tag{1.28}$$

where the bulk and the brane variations – respectively the volume and surface terms – must vanish separately due to independent field variations (no reason to be linked). Besides, all those field variations are not vanishing (unknown fields) so that we obtain the bulk EOM,

$$\forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,, \quad i\,\Gamma^M \partial_M F = 0\,, \tag{1.29}$$

with it's chiral formula after the chiral projection (1.17),

$$\left\{ \begin{array}{rcl} i\bar{\sigma}^\mu \partial_\mu F_L + \partial_4 F_R &=& 0\,, \\ i\sigma^\mu \partial_\mu F_R - \partial_4 F_L &=& 0\,, \end{array} \right. \tag{1.30}$$

and the corresponding NBC derived via non-vanishing boundary variations $\delta F_{L/R}^\dagger \big|_{0,L} \neq 0$,

$$F_L|_0 = F_R|_0 = F_L|_L = F_R|_L = 0\,. \tag{1.31}$$

At this level, we can first solve Eq. (1.30) together with Eq. (1.31) to find out $F$ fields over the domain, $\mathcal{I} = [0, L]$. Inserting the KK decomposition (1.24) and the 4D Dirac equations (1.25) into the 5D Euler-Lagrange equations in Eq. (1.30) and BC in Eq. (1.31), one can directly extract the differential equations for KK wave functions [thanks to the linear independence of 4D fields, which are mass eigenstates of the 4D Dirac-Weyl equations in Eq. (1.25)],

$$\forall\, n \in \mathbb{N}, \; \left\{ \begin{array}{rcl} \partial_4 f_L^n(y) - m_n^F\, f_R^n(y) &=& 0\,, \\ \partial_4 f_R^n(y) + m_n^F\, f_L^n(y) &=& 0\,, \end{array} \right. \tag{1.32}$$

and the following Dirichlet boundary conditions for profiles of all KK modes,

$$\forall\, n \in \mathbb{N},\ f_{L/R}^n \Big|_0 = f_{L/R}^n \Big|_L = 0\,. \tag{1.33}$$

Then, we would argue for the profile vanishing. For the massless mode, the zero mass $m_0^F = 0$, combined with the coupled EOM (1.32), would induce the Neumann BC for KK zero modes both in the left and the right chirality, i.e. $\partial_4 f_{L/R}^0 \Big|_{0,L} = 0$. For massive modes, the Dirichlet BC for KK wave functions associated to one chirality (1.33), combined with the coupled EOM (1.32), would give the Neumann BC for KK wave functions of the other chirality again.

$$\forall\, n \in \mathbb{N},\ \partial_4 f_{L/R}^n \Big|_0 = \partial_4 f_{L/R}^n \Big|_L = 0\,. \tag{1.34}$$

Hence, all KK wave functions have both the Dirichlet and the Neumann BC at each boundary.

The two first order coupled EOM in Eq. (1.32) can be combined into two second order decoupled equations,

$$\forall\, n \in \mathbb{N},\ \left[\partial_4^2 + \left(m_n^F\right)^2\right] f_{L/R}^n(y) = 0\,, \tag{1.35}$$

which are the equations for independent harmonic oscillators whose solutions have the general formalism in $y \in [0, L]$,

$$\forall\, n \in \mathbb{N},\ f_{L/R}^n(y) = A_{L/R}^n \cos(m_n^F\, y) + B_{L/R}^n \sin(m_n^F\, y)\,, \tag{1.36}$$

where $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients, which are related by the coupled equations in Eq. (1.32) as

$$\begin{cases} A_L^n &= -B_R^n\,, \\ A_R^n &= B_L^n\,. \end{cases} \tag{1.37}$$

for all non-zero modes with $m_n^F \neq 0$, $\forall\, n \in \mathbb{N}^*$. After considering the factor relations, we obtain the general solutions (1.36) in a revised formalism,

$$\forall\, n \in \mathbb{N},\ \begin{cases} f_L^n(y) &= -B_R^n \cos(m_n^F\, y) + B_L^n \sin(m_n^F\, y)\,, \\ f_R^n(y) &= B_L^n \cos(m_n^F\, y) + B_R^n \sin(m_n^F\, y)\,. \end{cases} \tag{1.38}$$

The Dirichlet (1.33) and the Neumann BC (1.34) would provide additional constraints for the coefficients in the general formalism (1.36) [6], so that we obtain

$$\forall n \in \mathbb{N}\,,\ f_{L/R}^n(y) = 0\ ,\quad \forall\, y \in \mathcal{I} = [0, L]\,, \tag{1.39}$$

which conflicts to the ortho-normalization condition in Eq.(1.24) and suspend all KK modes on the whole $\mathcal{I}$ region. Hence, we can conclude the solutions of 5D fields obtained through this naive method are not physically consistent.

We can do a further mathematical analysis. This problem comes from the fact that the system is over-constrained at the boundaries. Indeed, the equation set of KK wave functions in Eq. (1.32) relates $f_L^n(y)$ and $f_R^n(y)$ on-shell for massive modes: a boundary condition for $f_L^n(y)$ is also a constraint on $f_R^n(y)$ and vice versa. Therefore, for zero modes, $f_{L/R}^0(y)$ depend on one complex coefficient $B_{R/L}^0$ respectively in Eq. (1.38). With respect to massive KK wave functions, $f_{L/R}^n(y)$ with $m_n^F \neq 0$ depend on the same three parameters in Eq. (1.38): the mass $m_n^F$ and the two complex coefficients $B_{L/R}^n$. The variation of the action at the boundaries in Eq. (1.28) involves the variations of both $F_L$ and $F_R$ so there are two NBC at each boundary. The system is thus over-constrained, it is why KK wave functions vanish everywhere for all modes, which has been well analyzed in the literature [1].

### 1.3.2 Bulk Massive Fermion Fields

After the investigation of bulk massless fields in the last section, we turn to bulk terms (1.27) including bulk mass terms (1.20),

$$\mathcal{L}_{\text{bulk}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}}\,. \tag{1.40}$$

6. The EOM of $f_{L/R}^n(y)$ Eq. (1.35) is a second order linear differential equation with constant coefficients with respect to the extra dimension $y$ for any particular KK mass $m_n^F$, whose solution can be fixed by the value of the field and its derivative at one point. So, in fact, the existence of both Dirichlet and the Neumann BC at one point is enough to force the field to vanish.

The least action principle is applied by the similar process in Eq. (1.28) - leads to the bulk EOM with the bulk mass $\widetilde{m}_F$,

$$\left(i\,\Gamma^M \partial_M - \widetilde{m}_F\right) F = 0\,, \qquad \forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,, \tag{1.41}$$

and the NBC remain identical to Eq. (1.31). Then, inserting the KK decomposition (1.24) and the 4D Dirac equations (1.25) into the 5D EOM (1.41) and NBC (1.31), one directly obtains the EOM for the profiles on $[0, L]$:

$$\forall\, n \in \mathbb{N},\ \left\{ \begin{array}{rcl} \left(\partial_4 + \widetilde{m}_F\right) f_L^n(y) - m_n^F\, f_R^n(y) & = & 0\,, \\ \left(\partial_4 - \widetilde{m}_F\right) f_R^n(y) + m_n^F\, f_L^n(y) & = & 0\,, \end{array} \right. \tag{1.42}$$

and the Dirichlet BC for profiles in Eq. (1.33) would be derived again.

The two first order coupled EOM in Eq. (1.42) can be combined into the decoupled second order equations,

$$\forall\, n \in \mathbb{N},\ \partial_4^2 f_{L/R}^n(y) + \left[\left(m_n^F\right)^2 - \left(\widetilde{m}_F\right)^2\right] f_{L/R}^n(y) = 0\,, \tag{1.43}$$

which are regular Sturm-Liouville equations on the interval $\mathcal{I} = [0, L]$, and more complicated than that in the bulk massless case of Eq. (1.35). For $\left(m_n^F\right)^2 - \left(\widetilde{m}_F\right)^2 < 0$ in Eq. (1.43), the solutions have the general form on $y \in [0, L]$,

$$f_{L/R}^n(y) = A_{L/R}^n \exp\left(\Delta m_n^F\, y\right) + B_{L/R}^n \exp\left(\Delta m_n^F\, y\right)\,, \tag{1.44}$$

where $\Delta m_n^F \,\hat{=}\, \sqrt{\left|\left(\widetilde{m}_F\right)^2 - \left(m_n^F\right)^2\right|}$ and $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients, which are related by the coupled EOM in Eq. (1.42) as,

$$n = 0\,,\ \left\{ \begin{array}{rcl} A_L^0 & = & 0\,, \\ B_R^0 & = & 0\,, \end{array} \right. \tag{1.45}$$

$$n \neq 0\,,\ \left\{ \begin{array}{rcl} \left(\Delta m_n^F + \widetilde{m}_F\right) A_L^n & = & m_n^F\, A_R^n\,, \\ \left(-\Delta m_n^F + \widetilde{m}_F\right) B_L^n & = & m_n^F\, B_R^n\,. \end{array} \right. \tag{1.46}$$

Then, for $\left(m_n^F\right)^2 - \left(\widetilde{m}_F\right)^2 \geq 0$ in Eq. (1.43), the solutions would have the general form as,

$$f_{L/R}^n(y) = A_{L/R}^n \cos\left(\Delta m_n^F\, y\right) + B_{L/R}^n \sin\left(\Delta m_n^F\, y\right)\,, \tag{1.47}$$

where $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients, which satisfy the relationship generated from the coupled EOM in Eq. (1.42) as,

$$\left\{ \begin{array}{rcl} \Delta m_n^F\, B_L^n + \widetilde{m}_F\, A_L^n & = & m_n^F\, A_R^n\,, \\ -\Delta m_n^F\, A_L^n + \widetilde{m}_F\, B_L^n & = & m_n^F\, B_R^n\,. \end{array} \right. \tag{1.48}$$

Following the exact same analysis as the bulk massless case, the Dirichlet BC (1.33), combined with the coupled EOM (1.32), would lead to the Neumann BC(1.34) for all KK modes both in the left and the right chirality. Hence, all KK wave functions must vanish via the over-constrained BC – the Dirichlet and the Neumann BC at each boundary. Finally, this naive bulk massive model can't provide physical solutions, which is disappointing but consistent with superficial predictions after the bulk massless case in Section 1.3.1.

# Chapter 2

# The Warped Background

In this chapter, we continue our treatments of 5D bulk fermions but in a particularly warped scenario – the Randall Sundrum Model, which is a direct extension of the flat geometry in Chapter 1 but with more abundant physical ingredients.

## 2.1 The Randall-Sundrum Metric

Consider a 5D spacetime $x^M \hat{=} (x^\mu, y)$, $M \in [\![0,4]\!]$ embedded in a warped geometry – $AdS_5$ space,

$$\mathcal{E}^5 = \mathcal{M}^4 \times \mathcal{I}\,,$$

with an extra dimension compactified on an interval $\mathcal{I} = [0, L]$, but now the two 3-branes have opposite tensions. The gravitational back reaction of the brane tensions is balanced by the introduction of a negative bulk cosmological constant, $\Lambda_5 < 0$. A zero effective 4D cosmological constant is preserved by a fine tuning between opposite tensions at 3-branes and $\Lambda_5 < 0$. The 5D metric solution of Einstein's equations reads,

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \hat{=} g_{MN}dx^M dx^N\,, \tag{2.1}$$

where $k$ is the AdS curvature scale and $\eta_{\mu\nu}$ with $\mu$, $\nu \in [\![0,3]\!]$ is the 4D Minkowski metric in Eq. (A.1). The determinant of the AdS metric $g_{MN}$ is denoted as,

$$g \hat{=} |\det g_{MN}| = e^{-8ky} > 0\,. \tag{2.2}$$

The vierbein $e_M{}^A$ , with $A \in [\![0,4]\!]$ is defined via the relation with the 5D Minkowski metric $\eta_{AB}$ in Eq. (A.2),

$$g_{MN} = e_M{}^A e_N{}^B \eta_{AB}\,, \tag{2.3}$$

with an explicit formalism,

$$e_M{}^A = \left(e^{-ky}\delta_\mu{}^\alpha,\, 1\right)\,, \tag{2.4}$$

and its inverse is denoted by $e^M{}_A e_M{}^B = \delta^B_A$,

$$e^M{}_A = \left(e^{ky}\delta^\mu{}_\alpha,\, 1\right)\,. \tag{2.5}$$

For practical reasons, we also define a determinant as $e = \left|\det e_M{}^A\right|$, which satisfies the relation, $e = \sqrt{g}$. In this framework, the UV-brane is located at $y = 0$ and the IR-brane is at $y = L$, where the electroweak symmetry breaking occurs. In some cases, it is sometimes more convenient to work in a conformally flat framework by making the coordinate transformation,

$$z = \frac{\mathrm{e}^{ky}}{k}\,, \tag{2.6}$$

such that the conformally flat vierbein is

$$e^M{}_A = kz\,\delta^M{}_A \quad \text{and} \quad e = (kz)^{-5}\,, \tag{2.7}$$

so

$$g_{MN} = \left(\frac{1}{kz}\right)^2 \eta_{MN}\,. \tag{2.8}$$

while, in this frame by $z$, the UV-brane is setup at $z = 1/k$ and the IR-brane is at $z = \mathrm{e}^{kL}/k$, where the electroweak symmetry breaking occurs. If $M_*$ is the 5D gravity scale, one should have $M_*/k \gtrsim \mathcal{O}(10)$ such that we are in the classical regime where one can use the Einstein equations: the $\mathrm{AdS}_5$ background metric is well defined. The cut-off on the UV-brane is usually taken $\Lambda_{UV} \sim M_*$ and the one on the IR-brane is $\Lambda_{IR} = \mathrm{e}^{-kL}\Lambda_{UV}$, due to the gravitational redshift induced by the exponential warp factor. If one stays at $\Lambda_{IR} \sim \mathcal{O}(1)$ TeV and manipulates an IR-brane localized (or peaked) Higgs field, one can solve the gauge hierarchy problem of the Higgs sector in the SM.

## 2.2 Bulk Scalar Fields

Following the methodology in the flat geometry in Chapter 1, we start from the exploration of a 5D bulk real scalar field, which contains quadratic mass terms and potential terms at 3-branes. The 5D action of the field $H$ should be modified by the AdS metric,

$$S_H = \int d^4x \left[\left(\int_0^L dy\ \sqrt{g}\,\mathcal{L}_H\right) - \sqrt{g}\big|_0 \times V_0(H) - \sqrt{g}\big|_L \times V_L(H)\right], \tag{2.9}$$

while the 5D bulk scalar Lagrangian of $H$ should also contain addition physical contributions from the warped curvature [1],

$$\mathcal{L}_H = \frac{1}{2}\,g^{MN}\partial_M H \partial_N H - \frac{M_H^2}{2}\,H^2\,, \tag{2.10}$$

and the boundary-localized potentials $V_{0,L}(H)$ in Eq. (1.3) would still make sense.

The Hamilton's principle by varying the field $H$ should be performed as in Eq. (1.4), but the bulk and the boundary terms should take into account of the contribution from the volume element $\sqrt{g}$ in Eq. (2.2),

$$\mathcal{L}_H \to \sqrt{g}\,\mathcal{L}_H\,, \quad V|_{0,L} \to \sqrt{g}\,V|_{0,L}\,.$$

Thus, the independent bulk and the boundary variations should induce the Euler-Lagrange equation of $H$ in the bulk [2]:

$$\forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,, \quad \left(\partial_M \partial^M + M_H^2\right) H + \left(\frac{1}{\sqrt{g}}\,\partial_4 \sqrt{g}\right)\partial^4 H = 0\,, \tag{2.11}$$

1. $g^{MN}$ is conventionally defined by $g^{MN} g_{NP} = \delta^M_P$.
2. $\partial^M \hat{=}\, g^{MN}\partial_N$.

which can be rewritten by $\sqrt{g}$ in Eq. (2.2) [3],

$$e^{2ky}\,\eta^{\mu\nu}\,\partial_\mu\partial_\nu H-\left(\partial_4^2-4k\,\partial_4\right)H+M_H^2H=0\,, \tag{2.12}$$

and the NBC (1.6) on the branes keep valid. Due to the quartic potential terms (1.3) on the IR brane, a non-vanishing VEV, $v(y)$ in Eq. (1.7) is still expected. The EOM of $v(y)$ should be abstracted from the EOM (2.11) of $H$ in the bulk according to the 4D spacetime dependence and asymptotic principle through a similar procedure in Appendix B,

$$\left(\partial_4^2-4k\,\partial_4-M_H^2\right)v(y)=0\,, \tag{2.13}$$

and its BC keep consistent with Eq. (1.9). The bulk EOM of the VEV in Eq. (2.13) would generate a general formalism of solutions,

$$v(y)=\mathcal{N}_v\,e^{2k\,y}\left(e^{\alpha k\,y}+Be^{-\alpha k\,y}\right)\,,\quad \alpha=\frac{\sqrt{4k^2+M_H^2}}{k}\,, \tag{2.14}$$

where $\mathcal{N}_v$ and $B$ are constants to be determined by the BC for the VEV in Eq. (1.9),

$$\left\{\begin{array}{rcl}(2k+\alpha k-M_0)+B\,(2k-\alpha k-M_0) & = & 0\,,\\ (2k+\alpha k-M_0)\,\Omega^{\alpha}+B\,(2k-\alpha k-M_0)\,\Omega^{-\alpha}+\dfrac{\lambda_H}{12}\left(\mathcal{N}_v\Omega^2\right)^2\left(\Omega^{\alpha}+B\Omega^{-\alpha}\right)^3 & = & 0\,,\end{array}\right. \tag{2.15}$$

so that one can derive,

$$\left\{\begin{array}{rcl}B & = & -\dfrac{(2+\alpha)\,k-M_0}{(2-\alpha)\,k-M_0}\,,\\ \mathcal{N}_v & = & \sqrt{\dfrac{[M_0-(2+\alpha)\,k]\,\Omega^{2+\alpha}+B\,[M_0-(2-\alpha)\,k]\,\Omega^{2-\alpha}}{(\lambda_H/12)\left(\Omega^{2+\alpha}+B\Omega^{2-\alpha}\right)^3}}\,,\end{array}\right. \tag{2.16}$$

where $\Omega\mathrel{\hat=}e^{kL}$ is denoted as the warp factor. As the flat case in Eq. (1.10), we can select $M_0=2k+\sqrt{4k^2+M_H^2}$ to simplify the profile of the VEV (2.14), such that

$$v(y)=\mathcal{N}_v\,e^{(2+\alpha)k\,y}\,, \tag{2.17}$$

where

$$\left\{\begin{array}{rcl}B & = & 0\,,\\ \mathcal{N}_v & = & \sqrt{\dfrac{12}{\lambda_H}\,[M_0-(2+\alpha)\,k]}\,\Omega^{-\frac{4+\alpha}{2}}\,,\end{array}\right. \tag{2.18}$$

and the VEV would be deformed to an exponential profile peaked at $y=L$ again. Note that the exponential VEV in Eq. (2.17) would exactly recover that in the flat geometry (1.11) if we take the limit $k\to 0$.

Here, we need to argue for the simplified quartic potential setup, which has been picked up in Chapter 1.2. The choice of our simple potential is inspired by the case of models with a warped extra dimension, which is discussed in Ref. [110]. Indeed, in a RS model, the AdS geometry implies that scales are red-shifted by the warp factor relying on the extra dimensional position. This effect is important for the quartic couplings of $H^4$ on the 4D slides along the extra dimension, which are highly suppressed by powers of the

3. $\eta^{\mu\nu}$ is conventionally defined by $\eta^{\mu\nu}\eta_{\nu\rho}=\delta^\mu_\rho$.

curvature related to the Plank scale $\sim \mathcal{O}\left(\Lambda_{Planck}^{-2}\right)$ in the bulk and at the UV brane. While $\lambda_H$ at IR brane is significantly warped down to the KK scale $\sim \mathcal{O}\left(m_{KK}^{-2}\right)$ which would play a dominant role in the entire space, so authors always can neglect the other ones for simplicity.

Besides, it is also interesting to comment on the effect of the warp factor on the localization of the VEV, which has the typical behavior discussed in Ref.[110]. Compare the VEV in RS model (2.17) and that in the flat scenario (1.11), we can clearly see that the curvature $k$ provides addition contributions to the exponential behavior and makes the scalar profile focus on the IR brane more rapidly.

From the Euler-Lagrange equation (2.11) of $H$ in the bulk, and its BC (1.6), one also derives the EOM for $h(x^\mu, y)$ following a similar procedure in Chapter 1.2,

$$e^{2ky}\,\eta^{\mu\nu}\,\partial_\mu\partial_\nu h - \left(\partial_4^2 - 4k\,\partial_4\right)h + M_H^2 h = 0\,, \qquad \forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,, \tag{2.19}$$

and the NBC on the branes is identical to Eq. (1.14).

## 2.3 Bulk Fermion Fields

A general bulk action of a 5D Fermion field in the AdS is developed as:

$$S_{\text{bulk}} = \int d^4x \int_0^L dy\ \sqrt{g}\,\mathcal{L}_{\text{bulk}}\,, \tag{2.20}$$

where $\mathcal{L}_{\text{bulk}}$ includes the fermion kinetic and the mass terms of the Lagrangian density, which is integrable over the entire AdS. The 5D AdS spinor representation would be constructed from a revised gamma matrices by the inverse vierbein $e^M{}_A$,

$$\left\{e^M{}_A\,\Gamma^A,\, e^N{}_B\,\Gamma^B\right\} = 2\,e^M{}_A\,e^N{}_B\,\eta^{AB} = 2g^{MN}\,. \tag{2.21}$$

One also needs the covariant derivative to develop kinetic terms,

$$D_M \mathrel{\hat{=}} \partial_M + \omega_M\,, \tag{2.22}$$

where $\omega_M$ is the spin connection [derived precisely in Appendix D],

$$\omega_M = \left(i\,\frac{k}{2}\,e^{-ky}\gamma_\mu\gamma^5,\, 0\right)\,, \qquad \gamma_\mu = \eta_{\mu\nu}\gamma^\nu\,. \tag{2.23}$$

Now we can replace common derivatives $\partial_M$ in Eq. (1.16) by covariant derivatives $D_M$ in Eq. (2.22) and obtain kinetic terms in the AdS,

$$\mathcal{L}_{\text{kin}} = \frac{i}{2}\,\bar{F}e^M{}_A\,\Gamma^A\overleftrightarrow{D_M}F\,, \tag{2.24}$$

where

$$\bar{F}e^M{}_A\,\Gamma^A\overleftrightarrow{D_M}F \mathrel{\hat{=}} \bar{F}e^M{}_A\,\Gamma^A D_M F - \overline{D_M F}e^M{}_A\,\Gamma^A F\,,$$

using the standard notations $\overline{D_M F} = (D_M F)^\dagger \gamma^0$, and the Hermiticity of which can be proved [4],

$$\left(\frac{i}{2}\bar{F}e^M{}_A\Gamma^A D_M F\right)^\dagger = -\frac{i}{2}(D_M F)^\dagger \left(e^M{}_A\Gamma^A\right)^\dagger \bar{F}^\dagger = -\frac{i}{2}(D_M F)^\dagger \gamma^0\gamma^0 \left(e^M{}_A\Gamma^A\right)^\dagger \gamma^0 F$$
$$= -\frac{i}{2}\overline{D_M F}e^M{}_A\left(\gamma^0\Gamma^{A\dagger}\gamma^0\right)F$$
$$= -\frac{i}{2}\overline{D_M F}e^M{}_A\Gamma^A F\,.$$

It's interesting that in the RS metric (2.1), the contribution of the spin connection $\omega_M$ cancels in the kinetic Lagraangian of Eq. (2.24),

$$\begin{aligned}
\bar{F}e^M{}_A\Gamma^A\overleftrightarrow{D_M}F &= \bar{F}e^\mu{}_A\Gamma^A\overleftrightarrow{D_\mu}F + \bar{F}e^4{}_A\Gamma^A\overleftrightarrow{\partial_4}F\\
&= \bar{F}e^\mu{}_A\Gamma^A D_\mu F - \overline{D_\mu F}e^\mu{}_A\Gamma^A F + \bar{F}e^4{}_A\Gamma^A\overleftrightarrow{\partial_4}F\\
&= \bar{F}e^{ky}\gamma^\mu\left(\partial_\mu + i\,\frac{k}{2}\,e^{-ky}\gamma_\mu\gamma^5\right)F - \left(\partial_\mu F + i\,\frac{k}{2}\,e^{-ky}\gamma_\mu\gamma^5 F\right)^\dagger \gamma^0 e^{ky}\gamma^\mu F\\
&\quad + \bar{F}e^4{}_A\Gamma^A\overleftrightarrow{\partial_4}F\\
&= \bar{F}e^{ky}\gamma^\mu\partial_\mu F + i\,\frac{k}{2}\,\bar{F}\gamma^\mu\gamma_\mu\gamma^5 F - \left(\partial_\mu\bar{F}\right)e^{ky}\gamma^\mu F - i\,\frac{k}{2}\,\bar{F}\gamma^5\gamma_\mu\gamma^\mu F\\
&\quad + \bar{F}e^4{}_A\Gamma^A\overleftrightarrow{\partial_4}F\\
&= \bar{F}e^M{}_A\Gamma^A\overleftrightarrow{\partial_M}F\,,
\end{aligned}$$

and we can now safely return to the common derivative $\partial_M$,

$$\mathcal{L}_{\text{kin}} = \frac{i}{2}\,\bar{F}e^M{}_A\Gamma^A\overleftrightarrow{\partial_M}F\,, \tag{2.25}$$

where

$$\bar{F}e^M{}_A\Gamma^A\overleftrightarrow{\partial_M}F = \bar{F}e^M{}_A\Gamma^A\partial_M F - \left(\partial_M\bar{F}\right)e^M{}_A\Gamma^A F\,.$$

Based on the chiral decomposition (1.17), we can rewrite the kinetic terms, $\mathcal{L}_{\text{kin}}$ of Eq. (2.24) in the following form where the Left/Right chiralities (1.17) are obvious:

$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= \frac{1}{2}\left(iF_R^\dagger e^{ky}\sigma^\mu\overleftrightarrow{\partial_\mu}F_R + iF_L^\dagger e^{ky}\bar{\sigma}^\mu\overleftrightarrow{\partial_\mu}F_L - F_R^\dagger\overleftrightarrow{\partial_4}F_L + F_L^\dagger\overleftrightarrow{\partial_4}F_R\right)\\
&= \frac{1}{2}\left(i\overline{\mathcal{F}_R}e^{ky}\gamma^\mu\overleftrightarrow{\partial_\mu}\mathcal{F}_R + i\overline{\mathcal{F}_L}e^{ky}\gamma^\mu\overleftrightarrow{\partial_\mu}\mathcal{F}_L - \overline{\mathcal{F}_R}\overleftrightarrow{\partial_4}\mathcal{F}_L + \overline{\mathcal{F}_L}\overleftrightarrow{\partial_4}\mathcal{F}_R\right)\,,
\end{aligned} \tag{2.26}$$

using the matrices $\sigma^\mu, \bar{\sigma}^\mu$ defined in Appendix A. Then, we can still add bulk mass terms $\mathcal{L}_{\text{mass}}$ of Eq. (1.20) [and its chiral form in Eq. (1.21)] to the bulk terms $\mathcal{L}_{\text{bulk}}$ (2.20).

The least action principle by varying the field $F$ (and the corresponding $\bar{F}$) should be performed as in Eq. (1.4), but the bulk terms should take account of the contribution from the volume element $\sqrt{g}$ in Eq. (2.2),

$$\mathcal{L}_{\text{bulk}} \to \sqrt{g}\,\mathcal{L}_{\text{bulk}}\,.$$

4. The conjugate equation has been inserted,

$$\Gamma^{A\dagger} = \gamma^0\Gamma^A\gamma^0\,.$$

Then, a generic 5D Fermion field can be decomposed by the following KK decomposition in Eq. (1.24). However, the ortho-normalization conditions should be modeified by the warp factor,

$$\forall\, n, m \in \mathbb{N},\ \ \frac{1}{L}\int_0^L dy\ \sqrt{g}\, e^{ky}\, f^{n*}_{L/R}(y)\, f^{m}_{L/R}(y) = \delta_{nm}\,, \tag{2.27}$$

originating from the condition of a canonical form for the 4D effective kinetic terms.

### 2.3.1 Bulk Massless Fermion Fields

Following the guideline in Chapter 1.3, we still start from the bulk massless case where

$$\mathcal{L}_{\text{bulk}} = \mathcal{L}_{\text{kin}}\,. \tag{2.28}$$

Based on the Lagrangian $\mathcal{L}_{\text{kin}}$ of Eq. (2.25), bulk terms and remaining brane terms can be calculated:

$$\begin{aligned}\delta_{\bar F} S_{\text{bulk}} &= \int d^4x \int_0^L dy\ \delta\bar F\left\{i\sqrt{g}\left[e^M{}_A\,\Gamma^A D_M F\right]\right\} + \int d^4x\ \ \delta\bar F\left[-\sqrt{g}\,\frac{\gamma^5}{2}F\right]\Bigg|_0^L \\ &= \int d^4x \left\{\int_0^L dy\ \delta\bar F\,\sqrt{g}\left[i\,e^M{}_A\,\Gamma^A D_M F\right] + \frac{\sqrt{g}}{2}\left[\delta F_R^\dagger F_L - \delta F_L^\dagger F_R\right]\Bigg|_0^L\right\}\,,\end{aligned} \tag{2.29}$$

where the independent vanishing of the volume terms would lead to the bulk EOM in the covariant form,

$$\forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,,\ \ i\,e^M{}_A\,\Gamma^A D_M F = 0\,, \tag{2.30}$$

and it's explicit form with common derivatives $\partial_M$,

$$i\,e^{ky}\gamma^\mu\partial_\mu F + \gamma^5\left(\partial_4 - 2k\right)F = 0\,, \tag{2.31}$$

with it's chiral formula after the chiral projection,

$$\left\{\begin{array}{rcl} i\,e^{ky}\bar\sigma^\mu\partial_\mu F_L + (\partial_4 - 2k)\,F_R &=& 0\,, \\ i\,e^{ky}\sigma^\mu\partial_\mu F_R - (\partial_4 - 2k)\,F_L &=& 0\,, \end{array}\right. \tag{2.32}$$

and surface and the corresponding NBC is dentical to Eq. (1.31).

At this level, we can first solve Eq. (2.32) together with Eq. (1.31) to find out $F$ fields over the domain, $\mathcal{I} = [0, L]$. Inserting the KK decomposition (1.24) and the 4D Dirac equations (1.25) into the 5D Euler-Lagrange equations in Eq. (2.32) and BC in Eq. (1.31), one can directly extract the differential equations for KK wave functions,

$$\forall\, n \in \mathbb{N},\ \left\{\begin{array}{rcl} (\partial_4 - 2k)\, f_L^n(y) - m_n^F\, e^{ky} f_R^n(y) &=& 0\,, \\ (\partial_4 - 2k)\, f_R^n(y) + m_n^F\, e^{ky} f_L^n(y) &=& 0\,, \end{array}\right. \tag{2.33}$$

and the Dirichlet boundary conditions for profiles of all KK modes in Eq. (1.33).

Here, we would argue for the profile vanishing again. Since the boundary conditions keep identical to that in the flat case, the Dirichlet and the Neumann BC for the two chiralities of all KK wave functions would exist at each boundary, through an exactly same comment in Chapter 1.3.1.

For the zero mode with $m_0^F = 0$, the EOM (2.33) would be simplified as,

$$\left\{\begin{array}{rcl} (\partial_4 - 2k)\, f_L^0(y) &=& 0\,, \\ (\partial_4 - 2k)\, f_R^0(y) &=& 0\,, \end{array}\right. \tag{2.34}$$

whose solutions have the general formalism in $y \in [0, L]$,

$$f^0_{L/R}(y) = A^0_{L/R}\, e^{2ky}\,, \tag{2.35}$$

where $A^0_{L/R}$ are complex coefficients, which would be clearly suspended to zero by the Dirichlet (1.33) and the Neumann BC (1.34). For non-zero modes with $m^F_n \neq 0$, $n \in \mathbb{N}^*$, the two first order coupled EOM in Eq. (2.33) can be combined into two second order decoupled equations,

$$n \in \mathbb{N}^*,\ \partial_4 \left( e^{-5ky} \partial_4 f^n_{L/R} \right) + \left[ 6k^2 e^{-5ky} + \left( m^F_n \right)^2 e^{-3ky} \right] f^n_{L/R} = 0\,, \tag{2.36}$$

which are regular Sturm-Liouville equations and we can solve the general formalism on $y \in [0, L]$ [precise calculations are presented in Appendix E.1] [5],

$$f^n_{L/R}(y) = e^{\frac{5}{2}ky} \left[ A^n_{L/R}\, J_{-/+\frac{1}{2}} \left( \frac{m^F_n}{k} e^{ky} \right) + B^n_{L/R}\, Y_{-/+\frac{1}{2}} \left( \frac{m^F_n}{k} e^{ky} \right) \right]\,, \tag{2.37}$$

where $A^n_{L/R}$, $B^n_{L/R}$ are complex coefficients, which are related by the coupled equations in Eq. (2.33),

$$\begin{cases} A^n_L &= A^n_R\,, \\ B^n_L &= B^n_R\,, \end{cases} \tag{2.38}$$

which is precisely calculated in Appendix E.1. The Dirichlet (1.33) and the Neumann BC (1.34) would provide additional constraints for the coefficients in the general formalism (2.37),

$$\begin{cases} A^n_R\, J_{\frac{1}{2}}(\xi)\Big|_{y=0} + B^n_R\, Y_{\frac{1}{2}}(\xi)\Big|_{y=0} &= 0\,, \\ A^n_R\, \partial_\xi J_{\frac{1}{2}}(\xi)\Big|_{y=0} + B^n_R\, \partial_\xi Y_{\frac{1}{2}}(\xi)\Big|_{y=0} &= 0\,, \end{cases} \tag{2.39}$$

with $\xi \mathrel{\hat{=}} \dfrac{m^F_n}{k}\, e^{ky}$, while the linear independence of $J_{\frac{1}{2}}(\xi)$ and $Y_{\frac{1}{2}}(\xi)$ would induce the non-zero Wronskian determinant [6] anywhere on $y \in [0, L]$,

$$W = \begin{vmatrix} J_{\frac{1}{2}}(\xi) & Y_{\frac{1}{2}}(\xi) \\ \partial_\xi J_{\frac{1}{2}}(\xi) & \partial_\xi Y_{\frac{1}{2}}(\xi) \end{vmatrix} \neq 0\,, \qquad \forall\, y \in \mathcal{I} = [0, L]\,, \tag{2.40}$$

which forces the set of equation (2.39) to generate zero solutions.

Combing the analysis of the zero and non-zero modes above, we obtain the vanishing of all KK modes on the whole interval $\mathcal{I}$, which is presented explicitly in Eq. (1.39), which conflicts to the ortho-normalization condition in Eq.(2.27). Hence, we can conclude the solutions of 5D fermion fields obtained through this naive method are not physically consistent.

5. $J_\nu$ and $Y_\nu$ are the Bessel functions of the first and the second kind respectively.

6. To be rigorous, the Wronskian determinant in Eq. (2.40) is originally developed for the variable $\xi = \frac{m^F_n}{k} e^{ky}$ satisfying the Eq. (E.5) $[\nu_R = \frac{1}{2}]$ with respect to the function $e^{-\frac{5}{2}ky} f^n_R$.

### 2.3.2 Bulk Massive Fermion Fields

Based on the tricks and methodology developed for the bulk massless case in Chapter 2.3.1, we turn to bulk terms (2.28) including bulk mass terms (1.20),

$$\mathcal{L}_{\text{bulk}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} \,. \tag{2.41}$$

The least action principle is applied by the similar process in Eq. (2.29) - leads to the covariant form of the bulk EOM with the bulk mass $\widetilde{m}_F$,

$$\forall\, x^\mu,\ \forall\, y \in \mathcal{I} = [0, L]\,, \quad \left(i\, e^M{}_A\, \Gamma^A D_M - \widetilde{m}_F\right) F = 0\,, \tag{2.42}$$

and its explict form with common derivatives $\partial_M$,

$$i\, e^{ky} \gamma^\mu \partial_\mu F + \gamma^5 \left(\partial_4 - 2k\right) F - \widetilde{m}_F\, F = 0\,, \tag{2.43}$$

which generates the chiral formula after the chiral projection,

$$\begin{cases} i\, e^{ky} \bar{\sigma}^\mu \partial_\mu F_L + \partial_4 F_R - (c+2)\, k F_R &=& 0\,, \\ i\, e^{ky} \sigma^\mu \partial_\mu F_R - \partial_4 F_L - (c-2)\, k F_L &=& 0\,, \end{cases} \tag{2.44}$$

with $c \,\hat{=}\, \widetilde{m}_F / k$, while the corresponding NBC derived by the suface term vanishing are dentical to Eq. (1.31). Then, inserting the KK decomposition (1.24) and the 4D Dirac equations (1.25) into the 5D EOM (2.44) and NBC (1.31), one directly obtains the EOM for the profiles on $[0, L]$:

$$\forall\, n \in \mathbb{N}, \begin{cases} \left[\partial_4 + (c-2)\, k\right] f_L^n(y) - m_n^F\, f_R^n(y) &=& 0\,, \\ \left[\partial_4 - (c+2)\, k\right] f_R^n(y) + m_n^F\, f_L^n(y) &=& 0\,, \end{cases} \tag{2.45}$$

and the Dirichlet BC for profiles in Eq. (1.33) would be derived again, combining the the EOM (2.44), which would induce the Neumann BC for the two chiralities at each boundary through the comment mentioned in Chapter 1.3.1.

For the zero mode with $m_0^F = 0$, the EOM (2.45) would be simplified as,

$$\begin{cases} \left[\partial_4 + (c-2)\, k\right] f_L^0(y) &=& 0\,, \\ \left[\partial_4 - (c+2)\, k\right] f_R^0(y) &=& 0\,, \end{cases} \tag{2.46}$$

whose solutions have the general formalism in $y \in [0, L]$,

$$f^0_{L/R}(y) = A^0_{L/R}\, e^{(2-/+c)ky}\,, \tag{2.47}$$

where $A^0_{L/R}$ are complex coefficients, which would be clearly suspended to zero by the Dirichlet (1.33) and the Neumann BC (1.34). For non-zero modes with $m_n^F \neq 0$,, the two first order coupled EOM in Eq. (2.45) can be combined into two second order decoupled equations,

$$\partial_4 \left(e^{-5ky} \partial_4 f^n_{L/R}\right) + \left[\left(6 - / + c - c^2\right) k^2 e^{-5ky} + \left(m_n^F\right)^2 e^{-3ky}\right] f^n_{L/R} = 0\,, \tag{2.48}$$

which are regular Sturm-Liouville equations and we can solve the general formalism on $y \in [0, L]$ [precise calculations are presented in Appendix E.2],

$$f^n_{L/R}(y) = e^{\frac{5}{2}ky} \left[A^n_{L/R}\, J_{c-/+\frac{1}{2}} \left(\frac{m_n^F}{k} e^{ky}\right) + B^n_{L/R}\, Y_{c-/+\frac{1}{2}} \left(\frac{m_n^F}{k} e^{ky}\right)\right]\,, \tag{2.49}$$

where $A^n_{L/R}$, $B^n_{L/R}$ are complex coefficients, which are related by the coupled equations in Eq. (2.45),

$$\begin{cases} A^n_L &= A^n_R\,, \\ B^n_L &= B^n_R\,, \end{cases} \tag{2.50}$$

which is precisely calculated in Appendix E.2. The Dirichlet (1.33) and the Neumann BC (1.34) would provide additional constraints for the coefficients in the general formalism (2.49),

$$\begin{cases} A^n_R\, J_{c+\frac{1}{2}}\,(\xi)\Big|_{y=0} + B^n_R\, Y_{c+\frac{1}{2}}\,(\xi)\Big|_{y=0} &= 0\,, \\ A^n_R\, \partial_\xi J_{c+\frac{1}{2}}\,(\xi)\Big|_{y=0} + B^n_R\, \partial_\xi Y_{c+\frac{1}{2}}\,(\xi)\Big|_{y=0} &= 0\,, \end{cases} \tag{2.51}$$

with $\xi \hat{=} \dfrac{m^F_n}{k}\, e^{ky}$, while the linear independence of $J_{c+\frac{1}{2}}\,(\xi)$ and $Y_{c+\frac{1}{2}}\,(\xi)$ would induce the non-zero Wronskian determinant [7] anywhere on $y \in [0, L]$,

$$W = \begin{vmatrix} J_{c+\frac{1}{2}}\,(\xi) & Y_{c+\frac{1}{2}}\,(\xi) \\ \partial_\xi J_{c+\frac{1}{2}}\,(\xi) & \partial_\xi Y_{c+\frac{1}{2}}\,(\xi) \end{vmatrix} \neq 0\,, \qquad \forall\, y \in \mathcal{I} = [0, L]\,, \tag{2.52}$$

which forces the set of equation (2.51) to generate zero solutions.

Combing the analysis of the zero and non-zero modes above, we obtain the vanishing of all KK modes on the whole interval $\mathcal{I}$, which is presented explicitly in Eq. (1.39), which conflicts to the ortho-normalization condition in Eq.(2.27). Hence, we can conclude the solutions of 5D fermion fields obtained through this naive method are not physically consistent.

7. To be rigorous, the Wronskian determinant in Eq. (2.52) is originally developed for the variable $\xi = \frac{m^F_n}{k} e^{ky}$ satisfying the Eq. (E.5) $[\nu_R = c + \frac{1}{2}]$ with respect to the function $e^{-\frac{5}{2}ky} f^n_R$.

# Part II

# Original Works on Extra Dimensions

# Chapter 3

# Beyond Regularization of a Brane-Localized Higgs Field

This chapter is a personal adaptation of Ref. [1] written in collaboration with Andrei ANGELESCU, Grégory MOREAU and Florian NORTIER.

## 3.1 Introduction and Motivation

In the present paper, we discuss the rigorous treatment of the other case of a boundary-localized Higgs scalar field, interacting with bulk quark/leptons propagating on a finite interval, which presents subtleties that deserve to be looked at more deeply. Such a field configuration occurs in realistic warped models, potentially addressing the fermion mass and gauge hierarchy simultaneously. The case of free bulk matter without interactions will also be studied.

Let us recall some subtle aspects as our basic motivations. First, a question arises about the correct treatment of the specific object that is the Dirac peak ("function") entering each Lagrangian term which involves the brane-Higgs boson. Secondly, this Dirac peak may induce an unusual discontinuity [1] in the wave function along the extra dimension (at the Higgs boundary where further conditions arise from the Lagrangian variations) for some of the bulk fermions: the so-called jump problem [97, 101]. These 5D aspects have motivated the introduction [97, 101] of a process of regularization of the Higgs Dirac peak (smoothing the peak or shifting it from the boundary) in the calculation of Kaluza-Klein fermion mass spectra and effective 4D Yukawa couplings. Although there is no profound theoretical reason to apply such a regularization procedure (forcing interaction-free boundary conditions for fermions), nowadays all the theoretical and phenomenological studies of the warped models with brane-Higgs (see *e.g.* Ref. [95, 100, 102, 105–107]) are relying on this Higgs peak regularization.

In this paper, we first present the mathematical inconsistencies of this regularization procedure used in the literature. Then, instead of regularizing, we develop the rigorous determination of the profiles – taking into account the mathematical nature of the Dirac peak in the Higgs coupling – which leads to bulk fermion wave functions without discontinuities on the considered extra space. We conclude from this whole approach that neither profile jump nor particular problem arises when a proper mathematical framework is used so that there is in fact no motivation to introduce a brane-Higgs regularization.

As a consequence, we can now interpret two non-commutativities of calculation steps

1. Field jumps may arise in other frameworks [113].

for Higgs production and decay rates [102, 103, 105, 106] or for fermion masses and 4D Yukawa couplings [96], previously studied in the literature, to be similar effects and confirmations of the mathematical inconsistencies in the Higgs peak regularization. Besides, the debate in the literature about those two non-commutativities is thus closed by the useless nature of this regularization.

The correct methods without regularization, together with their results, are illustrated here in the derivation of the KK fermion mass spectrum – same ideas apply to the calculation of effective 4D Yukawa couplings. This spectrum calculation is done in a simplified model with a flat extra dimension, the minimal field content (to write down a Yukawa interaction) and without gauge symmetry. Nevertheless, this toy model already possesses all the key ingredients to study the delicate brane-Higgs aspects. Hence, our conclusions can be directly extended to realistic warped models with the bulk SM matter addressing the fermion flavor and gauge hierarchy.

Several new methods of spectrum calculation are proposed, which further allow confirmation of the analytical results. These methods provide alternative implements like the 4D or the 5D approach (one extra dimension case), and the determination of fermion currents comes from the action variations – we generalize the Noether's theorem to include brane-localized terms like the Yukawa couplings – or by manipulating the equations of motion. Besides, the correct derivation of the standard free fermion mass spectrum (in the absence of Yukawa interactions) turns out to be a useful starting guide in particular for the 4D approach or more generically for a solid comprehension of such higher-dimensional scenarios.

From a historical point of view, the correct method established here arises naturally in the theory of variational calculus as the Lagrangian boundary term (brane-Higgs coupling to fermions) is included in a new boundary condition instead of entering the equations of motion [114] (via a regularization). Furthermore, the present analysis follows the prescription of considering the Dirac delta as a distribution. By the way, the Dirac peak (distribution formalism) and distributions were formalized and validated mathematically during the 1940s by L. Schwartz [115, 116] precisely to solve consistently physical problems. Hence, today it should not be avoided to respect the distribution formalism when facing a physical problem involving an object like the Dirac delta, as it occurs in the present higher-dimensional context.

The rigorous results obtained for the KK mass spectrum and effective 4D Yukawa couplings are different from the ones derived in general through the Higgs peak regularization (see Section 3.5). This difference is physical, affecting then phenomenological studies on indirect searches of KK states at high-energy colliders (in particular via the Higgs production and flavor changing neutral currents), and analytical (vanishing of the Yukawa coupling with 'wrong' fermion chiralities relatively to the SM), which improves the precise theoretical understanding of the higher-dimensional setup with a brane-localized Higgs field.

Furthermore, the correct mass spectrum obtained here allows to point out the necessity for bulk fermions (with or without coupling to a brane-localized scalar field) to have specific bilinear brane terms at boundaries which are fermion mass-like terms from the point of view of the spinorial structure but do not introduce new (bulk) mass parameters. Indeed, such terms guarantee the existence of physical solutions (with correct profile normalizations, Hermitian conjugate boundary conditions and satisfying the decoupling limit argument) derived via the least action principle through the variation calculus. Their necessary presence is confirmed by the non-trivial exact matching between the 5D and 4D analytical calculations of the mass spectrum.

At a brane without Yukawa coupling, instead of including such bilinear brane terms, we find that one can alternatively impose *essential boundary conditions* (in contrast with *natural boundary conditions* coming from the Lagrangian variations) induced from the vanishing of fermion currents along the extra dimension at this brane – and exclusively within the 4D approach in case of a brane with localized Yukawa interactions. Indeed, the generic reason for the presence of bilinear brane terms is the consistent and complete geometrical definition of models with a finite extra spatial interval in which the fermionic matter is stuck. Notice that the choice between the presence of bilinear brane terms and the vanishing condition of fermion currents relies on the Ultra-Violet completion of the model. Indeed the vanishing condition of fermion currents permits the existence of physical solutions alternatively.

Therefore, there are two possible cases for the UV completion:

1) The UV completion generates bilinear brane terms for the fermions on both boundaries (those with and without localized Yukawa coupling) of the interval. Then, the geometrical interval definition (interval boundaries and vanishing 5D fermion currents at these boundaries) would be completely contained in the action expression as boundary terms.
2) The UV completion would not induce bilinear brane terms on both boundaries. Such essential boundary conditions should be imposed at the brane(s) (without bilinear brane terms) in order to define well the geometrical configuration and to have acceptable physical solutions.

We can thus conclude that whether the geometrical setup is defined exclusively through the action expression [leading to the natural boundary conditions] or (also) via additional essential boundary conditions, depends on the origin of the model at high energies.

In the case 1), at low-energies, the chiral nature of the SM as well as its field chirality distribution (Left-handed $\mathrm{SU}(2)_\mathrm{L}$ doublets and Right-handed singlets) are entirely induced by the signs in front of these bilinear brane terms. This new relation shows how the particular chiral properties of the SM could be explained by an underlying theory through the bilinear brane term signs. We complete the analysis by a discussion, in this context, on the appropriate treatment of the cut-off in energy due to the framework of higher-dimensional models in a non-renormalizable theory.

This chapter is organized as follows. Firstly, we describe the minimal model in Section 3.2, before presenting the free case and the 4D treatment of the coupled fermions in Section 3.4. The 5D approaches are exposed as well, with (Section 3.5) and without (Section 3.6) regularization. Finally, an overview is provided in Section 3.7, together with a brief description of phenomenological impacts. We finalize this chapter with a discussion of generic bilinear brane terms.

## 3.2 Minimal Consistent Model

### 3.2.1 Spacetime Structure

We consider a simplified 5D toy model with a flat spacetime $\mathcal{E}^5 = \mathcal{M}^4 \times \mathcal{I}$:

(i) $\mathcal{M}^4$ is the usual 4D Minkowski spacetime manifold, which is characterized by 4-vector coordinates $x^\mu$ where $\mu = 0, 1, 2, 3$ is the Lorentz index. The metric and conventions used are given in Appendix A.

(ii) $\mathcal{I}$ is a compact 1D flat extra space. For our purpose, we consider the following simple case: the interval $\mathcal{I} \,\hat{=}\, [0, L]$, with a length $L \in \mathbb{R}^*$, parametrized by the continuous extra coordinate $y$ and bounded by two flat 3-branes at $y = 0, L$.

(iii) A point of the whole A point of the 5D spacetime $\mathcal{E}^5$ is labeled by the coordinates, $x^M \hat{=} (x^\mu, y)$, $M \in [\![0,4]\!]$ with the 5D metric is given by,

$$ds^2 = \eta_{MN} dx^M dx^N \,,$$

where $\eta_{MN}$ with $M,\, N \in [\![0,4]\!]$ is the 5D Minkowski metric in Eq. (A.2).

### 3.2.2 Bulk Fermions

We consider the minimal spin-1/2 fermion field content allowing to write down the 4D effective renormalizable SM Yukawa-like coupling between zero-mode fermions (of different chiralities) and a scalar field (see Section 3.2.5): a pair of fermions – of mass dimension 2 – $Q$ and $D$. Both are propagating along the extra dimension, as we have in mind a model extension to a realistic scenario with bulk matter (cf. Section 3.2.6) where $Q, D$ will be respectively $\mathrm{SU}(2)_\mathrm{L}$ doublet down-component and singlet quark fields in the decoupling limit [2].

The bulk action $S_{\text{bulk}}$ is developed via the bulk Lagrangian density $\mathcal{L}_{\text{bulk}}$ in Eq. (1.15). The 5D fields $Q(x^\mu, y)$ and $D(x^\mu, y)$ have thus the following kinetic terms in the covariant 5D action,

$$\mathcal{L}_{\text{kin}} = \sum_{F=Q,D} \frac{i}{2} \bar{F} \Gamma^M \overleftrightarrow{\partial_M} F \,, \tag{3.1}$$

which have a similar form as generic kinetic terms in Eq. (1.16) but contain two fermion fields $Q$ and $D$. In this chapter, we only consider the bulk massless fermions, such that

$$\mathcal{L}_{\text{bulk}} = \mathcal{L}_{\text{kin}} \,. \tag{3.2}$$

Let us rewrite the bulk action of Eq. (3.1) in a convenient form with the chiral decomposition in Eq. (1.17) as in Eq. (1.19),

$$\mathcal{L}_{\text{kin}} = \sum_{F=Q,D} \frac{1}{2} \left( i F_R^\dagger \sigma^\mu \overleftrightarrow{\partial_\mu} F_R + i F_L^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F_L - F_R^\dagger \overleftrightarrow{\partial_4} F_L + F_L^\dagger \overleftrightarrow{\partial_4} F_R \right) . \tag{3.3}$$

### 3.2.3 Bilinear Brane Terms

Interestingly, in the absence of vanishing fermion current condition at a boundary of the considered interval $[0, L]$, the presence at this 3-brane of some bilinear brane terms, for bulk fermions being either free or coupled to a scalar field on this brane, turns out to be necessary. Indeed, these bilinear terms ensure the existence of physical solutions [see Section 3.4 for the 4D approach and Section 3.6 for the 5D one] deduced from the least action principle. The theoretical reason for the presence of the BBT at the boundaries of the interval is the correct geometrical configuration definition for models where fermions cannot propagate beyond the two boundaries, as will also be precisely described in Section 3.4 and 3.6. These sections will also point out the 4D/5D approach matching of the mass spectrum exact result, which constitutes in particular a confirmation for the necessary presence and the explicit form (including coefficients) of the BBT [3]. In summary, the presence of the BBT has several following justifications:

(i) They allow to avoid physical consistency problems both in the free case (see Sections 3.3 and 3.3.3) and with Yukawa couplings (Sections 3.6.1 and 3.6.3).

2. From the theoretical consistency and phenomenological points of view, the SM must be approximately recovered at low energies in the limit of infinitely heavy KK excitations.

3. Here, the opposite sign of BBT in Ref. [1] comes from the different conventions of $\Gamma^4$ in Appendix A.

(ii) They play the role of defining well the model compact at the two boundaries both in the free case (see Sections 3.3.2 and 3.3.3) and with Yukawa couplings (Section 3.6.3).

(iii) They induce the expected matching of the analytical results on the spectrum derived through the 4D and 5D approaches (see Sections 3.4 and 3.6.3).

To realize the SM configuration, the appropriate formula of the necessary BBT reads as [4 5],

$$S_B = \int d^4x \left(\mathcal{L}_B|_L - \mathcal{L}_B|_0\right) ,$$
$$\text{with } \mathcal{L}_B = \sum_{F=Q,D} \frac{\sigma^F(y)}{2} \bar{F}F = \sum_{F=Q,D} \frac{\sigma^F(y)}{2} \left(F_L^\dagger F_R + F_R^\dagger F_L\right) , \tag{3.4}$$

where we impose the chiral decomposition (1.17) and $\sigma^F(y)$ are generic parameters for the field $F$ $(F = Q, D)$ at $y$ and

$$\sigma^Q_{0,L} = -\sigma^D_{0,L} = -1 , \tag{3.5}$$

using compact notations

$$\sigma^F_{0,L} \mathrel{\hat{=}} \sigma^F\Big|_{0,L} . \tag{3.6}$$

The BBT under the configuration (3.5) will indeed lead to the set of boundary conditions in Eq. (3.28) for the wave functions $q^n(y), d^n(y)$ of the 5D fields $Q, D$, which then possess a non-vanishing normalizable zero-mode ($m_{[n=0]} = 0$) for only one chirality [$L$ or $R$ as $\sin(m_{[n=0]}\, y) = 0$]; hence at low-energies (below the first KK mass eigenvalue $m_1$), only one chirality of a given 4D field arises in the KK decomposition (1.24) so that one recovers the chiral nature of the SM.

Furthermore, within an extended realistic model (as described in Section 3.2.6) where the $Q^{(D)}$ field would be the down-component of an $\mathrm{SU}(2)_\mathrm{L}$ gauge doublet in the SM, the unique existing chiralities of the zero-mode 4D fields $Q_L^{(D)\,0}(x^\mu)$ and $D_R^0(x^\mu)$ predicted by Eq. (3.28) via Eq. (1.24) would well correspond to the SM chirality configuration [6]. Notice that Eq. (1.24) [involving KK modes rather than mass eigenstates] and Eq. (3.28) are valid within the relevant 4D treatment of the localized Yukawa interaction, where it is explicit that the SM particles (whose mass mainly originates from the EW symmetry breaking) are indeed mainly composed of the zero-modes (small mixings with the massive KK states), as imposed by small experimental deviations generally observed with respect to the theoretical SM predictions.

Therefore, it is remarkable that the BBT allow to make a step towards the UV explanation of the well-known SM chiral properties (chiral nature and chirality configuration) by directly linking these chiral aspects to explicit signs in front of Lagrangian terms (BBT signs), as described right above. Then the last step would be to build a UV completion of

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4. Similar terms, leading in particular to $\mathcal{L}_B = \frac{1}{2}(\bar{D}D - \bar{Q}^D Q^D)$, would hold in a model version extended to the EW symmetry of the SM, with the $Q$ field promoted to an $\mathrm{SU}(2)_\mathrm{L}$ doublet. In contrast, terms of the kind $\bar{Q}^U D$ (or $\bar{Q}D$), $\bar{Q}^U Q^D$ or $\bar{U}D$ would obviously not belong to a gauge invariant form.

5. The BBT of Eq. (3.4) is based on natural units. If we return to the unit system of meter, kilogram, and second (MKS), the BBT read as

$$S_B \ni \frac{1}{c} \int d^4x \, \frac{1}{2} (\hbar c)\, \sigma^F_P \; \bar{F}F\big|_P , \quad P = 0, L ,$$

and an explicit dimensional analysis is presented in Appendix I.

6. Taking the opposite sign for the bilinear terms in $\mathcal{L}_B$ (3.4) would lead to exchanged boundary conditions between $q^n(y)$ and $d^n(y)$ relatively to Eq. (3.28) and in turn to another chirality configuration.

the model to generate these BBT signs. In other words, the absolute control of the chiral structure by the BBT signs is a new feature that shows how an underlying theory could produce the SM chiral structure.

For completeness, we mention that the two other BBT sign configurations [7],

$$S'_B = \int_0^L d^4x \left( \mathcal{L}'_B|_L - \mathcal{L}'_B|_0 \right) \,, \quad \text{with} \;\; \mathcal{L}'_B = \frac{\sigma^F(y)}{2} \, \bar{F}F = \frac{\sigma^F(y)}{2} \left( F_L^\dagger F_R + F_R^\dagger F_L \right) \,, \tag{3.7}$$

where we impose the chiral decomposition (1.17) and

$$\sigma_L^F = -\sigma_0^F = \pm 1 \,, \tag{3.8}$$

via compact notations defined in Eq. (3.6), for 5D fields of the form (1.24) lead to the two sets (3.29) of boundary conditions and in turn to a vector-like field content, as for the so-called custodian fermions in custodially protected warped models [117]. Indeed, Eq. (3.29) leads to the absence of zero-modes ($m_{[n=0]} \neq 0$) and hence any KK state has both Left and Right chiralities. Once again, the control of the vectorial structure by the BBT signs is a novel characteristic that shows how a UV completion could produce a vector-like field content. Such massive vector-like states [8] can be used to build custodially protected warped models [117] and are then called custodians (see for instance Ref. [123]).

What is the direct effect of the BBT (3.4) on the final fermion mass eigenvalues? In the 4D approach and the case without Yukawa interaction (see Section 3.4), these BBT have no effect on the 4D fermion mass matrix in Eq. (3.43): after injecting the profile solutions, those BBT vanish due to the induced boundary conditions of Eq. (3.28) which impose that one of the two wave functions ($L$ or $R$) [9] entering the BBT 5D fields [cf. Eq. (1.24)] is equal to zero, at $y = 0$ [$\sin(m_n\, 0) = 0$] and $y = L$ [$\sin(m_n\, L) = 0$], systematically for each one of the two Lagrangian BBT (3.4). In contrast, in the 5D approach, the BBT (3.4) play a numerical and direct rôle in the fermion mass spectrum [and guarantee the diagonal formula of the 4D effective Lagrangian density], through the boundary conditions coming from the action variations (see Section 3.6).

In history, this kind of bilinear fermion brane terms (3.4)-(3.7) was first introduced by hand to derive the more specific AdS/CFT correspondence in the calculation of correlation functions for spinors [124, 125] – the exact AdS/CFT duality being possibly realized in the UV completion of warped models (from which the present simplified scenario is inspired). Then, within this AdS/CFT paradigm, similar boundary terms have been added at the UV-brane only ($y = 0$) to guarantee the minimization of the action in the holographic version of the warped model with bulk fermions [109]. The least action principle was also invoked in Ref. [126] to justify such bilinear fermion brane terms in the AdS/CFT context and through the path integral formalism. Equivalently, still in the AdS/CFT framework, these terms have been motivated in the Lagrangian density from an action form involving explicitly the Hamiltonian (to obtain a consistent Hamiltonian formulation when performing the Legendre transformation) [127]. Other boundary-localized terms were also introduced in a field theory defined on a manifold with boundaries within the context of gravity: the Gibbons-Hawking boundary terms [128–131]. Those terms are needed to cancel the variation of the Ricci tensor at the boundaries of the manifold.

The finite geometry setup is defined via either the BBT inclusion or the vanishing fermion current condition, depending on the considered UV completion of the model.

7. In contrast to the BBT of Eq. (3.4) containing complete fields of the minimal model, here, we just take one 5D field $F$ as an example in Eq. (3.7).

8. Extensive phenomenology at colliders has been developed about such vector-like particles [118–122].

9. For instance, $\bar{D}D = D_L^\dagger D_R + D_R^\dagger D_L$.

From the point of view of the effective field theory, it means that it can happen that the underlying theory does not forbid (through a short-distance mechanism or a residual symmetry) any possible non-renormalizable Lorentz-invariant operator involving the 5D fields $Q, D$ (including covariant derivatives) up to dimension 5 – this dimension choice being motivated in Section 3.2.5 – in the low-energy effective model described in this Section 3.2. Then, the present fermionic operators would be those included in the considered actions (3.3) (dimension 5 operators) and (3.4) (dimension 4 operators): the BBT part.

Notice that bulk mass terms, usually modifying the bulk fermion profiles [see Chapter 1.3.2, 2.3.2 and 6.2.6], bring useless complications [at least for schematic purpose] so we will not consider them in our present calculations, as the paper conclusions on fermion couplings to a brane-field can be easily extended [132].

### 3.2.4 Brane-Localized Scalar Field

In contrast to the bulk scalar field in Section 1.2, to be simplified, here we consider a 4D real scalar field, $H$ (mass dimension 1), confined on a boundary taken here to be at $y = L$ (as inspired by the warped scenario addressing the gauge hierarchy problem). However, the subtle aspects would arise when the fermions couple to this brane-localized scalar field. The action of this scalar field has the generic form,

$$S_H = \int d^4x \; \mathcal{L}_H \,, \quad \text{with} \quad \mathcal{L}_H = \frac{1}{2}\, \partial_\mu H \partial^\mu H - V(H) \,, \tag{3.9}$$

where the potential $V(H)$ possesses a minimum which generates a non-vanishing vacuum expectation value for the field developed as

$$H(x^\mu) = \frac{v + h(x^\mu)}{\sqrt{2}} \,, \tag{3.10}$$

in analogy with the SM Higgs boson. Note that the VEV $v$ in Eq. (3.10) doesn't depend on the extra dimension, which is definitely different from that in Eq. (1.7).

### 3.2.5 Yukawa-like Interactions

We focus on the following basic interaction in order to study the subtleties induced by the brane-scalar field coupling to bulk fermions,

$$S_Y = \int d^4x \;\; \mathcal{L}_Y|_L \,, \quad \text{with} \quad \mathcal{L}_Y = -Y_5 \; Q_L^\dagger H D_R - Y_5' \; Q_R^\dagger H D_L + \text{H.c.} \,, \tag{3.11}$$

which involves $H$, $Q$ and $D$, and is up to dimension 5. Note that the coupling constants $Y_5$ and $Y_5'$ of Yukawa type, entering these two distinct terms, are independent [i.e. parameters with possibly different values] as a well-defined 4D chirality holds for the fermion fields on the 3-brane strictly at $y = L$ (see for instance Ref. [1, 101]),

$$Y_5^{(\prime)} \triangleq \left| Y_5^{(\prime)} \right| e^{i\alpha_{Y^{(\prime)}}} \quad \text{with} \quad \alpha_{Y^{(\prime)}} \in \mathbb{R} \,. \tag{3.12}$$

In order to avoid the introduction of a new energy scale, one could define the 5D Yukawa coupling constants by giving their explicit dependence in $L$:

$$Y_5 = y_4 \times L \;\; \text{and} \;\; Y_5' = y_4' \times L \,, \tag{3.13}$$

where $y_4, y_4'$ are dimensionless coupling constants of $\mathcal{O}(1)$. Then $y_4$ can be identified with the SM Yukawa coupling constant, as shown when applying the decoupling limit (infinitely heavy KK masses and any new physics energy scale) [10].

From now on, we restrict our considerations to the VEV of $H$ as the aim is to calculate the KK fermion mass spectrum, which is unaffected by the interactions of the $h(x^\mu)$ fluctuation field with fermions. Hence, we concentrate on the following action issued from Eq. (3.11),

$$S_X = \int d^4x \ \ \mathcal{L}_X|_L \ , \quad \text{with} \ \ \mathcal{L}_X = -X Q_L^\dagger D_R - X' Q_R^\dagger D_L + \text{H.c.} \, , \tag{3.14}$$

with the compact notations

$$X \mathrel{\hat{=}} \frac{vY_5}{\sqrt{2}} \quad \text{and} \quad X' \mathrel{\hat{=}} \frac{vY_5'}{\sqrt{2}} \, . \tag{3.15}$$

Based on Eq. (3.10), the complete action reads as, $S_Y = S_X + S_{hQD}$, with the localized fermion-scalar interaction terms:

$$S_{hQD} = \int d^4x \ \ \mathcal{L}_{hQD}|_L \ , \quad \text{with} \ \ \mathcal{L}_{hQD} = -\frac{Y_5}{\sqrt{2}} \, hQ_L^\dagger D_R - \frac{Y_5'}{\sqrt{2}} \, hQ_R^\dagger D_L + \text{H.c.} \, , \tag{3.16}$$

that allow to work out the 4D effective Yukawa coupling constants.

### 3.2.6 Model Extension

The toy model considered is thus characterized by the Lagrangian

$$S_{\text{5D}} = S_{\text{bulk}} + S_{\text{branes}} \, , \tag{3.17}$$

where $S_{\text{branes}}$ represents action terms located at the branes,

$$S_{\text{branes}} = S_B + S_H + S_X + S_{hQD} \, . \tag{3.18}$$

Nevertheless, the conclusions of the present paper can be directly generalized to realistic warped models with bulk SM matter solving the fermion mass and gauge hierarchies. Indeed, working with a warped extra dimension instead of a flat one would not affect the conceptual subtleties about coupling bulk fermions to a brane-localized scalar field [132]. The boundaries at $y = 0$ and $y = L$ could then become the Planck and the TeV branes respectively. Similarly, the scalar potential, $V(H)$, can be extended to any potential [like the SM Higgs potential breaking the EW symmetry] as long as it still generates a VEV for the scalar field as here. In this context, the $H$ singlet can be promoted to the Higgs doublet under the SM $\text{SU}(2)_\text{L}$ gauge group, simply by inserting doublets in the kinetic term of Eq. (3.9). The whole structure of the coupling of Eq. (3.14) between bulk fermions and the localized VEV would still remain identical in the case of fermions promoted to SM $\text{SU}(2)_\text{L}$ doublets: after group contraction of the doublet $(Q^U, Q^D)^t$ with down/up-quark singlets $D, U$, one would obtain two replica of the structure (3.14) with the forms $Q_\mathcal{C}^{D\dagger} D_{\mathcal{C}'}$ and $Q_\mathcal{C}^{U\dagger} U_{\mathcal{C}'}$ where $\mathcal{C}^{(\prime)} \mathrel{\hat{=}} L, R$ denotes the chirality. Hence, the procedure described in this chapter should just be applied to both terms separately [11]. The same comment holds for the SM color triplet contraction and the field content extension to the three flavors of quarks and leptons of the SM. Notice that the flavor mixing would be combined with the mixing among fermion modes of the KK towers, without any impact on the present considerations about brane-localized couplings.

10. Note that in the decoupling limit where in particular $L \to 0$, generally $Y_5 \to 0$ due to the dimension of the 5D Yukawa coupling constants.

11. The fermion actions in Eq. (3.3) and (3.4) would be trivially generalized as well to a scenario with a gauge symmetry.

## 3.3 5D Free Bulk Fermions on an Interval

In this part, we calculate the free fermionic mass spectrum in the basic case without Yukawa interactions in Eq. (3.11) (studied in various references [96–99, 109, 133–138]). Let us also remark that in this case, there is no need for 4D/5D matching (pure 5D calculation of the masses). The main interest of this section is to develop a rigorous procedure for applying the boundary conditions.

### 3.3.1 Natural Boundary Condition Only

We start by considering simply the bulk action part (1.15),

$$S_{\text{bulk}}\,,$$

developed from the kinetic terms of Eq. (3.1)-(3.3). Thus, the equations of motion and the natural boundary conditions for the bulk fermions would be obtained via the least action principle for each of them ($F = Q, D$). The stationary action condition can be split, without loss of generality (the functional variations are generic so that $\delta\bar{Q}$ and $\delta\bar{D}$ are independent), into action variations with respect to each field $\bar{Q}$ and $\bar{D}$

$$\delta_{\bar{F}} S_{\text{bulk}} = 0\,,$$

via the treatment in Eq. (1.23) and (1.28) ($F = Q, D$). Then, the EOM (1.29)-(1.30) and the NBC (1.31) would be respectively deduced for $F = Q, D$.

To develop a 4D effective picture, let us replace the 5D fields by their standard solutions in the form of a KK decomposition in Eq. (1.24) satisfying the ortho-normalization conditions (1.26), where $f^n_{L/R} = q^n_{L/R}$ or $d^n_{L/R}$ are the dimensionless wave functions along the extra dimension associated respectively to the 4D fields $F^n_{L/R} = Q^n_{L/R}$ or $D^n_{L/R}$ of the KK excitations tower [12] with the KK masses $m^F_n$ [13] labeled by the non-negative integer $n$.

Inserting the KK decomposition (1.24) into the 5D field EOM (1.30), one can directly extract this set of differential equations for free profiles:

$$\forall n \in \mathbb{N}\,, \quad \begin{cases} \partial_4 q^n_L(y) &= \; m^Q_n \, q^n_R(y)\,, \\ \partial_4 q^n_R(y) &= \; -m^Q_n \, q^n_L(y)\,, \\ \partial_4 d^n_L(y) &= \; m^D_n \, d^n_R(y)\,, \\ \partial_4 d^n_R(y) &= \; -m^D_n \, d^n_L(y)\,, \end{cases} \tag{3.19}$$

which is a double replica ($f = q, d$) for the generic EOM of profiles in Eq. (1.32). The set of the first order differential equations (3.19) has been solved via a generic treatment in Section 1.3, so that the profiles $f^n_{L/R}(y) = 0$ ($\forall n \in \mathbb{N}$, $f = q, d$) vanish again, which can't provide physical solutions and aren't compatible with the two ortho-normalization conditions in Eq. (1.26) for $f^n_{L/R}(y)$ ($f = q, d$) [14].

The theoretical inconsistency obtained here for the considered free model reveals a problem in the treatment of a simple boundary without localized couplings to bulk matter (which is the case of both boundaries here). The correct treatments, based on either fermion current conditions at the boundaries or boundary-localized terms (the BBT), are exposed respectively in the two following sections.

12. Not yet the mass eigenstates in the case of Yukawa interactions.
13. Also mass eigenvalues in the absence of Yukawa interactions.
14. The analysis of the over-constraints has been presented in the end of Section 1.3.

### 3.3.2 Introducing the Fermion Current Condition [EBC]

In fact, the free version of the model defined in Section 3.2 (and finite extra dimension scenario in general) does impose conditions for the bulk fermions at the extra dimension boundaries, which were not included in the above naive analysis in Section 3.3. These conditions contribute to define the geometrical field configuration of the considered model, which will constitute the so-called essential boundary conditions, as imposed by the model definition, complementary to the NBC already defined in Eq. (1.29) via the least action principle. Indeed, the NBC come from an integration by part of the initial action with respect to the fifth dimension over the interval $[0, L]$ and thus take into account the spacetime structure itself.

Regarding the geometrical field configuration within the present free model, each fermion field is defined only along the interval $[0, L]$. This model-building hypothesis, that fermions neither propagate towards nor come from the outside of a finite range, translates into the condition of vanishing probability currents at both boundaries for each independent fermion species separately (without possible compensations).

Formally speaking, after having varied the bulk action constituted by kinetic terms (3.1) [see Eq. (1.28)] and in turn derived the bulk EOM (1.30) as well as the brane terms in Eq. (1.28), the application of the Noether's theorem demonstrated in Appendix F would suspend the boundary variations $\delta F^{\dagger}_{L/R}$ and modify the Dirichlet NBC in Eq. (1.31), which provides a reasonable solution to avoid the disaster in the naive approach in Section 3.3.

The Noether's theorem (by using the EOM) [15] gives rise to the two probability currents (F.5) defined independently for the two bulk fermions [16] represented by the 5D fields $F = Q, D$:

$$j^M_Q = -\alpha \bar{Q} \Gamma^M Q\,, \quad j^M_D = -\alpha' \bar{D} \Gamma^M D \quad , \tag{3.20}$$

associated to the two global $\mathrm{U}(1)_\mathrm{F}$ $(F = Q, D)$ symmetries of the bulk action (1.15)

$$S_{\text{bulk}}\,,$$

inserted by kinetic terms (3.1)-(3.3) corresponding respectively to the distinct transformations,

$$\left\{ \begin{array}{ccc} Q & \mapsto & e^{i\alpha} Q, \\ \bar{Q} & \mapsto & e^{-i\alpha} \bar{Q}, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} D & \mapsto & e^{i\alpha'} D, \\ \bar{D} & \mapsto & e^{-i\alpha'} \bar{D}, \end{array} \right. \tag{3.21}$$

where $\alpha, \alpha'$ $(\in \mathbb{R})$ are continuous parameters entering for instance the infinitesimal field variations [17]:

$$\underline{\delta} Q = i\alpha Q\,, \quad \underline{\delta} \bar{Q} = -i\alpha \bar{Q}\,. \tag{3.22}$$

Now the four conditions of vanishing probability currents for $F = Q, D$ are thus,

$$j^4_F \Big|_{0,L} = -\alpha^{(\prime)}\, \bar{F} \Gamma^4 F \Big|_{0,L} = i\alpha^{(\prime)} \left( F^\dagger_L F_R - F^\dagger_R F_L \right) \Big|_{0,L} = 0\,, \quad \forall x^\mu\,, \tag{3.23}$$

where we have used the chiral decomposition (1.17). The most general way out is to make of Eq. (3.23) a trivial equality by having

$$\left\{ \begin{array}{ccc} F_L|_0 & = & 0, \\ & \text{or} & \\ F_R|_0 & = & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} F_L|_L & = & 0, \\ & \text{or} & \\ F_R|_L & = & 0, \end{array} \right. \quad \text{[EBC]} \tag{3.24}$$

15. Valid trivially in the absence of BBT as well.
16. See Ref. [133] for scalar field currents.
17. We use different notations for the infinitesimal field variations under specific transformations, $\underline{\delta} F$, and generic field variations in the variation calculus, $\delta F$.

corresponding to a vanishing coefficient in each term of the condition (3.23):

$$\forall n \in \mathbb{N}\,, \left\{ \begin{array}{ccc} f_L^n|_0 & = & 0, \\ & \text{or} & \\ f_R^n|_0 & = & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} f_L^n|_L & = & 0, \\ & \text{or} & \\ f_R^n|_L & = & 0, \end{array} \right. \tag{3.25}$$

which consider the linear independence of 4D mass eigenstates, as discussed below Eq. (3.19).

These necessary conditions (3.24) of vanishing fields at boundaries are the EBC and correspond to some fields initially fixed at boundaries. Having such known fields at boundaries imposes [139] to have vanishing functional variations,

$$\left\{ \begin{array}{ccc} \delta F_L|_0 & = & 0, \\ & \text{or} & \\ \delta F_R|_0 & = & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} \delta F_L|_L & = & 0, \\ & \text{or} & \\ \delta F_R|_L & = & 0. \end{array} \right. \tag{3.26}$$

which would the contribute to brane terms in Eq. (1.28), since no action minimization with respect to a field $F_{L/R}\Big|_{0,L}$ (relying on $\delta F_{L/R}\Big|_{0,L} \neq 0$) is needed for such a known fermion field at a boundary, in contrast to the naive treatment in Section 3.3.1 where the boundary fields $F|_{0,L}$ were assumed to be initially unknown and then found out the NBC (1.31) ($F = Q, D$) through the least action principle.

Some brane terms in Eq. (1.28) would vanish due to the absence of boundary variations (3.26) which read as,

$$\left(\delta F_L^\dagger F_R\right)\Big|_0 - \left(\delta F_R^\dagger F_L\right)\Big|_0 = \left(\delta F_L^\dagger F_R\right)\Big|_L - \left(\delta F_R^\dagger F_L\right)\Big|_L = 0\,. \tag{3.27}$$

In other words, when deriving the NBC, before knowing the EBC, one would consider generically in the action variations (1.28) with all non-vanishing field variations at boundaries. However, once the EBC (3.24) are determined and selected (fixing some fields at boundaries accordingly to Eq. (3.26)), one could keep only non-vanishing variations for unknown boundary values (i.e. omit to vary the action with respect to known fields). Then, the resulting NBC and EBC can be combined [18].

Now the general solutions (1.36) of the decoupled equations derived from the EOM (1.30), once re-injected into the initial equations (3.19) on the profiles, become general solutions in Eq. (1.38). These solutions are taken to be continuous at the boundaries in order to have well-defined derivatives appearing in the consistent action (kinetic) (1.15)-(3.1), as also described in detail in Section 3.5.2.1. Applying the four sets of EBC from Eq. (3.24)-(3.25) to the solution forms (1.38), it appears that certain constant parameters must be equal to zero and thus we obtain the following four possible sets of profiles and KK mass spectrum equation ($\forall n \in \mathbb{N}$),

$$\begin{array}{ll} 1) \; (--): \; f_L^n(y) = B_L^n \sin(m_n^F\, y)\,, \; (++): \; f_R^n(y) = B_L^n \cos(m_n^F\, y)\,; \; \sin(m_n^F\, L) = 0\,, \\ 2) \; (++): \; f_L^n(y) = B_R^n \cos(m_n^F\, y)\,, \; (--): \; f_R^n(y) = -B_R^n \sin(m_n^F\, y)\,; \; \sin(m_n^F\, L) = 0\,, \end{array} \tag{3.28}$$

and,

$$\begin{array}{ll} 3) \; (-+): \; f_L^n(y) = B_L^n \sin(m_n^F\, y)\,, \; (+-): \; f_R^n(y) = B_L^n \cos(m_n^F\, y)\,; \; \cos(m_n^F\, L) = 0\,, \\ 4) \; (+-): \; f_L^n(y) = B_R^n \cos(m_n^F\, y)\,, \; (-+): \; f_R^n(y) = -B_R^n \sin(m_n^F\, y)\,; \; \cos(m_n^F\, L) = 0\,. \end{array} \tag{3.29}$$

18. The brane condition in Eq. (3.27) is a special case where no NBC would be deduced. In a more general situation, some residue NBC would be derived after inserting the EBC.

Here, we have used the standard BC notations, i.e. $-$ or $+$ for example at $y = 0$ stands respectively for the Dirichlet or Neumann BC of wave functions: $f^n_{L/R}(0) = 0$ or $\partial_y f^n_{L/R}(0) = 0$. For instance, the symbolic notation $(-+)$ denotes Dirichlet (Neumann) BC at $y = 0$ $(y = L)$.

The SM-like profile $d^n_{L/R}(y)$ $(q^n_{L/R}(y))$ taken from line 1 (2) of Eq. (3.28) assigned to the (singlet/doublet component) quark fields give rise to the chiral nature of the SM and to its correct chirality configuration, as described in Section 3.2.3. The other solutions (3.29) lead to KK towers without zero-modes like custodian states [see also the discussion on Eq. (3.29) in Section 3.2.3].

Notice that the used BC (3.25) must be injected into the equations (3.19) issued from the EOM as those are valid for any point of the extra dimension including the boundaries [see the original Eq. (1.29)]. This leads to a new set of BC that we call the complete BC. These complete BC are well satisfied by the final solutions (3.28) and (3.29).

The constants $B^n_L = \sqrt{2}\, e^{i\alpha^n_L}$ and $B^n_R = \sqrt{2}\, e^{i\alpha^n_R}$ $(\forall n \in \mathbb{N}^*)$ [in the special case $n = 0$, the $\sqrt{2}$ factors must all be replaced by the unity] [19], where $\alpha^n_{L/R}$ are real angles, are fixed by the ortho-normalization condition (1.26). The relation $\sin(m^F_n\, L) = 0$ $(F = Q, D)$ has the following chosen solutions for the KK mass spectrum,

$$|m_n| = \frac{n\pi}{L}\,, \quad n \in \mathbb{N}\,, \tag{3.30}$$

where we define the notation of the common mass spectrum $m_n$ as

$$m_n \,\hat{=}\, m^Q_n = m^D_n\,. \tag{3.31}$$

Similarly, the relation $\cos(m^F_n\, L) = 0$ has the possible solutions:

$$\left|m^F_n\right| = \frac{(2n+1)\pi}{2L}\,, \quad n \in \mathbb{N}\,. \tag{3.32}$$

For instance as the boundary condition 1) and 3) in Eq. (3.28)-(3.29), we would like to know if the phase of $B^n_L$ is physical. For that purpose, we perform the transformations:

$$B^n_L \mapsto e^{i\theta_n} B^n_L \quad \Longrightarrow \quad (f^n_L, f^n_R) \mapsto (e^{i\theta_n} f^n_L, e^{i\theta_n} f^n_R)\,,$$

which let the KK wave functions equations (3.19) and the ortho-normalization conditions (1.26) invariant, thus the phase of $B^n_L$ is not physical and one can take $B^n_L = |B^n_L|$. For the boundary conditions 2) and 4), the same method is applied to conclude that the phase of $B^n_R$ is not physical. We can take $B^n_R = |B^n_R|$. The constants $|B^n_L|$ and $|B^n_R|$ are fixed by the ortho-normalization conditions (1.26). The boundary conditions 1) and 2) (3.28) have the non-negative solutions for the KK mass spectrum in Eq. (3.30), the negative branch is also a reasonable mass spectrum. We will show that the sign of $m_n$ is not physical. One can perform the transformations:

$$\begin{cases} m^F_n \mapsto -m^F_n\,, \\ F/f^n_R \mapsto -F/f^n_R \quad \text{or} \quad F/f^n_L \mapsto -F/f^n_L\,. \end{cases}$$

Then, the 4D Dirac equations (1.25), the KK wave functions equations (3.19) and the ortho-normalization conditions (1.26) are invariant. By using the same method as above, for the boundary conditions 3) and 4) (3.29), one can show that the sign of $m^F_n$ is not physical and take $m^F_n \geq 0$. For schematic purpose, in Figure 3.1, we give a plot of one

19. For the solution 1), we find $B^0_L = e^{i\alpha^0_L}$ while $B^0_R = e^{i\alpha^0_R}$ for the solution 2) [cf. Eq. (3.28)].

possible set of the KK wave functions along the extra dimension with the real solution Eq. (3.28).

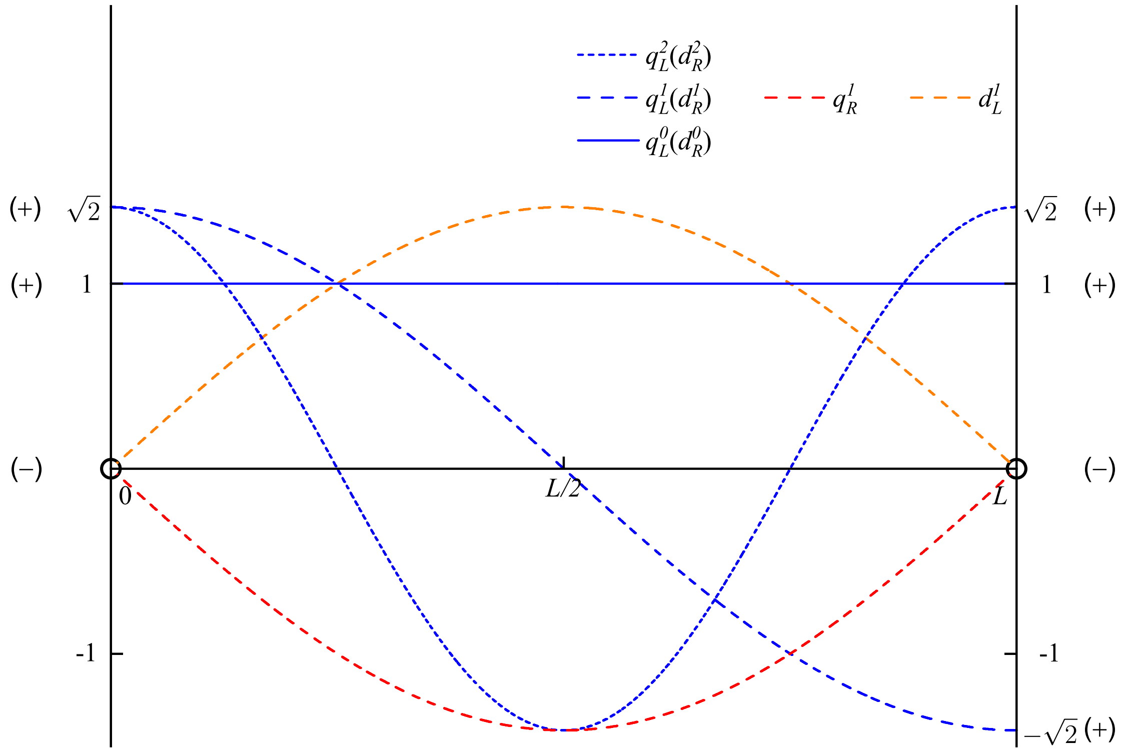


Figure 3.1 – Zero-mode and KK dimensionless wave functions $q(d)^{0,1,2}_{L(R)}(y)$, $q(d)^{1}_{R(L)}(y)$, along the interval domain, $y \in [0, L]$, free solutions of Eq. (3.28) in the simplified case, $\forall\, n \in \mathbb{N}, \alpha^n_L = \alpha^n_R = 0$ with non-negative KK masses in Eq. (3.30). The two ending points at $y = 0, L$, the BBT, and Dirichlet/Neumann BC, $(-)/(+)$, are indicated on the graph.

We finalize this section with an additional remark on the probability current, which plays a crucial role of the EBC. Alternatively, one can determine the explicit formula of the probability current (3.20) directly (without applying the Noether's theorem to the Lagrangian density) from a rewriting [20] of each free 5D Dirac equation (1.29)-(1.30) $(F = Q, D)$ in the bulk.

### 3.3.3 Introducing the Bilinear Brane Terms [NBC]

As announced at the end of Section 3.3.1, an alternative method [21] with respect to previous section for finding out the same consistent physical solutions, for the mass spectrum and the profiles, is to add the BBT (3.4) to the kinetic terms (3.3) so that the initial free fermionic action becomes,

$$S_{\text{bulk}} + S_B \,. \tag{3.33}$$

20. Subtracting the Dirac equation to its Hermitian conjugate form, with the relevant 5D field and $\gamma^0$ factors, and using the 5D Dirac matrix rules.

21. One could simultaneously impose the EBC (3.23) above and add the BBT to the action, but this method would contain some redundancy.

Let us apply the least action principle using this action as the starting point. We add the BBT piece (3.4) to the bulk kinetic action based on Eq. (3.3). Without loss of generality, the stationary action condition

$$0 = \delta_{\bar{F}} \left( S_{\text{bulk}} + S_B \right) ,$$

can be split into these two conditions with respect to the two field variations respectively,

$$\delta_{\bar{F}} \left( S_{\text{bulk}} + S_B \right) = \int d^4x \left\{ \int_0^L dy \left[ \delta F_L^\dagger \left( i\bar{\sigma}^\mu \partial_\mu F_L + \partial_4 F_R \right) + \delta F_R^\dagger \left( i\sigma^\mu \partial_\mu F_R - \partial_4 F_L \right) \right] \right.$$
$$\left. + \frac{\sigma_L^F - 1}{2} \, \delta F_L^\dagger F_R \Big|_L + \frac{\sigma_L^F + 1}{2} \, \delta F_R^\dagger F_L \Big|_L - \frac{\sigma_0^F - 1}{2} \, \delta F_L^\dagger F_R \Big|_0 - \frac{\sigma_0^F + 1}{2} \, \delta F_R^\dagger F_L \Big|_0 \right\}, \quad (3.34)$$

using $\sigma_{0,L}^F$ $(F = Q, D)$ (3.5) for the SM configuration. For generic field variations $\delta F_{L/R}^\dagger$ and $\delta F_{L/R}^\dagger \Big|_{0,L}$, the sum of the first two terms, both in Eq. (3.34), must vanish separately, leading to the same equations as the EOM (1.30) [22] and in turn via Eq. (1.25) to the profile equations (3.19) with solutions (1.36). The general solutions (1.36), once injected into the initial equations (3.19), take the specific forms (1.38). We are thus left with the NBC:

$$\left\{ \begin{array}{lcl} \dfrac{\sigma_L^F - 1}{2} \, \delta F_L^\dagger F_R \Big|_L & = & 0, \\ \dfrac{\sigma_0^F - 1}{2} \, \delta F_L^\dagger F_R \Big|_0 & = & 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \dfrac{\sigma_L^F + 1}{2} \, \delta F_R^\dagger F_L \Big|_L & = & 0, \\ \dfrac{\sigma_0^F + 1}{2} \, \delta F_R^\dagger F_L \Big|_0 & = & 0. \end{array} \right. \quad (3.35)$$

Then, using the appropriate constants $\sigma_{0,L}^F$ $(F = Q, D)$ (3.5) for each field and generic variations $\delta F_{L,R}^{(\dagger)} \Big|_{0,L} \neq 0$, it appears clearly that those BC belong to the set of BC (3.24)-(3.25) whose application on the solution forms (1.38) leads to the two respective sets of profiles and KK mass spectrum (3.28), as already derived. The structure of the profile solutions (3.28) corresponds to the chiral nature and configuration of the SM as already explained in Section 3.2.3.

For completeness, beyond the SM configuration, we take the custodian BBT (3.7) in the initial action for a given field $F$,

$$S_{\text{bulk}} + S'_B \, , \quad (3.36)$$

and the explicit chiral formula reads,

$$S_{\text{bulk}} + S'_B = \int d^4x \left\{ \int_0^L dy \left[ iF_R^\dagger \sigma^\mu \partial_\mu F_R + iF_L^\dagger \bar{\sigma}^\mu \partial_\mu F_L - F_R^\dagger \partial_4 F_L + F_L^\dagger \partial_4 F_R \right] \right.$$
$$\left. + \left[ \frac{\sigma^F + 1}{2} F_L^\dagger F_R + \frac{\sigma^F - 1}{2} F_R^\dagger F_L \right] \Bigg|_0 + \left[ \frac{\sigma^F - 1}{2} F_L^\dagger F_R + \frac{\sigma^F + 1}{2} F_R^\dagger F_L \right] \Bigg|_L \right\} . \quad (3.37)$$

22. We obtain the Hermitian conjugate EOM and NBC by integrating by part the bulk piece of the relation $\delta_{F_{L,R}} \left( S_{\text{bulk}} + S_B \right) = 0$ (non-vanishing boundary terms appear due to the integration over the extra dimension) in order to get rid of the field factors $\partial_M \delta F_{L,R}$.

The stationary action condition can be split into the two following conditions,

$$
\begin{aligned}
0 = \delta_{F_L^\dagger} \left( S_{\text{bulk}} + S'_B \right) \quad &= \quad \int d^4x \left\{ \int_0^L dy \left[ \delta F_L^\dagger \, i\bar{\sigma}^\mu \partial_\mu F_L + \delta F_L^\dagger \, \partial_4 F_R \right] \right. \\
&\quad + \left. \left[ \frac{\sigma^F - 1}{2} \delta F_L^\dagger F_R \bigg|_L + \frac{\sigma^F + 1}{2} \delta F_L^\dagger F_R \bigg|_0 \right] \right\}, \qquad (3.38) \\
0 = \delta_{F_R^\dagger} \left( S_{\text{bulk}} + S'_B \right) \quad &= \quad \int d^4x \left\{ \int_0^L dy \left[ \delta F_R^\dagger \, i\sigma^\mu \partial_\mu F_R - \delta F_R^\dagger \, \partial_4 F_L \right] \right. \\
&\quad + \left. \left[ \frac{\sigma^F + 1}{2} \delta F_R^\dagger F_L \bigg|_L + \frac{\sigma^F - 1}{2} \delta F_R^\dagger F_L \bigg|_0 \right] \right\}. \qquad (3.39)
\end{aligned}
$$

Once more, the sum of the first two terms in Eq. (3.38) and (3.39), respectively, must vanish, leading to the same profile equations as the ones deduced from Eq. (3.34) and hence to the identical bulk solution forms (1.38). Nevertheless, we are now left with the new NBC:

$$
\frac{\sigma^F - 1}{2} \delta F_L^\dagger F_R \bigg|_L = \frac{\sigma^F + 1}{2} \delta F_L^\dagger F_R \bigg|_0 = \frac{\sigma^F + 1}{2} \delta F_R^\dagger F_L \bigg|_L = \frac{\sigma^F - 1}{2} \delta F_R^\dagger F_L \bigg|_0 = 0\,. \quad (3.40)
$$

Then, for generic variations $\delta F_{L,R}^{(\dagger)}\Big|_{0,L} \neq 0$, it is clear that those BC belong to the set of BC (3.24)-(3.25) whose application on the solution forms (1.38) leads, for $\sigma^F = +1$, to the set 4) of profiles and KK mass spectrum in Eq. (3.29), and, for $\sigma^F = -1$, to the set 3) of solutions in Eq. (3.29), as already derived. The control of the BBT sign factor $\sigma^F$, in Eq. (3.7), on the final solution structure appears here clearly. The profile solutions (3.29) have a custodian chiral structure as already described in Section 3.2.3. Note that one could as well combine the two approaches to define the model: add a BBT only on an interval boundary for a given 5D field (as in this Section 3.3.3) and apply the current vanishing condition only on the other boundary (as in Section 3.3.2).

In the end of this section, let us now discuss the probability currents. In the presence of the BBT (3.4) or (3.7) [invariant under the transformations (3.21)], as demonstrated in the beginning of Appendix F, the application of the Noether's theorem based on the bulk EOM (1.30) – derived from the variation of the bulk (kinetic) action (3.1)-(3.3) invariant under the global $\mathrm{U}(1)_\mathrm{F}$ transformations (3.21) – leads to the same probability currents (F.5) defined separately for the bulk fermions represented by the 5D fields $F = Q, D$, as in Eq. (3.20). Now the NBC (3.35) or (3.40) induced by the BBT, as both satisfying the BC (3.24), lead to fours conditions of vanishing probability currents of the exact form (3.23). In other words, the presence of the BBT guarantees (without imposing any condition) the vanishing of the currents at both boundaries for each independent fermion species. These BBT-induced conditions contribute to the consistent and complete definition of the geometrical field configuration for the considered model with a finite extra spatial interval in which fermionic matter is stuck.

Alternatively, we can derive directly (without the Noether's theorem) the conservation relations, $\partial_M j^M_F = 0$, for the probability currents (3.20) from a rewriting [23] of each free 5D Dirac equation (1.30) in the bulk. The BBT (3.4) or (3.7) affect only the NBC derived from variation of the action (3.1)-(3.3).

23. Subtracting the Dirac equation to its Hermitian conjugate form.

## 3.4 Brane-localized Yukawa Couplings: 4D Approach

Once the free case is addressed, via the EBC (3.24) in Section 3.3.2 or the NBC (3.35) [or Eq. (3.40) for custodians] induced by the BBT in Section 3.3.3, the free fermion mass spectrum and profiles are known. Then, we take into account of the action part $S_X$ (3.14) [induced by the Yukawa interaction $S_Y$ (3.11)] in the mass spectrum. In this section, we only describe the two steps of a first method [55, 66, 68, 95, 96, 140–147], that will turn out to be a correct approach, for including the effects of the Yukawa terms (3.14) on the final fermion spectrum. The considered action reads thus as,

$$S_{\text{bulk}} + S_X \ (+S_B)\,, \tag{3.41}$$

where $S_B$ (existing if no EBC are applied) has no direct effect on the mass matrix (3.43) as explained in Section 3.2.3. First, the free profiles and free spectrum are calculated within a strict approach whose correct treatment was exposed in detail in Sections 3.3.2 and 3.3.3. Secondly, one can write a mass matrix for the 4D fermion fields involving the pure KK masses [the free spectrum of the first step] as well as the masses induced by the Higgs VEV in the Yukawa terms (3.14) [with free profiles of the first step], which mix the KK modes. The bi-diagonalization of this matrix gives rise to an infinite set of eigenvalues constituting the physical masses.

The action (3.41) leads – after insertion of the KK decomposition (1.24), use of free EOM (3.19), the ortho-normalization condition (1.26) and integration over the fifth dimension – to the canonical kinetic terms for the 4D fermion fields as well as to the following fermionic 4D effective mass terms in the Lagrangian density (and to independent 4D effective Higgs-fermion couplings not discussed here),

$$-\chi_L^\dagger\, \mathcal{M}\, \chi_R + \text{H.c.}\,,$$

in the combined basis for the left and right-hand (transposed) 4D fields:

$$\begin{cases} \chi_L^t(x^\mu) &= \left(Q_L^{0t}, Q_L^{1t}, D_L^{1t}, Q_L^{2t}, D_L^{2t}, \cdots\right)\,, \\ \chi_R^t(x^\mu) &= \left(D_R^{0t}, Q_R^{1t}, D_R^{1t}, Q_R^{2t}, D_R^{2t}, \cdots\right)\,. \end{cases} \tag{3.42}$$

Notice that there exists only one chirality for the zero-modes as explained below Eq. (3.4). The infinite mass matrix reads as,

$$\mathcal{M} = \begin{pmatrix} \alpha_{00} & 0 & \alpha_{01} & 0 & \alpha_{02} & \cdots \\ \alpha_{10} & m_1 & \alpha_{11} & 0 & \alpha_{12} & \cdots \\ 0 & \beta_{11} & m_1 & \beta_{12} & 0 & \cdots \\ \alpha_{20} & 0 & \alpha_{21} & m_2 & \alpha_{22} & \cdots \\ 0 & \beta_{21} & 0 & \beta_{22} & m_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\,, \tag{3.43}$$

where $m_n$ is the free spectrum (3.30) and the free wave function overlaps with the Higgs-brane are defined by [24],

$$\begin{cases} \forall (i,j) \in \mathbb{N}^2,\ \alpha_{ij} = \dfrac{X}{L} q_L^i(L) d_R^j(L)\,, \\ \forall (i,j) \in \mathbb{N}^{*2},\ \beta_{ij} = \dfrac{X'}{L} d_L^i(L) q_R^j(L)\,. \end{cases} \tag{3.44}$$

24. To simplify, drop non-physical phases discussed in the end of Section 3.3.2, here we only consider real wave functions.

As the profiles are the free ones [profiles and KK mass spectrum solutions (3.28) with SM chiral structure], only the $\alpha_{ij}$ coefficients do not vanish.

The physical fermion mass spectrum is obtained by bi-diagonalizing the mass matrix (3.43). This method is called the perturbation method in the sense that truncating the mass matrix at a given KK level corresponds to keeping only the dominant contributions to the lightest mass eigenvalue being the measured fermion mass (higher KK modes tend to mix less to the zero-mode due to larger mass differences).

Extracting the mass spectrum equation from the characteristic equation for the Hermitian-squared mass matrix (3.43), in the case of infinite KK towers, is not trivial. This useful exercise was addressed analytically in Ref. [96] for the present toy model but with a 5D Yukawa coupling constant $Y_5$ (and in turn a $X$ quantity) taken real, i.e. $\alpha_Y = 0$ in Eq. (3.12). The resulting exact equation – without any approximation – was found to be:

$$\forall n \in \mathbb{N}\,, \quad \tan^2(\sqrt{|M_n|^2}L) = X^2 \;\Leftrightarrow\; \tan(\sqrt{|M_n|^2}L) = \pm X\,. \tag{3.45}$$

in the case of a real $X$ parameter and positive $m_n$ branch from Eq. (3.30). Let us present here the absolute values of the solutions (physical masses) of this equation:

$$\forall n \in \mathbb{N}\,, \quad |M_n| = \left|\frac{\arctan(X) + (-1)^n \tilde{n}(n)\pi}{L}\right|\,, \tag{3.46}$$

where the function $\tilde{n}(n)$ is defined by

$$\tilde{n}(n) = \begin{cases} \frac{n}{2} & \text{for } n \text{ even}\,, \\ \frac{n+1}{2} & \text{for } n \text{ odd}\,, \end{cases} \tag{3.47}$$

so that the positive integer $n$ labelling the mass eigenvalues remains as well the label of the associated [as in the free case (1.25)] 4D mass eigenstates $\psi^n(x^\mu)$ [like in Eq. (1.24)]. Besides, we need to demonstrate that the two different classes $\pm X$ in Eq. (3.45) generates the unique mass spectrum (3.46):

$$\tan(\sqrt{|M_n^+|^2}\,L) = +X \quad\Rightarrow\quad |M_n^+| = \frac{\arctan(X) + n\,\pi}{L},\ \forall n \in \mathbb{N} \tag{3.48}$$
$$= \boxed{\left|\frac{\arctan(X) + (-1)^n \tilde{n}(n)\pi}{L}\right|\,,\ \text{for } n \text{ even}},$$

$$\tan(\sqrt{|M_n^-|^2}\,L) = -X \quad\Rightarrow\quad |M_n^-| = \frac{-\arctan(X) + n\,\pi}{L},\ \forall n \in \mathbb{N}^* \tag{3.49}$$
$$= \boxed{\left|\frac{\arctan(X) + (-1)^n \tilde{n}(n)\pi}{L}\right|\,,\ \text{for } n \text{ odd}},$$

so that the two branches of solutions in Eq. (3.48) and (3.49) combine together into a complete mass spectrum $|M_n|$ in Eq. (3.46), which is presented visually in Figure 3.2.

To check that the counting of states is correct, we observe that, in the realistic case $|X| \ll 1$ (typically small SM masses compared to the KK scale), two consecutive absolute masses $|M_n|$ (for one odd $n$ and the following even $n$, with $n \in \mathbb{N}^*$) of Eq. (3.46) are equal at leading order to the corresponding [unique $\tilde{n}$ value] absolute mass $\tilde{n}\,\pi/L$ as in the free spectrum (3.30). Hence, in the vanishing mixing limit [see matrix (3.43)], the two associated consecutive mass eigenstates $\psi^n(x^\mu)$ tend well to the two free 4D field components $Q^{\tilde{n}}(x^\mu)$ and $D^{\tilde{n}}(x^\mu)$ [of Eq. (1.25)].

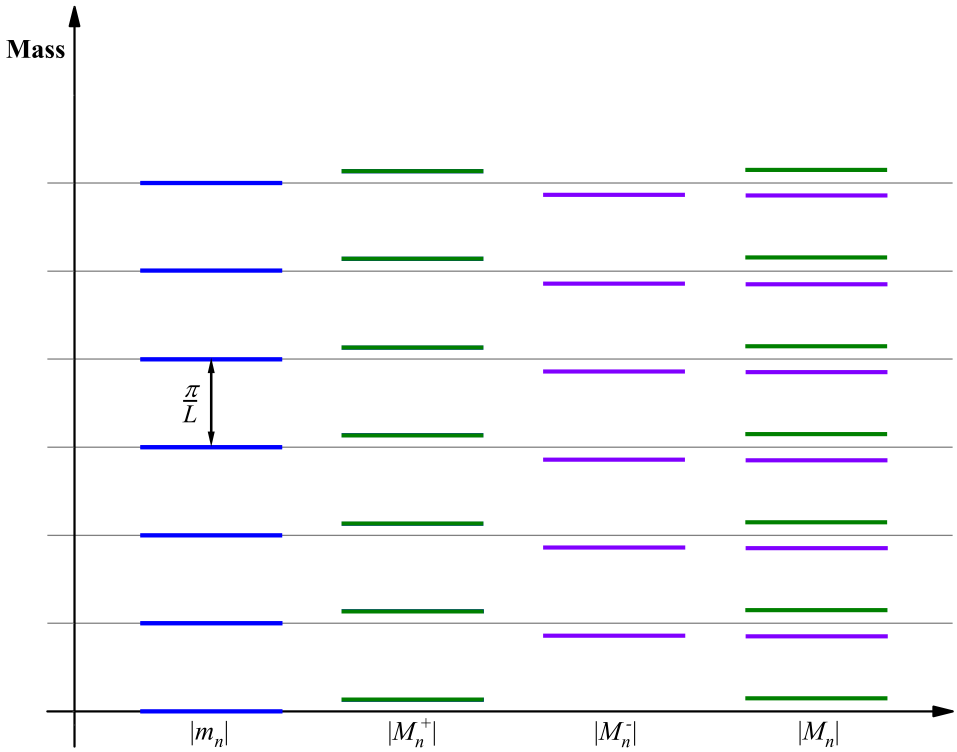


Figure 3.2 – The KK towers with [$|M_n|$ (3.46), $|M_n^+|$ (3.48), $|M_n^-|$ (3.49)] and without [$|m_n|$ (3.30)] a brane-localized Yukawa coupling.

## 3.5 5D Treatment: The Regularization Doom

In this part, we work out the fermion mass spectrum in the defined model with the extended 5D action (3.17) using the alternative 5D approach based on the brane-Higgs regularization [96–102] and we point out non rigorous patterns of this method.

### 3.5.1 Mixed Kaluza-Klein Decomposition

As we have just seen in Eq. (3.42)-(3.43), after the EW symmetry breaking, the infinite $Q_L^n$ and $D_L^n$ field towers mix together (as do the $Q_R^n$ and $D_R^n$) to form 4D fields $\psi_L^n$ (and $\psi_R^n$) representing mass eigenstates. In order to take into account this mixing within the 5D approach, these common 4D fields $\psi_L^n$ are defined instead of the $Q_L^n$ and $D_L^n$ fields (and similarly for the right-hand fields) in the whole KK decomposition, then called a *mixed*

KK decomposition [instead of the *free* one in Eq. (1.24)] [101], as follows,

$$\left[\begin{array}{lcl} Q_L\,(x^\mu,y) &=& \dfrac{1}{\sqrt{L}}\displaystyle\sum_{n=0}^{+\infty} q_L^n(y)\,\psi_L^n\,(x^\mu)\,, \\ Q_R\,(x^\mu,y) &=& \dfrac{1}{\sqrt{L}}\displaystyle\sum_{n=0}^{+\infty} q_R^n(y)\,\psi_R^n\,(x^\mu)\,, \\ D_L\,(x^\mu,y) &=& \dfrac{1}{\sqrt{L}}\displaystyle\sum_{n=0}^{+\infty} d_L^n(y)\,\psi_L^n\,(x^\mu)\,, \\ D_R\,(x^\mu,y) &=& \dfrac{1}{\sqrt{L}}\displaystyle\sum_{n=0}^{+\infty} d_R^n(y)\,\psi_R^n\,(x^\mu)\,, \end{array}\right. \tag{3.50}$$

where the 4D fields $\psi^n_{L/R}$ ($\forall n \in \mathbb{N}$) must satisfy the Dirac-Weyl equations,

$$\left\{\begin{array}{lcl} i\bar{\sigma}^\mu\partial_\mu\psi_L^n\,(x^\mu) - M_n\,\psi_R^n\,(x^\mu) &=& 0\,, \\ i\sigma^\mu\partial_\mu\psi_R^n\,(x^\mu) - M_n\,\psi_L^n\,(x^\mu) &=& 0\,, \end{array}\right. \tag{3.51}$$

where the spectrum $M_n$ includes the mass contribution whose origin is the Yukawa couplings (3.14). Note that in contrast with the free case, there is a unique mass spectrum $M_n$ for a unique 4D field tower $\psi^n_{L/R}(x^\mu)$. In order to guarantee the existence of diagonal and canonical kinetic terms for those 4D fields $\psi^n_{L/R}$, the associated new profiles must now obey the two following ortho-normalization conditions,

$$\forall\, n,m \in \mathbb{N},\ \frac{1}{L}\int_0^L dy\,[q_{\mathcal{C}}^{n*}(y)q_{\mathcal{C}}^m(y) + d_{\mathcal{C}}^{n*}(y)d_{\mathcal{C}}^m(y)] = \delta_{nm}\,, \tag{3.52}$$

for a chirality index $\mathcal{C} \,\hat{=}\, L/R$. These two conditions are different from the four ones of Eq. (1.26) due to the new mixed KK decomposition (3.50).

### 3.5.2 Inconsistencies of the Higgs Shift Procedure

Here we highlight the formal problems of the 5D process of shifting the brane-Higgs field [96, 97, 102] to get the fermion mass tower. Once more the considered fermion terms of the extended 5D action (3.17) are $S_{\text{bulk}}$ and $S_X$ (without $S_B$ which was missed in the relevant literature and that will be taken into account in Section 3.6). The variations of the studied action lead to the same free BC of Eq. (1.31) ($F = Q, D$) and to the following bulk EOM including the Yukawa coupling constants [instead of the free ones in Eq. (1.30)],

$$\left\{\begin{array}{lcl} i\bar{\sigma}^\mu\partial_\mu Q_L + \partial_4 Q_R - \delta(y-L)\,X D_R &=& 0\,, \\ i\sigma^\mu\partial_\mu Q_R - \partial_4 Q_L - \delta(y-L)\,X' D_L &=& 0\,, \\ i\bar{\sigma}^\mu\partial_\mu D_L + \partial_4 D_R - \delta(y-L)\,X' Q_R &=& 0\,, \\ i\sigma^\mu\partial_\mu D_R - \partial_4 D_L - \delta(y-L)\,X Q_L &=& 0\,. \end{array}\right. \tag{3.53}$$

where $X$ and $X'$ are taken real. Indeed, in view of regularizing the brane-Higgs field, the Yukawa interactions must be included in the bulk EOM [96] – as done in the literature. Inserting the mixed KK decomposition (3.50) in these 5D field EOM (3.53) allows to factorize out the 4D fields, obeying the 4D Dirac equations (3.51), and obtain the profile equations for each excited mode [instead of the free ones in Eq. (3.19)]:

$$\forall\, n \in \mathbb{N}, \quad \left\{ \begin{array}{lcl} \partial_4 q_R^n(y) + M_n\, q_L^n(y) & = & \delta(y-L)\, X d_R^n(y)\,, \\ \partial_4 q_L^n(y) - M_n\, q_R^n(y) & = & -\,\delta(y-L)\, X' d_L^n(y)\,, \\ \partial_4 d_R^n(y) + M_n\, d_L^n(y) & = & \delta(y-L)\, X' q_R^n(y)\,, \\ \partial_4 d_L^n(y) - M_n\, d_R^n(y) & = & -\,\delta(y-L)\, X q_L^n(y)\,. \end{array} \right. \tag{3.54}$$

Here we underline the first mathematical issue of this usual approach: introducing $\delta(y-L)$ Dirac peaks [25] in these profile equations leads to relations between distributions [26] and functions which are thus not mathematically consistent [115, 116].

The apparent "ambiguity" noticed in the literature (in the context of a warped extra dimension) was that the Yukawa terms in Eq. (3.54) are present only at the $y = L$ boundary and might thus affect the fermion boundary conditions. In order to avoid this potential problem (like a field vagueness), a regularization of the brane-Higgs coupling was suggested forcing to maintain the free fermion boundary conditions in the presence of Yukawa interactions.

#### 3.5.2.1 Regularization I Drawbacks

In the first type of regularization applied in the literature [96, 97, 100], called Regularization I, the BC are considered at the first level of the procedure to be injected in the EOM (3.54) [96]. The free BC impose $d_L^n(L) = q_R^n(L) = 0$ [27] [see respectively the first and fourth solutions in Eq. (3.28)] so that the EOM (3.54) is supposed to become

$$\forall\, n \in \mathbb{N}, \quad \left\{ \begin{array}{lcl} \partial_4 q_R^n(y) + M_n\, q_L^n(y) & = & \delta(y-L)\, X d_R^n(y)\,, \\ \partial_4 q_L^n(y) - M_n\, q_R^n(y) & = & 0\,, \\ \partial_4 d_R^n(y) + M_n\, d_L^n(y) & = & 0\,, \\ \partial_4 d_L^n(y) - M_n\, d_R^n(y) & = & -\,\delta(y-L)\, X q_L^n(y)\,. \end{array} \right. \tag{3.55}$$

At this level, we point out a second lack of strictness in this common "standard treatment": the two vanishing RHS of Eq. (3.55) originate from the assumption that $0 \times \delta(0) = 0$ whereas the quantity $0 \times \delta(0)$ is rigorously undefined [28] which should forbid to continue this standard method [29]. In the next step of this method, the usual mathematical trick is to shift the brane-Higgs coupling from the brane at $y = L$ (TeV-brane in a warped

25. Strictly speaking, a Dirac peak is a distribution although its historical name is "Dirac delta function", which is rigorously treated in Chapter 5.

26. Also called "generalized functions" in mathematical analysis.

27. In the literature, the BC $d_L^n|_{0,L} = q_R^n|_{0,L} = 0$ are selected by hand from the NBC (1.33) to realize the SM chiral configuration but without a clear argument based on the EBC (3.25) [see Section 3.3.2] or the BBT (3.4)-(3.7) [see Section 3.3.3].

28. This quantity corresponds also to an undefined product, namely $0 \times \infty$, within the original simplified description [148] still used in physics textbooks (together with normalisation conditions):

$$\delta(y-L) \mathrel{\hat{=}} \left\{ \begin{array}{lll} 0 & \text{if} & y \neq L\,, \\ \infty & \text{if} & y = L\,. \end{array} \right.$$

29. Such $\delta(0)$ divergences are automatically regulated – by the exchange of infinite towers of KK scalar modes – for a brane-Higgs coupled to bulk scalar fields within a minimal supersymmetric scenario [149].

framework) by an amount $\epsilon$:

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} \partial_4 q_R^n(y) + M_n\, q_L^n(y) &= \delta(y - [L-\epsilon])\, X d_R^n(y)\,, \\ \partial_4 q_L^n(y) - M_n\, q_R^n(y) &= 0\,, \\ \partial_4 d_R^n(y) + M_n\, d_L^n(y) &= 0\,, \\ \partial_4 d_L^n(y) - M_n\, d_R^n(y) &= -\,\delta(y - [L-\epsilon])\, X q_L^n(y)\,. \end{cases} \tag{3.56}$$

Then the integration of the four relations of Eq. (3.56) over an infinitesimal range, tending to zero and centered at $y = L - \epsilon$, leads to [30]

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} q_R^n([L-\epsilon]^+) - q_R^n([L-\epsilon]^-) &= X d_R^n(L-\epsilon)\,, \\ q_L^n([L-\epsilon]^+) - q_L^n([L-\epsilon]^-) &= 0\,, \\ d_R^n([L-\epsilon]^+) - d_R^n([L-\epsilon]^-) &= 0\,, \\ d_L^n([L-\epsilon]^+) - d_L^n([L-\epsilon]^-) &= -\,X q_L^n(L-\epsilon)\,. \end{cases} \tag{3.57}$$

Another inconsistency arising here in the regularization process is the following one. The first and fourth relations in Eq. (3.57) show that the wave functions $q_R^n(y)$ and $d_L^n(y)$ possess a discontinuity at $y = L - \epsilon$. Hence the functions $\partial_4 q_R^n(y)$ and $\partial_4 d_L^n(y)$ are not defined at $y = L - \epsilon$. Two of the integrations performed on Eq. (3.56) to get Eq. (3.57) are thus not well defined. The fundamental theorem of analysis [31] [150] cannot be applied for functions undefined on the whole interval of integration. Let us express this problem in other terms; the functions $\partial_4 q_R^n(y)$ and $\partial_4 d_L^n(y)$ being not defined at $y = L$ (in the limit $\epsilon \to 0$), the last two terms of the 5D kinetic action (3.3) – defined along the interval $\mathcal{I} \,\hat{=}\, [0, L]$ – are not well defined [32]. Another definition problem appears in this regularization: the brane-localized Yukawa action (3.14) is ill-defined [115, 116] since the Dirac peak $\delta(y - L)$ enters in particular as a factor of the profiles $q_R^n(y)$ and $d_L^n(y)$ being not continuous at $y = L$, as deduced from Eq. (3.57) – in the limit $\epsilon \to 0$ – combined with the free BC imposing $d_R^n(L) \neq 0$, $q_L^n(L) \neq 0$ [see respectively the first and fourth solutions in Eq. (3.28)] [33]. Finally, the $q_R^n(y)$ and $d_L^n(y)$ jump at $y = L$, obtained when regularizing the brane-Higgs coupling, which conflict to the field continuity axiom of the invoked theory of variation calculus and hence to the Hamilton's variational principle [114].

In the following steps of this Regularization I, one solves the shifted EOM (3.56) first in the interval $[0, L - \epsilon]$ (bulk EOM without Yukawa couplings) and applies the free BC at $y = 0$ on the obtained profiles. Then, one solves similarly on $[L - \epsilon, L]$ before applying the jump and continuity conditions (3.57) at $y = L - \epsilon$ on the resulting profiles. The last step is to apply the free BC at $y = L$ on these profiles and take the limit $\epsilon \to 0$ (to recover the studied brane-Higgs model) on the written BC. The obtained BC give rise to the equation whose solutions constitute the fermion mass spectrum:

$$\forall n \in \mathbb{N}\,, \quad \tan^2(M_n L) = X^2\,, \tag{3.58}$$

30. The integration of Eq. (3.56) could also be performed over the interval $[L - \epsilon, L]$; this variant of the calculation, suggested in an Appendix of Ref. [97], represents in fact an equivalent regularization process leading to the same physical results and with identical mathematical inconsistencies.

31. Let $(a, b) \in \mathbb{R}^2$ and $g$ be a continuous function on $[a, b]$, then $g$ admits continuous primitives on $[a, b]$. Let $G$ be one of them, then one has: $\int_a^b dy\, g(y) = G(b) - G(a)$.

32. From the current point of view, the conservation condition (F.10) – involving in particular the 5D probability current component (3.79) – cannot be properly written at any point along the fifth dimension since $q_R^n(y)$ and $d_L^n(y)$ have discontinuities at $y = L$ so that derivatives in $\partial_4 j^4$ are not well defined there.

33. The profiles $q_L^n(y)$, $d_R^n(y)$ are usually assumed to be continuous at $y = L - \epsilon$ while $q_R^n(y)$, $d_L^n(y)$ remain unknown exactly at this point.

which induces the exactly identical mass spectrum in the 4D approach result of Eq. (3.45)-(3.46).

#### 3.5.2.2 Regularization II Drawbacks

Within the Regularization II [96, 97, 101, 102], the Yukawa coupling is firstly shifted into the bulk equations (3.54) which become

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} \partial_4 q_R^n(y) + M_n\, q_L^n(y) &= \;\; \delta(y - [L-\epsilon])\, X d_R^n(y)\,, \\ \partial_4 q_L^n(y) - M_n\, q_R^n(y) &= -\, \delta(y - [L-\epsilon])\, X' d_L^n(y)\,, \\ \partial_4 d_R^n(y) + M_n\, d_L^n(y) &= \;\; \delta(y - [L-\epsilon])\, X' q_R^n(y)\,, \\ \partial_4 d_L^n(y) - M_n\, d_R^n(y) &= -\, \delta(y - [L-\epsilon])\, X q_L^n(y)\,. \end{cases} \tag{3.59}$$

Integrating these four relations over an infinitesimal range centered at $y = L - \epsilon$ gives:

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} q_R^n([L-\epsilon]^+) - q_R^n([L-\epsilon]^-) &= \;\; X d_R^n(L-\epsilon)\,, \\ q_L^n([L-\epsilon]^+) - q_L^n([L-\epsilon]^-) &= -\, X' d_L^n(L-\epsilon)\,, \\ d_R^n([L-\epsilon]^+) - d_R^n([L-\epsilon]^-) &= \;\; X' q_R^n(L-\epsilon)\,, \\ d_L^n([L-\epsilon]^+) - d_L^n([L-\epsilon]^-) &= -\, X q_L^n(L-\epsilon)\,. \end{cases} \tag{3.60}$$

which show that the four wave functions undergo a jump at $y = L-\epsilon$ so that their derivative with respect to $y$ are not well-defined at this point [especially for $q_L^n$ and $d_R^n$, in contrast to that in Eq. (3.57)]. Hence, the four integrations performed on Eq. (3.59) to obtain Eq. (3.60) are not well defined in this regularization, which is a repeated inconsistency in the Regularization I. Since that, a regularization at $y = L - \epsilon$ must be implied and one can select a standard mean value weighted thanks to a real number, $c$,

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} q_R^n([L-\epsilon]^+) - q_R^n([L-\epsilon]^-) &= \;\; X \frac{d_R^n([L-\epsilon]^-) + c\; d_R^n([L-\epsilon]^+)}{1+c}\,, \\ q_L^n([L-\epsilon]^+) - q_L^n([L-\epsilon]^-) &= -\, X' \frac{d_L^n([L-\epsilon]^-) + c\; d_L^n([L-\epsilon]^+)}{1+c}\,, \\ d_R^n([L-\epsilon]^+) - d_R^n([L-\epsilon]^-) &= \;\; X' \frac{q_R^n([L-\epsilon]^-) + c\; q_R^n([L-\epsilon]^+)}{1+c}\,, \\ d_L^n([L-\epsilon]^+) - d_L^n([L-\epsilon]^-) &= -\, X \frac{q_L^n([L-\epsilon]^-) + c\; q_L^n([L-\epsilon]^+)}{1+c}\,. \end{cases} \tag{3.61}$$

Scrutinizing the left-hand sides of those four equations, one observes that jumps may arise at $y = L$ (under the limit $\epsilon \to 0$) for the four profiles [for each excited $n^{th}$ mode]. Determining which profiles are discontinuous requires to consider the free BC at $y = L$ (before applying the limit $\epsilon \to 0$), the various $c$ values (including infinity) [34] and the four profiles simultaneously [as they are related through Eq. (3.61)].

The hypothesis that all of the four profiles are continuous at $y = L - \epsilon$ (in the limit $\epsilon \to 0$) would lead to the EOM and the vanishing BC for all fields as in the absence of Yukawa interactions [35] and in turn force all fields to vanish on the interval $\mathcal{I} = [0, L]$ (see Section 1.3.1). This kind of solution was not considered in the literature since it

34. Different values of $c$ correspond to physically equivalent regularizations based on different input values of the Yukawa coupling constants (different coupling definitions).

35. Free BC for continuous profiles and free version of the bulk equations (3.59) without the jump conditions (3.61) at $y = L - \epsilon$ involving effectively the Yukawa couplings.

does not provide a physical configuration. Therefore, there must exist at least one profile discontinuous at $y = L$, which in turn cannot be derived at this point and leads to an undefined kinetic term [in the last two terms of 5D kinetic Lagrangian density (3.3)]. Furthermore, the obtained discontinuous [at $y = L$] profile comes in factor of $\delta(y - L)$ in Eq. (3.14), spoiling the mathematical validity of this action. Besides, once more, this jump obtained at $y = L$ within the regularization process is not compatible with the field continuity axiom implicitly used when applying the Hamilton's variational principle.

In the next steps of Regularization II, the EOM (3.59) is first solved over the domain $[0, L-\epsilon]$ (free bulk EOM) and the free BC at $y = 0$ are applied on the resulting wave functions. Eq. (3.59) is then solved over $[L-\epsilon, L]$ before the jump/continuity conditions (3.61) at $y = L - \epsilon$ are applied on the obtained profiles. Finally, the free BC at $y = L$ are implemented on those profiles and one applies the limit $\epsilon \to 0$ on the expressed BC. These BC make appear the following fermion mass spectrum equation for $c = 1$:

$$\forall n \in \mathbb{N}\,, \quad \tan^2(M_n L) = \left(\frac{4X}{4 + XX'}\right)^2 , \tag{3.62}$$

which can still be shown [96] to be the same as the result of regularization I in Eq. (3.58) [with the redefined Yukawa couplings].

### 3.5.3 Inconsistencies of the Softened Brane-Higgs Coupling

Another type of regularization used in the literature (for warped models) [96, 97, 101, 102, 105–107] consists in replacing the Dirac peak $\delta(y - L)$ of Eq. (3.14) by a normalized square function [a so-called nascent delta function (or delta sequence)],

$$\delta^\epsilon(y - L) = \begin{cases} \dfrac{1}{\epsilon}\,, & y \in [L - \epsilon, L]\,, \\[2ex] 0\,, & \text{otherwise,} \end{cases} \tag{3.63}$$

which has a vanishing width ($\epsilon > 0$) and an infinite value ($1/\epsilon$) in the limit $\epsilon \to 0$ [which is not a true function in the mathematical sense] [36] where one expects to recover the considered model with a brane-Higgs coupling. Nevertheless, we point out here that the Dirac peak $\delta(y - L)$ at the Higgs brane, and in turn the original model, is not rigorously recovered via a limit, $\delta(y - L) = \lim_{\epsilon \to 0} \delta^\epsilon(y - L)$, which is only symbolic since a distribution cannot be defined as the simple direct limit of a basic function [37]. Hence, this would-be regularization is not satisfactory in the sense that it does not strictly reproduce the studied brane-Higgs scenario. By the way, to give a well-definition to this regularization procedure, $\delta(y - L)$ in Eq. (3.54) should multiply only continuous wave functions.

In addition, the two schemes of Regularization I and II still hold in this framework of a softened coupling and in the case of Regularization I a problem arises again: some terms of the profile EOM are taken at zero based on the assumption that $0 \times \delta(0) = 0$ whereas the quantity $0 \times \delta(0)$ is undefined.

---

36. The rigorous treatment based on the Dirac distribution formalism is presented in Chapter 5.

37. Strictly speaking, it is the effect of the Dirac peak in the integration of a function $f(y)$ over an interval covering the point $y = L$, $\int \delta(y - L) f(y) dy = f(L)$, which can be reproduced via an integration of the type, $\lim_{\epsilon \to 0} \int \delta^\epsilon(y - L) f(y) dy = f(L)$, but not the implementation in the present regularization.

### 3.5.4 Two Non-Commutativities of Calculation Steps

The analytical differences of the mass spectra found in the Regularisations I and II, as well as via the softened and shifted brane-Higgs peaks, could be compensated by the different input values of the Yukawa coupling constant parameters ($Y_5$ and $Y'_5$) to get identical physical mass values.

Nevertheless, the Regularizations I and II are in fact physically different as inducing the existence of measurable flavor violating effective 4D Yukawa couplings at leading order in $v^2/|m_1|^2$, which are generated by the $Y'_5$ couplings [101] presented exclusively within Regularization II (as appears clearly in the 4D approach). This physical difference between the two schemes of regularization raises the paradoxical question, of which one is the sole correct analytical scheme to use, and represents thus as a confirmation of the inconsistency of regularizing the Higgs peak.

These two schemes of regularization are obtained [96] by commuting in the 4D calculation (of masses and couplings) the ordering of implementation of the two limits $\epsilon \to 0$ [the regularizing parameter $\epsilon$ defined in Eq. (3.56)] and $N \to \infty$ [the upper value $N$ of the KK level $n$ in Eq. (1.24)]. Therefore, this physical non-commutativity of calculation steps reflects the inconsistency of the Higgs peak regularization.

Another paradoxical non-commutativity of calculation steps arising in the context of regularization of a brane-Higgs coupled to bulk fermions was discussed in Ref. [105, 106]: different results of Higgs production/decay rates when taking $\epsilon \to 0$ and then $N_{KK} \to \infty$ [38] [102] or the inverse order [103] in their calculation. We can thus interpret now this second non-commutativity of calculation steps as being another effect, and in turn another confirmation, of the problematic Higgs regularization (also expected with a warped extra dimension). Hence, the theoretical debate in the literature about the origins of those two non-commutativities (involving $\epsilon$) finds its solution in the mathematically ill-defined (see above) and unnecessary (see below) Higgs regularization (introducing $\epsilon$).

## 3.6 New 5D Treatment

In this part, we consider the presence of the Yukawa couplings (3.14) and present the rigorous 5D method to calculate the fermionic mass spectrum – which does not require any kind of regularization. We follow the main lines of the methodology developed for the free case in Section 3.3.

### 3.6.1 The Naive Approach

For the fermion masses, the relevant part of the considered action (3.17) to start with is

$$S^{\mathrm{m}}_{\mathrm{5D}} = S_{\mathrm{bulk}} + S_X + S^0_B\,, \quad \text{with} \quad S^0_B = -\int d^4x \, \mathcal{L}_B|_0 \,, \tag{3.64}$$

where the first term is based on kinetic terms (3.3) and $\mathcal{L}_B$ introduced by the BBT of Eq. (3.4) is imposed only at the brane $y = 0$ where the Yukawa interaction is absent. Regarding the free brane at $y = 0$, we could equivalently apply the EBC (3.23) instead of including these BBT, as we have exposed in details in Section 3.3.2-3.3.3. Now without

38. Here $N_{KK}$ stands for the number of excited fermion eigenstates exchanged at the loop-level.

loss of generality, the least action principle leads to the four following conditions,

$$
\begin{aligned}
0 = \delta_{Q_L^\dagger} S_{\rm 5D}^{\rm m} &= \int d^4x \left\{ \int_0^L dy\, \delta Q_L^\dagger \left[ i\bar\sigma^\mu \partial_\mu Q_L + \partial_4 Q_R \right] \right. \\
&\qquad \left. + \left[ \delta Q_L^\dagger \left( -X D_R - \frac{1}{2} Q_R \right) \right] \Big|_L + \left( \delta Q_L^\dagger Q_R \right) \Big|_0 \right\} , \\
0 = \delta_{Q_R^\dagger} S_{\rm 5D}^{\rm m} &= \int d^4x \left\{ \int_0^L dy\, \delta Q_R^\dagger \left[ i\sigma^\mu \partial_\mu Q_R - \partial_4 Q_L \right] + \left[ \delta Q_R^\dagger \left( -X' D_L + \frac{1}{2} Q_L \right) \right] \Big|_L \right\} , \\
0 = \delta_{D_L^\dagger} S_{\rm 5D}^{\rm m} &= \int d^4x \left\{ \int_0^L dy\, \delta D_L^\dagger \left[ i\bar\sigma^\mu \partial_\mu D_L + \partial_4 D_R \right] + \left[ \delta D_L^\dagger \left( -X'^* Q_R - \frac{1}{2} D_R \right) \right] \Big|_L \right\} , \\
0 = \delta_{D_R^\dagger} S_{\rm 5D}^{\rm m} &= \int d^4x \left\{ \int_0^L dy\, \delta D_R^\dagger \left[ i\sigma^\mu \partial_\mu D_R - \partial_4 D_L \right] \right. \\
&\qquad \left. + \left[ \delta D_R^\dagger \left( -X^* Q_L + \frac{1}{2} D_L \right) \right] \Big|_L - \left( \delta D_R^\dagger D_L \right) \Big|_0 \right\} . \qquad (3.65)
\end{aligned}
$$

Analogy to the studied free case, the non-vanishing field variations $\delta F_{L/R}^\dagger$, $\delta F_{L/R}^\dagger\big|_{0,L}$ are generic, so that the bulk terms in each of those four relations, must vanish separately, which leads to the same equations as the 5D EOM (1.30) and hence – via the mixed KK decomposition (3.50) and 4D Dirac-Weyl equations (3.51) – the profile equations,

$$
\forall n \in \mathbb{N} , \quad \left\{ \begin{array}{rcl} \partial_4 q_R^n(y) - M_n\, q_L^n(y) &=& 0 \, , \\ \partial_4 q_L^n(y) + M_n\, q_R^n(y) &=& 0 \, , \\ \partial_4 d_R^n(y) - M_n\, d_L^n(y) &=& 0 \, , \\ \partial_4 d_L^n(y) + M_n\, d_R^n(y) &=& 0 \, , \end{array} \right. \tag{3.66}
$$

which are solved in Eq. (1.38) ($f = q, d$) but via (distinct) KK masses $M_n$,

$$
\forall\, n \in \mathbb{N}, \quad \left\{ \begin{array}{rcl} f_L^n(y) &=& -B_R^n \cos(M_n\, y) + B_L^n \sin(M_n\, y) \, , \\ f_R^n(y) &=& B_L^n \cos(M_n\, y) + B_R^n \sin(M_n\, y) \, , \end{array} \right. \tag{3.67}
$$

and the NBC resulting from Eq. (3.65) read as:

$$
\left\{ \begin{array}{lll} (Q_R + 2X D_R)|_L = 0, & (D_R + 2X'^* Q_R)|_L = 0, & Q_R|_0 = 0, \\ (Q_L - 2X' D_L)|_L = 0, & (D_L - 2X^* Q_L)|_L = 0, & D_L|_0 = 0. \end{array} \right. \tag{3.68}
$$

Note that $Q_R$ and $D_L$ can't vanish at $y = L$. Otherwise, one would obtain the free mass spectrum independent of Yukawa couplings $X^{(\prime)}$[see Eq. (3.30)] via the solutions of Eq. (3.67) as in Section (3.3.2). Then, the NBC (3.68) leads to the following consistency conditions on the Lagrangian parameters,

$$
4 X X'^* = 4 X^* X' = 1 \, , \tag{3.69}
$$

and in turn to

$$
4 |X X'| = 1 \, , \quad \text{and} \quad \alpha_{Y'} = \alpha_Y + 2k\pi \, , \; k \in \mathbb{Z} \, . \tag{3.70}
$$

We should emphasize that the Yukawa coupling relation (3.69) prevents us from taking the free limit where $X \to 0$ and $X' \to 0$.

The boundary conditions (3.68), combined with the bulk profile EOM (3.66) [with solutions (3.67)] taken at $y = L$, constitute the complete BC. Referring to the dependence on the quantity $X^{(\prime)}$, we denote $(\times)$ this new class of complete BC at the brane with a Yukawa coupling (here at $y = L$) to distinguish them from the Dirichlet BC usually noted $(-)$ or the Neumann BC noted $(+)$. The BC (3.68) on the 5D fields give rise to the following conditions on the profiles, through the mixed KK decomposition (3.50),

$$\forall n \in \mathbb{N}\,, \left\{ \begin{array}{lll} q_R^n(L) + 2X d_R^n(L) = 0, & d_R^n(L) + 2X'^* q_R^n(L) = 0, & q_R^n(0) = 0, \\ q_L^n(L) - 2X' d_L^n(L) = 0, & d_L^n(L) - 2X^* q_L^n(L) = 0, & d_L^n(0) = 0, \end{array} \right. \tag{3.71}$$

since the 4D fermion fields for the mass eigenstates cannot be linearly related – as discussed below Eq. (3.19). Those profile conditions, once applied on the solutions (3.67), lead to the form,

$$\forall n \in \mathbb{N}\,, \left\{ \begin{array}{ll} q_L^n(y) = -C_R^n \cos(M_n\, y), & q_R^n(y) = C_R^n \sin(M_n\, y), \\ d_L^n(y) = D_L^n \sin(M_n\, y), & d_R^n(y) = D_L^n \cos(M_n\, y), \end{array} \right. \tag{3.72}$$

together with the relations,

$$\begin{aligned} \tan(M_n\, L) &= -2X\, \frac{D_L^n}{C_R^n} = -2X^*\, \frac{C_R^n}{D_L^n} \quad \Rightarrow \quad \tan^2(M_n\, L) = 4|X|^2\,, \\ \cot(M_n\, L) &= -2X'\, \frac{D_L^n}{C_R^n} = -2X'^*\, \frac{C_R^n}{D_L^n} \quad \Rightarrow \quad \cot^2(M_n\, L) = 4|X'|^2\,. \end{aligned} \tag{3.73}$$

where the last two mass spectrum relations induced are strictly equivalent thanks to Eq. (3.69). The obtained mass spectrum allows to further determine for instance the BC $(-\times)$ of the profile $d_L^n(y)$: $d_L^n(0) = 0$ and $d_L^n(L) = D_L^n \sin(M_n\, L)$.

Let us check the validity of the obtained solutions by physical consistencies. In the decoupling limit of high KK masses (compared to the typical SM energy scale) applied to the present model, one expects to recover approximately the SM setup at low-energies. This decoupling condition is necessary for the theoretical consistency of the model, and it is generically imposed by the experimental constraints. Firstly, according to Eq. (3.73), the lightest mode mass is,

$$|M_0| = \frac{1}{L} \arctan(2|X|) = \frac{1}{L} \arctan(\sqrt{2}|y_4 L v|) \underset{|m_1| \gg |v|}{\sim} \sqrt{2}|y_4 v|\,, \tag{3.74}$$

where we have imposed the relation of $X$, $Y_5$ and $y_4$ in Eq. (3.13)-(3.15) as well as the first excited KK mass $|m_1| = \pi/L$ [cf. Eq. (3.30)]. This 4D effective fermion mass [cf. Eq. (3.51)] is well proportional to the Higgs VEV as in the SM. Secondly, the effective 4D Yukawa coupling constants in the 4D action term involving the lightest modes, $-\int d^4x\, Y_{00} H \psi_L^{0\dagger} \psi_R^0 + \text{H.c.}$, is obtained by injecting the mixed KK decomposition (3.50) to the Yukawa action (3.11) and then integrating over $y$ by using the wave functions (3.72) to take into account the mass mixings induced by the Yukawa couplings (5D method):

$$\begin{aligned} y_{00} &= \frac{Y_5}{\sqrt{2}L}\, q_L^{0*}(L)\, d_R^0(L) + \frac{Y_5'^*}{\sqrt{2}L}\, d_L^{0*}(L)\, q_R^0(L) \\ &= \left[ \frac{Y_5}{\sqrt{2}L} - \frac{Y_5'^*}{\sqrt{2}L}\, (2X)^2 \right] q_L^{0*}(L)\, d_R^0(L) \\ &= \left[ \frac{Y_5}{\sqrt{2}L} - \frac{Y_5}{\sqrt{2}L} \right] q_L^{0*}(L)\, d_R^0(L) \\ &= 0\,, \end{aligned} \tag{3.75}$$

where we have invoked the BC (3.71) and Eq. (3.69). So, $y_{00}$ vanishes, which differs from the SM framework and in turn breaks the decoupling condition,

The problematic vanishing of the effective 4D Yukawa coupling constant $y_{00}$ reveals a problem in the present treatment of the studied model, which results from the invariance of the action (3.64) under the exchange transformation, $Q \leftrightarrow D$ together with $Y_5^* \leftrightarrow Y_5'$ at $y = L$ [symmetry also explicit in EOM (3.66) and the NBC (3.71) of profiles]; this symmetry will be broken in the correct treatments presented below. A confirmation of the failure of the present 5D treatment is the non-matching of the obtained spectrum equation (3.73) with the 4D matrix method result (3.45). Therefore, the naive treatment of the brane-Higgs coupling in this section should be reconsidered: we present the other methods in the following two sections.

### 3.6.2 Introducing the Fermion Current Condition [EBC]

Like in the free case treated in Section 3.3.2, we now try to define well the geometrical field configuration of the considered scenario based on the action $S_{5\text{D}}^{\text{m}}$ of Eq. (3.64) where the boundary at $y = 0$ has been constrained by the BBT and additional boundary conditions would be applied at $y = L$ [EBC] [39]. In this scenario, the two 5D fields $Q, D$ propagate only inside the interval $\mathcal{I} \,\hat{=}\, [0, L]$. This setup translates into a condition of vanishing probability current at both boundaries. Here, the current is the sum of the two individual currents of type (3.20) for the two species $Q, D$ since those fermions are mixed through the Yukawa terms (3.14). To find out this current form rigorously, we first vary the action as at the beginning of Section 3.6.1 and deduce the 5D EOM (1.30) whose profile solutions were given in Eq. (3.67). Then using the obtained EOM (1.30), we apply in Appendix F the Noether's theorem to work out the probability current (F.10) [40] which reads as,

$$j^M = -\alpha \sum_{F=Q,D} \bar{F}\,\Gamma^M F\,, \quad \text{with the local conservation relation } \partial_M j^M = 0\,, \tag{3.76}$$

as dictated by the global $U(1)$ symmetry of the action (3.64) relying on the transformations,

$$\begin{aligned} Q \mapsto e^{i\alpha} Q\ ,\ \ D \mapsto e^{i\alpha} D\ , \\ \bar{Q} \mapsto e^{-i\alpha} \bar{Q}\ ,\ \ \bar{D} \mapsto e^{-i\alpha} \bar{D}\,. \end{aligned} \tag{3.77}$$

where $\alpha$ ($\in \mathbb{R}$) is a continuous parameter [now forced by the invariant terms– Yukawa couplings (3.14) – to be common for the two fields $Q, D$] involved for example in the infinitesimal field variations ($F = Q, D$),

$$\underline{\delta} F_L = i\alpha F_L\,, \quad \underline{\delta} F_L^\dagger = -i\alpha F_L^\dagger\,. \tag{3.78}$$

We thus find that the effect of the Yukawa interactions is not to modify the currents but rather to force one to add them up for having a probability conservation relation (due to the induced mixing among the $Q$ and $D$ fields). Finally, the condition of vanishing probability current at the boundary where is located the Yukawa coupling reads as [41],

$$j^4\Big|_L = -\alpha \sum_{F=Q,D} \bar{F}\,\Gamma^4 F\Bigg|_L = i\alpha \sum_{F=Q,D} \left(F_L^\dagger F_R - F_R^\dagger F_L\right)\Bigg|_L = 0\,. \tag{3.79}$$

39. The boundary conditions at $y = 0$ can be alternatively realized by EBC induced by the probability currents instead of the BBT at $y = 0$ included in $S_{5\text{D}}^{\text{m}}$ (3.64), which has been presented in Section 3.3.2-3.3.3.

40. This result holds as well in the case without BBT.

41. The current condition at the other boundary is taken into account through the BBT in the last term of the action (3.64).

For a non-trivial transformation with $\alpha \neq 0$, the field variation of this relation is

$$\sum_{F=Q,D} \left( \delta F_L^\dagger F_R + F_L^\dagger \delta F_R - \delta F_R^\dagger F_L - F_R^\dagger \delta F_L \right) \Big|_L = 0 \,. \tag{3.80}$$

The variation calculus chronology here is quite simple as no field is fixed by the EBC (3.79): the fields [and their respective variations] are instead related via this Eq. (3.79) [and Eq. (3.80)]. Now the brane part of the variation of the action $S_{5\text{D}}^{\text{m}}$ (3.64), containing the boundary terms is written explicitly in Eq. (G.1) of the Appendix G. The complementary variation of the bulk action vanishing separately was already used just above to derive the 5D EOM (1.30). Notice that this variation of the bulk action with respect to the non-conjugate 5D fields in $\delta_{F_{L,R}} S_{5\text{D}}^{\text{m}}$ requires an integration by part to recover the Hermitian conjugate form of the EOM (1.30) [visible in Eq. (3.65)] and the boundary terms in $\delta_{F_{L,R}} S_{5\text{D}}^{\text{m}}$ [visible in Eq. (G.1)]. One could think of obtaining NBC and their Hermitian conjugate form respectively from $\delta_{F_{L,R}} S_{5\text{D}}^{\text{m}}$ and $\delta_{F_{L,R}^\dagger} S_{5\text{D}}^{\text{m}}$ [as obtained in Eq. (3.68)], in Eq. (G.1), but in fact all the field variations are connected via the relation of Eq. (3.80) so that one can not get rid of those directly.

In order to get some set of boundary conditions, the most trivial way of combining the NBC (G.1) with the EBC (3.80) is,

$$Q_{L/R}\Big|_L = D_{L/R}\Big|_L = 0 \,. \tag{3.81}$$

However, in this case, the dependence of the Yukawa couplings disappears from the EOM, the BC as well as the mass spectrum so that without going into the details, one can conclude that we will not recover the SM in the decoupling limit. To satisfy the EBC (3.79), we can also try another tricky mixed boundary condition [42],

$$X' = 0 \,, \quad (Q_R + X D_R)|_L = (D_L - X^* Q_L)|_L = 0 \,, \tag{3.82}$$

which will also be induced by the BBT approach in Section 3.6.3. However, the remaining NBC at $y = L$ induced via the least action principle of $S_{5\text{D}}^{\text{m}}$ (3.64) [injecting the EBC (3.82)],

$$\sum_{F_C = Q_{L/R}, D_{L/R}} \delta_{F_C^\dagger} S_{5\text{D}}^{\text{m}} \ni - \int d^4x \, \left( \delta D_L^\dagger D_R + \delta D_R^\dagger D_L \right) \Big|_L \,,$$

would lead to the vanishing of $D_{L/R}\Big|_L$ [43] and in turn to the boundary conditions (3.81) again. So, this particular trick (3.82) can't rescue the situation.

In conclusion, there is no consistent way of combining the NBC (G.1) [even by splitting it into several vanishing expressions] with the EBC (3.80), except in the particular but excluded case of Eq. (3.81). The current approach of the configuration with a Yukawa coupling located at a boundary, based on the vanishing of the fermion current taken as the EBC, is not yet the correct one. The origin of the problem is that the current (3.76) does not contain an explicit term that involves the Yukawa coupling constants.

42.

$$j^4\big|_L = i\alpha \sum_{F=Q,D} \left( F_L^\dagger F_R - F_R^\dagger F_L \right) \Big|_L = i\alpha \left( -Q_L^\dagger X D_R + X^* D_R^\dagger Q_L + D_L^\dagger D_R - D_R^\dagger D_L \right) \big|_L$$
$$= i\alpha \left( -D_L^\dagger D_R + D_R^\dagger D_L + D_L^\dagger D_R - D_R^\dagger D_L \right) \big|_L = 0 \,.$$

43. $\delta D_{L/R}^\dagger \Big|_L \neq 0$ for unknown fields $D_{L/R}^\dagger$.

### 3.6.3 Introducing the Bilinear Brane Terms [NBC]

In order to overcome the drawbacks discussed in Section 3.5, analogy to the free case in Section 3.3.3, we try here to develop a consistent approach, based on the introduction of the BBT (3.4) at $y = 0, L$. We consider the fermion part of the action (3.17):

$$S_{\text{bulk}} + S_B + S_X \,, \tag{3.83}$$

based on the kinetic Lagrangian density (3.3), the BBT (3.4) and the Yukawa terms (3.14). The boundary fields $F|_{0,L}$ are initially unknown so that their functional variations will be taken non-vanishing: $\delta F|_{0,L} \neq 0$. Without loss of generality, the stationary action condition

$$\delta_{\bar{F}} \left( S_{\text{bulk}} + S_B + S_X \right) = 0 \,,$$

can be split into the two following conditions for each field $F = Q, D$ [extending Eq. (3.34) to include the Yukawa terms],

$$\begin{aligned} \delta_{\bar{Q}}(S_{\text{bulk}} + S_X + S_B) \;&=\; \int d^4x \left\{ \int_0^L dy \; \delta\bar{Q}\, i\Gamma^M \partial_M Q \right. \qquad\qquad (3.84) \\ &\left. + \left[ -\delta Q_L^\dagger \left( Q_R + X\, D_R \right) - X' \delta Q_R^\dagger D_L \right] \Big|_L + \left( \delta Q_L^\dagger Q_R \right) \Big|_0 \right\} , \\ \delta_{\bar{D}}(S_{\text{bulk}} + S_X + S_B) \;&=\; \int d^4x \left\{ \int_0^L dy \; \delta\bar{D}\, i\Gamma^M \partial_M D \right. \qquad\qquad (3.85) \\ &\left. + \left[ -X'^* \delta D_L^\dagger Q_R + \delta D_R^\dagger \left( D_L - X^* Q_L \right) \right] \Big|_L - \left( \delta D_R^\dagger D_L \right) \Big|_0 \right\} . \end{aligned}$$

Once more, the non-vanishing field variations $\delta F^\dagger_{L/R}$, $\delta F^\dagger_{L/R}\Big|_{0,L}$ being generic, the bulk terms (first line) in Eq. (3.84)-(3.85) must vanish separately, which brings in the 5D EOM (1.30) and in turn – through the mixed KK decomposition (3.50) and 4D Dirac-Weyl equations (3.51) – the wave function equations (3.66) with solutions as in Eq. (3.67):

$$\forall n \in \mathbb{N} \,, \quad \left\{ \begin{array}{lcl} q_L^n(y) &=& -B_R^n \cos(M_n\, y) + B_L^n \sin(M_n\, y) \,, \\ q_R^n(y) &=& B_L^n \cos(M_n\, y) + B_R^n \sin(M_n\, y) \,, \\ d_L^n(y) &=& -D_R^n \cos(M_n\, y) + D_L^n \sin(M_n\, y) \,, \\ d_R^n(y) &=& D_L^n \cos(M_n\, y) + D_R^n \sin(M_n\, y) \,, \end{array} \right. \tag{3.86}$$

using here new constant parameters $B_{L/R}^n$, $D_{L/R}^n$. The NBC result from the vanishing of boundary terms in Eq. (3.84)-(3.85) [44]:

$$\left\{ \begin{array}{l} (Q_R + X\; D_R)|_L = 0 \,, \quad X'^*\; Q_R|_L = 0 \,, \quad Q_R|_0 = 0 \,, \\ X'\; D_L|_L = 0 \,, \quad (D_L - X^*\; Q_L)|_L = 0 \,, \quad D_L|_0 = 0 \,, \end{array} \right. \tag{3.87}$$

which can be rewritten without loss of generality as,

$$\left\{ \begin{array}{l} \{ (Q_R + X D_R)|_L = 0 \,, \quad (D_L - X^* Q_L)|_L = 0 \,, \quad X' = 0 \,, \} \quad \text{or} \quad \{ Q_R|_L = 0 \,, \quad D_L|_L = 0 \,, \} \\ Q_R|_0 = 0 \,, \quad D_L|_0 = 0 \,, \end{array} \right.$$

44. Integrating by part the bulk terms in the other relations

$$\delta_{F_L} \left( S_{\text{bulk}} + S_B + S_X \right) = \delta_{F_R} \left( S_{\text{bulk}} + S_B + S_X \right) = 0 \,,$$

allows to recover the Hermitian conjugate form of the EOM (1.30) as well as the Hermitian conjugate form of the NBC (3.87).

and in turn as,

$$\begin{aligned}&\underline{\text{BC 1 :}}\quad XD_R|_L=0\,,\quad X^*Q_L|_L=0\,,\quad Q_R|_L=0\,,\quad D_L|_L=0\,,\quad Q_R|_0=0\,,\quad D_L|_0=0\,,\\&\quad\text{or}\\&\underline{\text{BC 2 :}}\quad (Q_R+XD_R)|_L=0\,,\quad (D_L-X^*Q_L)|_L=0\,,\quad X'=0\,,\quad Q_R|_0=0\,,\quad D_L|_0=0\,.\end{aligned}\tag{3.88}$$

The lightest fermionic state possesses a mass equal to the $\alpha_{00}$ element of the 4D mass matrix (3.43) in the decoupling limit $m_1 \to \infty$ of the studied high-energy scenario, which allows to reproduce well the SM mass expression at the low-energy scales. For this purpose, one must have in particular a non-vanishing Yukawa coupling constant, i.e. $X \neq 0$ so that the BC 1 read as,

$$\underline{\text{BC 1 :}}\quad D_R|_L=0\,,\quad Q_L|_L=0\,,\quad Q_R|_L=0\,,\quad D_L|_L=0\,,\quad Q_R|_0=0\,,\quad D_L|_0=0\,,$$

BC at $y = L$ exactly similar to those in Eq. (3.81) which have been ruled out. Hence we exclude the BC 1. Let us move to the BC 2 which can be expressed in terms of the profiles, thanks to the relevant mixed KK decomposition (3.50), as follows (together with the condition $X' = 0$),

$$\underline{\text{BC 2 :}}\ \forall n\in\mathbb{N}\,,\quad \begin{cases} q_R^n(L)+X\,d_R^n(L)=0\,,\quad d_L^n(L)-X^*q_L^n(L)=0\,,\\ q_R^n(0)=0\,,\quad d_L^n(0)=0\,.\end{cases}$$

So these BC 2 at $y = 0$ applied on the solutions (3.86) produce the following profiles,

$$\forall n\in\mathbb{N}\,,\quad \begin{cases} (+\times):\ q_L^n(y)=A_q^n\,\cos(M_n\,y)\,,\quad (-\times):\ q_R^n(y)=-A_q^n\,\sin(M_n\,y)\,,\\ (-\times):\ d_L^n(y)=A_d^n\,\sin(M_n\,y)\,,\quad (+\times):\ d_R^n(y)=A_d^n\,\cos(M_n\,y)\,,\end{cases}\tag{3.89}$$

with the redefined normalization factors $A^n_{q,d}$ respect to Eq. (3.86),

$$A_q^n \mathrel{\hat{=}} -\,B_R^n\,,\quad A_d^n \mathrel{\hat{=}} D_L^n\,.$$

One must be careful to avoid some of the mathematical inconsistencies also encountered in the regularization procedures of Section 3.5: in particular, the existence of any profile jump at the interval boundaries which would induce an undefined derivative term in the 5D kinetic action in Eq. (3.3) [last two terms], an ill-defined term in the Yukawa couplings (3.14) – where the Dirac peak $\delta(y-L)$ would come in factor of a profile discontinuous at $y = L$ – and finally would conflict with the field continuity axiom of the invoked theory of variation calculus. Therefore, we are taking all the profiles continuous at both boundaries, which is the reason why we have applied the BC 2 at $y = 0$ on the bulk profiles (3.86). The application of the BC 2 at $y = L$ on the resulting bulk profiles (3.89) gives rise to the relations [using $M_n$, $B_R^n$, $D_L^n \neq 0, \forall n \in \mathbb{N}$, to be checked a posterior],

$$\forall n\in\mathbb{N},\ \tan(M_n\,L)=X\frac{A_d^n}{A_q^n}=X^*\frac{A_q^n}{A_d^n}\quad\Rightarrow\quad \tan^2(M_n\,L)=|X|^2\,,\tag{3.90}$$

which induces the mass spectrum as

$$\forall n\in\mathbb{N},\quad |M_n|=\left|\frac{\arctan(|X|)+(-1)^n\,\tilde{n}(n)\,\pi}{L}\right|\,,\tag{3.91}$$

using the $\tilde{n}(n)$ function already defined in Eq. (3.47). Note that the boundary relation in Eq. 3.90 shows $|A_q^n| = |A_d^n|$, which – combining with the ortho-normalization conditions of Eq. (3.52) ($n = m$) – give the solutions:

$$\forall n \in \mathbb{N}, \ \int_0^L dy\, |A_q^n|^2 \left[\sin^2(M_n\, y) + \cos^2(M_n\, y)\right] = L\,, \tag{3.92}$$

so that

$$A_q^n = e^{i\beta_q^n} \quad \text{and} \quad A_d^n = e^{i\alpha_0^n}\,,$$

with $\beta_q^n$, $\alpha_0^n \in \mathbb{R}$. Moreover, the boundary relations (3.90), together with the mass spectrum equation in Eq. (3.90), lead to the following two branches [45]:

$$I \ \ : \ \ \tan(M_n\, L) = |X| \quad \Rightarrow \quad A_q^n = e^{i(\alpha_0^n + \alpha_Y)}\,, \quad A_d^n = e^{i\alpha_0^n}\,, \tag{3.93}$$

$$II \ \ : \ \ \tan(M_n\, L) = -|X| \quad \Rightarrow \quad A_q^n = e^{i(\alpha_0^n + \alpha_Y \pm \pi)}\,, \quad A_d^n = e^{i\alpha_0^n}\,, \tag{3.94}$$

assuming that the generic phase $\alpha_Y$ of the 5D Yukawa coupling constant defined in Eq. (3.12). Analogy to the free case in Section 3.3.2, when one changes the sign of $M_n$, one goes from the spectrum of Eq. (3.93) to that of Eq. (3.94), which means to perform the transformation

$$\begin{cases} M_n \ \mapsto \ -M_n\,, \\ \psi/f_R^n \ \mapsto \ -\psi/f_R^n \quad \text{or} \quad \psi/f_L^n \mapsto -\psi/f_L^n\,, \quad f = q, d\,, \end{cases} \tag{3.95}$$

while the 4D Dirac equations (3.51), the KK wave functions equations (3.66), the ortho-normalization conditions (3.52) and the NBC (3.87) are invariant. One can conclude that the sign of $M_n$ is not physical and consider only the positive branch spectrum of Eq. (3.91). The phase $\alpha_0^n$ is not fixed yet. To see if it is physical, we perform the shift $\alpha_0^n \mapsto \alpha_0^n + \theta_n$, and check that the KK wave function equations (3.66) and the ortho-normalization conditions (3.52) are invariant, so one can take $\alpha_0^n = 0$ since it is not physical. What about $\alpha_Y$? By performing the shift $\alpha_Y \mapsto \alpha_Y + \theta$, we check also that the EOM (3.66) and the ortho-normalization conditions (3.52) are invariant so we can fix $\alpha_Y = 0$. For schematic purpose, in Figure 3.3, we give a plot of one possible set of the KK wave functions along the extra dimension with the real solution Eq. (3.89) ($\forall\, n \in \mathbb{N}, \alpha_Y = \alpha_0^n = 0$) [46].

Within the simplified case of a real 5D Yukawa coupling constant ($|X| = X$), we thus find that the unique tower (3.91) of absolute values of the physical fermion masses is matching the one obtained in the 4D approach of Eq. (3.46). This exact 4D-5D matching confirms the overall consistency of our calculations – without regularizations – and is of course expected to be reached as well for a complex 5D Yukawa coupling constant.

In particular, the insensitivity of the 4D fermion mass matrix (3.43) to the $Y_5'$ coupling constant [described below Eq. (3.44)] matches interestingly the spectrum independence on $Y_5'$ induced by the result $Y_5' = 0$ obtained in the BC 2 [see Eq. (3.88)] used for the 5D point of view.

Let us give an intuitive interpretation of the absence of rôle for the $Y_5'$ coupling (involved in $X'$) in the final spectrum (3.91) which depends only on the $X$ quantity. Starting with the free action $S_{\text{bulk}} + S_B$, the profiles $d_L^n(y)$ and $q_R^n(y)$ [$\forall n \in \mathbb{N}$], defined by Eq. (1.24)

45. It's just one pattern of splitting the whole mass spectrum.

46. $M_0 = \dfrac{1}{L} \arctan|X| > 0$, $M_1 = \dfrac{1}{L}\left[\arctan|X| - \pi\right] < 0$, $M_2 = \dfrac{1}{L}\left[\arctan|X| + \pi\right] > 0$.

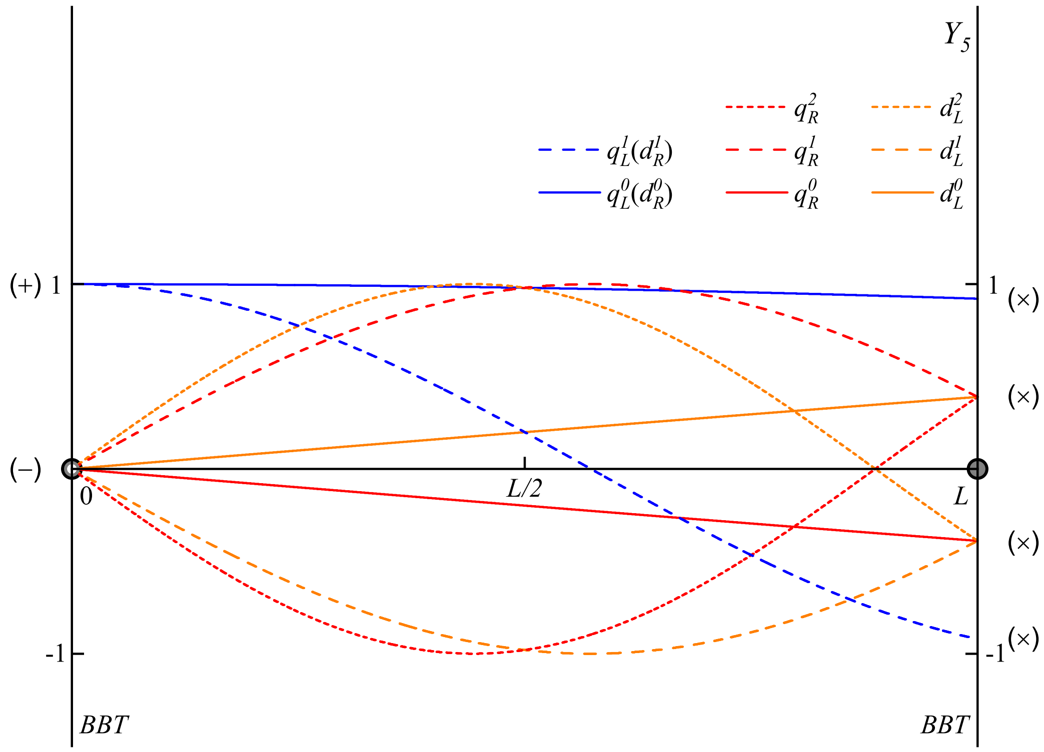


Figure 3.3 – Zero-mode and KK dimensionless wave functions $q^n_{L/R}(y)$, $d^n_{L/R}(y)$, with $n = 0, 1, 2$, along the interval domain, $y \in [0, L]$, corresponding to the brane Yukawa coupled solutions of Eq. (3.89) in the simplified case, $\forall\, n \in \mathbb{N}, \alpha_Y = \alpha_0^n = 0$ in Eq. (3.93). The two ending points at $y = 0, L$, the BBT, the $(\times)$ BC and Dirichlet/Neumann BC, $(-)/(+)$, are indicated on the graph.

and with solutions (3.28), vanish in particular at the boundary $y = L$. Hence the term with a $X'$ coefficient in the brane-localized Yukawa action piece $S_X$ of Eq. (3.14), once added to $S_{\text{bulk}} + S_B$, is expected to have a vanishing contribution. This argument is only intuitive as it does not include the possible 'back reaction' effect of the $X'$ term on the profiles via modified BC.

Then, let us review the condition for the fermion currents at the boundary $y = L$. The BC 2 of Eq. (3.88) exactly recover the tricky BC of Eq. (3.82), which means that the NBC (3.88) imply the EBC (3.79) [via the current vanishing condition] so that the geometrical field setup of the present model with matter stuck on an interval is well defined.

Finally, let us calculate, still without any kind of Higgs field regularization, the 4D effective Yukawa coupling constants between the mass eigenstates $\psi_L^n(x^\mu)$ and $\psi_R^m(x^\mu)$ as generated by inserting the mixed KK decomposition (3.50) into the Higgs interaction (3.16), based on the obtained profile solutions (3.89), (3.93)-(3.94):

$$S_{hQD} = \int d^4x \sum_{n,m=0}^{+\infty} \left( -y_{nm}\, h \psi_L^{n\dagger} \psi_R^m + \text{H.c.} \right), \tag{3.96}$$

thus

$$
\begin{aligned}
y_{nm} &\hat{=} \frac{Y_5}{\sqrt{2}L}\, q_L^{n*}(L)\, d_R^m(L) \\
&= \pm \frac{|Y_5|}{\sqrt{2}L}\, e^{i(\alpha_0^m - \alpha_0^n)} \cos(M_n\, L) \cos(M_m\, L) \\
&= \pm(-1)^{\tilde{n}(n)+\tilde{n}(m)}\, e^{i(\alpha_0^m - \alpha_0^n)}\, \frac{|Y_5|}{\sqrt{2}L(1+|X|^2)}\,, 
\end{aligned} \tag{3.97}
$$

where $\pm$ corresponds to the two branches in Eq. (3.93)-(3.94) respectively with respect to $q_L^n$ and we have used a trigonometric identity [47] to get the last equality. Note that the modulus of the 4D effective Yukawa couplings reads

$$
|y_{nm}| = \frac{|Y_5|}{\sqrt{2}L(1+|X|^2)}\,, \tag{3.98}
$$

which is independent of the KK mixing indexes $nm$.

In the decoupling limit of extremely heavy KK modes, $L \to 0$, we can then write the modulus of the lightest mode coupling constant [which recovers the SM content], using Eq. (3.13)-(3.15), as,

$$
|y_{00}| \underset{L\to 0}{\longrightarrow} \frac{|Y_5|}{\sqrt{2}L} = \frac{|y_4|}{\sqrt{2}}\,, \tag{3.99}
$$

and the absolute mass eigenvalue of the lightest eigenstates [from Eq. (3.91)] via Eq. (3.15) as,

$$
|M_0| \underset{L\to 0}{\longrightarrow} \frac{|X|}{L} = \frac{v|Y_5|}{\sqrt{2}L} \underset{L\to 0}{\longrightarrow} v\,|y_{00}|\,, \tag{3.100}
$$

so that the SM fermion setup – for the assumed single family – is recovered as expected from the decoupling condition.

In conclusion, adding the BBT at the brane with the Yukawa coupling to bulk fermions permits a consistent treatment of the considered scenario and a correct calculation of the mass spectrum.

## 3.7 Overview and Implications

### 3.7.1 The Action Content

In Table 3.1, we summarize the results for the obtained fermion BC at a single 3-brane. We conclude from this table that for fermions on an interval and coupled or not to a brane-localized Higgs field, either the BBT should be generated in the action or conditions should arise on the fermion current (forcing then the 4D treatment in case of a brane Yukawa coupling) depending on the origin of the model at high energies. In the same spirit, notice that the UV completion will determine whether the selection of fermion boundary conditions is imposed or deduced from the action form. The UV completion should not bring simultaneously the EBC (imposing vanishing currents) and the BBT (guaranteeing current vanishing) because it would be possible but redundant. It is interesting to observe anyway that the necessary additional fermionic ingredient, with respect to the kinetic terms, reveals that limiting the integration domain of the action does not suffice to define consistently and completely the basic field configuration

47. For $n \in \mathbb{Z}$, one has, $\cos(\theta + n\pi) = (-1)^n \cos(\theta)$, and for $T \in \mathbb{R}$, $\cos[\arctan(T)] = \frac{1}{\sqrt{1+T^2}}$.

along the interval (or, more generally, over a compactified space). Indeed, without having a vanishing fermion current at a boundary, one could imagine a source of creation or a mechanism of absorption for particles at the boundary. Therefore, the present status, resulting from this analysis and its synthesis, is that the action expression may not contain all the information (*e.g.* current conditions) needed to define a higher-dimensional model, regarding the geometrical setup and field configurations.

| | **No boundary characteristic** | **Vanishing current condition [EBC]** | **Bilinear brane terms [NBC]** |
|---|---|---|---|
| **4D approach** | (*Impossible*) | BC ($\pm$) | BC ($\pm$) |
| **5D approach** | (*Impossible*) | (*Impossible*) | BC ($\times$) |

Table 3.1 – Bulk fermion BC (when a consistent determination exists) at a 3-brane where the Higgs boson coupled to these fermions is located, for different boundary treatments: presence of the BBT, vanishing of the probability current or nothing specific. The 4D line holds as well for the 5D approach of the free brane. As usual, the Dirichlet BC is noted ($-$), the Neumann BC ($+$), and we denote ($\times$) the new BC depending on the Yukawa coupling constant.

Based on the above results, now we describe the generic methodology to find out the mass spectrum and KK wave functions (allowing to calculate the 4D effective couplings) along the extra spatial dimension(s) of a given scenario. For this purpose, we present in Figure 3.4 a schematic description of the main principles. The figure must be understood as follows. A given extra-dimensional model must be first defined by its geometrical setup [spacetime structure and field location configuration], its field content and its internal (gauge groups,. . . ) as well as other types (the Poincaré group here) of symmetries. These three types of information determine entirely the action form [48] whose minimization gives rise to the 5D EOM and the NBC. Besides the geometrical hypotheses of the model, concerning for instance, the space limits for field propagation may produce probability current conditions translating into the EBC [49] which must be combined with these NBC. At this level, a choice of the combined BC obtained can be required (if not determined automatically by the action structure itself). Then, the KK decomposition (together with the EOM on the 4D fields) allows us to derive the EOM and the BC for the KK profiles along the extra dimension(s). The last step is obviously to solve these profile EOM, coupled to the complete BC, in order to work out the mass spectra.

48. Within a well-understood scenario, all terms of the Lagrangian density should be deduced from the model definition exclusively: absence of couplings from symmetries, presence of the BBT from the geometrical setup, etc.

49. The EBC could also originate from the definition of the symmetry of orbifold scenarios (see Section 4.3.2).

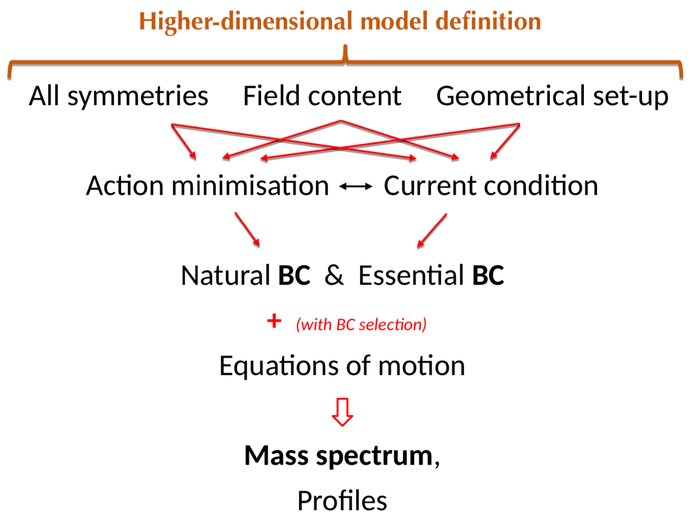


Figure 3.4 – Inverse pyramidal picture illustrating the general principles for determining the wave functions and masses of mixed KK modes within a given model based on extra dimension(s). Same notations as in the main text are used.

### 3.7.2 Implementation of the Cut-off on Energy

We have to discuss the cut-off treatment as the framework of higher-dimensional models is non-renormalizable theories that are valid in a limited domain of energy, up to a particular scale, set by perturbativity conditions on effective dimensionless couplings. If the UV completion of such models affects the KK excitation towers and thus the fermion mass spectrum in an unknown way, then its calculation must include the KK state masses only up to the cut-off value typically (the UV corrections at low-energies can be parametrized via higher-dimensional operators). In a case of the absence of (obvious) UV effects on the specific mass spectrum sector, the whole KK towers should be considered at the mass calculation level since even the smallest mass eigenvalues can be affected by the mixing effects of the infinite towers. Now in both situations, only the eigenstates with masses up to the cut-off scale should be considered for the phenomenological observables (reaction amplitudes, rates,...) due to the non-renormalizable nature of the theory. Technically, the implementation of an energy cut-off in the bulk fermion mass calculation and tree-level Lagrangian construction forces the use of the 4D approach. Indeed, the mixed KK decomposition (3.50), used in the 5D approach, includes the mixing of the whole tower: the fields $\psi^n_{L,R}(x^\mu)$ are mass eigenstates.

### 3.7.3 Phenomenological Impacts

In the appropriate treatment developed in the above sections, without regularization, the obtained mass spectrum and effective 4D Yukawa couplings depend on $Y_5$ but not on the $Y_5'$ coupling constant. For instance, in Eq. (3.97), one should in fact apply the result $Y_5' = 0$ as dictated by the relevant BC 2 in Eq. (3.88). Applying an energy cut-off in the process of mass calculation would not affect this independence on $Y_5'$, as is clear from the point of view of the 4D approach.

The results for fermion masses and profiles are also correct when one invokes the Higgs peak Regularization I, which cancels out the $Y_5'$ dependence. Hence, the phenomenological analyses of the literature based on such results are still valid: see for instance Ref. [76, 95, 100, 121, 147, 151–154]. Those studies apply to the geometrical background with warped extra dimensions where the KK spectrum independence on $Y_5'$ is expected to occur as well.

Note that the results from Regularization I and the correct ones in the approach without regularization at all, are precisely identical only by accident. Indeed in the Regularization I, considering first the 5D treatment, the mass spectrum calculation in the presence of Yukawa couplings suffers from two errors that exactly cancel out each other: there is no BBT, which affects the resulting spectrum equation by a factor 2 [as seen when comparing the spectra with the BBT in Eq. (3.91) and without the BBT in Eq. (3.73)], and a regularization is applied. Now starting from the 4D treatment of Regularization I and adding the BBT (or the current vanishing conditions) would have no effect on the 4D mass matrix [as described in Section 3.2.3] like avoiding the regularization process [as there is no analytical effect of Regularization I in which the limit $\epsilon \to 0$ is taken at the first step [96]]: the results of Regularization I are thus the same as in the correct approach.

In contrast, if the Higgs peak Regularization II is used, the obtained fermion masses and 4D Yukawa couplings depend on both $Y_5$ and $Y_5'$ so that the results differ effectively from the correct ones. Hence, the phenomenological studies based on these analytical results (for example Ref. [101, 102, 105–107]) should be reconsidered or redone.

For example, the effective 4D Yukawa couplings to fermions and their KK excitations affect the main Higgs production mechanism at the LHC: for instance, the gluon-gluon fusion via triangular loops of (KK) fermions. Hence, the effect of the realistic limit [96] of vanishing $Y_5'$ on the constraints on KK masses derived in the studies [102, 105–107], within the warped background and based on the Regularization II, should be calculated precisely.

Besides, the rotation matrices diagonalizing the 4D fermion mass matrix (3.43) do not diagonalize the effective 4D Yukawa coupling matrix simultaneously since the latter does not contain matrix elements made of the pure KK masses. Thus, $Y_5'$ would also contribute to the fermion mass-Yukawa shift via mass insertion approximation as shown diagrammatically in Figure 3.5 [101].

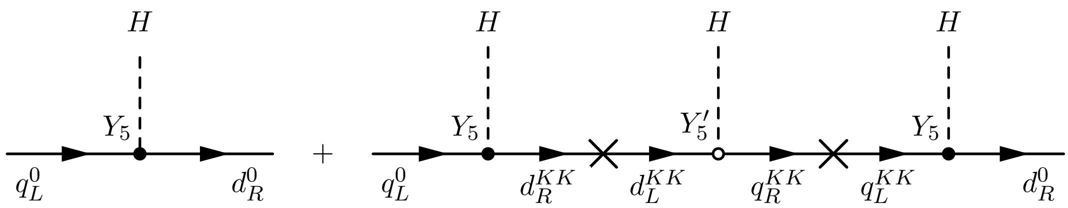


Figure 3.5 – Shift in masses and Yukawa couplings of SM fermions contributed by KK modes.

Particularly, if we extend to the case of three generations [50], we can see that the induced flavor violating 4D Yukawa couplings are contributed at leading order by $Y_5'$ contributions. Hence, there exist significant FCNC effects in measured $\Delta F = 2$ processes such as $\bar{K} - K$, $\bar{B} - B$ and $\bar{D} - D$ mixings, mainly produced by tree-level exchanges of the Higgs boson

50. Generalize the Lagrangian density to 3 replicas for 3 fermion generations.

via $Y_5'$ couplings, which lead to considerable lower bounds on the KK boson mass scale (in balance via opposite Yukawa coupling dependence with the ones from the tree-level contribution of the KK gluon exchange) found to be around $6-9$ TeV in the analysis [101] on warped extra dimensions using the Regularization II indeed. Hence, these bounds should be significantly suppressed in the realistic situation where $Y_5' \to 0$; this limit should undoubtedly be applied since the independence found in the above sections upon $Y_5'$ (extended via flavor indices) remains valid for the case of three flavors, as well as for fermion bulk masses, as is clear in the 4D approach where the $\beta_{ij}$-elements (3.44) of the mass matrix still vanish. The predictions in Ref [101], based on Regularization II, that FCNC reactions involving Yukawa couplings, like the rare top quark decay $t \to ch$ and exotic Higgs boson decay to charged leptons $h \to \mu\tau$ observable at the LHC, deserve reconsiderations as well when $Y_5' = 0$.

## 3.8 Summary & Conclusions

For bulk fermions coupled to a brane-Higgs boson, we have shown that the proper approach of the fermion masse spectrum and effective 4D Yukawa couplings does not rely on Higgs peak regularizations. The justifications are the following ones:

(i) There are no fermion wave function jumps at the Higgs boundary so that there's no motivation to introduce an arbitrary regularization;

(ii) The regularizations suffer from several mathematical discrepancies confirmed by two known non-commutativities of calculation steps;

(iii) The correct method without any regularization is validated in particular by the converging results of the 4D versus the 5D treatments.

In the rigorous methods developed for both free and brane-coupled bulk fermions, we have also pointed out the necessity to either include the BBT in the Lagrangian density, or alternatively impose vanishing conditions for probability currents at the interval boundaries. Here the arguments go as follows:

(i) The presence of the BBT guarantees the current vanishing conditions which define the geometrical field configuration of the model;

(ii) The BBT and the current conditions allow for finding physically consistent fermion masses, bulk profiles, and effective 4D Yukawa couplings (solutions fulfilling the ortho-normalization constraints, the Hermitian conjugate BC and the decoupling limit condition);

(iii) The BBT lead to an expected matching between the 4D and the 5D calculation results.

The BBT represent a possible origin of the chiral nature of the SM as well as of its chirality distribution among quark/lepton $SU(2)_L$ doublets and singlets. Those terms could thus provide new clues about the UV completion of the SM.

Depending on the UV completion, the general methodology worked out reveals that the information regarding the definition of a higher-dimensional model is not necessarily fully contained in the action itself – through the deduced EOM and the NBC – but might also be partly included in the EBC.

We have finished the analysis by the descriptions of the appropriate energy cut-off procedure in the present framework and of the phenomenological impacts of the new calculation method, which predicts the independence of the fermion masses and effective 4D Yukawa couplings on the $Y_5'$ parameter of the Lagrangian. This different coupling feature, with respect to the Regularization II usually applied in the literature, should in particular suppress significantly the previously obtained severe bounds on KK masses induced by FCNC processes generated via flavor violating couplings of the Higgs boson.

## 3.9 Unique BBT Factors

**This section is supplementary content to the published paper [1] in collaboration with Grégory MOREAU and Florian NORTIER.**

In Section 3.2.3, we have considered a specific BBT form in Eq. (3.4) with a given numerical factor, in this section, we will answer the question of the unicity of this BBT form. For this propose, we introduce a BBT with generic factors that we call GBBT:

$$S_{GB} = \int d^4x \left( \mathcal{L}_{GB}|_L - \mathcal{L}_{GB}|_0 \right) ,$$
$$\text{with } \mathcal{L}_{GB} = \sum_{F=Q,D} \frac{\mu^F(y)}{2} \bar{F}F = \sum_{F=Q,D} \frac{\mu^F(y)}{2} \left( F_L^\dagger F_R + F_R^\dagger F_L \right) , \tag{3.101}$$

where we impose the chiral decomposition (1.17) and $\mu^F(y)$ are generic parameters for the field $F$ $(F = Q, D)$ at $y$ and define compact notations

$$\mu^F_{0,L} \mathrel{\hat{=}} \mu^F \Big|_{0,L} . \tag{3.102}$$

Note that $\mu^F \in \mathbb{R}$ $(F = Q, D)$ is guaranteed since the Lagrangian density must be Hermitian.

### 3.9.1 5D Free Bulk Fermions with the GBBT

In this part, we discuss the unicity of the BBT, using the GBBT to realize the profile solutions (3.28) and the mass spectrum (3.30) presented in Section 3.3.3. We add the GBBT (3.101) to the kinetic terms (3.3) so that the initial free fermionic action becomes,

$$S_{\text{bulk}} + S_{GB} . \tag{3.103}$$

Let us apply the least action principle using this action as the starting point. The stationary action condition $\delta_{\bar{F}} \left( S_{\text{bulk}} + S_{GB} \right) = 0$ can be calculated,

$$\begin{aligned} \delta_{\bar{F}} \left( S_{\text{bulk}} + S_{GB} \right) &= \int d^4x \left\{ \int_0^L dy\, \delta\bar{F} \left[ i\Gamma^M \partial_M F \right] + \delta\bar{F} \left[ \frac{\mu^F - \gamma^5}{2} F \right] \Bigg|_0^L \right\} \\ &= \int d^4x \left\{ \int_0^L dy\, \delta\bar{F} \left[ i\Gamma^M \partial_M F \right] + \left[ \delta F_R^\dagger \frac{\mu^F + 1}{2} F_L + \delta F_L^\dagger \frac{\mu^F - 1}{2} F_R \right] \Bigg|_0^L \right\} , \end{aligned} \tag{3.104}$$

where the bulk variation would recover the bulk EOM in Eq. (1.29)-(1.30) for $F = Q, D$. The brane terms of Eq. (3.104) would vanish,

$$\frac{\mu_L^F + 1}{2} \delta F_R^\dagger F_L \Bigg|_L = \frac{\mu_0^F + 1}{2} \delta F_R^\dagger F_L \Bigg|_0 = \frac{\mu_L^F - 1}{2} \delta F_L^\dagger F_R \Bigg|_L = \frac{\mu_0^F - 1}{2} \delta F_L^\dagger F_R \Bigg|_0 = 0 , \tag{3.105}$$

and the new NBC would be obtained via non-vanishing boundary variations $\delta F^\dagger_{L/R}\Big|_{0,L} \neq 0$,

$$\frac{\mu^F_L+1}{2} F_L\bigg|_L = \frac{\mu^F_0+1}{2} F_L\bigg|_0 = \frac{\mu^F_L-1}{2} F_R\bigg|_L = \frac{\mu^F_0-1}{2} F_R\bigg|_0 = 0\,, \tag{3.106}$$

which would lead to the Dirichlet BC for $F_{L/R}$ simultaneously at the brane where $\mu^F \neq \pm 1$ and in turn to the vanishing of all fields on the whole interval $\mathcal{I} = [0, L]$ as described in Section 1.3.1.

Therefore, the GBBT different from the BBT are excluded by the wave function normalization. So, $\mu^F_{0,L} = \pm 1$ for the factor of the GBBT must be fixed to $S_B$ in Eq. (3.4) with $\mu^F_0 = \mu^F_L$ [the SM configuration] or to $S'_B$ in Eq. (3.7) with $\mu^F_0 = -\mu^F_L$ [custodians].

### 3.9.2 5D Approach: Introducing the GBBT

Analogy to the free case in Section 3.9.1, based on the introduction of the GBBT (3.101) at $y = 0, L$, we consider the fermion part of the action (3.17):

$$S_{\text{bulk}} + S_{GB} + S_X\,, \tag{3.107}$$

based on the kinetic Lagrangian density (3.3), the GBBT (3.101) and the Yukawa terms (3.14). The least action principle $\delta_{\bar F}\left(S_{\text{bulk}} + S_{GB} + S_X\right) = 0$ for each field $F = Q, D$,

$$\begin{aligned}\delta_{\bar Q}\left(S_{\text{bulk}} + S_{GB} + S_X\right) = \int d^4x \Bigg\{ &\int_0^L dy\, \delta\bar Q \left[ i\Gamma^M \partial_M Q \right] \\ &+ \left[ \delta Q^\dagger_R \left( \frac{\mu^Q_L+1}{2} Q_L - X' D_L \right) + \delta Q^\dagger_L \left( \frac{\mu^Q_L-1}{2} Q_R - X D_R \right) \right]\Bigg|_L \\ &- \left[ \delta Q^\dagger_R \frac{\mu^Q_0+1}{2} Q_L + \delta Q^\dagger_L \frac{\mu^Q_0-1}{2} Q_R \right]\Bigg|_0 \Bigg\}\,,\end{aligned} \tag{3.108}$$

$$\begin{aligned}\delta_{\bar D}\left(S_{\text{bulk}} + S_{GB} + S_X\right) = \int d^4x \Bigg\{ &\int_0^L dy\, \delta\bar D \left[ i\Gamma^M \partial_M D \right] \\ &+ \left[ \delta D^\dagger_R \left( \frac{\mu^D_L+1}{2} D_L - X^* Q_L \right) + \delta D^\dagger_L \left( \frac{\mu^D_L-1}{2} D_R - X'^* Q_R \right) \right]\Bigg|_L \\ &- \left[ \delta D^\dagger_R \frac{\mu^D_0+1}{2} D_L + \delta D^\dagger_L \frac{\mu^D_0-1}{2} D_R \right]\Bigg|_0 \Bigg\}\,,\end{aligned} \tag{3.109}$$

would lead to the bulk EOM in Eq. (1.29)-(1.30) for $F = Q, D$ and the vanishing of the brane terms reads,

$$\left\{\begin{aligned} \delta Q^\dagger_R \left( \frac{\mu^Q_L+1}{2} Q_L - X' D_L \right)\bigg|_L &= \delta Q^\dagger_L \left( \frac{\mu^Q_L-1}{2} Q_R - X D_R \right)\bigg|_L = 0\,, \\ \delta Q^\dagger_R \frac{\mu^Q_0+1}{2} Q_L\bigg|_0 &= \delta Q^\dagger_L \frac{\mu^Q_0-1}{2} Q_R\bigg|_0 = 0\,, \end{aligned}\right. \tag{3.110}$$

$$\left\{\begin{aligned} \delta D^\dagger_R \left( \frac{\mu^D_L+1}{2} D_L - X^* Q_L \right)\bigg|_L &= \delta D^\dagger_L \left( \frac{\mu^D_L-1}{2} D_R - X'^* Q_R \right)\bigg|_L = 0\,, \\ \delta D^\dagger_R \frac{\mu^D_0+1}{2} D_L\bigg|_0 &= \delta D^\dagger_L \frac{\mu^D_0-1}{2} D_R\bigg|_0 = 0\,, \end{aligned}\right. \tag{3.111}$$

which induce the NBC at $y = 0$,

$$\frac{\mu_0^Q + 1}{2} Q_L \bigg|_0 = \frac{\mu_0^Q - 1}{2} Q_R \bigg|_0 = \frac{\mu_0^D + 1}{2} D_L \bigg|_0 = \frac{\mu_0^D - 1}{2} D_R \bigg|_0 = 0\,, \tag{3.112}$$

where

$$\mu_0^D = -\mu_0^Q = 1\,, \tag{3.113}$$

must be fixed at $y = 0$ [the BBT (3.4) sign at $y = 0$] to realize the SM configuration via as the boundary conditions at $y = 0$ of Eq. (3.88) and avoid the vanishing of fields [see Section 3.9.1]. Note that the NBC at $y = L$ including the Yukawa couplings $X$ and $X'$,

$$\left\{\begin{array}{rcl} \left(\frac{\mu_L^Q - 1}{2} Q_R - X D_R\right)\Big|_L &=& 0, \\ \left(-X'^* Q_R + \frac{\mu_L^D - 1}{2} D_R\right)\Big|_L &=& 0, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \left(\frac{\mu_L^Q + 1}{2} Q_L - X' D_L\right)\Big|_L &=& 0, \\ \left(-X^* Q_L + \frac{\mu_L^D + 1}{2} D_L\right)\Big|_L &=& 0, \end{array}\right. \tag{3.114}$$

must provide non-zero boundary conditions for $Q_R$ and $D_L$ at $y = L$. Otherwise, combining with the Dirichlet BC for $Q_R$ and $D_L$ at $y = 0$, it would lead to a fermion mass spectrum independent of the Yukawa coupling $Y_5$, which is non-realistic and incompatible with the decoupling limit. Thus, the two determinants of the Eq. (3.114),

$$\left\{\begin{array}{lcl} \mathscr{D}_1 & \hat{=} & \begin{vmatrix} \frac{\mu_L^Q - 1}{2} & -X \\ -X'^* & \frac{\mu_L^D - 1}{2} \end{vmatrix} = \frac{\mu_L^Q - 1}{2} \frac{\mu_L^D - 1}{2} - X X'^* = 0\,, \\ \mathscr{D}_2 & \hat{=} & \begin{vmatrix} \frac{\mu_L^Q + 1}{2} & -X' \\ -X^* & \frac{\mu_L^D + 1}{2} \end{vmatrix} = \frac{\mu_L^Q + 1}{2} \frac{\mu_L^D + 1}{2} - X' X^* = 0\,, \end{array}\right. \tag{3.115}$$

must be zero [51], which – combining with $\mu^F = \mu^{F*}$ – lead to

$$\frac{\mu_L^Q - 1}{2} \frac{\mu_L^D - 1}{2} = \frac{\mu_L^Q + 1}{2} \frac{\mu_L^D + 1}{2}\,,$$

so that the relation of $\mu_L^Q$ and $\mu_L^D$ is fixed as,

$$\mu_L^Q = -\mu_L^D\,, \tag{3.116}$$

which – inserting to Eq. (3.115) – provides an explicit constraint for $X$ and $X'$ as,

$$4 X X'^* = 1 - \left(\mu_L^Q\right)^2\,, \tag{3.117}$$

and $X, X'$ have the same phase. Insert Eq. (3.115)-(3.116) to the NBC (3.114), one would obtain the reduced NBC only including the effective Yukawa coupling $\tilde{X}$ (3.119),

$$\left\{\begin{array}{rcl} \left(Q_R + \tilde{X} D_R\right)\Big|_L &=& 0, \\ \left(D_L - \tilde{X}^* Q_L\right)\Big|_L &=& 0, \end{array}\right. \tag{3.118}$$

51. It is necessary but not sufficient for $Q_R|_L\,,\ D_L|_L \neq 0$

where

$$\tilde{X} \hat{=} \frac{2}{1-\mu_L^Q} X \,, \ \tilde{Y_5} \hat{=} \frac{2}{1-\mu_L^Q} Y_5 \,. \tag{3.119}$$

So far so good, the NBC in Eq. (3.112)-(3.118) induced by the GBBT (3.101) recover the similar formula of that deduced by the BBT (3.4) in Eq. (3.88) [BC 2]. Here, we can directly apply the formula of the mass spectrum in Eq. (3.90) with the effective Yukawa coupling $\tilde{X}$ (3.119),

$$\tan^2(M_n\, L) = \left|\tilde{X}\right|^2 \,, \tag{3.120}$$

which induces the mass spectrum as

$$\forall n \in \mathbb{N}, \quad |M_n| = \left|\frac{\arctan(\left|\tilde{X}\right|) + (-1)^n\, \tilde{n}(n)\, \pi}{L}\right| \,, \tag{3.121}$$

using the $\tilde{n}(n)$ function already defined in Eq. (3.47). Then, we need to check the decoupling limit, i.e. the consistency to the SM mass-Yukawa relation in Eq. (3.100). The 4D effective Yukawa couplings $y_{nm}$ defined in Eq. (3.96) can be derived as [52],

$$\begin{aligned} y_{nm} &= \frac{Y_5}{\sqrt{2}L}\, q_L^{n*}(L)\, d_R^m(L) + \frac{Y_5'^*}{\sqrt{2}L}\, d_L^{n*}(L)\, q_R^m(L) \\ &= \left[\frac{Y_5}{\sqrt{2}L} - \frac{Y_5'^*}{\sqrt{2}L}\left(\tilde{X}\right)^2\right] q_L^{n*}(L)\, d_R^m(L) \\ &= \left(1 - \frac{1+\mu_L^Q}{1-\mu_L^Q}\right) \frac{Y_5}{\sqrt{2}L}\, q_L^{n*}(L)\, d_R^m(L) \\ &= \left(-\mu_L^Q\right) \times \frac{\tilde{Y_5}}{\sqrt{2}L}\, q_L^{n*}(L)\, d_R^m(L) \,, \end{aligned} \tag{3.122}$$

which leads to the modulus of the lightest mode coupling constant [which corresponds to the SM particle content], using Eq. (3.15)-(3.119), as,

$$|y_{00}| \underset{L\to 0}{\longrightarrow} \left|-\mu_L^Q\right| \frac{|\tilde{Y_5}|}{\sqrt{2}L} = \left|-\mu_L^Q\right| \times \frac{|\tilde{X}|}{v} \,, \tag{3.123}$$

and the absolute mass eigenvalue of the lightest eigenstates [from Eq. (3.121)] via Eq. (3.15) as,

$$|M_0| \underset{L\to 0}{\longrightarrow} \frac{|\tilde{X}|}{L} = \frac{v|\tilde{Y_5}|}{\sqrt{2}L} \underset{L\to 0}{\longrightarrow} \left|-\left.\frac{1}{\mu^Q}\right|_L\right| \times v\, |y_{00}| \,, \tag{3.124}$$

which is proportional to the the lightest mode coupling constant $|y_{00}|$ with an additional factor $\left|-\left.\frac{1}{\mu^Q}\right|_L\right|$ ($\left.\mu^Q\right|_L \neq 0$). Only the BBT case with $\left.\mu^Q\right|_L = 1$ allows to the SM proportionality between $|y_{00}|$ and $|M_0|$ as imposed by the decoupling criteria. Then, this conclusion is concluded by the fact that for the GBBT (different from the BBT $\left.\mu^Q\right|_L = 1$), Eq. (3.117) prevents us to apply the simultaneous limit where $X \to 0$ and $X' \to 0$. So, the decoupling criteria rules out the GBBT different from the BBT.

Another method to show the GBBT can't be different from the BBT is to use the 4D approach. Indeed, the 4D approach replies on the free profiles, which have no consistent solutions in the GBBT different from the BBT.

52. If we set $\mu_L^Q = \mu_L^D = 0$ at $y = L$ and Eq. (3.113) at $y = 0$, we can recover the results through the naive approach in Section 3.6.1 and the vanishing $y_{00}$ in Eq. (3.75).

Finally, we add a complementary remark regarding why $X'$ has no impact. In the case of the GBBT, we find that $Y_5'$ has no impact, which is a generalization of what we found in the BBT case. There are two demonstrations for that. First, the new BC in Eq. 3.118-(3.119) doesn't include $X'$. Neither do the bulk EOM. This is also confirmed by the $X'$ disappearing in the action. Inserting the NBC (3.118) to $S_X + S_{GB}$ in Eq. (3.107) and replace $Q_R|_L$, $D_L|_L$ with $D_R|_L$, $Q_L|_L$ respectively, one can obtain the effective Yukawa sector using Eq. (3.116),

$$S_X + S_{GB} = \int d^4x \left\{ -\tilde{X} Q_L^\dagger D_R \Big|_L + \text{H.c.} \right\} , \tag{3.125}$$

where

$$\begin{aligned} S_X &= \int d^4x \left\{ \mu_L^Q \tilde{X} Q_L^\dagger D_R \Big|_L + \text{H.c.} \right\} , \\ S_{GB} &= \int d^4x \left\{ - \left( \mu_L^Q + 1 \right) \tilde{X} Q_L^\dagger D_R \Big|_L + \text{H.c.} \right\} . \end{aligned}$$

# Chapter 4

# Rigorous Treatment of the $S^1/\mathbb{Z}_2$ Orbifold and Profile Jumps

This chapter is a personal adaptation of Ref. [2] written in collaboration with Grégory MOREAU and Florian NORTIER.

## 4.1 Introduction

An orbifold $\mathcal{O}$ being defined as an extra compact manifold $\mathcal{C}$ with so-called fixed points where the introduced spatial transformation (element from a discrete group $G$) – letting the Lagrangian invariant – is just equivalent to the identity. It is noted as $\mathcal{O} = \mathcal{C}/G$ and possesses thus singularities, not like a smooth manifold [155, 156]. One can for instance wind an infinite [non-compact] 1D real axis $\mathbb{R}^1$ around a compact $\mathcal{S}^1$ circle and define winding numbers, which indicates a spatial compactification of the Lagrangian density on $\mathbb{R}^1$ as

$$\mathcal{L}(y + n \times 2\pi R) = \mathcal{L}(y)\,,$$

with winding numbers $n \in \mathbb{Z}$. Note that one must distinguish a compactification of the Lagrangian density on $\mathbb{R}^1$ from a discrete symmetry on a compact space [*e.g.* the $\mathcal{S}^1$ circle].

In this section, we will study the original version [58] of the warped dimension scenario based on the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold [157, 158] where the extra space is compactified on a circle respecting a spatial parity of the Lagrangian density. Focusing our attention on the subtle bulk fermion interactions with the brane-Higgs field localized at a fixed point, we will analyze the toy model with a flat extra dimension and minimal field content: the results obtained on the fermion-Higgs coupling structure are directly applicable to the realistic warped model.

We will clarify the treatment of the bulk fermion couplings to the brane-localized Higgs boson, within the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold background, by building rigorously the four-dimensional [1] effective Lagrangian of the minimal model, that is by calculating consistently the Kaluza-Klein tower spectrum of fermion mass eigenvalues and the 4D effective Yukawa couplings (via the fermion wave functions along the extra dimension).

In particular, following the same methodology of the finite interval scenario in Chapter 3, we will demonstrate that no brane-Higgs regularization [like smoothing the Higgs Dirac peak] should be applied (not necessary and no theoretical argument for it) in contrast with the usual regularization procedure of literature (see Ref. [1] and references therein)

1. Including time.

and that, instead, one must introduce either Essential Boundary Conditions on 5D fields [2], originating from the $\mathbb{Z}_2$ symmetry, or equivalently some BBT in the fundamental 5D Lagrangian density. The exact matching of the fermion mass spectra derived respectively through the 4D and the 5D treatments will be used in order to confirm our analytical results. All those statements (except the 4D approach) hold as well for the free case i.e. without the Yukawa interactions.

This necessity of the presence of the EBC or the BBT (terms with the same form as in Chapter 3 [see Ref. [1]], in the 4D or the 5D approach, has been found as well in Section 3.2.3 [see Ref. [1]] in the finite interval scenario (the higher-dimensional framework of the other warped model version) with identical brane-Higgs couplings to bulk fermions: this conclusion confirms that a specific treatment is required for point-like interactions between bulk fermions and brane-Higgs bosons in higher-dimensional spaces.

Besides, we will strictly describe and work out the entire known duality: identical physical quantities, namely the mass eigenvalues and 4D effective Yukawa couplings, are obtained in the different $\mathcal{S}^1/\mathbb{Z}_2$ orbifold scenario with the Higgs boson localized at a fixed point and finite interval geometrical setup with the Higgs field stuck at a boundary.

The choice of The EBC and the BBT (forms including signs), which should originate from the UV completion of the theory, turns out to induce the chiral nature of the low-energy effective theory as well as realizing the specific SM fermion chiralities. Indeed, all these chirality properties are in fact not selected by the remaining sign choices for the 5D fields transformed via the spatial $\mathbb{Z}_2$ group – as the solutions we find within this orbifold configuration can exhibit twist transformations (sign modification here) of the 5D fields, à la Scherk-Schwarz [159, 160], through the extra space reflection. We will even show that the transformation sign choices are just mathematical conventions without physical impact on the SM field chiralities, the fermion mass spectrum and the 4D effective Yukawa couplings.

Nevertheless, in order to clarify the chirality aspects, we will also study a different scenario – considered for example in Ref. [161] – where the $\mathbb{Z}_2$ transformation definitions on the fields cover as well the fixed points themselves. It turns out that the associated transformation sign choices precisely at these fixed points constitute here additional EBC, noted as EBC', that have the capacity to select some of the previous EBC and hence to fix the chirality setup. Once more, the role of these EBC' can be played instead by certain of the above BBT. Interestingly, such an inclusive $\mathbb{Z}_2$ symmetry definition can induce by itself the chiral nature of the theory as well as the SM chirality distribution over the various fields. This origin for the whole chirality configuration is not offered within the simpler interval model for instance. In the presence of brane-localized Yukawa couplings, such an inclusive $\mathbb{Z}_2$ scenario can only be treated through the 4D treatment. The fermion masses and couplings are also affected by this inclusive $\mathbb{Z}_2$ symmetry.

The action integral definition and integral domain end-points will be treated carefully. In particular the decomposition of the action to introduce improper integrals will appear to be required in the presence of orbifold fixed points or point-like fermion-boson interactions (not located at the boundary of a finite extra space like an interval). Within this new and appropriate approach of the specific points along the extra dimension of the orbifold, we find in the free or Yukawa case that some of the obtained consistent solutions exhibit certain field jumps at these fixed points and localized-interaction point. This interesting result of the possible existence of consistent profile jumps stands against one's first intuition [97, 101], but those jumps are only induced by sign flipping and not by point-like

2. Directly imposed by the model definition, in contrast with the Natural Boundary Conditions deduced from the least action principle.

changes of the absolute value of the wave function amplitudes.

The analysis of the present orbifold background with brane-localized fermion-scalar interactions, as well as the previous results on the interval background in Chapter 3, show that generally speaking the action expression does not systematically contain all the information allowing to fully define the model: in particular some EBC may be used (in contrast to the BBT, which are terms in the action) depending on the brane treatment adopted or on the UV completion of the theory (which could introduce the BBT).

## 4.2 Minimal $\mathcal{S}^1/\mathbb{Z}_2$ Consistent Model

### 4.2.1 Geometry and Symmetries: the Proper Action

We consider a 5D spacetime model with the factorizable geometry $\mathcal{M}^4 \times \mathcal{S}^1/\mathbb{Z}_2$ described just below:

— $\mathcal{M}^4$ represents the usual 4D Minkowski spacetime manifold whose coordinates are noted as $x^\mu$ where $\mu \in [\![0,3]\!]$ is the Lorentz index of the covariant formalism. The metric conventions are given in Appendix A.

— $\mathcal{S}^1/\mathbb{Z}_2$ stands for the extra space orbifold obtained from modding out the circle $\mathcal{S}^1$ by the discrete group[3] symmetry $\mathbb{Z}_2$.
This circle $\mathcal{S}^1$ is characterized by a radius $R$ and its coordinate is $y \in (-\pi R, \pi R]$, not double-counting the point $y = \pi R$ since it is this point, by pure convention, which is chosen to be the junction point geometrically identified with the point $y = -\pi R$ (which we note: $-\pi R \equiv \pi R$) in order to implement the circle periodicity. The circle could be constructed from the real axis by imposing a periodicity, that is by identifying geometrically an infinite number of translated regions of size $2\pi R$ and hence by limiting the 1D space to the fundamental domain $(-\pi R, \pi R]$ in Figure 4.1.

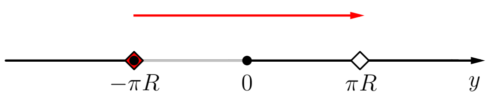


Figure 4.1 – Translations (solid red arrows) as 1D space group generator. Fundamental domain of the orbifold (thick gray line). Two end points (red and white diamond) are identical. Orbifold fixed points (black dots).

The $\mathbb{Z}_2$ symmetry on space, $y \to -y$[4], has a representation on a generic 5D field,

$$\forall y \in (-\pi R, 0) \cup (0, \pi R)\,,\ \Phi(x^\mu, -y) = \mathcal{T}\Phi(x^\mu, y)\,, \quad \text{with}\, \mathcal{T}^2 = \mathbb{1}\,, \tag{4.1}$$

which must let the Lagrangian density invariant, by the definition of a symmetry:

$$\forall y \in (-\pi R, 0) \cup (0, \pi R)\,,\ \mathcal{L}\left[\Phi(x^\mu, -y)\right] = \mathcal{L}\left[\Phi(x^\mu, y)\right]\,. \tag{4.2}$$

We mention that this equation can define an equivalence class with respect to a given coordinate $y_0$, defined as $[y_0] = \{y \in \mathcal{S}^1 \,|\, y \sim y_0\}$ with $y \sim \pm y$, as illustrated schematically in Figure 4.2. Note that two fixed points arise: $(y = 0) \to (-0 = 0)$

3. Factor element, $e^{\pm i \frac{2\pi}{2}} = -1$, and the identity element, 1.
4. The convention above, of having taken the coordinate origin at the strict middle of the circle domain (or fundamental domain), renders the $\mathbb{Z}_2$ parity with respect to the origin more explicit and convenient to study.

and $(y = \pi R) \to (-\pi R \equiv \pi R)$. At these fixed points, the Lagrangian condition of Eq. (4.2) is automatically satisfied:

$$\begin{cases} \mathcal{L}\left[\Phi(x^\mu, -0(\pi R))\right] &= \mathcal{L}\left[\Phi(x^\mu, 0(\pi R))\right], \\ \Phi(x^\mu, -0(\pi R)) &= \Phi(x^\mu, 0(\pi R)), \end{cases} \tag{4.3}$$

since no transformation needs to apply on the fields there [two fixed points are eliminated in the $\mathbb{Z}_2$ symmetry representation of Eq. (4.1)]. In contrast, another scenario will be analyzed in Section 4.6.

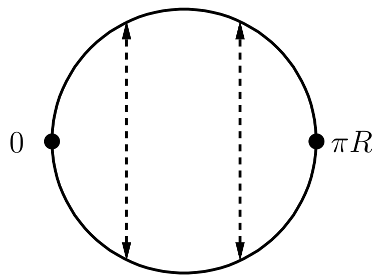


Figure 4.2 – $\mathcal{S}^1/\mathbb{Z}_2$ orbifold picture. The fixed points at $y = 0$ and $y = \pi R$ are indicated by the two black points. The two examples of pairs of points with opposite coordinates, respectively indicated by the double dashed arrows, correspond to an identical Lagrangian density (for each pair).

In order to properly write down the initial action, we urge the importance of taking care of possible field jumps along the extra dimension upon the reader. We are going to show that the existence of a field jump in field theory can make sense mathematically if the action integration domain is properly divided at the jump location. Different discontinuity configurations must be considered. First, the hypothesis of a possible jump at any point of the bulk would lead to an infinite number of cuts in the action integration region which would obviously not be treatable leading to unpredictable observables: this assumption is thus excluded. Secondly, assuming an arbitrary finite number of possible jumps and hence of mathematical separations in the action domain, outside the fixed points, is not expected to affect the unique physical results – like the fermion mass spectrum – since none of those jump points exhibit some specific property: it is thus useless to explore this direction. Thirdly, the case of possible profile jumps at the two specific points that are the fixed points of the orbifold – one of those two, $y = \pi R$, corresponding as well to the Yukawa coupling location (see Section 4.2.4) – remains to be studied. The effective presence of such profile jumps in some of the obtained solutions (see Figures 4.3 and 4.4 respectively for the free and coupled fermion situations) confirms this possibility. For example in the case of a profile jump at $y = 0$ (an identical discussion holds for the other fixed point at $y = \pi R$), regarding a well-defined integral of the Lagrangian density involving 5D fields over the whole action domain, we simply have to choose between the mathematical definitions of the left or right continuity for a generic profile function along the extra dimension:

$$f(0) = f(0^-) \hat{=} \lim_{\epsilon \to 0} f(0 - \epsilon)\,, \quad \text{with}\, \epsilon > 0\,,$$

or

$$f(0) = f(0^+) \hat{=} \lim_{\epsilon \to 0} f(0+\epsilon)\,, \quad \text{with}\, \epsilon > 0\,.$$

This choice is conventional and hence cannot affect numerical results, so let us choose conveniently

$$\begin{cases} f(0) &= f(0^+), \\ f(\pi R) &= f(\pi R^-), \end{cases} \tag{4.4}$$

throughout this chapter, in case of jumps at the fixed points. Then, the well-defined global action of this model must be written as a sum of bulk terms and some brane terms,

$$S_{5\mathrm{D}} = S_{\mathrm{bulk}} + S_{\mathrm{branes}}\,, \tag{4.5}$$

which keeps the same general formula as the interval scenarion in Eq. (3.17). In this chapter, we only consider bulk massless fermions, so the bulk terms only consist of kinetic terms as

$$S_{\mathrm{bulk}} = \int d^4x \left( \int_{-\pi R^+}^{0^-} dy\ \mathcal{L}_{\mathrm{kin}} + \int_0^{\pi R} dy\ \mathcal{L}_{\mathrm{kin}} \right)\,, \tag{4.6}$$

which contains an improper integral [in contrast to the simple interval case in Eq. (1.15)] due to the discontinuity argument above,

$$\int_{-\pi R^+}^{0^-} dy\ \mathcal{L}_{\mathrm{kin}} \hat{=} \lim_{a\to 0^-,\, b\to -\pi R^+} \int_b^a dy\ \mathcal{L}_{\mathrm{kin}} = \lim_{\epsilon\to 0} \int_{-\pi R+\epsilon}^{0-\epsilon} dy\ \mathcal{L}_{\mathrm{kin}}\,, \quad \text{with}\, \epsilon > 0\,, \tag{4.7}$$

and a standard integration over different regions covering the whole physical domain of the circle. $S_{\mathrm{branes}}$ represents action terms located at the orbifold fixed points, e.g. the brane-localized BBT (4.18) and the Yukawa interactions (4.14) at the brane $y = \pi R$.

Indeed, all the considered fields must be well-defined at the two fixed points via Eq. (4.4). Besides, the Lagrangian density of kinetic terms $\mathcal{L}_{\mathrm{kin}}$ (4.6) constructed by profile derivatives $f'(y)$, should be integrable over the entire region $y \in [0, \pi R] \cup [-\pi R^+, 0^-]$ [5], and profile derivatives $f'(y)$ must be well-defined over the two regions $[-\pi R^+, 0^-]$ and $[0, \pi R]$ (see Sections 4.3.2 and 4.5.3 respectively for the free and coupled fermion situations), which is compatible with Eq. (4.4). For example, $f(y)$ is derivable in the region $[0, \pi R]$ at $y = 0$ if and only if $f(y)$ is right-derivable at $y = 0$, and the corresponding right-derivative does not diverge thanks to the first equality of Eq. (4.4).

Notice that from the point of view of the integration by pieces of the action in Eq. (4.6) precisely over the physical domain, the inclusion (or not) of the single points at $y = 0$ or $y = \pi R \equiv -\pi R$ does not affect the integral results – given the continuous form of the even $\mathcal{L}_{\mathrm{kin}}$ over the two regions [see Eq.(4.2)] – so that only consistent action definition argues were considered here.

Finally, the Lagrangian densities of the whole expression (4.5) will respect the $\mathbb{Z}_2$ symmetry since the bulk kinetic terms $\mathcal{L}_{\mathrm{kin}}$ (4.6) will fulfill the condition (4.2) and the brane action $S_{\mathrm{branes}}$ will exclusively involve Lagrangian terms taken at fixed points.

5. To be clear, the integration domain $[-\pi R^+, 0^-]$ corresponds to the spatial region along the extra dimension $]-\pi R, 0[ \Leftrightarrow (-\pi R, 0)$ – respectively the Francophone and Anglophone notations – which does not include the fixed points at $y = 0$ and $y = -\pi R$.

### 4.2.2 Bulk Fermion Fields

Analogy to the interval case in Section 3.2.2, let us introduce the minimal spin-1/2 field content which allows to write down a SM Yukawa-like coupling between zero mode fermions (of different chiralities) and a spin-0 field (see Section 4.2.4). It is constituted by a pair of fermion fields called $Q$ and $D$. Those particles propagate along the circle $\mathcal{S}^1$, as we have in mind an extension of this toy model to a realistic scenario with bulk matter (cf. Section 4.2.5) where $Q, D$ will represent respectively the $\mathrm{SU}(2)_\mathrm{L}$ gauge doublet down-component quark and the singlet down-quark.

The 5D spinor fields $Q(x^\mu, y)$ and $D(x^\mu, y)$ – of mass dimension 2 – have the following kinetic terms [entering Eq. (4.6)] which allow to recover canonical covariant kinetic terms for the associated fermions in the 4D effective action (as imposed by the argument of decoupling limit):

$$\mathcal{L}_\mathrm{kin} = \sum_{F=Q,D} \frac{i}{2} \bar{F} \Gamma^M \overleftrightarrow{\partial_M} F \,, \tag{4.8}$$

which keeps the identical formula as the interval case of Eq. (3.1) [chiral formula of Eq. (3.3) via the chiral decomposition (1.17) ($F = Q, D$)] but with the discontinuity argument (see Section 4.2.1) on the domain $x^M \in \mathcal{M}^4 \times \mathcal{S}^1/\mathbb{Z}_2$.

As stated at the start of Section 4.2.1, the bulk Lagrangian density $\mathcal{L}_\mathrm{kin}$ must obey the $\mathbb{Z}_2$ symmetry condition (4.2). For this purpose, the $\mathbb{Z}_2$ symmetry representation (4.1) on the 5D Dirac spinor fields $Q(x^\mu, y)$ and $D(x^\mu, y)$ can take four different forms which constitute Essential Conditions issued from the model definition:

$$\text{Type I} \begin{cases} Q\,(x^\mu, -y) &=& -\gamma^5\, Q\,(x^\mu, y) \implies Q_L \text{ even},\ Q_R \text{ odd}, \\ D\,(x^\mu, -y) &=& \gamma^5\, D\,(x^\mu, y) \implies D_L \text{ odd},\ D_R \text{ even}, \end{cases} \tag{4.9}$$

$$\text{Type II} \begin{cases} Q\,(x^\mu, -y) &=& \gamma^5\, Q\,(x^\mu, y) \implies Q_L \text{ odd},\ Q_R \text{ even}, \\ D\,(x^\mu, -y) &=& -\gamma^5\, D\,(x^\mu, y) \implies D_L \text{ even},\ D_R \text{ odd}, \end{cases} \tag{4.10}$$

$$\text{Type III} \begin{cases} Q\,(x^\mu, -y) &=& -\gamma^5\, Q\,(x^\mu, y) \implies Q_L \text{ even},\ Q_R \text{ odd}, \\ D\,(x^\mu, -y) &=& -\gamma^5\, D\,(x^\mu, y) \implies D_L \text{ even},\ D_R \text{ odd}, \end{cases} \tag{4.11}$$

$$\text{Type IV} \begin{cases} Q\,(x^\mu, -y) &=& \gamma^5\, Q\,(x^\mu, y) \implies Q_L \text{ odd},\ Q_R \text{ even}, \\ D\,(x^\mu, -y) &=& \gamma^5\, D\,(x^\mu, y) \implies D_L \text{ odd},\ D_R \text{ even}, \end{cases} \tag{4.12}$$

under which the kinetic Lagrangian density (4.8) is indeed invariant, as appears by using the properties of the $\gamma_5$ Dirac matrix and the odd parity of the fifth partial derivative $\partial_4$ [6]. Based on the chiral formula of Eq. (3.3), it's more convenient to see at a glance the even parity of the kinetic Lagrangian density (4.8), simply by using the occurrence of fixed 5D field parities, different for the Left/Right chiralities [cf. Eq. (4.9)-(4.12)], and the $\partial_4$ odd parity.

6. The parity of the derivative can be derived explicitly as,

$$\begin{aligned} F|_y &= \pm\gamma^5 F\big|_{-y} \,, \\ \partial_4 F|_y &= \lim_{\epsilon\to 0} \frac{F(y+\epsilon) - F(y)}{\epsilon} = \lim_{\epsilon\to 0} \pm\gamma^5 \frac{F(-y-\epsilon) - F(-y)}{\epsilon} = \mp\gamma^5 \lim_{\epsilon\to 0} \frac{F(-y) - F(-y-\epsilon)}{\epsilon} \\ &= \mp\gamma^5 \partial_4 F\big|_{-y} = \mp\gamma^5 \partial_4 F\big|_{-y} = (-\partial_4)\left(\pm\gamma^5 F\right)\big|_{-y} \,. \end{aligned}$$

Notice that the $\mathbb{Z}_2$ parity (second order cyclic group) does not allow for complex phase factors in the transformations:

$$F|_y = e^{i\theta_F}\gamma^5 F|_{-y} = e^{i\theta_F}\gamma^5 e^{i\theta_F}\gamma^5 F|_y = \left(e^{i\theta_F}\right)^2 F|_y \,,$$

which indicates $e^{i\theta_F} = \pm 1$ as in Eq. (4.9)-(4.12).

### 4.2.3 Brane-Localized Scalar Field

The questions about the mass calculation arise when the bulk fermions couple to a single 4D real scalar field $H$ (mass dimension 1) which is confined at a fixed point of the orbifold, as in the studied interval model in Section 3.2.4 (inspired by the warped scenario addressing the gauge hierarchy problem). We simply choose this fixed point to be at $y = \pi R$, rather than $y = 0$, which is a purely mathematical convention since these two points belong to a circle. The scalar field has an action of the generic form,

$$S_H = \int d^4x \; \mathcal{L}_H \,, \quad \text{with} \quad \mathcal{L}_H = \frac{1}{2}\,\partial_\mu H \partial^\mu H - V(H)\,, \tag{4.13}$$

with a potential $V(H)$ possessing a minimum which generates a non-vanishing VEV for the field $H$ expanded as in Eq. (3.10). Note that it has an identical formula in Eq. (3.9) but localizes at the fixed point $y = \pi R$ of the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold rather than that at the end of an interval.

### 4.2.4 Yukawa Interactions

We consider the following Yukawa interactions allowing to study the subtleties induced by the coupling of the above brane-scalar field (at $y = \pi R$) to the introduced bulk fermions,

$$S_Y = \int d^4x \;\, \mathcal{L}_Y|_{\pi R} \;, \text{with} \;\; \mathcal{L}_Y = -Y_5 \; H Q_L^\dagger D_R - Y_5' \; H Q_R^\dagger D_L + \text{H.c.}\,, \tag{4.14}$$

where the complex phases $\alpha_{Y^{(\prime)}}$ of the two independent Yukawa couplings $Y_5^{(\prime)}$ at the 3-brane $y = \pi R$ are defined in Eq. (3.12). Notice that considering operators involving the fields $H$, $Q$, $D$ up to dimension 5 allows to include such a Yukawa coupling as indicated in Section 3.2.5 for the interval case. Let us recall here that in case of profile jumps at the fixed point at $y = \pi R$, the 5D fields $Q_{L/R}(x^\mu, \pi R)$, $D_{L/R}(x^\mu, \pi R)$ are defined through the profile convention (4.4), as already described. The studied model with a Yukawa coupling at a fixed point will turn out to be dual to the interval model including a Yukawa coupling at a boundary (see Section 4.8).

To avoid the introduction of a new energy scale, in the spirit of the warped model, we can define the 5D Yukawa coupling constants as

$$Y_5 = y_4 \times 2\pi R \;\; \text{and} \;\; Y_5' = y_4' \times 2\pi R\,, \tag{4.15}$$

where $y_4, y_4'$ are dimensionless coupling constants of $\mathcal{O}(1)$. Then, $y_4$ can be approximately identified with the SM Yukawa coupling constant within the decoupling limit [same motivation of Eq. (3.13)], as will be described in Eq. (4.75)-(4.76).

When calculating the tower of excited fermion masses, we restrict our considerations to the VEV of $H$ via Eq. (3.10) and concentrate our attention on the following part of the action (4.14),

$$S_X = \int d^4x \;\, \mathcal{L}_X|_{\pi R} \;, \text{with} \;\; \mathcal{L}_X = -X Q_L^\dagger D_R - X' Q_R^\dagger D_L + \text{H.c.}\,, \tag{4.16}$$

with the effective couplings $X, X'$ defined in Eq. (3.15). Based on Eq. (3.10), the complete action reads as, $S_Y = S_X + S_{hQD}$, with the localized fermion-scalar interaction terms:

$$S_{hQD} = \int d^4x \; \mathcal{L}_{hQD}|_{\pi R} \, , \text{with} \;\; \mathcal{L}_{hQD} = -\frac{Y_5}{\sqrt{2}} \, hQ_L^\dagger D_R - \frac{Y_5'}{\sqrt{2}} \, hQ_R^\dagger D_L + \text{H.c.} \, , \quad (4.17)$$

that allow to work out the 4D effective Yukawa coupling constants.

### 4.2.5 Bilinear Brane Terms

Introducing all the covariant operators up to mass dimension 5 [like for the Yukawa couplings (4.14)] in this model, one should consider as well the dimension 4 operators given just below, that we call the BBT like in Ref. [1]. Furthermore, the presence of the BBT has several justifications:

(i) They allow to avoid physical consistency problems both in the free case (see Sections 4.3.1 and 4.3.3) and with Yukawa couplings (Sections 4.5.1 and 4.5.3).

(ii) They play the role of defining well the model at the two orbifold fixed points [particularly to treat the (possible) discontinuity] both in the free case (see Sections 4.3.2 and 4.3.3) and with Yukawa couplings (Section 4.5.3).

(iii) They induce the expected matching of the analytical results on the spectrum derived through the 4D and 5D approaches (see Sections 4.4 and 4.5.3).

The following BBT lead to the SM chirality configuration,

$$\begin{aligned} &S_B = \int d^4x \left( \mathcal{L}_B|_{\pi R} - \mathcal{L}_B|_0 \right) \, , \\ &\text{with} \;\; \mathcal{L}_B = \sum_{F=Q,D} \sigma^F(y) \, \bar{F}F = \sum_{F=Q,D} \sigma^F(y) \left( F_L^\dagger F_R + F_R^\dagger F_L \right) \, , \end{aligned} \quad (4.18)$$

where we impose the chiral decomposition (1.17) and $\sigma^F(y)$ are generic parameters for the field $F$ $(F = Q, D)$ at $y$ and

$$\sigma^Q_{0,\pi R} = -\sigma^D_{0,\pi R} = -1 \, , \quad (4.19)$$

using compact notations

$$\sigma^F_{0,\pi R} \mathrel{\hat{=}} \sigma^F \Big|_{0,\pi R} \, . \quad (4.20)$$

Indeed, without Yukawa couplings, these terms will induce only a non-vanishing profile $q_L^0(y)$ [see line 2 of Eq. (4.41) and Table 4.2 in case of the zero-mode with mass $m_0 = 0$] in the 5D field $Q_L(x^\mu, y)$ so that only the Left-handed 4D field $Q_L^0(x^\mu)$ will exist. This zero-mode $Q_L^0(x^\mu)$, without KK mass contribution, constitutes the lightest mode of the KK tower and also the SM state. Hence, we can well recover the SM configuration: a chiral field content and a Left-handed 4D field potentially representing the $\text{SU}(2)_\text{L}$ quark doublet in the direct extension to gauge symmetries (and three flavours). Given that, similarly, the BBT (4.18) will exclusively lead to a Right-handed 4D field $D_R^0(x^\mu)$ [line 1 of Eq. (4.41)] potentially representing the SM down quark type (gauge singlet). When adding the Yukawa couplings (4.14), this SM chirality setup remains although it is no more explicit due to the $Q^n(x^\mu)$-$D^n(x^\mu)$ mixing, via vector-like KK state mixings, which induces some vector-like mass eigenstates $\psi^0_{L/R}(x^\mu)$ for the lightest modes of the tower (see Sections 4.4 and 4.5.3). In the decoupling limit where heavy KK state mixings tend to vanish, the SM chirality configuration is recovered as expected.

For completeness, let us underline that in the free case, the different BBT signs, $\sigma^Q_{0,\pi R}$ and $\sigma^D_{0,\pi R}$, would lead to a chiral setup for the zero-modes but different from the potential SM chirality configuration, which are listed in Table 4.1.

| $\sigma^Q_{0,\pi R}$ | $\sigma^D_{0,\pi R}$ | $Q$ | $D$ |
|---|---|---|---|
| 1 | 1 | $Q^0_R(x^\mu)$ | $D^0_R(x^\mu)$ |
| 1 | $-1$ | $Q^0_R(x^\mu)$ | $D^0_L(x^\mu)$ |
| $-1$ | 1 | $Q^0_L(x^\mu)$ | $D^0_R(x^\mu)$ |
| $-1$ | $-1$ | $Q^0_L(x^\mu)$ | $D^0_L(x^\mu)$ |

Table 4.1 – Chiral setups for the zero-modes of fields $Q$ and $D$ from various different BBT signs $\sigma^{Q,D}_{0,\pi R}$ in Eq. (4.18).

Finally, analogy to the vector-like states of custodians introduced in Section 3.2.3, such massive vector-like states can be realized by $\sigma^F_0 = -\sigma^F_{\pi R} = \pm 1$ (opposite sign for 0 and $\pi R$) of the BBT (4.18) [in contrast to that of the chiral solutions in Eq. (4.19)], which would instead lead to the profile solutions (4.42) with two non-vanishing profiles for the lightest modes (as $m_0 \neq 0$) in Sections 4.3.2 and 4.3.3. Of course there exist 8 remaining cases combining the above Lagrangian sign configurations: $\sigma^Q_{0(\pi R)} = \underset{(-)}{+}1, \overset{(+)}{-}1$, $\sigma^D_{0(\pi R)} = \pm 1$, and, $\sigma^Q_{0(\pi R)} = \pm 1$, $\sigma^D_{0(\pi R)} = \underset{(-)}{+}1, \overset{(+)}{-}1$.

Therefore, it appears clearly that the BBT control the chiral configurations of the model. The UV completion of the theory can be at the origin of the BBT and hence of the chirality setup: chiral nature of the theory and specific chiralities of the various fields.

To end up this section, we note that the complete toy model studied in Eq. (4.5) will respect the $\mathbb{Z}_2$ symmetry and the brane action $S_{\text{branes}}$ will exclusively be characterized by the action taken at fixed points as

$$S_{\text{branes}} = S_B + S_H + S_X + S_{hQD}\,. \tag{4.21}$$

The conclusions that will be derived in the present work can be directly extended to the realistic warped model with SM bulk matter addressing the fermion mass and gauge hierarchies, along the same lines as the flavor and gauge symmetry generalizations described in details in Section 3.2.6 (also see Ref. [1]).

## 4.3 Free Bulk Fermions on the Orbifold

Following the methodology in the interval scenario in Section 3.3, in this section, we calculate the fermionic mass spectrum for the free case without the Yukawa action piece $S_Y$ given by Eq. (4.14). It would provide us a main procedure and technical tools, which is also suitable for the 5D approach in the Yukawa case in Section 4.5.

### 4.3.1 Applying the NBC

We start by considering the bulk action part of Eq. (4.6),

$$S_{\text{bulk}}\,,$$

from the complete action $S_{5\text{D}}$ (4.5). We apply the least action principle to it, which leads to two relations of the kind,

$$\delta_{\bar{F}} S_{\text{bulk}} = 0\,,$$

one for each of the unknown 5D fields $F = Q, D$, and two corresponding ones, $\delta_F S_{\rm bulk} = 0$, involving the complex conjugate fields [7], since the elementary field variations $\delta Q_\alpha$, $\delta\bar{Q}_\alpha$, $\delta D_\alpha$ and $\delta\bar{D}_\alpha$ (see Appendix C.1) are generic and hence independent from each other. Using compact notations defined in Eq. (1.22), we can write in particular

$$\begin{aligned}
\delta_{\bar{F}} S_{\rm bulk} &= \int d^4x \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left\{ \delta\bar{F} \frac{\partial \mathcal{L}_{\rm kin}}{\partial \bar{F}} + \delta\left(\partial_M \bar{F}\right) \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_M \bar{F}} \right\} \\
&= \int d^4x \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left\{ \delta\bar{F} \frac{\partial \mathcal{L}_{\rm kin}}{\partial \bar{F}} + \partial_M \left[ \delta\bar{F} \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_M \bar{F}} \right] - \delta\bar{F}\, \partial_M \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_M \bar{F}} \right\} \\
&= \int d^4x \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left\{ \delta\bar{F} \left[ \frac{\partial \mathcal{L}_{\rm kin}}{\partial \bar{F}} - \partial_M \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_M \bar{F}} \right] \right\} \\
&\qquad + \int d^4x \ \left( \delta\bar{F} \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_4 \bar{F}} \bigg|_{-\pi R^+}^{0^-} + \delta\bar{F} \frac{\partial \mathcal{L}_{\rm kin}}{\partial\, \partial_4 \bar{F}} \bigg|_0^{\pi R} \right), \qquad (4.22)
\end{aligned}$$

where we omit the global 4-divergence which vanishes in the action integration due to vanishing fields at the boundaries at infinities via the same comments for Eq. (1.23). Based on the Lagrangian $\mathcal{L}_{\rm kin}$ of Eq. (4.8), these two bulk terms take the same form (the first one being calculated explicitly in Eq. (C.6) to clarify the spinor component treatment) and the two remaining brane terms can be calculated as well:

$$\begin{aligned}
\delta_{\bar{F}} S_{\rm bulk} &= \int d^4x \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left\{ \delta\bar{F} \left[ i\Gamma^M \partial_M F \right] \right\} \\
&\qquad + \int d^4x \ \left( \delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \bigg|_{0^+}^{\pi R^-} + \delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \bigg|_0^{\pi R} \right), \qquad (4.23)
\end{aligned}$$

where we have further invoked the $\mathbb{Z}_2$ transformations (4.9)-(4.12) for the generic 5D field, Eq. (C.7) for its variation and $\gamma^5$ properties:

$$\delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \bigg|_{0^-, -\pi R^+} = \left( \mp \delta\bar{F} \gamma^5 \right) \left[ -\frac{\gamma^5}{2} \left( \pm \gamma^5 F \right) \right] \bigg|_{0^+, \pi R^-} = -\, \delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \bigg|_{0^+, \pi R^-} .$$

Then, thanks to Eq. (4.4) [8] and Eq. (C.4)-(C.5) respectively, the expression (4.23) simplifies to,

$$\begin{aligned}
\delta_{\bar{F}} S_{\rm bulk} &= \int d^4x \left\{ \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left[ \delta\bar{F} \left( i\Gamma^M \partial_M F \right) \right] + 2\, \delta\bar{F} \left[ -\frac{\gamma^5}{2} F \right] \bigg|_0^{\pi R} \right\} \\
&= \int d^4x \left\{ \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \ \left[ \delta\bar{F} \left( i\Gamma^M \partial_M F \right) \right] + \left[ \delta F_R^\dagger F_L - \delta F_L^\dagger F_R \right] \bigg|_0^{\pi R} \right\}, \quad (4.24)
\end{aligned}$$

where the bulk and the brane variations – respectively the volume and surface terms – must vanish separately due to independent field variations (no reason to be linked). Besides all those field variations are not vanishing (unknown fields) so that we get the bulk EOM,

$$\forall\, x^\mu, \ \forall\, y \in [-\pi R^+, 0^-] \cup [0, \pi R] \ , \quad i\, \Gamma^M \partial_M F = 0\, , \qquad (4.25)$$

7. The equations of motion and boundary conditions derived from the least action principle for the fields and their conjugates are trivially related through Hermitian conjugation.

8. Those continuity relations lead to $\bar{F}|_0 = \bar{F}|_{0^+}$, i.e. $\bar{F}_\alpha|_0 = \bar{F}_\alpha|_{0^+}$ [cf. Eq. (C.2)], and in turn to $\delta\bar{F}_\alpha|_0 = \delta\bar{F}_\alpha|_{0^+}$ which can be written as $\delta\bar{F}|_0 = \delta\bar{F}|_{0^+}$ via Eq. (C.3). Similarly we get $\delta\bar{F}|_{\pi R} = \delta\bar{F}|_{\pi R^-}$.

and it's chiral formula after the chiral projection (1.17),

$$\forall\, x^\mu,\ \forall\, y \in [-\pi R^+, 0^-] \cup [0, \pi R]\ ,\ \left\{ \begin{array}{rcl} i\bar{\sigma}^\mu \partial_\mu F_L + \partial_4 F_R & = & 0\,, \\ i\sigma^\mu \partial_\mu F_R - \partial_4 F_L & = & 0\,, \end{array} \right. \tag{4.26}$$

with the corresponding NBC,

$$F_L|_0 = F_R|_0 = F_L|_{\pi R} = F_R|_{\pi R} = 0\,. \tag{4.27}$$

At this level, we can first solve Eq. (4.26) together with Eq. (4.27) to find out the $F$ fields over the domain, $y \in [0, \pi R]$. This is precisely what has been done in the interval scenario where the two exactly identical Eq. (1.30) and (1.31) have been solved over the interval, $y \in [0, L]$ [also cf. Ref. [1]]. Since the fields are continuous over $y \in [0, \pi R]$ [cf. Eq. (4.4)] like there over $y \in [0, L]$, we can thus apply here the results obtained in this reference: the solutions found for Eq. (4.26)-(4.27) are expressed through the KK decomposition (with a similar choice of global dimensional factor),

$$F_{L/R}\,(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=0}^{+\infty} f^n_{L/R}(y)\, F^n_{L/R}\,(x^\mu)\ , \tag{4.28}$$

where the 4D fields $F^n_{L/R} = Q^n_{L/R}, D^n_{L/R}$ represent the KK states and satisfy the Dirac-Weyl equations in Eq. (1.25) with the KK masses $m^F_n$.

However, the only resulting profiles $f^n_{L/R}(y) = q^n_{L/R}(y)$, $d^n_{L/R}(y)$, included respectively into $F = Q, D$, are vanishing over $y \in [0, \pi R]$. Now let us study the profile solutions in the complementary region, $y \in [-\pi R^+, 0^-]$. Inserting the KK decomposition (4.28) into the first type of $\mathbb{Z}_2$ transformation (4.9), one obtains the $\mathbb{Z}_2$ transformations directly on the $f^n_{L/R}(y)$ profiles ($\forall n \in \mathbb{N}$):

$$\text{Type I}\ \left\{ \begin{array}{l} \displaystyle\sum_{n=0}^{+\infty} \left[ q^n_{L(R)}(-y) \overset{(+)}{-} q^n_{L(R)}(y) \right] Q^n_{L(R)}\,(x^\mu) = 0 \ \Rightarrow\ q^n_{L(R)}(-y) = \underset{(-)}{+} q^n_{L(R)}(y) \\ \displaystyle\sum_{n=0}^{+\infty} \left[ d^n_{L(R)}(-y) \underset{(-)}{+} d^n_{L(R)}(y) \right] D^n_{L(R)}\,(x^\mu) = 0 \ \Rightarrow\ d^n_{L(R)}(-y) = \overset{(+)}{-} d^n_{L(R)}(y) \end{array} \right. \tag{4.29}$$

where the implications come from the linear independence of mass eigenstates $F^n_{L/R}\,(x^\mu)$. Similarly, for the three other types of $\mathbb{Z}_2$ transformations (4.10)-(4.12), we have the following profile parities:

$$\text{Type II}\ \left\{ \begin{array}{l} q^n_{L(R)}(-y) = \overset{(+)}{-} q^n_{L(R)}(y) \\ d^n_{L(R)}(-y) = \underset{(-)}{+} d^n_{L(R)}(y) \end{array} \right. \tag{4.30}$$

$$\text{Type III}\ \left\{ \begin{array}{l} q^n_{L(R)}(-y) = \underset{(-)}{+} q^n_{L(R)}(y) \\ d^n_{L(R)}(-y) = \underset{(-)}{+} d^n_{L(R)}(y) \end{array} \right. \tag{4.31}$$

$$\text{Type IV}\ \left\{ \begin{array}{l} q^n_{L(R)}(-y) = \overset{(+)}{-} q^n_{L(R)}(y) \\ d^n_{L(R)}(-y) = \overset{(+)}{-} d^n_{L(R)}(y) \end{array} \right. \tag{4.32}$$

Therefore, all the $f^n_{L/R}(y)$ profiles are systematically vanishing on the whole $\mathcal{S}^1/\mathbb{Z}_2$ orbifold region, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$. Such profiles conflict with the two (for $L/R$) orthonormalization conditions over the full domain,

$$\forall\, n, m \in \mathbb{N},\ \frac{1}{2\pi R} \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\ f^{n*}_{L/R}(y)\, f^m_{L/R}(y) = \delta_{nm}\,, \tag{4.33}$$

originating from the condition of a canonical form for the 4D effective kinetic terms. Hence the solutions for the fields obtained through this first method are not physically consistent.

### 4.3.2 Introducing the EBC

In fact, one necessary ingredient was missing in the naive approach in Section 4.3.1. In order to identify it, we have to study the conserved fermion probability currents corresponding, via the Noether's theorem, to the global $\mathrm{U}(1)_{\mathrm{Q}}$ and $\mathrm{U}(1)_{\mathrm{D}}$ symmetries of the action,

$$S_{\text{bulk}}\,,$$

involving the bulk kinetic Lagrangian $\mathcal{L}_{\text{kin}}$ (4.8). The two independent global $\mathrm{U}(1)_{\mathrm{Q,D}}$ transformations of the fields, letting $\mathcal{L}_{\text{kin}}$ invariant, act respectively as,

$$\begin{cases} Q & \mapsto & e^{i\alpha} Q, \\ \bar{Q} & \mapsto & e^{-i\alpha}\bar{Q}, \end{cases} \quad \text{and} \quad \begin{cases} D & \mapsto & e^{i\alpha'} D, \\ \bar{D} & \mapsto & e^{-i\alpha'}\bar{D}, \end{cases} \tag{4.34}$$

where $\alpha, \alpha' \in \mathbb{R}$ are continuous constants entering for instance the infinitesimal field variations [9]:

$$\underline{\delta} Q = i\alpha Q\,, \quad \underline{\delta}\bar{Q} = -i\alpha\bar{Q}\,.$$

Choosing instead to consider a unique symmetry ($\alpha = \alpha'$ for any field $F$) would correspond to a particular case only, among the general Lagrangian symmetry possibilities. Besides, this particular case would not provide the maximal information, since one symmetry would be associated to only one conserved probability current. We thus will consider, in this section, the transformations (4.34) with independent $\alpha$ and $\alpha'$, leading to the two independent $\mathrm{U}(1)_{\mathrm{Q,D}}$ symmetries. Based on these two symmetries, and the bulk EOM whose standard structure appears in Eq. (4.22), the Noether's theorem predicts the local conservation relation,

$$\partial_M j^M_F = 0\,, \tag{4.35}$$

for the two probability currents,

$$j^M_Q = -\alpha\bar{Q}\,\Gamma^M Q\,, \quad j^M_D = -\alpha'\bar{D}\,\Gamma^M D\,, \tag{4.36}$$

as derived in details within the Appendix F. This relation holds over the whole $\mathcal{S}^1/\mathbb{Z}_2$ orbifold domain, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$, since the sole bulk terms in the action infinitesimal variation – under $\mathrm{U}(1)_{\mathrm{Q,D}}$ transformation – must vanish for any integration sub-region included inside the entire integration domain of the action precisely defined for the model. The mathematical consistency of the condition (4.35) imposes necessarily (left/right) continuous 5-current components over all the model spacetime and in particular a (left/right) continuous $j^4_F$ along $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$ [10].

Notice that a jump of the form, $j^4_F|_{0^-} \neq j^4_F|_0$, would not determine any field at the fixed point and thus would not lead to any vanishing variation in Eq. (4.23). Thus, the Dirichlet BC (4.27) would be preserved and in turn induce non-physical solutions. A similar argue applies at the other fixed point, $y = \pi R \equiv -\pi R$.

9. Different clear notations are used here for the infinitesimal field variations under specific transformations, $\underline{\delta} F$, and the above generic field variations in the variation calculus context of the least action principle, $\delta F$ [see typically Eq. (C.3)].

10. Notice that this condition is in agreement with Eq. (4.4) which guarantees continuous fields along $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$.

Hence, one has to consider the remaining model possibility, $j^4_F|_{0^-} = j^4_F|_0$ and [11] $j^4_F|_{-\pi R^+} = j^4_F|_{\pi R}$, so that this current component is continuous over all the range, $y \in [-\pi R, \pi R]$. In particular, we can now write,

$$\begin{cases} j^4_F|_{0^-} = j^4_F|_0 = j^4_F|_{0^+} \,, \\ j^4_F|_{\pi R^-} = j^4_F|_{\pi R} = j^4_F|_{-\pi R^+} \,. \end{cases} \tag{4.37}$$

This obtained relation must be compared with the following one, coming directly from the $\mathbb{Z}_2$ transformations of type (4.9)-(4.12) and $\gamma^5$ properties,

$$\begin{aligned} j^4_F\Big|_{0^-(\pi R^-)} &= -\alpha^{(\prime)}\, \bar{F}\, \Gamma^4 F\Big|_{0^-(\pi R^-)} = -\alpha^{(\prime)} \left(\pm\gamma^5 F\right)^\dagger \gamma^0 \left[-i\gamma^5\right] \left(\pm\gamma^5 F\right)\Big|_{0^+(-\pi R^+)} \\ &= \alpha^{(\prime)} F^\dagger \gamma^0 \gamma^5 \left[-i\gamma^5\right] \left(\gamma^5 F\right)\Big|_{0^+(-\pi R^+)} = \alpha^{(\prime)}\, \bar{F}\, \Gamma^4 F\Big|_{0^+(-\pi R^+)} = -\, j^4_F\Big|_{0^+(-\pi R^+)} \,. \end{aligned} \tag{4.38}$$

The combination of Eq. (4.37) and Eq. (4.38) gives rise to a vanishing current component at the fixed point:

$$\begin{cases} j^4_F|_{0^-} = j^4_F|_0 = j^4_F|_{0^+} = 0 \,, \\ j^4_F|_{\pi R^-} = j^4_F|_{\pi R} = j^4_F|_{-\pi R^+} = 0 \,, \end{cases}$$

so that, using the generic chiral decomposition (C.5), we get the following current conditions,

$$j^4_F\Big|_{0,\pi R} = i\alpha^{(\prime)} \left(F_L^\dagger F_R - F_R^\dagger F_L\right)\Big|_{0,\pi R} = 0 \,, \tag{4.39}$$

leading to the minimal boundary conditions,

$$\left\{ \begin{array}{ccc} F_L|_0 & = & 0\,, \\ & \text{or} & \\ F_R|_0 & = & 0\,, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} F_L|_{\pi R} & = & 0\,, \\ & \text{or} & \\ F_R|_{\pi R} & = & 0\,. \end{array} \right. \qquad \text{[EBC]} \tag{4.40}$$

These BC induce systematically the vanishing of all the brane terms in the varied action obtained in Eq. (4.24). Indeed, for example, the fixed value $F_L|_0 = 0$ implies $F_L^\dagger|_0 = 0$ and in turn $\delta F_L^\dagger|_0 = 0$ [12] [considering more precisely their two respective components as is clear from Appendix C.1]. Therefore the sole remaining BC are those of Eq. (4.40): there are no more NBC generated from the brane terms of Eq. (4.24) and we name the BC (4.40) as EBC since they are imposed by the $\mathbb{Z}_2$ transformations (4.38) which contribute to define the studied model. From the point of view of the methodology, notice interestingly that it was necessary to consider the fermion probability currents to reveal the existence of the EBC.

Now, solving the new EBC (4.40) together with the unchanged bulk EOM (4.26) over the domain, $y \in [0, \pi R]$, was precisely realized in the interval scenario $y \in [0, L]$ (see Section 3.3.2). Once more, since the fields are continuous over $y \in [0, \pi R]$ [see Eq. (4.4)] like there over $y \in [0, L]$, we can apply here the results derived in this previous work: the profile solutions – inserting the KK decomposition (4.28) to the EOM (4.26) and the

11. A change must occur at both fixed points to cure the problems of the solutions worked out in previous subsection.

12. Rigorously speaking, the action should not be minimized with respect to the known fixed fields so that the terms with vanishing field variations should not even appear. In fact, the brane terms of Eq. (4.24) should originally be written as a generic sum over unfixed fields.

EBC (4.40) – are given by the following four possible sets of profiles over $y \in [0, \pi R]$ together with the associated KK mass spectrum equations ($\forall\, n \in \mathbb{N}$),

$$
\begin{array}{ll}
1) \quad (--): \ f_L^n(y) = B_L^n \ \sin(m_n^F\, y)\,, \ (++): \ f_R^n(y) = B_L^n \ \cos(m_n^F\, y)\,; \ \sin(m_n^F\, \pi R) = 0\,, \\
2) \quad (++): \ f_L^n(y) = B_R^n \ \cos(m_n^F\, y)\,, \ (--): \ f_R^n(y) = -B_R^n \ \sin(m_n^F\, y)\,; \ \sin(m_n^F\, \pi R) = 0\,,
\end{array}
\tag{4.41}
$$

and,

$$
\begin{array}{ll}
3) \quad (-+): \ f_L^n(y) = B_L^n \ \sin(m_n^F\, y)\,, \ (+-): \ f_R^n(y) = B_L^n \ \cos(m_n^F\, y)\,; \ \cos(m_n^F\, \pi R) = 0\,, \\
4) \quad (+-): \ f_L^n(y) = B_R^n \ \cos(m_n^F\, y)\,, \ (-+): \ f_R^n(y) = -B_R^n \ \sin(m_n^F\, y)\,; \ \cos(m_n^F\, \pi R) = 0\,.
\end{array}
\tag{4.42}
$$

where we use the standard BC notations $-$ or $+$ at $y = 0, \pi R$ defined below Eq. (3.29), which make explicit the correspondence between the four EBC (4.40) and the four solutions (4.41)-(4.42). The SM-like profile $d^n_{L/R}(y)$ ($q^n_{L/R}(y)$) are taken from line 1 (2) of Eq. (4.41) for the field $D$ ($Q$), as described in Section 4.2.5.

The equation $\sin(m_n^F\, \pi R) = 0$ possesses the following solutions for the KK mass spectrum,

$$
|m_n| = \frac{n}{R}\,, \quad n \in \mathbb{N}\,,
\tag{4.43}
$$

where we define the notation of the common mass spectrum $m_n$ as

$$
m_n \mathrel{\hat{=}} m_n^Q = m_n^D\,.
\tag{4.44}
$$

Similarly, the equation $\cos(m_n^F\, \pi R) = 0$ has the solutions:

$$
\left| m_n^F \right| = \frac{2n+1}{2R}\,, \quad n \in \mathbb{N}\,.
\tag{4.45}
$$

The remaining part of the general $f^n_{L/R}(y)$ solutions in the complementary domain $y \in [-\pi R^+, 0^-]$, is now obtained via the four types of $\mathbb{Z}_2$ transformations (4.29)-(4.32). Therefore, the inclusion of the EBC based on the vanishing probability currents allows to obtain consistent fermion profile and mass solutions.

In Table 4.2, we present the explicit solutions over the whole orbifold domain for the SM-like profiles $d^n_{L/R}(y)$ ($q^n_{L/R}(y)$) taken from line 1 (2) of Eq. (4.41): see the discussion on SM chirality configuration in Section 4.2.5. The mass spectrum for the 4D KK states is defined by Eq. (1.25) and it is already determined by Eq. (4.43)-(4.45). Notice in Table 4.2 that the same $m_n$ spectrum enters the profile solutions in both regions, $y \in [0, \pi R]$, and, $y \in [-\pi R^+, 0^-]$. In this table, we also give the general values of the $B^n_{L/R}$ complex constants in Eq. (4.41), obtained from the ortho-normalization conditions (4.33) [13]. We observe in Table 4.2 that the choice of type of $\mathbb{Z}_2$ transformation is just a convention since it can modify the profile signs but it affects neither the mass spectrum nor the fermion chirality configuration – as a certain chiral zero-mode profile vanishing on the region $[0, \pi R]$ is also systematically vanishing over $y \in [-\pi R^+, 0^-]$. In contrast, the chirality configuration and mass spectrum are fixed by the choice of EBC (4.40) which can lead either to the two kinds of chiral solutions in Eq. (4.41) or to the vector-like solutions (4.42).

13. Here, thanks to the parity symmetry of profiles, a change of variable, $y \to -y$, could be applied to recover exclusively the integration domain $[0, \pi R]$.

| Continuity domains | $\mathbb{Z}_2$ | Fields | | | |
|---|---|---|---|---|---|
| | | $Q_{L/R}$ | | $D_{L/R}$ | |
| | | $q_L^n(y)/e^{i\alpha_Q^n}$ | $q_R^n(y)/e^{i\alpha_Q^n}$ | $d_L^n(y)/e^{i\alpha_D^n}$ | $d_R^n(y)/e^{i\alpha_D^n}$ |
| $[0, \pi R]$ | Any | $\sqrt{2}\,\cos(m_n\,y)$ | $-\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\cos(m_n\,y)$ |
| $[-\pi R^+, 0^-]$ | I | $\sqrt{2}\,\cos(m_n\,y)$ | $-\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\cos(m_n\,y)$ |
| | II | $-\sqrt{2}\,\cos(m_n\,y)$ | $\sqrt{2}\,\sin(m_n\,y)$ | $-\sqrt{2}\,\sin(m_n\,y)$ | $-\sqrt{2}\,\cos(m_n\,y)$ |
| | III | $\sqrt{2}\,\cos(m_n\,y)$ | $-\sqrt{2}\,\sin(m_n\,y)$ | $-\sqrt{2}\,\sin(m_n\,y)$ | $-\sqrt{2}\,\cos(m_n\,y)$ |
| | IV | $-\sqrt{2}\,\cos(m_n\,y)$ | $\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\sin(m_n\,y)$ | $\sqrt{2}\,\cos(m_n\,y)$ |
| KK Masses | $\lvert m_n\rvert = n/R,\ n \in \mathbb{N}$ | | | | |

Table 4.2 – SM-like free fermionic $f^n_{L/R}(y)$ profiles – normalized to the indicated complex phases – on the two orbifold domains $[-\pi R^+, 0^-]$ and $[0, \pi R]$, corresponding to the solution of line 1 (2) in Eq. (4.41) for the field $D$ ($Q$). The associated mass spectrum (4.43) is included as well for completeness. The profiles are given for the four types of $\mathbb{Z}_2$ transformations (4.29)-(4.32). The phases $\alpha^n_{Q/D}$ belong to $\mathbb{R}$. In the special case, $n = 0$, the $\sqrt{2}$ factors must all be replaced by the unity.

In Figure 4.3, we draw the first two excitation profiles for each free solution with non-negative KK masses presented in Table 4.2 within the simple real case, $\alpha^n_{Q,D} = 0$, and for two different types of $\mathbb{Z}_2$ transformations (4.29)-(4.32). We see clearly in Figure 4.3 that for example under the Type II $\mathbb{Z}_2$ transformation, jumps appear for the profiles $q_L^{0,1,2}(y)$ and $d_R^{0,1,2}(y)$ at the two fixed points at, $y = 0$, $y = \pi R \equiv -\pi R$, in the scenario without Yukawa couplings. The presence of profile discontinuities here already justifies the treatment exposed in Section 4.2.1. The precise prescription (4.4) regarding the action integration domain, described in this section, renders the jumps of Figure 4.3 consistent mathematically: the difference, e.g. $q_L^1(0^-) \neq q_L^1(0)$, is compatible with a well defined Lagrangian integrand over the action integration domain, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$, where the profiles are continuous.

### 4.3.3 Introducing the BBT

As suggested in Section 4.2.5, we can alternatively introduce the dimension 4 operators of Eq. (4.18) to study their effects with respect to the inconsistencies raised in Section 4.3.1. Hence, to the action $S_{\text{bulk}}$ from Eq. (4.6), we add now another part and consider:

$$S_{\text{bulk}} + S_B \,.$$

The variations of $S_B$ with respect to the generic field $\bar{F}$ [using Eq. (C.4)],

$$\delta_{\bar{F}} S_B = \int d^4x \, \left( \sigma^F_{\pi R} \, \delta F_R^\dagger F_L \Big|_{\pi R} + \sigma^F_{\pi R} \, \delta F_L^\dagger F_R \Big|_{\pi R} - \sigma^F_0 \, \delta F_R^\dagger F_L \Big|_0 - \sigma^F_0 \, \delta F_L^\dagger F_R \Big|_0 \right),$$

together with Eq. (4.24) allow to write down the variations of the free fermion action:

$$\delta_{\bar{F}} \left(S_{\text{bulk}} + S_B\right) = \int d^4x \, \Bigg\{ \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \; \delta\bar{F} \left( i\Gamma^M \partial_M F \right) + \left( \sigma^F_{\pi R} + 1 \right) \delta F_R^\dagger F_L \Big|_{\pi R}$$
$$+ \left( \sigma^F_{\pi R} - 1 \right) \delta F_L^\dagger F_R \Big|_{\pi R} - \left( \sigma^F_0 + 1 \right) \delta F_R^\dagger F_L \Big|_0 - \left( \sigma^F_0 - 1 \right) \delta F_L^\dagger F_R \Big|_0 \Bigg\} . \qquad (4.46)$$

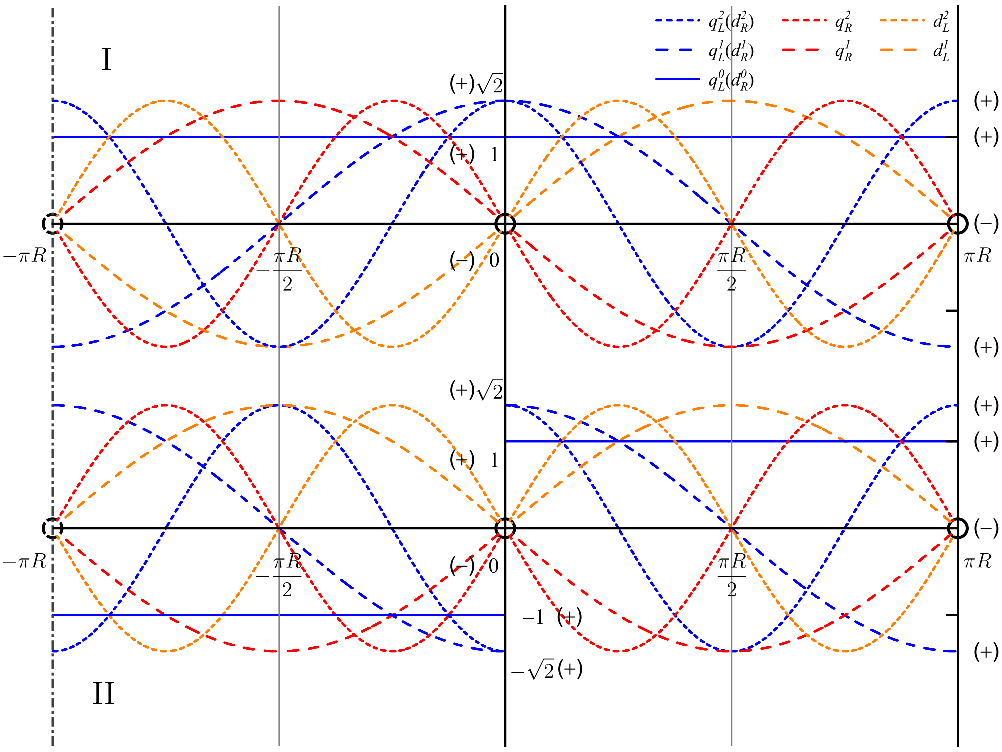


Figure 4.3 – Zero-mode and KK dimensionless wave functions $q^n_{L/R}(y)$, $d^n_{L/R}(y)$, with $n = 0, 1, 2$, along the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold domain, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$, corresponding to the free solutions of Table 4.2 in the simplified case, $\alpha^n_{Q,D} = 0$, $m_n \geq 0$, and for the two different types of $\mathbb{Z}_2$ transformations, I, II from Eq. (4.29)-(4.32). The two fixed points at, $y = 0$, $y = \pi R \equiv -\pi R$, and Dirichlet/Neumann BC, $(-)/(+)$, are indicated on the graph.

The individual vanishing of those volume and surface terms lead to the EOM (4.26) together with the four following NBC, depending on the two $\sigma^F_{0,\pi R}$ choices,

$$\left\{\begin{array}{rcl} F_L|_0 & = & 0\ (\sigma^F_0 = +1), \\ & \text{or} & \\ F_R|_0 & = & 0\ (\sigma^F_0 = -1), \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} F_L|_{\pi R} & = & 0\ (\sigma^F_{\pi R} = +1), \\ & \text{or} & \\ F_R|_{\pi R} & = & 0\ (\sigma^F_{\pi R} = -1). \end{array}\right. \qquad \text{[NBC]} \tag{4.47}$$

At this level, the EOM and NBC are effectively the same as the EOM (4.26) and EBC (4.40) of previous subsection, in the domain $y \in [0, \pi R]$, so that we find again the solutions (4.41)-(4.42) together with the mass spectra (4.43)-(4.45). For instance, the SM-like choice $\sigma^Q_{0,\pi R)} = -1$ of Eq. (4.18) leads via Eq. (4.47) to the solution of line 2 in Eq. (4.41). Then the parts of the general profile solutions in the complementary region, $y \in [-\pi R^+, 0^-]$, are found out via the different types of $\mathbb{Z}_2$ transformations (4.29)-(4.32) in the free case, as in Section 4.3.2, so that the complete solutions are once more identical and can also be illustrated by the Table 4.2 and Figure 4.3 both based on the ortho-normalization conditions (4.33). In conclusion, introducing the BBT permits to rigorously work out profile and mass solutions. A second conclusion in this approach is that the chirality setup – one of the two chiral solutions (4.41) or of the vector-like ones (4.42) – and associated mass spectrum are fixed by the choice of NBC (4.47) and in turn by the choices of $\sigma^F_{0,\pi R}$ BBT signs in Eq. (4.18). In simpler words, the BBT (like the EBC previously) control the chiral nature of the theory as well as each field chirality.

Let us now discuss the probability currents. The addition of the $S_B$ part in Eq. (4.18) to $S_{\text{bulk}}$ is not affecting the current equations (4.35)-(4.36) since the new brane terms so induced in the infinitesimal action variation – under the $\mathrm{U}(1)_{\mathrm{Q,D}}$ transformations (4.34) – vanish due to their $\mathrm{U}(1)_{\mathrm{Q,D}}$ invariant form. In contrast with previous subsection and with the interval model in the free case with BBT (see Sections 3.3.2-3.3.3), there exists no demonstration there of Eq. (4.39) from discrete symmetries. Nevertheless, we can check that $j^4_F|_{0,\pi R}$ is well vanishing by using the obtained solutions (4.41)-(4.42): the product $f^n_L(y) f^m_R(y)$ systematically vanishes at $y = 0, \pi R$. Therefore, the BBT play the role of making $j^4_F|_{0,\pi R}$ vanish ($\mathbb{Z}_2$ transformation consequence) like the EBC were guaranteeing it in Section 4.3.2. Note that we could simultaneously apply the EBC and introduce the BBT but those two processes would be physically redundant to define the model.

## 4.4 Brane-Localized Yukawa Couplings on the Orbifold: 4D Approach

Once the free case is addressed, via the EBC (4.40) in Section 4.3.2 or the NBC (4.47) induced by the BBT in Section 4.3.3, the fermion mass spectrum and profiles are known. Then how to take into account the effects of the action part $S_X$ (4.16) in the mass spectrum, induced by the Yukawa interaction between a brane-localized scalar field and bulk fermions in Eq. (4.14)? The considered action reads thus as,

$$S_{\text{bulk}} + S_X \; (+S_B) \,. \tag{4.48}$$

A first method called the perturbation method, described in the present section, is performed at the level of the 4D effective Lagrangian, that is by calculating the mass mixings between the different levels of the KK towers. Considering the SM-like profile solutions $d^n_{L/R}(y)$ ($q^n_{L/R}(y)$) and associated free KK mass spectrum from line 1 (2) of Eq. (4.41), all the initial 4D effective masses for the KK modes of Eq. (4.28) in the interaction basis can be classified into two species: the pure KK masses (4.43) and the mass contributions from the Yukawa interaction given by the overlap between the wave functions and Higgs-brane,

$$\begin{cases} \forall (i,j) \in \mathbb{N}^2, \; \alpha_{ij} = X \, \dfrac{q^i_L(\pi R)}{\sqrt{2\pi R}} \, \dfrac{d^j_R(\pi R)}{\sqrt{2\pi R}} \,, \\[2ex] \forall (i,j) \in \mathbb{N}^{\star 2}, \; \beta_{ij} = X' \, \dfrac{d^i_L(\pi R)}{\sqrt{2\pi R}} \, \dfrac{q^j_R(\pi R)}{\sqrt{2\pi R}} \,. \end{cases} \tag{4.49}$$

In particular, $\beta_{ij} = 0$ as imply the respective SM solutions (4.41) so that the coupling constant $X'$ disappears from the mass dependences. Note that for similar reasons [cf. Eq. (C.5)], in the case of the presence of the BBT (4.18), those do not generate 4D mass terms. All the 4D mass terms enter the 4D effective Lagrangian through the following mass matrix [similar to the interval case in Section 3.4],

$$-\chi^\dagger_L \, \mathcal{M} \, \chi_R + \text{H.c.}$$

within the field basis noted,

$$\begin{cases} \chi^t_L(x^\mu) &= \left( Q^0_L, Q^1_L, D^1_L, Q^2_L, D^2_L, \cdots \right) , \\[1ex] \chi^t_R(x^\mu) &= \left( D^0_R, Q^1_R, D^1_R, Q^2_R, D^2_R, \cdots \right) . \end{cases} \tag{4.50}$$

The texture of this infinite mass matrix $\mathcal{M}$ involving the diagonal $m_n$ (3.30), off-diagonal $\alpha_{ij}$ (4.49) and mixing the $Q$, $D$ fields together can be precisely taken from the interval model context in Eq. (3.43), with two substitutions:

(i) $L \leftrightarrow \pi R$, since the KK masses and bulk profile solutions are then identical (up to extensions over $[-\pi R^+, 0^-]$ as seen in Section 4.3.2 here) like the Yukawa interactions localized at $y = \pi R$ [for any $\mathbb{Z}_2$ transformation (4.58)-(4.32)].

(ii) $X \leftrightarrow X/2$, since the two present profiles (even or odd) entering $\alpha_{ij}$ (4.49) are normalized via Eq. (4.33) over a domain of double size $2L \leftrightarrow 2\pi R$ compared to the interval case.

Now we can apply the results for the mass eigenvalues $M_n$ of the 4D eigenstates $\psi^n_{L/R}(x^\mu)$ obtained through the bi-diagonalization performed in Eq. (3.45), based on the calculations in Ref. [96]. Then, the obtained exact mass eigenvalues are determined by the following equation, coming from the characteristic equation,

$$\forall\, n \in \mathbb{N},\ \tan^2(\sqrt{|M_n|^2}\,\pi R) = \left(\frac{X}{2}\right)^2, \tag{4.51}$$

in the case of a real $X$ parameter, i.e. $\alpha_Y = 0$ (3.12) and positive $m_n$ branch from Eq. (4.43). Hence, the physical absolute value of the mass spectrum reads as:

$$|M_n| = \frac{1}{\pi R}\left|\arctan\left(\frac{X}{2}\right) + (-1)^n\,\tilde{n}(n)\,\pi\right|,\ n \in \mathbb{N}\,, \tag{4.52}$$

using the $\tilde{n}(n)$ function already defined in Eq. (3.47).

## 4.5 Brane-Localized Yukawa Couplings on the Orbifold: 5D Approach

### 4.5.1 Applying the NBC

Let us now study the presence of Yukawa couplings at the fixed point, $y = \pi R$, through the action,

$$S^{\rm m}_{\rm 5D} = S_{\rm bulk} + S_X + S^0_B\,, \quad \text{with} \quad S^0_B = -\int d^4x\ \mathcal{L}_B|_0\,, \tag{4.53}$$

where the first bulk term is based on kinetic terms (4.8) and $\mathcal{L}_B$ introduced by the BBT of Eq. (4.18) is imposed only at the brane $y = 0$ where the Yukawa interaction is absent. Within the 5D approach, that is by considering the mixings among KK excitation states at the level of the 5D fields. The BBT introduced here at the fixed point at $y = 0$ are the ones of Eq. (4.18)-(4.19) leading to SM-like chirality configurations: $\sigma^Q_0 = -1$, $\sigma^D_0 = 1$. Those guarantee a correct treatment of the free brane, like EBC, as analyzed throughout Section 4.3. Using Eq. (4.46) and Eq. (4.16), one gets directly the following action variations with respect to the fields $\bar{Q}$ and $\bar{D}$,

$$\begin{aligned}
\delta_{\bar{Q}} S^{\rm m}_{\rm 5D} = \int d^4x\, \Bigg\{ &\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \delta\bar{Q}\left(i\Gamma^M \partial_M Q\right) \\
&+ \left[\delta Q^\dagger_L\left(-X D_R - Q_R\right) + \delta Q^\dagger_R\left(-X' D_L + Q_L\right)\right]\Big|_{\pi R} + 2\left(\delta Q^\dagger_L Q_R\right)\Big|_0 \Bigg\}\,, \\
\delta_{\bar{D}} S^{\rm m}_{\rm 5D} = \int d^4x\, \Bigg\{ &\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \delta\bar{D}\left(i\Gamma^M \partial_M D\right) + \\
&\left[\delta D^\dagger_L\left(-X'^* Q_R - D_R\right) + \delta D^\dagger_R\left(-X^* Q_L + D_L\right)\right]\Big|_{\pi R} - 2\left(\delta D^\dagger_R D_L\right)\Big|_0 \Bigg\}\,.
\end{aligned} \tag{4.54}$$

The separate vanishings of these volume and surface terms, induced by the least action principle, give rise respectively to the EOM (4.26) and the following NBC,

$$\left\{ \begin{array}{lll} (Q_R + X\, D_R)|_{\pi R} = 0\,, & (D_R + X'^*\, Q_R)|_{\pi R} = 0\,, & Q_R|_0 = 0\,, \\ (Q_L - X'\, D_L)|_{\pi R} = 0\,, & (D_L - X^*\, Q_L)|_{\pi R} = 0\,, & D_L|_0 = 0\,. \end{array} \right. \tag{4.55}$$

As usual, the 5D field solutions of the EOM (4.26) and NBC (4.55) have the form of the following mixed KK decomposition [instead of Eq. (4.28)] [100, 101],

$$\left[ \begin{array}{lll} Q_L\,(x^\mu, y) & = & \dfrac{1}{\sqrt{2\pi R}} \sum\limits_{n=0}^{+\infty} q_L^n(y)\, \psi_L^n\,(x^\mu)\,, \\ Q_R\,(x^\mu, y) & = & \dfrac{1}{\sqrt{2\pi R}} \sum\limits_{n=0}^{+\infty} q_R^n(y)\, \psi_R^n\,(x^\mu)\,, \\ D_L\,(x^\mu, y) & = & \dfrac{1}{\sqrt{2\pi R}} \sum\limits_{n=0}^{+\infty} d_L^n(y)\, \psi_L^n\,(x^\mu)\,, \\ D_R\,(x^\mu, y) & = & \dfrac{1}{\sqrt{2\pi R}} \sum\limits_{n=0}^{+\infty} d_R^n(y)\, \psi_R^n\,(x^\mu)\,, \end{array} \right. \tag{4.56}$$

with the 4D fields $\psi^n_{L/R}$, already mentioned in Section 4.4, satisfying the Dirac-Weyl equations in Eq. (3.51), where the the fermion mass eigenvalues $M_n$ include the contributions from the Yukawa terms and these 4D fields the mass eigenstates including the effects of mixings among the $Q$, $D$ fields as well as (infinite) KK levels. Besides, the two (for $L/R$) following ortho-normalization conditions over the full $\mathcal{S}^1$ domain [replacing Eq. (4.33)],

$$\forall\, n, m \in \mathbb{N},\ \frac{1}{2\pi R} \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\, \left[ q^{n*}_{L/R}(y)\, q^m_{L/R}(y) + d^{n*}_{L/R}(y)\, d^m_{L/R}(y) \right] = \delta_{nm}\,. \tag{4.57}$$

Indeed, injecting the mixed KK decomposition (4.56) into the first type of $\mathbb{Z}_2$ transformation (4.9), we get the $\mathbb{Z}_2$ transformations directly on the profiles:

$$\text{Type I} \left\{ \begin{array}{l} \sum\limits_{n=0}^{+\infty} \left[ q^n_{L(R)}(-y) \overset{(+)}{-} q^n_{L(R)}(y) \right] \psi^n_{L(R)}\,(x^\mu) = 0 \;\Rightarrow\; q^n_{L(R)}(-y) = \underset{(-)}{+} q^n_{L(R)}(y) \\ \sum\limits_{n=0}^{+\infty} \left[ d^n_{L(R)}(-y) \underset{(-)}{+} d^n_{L(R)}(y) \right] \psi^n_{L(R)}\,(x^\mu) = 0 \;\Rightarrow\; d^n_{L(R)}(-y) = \overset{(+)}{-} d^n_{L(R)}(y) \end{array} \right. \tag{4.58}$$

In the same way, for the three other types of $\mathbb{Z}_2$ transformations (4.10)-(4.12), one obtains the same profile parities as in Eq. (4.32). The explicit profile solutions appearing in Eq. (4.56) over the domain, $y \in [0, \pi R]$, were found out for the interval model studied in Section 3.6.1 where the exactly identical EOM (1.30) and the NBC (3.68), up to a sign and a factor 2 in front of each $X^{(\prime)}$ parameter, have been solved over $y \in [0, L]$. Because the fields are continuous over $y \in [0, \pi R]$ [cf. Eq. (4.4)] like there over $y \in [0, L]$, one can apply here the conclusions obtained in this reference. Note that the ortho-normalization conditions (4.57) can be recast into the integration of Eq. (3.52) over the region $[0, \pi R]$ but with a global factor 2, thanks to the change of variable $y' = -y$, the fixed odd/even parities of the profiles and Eq. (4.4), so that the demonstration about profile solutions on the interval in Eq. (3.72) remains unchanged here, from this point of view as well. Meanwhile, the relative factors 2, at the same places in the NBC (4.55), come from the existence of surface terms both at $y = 0,\, 0^-$ and $y = \pi R,\, -\pi R^+$ as is clearly described in Eq. (4.23)-(4.24). These factors turn out not to modify the relations between the different profile solutions but to only change by a factor 4 in the final mass spectrum equation in Eq. (3.73).

As a conclusion, the same result as in Section 3.6.1 holds here for the orbifold: the 4D effective Yukawa coupling constant for the lightest modes ($\psi^0_{L,R}$), induced by the found profiles, tends to zero within the decoupling limit which is not compatible with the SM configuration expected. The problematic characteristics of the solutions obtained in this naive approach are confirmed by the final mass spectrum equation (independent from the profile normalization),

$$\tan^2(M_n \pi R) = |X|^2 \,,$$

which conflicts analytically with the one obtained through the 4D method in Eq. (4.51) for a real $X$ parameter. This failure motivates the alternative 5D methods of the next two subsections.

### 4.5.2 Introducing the EBC

Following the same idea as for the free case in Section 4.3.2, we try now to find consistent fermion mass solutions via considerations on their currents. The currents permit a priority to fully define the geometrical field configuration like here for the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold scenario. The complete relevant action including the brane-localized Yukawa terms (4.16),

$$S_{\text{bulk}} + S_X + S_B^0 \,, \tag{4.59}$$

like in Eq. (4.53), is invariant under the unique $\mathrm{U(1)_F}$ symmetry defined via Eq. (4.34) only for,

$$\alpha = \alpha' \,, \tag{4.60}$$

since the fermions $Q$ and $D$ are mixed on the brane at $y = \pi R$. Based on this symmetry involving both $Q$ and $D$ as well as on the bulk EOM [whose standard structure appears in the action variation (4.22)], the Noether's theorem predicts (cf. Appendix F) the new local probability conservation relation,

$$\partial_M j^M = 0 \,, \quad \text{with} \;\; j^M = \sum_{F=Q,D} j^M_F \,, \tag{4.61}$$

involving the individual currents given by Eq. (4.36)-(4.60) over the full orbifold domain, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$. The addition of the $S_X$ part to $S_{\text{bulk}}$ is not modifying the conservation relation (4.61) as the new brane terms entering the infinitesimal action variation – under the $\mathrm{U(1)_F}$ transformations – vanish because of their invariant form. The mathematical consistency of the relation (4.61) implies necessarily the (left/right) continuity of 5-current components over the whole spacetime and in particular a (left-/right) continuous $j^4$ along $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$. Besides, a discontinuity of the form, $j^4|_{-\pi R^+} \neq j^4|_{-\pi R} \equiv j^4|_{\pi R}$, would not fix any field at this fixed point and in turn would not induce vanishing variations in Eq. (4.54), which preserves the BC (4.55) and thus induce the drawbacks already pointed out in Section 4.5.1. As a consequence, we must consider the remaining model possibility:

$$j^4|_{-\pi R^+} = j^4|_{-\pi R} \equiv j^4|_{\pi R} = j^4|_{\pi R^-} \,, \tag{4.62}$$

where Eq. (4.4) is also invoked. On the other side, the current $j^4$ is odd under any type of $\mathbb{Z}_2$ transformation (4.9)-(4.12) as can be shown in a similar way as in Eq. (4.38):

$$j^4\Big|_{-\pi R^+} = -\, j^4\Big|_{\pi R^-} \,. \tag{4.63}$$

Combining Eq. (4.62) with Eq. (4.63) leads to,

$$j^4|_{\pi R^-} = j^4|_{\pi R} = j^4|_{-\pi R^+} = 0\,,$$

so that, using Eq. (4.39) and (4.60), we get the relation (inducing EBC),

$$j^4\Big|_{\pi R} = i\alpha\left(Q_L^\dagger Q_R - Q_R^\dagger Q_L + D_L^\dagger D_R - D_R^\dagger D_L\right)\Big|_{\pi R} = 0\,, \tag{4.64}$$

and its variation (for a non-trivial transformation with $\alpha \neq 0$),

$$\begin{aligned}\Big(\delta Q_L^\dagger Q_R + Q_L^\dagger \delta Q_R - \delta Q_R^\dagger Q_L - Q_R^\dagger \delta Q_L& \\ + \delta D_L^\dagger D_R + D_L^\dagger \delta D_R - \delta D_R^\dagger D_L - D_R^\dagger \delta D_L\Big)\Big|_{\pi R} &= 0\,.\end{aligned} \tag{4.65}$$

At this level, we can consider the search for field solutions of vanishing Eq. (4.54) and Eq. (4.64)-(4.65) first on the domain, $y \in [0, \pi R]$, which is equivalent to the search performed for the interval model in Section 3.6.2, $L \leftrightarrow \pi R$. Thus, we can directly apply the conclusion in Section 3.6.2 and claim that there exists no SM-like consistent solution for the fields (over $y \in [0, \pi R]$). As a conclusion, the introduction of EBC does not constitute the correct approach towards the treatment of point-like Yukawa interactions at a fixed point of the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold. Regarding the bulk fermion probability currents, both the cases of a $j^4$ jump and a $j^4$ continuity at the Yukawa coupling location $y = \pi R$, lead to inconsistent field solutions so that, at this stage of the study, there exists no theoretical proof of the $j^4$ continuity – and via Eq. (4.63) of its vanishing – at this fixed point.

### 4.5.3 Introducing the BBT

In order to get meaningful field solutions in the presence of brane-localized Yukawa couplings at the fixed point $y = \pi R$, let us finally try the introduction of the SM-like BBT (4.18) as in the free case described in Section 4.3.3 or as in the interval model in Section 3.6.3. We thus consider here the same action as in Eq. (4.53)-(4.59) but adding now the BBT at $y = \pi R$:

$$S_{\text{bulk}} + S_X + S_B\,.$$

Using Eq. (4.46) and Eq. (4.54), we find the following action variations with respect to $\bar{Q}$ and $\bar{D}$:

$$\begin{aligned}\delta_{\bar{Q}}(S_{\text{bulk}} + S_X + S_B) = \int d^4x \Bigg\{\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\; \delta\bar{Q}\, i\Gamma^M \partial_M Q +& \\ \left[-2\,\delta Q_L^\dagger\left(Q_R + \frac{X}{2} D_R\right) - X' \delta Q_R^\dagger D_L\right]\Big|_{\pi R} + 2\left(\delta Q_L^\dagger Q_R\right)\Big|_0 \Bigg\}&\,, \\ \delta_{\bar{D}}(S_{\text{bulk}} + S_X + S_B) = \int d^4x \Bigg\{\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\; \delta\bar{D}\, i\Gamma^M \partial_M D +& \\ \left[-X'^* \delta D_L^\dagger Q_R + 2\,\delta D_R^\dagger\left(D_L - \frac{X^*}{2} Q_L\right)\right]\Big|_{\pi R} - 2\left(\delta D_R^\dagger D_L\right)\Big|_0 \Bigg\}&\,.\end{aligned}$$

The individual vanishing of those volume and surface terms, due to the action minimisation, leads to the EOM (4.26) and the following NBC,

$$\begin{cases} \left(Q_R + \dfrac{X}{2}\, D_R\right)\Big|_{\pi R} = 0\,, \quad X'^*\, Q_R|_{\pi R} = 0\,, \quad Q_R|_0 = 0\,, \\ X'\, D_L|_{\pi R} = 0\,, \quad \left(D_L - \dfrac{X^*}{2}\, Q_L\right)\Big|_{\pi R} = 0\,, \quad D_L|_0 = 0\,, \end{cases} \tag{4.66}$$

which differ from the NBC (4.55) obtained without the BBT. As before, given the continuity region defined by Eq. (4.4), we can start by considering the search for profile solutions of 5D EOM (4.26) and 5D NBC (4.66) on the domain $y \in [0, \pi R]$, being equivalent to the search performed for the interval scenario (with the BBT) in Section 3.6.3 after two substitutions,

$$L \leftrightarrow \pi R \ \text{ and } \ X \leftrightarrow \frac{X}{2}\,,$$

which have been proposed in Section 4.4. The 4D field solutions in the mixed KK decomposition (4.56) obey the known Dirac-Weyl equations (1.25). Note that the factor 1/2 difference at the same places in NBC (4.66), compared to the interval NBC (3.87), comes from the existence of double numbers of surface terms (at $y = 0$, $0^-$ and $y = \pi R$, $-\pi R^+$) – like in Section 4.5.1 – and leads to the factor 1/2 in the final mass spectrum relations (4.70)-(4.71) through a re-normalization of the $X$ parameter as $X/2$. Besides, the necessary ortho-normalization condition (4.57) can be rewritten on the domain $[0, \pi R]$ only, as [the subscript $_C$ stands for $_L$ or $_R$],

$$\delta_{nm} = \frac{1}{\pi R} \int_0^{\pi R} dy \ [q_C^{n*}(y) q_C^m(y) + d_C^{n*}(y) d_C^m(y)] \,, \tag{4.67}$$

thanks to the change of variable, $y' = -y$, the fixed profile parities (4.58)-(4.32) and the continuity relations (4.4):

$$\int_{-\pi R^+}^{0-} dy \, [q_C^{n*}(y) q_C^m(y) + d_C^{n*}(y) d_C^m(y)] = \int_{0^+}^{\pi R^-} dy' \ [q_C^{n*}(-y') q_C^m(-y') + d_C^{n*}(-y') d_C^m(-y')]$$
$$= \int_{0^+}^{\pi R^-} dy' \ [q_C^{n*}(y') q_C^m(y') + d_C^{n*}(y') d_C^m(y')] = \int_0^{\pi R} dy \, [q_C^{n*}(y) q_C^m(y) + d_C^{n*}(y) d_C^m(y)] \,,$$

recovering thus exactly and conveniently the interval condition, if $L = \pi R$. Nevertheless, the dimensional wave functions $[\frac{1}{\sqrt{2\pi R}} f^n_{L/R}(y)]$ (4.56) in the orbifold are identical with that (3.50) in the interval framework only up to an additional normalization factor $(1/\sqrt{2})$ here, due to the double compact space size. Therefore, here we can finally apply the results in Section 3.6.3 so that

$$X' = 0\,,$$

leading to the new Yukawa coupling ($X$) dependent BC [denoted as ($\times$) at the brane located at $y = \pi R$]:

$$\begin{cases} \left(Q_R + \frac{X}{2}\, D_R\right)\Big|_{\pi R} = 0\,, \quad Q_R|_0 = 0\,, \\ \left(D_L - \frac{X^*}{2}\, Q_L\right)\Big|_{\pi R} = 0\,, \quad D_L|_0 = 0\,. \end{cases} \tag{4.68}$$

Referring to the SM-like consistent profile solutions in the free case in Eq. (3.89) over $y \in [0, \pi R]$, we obtain the dimensionless profiles,

$$\forall n \in \mathbb{N}\,, \quad \begin{cases} (+\times):\ q_L^n(y) = A_q^n \cos(M_n\, y)\,, & (-\times):\ q_R^n(y) = -A_q^n \sin(M_n\, y)\,, \\ (-\times):\ d_L^n(y) = A_d^n \sin(M_n\, y)\,, & (+\times):\ d_R^n(y) = A_d^n \cos(M_n\, y)\,, \end{cases} \tag{4.69}$$

for the two classes of real mass spectrum solutions ($\alpha_0^n \in \mathbb{R}$),

$$I \quad : \quad \tan(M_n\, L) = \left|\frac{X}{2}\right| \qquad \Rightarrow \quad A_q^n = e^{i(\alpha_0^n + \alpha_Y)}\,, \quad A_d^n = e^{i\alpha_0^n}\,, \tag{4.70}$$

$$II \quad : \quad \tan(M_n\, L) = -\left|\frac{X}{2}\right| \quad \Rightarrow \quad A_q^n = e^{i(\alpha_0^n + \alpha_Y \pm \pi)}\,, \quad A_d^n = e^{i\alpha_0^n}\,, \tag{4.71}$$

and for the absolute values of the fermion masses [based on Eq. (3.47)],

$$|M_n| = \frac{1}{\pi R} \left| \arctan \left| \frac{X}{2} \right| + (-1)^n \, \tilde{n}(n) \, \pi \right| . \tag{4.72}$$

Note that the sign of $M_n$ is not physical as demonstrated in Section 3.6.3 via the transformation in Eq. (3.95). At this stage, the part of the profile solutions on the complementary region, $y \in [-\pi R^+, 0^-]$, is deduced through the four types of $\mathbb{Z}_2$ transformations (4.58)-(4.32). Hence, the $M_n$ spectrum entering the profile solutions in both regions, $[0, \pi R]$ and $[-\pi R^+, 0^-]$, is the same.

As a first conclusion, the introduction of the BBT allows to obtain realistic fermion wave functions and consistent mass eigenvalues. The absolute mass spectrum obtained within the 5D approach in Eq. (4.72) is analytically matching the one derived via the 4D method in Eq. (4.52) for a real Yukawa coupling constant: this feature represents a non-trivial confirmation of the present exact results. In particular, the absence of $X'$ parameter in the fermion 4D mass matrix $\mathcal{M}$, described below Eq. (4.49), is interestingly recovered through the mass independence from $X'$, issued from the 5D NBC (4.68).

Regarding the probability current, the component $j^4|_{\pi R}$ at the Yukawa brane is still given by Eq. (4.64) since the BBT do not affect it, as explained at the end of Section 4.3.3. The relations found in the first line of the NBC (4.66), injected once into each term of this current component expression, give rise to,

$$j^4|_{\pi R} \; = \; 0 \, .$$

The BBT are thus found to induce NBC leading to a vanishing current component along the extra dimension at the fixed points of the orbifold, with (present section) or without (cf. Section 4.3.3) a brane-localized Yukawa coupling, and in turn to a continuous current component along the extra dimension at those points given the odd parities, demonstrated in Eq. (4.63) or (4.38) respectively.

| Continuity domains | $\mathbb{Z}_2$ | Fields | | | |
|---|---|---|---|---|---|
| | | $Q_{L/R}$ | | $D_{L/R}$ | |
| | | $q^n_L(y)/(\pm e^{i(\alpha^n_0+\alpha_Y)})$ | $q^n_R(y)/(\pm e^{i(\alpha^n_0+\alpha_Y)})$ | $d^n_L(y)/e^{i\alpha^n_0}$ | $d^n_R(y)/e^{i\alpha^n_0}$ |
| $[0, \pi R]$ | Any | $\cos(M_n\, y)$ | $-\sin(M_n\, y)$ | $\sin(M_n\, y)$ | $\cos(M_n\, y)$ |
| $[-\pi R^+, 0^-]$ | I | $\cos(M_n\, y)$ | $-\sin(M_n\, y)$ | $\sin(M_n\, y)$ | $\cos(M_n\, y)$ |
| | II | $-\cos(M_n\, y)$ | $\sin(M_n\, y)$ | $-\sin(M_n\, y)$ | $-\cos(M_n\, y)$ |
| | III | $\cos(M_n\, y)$ | $-\sin(M_n\, y)$ | $-\sin(M_n\, y)$ | $-\cos(M_n\, y)$ |
| | IV | $-\cos(M_n\, y)$ | $\sin(M_n\, y)$ | $\sin(M_n\, y)$ | $\cos(M_n\, y)$ |
| KK Masses | $\|M_n\| = \|\arctan\|X/2\| + (-1)^n\, \tilde{n}(n)\, \pi\| /\pi R,\ n \in \mathbb{N}$ | | | | |

Table 4.3 – SM-like coupled fermion profiles on the two orbifold continuity domains $[-\pi R^+, 0^-]$ and $[0, \pi R]$, corresponding to the solutions (4.69), (4.70)-(4.71), together with the associated absolute mass spectrum (4.72) for completeness. The profiles are given for the four types of $\mathbb{Z}_2$ transformations (4.58)-(4.32).

In Table 4.3 are exhibited the explicit profile functions over the entire orbifold domain for the SM-like solutions (4.69), (4.70)-(4.71), (4.72). We can see in this table that the choice of type of $\mathbb{Z}_2$ transformation is purely a convention because it can modify the profile signs but without any effect on the mass spectrum.

In Figure 4.4, we illustrate a set of excitation profiles, obeying the $\mathbb{Z}_2$ transformations of types I and II in Eq. (4.58)-(4.32), for the found solutions, which are explicitly presented in Table 4.3, within the simplified real case, $\alpha_Y = \alpha_0^n = 0$. We observe in this figure that all the wave function values at the Yukawa-brane (at the fixed point, $y = \pi R$) are modified due to the presence of this coupling. For example, under the Type I of $\mathbb{Z}_2$ transformation, the profile values $d_L^n(\pi R) = d_L^n(\pi R^-)$ are shifted from zero as well as from $d_L^n(-\pi R^+)$, in contrast to the free case shown in Figure 4.3. This shift creates profile jumps whose amplitude is depending on the Yukawa coupling constant through the $X$ parameter [BC $(\times)$ in Eq. (4.68)]. The presence of new possible profile discontinuities justifies once more mathematically the prescriptions about the field continuities and action integration domains introduced in Section 4.2.1.

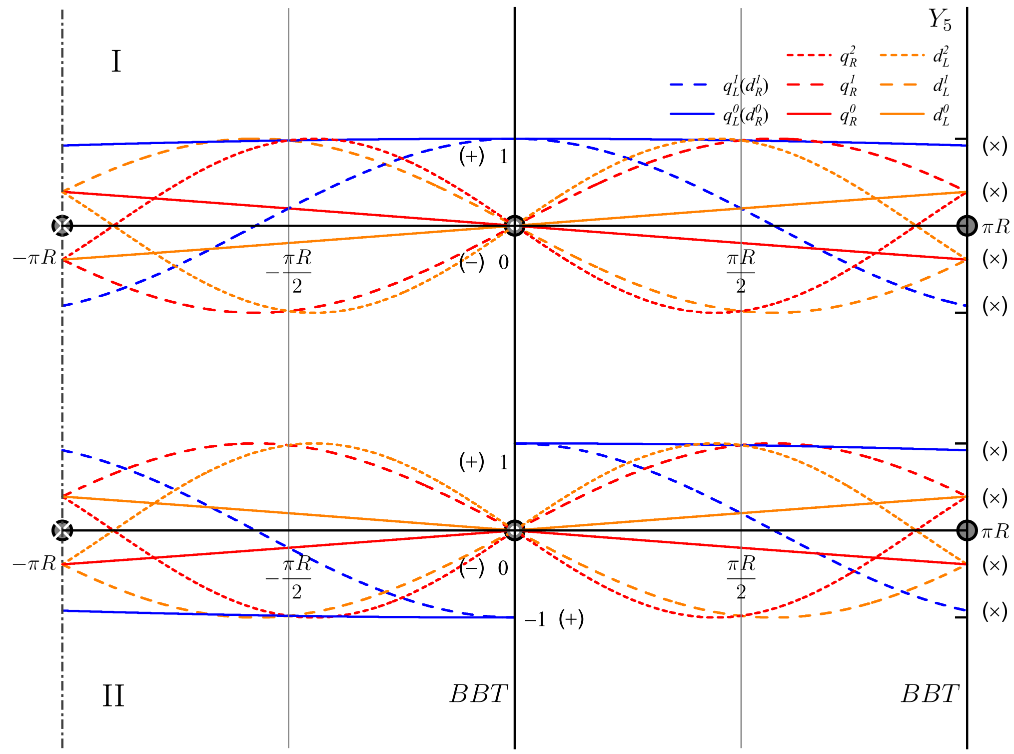


Figure 4.4 – Zero-mode and excitation wave functions $q_{L/R}^n(y)$, $d_{L/R}^n(y)$, with $n = 0, 1, 2$, along the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold domain, $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$, corresponding to the Yukawa-coupled solutions (4.70), presented in Table 4.3, for the simplified case, $\alpha_Y = \alpha_0^n = 0$, and the two different types of $\mathbb{Z}_2$ transformations, I, II from Eq. (4.58)-(4.32). The two fixed points at, $y = 0$, $y = \pi R \equiv -\pi R$, the BC, $(-)/(+)/(\times)$, the BBT and Yukawa coupling brane-locations are indicated on the graph.

Finally, let us calculate, still without any kind of Higgs field regularisation, the 4D effective Yukawa coupling constants [defined in Eq. (3.96)] between the mass eigenstates $\psi_L^n(x^\mu)$ and $\psi_R^m(x^\mu)$ as generated by the insertion of decompositions (4.56) into Eq. (4.17), based on the obtained profile expressions (4.69), (4.70)-(4.71), (4.72):

$$
\begin{aligned}
y_{nm} &\hat{=} \frac{Y_5}{2\sqrt{2}\pi R}\, q_L^{n*}(\pi R)\; d_R^m(\pi R) \\
&= \pm\frac{|Y_5|}{2\sqrt{2}\pi R}\, e^{i(\alpha_0^m-\alpha_0^n)} \cos(M_n\,\pi R)\cos(M_m\,\pi R) \\
&= \pm(-1)^{\tilde{n}(n)+\tilde{n}(m)}\, e^{i(\alpha_0^m-\alpha_0^n)}\, \frac{|Y_5|}{2\sqrt{2}\pi R(1+|X/2|^2)}\,,
\end{aligned}
\tag{4.73}
$$

where we have used the trigonometric identity [inserted in Eq. (3.97)] to get the last equality. Note that the modulus of the 4D effective Yukawa couplings reads

$$
|y_{nm}| = \frac{|Y_5|}{2\sqrt{2}\pi R(1+|X/2|^2)}\,,
\tag{4.74}
$$

which is independent of the KK mixing indexes $nm$. In the decoupling limit of extremely heavy KK modes, $R \to 0$, we can then write the modulus of the lightest mode coupling constant, using Eq. (4.15)-(3.15), as,

$$
|y_{00}| \underset{R\to 0}{\rightarrow} \frac{|Y_5|}{2\sqrt{2}\pi R} = \frac{|y_4|}{\sqrt{2}}\,,
\tag{4.75}
$$

and the absolute mass eigenvalue of the lightest eigenstates as [from Eq. (4.72)],

$$
|M_0| \underset{R\to 0}{\rightarrow} \frac{|X|}{2\pi R} = \frac{v|Y_5|}{2\sqrt{2}\pi R} \underset{R\to 0}{\rightarrow} v\,|y_{00}|\,,
\tag{4.76}
$$

so that the SM fermion setup – for the assumed single family – is recovered as expected from the decoupling condition. Besides, we can conclude that the choice of type of $\mathbb{Z}_2$ transformation among Eq. (4.58)-(4.32) affects neither the profile values taken at the point $y = \pi R$ – see Table 4.3 – nor their global ortho-normalization condition (4.57) – as described right below Eq. (4.67) – so that the 4D effective Yukawa coupling constants (4.73) are insensitive as well to this $\mathbb{Z}_2$ representation choice.

## 4.6 The Inclusive $\mathbb{Z}_2$ Parity

Let us study the alternative scenario whose definition is based on the $\mathbb{Z}_2$ transformation of 5D fields extended to include the two fixed points at $y = 0$ and $y = \pi R$:

$$
\Phi(x^\mu, -y) = \mathcal{T}\Phi(x^\mu, y)\,, \quad \forall y \in (-\pi R, \pi R]\,,
\tag{4.77}
$$

in contrast with Eq. (4.1). This generic transformation still lets the Lagrangian density invariant, exactly like in Eq. (4.2). At the two fixed points, this Lagrangian invariance is once more automatically satisfied without the need for any specific $\mathcal{T}$ transformation. Accordingly to the simple Eq. (4.77), the operator $\mathcal{T}$ for the fixed points is the same as the non-trivial one which must let the Lagrangian invariant in the bulk. Let us consider in particular the realistic $\mathbb{Z}_2$ transformation leading to the SM chirality setup: it is the bulk transformation in Eq. (4.9), defined now over the same range as in Eq. (4.77), which keeps well $\mathcal{L}_{\text{kin}}$ invariant in the bulk according to Eq. (4.2):

$$
\forall y \in (-\pi R, \pi R]\,, \quad
\begin{cases}
Q\,(x^\mu, -y) = -\gamma^5\, Q\,(x^\mu, y)\;, \\
D\,(x^\mu, -y) = \gamma^5\, D\,(x^\mu, y)\;.
\end{cases}
\tag{4.78}
$$

Focusing on the fixed points at $y = 0$ and $y = \pi R \equiv -\pi R$, we obtain the four non-trivial relations

$$\begin{cases} Q\,(x^\mu, 0) &=& -\gamma^5\, Q\,(x^\mu, 0) \;\Rightarrow\; Q_R|_0 = -\, Q_R|_0 = 0\,, \\ D\,(x^\mu, 0) &=& \gamma^5\, D\,(x^\mu, 0) \;\;\;\Rightarrow\; D_L|_0 = -\, D_L|_0 = 0\,, \end{cases}$$
$$\begin{cases} Q\,(x^\mu, \pi R) &=& -\gamma^5\, Q\,(x^\mu, \pi R) \;\Rightarrow\; Q_R|_{\pi R} = -\, Q_R|_{\pi R} = 0\,, \\ D\,(x^\mu, \pi R) &=& \gamma^5\, D\,(x^\mu, \pi R) \;\;\;\Rightarrow\; D_L|_{\pi R} = -\, D_L|_{\pi R} = 0\,, \end{cases} \qquad [\text{EBC}'] \quad (4.79)$$

representing the new EBC that we denote EBC' to distinguish them from those in Eq. (4.40).

In the free case, Section 4.3.1 has shown that the EBC(') or the BBT must be considered. Starting with the EBC('), in analogy to Section 4.3.2, the fixed $\mathbb{Z}_2$ transformations (4.78) in the bulk lead to the EBC (4.40) while the $\mathbb{Z}_2$ transformations (4.79) at the fixed points lead to the EBC'. Those EBC' select one general BC set among these four EBC sets for the 5D field $Q$, and same statement for $D$: the sets corresponding to the chiral solution of line 1 (2) in Eq. (4.41) for the field $D$ ($Q$), namely the SM-like chirality configuration. Finally, the complete profile solutions over the whole orbifold domain are found out as before via the bulk $\mathbb{Z}_2$ transformations (4.78).

Alternatively, the selected consistent BBT (4.18) can be included like in Section 4.3.3 to obtain the same SM-like solutions. The corresponding EBC' (4.79), part of the EBC (4.40) and required by the model, are checked to be satisfied afterwards, as consequences.

Once the free profiles are worked out as described right above – either through the EBC(') or the BBT – we can apply the 4D method prescription in Section 4.4, based on infinite matrix bi-diagonalization, in order to derive the mass spectrum in the presence of brane-localized Yukawa couplings. Even the 4D effective Yukawa coupling constants can be calculated in this way: the above EBC' selection of a specific chirality setup and mass spectrum for the free fields would affect as well these effective coupling constants, for instance via the KK mass mixings.

In contrast, we need to emphasize that the analysis of point-like Yukawa interactions cannot be achieved via the 5D approach within the present inclusive $\mathbb{Z}_2$ symmetry model.

First, motivated by Section 4.5.1, the essential boundary conditions can be classified into the EBC coming directly from the vanishing probability currents – indirectly from the fixed $\mathbb{Z}_2$ transformations (4.78) in the bulk – discussed in Section 4.5.2 and the EBC' (4.79). These EBC' combined with the surface terms at $y = \pi R$ in Eq. (4.54), including the Yukawa terms, give rise to the BC of type (4.55). Considering non-vanishing Yukawa coupling constant $X \neq 0$, all the fields are forced to vanish at $y = 0, \pi R$. Hence, the resulting mass spectrum looses its dependence on the Yukawa coupling constant [14] which conflicts with the decoupling limit argument [see Eq. (4.76)].

Secondly, the BBT (4.18) could be added like in Section 4.5.3 to try obtaining SM-like solutions. However the EBC' (4.79), expected to be recovered afterwards, are not compatible with the resulting BC (4.69) together with the spectrum equations (4.70)-(4.71).

## 4.7 Result Analysis

14. Actually, the independence of the Yukawa coupling constants can be concluded from the EBC' (4.79) directly via $Q_R$ (or $D_L$) since both of its EOM and BC are Yukawa independent.

### 4.7.1 The Higher-Dimensional Method

The present study confirms the general methodology depicted in Figure 3.4 presented in Section 3.7. Within the present model, the probability current condition on this schematic description is the vanishing of fermion currents (4.39) at the two fixed points, issued from $\mathbb{Z}_2$ symmetry criteria and inducing the EBC (4.40). For the interval model, the vanishing current condition is a direct implication of the existence of boundaries for the matter fields. This current vanishing holds both in the presence and absence of brane-localized Yukawa couplings.

In the framework of the orbifold version described in Section 4.6, the additional field condition (4.79), coming from the $\mathbb{Z}_2$ symmetry at the fixed points, accompanies the definition of the $\mathbb{Z}_2$ symmetry of the bulk action and leads to the new EBC' (4.79).

### 4.7.2 Discussion of the Action Content

In addition to the information contained in the action (4.5)-(4.21), the present orbifold model is defined in a complementary way by other elements like:

(i) The $\mathcal{S}^1$ junction point at $y = \pi R \equiv -\pi R$.

(ii) The choices of $\mathbb{Z}_2$ transformations for the fields in the bulk [see Eq. (4.9)-(4.12)] and possibly at the fixed points [cf. Eq. (4.79)].

(iii) The EBC (4.40) imposed by the model definition when those are used instead of the BBT. Table 4.4 summarizes the obtained cases where the EBC and the BBT can be used. This table is identical to the one 3.1 obtained in the interval model study (see Section 3.7).

| | **No boundary characteristic** | **Vanishing current condition [EBC]** | **Bilinear brane terms [NBC]** |
|---|---|---|---|
| **4D approach** | $(Impossible)$ | BC $(\pm)$ | BC $(\pm)$ |
| **5D approach** | $(Impossible)$ | $(Impossible)$ | BC $(\times)$ |

Table 4.4 – Types of boundary conditions for the bulk fermions at an orbifold fixed point where their interactions with the Higgs boson locate, in different brane treatments: presence of BBT, vanishing of probability current or nothing specific. The 4D line holds as well for the 5D approach of the free brane. As usually, the Dirichlet BC are noted $(-)$, the Neumann BC $(+)$ and we denote $(\times)$ the new BC depending on the Yukawa coupling constant [corresponding to Eq. (4.69) taken at $y = \pi R$].

## 4.8 About the Orbifold/Interval Duality

The present $\mathcal{S}^1/\mathbb{Z}_2$ orbifold model and the $[0, L]$ interval scenario studied in Chapter 3 are physically different in two crucial aspects:

(i) Geometrical setups;

(ii) Lagrangian symmetries.

Nevertheless, the respective theoretical predictions for the observables like the (brane-coupled) fermion mass spectra and 4D effective Yukawa coupling constants are identical up to factors [2], which may be called a duality. Indeed, for a comparable dimension size $L = \pi R$, although the mass absolute values (4.72) involve a new factor $1/2$ in front of $X$, with respect to the interval analytical result (3.91), the measurable range of values for $|M_n|$ is of the same order and the precise limits of this range rely on the approximate perturbative limits of the 4D effective Yukawa coupling constants proportional to $y_4$ (4.15) [see Section 4.2.4 and Eq. (4.73)-(4.76)]. Besides, the dependence of the analytical mass formula on the Lagrangian parameters is identical in the two models, up to this factor $1/2$ entering the coupling constant definition, as can be seen from Eq. (4.72) and Section 4.2.4 – including the free limiting case $X \to 0$. Similar comments hold for the 4D effective Yukawa coupling constants (4.73) which have additional factors $1/2$ in front of $X$ and as an overall factor (latter one induced by ortho-normalization considerations), with respect to the interval case.

The orbifold version of Section 4.6 contains additional information at the fixed point branes. It predicts thus a specific chirality configuration and mass spectrum [among chiral or vector-like solutions respectively of type (4.41)-(4.42)] so that it is not dual to the interval model.

Coming back to the case of duality, there exist similarities between the orbifold and interval models, as it appeared throughout this work when solving the EOM and the (N,E)BC to find out the fields. Let us now comment on the similarities at the Lagrangian level. First, the BBT (4.18) have the same form as in the interval framework (3.4) and the different factor 2 is related to the double size of the compactified space for the identification, $L = \pi R$.

In the global action (4.5), $S_{\text{bulk}}$ remains to be discussed, the other parts being identical in the orbifold and interval models. Thanks to the orbifold property – $\mathbb{Z}_2$ Lagrangian symmetry (4.2), the change of variable $y' = -y$, allows the following rewriting of the bulk action (4.6),

$$\begin{aligned} S_{\text{bulk}} &= \int d^4x \left\{ \int_{-\pi R^+}^{0^-} dy \ \mathcal{L}_{\text{kin}}(y) + \int_0^{\pi R} dy \ \mathcal{L}_{\text{kin}}(y) \right\} \\ &= \int d^4x \left\{ \int_{0^+}^{\pi R^-} dy' \ \mathcal{L}_{\text{kin}}(y') + \int_0^{\pi R} dy \ \mathcal{L}_{\text{kin}}(y) \right\} \\ &= 2 \times \int d^4x \ \int_0^{\pi R} dy \ \mathcal{L}_{\text{kin}}(y) \,, \end{aligned} \tag{4.80}$$

where the last step is based on the continuity condition in Eq. (4.4). Therefore, using the relevant dimension identification $L = \pi R$, we can express the orbifold action (4.5)-(4.21) in terms of the interval action pieces (3.17)-(3.18) (indicated by the $L$ exponent):

$$\begin{aligned} S_{5\text{D}} &= 2\, S^L_{\text{bulk}} + S^{(L)}_H + S^{(L)}_X + S^{(L)}_{hQD} + 2\, S^L_B \\ &= 2 \left\{ S^L_{\text{bulk}} + \frac{1}{2} \left[ S^{(L)}_X + S^{(L)}_{hQD} \right] + S^L_B \right\} + S^{(L)}_H \\ &= 2 \left\{ S^L_{\text{bulk}} + S^{(L)}_{X/2} + S^{(L)}_{hQD}|_{Y_5/2} + S^L_B \right\} + S^{(L)}_H \,. \end{aligned} \tag{4.81}$$

This re-expression reveals an alternative method to derive the fermion masses and couplings, which are independent from the pure scalar part, namely $S^{(L)}_H$. The idea is that, within the orbifold model now described by the action (4.81) importantly together with the description of the $\mathbb{Z}_2$ symmetry over $\mathcal{S}^1$, we can first search for the field parts along the limited domain $[0, \pi R]$. This search is in fact based on the action $[S^L_{\text{bulk}} + S^{(L)}_{X/2} + S^{(L)}_{hQD}|_{Y_5/2} + S^L_B]$, since the overall factor 2 in Eq. (4.81) affects neither the EOM (global factor) nor the BC (same factor in front of the surface terms and pure brane terms combined into BC) [15], and is in turn strictly equivalent to solving the interval model. Given this action, the solutions can be obtained for the 4D masses in Eq. (3.91) (and 4D effective Yukawa coupling constants from profile overlaps with the Higgs boson peak at $y = \pi R$ in Eq. (3.97)), but involving a re-normalized coupling parameter $X/2$. The last stage of this technique is the extension of the obtained profiles over the complete orbifold domain via the $\mathbb{Z}_2$ transformations, before applying the ortho-normalization condition. The 4D effective Yukawa coupling constants are then changed by an additional factor $1/2$, as is clear from the wave function normalization forms (4.57)-(4.67), which confirms the result (4.73). On the other side, we see as well that the fermion masses so obtained (unchanged by the spatial domain extension) involve only a new normalised parameter $X/2$, with respect to Eq. (3.91), which confirms the found spectrum (4.72).

Beyond these action correspondences, there are other elegant similarities. For example, as illustrated by Figure 3.4, both the interval and orbifold scenarios lead to the same vanishing probability current conditions at the two branes (and hence to identical EBC); those current conditions come, respectively, directly from the interval boundary criteria and indirectly from $\mathbb{Z}_2$ symmetry considerations. Besides, Table 4.4 shows that the same treatments of the two branes, at the fixed points or interval boundaries, must be adopted in identical situations and that the same BC are generated.

Finally, let us propose an intuitive description for understanding the orbifold versus interval model duality. The obtained wave functions for the bulk fermions on the interval are of the kind $\cos(M_n\, y) \propto (e^{iM_n\, y} + e^{-iM_n\, y})$, coming in factor (via the KK decomposition) of the energy coefficients $e^{\pm iE\, t}$ in the 4D Dirac fields, which gives rise to wave planes propagating in both $y$-directions of the interval with momenta $\pm p_n = \pm M_n$ – as for oscillations left-moving and right-moving along opposite directions in the world-sheet parameter space of strings. The associated particle, going in the direction $L \to 0$ and then coming back along $0 \to L$, reproduces the propagation along $\mathcal{S}^1$, following consecutively the two fundamental domains $-\pi R \to 0^-$ and $0^+ \to \pi R$ of the orbifold (effectively equivalent orientations of the circle in the bulk so a unique propagation direction chosen along it): exactly the same $\mathcal{L}\,[\Phi(x^\mu, y)]$ Lagrangian evolution is felt by this particle during those dual travelings along the extra $y$-dimension, in the two different models, as is clear from the Lagrangian $\mathbb{Z}_2$ symmetry depicted in the drawing 4.2.

## 4.9 Conclusions

In the study of the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold, the proper action definition through improper integrals has allowed to obtain consistent bulk profile solutions with possible discontinuities at the fixed points. In particular the point-like interaction of Yukawa creates a profile jump.

These solutions have been obtained without brane-Higgs regularization, by relying on the necessary EBC, coming from vanishing fermion probability currents, or alternatively on the introduction of BBT in the action. The associated calculations have been confirmed by the matching between the 4D and the 5D approaches of the analytical results for the

15. This search could also be constrained by vanishing currents at $y = 0, \pi R$ instead of the $S^L_B$ presence, in the free case, as shown in Sections 4.3.2 and 4.5.2.

fermion mass spectrum and 4D effective Yukawa coupling constants.

The orbifold version, with $\mathbb{Z}_2$ transformations of the fields extended to the fixed points, was shown to be able to generate the chiral nature of the theory and even to select the expected SM chirality configuration for the 4D states.

The duality between the interval and orbifold scenarios has been deeply described. It has also constituted the opportunity to point out an alternative method for calculating the tower of excitation masses and 4D Yukawa couplings.

# Chapter 5

# Distribution Formalism for the $\mathcal{S}^1/\mathbb{Z}_2$ Orbifold

This chapter is based on a work in progress in collaboration with Grégory MOREAU and Florian NORTIER.

## 5.1 Introduction

The field treatments in the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold model, with a brane-localized Higgs field, have been precisely described in Chapter 4 without any brane-Higgs regularization (see Section 3.5.2). In particular, the focus was put on the treatment of possible field jumps at fixed points, via the introduction of improper integrals, as well as on the necessity of fermionic bilinear brane terms: the so-called BBT. We emphasize that the fields were mathematically described there, as usual, as spacetime functions. In this chapter, we attempt to find an alternative rigorous treatment of such an orbifold model, also motivated by the presence of a Dirac peak localizing the Yukawa interactions.

Recall that the Dirac peak also appears frequently in quantum field theory. In the early 1950s, Arthur Wightman attempted to develop a mathematically rigorous formulation of quantum field theory with one crucial mission to treat properly the Dirac peak [162] (see Section 5.2). Motivated by Wightman's prescription (without anymore brane-Higgs regularization in this chapter), we reformulate the function formalism of Chapter 4 through a new rigorous formalism based on fields as distributions [1] (further distinguished from the regularized Dirac peak function (3.63) of Section 3.5).

Moreover, this re-expression procedure towards the language of distributions allows the automatic appearance of the BBT. Concretely, we define the Lagrangian density via regular and Dirac distributions for an introduction to the theory of distributions, which act on test functions whose supports are included in the compactified geometry $\mathcal{S}^1$ [see Appendix H]. We use the distribution theory to re-define the partial derivatives $\partial_y$ in the bulk kinetic terms as weak derivatives for the discontinuous odd fields. Indeed, the fields can be even (odd) with respect to each fixed point, and the branes at the fixed points are not boundaries of the covering space $\mathcal{S}^1$ so that we have to discuss if the fields are continuous or discontinuous across the branes. Then, the BBT [identical to the ones from the function formalism in Eq. (4.18)] turn out to originate spontaneously from these weak partial derivatives of the odd discontinuous fields at the fixed points. It is important to

1. See Refs. [115, 116] for an introduction to the theory of distributions by L. Schwartz and an application to quantum mechanics.

understand here that the BBT are not added by hand to the orbifold $\mathcal{S}^1/\mathbb{Z}_2$ description but come from those weak partial derivatives.

The consistency of the distribution formalism will be confirmed by the exact recovering of the Lagrangian density, expressed in terms of the fields as simple spacetime functions, and hence then of the fermion profile solutions as well as the analytical results for the fermion mass spectrum and 4D effective Yukawa coupling constants. Nevertheless, some conventions about the orbifold parity differ: in contrast to the function formalism in Chapter 4, it turns out that each $\mathbb{Z}_2$ transformation of the fields will be shown to fix the chirality configuration. Thus, the $\mathbb{Z}_2$ symmetry can generate the chiral nature of the theory and the SM chirality setup by itself, in the decoupling limit. This origin for the chirality configuration is not generated within the function prescription. Mathematically, we will have to build rigorously the five-dimensional models by introducing the Kurasov's distributions for the bulk fermions in order to handle boundary-localized interactions at the fixed points possibly inducing profile jumps.

This chapter is organized as follows. After having briefly recalled the basics of the Wightman's distribution theory, we describe the minimal model in Section 5.3. Then, the treatment of the free case is presented in Section 5.4.1 before a 4D and a 5D approach of the Yukawa case is exposed. An overview is provided in Section 5.6 and a brief description of the relationship with the well-described function treatment [in Chapter 4] is summarized in Section 5.7. To discuss the comparison with the function formalism, the specific inclusive $\mathbb{Z}_2$ transformation studied in Section 4.6 is also used.

## 5.2 Wightman's Distribution Theory

In order to improve the annoying treat the Dirac peak, Arthur Wightman promoted the quantized free field to a so-called Wightman field $\left\{\hat{\phi}\right\}$ – an *operator-valued tempered distribution*, satisfying the *Wightman Axioms* [162]. The smeared field $\left\{\hat{\phi}\right\}[f]$ [2]

$$\forall\, f \in \mathcal{S}(\mathbb{R}^4)\,, \quad \left\{\hat{\phi}\right\}[f] \,\hat{=} \int d^4x \;\hat{\phi}(x) f(x)\,, \tag{5.1}$$

is a well-defined operator on a domain in Fock space and $f \in \mathcal{S}(\mathbb{R}^4)$ is a test function. Thus, any product of such $\left\{\hat{\phi}\right\}[f]$ would get an operator. Moreover, causality is guaranteed by the commute condition $\left\{\hat{\phi}\right\}[f]\left\{\hat{\phi}\right\}[g] = \left\{\hat{\phi}\right\}[g]\left\{\hat{\phi}\right\}[f]$ if $\operatorname{supp} f$ is space-like to $\operatorname{supp} g$ [3]. Wightman also postulated $\left\{\hat{\phi}\right\}[f]$ is symmetric on its dense domain $D$ in the Hilbert space of states, i.e.

$$\forall\, \Phi,\, \Psi \in D\,, \quad \langle \Psi | \left\{\hat{\phi}\right\}[f] \Phi\rangle = \langle \left\{\hat{\phi}\right\}[f] \Psi | \Phi \rangle\,.$$

Based on the Schwartz's nuclear theorem, the *n-point Wightman distribution* $\widetilde{\mathcal{W}_n} \in \mathcal{S}'(\mathbb{R}^4 \times \mathbb{R}^4 \cdots \times \mathbb{R}^4)$ [mapping $n$ test functions to a complex number] is defined through the vacuum expectation value for the Wightman vacuum ($\Omega \in D$),

$$\widetilde{\mathcal{W}_n}[f(x_1, x_2, \ldots, x_n)] \,\hat{=}\, \langle \Omega | \left\{\hat{\phi}\right\}[f_1] \left\{\hat{\phi}\right\}[f_2] \cdots \left\{\hat{\phi}\right\}[f_n] \Omega \rangle\,, \tag{5.2}$$

2. The space $\mathcal{S}(\mathbb{R}^4)$ consists of infinitely differentiable real functions of four variables, which go to zero at infinite infinitely faster than any power of Euclidean distance, which is also furnished with furnished by Schwartz [115, 116].

3. The support of function $f$ is defined by $\operatorname{supp} f \,\hat{=}\, \left\{ x^\mu \in \mathbb{R}^4 \middle|\, f(x) \neq 0 \right\}$.

where $f(x_1, x_2, \dots, x_n) \in \mathcal{S}(\mathbb{R}^4 \times \mathbb{R}^4 \cdots \times \mathbb{R}^4)$ is defined by

$$f(x_1, x_2, \dots, x_n) \mathrel{\hat{=}} (f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x_1, x_2, \dots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n) \,.$$

Then, invoking the canonical quantized $n$-point correlation function $G_n(x_1, x_2, \dots, x_n)$ of the standard quantum field theory, the Wightman distribution (5.2) ($\widetilde{\mathcal{W}_n}$ assumed to be a regular distribution) can be rewritten as,

$$\widetilde{\mathcal{W}_n}[f(x_1, \dots, x_n)] = \int d^4x_1 \cdots d^4x_n \; G_n(x_1, \dots, x_n) f_1(x_1) \cdots f_n(x_n) \,. \tag{5.3}$$

Even if $G_n(x_1, \dots, x_n)$ is singular (like a Dirac peak), $\widetilde{\mathcal{W}_n}[f(x_1, \dots, x_n)]$ still has a rigorous definition but $\widetilde{\mathcal{W}_n}$ would be instead a singular distribution. Eq. (5.3) can thus be used as a new definition – via the unique correspondance between the $f$ functions and $\widetilde{\mathcal{W}_n}[f(x_1, \dots, x_n)]$ – of the Green function, a general rigorous definition: the interest of this formalism.

Similarly, we will develop a distribution formalism to treat discontinuous fields on a compact space. Compared with Wightman's field in Eq. (5.1), the kinetic Lagrangian $\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)$ of Eq. (5.8) will be defined as a (not operator-valued) distribution acting on certain test functions along $\mathcal{S}^1$, as will be described precisely in Section 5.3.1.

## 5.3 Minimal $\mathcal{S}^1/\mathbb{Z}_2$ Consistent Model

### 5.3.1 Geometry and the Proper Action via Distribution Formalism

We consider the 5D spacetime structure via the product geometry $\mathcal{M}^4 \times \mathcal{S}^1/\mathbb{Z}_2$ explicitly described in Section 4.2.1, which is labeled by the coordinates, $x^M \mathrel{\hat{=}} (x^\mu, y)$, $M \in [\![0, 4]\!]$ where the Lorentz index $x^\mu$, $\mu \in [\![0, 3]\!]$ represents the usual 4D Minkowski spacetime manifold whose coordinates and the circle $\mathcal{S}^1$ is characterized by the coordinate $y \in (-\pi R, \pi R]$ with a radius $R$ in Figure 4.2.

As we urged in Chapter 4, possible field jumps at the points $y = 0, \pi R$ must be taken into account. Thus, we maintain the left/right continuity of a generic profile at $y = 0, \pi R$ conventionally defined in Eq. (4.4) and we consider a 5D field noted generically $\Phi(x^\mu, y)$ being piece-wise smooth along the extra dimension. Hence the Kurasov's distributions must be introduced, instead of the Schwartz's distribution theory whose regular distributions rely exclusively on continuous functions. Along these lines, we also configure test K-functions (H.57) rather than test S-functions (H.16) of the Schwartz's distribution theory (cf. Appendix H.2) which just manipulates smooth test functions. The mathematical properties of the function associated to a regular K-distribution or of a test K-function (H.57) (cf. Appendix H.4) read as,

$$\forall\, x^\mu \in \mathcal{M}^4 \,, \quad \Phi(x^\mu, y) \in \mathcal{C}^\infty([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C}) = K(\mathcal{S}^1, \mathbb{C}) \,, \tag{5.4}$$

where $K(\mathcal{S}^1, \mathbb{C})$ is the test K-function space defined in Eq. (H.57). $\Phi$ can be embedded into a continuous linear functional $\widetilde{\Phi} \in K'(\mathcal{S}^1, \mathbb{C})$ on the test K-function space $K(\mathcal{S}^1, \mathbb{C})$, i.e. a regular K-distribution on $\mathcal{S}^1$ [see Eq. (H.58)], which will play a crucial role for the Lagrangian building.

Then, in contrast to the function formalism in Section 4.2.1 where the bulk action is decomposed via improper integrals over $[-\pi R^+, 0^-] \cup [0, \pi R]$, here the well-defined global action of this model should be written as a sum of bulk terms and some brane terms,

$$S_{5\text{D}} = S^{\mathcal{D}}_{\text{bulk}} + S^{\mathcal{D}}_{\text{branes}} \,, \tag{5.5}$$

which has an identical generic formula as in the function formalism [see Eq. (4.5)]. However, the 4D effective Lagrangian density is constituted by a K-distribution acting on a constant test K-function $\varphi(y) \equiv 1 \in K(\mathcal{S}^1, \mathbb{C})$,

$$S^{\mathcal{D}}_{\text{bulk}} \hat{=} \int d^4x \; \widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)\,[1] \;, \tag{5.6}$$

where the K-distribution $\widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)$ reads [4],

$$\begin{aligned} \forall\, x^\mu \in \mathcal{M}^4\,,\; \widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu) \;&:\; K(\mathcal{S}^1, \mathbb{C}) \to \mathbb{C}\,, \\ &\qquad\qquad \varphi \mapsto \widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)\,[\varphi]\,, \end{aligned}$$

so that the 4D Lagrangian density function $\widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)\,[1]$ on $x^\mu \in \mathcal{M}^4$ [5] is the result of the K-distribution $\widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)$ acting on the constant test K-function $\varphi(y) \equiv 1 \in K(\mathcal{S}^1, \mathbb{C})$ on the extra spatial dimension $\mathcal{S}^1$. In this chapter, we only consider bulk massless fermions, so the bulk terms only consist of kinetic terms as

$$\widetilde{\mathcal{L}_{\text{bulk}}} \;=\; \widetilde{\mathcal{L}_{\text{kin}}}\,. \tag{5.7}$$

Note that in contrast to the treatment in the function formalism in Chapter 4.2.1 where we have to operate on the profile functions directly, here we focus on the K-distributions where all fields considered are embedded, which would also provide additional information at the brane,

$$\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)\,[1] = \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\; \mathcal{L}_{\text{kin}}\,(x^\mu, y) + \mathcal{L}^{\text{kin}}_{B}\,(x^\mu)\;, \tag{5.8}$$

where $\mathcal{L}_{\text{kin}}\,(x^\mu, y)$ is the kinetic function induced by the K-distribution $\widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)$, which will fulfill the $\mathbb{Z}_2$ symmetry condition in Eq. (4.2). $\mathcal{L}^{\text{kin}}_B$ will turn out to the brane terms – BBT, derived from the weak derivative in $\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)$, which would be described in Section 5.3.2 precisely.

$S^{\mathcal{D}}_{\text{branes}}$ represents action terms located at the orbifold fixed points, which for instance exclusively involves Lagrangians with the Dirac K-distribution $\delta_{\pi R}[\varphi]$ (H.72) (e.g. the brane-localized Yukawa interactions (5.26) in Section 5.3.4) taken at the fixed point $y = \pi R$.

Finally, the Lagrangian densities of the whole expression (5.5) will respect the $\mathbb{Z}_2$ parity symmetry in the sense of induced functions of the 5D Lagrangian density and 5D fields [see Eq. (4.1)-(4.2)].

### 5.3.2 Bulk Fermion Fields

To keep consistent with the function formalism in Section 4.2.2 and write down a SM Yukawa-like coupling, we introduce a pair of bulk 5D fermion fields $Q(x^\mu, y)$ and $D(x^\mu, y)$ – of mass dimension 2 – as the minimal spin-1/2 field content, which will represent respectively the $\mathrm{SU(2)_L}$ gauge doublet down-component quark and the singlet down-quark respectively. Note that

$$\forall\, x^\mu \in \mathcal{M}^4\,,\; Q(x^\mu, y), D(x^\mu, y) \in K(\mathcal{S}^1, \mathbb{C})\,,$$

4. $\widetilde{\mathcal{L}_{\text{bulk}}}\,(x^\mu)\,[\varphi] \in L^1\left(\mathcal{M}^4\right) \hat{=} \left\{\mathcal{L} : \mathcal{M}^4 \to \mathbb{C} \,|\, \int d^4x\, |\mathcal{L}\,(x^\mu)| < \infty\right\}$, the Lebesgue integration on the 4D Minkowski spacetime.

5. Another part of the complete 4D effective Lagrangian density should be generated from $S^{\mathcal{D}}_{\text{branes}}$ in Eq. (5.5).

so that after the chiral decomposition in Eq. (1.17), we define the regular K-distributions $\widetilde{Q_{L/R}}(x^\mu), \widetilde{D_{L/R}}(x^\mu)$ associated to the 5D fields $Q_{L/R}(x^\mu, y)$ and $D_{L/R}(x^\mu, y)$ which have the following kinetic terms with the weak derivatives (cf. Appendix H.4.2) in the distribution formula [entering Eq. (5.6)]:

$$\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu) = \sum_{F=Q,D} \frac{i}{2} \widetilde{\bar{F}} \Gamma^M \overleftrightarrow{\partial_M} \widetilde{F}, \tag{5.9}$$

using the same standard notations in the function formalism [see Eq. (1.16)] that $\overleftrightarrow{\partial_M} = \overrightarrow{\partial_M} - \overleftarrow{\partial_M}$, $\partial_M = \partial/\partial x^M$, $x^M = (x^\mu, y)$ with $M \in [\![0,4]\!]$ for the coordinates $x^M \in \mathcal{M}^4 \times \mathcal{S}^1/\mathbb{Z}_2$ and $\Gamma^M$ for the 5D Dirac matrices (cf. Appendix A). Notice that the kinetic Lagrangian density $\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu)$ (5.9) has a similar formula as that in the function prescription [see Eq. (4.8)] but contains definitely different building blocks via regular K-distributions (H.58) combined with weak derivatives (H.77) as product K-distributions (H.65).

Inserting the chiral decomposition in Eq. (1.17), we can rewrite the kinetic terms $\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu)$ (5.9) as

$$\begin{aligned}\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu) &= \frac{1}{2}\left(i\widetilde{F_R^\dagger}\sigma^\mu \overleftrightarrow{\partial_\mu}\widetilde{F_R} + i\widetilde{F_L^\dagger}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu}\widetilde{F_L} - \widetilde{F_R^\dagger}\overleftrightarrow{\partial_4}\widetilde{F_L} + \widetilde{F_L^\dagger}\overleftrightarrow{\partial_4}\widetilde{F_R}\right)\\ &= \frac{1}{2}\left(i\widetilde{\overline{\mathcal{F}_R}}\gamma^\mu \overleftrightarrow{\partial_\mu}\widetilde{\mathcal{F}_R} + i\widetilde{\overline{\mathcal{F}_L}}\gamma^\mu \overleftrightarrow{\partial_\mu}\widetilde{\mathcal{F}_L} - \widetilde{\overline{\mathcal{F}_R}}\overleftrightarrow{\partial_4}\widetilde{\mathcal{F}_L} + \widetilde{\overline{\mathcal{F}_L}}\overleftrightarrow{\partial_4}\widetilde{\mathcal{F}_R}\right),\end{aligned} \tag{5.10}$$

using the matrices $\sigma^\mu, \bar{\sigma}^\mu$ defined in Appendix A and the weak derivatives can be written explicitly via Eq. (H.77),

$$\left\{\begin{array}{lcl} \partial_4\widetilde{F_L} &=& \{\partial_4 F_L\} + (\beta_{\pi R}[F_L]\beta_{\pi R} - \beta_0[F_L]\beta_0) + \displaystyle\sum_{y_0=0,\pi R} \beta_{y_0}[F_L]\delta_{y_0}, \\ \partial_4\widetilde{F_R} &=& \{\partial_4 F_R\} + (\beta_{\pi R}[F_R]\beta_{\pi R} - \beta_0[F_R]\beta_0) + \displaystyle\sum_{y_0=0,\pi R} \beta_{y_0}[F_R]\delta_{y_0}, \end{array}\right. \tag{5.11}$$

where $\left\{\partial_4 F_{L/R}\right\}$ are the regular K-distributions associated to the partial derivatives $\partial_4 F_{L/R}$. In contrast to the function formalism in Section 4.2.2 where the $\mathbb{Z}_2$ symmetry represents on the Lagrangian density and 5D fields obviously, here the $\mathbb{Z}_2$ parity should be revealed by the 5D fields $Q(x^\mu, y)$ and $D(x^\mu, y)$ induced by the regular K-distributions $\widetilde{Q_{L/R}}(x^\mu), \widetilde{D_{L/R}}(x^\mu)$ respectively [see Eq. (H.58)] so that $Q(x^\mu, y)$ and $D(x^\mu, y)$ can still take the four different forms in Eq. (4.9)-(4.12). Notice that one of $F_L$ and $F_R$ ($F = Q, D$) must be even and in turn leads to the associated vanishing jump at the branes (fixed points). Thus, certain terms induced by the second term of Eq. (5.11)

$$\begin{aligned}\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu)[1] &\ni \frac{1}{2}\widetilde{F_{R/L}^\dagger}\left(\beta_{\pi R}[F_{L/R}]\beta_{\pi R} - \beta_0[F_{L/R}]\beta_0\right)[1]\\ &\ni \frac{1}{2}\left(\beta_{\pi R}[F_{L/R}]\beta_{\pi R}[F_{R/L}] - \beta_0[F_{L/R}]\beta_0[F_{R/L}]\right),\end{aligned} \tag{5.12}$$

must vanish since $\beta_{0,\pi R}[F_L] = 0$ ($F_L$ is even) or $\beta_{0,\pi R}[F_R] = 0$ ($F_R$ is even) where $\beta_{y_0}[\varphi] \in K'(\mathcal{S}^1, \mathbb{C})$ is the Jump K-distribution [see Eq. (H.70)]. Moreover, the first term of Eq. (5.11) leads to

$$\begin{aligned}\widetilde{\mathcal{L}_{\text{kin}}}(x^\mu)[1] &\ni \frac{1}{2}\widetilde{F_{R/L}^\dagger}\left\{\partial_4 F_{L/R}\right\}[1]\\ &\ni \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\, \frac{1}{2} F_{R/L}^\dagger\, \partial_4 F_{L/R},\end{aligned} \tag{5.13}$$

combining with kinetic terms of Eq. (5.10) along ordinary 4D coordinates,

$$\begin{aligned}\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)\,[1] &\ni \frac{i}{2}\widetilde{F^\dagger_{R/L}}\sigma^\mu\overleftrightarrow{\partial_\mu}\widetilde{F_{R/L}}[1]\\ &\ni \left(\int_{-\pi R^+}^{0^-}+\int_0^{\pi R}\right)dy\ \frac{i}{2}F^\dagger_{R/L}\sigma^\mu\overleftrightarrow{\partial_\mu}F_{R/L}\,,\end{aligned}\tag{5.14}$$

will recover the $\mathbb{Z}_2$ symmetric kinetic terms $\mathcal{L}_{\text{kin}}\,(x^\mu,y)$ of Eq. (4.8) built via function formalism as expected in Eq. (5.8). The rest terms including $\delta_{y_0}$ will potentially lead to the SM configuration BBT (4.18)-(4.19) associated to each type of $\mathbb{Z}_2$ parity (4.9)-(4.12), which will be described precisely afterwards. Finally, we can conclude that the kinetic Lagrangian $\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)\,[1]$ (5.9) developed via distribution formalism satisfies the $\mathbb{Z}_2$ symmetry.

Then, let us deduce the brane-localized terms $\mathcal{L}_B^{\text{kin}}\,(x^\mu)$ (5.8) under each type of $\mathbb{Z}_2$ parity (4.9)-(4.12) respectively. In the Type I $\mathbb{Z}_2$ transformation (4.9), $Q_L$ and $D_R$ are continuous at the two fixed points since they are even fields, i.e. $\beta_{0,\pi R}[Q_L]=\beta_{0,\pi R}[D_R]$. Instead, $Q_R$ and $D_L$ are odd fields so they can be discontinuous at the fixed points. Thus, considering $\forall\, x^\mu\in\mathcal{M}^4\,,\ F_{L/R}(x^\mu,y)\in K(\mathcal{S}^1,\mathbb{C})$ $(F=Q,D)$, we can calculate the associated weak derivatives of $\widetilde{F_{L/R}}$ $(F=Q,D)$ via Eq. (5.11),

$$\text{Type I}\begin{cases}\partial_4\widetilde{Q_L} &=& \{\partial_4 Q_L\}\,,\\ \partial_4\widetilde{Q_R} &=& \{\partial_4 Q_R\}+(\beta_{\pi R}[Q_R]\beta_{\pi R}-\beta_0[Q_R]\beta_0)+\displaystyle\sum_{y_0=0,\pi R}\beta_{y_0}[Q_R]\delta_{y_0}\,,\\ \partial_4\widetilde{D_L} &=& \{\partial_4 D_L\}+(\beta_{\pi R}[D_L]\beta_{\pi R}-\beta_0[D_L]\beta_0)+\displaystyle\sum_{y_0=0,\pi R}\beta_{y_0}[D_L]\delta_{y_0}\,,\\ \partial_4\widetilde{D_R} &=& \{\partial_4 D_R\}\,,\end{cases}\tag{5.15}$$

which lead to the brane terms in $\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)\,[1]$ as

$$\begin{aligned}\widetilde{Q^\dagger_L}\,\partial_4\widetilde{Q_R}[1] &\ni \sum_{y_0=0,\pi R}\delta_{y_0}[Q^\dagger_L]\,\beta_{y_0}[Q_R]=Q^\dagger_L\Big|_0\ Q_R|^{0}_{0^-}+Q^\dagger_L\Big|_{\pi R}\ Q_R|^{-\pi R^+}_{\pi R}\\ &\ni 2\left(Q^\dagger_L\,Q_R\Big|_0-Q^\dagger_L\,Q_R\Big|_{\pi R}\right),\\ \widetilde{D^\dagger_R}\,\partial_4\widetilde{D_L}[1] &\ni \sum_{y_0=0,\pi R}\delta_{y_0}[D^\dagger_R]\,\beta_{y_0}[D_L]=D^\dagger_R\Big|_0\ D_L|^{0}_{0^-}+D^\dagger_R\Big|_{\pi R}\ D_L|^{-\pi R^+}_{\pi R}\\ &\ni 2\left(D^\dagger_R\,D_L\Big|_0-D^\dagger_R\,D_L\Big|_{\pi R}\right),\end{aligned}\tag{5.16}$$

where the odd $\mathbb{Z}_2$ parity of $Q_R,D_L\in K(\mathcal{S}^1,\mathbb{C})$ have been injected

$$\begin{cases}Q_R|_0 &=& Q_R|_{0^+}=-\,Q_R|_{0^-}\,,\\ Q_R|_{\pi R} &=& Q_R|_{\pi R^-}=-\,Q_R|_{-\pi R^+}\,,\end{cases}\qquad \begin{cases}D_L|_0 &=& D_L|_{0^+}=-\,D_L|_{0^-}\,,\\ D_L|_{\pi R} &=& D_L|_{\pi R^-}=-\,D_L|_{-\pi R^+}\,.\end{cases}$$

Inserting Lagrangian pieces (5.16) with its Hermitian conjugate and the other terms in Eq. (5.12)-(5.14), we can reformulate the kinetic terms $\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)$ (5.9) into the function formalism (5.8),

$$\widetilde{\mathcal{L}_{\text{kin}}}\,(x^\mu)\,[1]=\left(\int_{-\pi R^+}^{0^-}+\int_0^{\pi R}\right)dy\ \mathcal{L}_{\text{kin}}\,(x^\mu,y)+\mathcal{L}^{\text{kin}}_{B\text{I}}\,(x^\mu)\,,\tag{5.17}$$

where

$$\begin{aligned}
\mathcal{L}_{\text{kin}}\left(x^{\mu},y\right) &= \sum_{F=Q,D}\frac{1}{2}\left(iF_R^{\dagger}\sigma^{\mu}\overleftrightarrow{\partial_{\mu}}F_R+iF_L^{\dagger}\bar{\sigma}^{\mu}\overleftrightarrow{\partial_{\mu}}F_L-F_R^{\dagger}\overleftrightarrow{\partial_4}F_L+F_L^{\dagger}\overleftrightarrow{\partial_4}F_R\right)\\
&= \sum_{F=Q,D}\frac{i}{2}\bar{F}\Gamma^M\overleftrightarrow{\partial_M}F\,,\\
\mathcal{L}_{BI}^{\text{kin}}\left(x^{\mu}\right) &= \left(\bar{D}D-\bar{Q}Q\right)\Big|_{\pi R}-\left(\bar{D}D-\bar{Q}Q\right)\Big|_{0}\,.
\end{aligned} \tag{5.18}$$

Now we can clearly see that $\mathcal{L}_{\text{kin}}\left(x^{\mu},y\right)$ of Eq. (5.18) exactly recovers the 5D kinetic Lagrangian density of Eq. (4.8) within the function formalism. Similarly

$$S_{BI}\,\hat{=}\int d^4x\;\mathcal{L}_{BI}^{\text{kin}}\left(x^{\mu}\right)=\int d^4x\left[\left(\bar{D}D-\bar{Q}Q\right)\Big|_{\pi R}-\left(\bar{D}D-\bar{Q}Q\right)\Big|_{0}\right], \tag{5.19}$$

is exactly identical to the BBT corresponding to the SM configuration in Eq. (4.18)-(4.19) within the function formalism. Therefore, writing the initial Lagrangian as a distribution (5.9) is equivalent to express in a field function formalism.

Similarly, for the three other types of $\mathbb{Z}_2$ transformations (4.10)-(4.12), we have the following weak derivatives of $\widetilde{F_{L/R}}$ $(F=Q,D)$:

$$\text{Type II}\left\{\begin{array}{lcl}
\partial_4\widetilde{Q_L} &=& \{\partial_4 Q_L\}+\left(\beta_{\pi R}[Q_L]\beta_{\pi R}-\beta_0[Q_L]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[Q_L]\delta_{y_0}\,,\\
\partial_4\widetilde{Q_R} &=& \{\partial_4 Q_R\}\,,\\
\partial_4\widetilde{D_L} &=& \{\partial_4 D_L\}\,,\\
\partial_4\widetilde{D_R} &=& \{\partial_4 D_R\}+\left(\beta_{\pi R}[D_R]\beta_{\pi R}-\beta_0[D_R]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[D_R]\delta_{y_0}\,,
\end{array}\right. \tag{5.20}$$

$$\text{Type III}\left\{\begin{array}{lcl}
\partial_4\widetilde{Q_L} &=& \{\partial_4 Q_L\}\,,\\
\partial_4\widetilde{Q_R} &=& \{\partial_4 Q_R\}+\left(\beta_{\pi R}[Q_R]\beta_{\pi R}-\beta_0[Q_R]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[Q_R]\delta_{y_0}\,,\\
\partial_4\widetilde{D_L} &=& \{\partial_4 D_L\}\,,\\
\partial_4\widetilde{D_R} &=& \{\partial_4 D_R\}+\left(\beta_{\pi R}[D_R]\beta_{\pi R}-\beta_0[D_R]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[D_R]\delta_{y_0}\,,
\end{array}\right. \tag{5.21}$$

$$\text{Type IV}\left\{\begin{array}{lcl}
\partial_4\widetilde{Q_L} &=& \{\partial_4 Q_L\}+\left(\beta_{\pi R}[Q_L]\beta_{\pi R}-\beta_0[Q_L]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[Q_L]\delta_{y_0}\,,\\
\partial_4\widetilde{Q_R} &=& \{\partial_4 Q_R\}\,,\\
\partial_4\widetilde{D_L} &=& \{\partial_4 D_L\}+\left(\beta_{\pi R}[D_L]\beta_{\pi R}-\beta_0[D_L]\beta_0\right)+\sum\limits_{y_0=0,\pi R}\beta_{y_0}[D_L]\delta_{y_0}\,,\\
\partial_4\widetilde{D_R} &=& \{\partial_4 D_R\}\,,
\end{array}\right. \tag{5.22}$$

which would lead to the same function formalism of the 5D kinetic Lagrangian density $\mathcal{L}_{\text{kin}}\left(x^{\mu},y\right)$ in Eq. (5.18) but different configurations of BBT:

$$\left[\begin{array}{lcl}
S_{BII} &=& \int d^4x\;\mathcal{L}_{BII}^{\text{kin}}\left(x^{\mu}\right)=\int d^4x\left[\left(-\bar{D}D+\bar{Q}Q\right)\Big|_{\pi R}-\left(-\bar{D}D+\bar{Q}Q\right)\Big|_{0}\right],\\
S_{BIII} &=& \int d^4x\;\mathcal{L}_{BIII}^{\text{kin}}\left(x^{\mu}\right)=\int d^4x\left[\left(-\bar{D}D-\bar{Q}Q\right)\Big|_{\pi R}-\left(-\bar{D}D-\bar{Q}Q\right)\Big|_{0}\right],\\
S_{BIV} &=& \int d^4x\;\mathcal{L}_{BIV}^{\text{kin}}\left(x^{\mu}\right)=\int d^4x\left[\left(\bar{D}D+\bar{Q}Q\right)\Big|_{\pi R}-\left(\bar{D}D+\bar{Q}Q\right)\Big|_{0}\right],
\end{array}\right. \tag{5.23}$$

recovering the chiral setups for zero modes but different from the potential SM chirality configuration, which are listed in Table 4.1. It is important to see here that the BBT are not added by hand to the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold description (as in the function formalism in Section 4.2.5) but originate from the weak partial derivative of the odd field, which are discontinuous at the fixed points $y = 0, \pi R$. Moreover, the BBT configuration is fixed by the $\mathbb{Z}_2$ parity form in Eq. (4.9)-(4.12) and only the Type I $\mathbb{Z}_2$ parity symmetry (4.9) can realize the BBT under the SM configuration in Eq. (4.18)-(4.19). Besides, the vector-like BBT configuration [see Section 4.2.5] can't be realized in the distribution prescription.

### 5.3.3 Brane-Localized Scalar Field

To derive the mass spectrum by the bulk fermion coupling, we introduce the 4D real scalar Higgs field $H$ (mass dimension 1) confined at $y = \pi R$, a fixed point of the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold, as in the function description in Section 4.2.3. In contrast, the brane-localized scalar field is developed via Dirac K-distribution at $y = \pi R$,

$$S_H = \int d^4x \, \widetilde{\mathcal{L}_H}\,(x^\mu)\,[1]\,, \quad \text{with} \quad \widetilde{\mathcal{L}_H}\,(x^\mu) = \left[\frac{1}{2}\,\partial_\mu H \partial^\mu H - V(H)\right] \delta_{\pi R}\,, \tag{5.24}$$

with a potential $V(H)$ possessing a minimum which generates a non-vanishing VEV for the field $H$ expanded as in Eq. (3.10). The K-distribution formalism $\widetilde{\mathcal{L}_H}\,(x^\mu)\,[1]$ (5.24) can be reformulated to the function form,

$$\mathcal{L}_H \mathrel{\hat{=}} \widetilde{\mathcal{L}_H}\,(x^\mu)\,[1] = \left[\frac{1}{2}\,\partial_\mu H \partial^\mu H - V(H)\right] \delta_{\pi R}[1] = \frac{1}{2}\,\partial_\mu H \partial^\mu H - V(H)\,, \tag{5.25}$$

which is identical to the 4D Higgs Lagrangian density directly built via functions in Eq. (4.13).

### 5.3.4 Yukawa Interactions

The brane-localized Yukawa interactions between the bulk fermions and the above brane-scalar Higgs field (at $y = \pi R$) can be introduced via the Dirac K-distribution $\delta_{\pi R}[\varphi]$ (H.72) as

$$\begin{aligned} &S_Y = \int d^4x \, \widetilde{\mathcal{L}_Y}(x^\mu)[1]\,, \\ &\text{with} \;\; \widetilde{\mathcal{L}_Y}(x^\mu) = \left[-Y_5 \; H(x^\mu) Q_L^\dagger D_R - Y_5' \; H(x^\mu) Q_R^\dagger D_L + \text{H.c.}\right] \delta_{\pi R}\,, \end{aligned} \tag{5.26}$$

where the complex phases $\alpha_{Y^{(\prime)}}$ of the two independent Yukawa couplings $Y_5^{(\prime)}$ at the 3-brane $y = \pi R$ are defined in Eq. (3.12). Besides, the dimensionless Yukawa couplings $y_4, y_4' \sim \mathcal{O}(1)$ are defined in Eq. (4.15) and $y_4$ can be approximately identified with the SM Yukawa coupling constant within the decoupling limit. The K-distribution formalism $\widetilde{\mathcal{L}_Y}(x^\mu)[1]$ (5.24) can be reformulated to the function form,

$$\begin{aligned} \mathcal{L}_Y|_{\pi R} &\mathrel{\hat{=}} \widetilde{\mathcal{L}_Y}\,(x^\mu)\,[1] \\ &= H(x^\mu)\,\delta_{\pi R}\left[-Y_5 \; Q_L^\dagger D_R - Y_5' \; H(x^\mu) Q_R^\dagger D_L + \text{H.c.}\right] \\ &= -Y_5 \; H(x^\mu) Q_L^\dagger D_R - Y_5' \; H(x^\mu) Q_R^\dagger D_L + \text{H.c.}\,, \end{aligned} \tag{5.27}$$

which is identical to the brane-localized Yukawa interactions built via functions in Eq. (4.14).

When calculating the tower of excited fermion masses, we restrict our considerations to the VEV of $H$. In analogy to the SM Higgs, after inserting the VEV generation (3.10) relation to the Yukawa interactions (5.27), we should obtain [6]

$$S_Y = S_X + S_{hQD}\,,$$

where $S_X$ is the Yukawa mass sector in Eq. (4.16) with effective couplings $X, X'$ defined in Eq. (3.15). $S_{hQD}$ is the localized fermion-scalar interaction in Eq. (4.17), allowing to work out the 4D effective Yukawa coupling constants.

We notice that the complete toy model originally developed in the distribution formalism of Eq. (5.5) can be reformulated into function formalism in Eq. (4.5)-(4.6), (4.21), since

$$\begin{aligned} S^{\mathcal{D}}_{\text{bulk}} &= S_{\text{bulk}} + S_{B\mathcal{I}}\,, \\ S^{\mathcal{D}}_{\text{branes}} &= S_H + S_X + S_{hQD}\,, \end{aligned} \tag{5.28}$$

where $S_{\text{bulk}}$ is the bulk action via the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$ (5.18) since we consider bulk massless fermions (5.7),

$$S_{\text{bulk}} = \int d^4x \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\ \mathcal{L}_{\text{kin}}\left(x^\mu, y\right)\,, \tag{5.29}$$

recovering Eq. (4.6) and $S_{B\mathcal{I}}$ ($\mathcal{I} = \text{I}, \text{II}, \text{III}, \text{IV}$) is the $\mathbb{Z}_2$ parity dependent BBT in Eq. (5.19)-(5.23), which recovers the SM configuration BBT (4.18)-(4.19) only under the Type I $\mathbb{Z}_2$ parity symmetry (4.9).

## 5.4 Function Recovery

### 5.4.1 Free Bulk Fermions on the Orbifold

In order to reproduce the profile solutions and mass spectra derived in Section 4.3 (free case) and Section 4.4-4.5 (Yukawa case), we firstly investigate the free case via the bulk action part of $S_{5\text{D}}$ (5.5) built in distribution formalism,

$$S^{\mathcal{D}}_{\text{bulk}} = S_{\text{bulk}} + S_{B\mathcal{I}}\,, \quad \text{with} \ \ \mathcal{I} = \text{I}, \text{II}, \text{III}, \text{IV}\,,$$

which depends on the type of $\mathbb{Z}_2$ parity in Eq. (4.9)-(4.12) due to $S_{B\mathcal{I}}$ ($\mathcal{I} = \text{I}, \text{II}, \text{III}, \text{IV}$) in Eq. (5.19)-(5.23) and $S_{\text{bulk}}$ is the bulk action via the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$ (5.18). We apply the least action principle to it, which leads to two relations

$$\delta_{\bar{F}}\left(S_{\text{bulk}} + S_{B\mathcal{I}}\right) = 0\,,$$

one for each of the generic independent unknown 5D fields $F = Q, D$, and two corresponding ones, $\delta_F\left(S_{\text{bulk}} + S_{B\mathcal{I}}\right) = 0$, involving the complex conjugate fields. This is exactly what has been treated in Section 4.3.3, where identical bulk kinetic terms (4.8) and the BBT (4.18) have been solved for piece-wise smooth fields over $y \in [-\pi R^+, 0^-] \cup [0, \pi R]$. We can thus apply here the results obtained there. The EOM (4.25)-(4.26) and the NBC (4.47) would be respectively deduced for $F = Q, D$.

Inserting the KK decomposition (4.28), we would obtain the profile solutions in Eq. (4.41) in the domain $y \in [0, \pi R]$, together with the mass spectra (4.43). Combining with the $\mathbb{Z}_2$

6. The linear decomposition is guaranteed by the Dirac K-distribution $\delta_{\pi R}$ (see Appendix H.4).

parity of the profiles in Eq. (4.29)-(4.32), we can obtain profile solutions in the complementary region, $y \in [-\pi R^+, 0^-]$. Note that the configuration of the BBT can no longer be added by hand freely but fixed by the type of $\mathbb{Z}_2$ parity in Eq. (4.9)-(4.12) [see Eq. (5.19)-(5.23)], which would extend the BBT-chirality configuration Table 4.1 to Table 5.1 including the relation to the $\mathbb{Z}_2$ parity type. We can see that the Type I $\mathbb{Z}_2$ parity symmetry (4.9) would realize the SM configuration and induce the profile solution in Table. 4.2 [only the line of Type I $\mathbb{Z}_2$ parity], which is constrained by the ortho-normalization conditions in Eq. (4.33). In simpler words, the $\mathbb{Z}_2$ parity would fix the BBT (chiral) configuration and in turn controls the chiral nature of the theory as well as each field chirality.

| $\mathbb{Z}_2$ | $\sigma^Q_{0,\pi R}$ | $\sigma^D_{0,\pi R}$ | $Q$ | $D$ |
|---|---|---|---|---|
| IV | $1$ | $1$ | $Q^0_R(x^\mu)$ | $D^0_R(x^\mu)$ |
| II | $1$ | $-1$ | $Q^0_R(x^\mu)$ | $D^0_L(x^\mu)$ |
| I | $-1$ | $1$ | $Q^0_L(x^\mu)$ | $D^0_R(x^\mu)$ |
| III | $-1$ | $-1$ | $Q^0_L(x^\mu)$ | $D^0_L(x^\mu)$ |

Table 5.1 – Chiral setups for the zero-modes of fields $Q$ and $D$ from various different BBT signs $\sigma^{Q,D}_{0,\pi R}$ in Eq. (4.18), which is fixed by the type of $\mathbb{Z}_2$ parity in Eq. (4.9)-(4.12) in the distribution formalism via Eq. (5.19)-(5.23).

For completeness, let us now discuss the probability currents. Within the same symmetry analysis in Section 4.3.2, the two independent global $\mathrm{U}(1)_{\mathrm{Q,D}}$ transformations of the fields in Eq. (4.34), would leave the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$ (5.18) invariant. Based on these two symmetries, inserting the bulk EOM (4.25), the Noether's theorem predicts the local conservation relation in Eq. (4.35) with respect to two probability currents $j^M_F$ $(F = Q, D)$ (4.36) respectively. The addition of the $S_{B\mathcal{I}}$ ($\mathcal{I} = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) part in Eq. (5.19)-(5.23) to $S_{\text{bulk}}$ is not affecting the current conservation equations (4.35)-(4.36), which has been concluded in Section 4.3.3.

In contrast to the function formalism in Section 4.3.2, the weak derivatives of odd fields induce the BBT automatically, associated to the $\mathbb{Z}_2$ parity form so that there's no need to inject essential boundary conditions (4.40) via the (vanishing) probability currents (4.40) [using the continuity at the fixed points $y = 0, \pi R$ (4.37) and the odd parity (4.38)]. However, we can check that $j^4_F|_{0,\pi R}$ is well vanishing due to the chiral NBC (4.47) [see Table 5.1]. Therefore, the $\mathbb{Z}_2$ parity leads to the associated BBT and in turn suppresses the probability current $j^4_F|_{0,\pi R}$ to vanish. The $\mathbb{Z}_2$ parity configuration, chosen initially, leads to a specific setup of BBT signs as presented in Table 5.1.

### 5.4.2 Brane-Localized Yukawa Couplings on the Orbifold

In order to work out the fermion mass spectrum in the presence of Yukawa couplings $S_X$ (4.16), we would focus on the fermion terms of the complete toy model $S_{\text{5D}}$ (5.5) including brane-localized action terms $S^{\mathcal{D}}_{\text{branes}}$ (5.28) at the fixed points $y = \pi R$. The considered action reads as,

$$S^{\mathcal{D}}_{\text{bulk}} + S_X = S_{\text{bulk}} + S_{B\mathrm{I}} + S_X \,,$$

which exactly recovers the considered action in Section 4.4 [4D perturbation approach]-4.5 [5D approach]. Thus, we can reproduce all the results: profiles in Table 4.3 [only the line of Type I $\mathbb{Z}_2$ parity], mass spectra (4.73), 4D effective Yukawa couplings (4.73), via the identical procedure in both of the 4D and the 5D approaches and realize the decoupling limit in Eq. (4.76). Regarding the probability current, vanishing current component along the extra dimension at the fixed points of the orbifold $j^4|_{\pi R}$ (4.64) is achieved by the NBC induced by the BBT as described in Section 4.5. Then, the probability current turns to be continuous along the entire extra dimension as demonstrated in Eq. (4.63) or (4.38) respectively.

We need to emphasize that since we concentrate on the SM chiral configuration as in Section 4.4-4.5, the $\mathbb{Z}_2$ parity form must be fixed in the Type I (4.9) leading in to $S_{B\mathrm{I}}$ (5.19) [see Section 5.3.2], which has been indicated in Table 5.1.

## 5.5 The Inclusive $\mathbb{Z}_2$ Parity

Let us study the inclusive $\mathbb{Z}_2$ parity scenario (4.77) proposed in Section 4.6, which would lead to the Dirichlet BC for the odd fields [see SM chirality setup in Eq. (4.79)].

In the free case, the EBC' (4.79) and the chiral BBT $S_{B\mathcal{I}}$ ($\mathcal{I} = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$) in Eq. (5.19)-(5.23) would induce the identical Dirichlet BC (4.47) for the odd fields and in turn to the same chiral configuration with respect to the type of $\mathbb{Z}_2$ parity in Eq. (4.9)-(4.12) [see Table 5.1]. Thus, only the Type I $\mathbb{Z}_2$ parity (4.9) can induce the SM chiral configuration, which is consistent with the analysis in Section 4.6. We should also notice that the EBC' and the chiral BBT deduced from the distribution formalism are physically redundant to fix the chiral configuration. In the presence of Yukawa couplings, the analysis of point-like Yukawa interactions cannot be achieved via the 5D approach within the present inclusive $\mathbb{Z}_2$ symmetry model due to the decoupling limit, which has been mentioned in Section 4.6.

## 5.6 Result Analysis

### 5.6.1 Distribution Formalism

The present study indicates that the distribution formalism can provide additional information for discontinuities. In particular, when we go from a distributional to a functional treatment of the Lagrangian densities and fields, the BBT under a chiral configuration would be deduced from the weak partial derivative of the discontinuous fields automatically, which provides another motivation to introduce the BBT in Chapter 3-4 except for the UV complement. The vanishing probability current condition is thus preserved as dedicated in Section 5.4 both in the presence and absence of brane-localized Yukawa couplings. Note that the jump has the possibility to happen at each brane at the fixed points $y = 0, \pi R$ so that we don't need to insert any EBC or additional BBT.

Moreover, in the framework of the inclusive $\mathbb{Z}_2$ symmetry described in Section 5.5, the $\mathbb{Z}_2$ transformation form can fix the chiral configuration through the Dirichlet boundary conditions from the odd parity at the fixed points [EBC'] (4.79), which performs equivalently via the BBT originating from the weak derivatives of odd fields in the free case.

### 5.6.2 Distribution/Function Prescription

Although the weak derivative in the distribution formalism can introduce the BBT, only the BBT under the chiral configuration (5.19)-(5.23) can be induced, associated to the four $\mathbb{Z}_2$ parity form in Eq. (4.9)-(4.12) [see Table 5.1] and only the Type I $\mathbb{Z}_2$ parity symmetry (4.9) can realize the BBT under the SM configuration in Eq. (4.18)-(4.19).

However, it does not mean the other cases does not exist, which is just constrained by the distribution formalism itself. In contrast, the configuration of BBT can be injected freely and independent of the $\mathbb{Z}_2$ parity form, which is one of the crucial results in Chapter 4. Besides, the vector-like configuration BBT for custodians can only be constructed in the function formalism [see Section 4.2.5].

## 5.7 Conclusions

In the study of the proposed distribution formalism for the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold, the chiral BBT can be generated automatically by the weak derivative at the branes (fixed points $y = 0, \pi R$) where the jump of a fermionic field can happen, so that one does not need anymore to insert by hand any BBT or the equivalent essential boundary conditions. Through a proper action definition [particularly for the kinetic terms], the chiral SM configuration of fermions can be realized both in the free and the brane-localized Yukawa case, recovering the results in the usual function prescription (without brane-Higgs regularization) [see Chapter 4].

Instead of only deriving the action in terms of field-functions from the starting point distributions, via the weak derivatives, we are now working on the attempt to keep a distribution formalism up to 'the end', that is up to the full derivation and treatment of the equations of motion and boundary conditions leading to the final results on the fermion profiles and KK masses. The motivation being to develop in particular an alternative, and maybe instructive, method.

Furthermore, we are working on providing the general role to the test functions (and functions associated to the regular distributions) of implementing the model definition of the considered compactified space [its infinite or finite size, the boundary locations,...], for instance by choosing formal infinite extra spatial dimensions at the level of distribution applications while taking test functions as 'step' functions equal to unity but vanishing outside the finite physical regions. Such an achievement would also allow to treat the interval model via a distribution framework.

Finally, we are trying to merge Wightman's distribution theory with our extra-dimensional distribution formalism, through a 5D action based on the ordinary 4D integration implemented via the Wightman's smeared field and the extra dimension integrated via the K-distributions as we developed in this chapter.

# Chapter 6

# An Origin for Flavors: the Compact Space Partition

This chapter is based on a study that we are now finalizing, in collaboration with Grégory MOREAU and Florian NORTIER.

## 6.1 Introduction

Still nowadays, the non-trivial structure of the SM suggests that its flavor sector and gauge interactions may not be arbitrary but should have some underlying first-principle explanation. This outstanding theoretical open question represents one of the great mysteries in elementary particle physics. In particular in this community, one should admit that there exists today no unanimously celebrated model reproducing the fermion patterns of the flavor space, and, that one does not even know the energy scale at which the flavor dynamics sets in.

The first enigma about the SM fermion field content is the origin of the existence of flavors itself, namely the replication of each fermion in three copies with identical quantum numbers (i.e. spin and charges). There have been of course several attempts in the literature to interpret the presence of the SM fermion families.

The first class of scenarios explaining the fermion replication relies on GUT, introducing new gauge bosons and symmetry breaking mechanism. For instance, the SM gauge group could originate from the reduction pattern, $\mathrm{E}_8 \to \mathrm{E}_6 \times \mathrm{SU}(2)_\mathrm{F} \times \mathrm{U}(1)_\mathrm{F}$ followed by the subsequent breaking $\mathrm{E}_6 \to [\mathrm{SU}(3)]^3$, where $[\mathrm{SU}(3)]^3$ is associated to the gauge trinification symmetry and $\mathrm{U}(1)_\mathrm{F} \times \mathrm{SU}(2)_\mathrm{F}$ is a local group of the family symmetry whose respective singlet and doublet components constitute the three SM fermion families [163–166]. These three SM generations can emerge from other schemes of gauge symmetry group extensions [167–173].

The second class of models is based on extra spatial dimensions and has thus no counterpart in 4D field theories. First, one could simply take advantage of the 8 components of the vector-like spinor in flat 6D quantum field theories, in order to produce two 4-component 4D Dirac spinors corresponding to two fermion families [174]. Nevertheless, it is not possible to create more than these 2 flavors, even when systematically scrutinizing all the possible boundary conditions on a 6D compactified flat spacetime [175] (like for supersymmetric versions [176]). Then, another hope is to introduce a specific curved metric for a 6D spacetime giving rise to a finite number of mass-degenerate 0-modes (without Kaluza-Klein masses) identified as the three families of SM fermion fields (acquiring their mass mainly through a brane-localized Higgs boson coupling): this can be realized by generating three 5D fermion families in some 6D to 5D action reduction, and then recovering three 4D 0-modes being identical [up to masses] as in Ref. [177, 178]. In order to realize the three SM flavors by forcing the existence of several 4D 0-mode fields, one could try to build a model where those are contained in a higher-dimensional fermion field taking possibly different position states in an extra (compact) space: each 4D 0-mode would then correspond to a different state along extra dimensions. This can be achieved, via a topological defect mechanism, by coupling a 6D fermion field to a non-trivial solitonic object in a 2D (compact) space: a global vortex background [179–184], and the resulting fermion wave functions along this space – for the different position states [1] – are then overlapping with each other. Similarly, different angular momentum eigenstates [rotations around a 1+3 brane] in a 2D extra space, with a warp metric (acting as a potential well trapping fermions), can generate a 4D fermion 0-mode replication reproducing the SM flavors [186–188]. The origin of families could be understood as well through a topological property of a semi-finite extra dimension as the number of 4D massless surface modes is directly related to the quantized coefficient of the Chern-Simons operator obtained by integrating out the heavy bulk fermions [189].

Within the composite models, generally possibly related to the previous ones via the AdS/CFT duality, the origin of the three quark/lepton generations should typically rely on gauge dynamics. In this context, metacolor or hypercolor gauge forces [190] as well as strongly coupled supersymmetric theories [191] have been studied. To be exhaustive, let us mention at this level other kinds of approaches, to the interpretation of the presence of three fermion families, which are connected to the cosmological constant problem [192] or to the compactification with magnetic fluxes (see for example Ref. [193–196] and references therein): there the number of 4D chiral 0-modes (with non-trivially quasi-localized profiles) – corresponding to the generation number – is determined by the magnitude of magnetic flux. The three fermion families of quarks and leptons could also come from three chiral multiplets of a 6D supersymmetric gauge theory containing a specific vector multiplet and compactified on a $T^2/Z_3$ orbifold [197, 198].

The last main type of scenario, where the origin of SM fermion families has been explored, is based on string theories. First, within the 10D heterotic string theories, where 6 dimensions can be compactified on a Calabi-Yau manifold or on orbifolds, the flavor properties are strictly related to the features of the compact space. In Calabi-Yau compactifications, the number of chiral generations is proportional to the Euler characteristics of the manifold. In orbifold compactifications, matter in the twisted sector is localized around the fixed points, and, the Yukawa couplings – arising from world-sheet instantons – have a natural geometrical interpretation [199–201]. Similarly, in the more recent string realizations, where the light matter fields of the SM arise from intersecting branes (in superstring theories, intersecting D-brane models are T-dual of magnetized D-brane models), the flavor dynamics is controlled by topological properties of the geometrical construction: the generation number is determined by the intersecting number in the framework of intersecting D-branes (see for instance Ref. [202, 203] or rather Ref. [204, 205] for non-supersymmetric string model versions of intersecting D5-branes). Finally, the idea of a three-family configuration of the field content arising from GUT has also been investigated within the framework of perturbative heterotic superstring models (see Ref. [206] for a review).

In the present chapter, we propose a new simple geometrical mechanism, along a single

1. See Ref. [185] for a similar discussion with a unique extra dimension: a 5D fermion coupled to a specific domain wall (the introduction of an auxiliary fermionic field together with a certain background metric are then necessary).

flat extra dimension, which generates the fermion replication needed to reproduce the three SM flavors. Nevertheless, to be clear, our model does not predict why the number of SM flavors is equal to three, as done for instance in Ref. [207, 208] (using conditions coming from the not so trivial cancellation of 6D anomalies). However, the presented fermion replication tool will be extended to custodian-like particles, which allows to realize as well the promoted group multiplets invoked for the custodially-protected [117] warped extra dimension scenario addressing the gauge hierarchy problem [58].

The central and original mechanism of this model is based on the presence in the Lagrangian of bulk-fermion BBT at several intermediate points along the extra dimension, which is natural in the sense that those terms exist generally if no specific symmetry is applied on the dominant lowest-dimension operators. Furthermore, the BBT are needed [2] at the boundaries [3] of extra dimensions in order to well define the finite intervals (since the BBT induce vanishing probability currents along the new dimension), as shown in Chapter 3, so there is no obvious reason why BBT should not appear as well at several intermediate locations. The BBT, when located at such intermediate positions, are then called *partition terms* since, as we will demonstrate, they still induce a probability current component vanishing at these positions – as a point-like infinite potential – and hence can restrict the fermion field domain to the left or right side of those positions (the whole discussion in this paragraph holds for the free bulk fermion case as well as in the presence of brane or bulk Yukawa couplings for the fermions). As the fermion profiles may be discontinuous at these BBT points, we will apply, in a clear way, the rigorous procedure developed in Chapter 4 to treat jumps.

We will further clearly explain in this chapter the heart of the higher-dimensional mechanism of fermion replication: how a unique fundamental 5D fermion field can generate several massless 4D fields. At the two main steps of the explanation, an explicit quantum interpretation will be provided. The first step relies on the summation of solutions, of the differential linear homogeneous equations of motion, in order to obtain the generic higher-dimensional field expression [4]. This summation runs over the various spatial states in the compact space – states possibly lying on both sides of the BBT positions in the present model or e.g. angular momentum states in the 6D model mentioned above – exactly like it runs over the various momentum states along the extra dimensions in the usual Kaluza-Klein decomposition. From the quantum point of view, all these states are represented by ket elements defining a basis of the Hilbert space for the extra space and their formal sum corresponds to the physical state superposition. The second step is to inject the higher-dimensional field solutions into the initial action and then impose ortho-normalization conditions on their wave functions along the extra space, in order to recover the canonical kinetic terms for the associated 4D fermion fields in a possible 4D effective theory – as imposed by the decoupling criteria. Those conditions correspond to the standard ortho-normalizations of the quantum states constituting the Hilbert space basis. Finally, the several massless 4D fields, associated to the wave functions of the various position states along extra dimensions (in contrast with the heavy 4D KK modes), originate from the same higher-dimensional field and possess in turn identical quantum numbers: they constitute thus fermion copies which can realize the SM flavors (getting then different masses mainly

2. Or some equivalent essential boundary conditions as established in Chapter 3.

3. In fact, we will show in the present chapter that with BBT at intermediate positions (not boundaries), it is not necessary to have BBT precisely at the boundaries of an interval: in other words, BBT at two positions, at least, are needed to have non-vanishing normalizable wave functions in between those positions.

4. Keeping in mind the cut-off on the tower of massive 4D solution fields, which implements the energy limit of the physical domain of validity of the non-renormalizable theory.

through Higgs couplings as required by experimental data: see discussion below). Then the SM gauge interactions can be introduced to build a realistic model.

The intuition only of a higher-dimensional field written as a sum of individual flavor fields, lying in distinct intervals connected by some point(s) in an extra space, was already in the literature, but without explicit model-building realization and precise description [5] – as we have presented just above. First, an effective [6] approach of such a field configuration was performed in Ref. [111, 212–214]. Secondly, a geometrical extension of this field configuration, with the different flavor profiles possibly lying in distinct rose graph branches (N loop-intervals that begin and end up at a common boundary), has been studied [215]. The alternative star graph (N intervals connected by a common boundary: an Ultra-Violet brane) – representing a simple field theory limit of a "multi-throat" setup where one of the warped extra dimensions in each throat is much larger than the other ones – has also been analysed within the same flavor context [216]. In string theories, a generic type of situation arising from flux compactifications leads to geometries with such multi-throats hanging out from a "head", which constitute a compact Calabi-Yau manifold.

The other enigma of the SM flavor sector lies in the origin of the mass scale hierarchies among the three fermion families as well as between the quarks, charged leptons and neutrinos. For a 4D world, the main approach to this question is probably to decipher the observed mass hierarchies via spontaneously broken flavor symmetries [217] (and the relatively tiny neutrino mass scale via the seesaw mechanism), which could be discrete or continuous, global or local, as widely explored in the literature. In this context, a realistic reproduction of the measured fermion masses and mixing angles typically requires either a large number of parameters or a certain degree of complexity, so that today one is for sure unable to select the best model of this kind. The paradigm on higher-dimensional frameworks, appeared around the year 2000, has brought geometrical ideas for generating both the gauge hierarchy and the fermion mass hierarchies – including the small neutrino masses: the various 0-mode fermion wave functions (exponential or gaussian) along the compactified space can overlap differently either between the two 4D chiralities [63, 218] (see precise mass and phase reproductions respectively in Ref. [219, 220]) [see also e.g. Ref. [55, 66, 221–224] for the neutrinos] or with the Higgs boson profile [68, 225, 226] (see detailed mass reproductions e.g. in Ref. [76, 100, 123, 153, 227, 228]) [see e.g. Ref. [152, 229] for neutrinos], in a way that creates strong hierarchies in the 4D effective Yukawa couplings, and in turn in the 4D masses after electroweak symmetry breaking – the benefit being that the fundamental fermion parameters (solitonic bulk masses and sub-interval sizes) are all of the same order of magnitude.

Our present model with distinct flavor fermion wave functions separated by BBT points, or possibly having different non-vanishing amplitudes on each BBT side, further addresses, in a new way, the fermion mass hierarchy: for a 5D Higgs boson profile exponential along the entire flat extra dimension – easily obtained from a bulk scalar mass – its overlaps with the different flavor profiles, being maximal at various locations, are exponentially different and hence generate the observed strongly hierarchical fermion masses through hierarchical Yukawa couplings [similarly for generating the tiny neutrino mass scale]. For instance, compared to the above mass models with light fermion profiles ex-

5. Such an effective flavor field sum, but with overlapped exponential fields shifted along some extra dimension, is also mentioned in Ref. [209] (see also Ref. [210] for the dual composite Higgs approach, with multiple flavor scales, of a similar field configuration).

6. In other words, a generic point-like interaction model was simulated: without such interactions, the equality $\delta F = 0$ at a given point $y_0$, along some extra dimension, should come as the model hypothesis of a known 5D fermion field $F(x^\mu, y_0)$ at the calculation level of action variation, and cannot be deduced from a final result on this field (see Ref. [211] for the chronological aspects of the variational calculus).

ponentially spread along a warped extra dimension and only peaked towards the so-called Planck-brane, a warped version of the setup proposed here with light fermion profiles possibly partitioned around the Planck-brane (hence strictly vanishing in the other TeV-brane region) favors small light fermion couplings with the KK gauge bosons peaked at the TeV-brane which tends to soften dangerously constrained flavor changing neutral current effects. A warped version of the present model also possesses the attractive feature that the exponential Higgs profile (induced by the warped metric and a bulk mass) peaked towards the TeV-brane allows to address simultaneously the fermion mass hierarchies and the gauge hierarchy puzzle [58], which represents theoretically a kind of economy: in other words, the present model is connecting, via the curved Higgs profile, the flavor appearance, the quark/lepton mass hierarchies and the gauge hierarchy. The phenomenological study of the reproduction of SM fermion mass and mixing angle values, within the setup of fermion flavors in distinct intervals, was performed for the effective approach mentioned above [111, 212–214]. A comparable setup – with exponential profiles for the two bulk Higgs doublets, the first matter generation localized on an interval boundary, second fermion generation in the bulk and third one on the other boundary – has also been studied in the context of a supersymmetric 5D GUT based on a $S_1/(Z_2 \times Z_2')$ orbifold [230].

In the present work, we find out in particular the configuration for a given flavor fermion wave function with different amplitudes on each side of a BBT location (as exposed above) which was not studied in this Ref. [111, 212–214]. This configuration permits the needed mass mixings among different flavors thanks to non-vanishing wave function overlaps at the Yukawa coupling level, in contrast with the setup of distinct flavor wave functions partitioned respectively on the two sides of a BBT location which requires a shift of the BBT locations between the Left and Right-handed 0-mode fields to create such a mixing effect [111, 212–214]. We perform a phenomenological analysis of this profile configuration by working out examples of regions in the parameter space where SM fermion masses are reproduced, in order to illustrate the result that realistic numerical values can be easily reached.

The other puzzle in the SM fermion sector is the set of deviations from lepton flavor universality observed experimentally through neutral-current and charged-current semi-leptonic $B$ meson decays. Among the various scenarios built in the literature, the most successful mediator addressing both sets of anomalies has turned out to be a TeV-scale $U_1$ vector leptoquark [231–237]. The couplings of such a $U_1$ particle, being necessarily stronger with the third quark/lepton generation, as well as the needed multi-scale construction [238] find a concrete realization in the following SM fermion distribution along a compactified warped extra dimension: distinct up-type Right-handed fermion flavors localized on distinct 3-branes [7], one brane-localized down-type Right-handed fermion and their two other flavors respectively in two complementary sub-intervals, the second family of Left-handed doublet inside a sub-interval attached to the Infra-Red brane and their two other families spread on the whole interval [239].

It turns out that our present partition mechanism allows to simultaneously induce the flavor replication and arrange them accordingly to the above field distribution along the extra dimension, representing thus a natural theoretical framework for interpreting the measured flavor anomalies. In particular, for the up-type SM fermions, distinct flavors can indeed be strictly localized in distinct 1D sub-domains separated by intermediate vanishing profile sub-intervals [8], thanks to the BBT, and those sub-domain widths could be reduced to produce (thick) 3-branes [see next paragraph]. Regarding the SM doublets

7. With three spatial dimensions.
8. As long as the profiles are globally ortho-normalizable.

of Ref. [239], two 5D fermion doublets should be introduced with one of them splitting into two 4D fermion flavors.

An even more ambitious approach is to solve the $B$ meson anomalies using the gauge vector leptoquark – transforming as $U_1 \equiv (3, 1, 2/3)$ under the SM gauge group – of the Pati-Salam (PS) scenario which is unifying quarks and leptons in a fundamental representation of the SU(4) group. This attempt can be pursued within the context of a 4D model [240] or a 5D warped model [241]. The PS$^3$ model relies on the discrete language of three sites connected by nearest-neighbour interactions which implicitly admits the embedding of the theory into a higher-dimensional spacetime: this can be realized along a warped extra dimension [242] with, once more, the various SM doublet and singlet flavors distributed in complementary sub-intervals and 3-branes. Now, again, our present theoretical partition mechanism, based on BBT, allows to create the three flavors and shape those according exactly to the PS$^3$ model field configuration along the extra dimension.

Furthermore, we propose a new type of spin-1/2 fermion localization mechanism. Let us first recall the general context of this topics. For example, the wave function shapes of spin-1/2 fields can be modified along the extra dimensions by gravitational interactions but cannot be totally localized on a brane in five or six dimensions only via these interactions (see for instance Ref. [243, 244]). The idea of localizing fermions towards some wall in an higher-dimensional space which relies on the index theorem in soliton background [245, 246] goes back to Ref. [247]. There the fermions couple to the spatially varying Vacuum Expectation Value of an higher-dimensional scalar field. For instance, a kink function (step function) for this VEV leads to a spread profile with an angular peak for the (chiral) fermion 0-mode [248]. A more smooth domain wall with an hyperbolic tangent form creates instead a fermion Gaussian profile [249]. The fermions can also couple to extended functions of the scalars with varying VEVs [250, 251] (see also Ref. [252]), generically in $4 + d$ dimensions [253], and the additional effect of gravity on the wall may be included as well [254].

Within the string theory framework, the hypothesis itself of matter localization on a brane-world was studied, by restricting the matter Lagrangian to lie on a D3-brane thanks to a simple delta distribution with support at the brane location: it was demonstrated that in D3-brane scenarios, 4D fermion modes are not normalizable [255]. Nonetheless, when domain wall generalizations for D-branes are considered, adding an interaction between a fermion and the scalar field generating the domain wall, at least one chirality of the effective 4D fermion could be localized. Chiral fermions can be localized, e.g. at a D3/D7 brane intersection world-volume via a stringy defect mechanism [256]. More generally, fermions can be localized thanks to a string-like defect [257–260].

In the present work, the field theory partition mechanism of spin-1/2 fermion localization, along an extra dimension, is quite simple and possesses the following particularities:

(i) It has a point-like phenomena origin through the BBT – thus not relying on a fermion coupling to a spatially varying scalar field VEV.

(ii) It localizes a (chiral or vectorial) fermion [any KK-level mode] to a strict interval (i.e. exactly vanishing profile outside this interval).

(iii) The interval width, determined by the BBT points, can be chosen as small as wanted down to a point, which represents nothing else but a brane-localization.

In particular, the localization mechanism we proposed, based on certain BBT positions, also allows to have a compactified space domain where all fermion profiles are exactly vanishing. Taking the limit of vanishing width (instead of keeping a strict thick wall) for the complementary domain – where all the fermions spread – leads to the concrete realization of the types of warped or flat extra dimension models where all SM fermions are confined on a brane (instead of a strict thick brane), as proposed in the literature to address the gauge hierarchy puzzle. Of course, by taking different BBT among fermions, we can also build variants of such scenarios where only a sub-class of fermions is brane-localized, like for tiny neutrino mass models with all SM fermions stuck on a brane and a Right-handed neutrino field propagating in the bulk.

This chapter is organized as follows. First, we give the definition of the spacetime geometry and the complete toy model in Section 6.2.1. Then, in Sections 6.2.2-6.2.6, we study the free fermion profile through the least action principle, and give the detailed description of the fermion generation splitting mechanism. In Section 6.3, we realize the fermion mass hierarchy with the bulk Higgs VEV function on the extra dimension, via the bulk Yukawa interactions. Finally, an overview and a brief conclusion would be given in Section 6.4.

## 6.2 Flavor Model

In this section, we search for a flavor model including a fermion generation splitting mechanism in the basic free case without Yukawa interactions, deriving the associated profiles and mass spectra to each generation. The main target of this section is to realize the generation splitting in a rigorous procedure for revealing the SM mass hierarchy in Section 6.3.

### 6.2.1 Partition Model

#### 6.2.1.1 Spacetime Geometry

We consider a 5D toy model on the product spacetime geometry, $\mathcal{E}^5 = \mathcal{M}^4 \times \mathcal{I}_1$.

— $\mathcal{M}^4$ represents the usual 4D Minkowski spacetime whose coordinates are denoted by $x^\mu$ where $\mu \in [\![0, 3]\!]$ is the Lorentz index of the covariant formalism. The metric conventions are given in Appendix A.

— $\mathcal{I}_1$ is a compact 1D flat interval of the extra spatial dimension, which is denoted by $y \in [0, L]$, with a length, $L \in \mathbb{R}^*$, and bounded by two flat 3-branes at $y = 0$ and $y = L$. The interesting thing is that the brane-localized interactions – the BBT – play the role of extra intermediate branes between the two boundaries at $y = 0, L$ for the fermion, $F$, which may interrupt the continuity of the fermion profiles and force the fermion probability currents vanish at the intermediate branes at $y = L_F$ (cf. Section 6.2.2). Here, we firstly consider the toy model with one intermediate brane [9] to realize two fermion generations localizing in the first and the second regions respectively, as illustrated symbolically in Figure 6.1.

— A point of the 5D spacetime $\mathcal{E}^5$ is labeled by the coordinates, $x^M \hat{=} (x^\mu, y)$, with $M \in [\![0, 4]\!]$.

In order to write down the initial action, we urge the importance of taking care of possible field jumps along the extra dimension at intermediate branes by the treatment of the discontinuity precisely described in Chapter 4 [function formalism]. The bulk Lagrangian density will involve profile derivatives $\partial_y f(y)$, so that $\partial_y f(y)$ must be well-defined on $[0, L_F] \cup (L_F, L]$, i.e. on $[0, L_F]$ and $(L_F, L]$ respectively. Analogy to the treatment of jumps in Chapter 4 (also see Ref. [2]), the necessary (but not sufficient) condition for this last feature is that the profiles $f(y)$ have to be continuous on the three segments

9. It's the reason why we impose '1' as the subscript in $\mathcal{I}_1$.

respectively. For example, $f(y)$ is derivable in the region $[0, L_F]$ at $y = 0$ if and only if $f(y)$ is right-derivable at $y = 0$, and the corresponding right-derivative is convergent. Then, the complete 1D interval is decomposed by $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$ with respect to the piece-wise smoothness. The lengths of the two segments are denoted respectively as,

$$\Delta L_F^1 \hat{=} L_F - 0\,, \quad \Delta L_F^2 \hat{=} L - L_F\,. \tag{6.1}$$

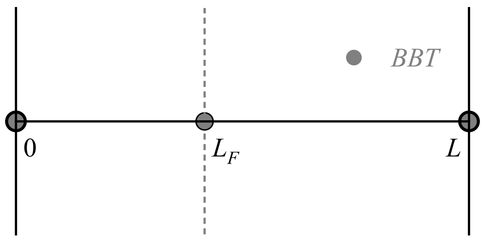


Figure 6.1 – A picture of the interval with one intermediate brane. Two 3-branes (two solid lines) on two boundaries (two black points) at $y = 0, L$. Unique intermediate brane (dashed line) at $y = L_F^{(+)}$ are induced by the BBT (three gray points).

Then, the well-defined global action covering the whole physical domain of the interval is developed including an improper integral:

$$\begin{aligned} S_{5\mathrm{D}}^F &= S_{\mathrm{bulk}}^F + S_{\mathrm{branes}}^F\,, \\ \text{with} \quad S_{\mathrm{bulk}}^F &= \int d^4x \left( \int_0^{L_F} + \int_{L_F^+}^{L} \right) dy\ \mathcal{L}_{\mathrm{bulk}}^F\,, \end{aligned} \tag{6.2}$$

where

- $\mathcal{L}_{\mathrm{bulk}}^F$ includes the fermion kinetic (6.3) (cf. Section 6.2.1.2) and the mass terms (6.58) (cf. Section 6.2.6) of the Lagrangian density , which is integrable over the entire region, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$. Here, we introduce a unique field content to achieve a simplest 2-gen example. Fermionic particles propagate along $\mathcal{I}_1$ as we have in mind that a direct extension of this toy model to a 3-gen scenario as the realistic SM.
- $S_{\mathrm{branes}}^F$ represents action terms localized at the two boundaries $(y = 0, L)$ and intermediate branes $[y = L_F^{(+)}]$ for the fermion field, e.g. the brane-localized BBT (cf. Section 6.2.1.3).

Note that this two-generation toy model can be easily extended to the realistic SM scenario with three generations, where two intermediate branes are induced by the BBT and profiles will be piece-wise smooth with respect to three sub-regions. The bulk action will preserve a similar formula to Eq. (6.2) but include a three-piece integration rather than two.

#### 6.2.1.2 Bulk Fermion Fields

The 5D fermion field $F(x^\mu, y)$ – of mass dimension 2 – has the following kinetic terms [entering Eq. (6.2)] which allow to recover canonical covariant kinetic terms for the associated fermions in the 4D effective action (as imposed by the argument of decoupling limit [10]):

$$\mathcal{L}_{\text{kin}}^{F} = \frac{i}{2}\,\bar{F}\Gamma^{M}\overleftrightarrow{\partial_{M}}F\,, \tag{6.3}$$

which keeps the identical formula as the interval case of Eq. (3.1) and its chiral formula can be derived via the chiral decomposition (1.17) as

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{F} &= \frac{1}{2}\left(iF_{R}^{\dagger}\sigma^{\mu}\overleftrightarrow{\partial_{\mu}}F_{R} + iF_{L}^{\dagger}\bar{\sigma}^{\mu}\overleftrightarrow{\partial_{\mu}}F_{L} - F_{R}^{\dagger}\overleftrightarrow{\partial_{4}}F_{L} + F_{L}^{\dagger}\overleftrightarrow{\partial_{4}}F_{R}\right)\\ &= \frac{1}{2}\left(i\bar{\mathcal{F}}_{R}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\mathcal{F}_{R} + i\bar{\mathcal{F}}_{L}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\mathcal{F}_{L} - \bar{\mathcal{F}}_{R}\overleftrightarrow{\partial_{4}}\mathcal{F}_{L} + \bar{\mathcal{F}}_{L}\overleftrightarrow{\partial_{4}}\mathcal{F}_{R}\right)\,,\end{aligned} \tag{6.4}$$

but with the discontinuity argument (see Section 6.2.1.1) at the intermediate branes at $y = L_F$. Particularly, the bulk mass terms $\mathcal{L}_{\text{mass}}$ will be treated in Section 6.2.6.

#### 6.2.1.3 Partition Terms

The BBT – of mass dimension 4 – introduced in Chapter 3 (also see Ref. [1]) turn out to be necessary here again for several reasons:

- They allow to avoid physical consistency problems (e.g. recovering the SM fermion chirality configuration).
- They play interestingly the role of intermediate branes at $y = L_F^{+}$ where the fermion probability currents are forced to vanish and each fermion generation is embedded into one segment individually (cf. Section 6.2.2).
- They define the compact model with the two boundaries at $y = 0, L$ and constrain all fermion fields on the physical domain, $\mathcal{I}_1$.

The following BBT lead to the SM chirality configuration [11]

$$\begin{aligned} S_B^F &= \int d^4x \left( \mathcal{L}_B^F\Big|_{L_F} - \mathcal{L}_B^F\Big|_{0} + \mathcal{L}_B^F\Big|_{L} - \mathcal{L}_B^F\Big|_{L_F^+} \right)\,,\\ \text{with}\quad \mathcal{L}_B^F &= \frac{\sigma^F(y)}{2}\,\bar{F}F = \frac{\sigma^F(y)}{2}\left(F_L^{\dagger}F_R + F_R^{\dagger}F_L\right)\,,\end{aligned} \tag{6.5}$$

where we impose the chiral decomposition (1.17) and $\sigma^F(y)$ are generic parameters for the field $F$ at $y$ with compact notations

$$\sigma^F_{0,L_F,L_F^+,L} \mathrel{\hat{=}} \sigma^F\Big|_{0,L_F,L_F^+,L}\,. \tag{6.6}$$

The two choices of dimensionless BBT signs [12]

$$\begin{cases} \sigma_1^F : \sigma^F_{0,L_F,L_F^+,L} &= -1\,,\\ \sigma_2^F : \sigma^F_{0,L_F,L_F^+,L} &= +1\,,\end{cases} \tag{6.7}$$

10. From the theoretical consistency and phenomenological points of view, the SM must be approximately recovered at low-energies in the limit of infinitely heavy KK excitations, which is also a crucial argument in other scenarios in Chapter 3-4.

11. One can also setup the BBT at $L_F^-$ instead of $L_F$ but it wouldn't change any physical results such as mass spectra.

12. The different BBT signs would lead to a different chirality configuration of the zero-modes from the SM or custodian fermions (cf. Section 3.3.3).

would lead to a set of natural boundary conditions localized at intermediate branes and boundaries in Eq. (6.12), which then induces two generations of normalizable profiles [clarified in two series in Eq. (6.35)] associated to the 5D fermionic field $F$ (6.36) with the non-vanishing zero modes [without KK mass contribution] for only one chirality [$L$ or $R$] respectively [see Eq. (6.44)-(6.51) with Table 6.1 for bulk massless case and Eq. (6.74)-(6.75) with Table 6.2 for bulk massive case].

Considering all the aspects above, the complete toy model (6.2) studied for the free fermions in Section 6.2 is characterized by the action,

$$S^F_{\text{5D}} = S^F_{\text{bulk}} + S^F_{\text{branes}} = S^F_{\text{bulk}} + S^F_B\,, \tag{6.8}$$

where the bulk action terms consist of the kinetic Lagrangian density of Eq. (6.3)-(6.4) in the bulk massless case, while the bulk massive case would be studied in Section 6.2.6. The conclusions that will be derived in the present work can be directly extended to the realistic warped model with SM bulk matter addressing the fermion mass and gauge hierarchies, along the same lines as the flavor and gauge symmetry generalizations described in details in Section 3.2.6.

### 6.2.2 Mass Spectra & Profiles

As a preparation for the presence of the Yukawa interactions, we firstly concentrate on the free bulk massless fermions, i.e.

$$\mathcal{L}^F_{\text{bulk}} = \mathcal{L}^F_{\text{kin}}\,,$$

to derive profiles, mass spectra and realize the fermion generation splitting in the free case. The least action principle is applied to the free action (6.8) and leads to two associated stationary equations,

$$\delta_{\bar{F}}\left(S^F_{\text{bulk}} + S^F_B\right) = 0\,,$$

for the unknown 5D field $F$ are generic and independent field variations. In particular, we can write the explicit variations of $S^F_{\text{bulk}} + S_B$ [13],

$$\begin{aligned}\delta_{\bar{F}}\left(S^F_{\text{bulk}} + S^F_B\right) = \int d^4x \,\Bigg\{ &\left(\int_0^{L_F} + \int_{L_F^+}^{L}\right) dy\; \delta\bar{F} i\Gamma^M\partial_M F \\
&+\frac{\sigma^F_{L_F}-1}{2}\,\delta F_L^\dagger F_R\Big|_{L_F} + \frac{\sigma^F_{L_F}+1}{2}\,\delta F_R^\dagger F_L\Big|_{L_F} - \frac{\sigma^F_0-1}{2}\,\delta F_L^\dagger F_R\Big|_0 - \frac{\sigma^F_0+1}{2}\,\delta F_R^\dagger F_L\Big|_0 \\
&+\frac{\sigma^F_L-1}{2}\,\delta F_L^\dagger F_R\Big|_L + \frac{\sigma^F_L+1}{2}\,\delta F_R^\dagger F_L\Big|_L - \frac{\sigma^F_{L_F^+}-1}{2}\,\delta F_L^\dagger F_R\Big|_{L_F^+} - \frac{\sigma^F_{L_F^+}+1}{2}\,\delta F_R^\dagger F_L\Big|_{L_F^+}\Bigg\}\,,\end{aligned} \tag{6.9}$$

where the the chiral decomposition (1.17) is inserted to brane terms. In this stationary action condition, the bulk and brane variations must vanish individually, which would lead to the bulk EOM,

$$\forall\, x^\mu,\; y \in \mathcal{I}_1 = [0, L_F] \cup (L_F, L]\,,\quad i\,\Gamma^M\partial_M F = 0\,, \tag{6.10}$$

13. We omit the global 4-divergence by the remarks in Section 1.3.

and it's chiral formula after the chiral projection (1.17),

$$\forall\, x^\mu,\, y \in \mathcal{I}_1 = [0, L_F] \cup (L_F, L]\,, \quad \begin{cases} i\bar{\sigma}^\mu \partial_\mu F_L + \partial_4 F_R &= 0\,, \\ i\sigma^\mu \partial_\mu F_R - \partial_4 F_L &= 0\,, \end{cases} \tag{6.11}$$

with the corresponding NBC under the SM $\sigma^F_{0,L_F,L_F^+,L}$ configuration mentioned before,

$$\begin{cases} \dfrac{\sigma^F_0 - 1}{2}\, F_R|_0 &= \dfrac{\sigma^F_0 + 1}{2}\, F_L|_0 = \dfrac{\sigma^F_{L_F} - 1}{2}\, F_R|_{L_F} = \dfrac{\sigma^F_{L_F} + 1}{2}\, F_L|_{L_F}\ , \\ \dfrac{\sigma^F_{L_F^+} - 1}{2}\, F_R|_{L_F^+} &= \dfrac{\sigma^F_{L_F^+} + 1}{2}\, F_L|_{L_F^+} = \dfrac{\sigma^F_L - 1}{2}\, F_R|_L = \dfrac{\sigma^F_L + 1}{2}\, F_L|_L\ . \end{cases} \tag{6.12}$$

where $\sigma^F_{0,L_F,L_F^+,L} = \pm 1$ leading to the associated NBC [14],

$$\begin{cases} F_L|_{0(L_F^+)} &= 0\ (\sigma^F_{0(L_F^+)} = +1), \\ & \text{or} \\ F_R|_{0(L_F^+)} &= 0\ (\sigma^F_{0(L_F^+)} = -1), \end{cases} \quad \text{and} \quad \begin{cases} F_L|_{L_F(L)} &= 0\ (\sigma^F_{L_F(L)} = +1), \\ & \text{or} \\ F_R|_{L_F(L)} &= 0\ (\sigma^F_{L_F(L)} = -1). \end{cases} \quad \text{[NBC]} \tag{6.13}$$

Regarding the 5D EOM of Eq. (6.11), the 5D spinors $F_{L/R}\,(x^\mu, y)$ are the generic solutions of Eq. (6.11), which is nothing else but the 5D Dirac equation (in its two-component chiral form) for massless fermions, so that we can write them accordingly to the following factorization,

$$F^{sol.}_{L/R}\,(x^\mu, y) = \frac{1}{\sqrt{L}}\ f_{L/R}(y)\, F^{\text{4D}}_{L/R}\,(x^\mu)\,, \tag{6.14}$$

with the convention of a dimensionless profile $f_{L/R}(y)$ (still in the natural unit system) and [15],

$$\begin{cases} f_{L/R}(y) &= C_{+L/R}(y)\, e^{ip^4 y} + C_{-L/R}(y)\, e^{-ip^4 y}\,, \\ F^{\text{4D}}_{L/R}\,(x^\mu) &= \displaystyle\int \frac{d^3 p_{\text{4D}}}{(2\pi)^3} \frac{1}{\sqrt{E_{\mathbf{p}}}} \sum_{s=1,2} \left\{ a^s_{\mathbf{p}_{\text{4D}}} u^s_{L/R}(\mathbf{p}_{\text{4D}}) e^{-ip_\mu x^\mu} + b^s_{\mathbf{p}_{\text{4D}}} v^s_{L/R}(\mathbf{p}_{\text{4D}}) e^{ip_\mu x^\mu} \right\}, \end{cases} \tag{6.15}$$

where $C_{+L/R}(y), C_{-L/R}(y)$ are piece-wise normalization constants on $[0, L_F] \cup (L_F, L]$ due to possible profile jumps at $y = L_F$,

$$\begin{cases} C_{+L/R}(y) &= C_{+L/R,1}\, \theta\,(y)\, \theta\,(L_F - y) + C_{+L/R,2}\, \theta\left(y - L_F^+\right) \theta\,(L - y)\ , \\ C_{-L/R}(y) &= C_{-L/R,1}\, \theta\,(y)\, \theta\,(L_F - y) + C_{-L/R,2}\, \theta\left(y - L_F^+\right) \theta\,(L - y)\ , \end{cases} \tag{6.16}$$

with the Heaviside step function,

$$\theta\,(y - y_0) \mathrel{\hat{=}} \begin{cases} 0 & ,\ y < y_0\,, \\ 1 & ,\ y \geq y_0\,. \end{cases} \tag{6.17}$$

14. In principle, different configurations on the two regions $[0, L_F]$and $(L_F, L]$ can be achieved if $\sigma^F_0 \neq \sigma^F_{L_F^+}$ or $\sigma^F_{L_F} \neq \sigma^F_L$.

15. The bold letter represents related spatial components.

Two opposite orientation momenta along the extra spatial dimension in $f_{L/R}(y)$ (6.15) are contributions from the reflection at the branes. $a^s_{\mathbf{p}_{4\mathrm{D}}}$ and $b^s_{\mathbf{p}_{4\mathrm{D}}}$ $(s = 1, 2)$ are coefficients associated to the 4D momentum. $u^s_{L/R}(\mathbf{p}_{4\mathrm{D}})e^{-ip_\mu x^\mu}$ and $v^s_{L/R}(\mathbf{p}_{4\mathrm{D}})e^{ip_\mu x^\mu}$ $(s = 1, 2)$ are (normalized) 4D spinor fields [8] satisfying the 4D Dirac-Weyl equations (1.25), corresponding to particle and anti-particle respectively. $p^M = (p^\mu, p^4)$ denote the relativistic 5-momentum which must obey the on-shell (massless) relation,

$$\left(p^0\right)^2 = \sum_{j=1}^{4} \left(p^j\right)^2 = \mathbf{p}^2_{4\mathrm{D}} + \left(p^4\right)^2 \,, \quad \text{and} \quad p^0 \,\hat{=}\, E_{\mathbf{p}} \,. \tag{6.18}$$

Notice that the Eq. (6.15)-(6.18) guarantee $F^{sol.}\left(x^\mu, y\right)$ (6.14) to satisfy the free massless 5D Klein-Gordon equation

$$\partial_M \partial^M F_{L/R}\left(x^\mu, y\right) = 0 \,, \tag{6.19}$$

derived from the 5D EOM (Dirac equation) as required [16]. Given the momentum operator $\hat{P}^j \,\hat{=}\, -\,i\partial_j$ $[j = 1, ..., 4]$ [17], we can rewrite the wave functions and spinors of Eq. (6.15) using (squared) momentum eigenstates:

$$\left\{ \begin{array}{rcl} f_{L/R}(y) & = & \left\langle y \middle| p^2_4 \right\rangle_{L/R} \,, \\ F^{4\mathrm{D}}_{L/R}\left(x^\mu\right) & = & \displaystyle\int \frac{d^3 p_{4\mathrm{D}}}{(2\pi)^3} \frac{1}{\sqrt{E_{\mathbf{p}}}} \\ & & \times \displaystyle\sum_{s=1,2} \left\{ a^s_{\mathbf{p}_{4\mathrm{D}}} u^s_{L/R}(\mathbf{p}_{4\mathrm{D}}) \langle \mathbf{x} | p_{4\mathrm{D}}(t) \rangle + b^s_{\mathbf{p}_{4\mathrm{D}}} v^s_{L/R}(\mathbf{p}_{4\mathrm{D}}) \langle \mathbf{x} | - p_{4\mathrm{D}}(t) \rangle \right\} , \end{array} \right. \tag{6.20}$$

where

$$\begin{array}{c} \left(\hat{P}^4\right)^2 \left| p^2_4 \right\rangle_{L/R} = \left(p^4\right)^2 \left| p^2_4 \right\rangle_{L/R} \,, \\ \hat{P}^j \left| p_{4\mathrm{D}}(t) \right\rangle = p^j \left| p_{4\mathrm{D}}(t) \right\rangle \,, \quad \text{with} \quad j = 1, 2, 3 \,, \end{array} \tag{6.21}$$

and the compact notations,

$$\begin{array}{rcl} |\mathbf{x}\rangle & = & |x^1\rangle \otimes |x^2\rangle \otimes |x^3\rangle \,, \\ |p_{4\mathrm{D}}(t)\rangle & = & e^{-\,iE_{\mathbf{p}}t} \, |\mathbf{p}_{4\mathrm{D}}\rangle = e^{-\,iE_{\mathbf{p}}t} \, |p^1\rangle \otimes |p^2\rangle \otimes |p^3\rangle \,. \end{array} \tag{6.22}$$

Besides, $F^{4\mathrm{D}}_{L/R}\left(x^\mu\right)$ (6.15) must constitute a 4D fermion field satisfying the 4D Dirac-Weyl equation (1.25), involving a effective 4D mass $m^F$, leading to the 4D Klein-Gordon equation,

$$\left[\partial_\mu \partial^\mu + \left(m^F\right)^2\right] F^{4\mathrm{D}}_{L/R}\left(x^\mu\right) = 0 \,. \tag{6.23}$$

Inserting the factorization (6.14)-(6.15) to the 5D Klein-Gordon equation (6.19) and comparing to the 4D Klein-Gordon equation (6.23), one can obtain the mass-momentum relation,

$$m^F = \pm p^4 \,. \tag{6.24}$$

16. $0 = -i^2\, \Gamma^M \Gamma^N \partial_M \partial_N F = \frac{1}{2} \left\{\Gamma^M, \Gamma^N\right\} \partial_M \partial_N F = \eta^{MN} \partial_M \partial_N F = \partial_M \partial^M F$, using the Christoffel algebra $\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN}$ in Appendix A.

17. Notice the subtlety due to the compact spatial geometry that the extra momentum operator $\left(\hat{P}^4\right)^2$ is not Hermitian, even if its real eigenvalues and ortho-normalized eigenstates can still be defined [261].

Hence, the factorization (6.14) allows to fulfill the decoupling limit criteria within a realistic model at the field theory level [for theoretical and phenomenological reasons]: in the heavy limit for the new particles of the extended scenario (that will turn out here to be the Kaluza-Klein excitations), the SM Lagrangian involving 4D fermion fields $F_{L/R}\,(x^\mu)$ must be recovered at low energies (the profiles $f_{L/R}(y)$ being integrated out over $y$ in the 4D effective action, ending up in global factors). Moreover, since the profile solution $f_{L/R}(y)$ (6.15) is even with respect to $p^4$ [18], it will turn out that one can fix the mass-momentum relation (6.24) as,

$$m^F = p^4\,. \tag{6.25}$$

If we insert the 5D factorization (6.14)-(1.25) into the 5D EOM (6.11), we would obtain the EOM of profiles explicitly,

$$\forall\, y \in [0, L_F] \cup (L_F, L]\,, \quad \begin{cases} \partial_4 f_L\,(y) - m^F\, f_R\,(y) & = & 0\,, \\ \partial_4 f_R\,(y) + m^F\, f_L\,(y) & = & 0\,, \end{cases}$$

which leads to general profile solutions

$$\begin{cases} f_L\,(y) & = & B_R(y)\cos(m^F y) + B_L(y)\sin(m^F y)\,, \\ f_R\,(y) & = & B_L(y)\cos(m^F y) - B_R(y)\sin(m^F y)\,, \end{cases} \tag{6.26}$$

where $B_L(y), B'(y)$ are piece-wise normalization constants via the Heaviside step function (6.17) due to possible profile jumps at $y = L_F$,

$$\begin{cases} B_L(y) & = & B_{L,1}\,\theta\,(y)\,\theta\,(L_F - y) + B_{L,2}\,\theta\left(y - L_F^+\right)\theta\,(L - y)\;, \\ B_R(y) & = & B_{R,1}\,\theta\,(y)\,\theta\,(L_F - y) + B_{R,2}\,\theta\left(y - L_F^+\right)\theta\,(L - y)\;. \end{cases} \tag{6.27}$$

with the two redefined constants, comparing with the momentum solution (6.15) and the mass-momentum relation (6.25):

$$\begin{cases} B_L & = & i\,(C_{+L} - C_{-L})\;, \\ B_R & = & C_{+L} + C_{-L}\,, \end{cases} \quad \text{with} \quad \begin{cases} C_{+R} & = & iC_{+L}\,, \\ C_{-R} & = & -iC_{-L}\,, \end{cases} \tag{6.28}$$

can be derived. Similarly, the reverse constant transformation can be derived,

$$\begin{cases} C_{+L} & = & \dfrac{B_R - iB_L}{2}\,, \\ C_{-L} & = & \dfrac{B_R + iB_L}{2}\,, \end{cases} \quad \text{with} \quad \begin{cases} C_{+L} & = & iC_{+R}\,, \\ C_{-L} & = & -iC_{-R}\,. \end{cases} \tag{6.29}$$

Note that the solution (6.26)-(6.28) includes the massless case $m^F = p_4 = 0$ via piece-wise constant profiles $f_L\,(y) = B_R(y)$, $f_R\,(y) = B_L(y)$. On the contrary, $(m^F)^2 \neq p_4^2$ would induce vanishing profile solutions (i.e. $f_L\,(y) = f_R\,(y) = 0$) and hence a vanishing 5D field $F(x^\mu, y) = 0$ [in the Lagrangian], which is ruled out because the field content of the SM (and its present realistic extension) must include fermion spinors.

Then, let us derive all possible profile solutions $f_{L/R}\,(y)$ (6.15)-(6.26) related to the NBC (6.13) explicitly. Since $f_{L/R}\,(y)$ are piece-wise smooth on $[0, L_F] \cup (L_F, L]$, we can

18. For any appreciate profile solution $f_{L/R}(y)$ realized by $p^4$, we can reproduce it with $-p^4$ and exchange $C_{+L/R}(y) \leftrightarrow C_{-L/R}(y)$.

solve the profile on each region respectively. The solutions of the profile EOM (1.32) on the piece-wise smooth domain $y \in [0, L_F]$ ($(L_F, L]$) [normalized via Eq. (1.26)] have been precisely solved in Section 3.3.3 but we need to the modify the length as $\Delta L_F^{1(2)}$ (6.1). Moreover, the lack of BBT at any (intermediate) brane would induce related vanishing profiles and a trivial mass spectrum equation [see Section 3.3.1]. Non-trivial solutions found with particular BBT configurations for Eq. (6.11)-(6.12) are factorized in Eq. (6.14)-(6.26), following four possible sets of profiles over $y \in [0, L_F]$ together with the associated (KK) mass spectrum equations ($\forall\, n \in \mathbb{N}$),

$$
\begin{aligned}
&1)\quad (--):\quad f_L^n(y) = B_{L,1}^n \sin(m_n^F\, y)\,,\quad (++):\quad f_R^n(y) = B_{L,1}^n \cos(m_n^F\, y)\,,\\
&\qquad\qquad\qquad\qquad\qquad \sin(m_n^F\, \Delta L_F^1) = 0\,;\\
&2)\quad (++):\quad f_L^n(y) = B_{R,1}^n \cos(m_n^F\, y)\,,\quad (--):\quad f_R^n(y) = -B_{R,1}^n \sin(m_n^F\, y)\,,\\
&\qquad\qquad\qquad\qquad\qquad \sin(m_n^F\, \Delta L_F^1) = 0\,;
\end{aligned}
\tag{6.30}
$$

and,

$$
\begin{aligned}
&3)\quad (-+):\quad f_L^n(y) = B_{L,1}^n \sin(m_n^F\, y)\,,\quad (+-):\quad f_R^n(y) = B_{L,1}^n \cos(m_n^F\, y)\,,\\
&\qquad\qquad\qquad\qquad\qquad \cos(m_n^F\, \Delta L_F^1) = 0\,;\\
&4)\quad (+-):\quad f_L^n(y) = B_{R,1}^n \cos(m_n^F\, y)\,,\quad (-+):\quad f_R^n(y) = -B_{R,1}^n \sin(m_n^F\, y)\,,\\
&\qquad\qquad\qquad\qquad\qquad \cos(m_n^F\, \Delta L_F^1) = 0\,;
\end{aligned}
\tag{6.31}
$$

where we use the standard BC notations $-$ or $+$ at $y = 0, \pi R$ defined below Eq. (3.29), which make explicit the correspondence between the correspondence between the four NBC (6.12) with non-zero BBT and the four solutions (6.30)-(6.31). The equation

$$\sin(m_n^F\, \Delta L_F^1) = 0\,,$$

possesses the following solutions for the KK mass spectrum,

$$\left| m_n^F \right| = \frac{n\pi}{\Delta L_F^1}\,,\quad n \in \mathbb{N}\,. \tag{6.32}$$

Similarly, the equation

$$\cos(m_n^F\, \Delta L_F^1) = 0\,,$$

has the solutions:

$$\left| m_n^F \right| = \frac{(2n+1)\pi}{2\Delta L_F^1}\,,\quad n \in \mathbb{N}\,. \tag{6.33}$$

We can clearly see that the mass spectrum (6.32)-(6.33) deduced by the piece-wise smooth profile solutions of the EOM (1.32) and the BC (6.12) is parameterized by the length of associated sub-region. So, let us firstly consider the non-zero KK modes [$m_n^F \neq 0$]. The generic profile solution $f_{L/R}(y)$ (6.15)-(6.26), defined on the entire domain $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$, must have the unique momentum $p_4$ (mass $m^F$) on the entire extra spatial dimension (both of two sub-regions), which corresponds to the unique effective 4D mass as discussed in the comparasion between the 5D (6.19) and the 4D-(6.23) Klein-Gordon equation. Hence, the (non-vanishing) profile solutions of Eq. (6.30)-(6.31) (similar profile solutions on the second sub-region $(L_F, L]$ but with the interval length $\Delta L_F^2$) must only exist on one sub-region ($[0, L_F]$ or $(L_F, L]$) and vanish on the other, since the two sub-region lengths induce two different mass spectra. We denote two generations (families) of fermion profile solutions as $f^n_{\mathcal{G}_i L/R}(y)$ ($\forall\, n \in \mathbb{N}^*$, $i = 1, 2$), where $n \in \mathbb{N}^*$ is the KK mode indices. $\mathcal{G}_i$ ($i = 1, 2$) is the generation label corresponding to the non-vanishing solutions on the first and the second sub-region respectively. Furthermore, the associated (SM-like) mass spectrum can be derived explicitly via the sub-interval length $\Delta L^i_F$ ($i = 1, 2$) (6.1) in analogy to Eq. (6.32),

$$\left| m^F_{\mathcal{G}_{1(2)} n} \right| = \frac{n\pi}{\Delta L^{1(2)}_F}\,, \quad n \in \mathbb{N}\,. \tag{6.34}$$

Then, let us turn our focus to the zero mode [$m^F_n = 0$]. We can see from Eq. (6.30)-(6.31) that the zero modes are piece-wise constants on the two sub-regions $[0, L_F]$ and $(L_F, L]$. The zero mode is the unique mode whose mass is interval length independent (vanishing on both of two sub-regions), which leads to a possible non-vanishing profile solution of the EOM (1.32) and the BC (6.12), propagating on the whole extra spatial dimension. Moreover, the two sub-regions will induce a two-dimensional Hilbert space for zero modes [discussed later in Eq. (6.44)-(6.50)], so we denote the two 0-mode profiles (basis states) as $f^0_{\mathcal{G}_i L/R}(y)$ ($i = 1, 2$).

We sum two kinds of flavors and corresponding KK modes, constituting all individual solutions of *both* homogeneous differential equations [5D EOM (6.11)] *and* 5D homogeneous NBC (6.12)-(6.13) (associated to a certain BBT configuration). From the quantum point of view, it is based on the quantum superposition principle [19]. We obtain

$$F_{L/R}\,(x^\mu, y) = \sum_{i=1,2} F_{\mathcal{G}_i L/R}\,(x^\mu, y)\,, \tag{6.35}$$

which consists of two flavor generations [20] $F_{\mathcal{G}_i L/R}\,(x^\mu, y)$ ($i = 1, 2$),

$$F_{\mathcal{G}_i L/R}\,(x^\mu, y) = \frac{1}{\sqrt{L}}\, f^0_{\mathcal{G}_i L/R}\Big|_{\Omega_F}(y)\, F^0_{\mathcal{G}_i L/R}\,(x^\mu) + \frac{1}{\sqrt{L}} \sum_{n=1}^{+\infty} f^n_{\mathcal{G}_i L/R}(y)\, F^n_{\mathcal{G}_i L/R}\,(x^\mu)\,, \tag{6.36}$$

where zero mode profile $f^0_{\mathcal{G}_i L/R}\Big|_{\Omega_F}(y)$ ($i = 1, 2$) is labeled by a rotation angle $\Omega_F \in [0, 2\pi)$, which will be described later in the precise solutions (6.44)-(6.50). Each profile solution $f^n_{\mathcal{G}_i L/R}\,(y)$ ($i = 1, 2$) is related to its 4D field $F^n_{\mathcal{G}_i L/R}\,(x^\mu)$ ($i = 1, 2$, $n \in \mathbb{N}$) with the KK mass $m^F_{\mathcal{G}_i n}$ ($i = 1, 2$, $n \in \mathbb{N}$) (6.34).

In analogy to Eq. (6.20), each KK mode profile $f^n_{\mathcal{G}_i L/R}\,(y)$ ($\forall\, n \in \mathbb{N}$, $i = 1, 2$) can still be written by the squared momentum eigenstates $\left|p^2_{4,n\mathcal{G}_i}\right\rangle_{L/R}$ ($\forall\, n \in \mathbb{N}$, $i = 1, 2$),

$$\begin{cases} f^0_{\mathcal{G}_i L/R}\Big|_{\Omega_F}(y) &= \left\langle y \middle| p^2_{4,0\mathcal{G}_i}(\Omega_F) \right\rangle_{L/R}, \quad n = 0\,, \\ f^n_{\mathcal{G}_i L/R}\,(y) &= \left\langle y \middle| p^2_{4,n\mathcal{G}_i} \right\rangle_{L/R} \quad, \quad n \in \mathbb{N}^*\,, \end{cases} \tag{6.37}$$

where $n \in \mathbb{N}$ denoted the KK order corresponding to family $\mathcal{G}_i$ ($i = 1, 2$) with the KK mass $m^F_{\mathcal{G}_i n}$ ($i = 1, 2$) (6.32)-(6.33) and $\Omega_F \in [0, 2\pi)$ is a rotation angle for the 0-mode Hilbert space, which will be described later in the precise solutions (6.44)-(6.50).

19. In the mathematical point of view, the generic solution $F_{L/R}\,(x^\mu, y)$ (6.35) – the summation of all possible solutions – is induced by the linearity and homogeneity of the EOM (6.11) and the BC (6.13) [fixed BBT configuration]. It is similar to the integration over momentum 3D, the summation of spin states, of particle and anti-particle (6.20). Besides, the ortho-normalized basis made of flavor states, KK states, 4D momentum eigenstates, particle/anti-particle sates, up/down spin states are also all motivated by the quantum point of view.

20. A further justification is presented above the ortho-normalization conditions (6.38).

In the decoupling limit, $F_{L/R}\,(x^\mu, y)$ (6.35) must recover canonical 4D fields with independent (diagonal) kinetic terms, i.e. the effective 4D fields must be orthogonal [see Eq. (6.40)-(6.42)], and one cannot build more than 2 (families of) orthogonal fields satisfying the ortho-normalization conditions,

$$\begin{aligned}
&\forall\, n, m \in \mathbb{N},\ \forall\, i, j \in \{1,2\}\,,\\
&\frac{1}{L}\left(\int_0^{L_F} + \int_{L_F^+}^{L}\right) dy\ f^{n*}_{\mathcal{G}_i L/R}(y)\, f^{m}_{\mathcal{G}_j L/R}(y) = \delta_{ij}\delta_{nm}\,. 
\end{aligned} \tag{6.38}$$

Indeed, this crucial physical condition can be justified by firstly injecting $F_{L/R}\,(x^\mu, y)$ back to the kinetic terms $\mathcal{L}^F_{\text{kin}}$ (6.4) [21],

$$\begin{aligned}
&\sum_2 \int dy\ \mathcal{L}^F_{\text{kin}} \rightarrow \sum_{i=1,2}\sum_2 \int dy\ \mathcal{L}^F_{\mathcal{G}_i\text{kin}}\,,\\
&\text{with,}\ \ \mathcal{L}^F_{\mathcal{G}_i\text{kin}} = \frac{1}{2}\left(iF^\dagger_{\mathcal{G}_iR}\sigma^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_iR} + iF^\dagger_{\mathcal{G}_iL}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_iL} - F^\dagger_{\mathcal{G}_iR}\overleftrightarrow{\partial_4} F_{\mathcal{G}_iL} + F^\dagger_{\mathcal{G}_iL}\overleftrightarrow{\partial_4} F_{\mathcal{G}_iR}\right)\,,
\end{aligned} \tag{6.39}$$

where $\mathcal{L}^F_{\mathcal{G}_i\text{kin}}$ is the kinetic terms for the $i^{\text{th}}$ ($i = 1, 2$) generation. Note that non-diagonal terms vanish by $\forall\, i \neq j \in \{1, 2\}$,

$$\begin{aligned}
&\sum_2 \int dy\ F^\dagger_{\mathcal{G}_iL/R}\sigma^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_jL/R}\\
&= \frac{1}{L}\sum_{n,m=0}^{+\infty}\sum_2 \int dy\ f^{n*}_{\mathcal{G}_iL/R}(y)\, F^{n\dagger}_{\mathcal{G}_iL/R}\,(x^\mu)\,\sigma^\mu \overleftrightarrow{\partial_\mu}\left[f^m_{\mathcal{G}_jL/R}(y)\, F^m_{\mathcal{G}_jL/R}\,(x^\mu)\right] = 0\,,\\
&\sum_2 \int dy\ F^\dagger_{\mathcal{G}_iL/R}\overleftrightarrow{\partial_4} F_{\mathcal{G}_jR/L}\\
&= \frac{1}{L}\sum_{n,m=0}^{+\infty}\sum_2 \int dy\ f^{n*}_{\mathcal{G}_iL/R}(y)\, F^{n\dagger}_{\mathcal{G}_iL/R}\,(x^\mu)\,\overleftrightarrow{\partial_4}\left[f^m_{\mathcal{G}_jR/L}(y)\, F^m_{\mathcal{G}_jR/L}\,(x^\mu)\right] = 0\,,
\end{aligned}$$

with $\forall\, i \neq j \in \{1, 2\}$ , $n, m \in \mathbb{N}$ , $y \in [0, L_F] \cup (L_F, L]$ from the orthogonality relations in Eq. (6.38),

$$\begin{aligned}
\sum_2 \int dy\ f^{n*}_{\mathcal{G}_iL/R}(y) f^m_{\mathcal{G}_jL/R}(y) = 0\,,\\
\sum_2 \int dy\ f^{n*}_{\mathcal{G}_iL/R}(y)\overleftrightarrow{\partial_4} f^m_{\mathcal{G}_jR/L}(y) = 0\,,
\end{aligned} \tag{6.40}$$

where we use the equation from the EOM (6.11) [cf. Eq. (1.32)],

$$\begin{aligned}
f^{n*}_{\mathcal{G}_iL/R}\overleftrightarrow{\partial_4} f^m_{\mathcal{G}_jR/L} &= f^{n*}_{\mathcal{G}_iL/R}\left[(-/+)m^F_{\mathcal{G}_jm} f^m_{\mathcal{G}_jL/R}\right] - \left[(+/-)m^F_{\mathcal{G}_in} f^{n*}_{\mathcal{G}_iR/L}\right] f^m_{\mathcal{G}_jR/L}\\
&= (-/+)\left(m^F_{\mathcal{G}_jm} f^{n*}_{\mathcal{G}_iL/R} f^m_{\mathcal{G}_jL/R} + m^F_{\mathcal{G}_in} f^{n*}_{\mathcal{G}_iR/L} f^m_{\mathcal{G}_jR/L}\right)\,.
\end{aligned} \tag{6.41}$$

Considering the absence of bulk mass terms, the kinetic splitting (6.39) confirms the initial scheme that $F_{\mathcal{G}_iL/R}\,(x^\mu, y)$ ($i = 1, 2$) contains two different 4D particles, such that the bulk

21. To be compact, $\sum_2 \int dy \,\hat{=} \left(\int_0^{L_F} + \int_{L_F^+}^{L}\right) dy$.

terms, $\mathcal{L}^F_{\text{bulk}}$ (6.3), finally split into two generations,

$$\sum_2 \int dy\ \mathcal{L}^F_{\text{bulk}} \to \sum_{i=1,2} \sum_2 \int dy\ \mathcal{L}^F_{\mathcal{G}_i\text{bulk}}\,, \quad \text{with} \ \ \mathcal{L}^F_{\mathcal{G}_i\text{bulk}} = \mathcal{L}^F_{\mathcal{G}_i\text{kin}}\,. \tag{6.42}$$

Then, let us integrate out the extra dimension and realize an effective 4D scenario. Inserting the free KK decomposition Eq. (6.36), the orthonormalization conditions (6.38) and the EOM of free profiles for each generation [cf. Eq. (1.32)], the bulk action terms, $S^F_{\text{bulk}}$ (6.2) leads to the canonical kinetic terms for each generation ($\forall\, i \in \{1,2\}$) of the 4D fermion fields by the generation splitting (6.39)-(6.42),

$$\begin{aligned}
\sum_2 \int dy\ \mathcal{L}^F_{\mathcal{G}_i\text{kin}} &= \sum_2 \int dy\ \frac{1}{2}\Big(iF^\dagger_{\mathcal{G}_iR}\sigma^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_iR} + iF^\dagger_{\mathcal{G}_iL}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_iL} - F^\dagger_{\mathcal{G}_iR}\overleftrightarrow{\partial_4} F_{\mathcal{G}_iL} + F^\dagger_{\mathcal{G}_iL}\overleftrightarrow{\partial_4} F_{\mathcal{G}_iR}\Big) \\
&= \sum_{n=n^F_L}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_iL}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_iL} + \sum_{n=n^F_R}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_iR}\sigma^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_iR} \\
&\qquad\qquad + \sum_2 \int dy\ \frac{1}{2L} \sum_{n,m=0}^{+\infty} \Big( f^{n*}_{\mathcal{G}_iL} \overleftrightarrow{\partial_4} f^m_{\mathcal{G}_iR} F^{n\dagger}_{\mathcal{G}_iL} F^m_{\mathcal{G}_iR} + \text{H.c.}\Big) \\
&= \sum_{n=n^F_L}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_iL}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_iL} + \sum_{n=n^F_R}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_iR}\sigma^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_iR} \\
&\qquad\qquad - \sum_{n=1}^{+\infty} m^F_{\mathcal{G}_in} \Big( F^{n\dagger}_{\mathcal{G}_iL} F^n_{\mathcal{G}_iR} + F^{n\dagger}_{\mathcal{G}_iR} F^n_{\mathcal{G}_iL}\Big)\,,
\end{aligned} \tag{6.43}$$

with $n^F_{L\langle R\rangle} = 0\langle 1\rangle$ or $1\langle 0\rangle$ through two SM-like BBT configurations (6.7) respectively [the lowest non-vanishing KK mode] [22],

$$\begin{cases} \sigma^F_1 \ :\ n^F_{L\langle R\rangle} = 0\langle 1\rangle\,, \\ \sigma^F_2 \ :\ n^F_{L\langle R\rangle} = 1\langle 0\rangle\,, \end{cases}$$

using diagonal relations derived from the ortho-normalization conditions (6.38) and the relation in Eq. (6.41),

$$\begin{aligned}
\sum_2 \int dy\ F^\dagger_{\mathcal{G}_iL\langle R\rangle}\bar{\sigma}^\mu\langle\sigma^\mu\rangle \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_iL\langle R\rangle} &= \sum_2 \int dy\ \frac{1}{L}\sum_{n,m=0}^{+\infty} f^{n*}_{\mathcal{G}_iL\langle R\rangle} f^m_{\mathcal{G}_iL\langle R\rangle} F^{n\dagger}_{\mathcal{G}_iL\langle R\rangle}\bar{\sigma}^\mu\langle\sigma^\mu\rangle \overleftrightarrow{\partial_\mu} F^m_{\mathcal{G}_iL\langle R\rangle} \\
&= \sum_{n=n^F_{L\langle R\rangle}}^{+\infty} F^{n\dagger}_{\mathcal{G}_iL\langle R\rangle}\bar{\sigma}^\mu\langle\sigma^\mu\rangle \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_iL\langle R\rangle}\,,
\end{aligned}$$

$$\sum_2 \int dy\ \frac{1}{2L}\sum_{n,m=0}^{+\infty} f^{n*}_{\mathcal{G}_iL}\overleftrightarrow{\partial_4} f^m_{\mathcal{G}_iR} F^{n\dagger}_{\mathcal{G}_iL} F^m_{\mathcal{G}_iR} = -\sum_{n=1}^{+\infty} m^F_{\mathcal{G}_in} F^{n\dagger}_{\mathcal{G}_iL} F^n_{\mathcal{G}_iR}\,,$$

which contains a canonical formalism for the 4D effective kinetic terms and diagonal KK mass terms as expected.

### 6.2.3 Quantum Description of 0-Modes

To end up the study of mass spectra and profiles, in this section, we study the explicit 0-mode profile solutions, satisfying the ortho-normalization conditions (6.38).

22. $n^F_{L/R} = 0$ in the custodian configurations (6.31)-(6.33).

First, zero modes over the whole domain for the SM-like profile $\left. f^0_{\mathcal{G}_i L/R}\right|_{\Omega_F}(y)$ $(i=1,2)$ are taken from Eq. (6.30) via the Heaviside step function (6.17), leading to non-vanishing zero modes in the $\sigma_1^F$ BBT configuration (6.7),

$$\left\{\begin{aligned} \left. f^0_{\mathcal{G}_1 L}\right|_{\Omega_F}(y) &= \theta(y)\,\theta(L_F-y)\sqrt{\frac{L}{\Delta L_F^1}}\cos\Omega_F\, e^{i\alpha_F^{10}} \\ &\qquad +\theta\left(y-L_F^+\right)\theta(L-y)\sqrt{\frac{L}{\Delta L_F^2}}\sin\Omega_F\, e^{i\alpha_F^{20}}\,, \\ \left. f^0_{\mathcal{G}_2 L}\right|_{\Omega_F}(y) &= -\theta(y)\,\theta(L_F-y)\sqrt{\frac{L}{\Delta L_F^1}}\sin\Omega_F\, e^{i\left(\alpha_F^{10}+\delta_F\right)} \\ &\qquad +\theta\left(y-L_F^+\right)\theta(L-y)\sqrt{\frac{L}{\Delta L_F^2}}\cos\Omega_F\, e^{i\left(\alpha_F^{20}+\delta_F\right)}\,, \end{aligned}\right. \tag{6.44}$$

and that in the $\sigma_2^F$ BBT configuration (6.7),

$$\left\{\begin{aligned} \left. f^0_{\mathcal{G}_1 R}\right|_{\Omega_F}(y) &= \theta(y)\,\theta(L_F-y)\sqrt{\frac{L}{\Delta L_F^1}}\cos\Omega_F\, e^{i\alpha_F^{10}} \\ &\qquad +\theta\left(y-L_F^+\right)\theta(L-y)\sqrt{\frac{L}{\Delta L_F^2}}\sin\Omega_F\, e^{i\alpha_F^{20}}\,, \\ \left. f^0_{\mathcal{G}_2 R}\right|_{\Omega_F}(y) &= -\theta(y)\,\theta(L_F-y)\sqrt{\frac{L}{\Delta L_F^1}}\sin\Omega_F\, e^{i\left(\alpha_F^{10}+\delta_F\right)} \\ &\qquad +\theta\left(y-L_F^+\right)\theta(L-y)\sqrt{\frac{L}{\Delta L_F^2}}\cos\Omega_F\, e^{i\left(\alpha_F^{20}+\delta_F\right)}\,, \end{aligned}\right. \tag{6.45}$$

with arbitrary phases $\alpha_F^{1(2)0}$, $\delta_F$ and the relative phase angle $\Omega_F\in[0,2\pi)$ [23]. The ortho-normalizations (6.38) are basically guaranteed by two mathematical relations: $\cos\Omega_F\sin\Omega_F - \sin\Omega_F\cos\Omega_F = 0$ and $\cos^2\Omega_F+\sin^2\Omega_F=1$.

From the ortho-normality (6.38) of above wave functions $\left. f^0_{\mathcal{G}_i L/R}\right|_{\Omega_F}(y)$ $(i=1,2)$ and the completeness relation

$$\frac{1}{L}\sum_2\int dy\ |y\rangle\langle y| = 1\,, \tag{6.46}$$

we can show that $(\forall\, i,j\in\{1,2\})$

$$\begin{aligned} \left\langle p^2_{4,0\mathcal{G}_j}(\Omega_F)\middle| p^2_{4,0\mathcal{G}_i}(\Omega_F)\right\rangle_{L/R} &= \frac{1}{L}\sum_2\int dy\ \left\langle p^2_{4,0\mathcal{G}_j}(\Omega_F)\middle| y\right\rangle\left\langle y\middle| p^2_{4,0\mathcal{G}_i}(\Omega_F)\right\rangle_{L/R} \\ &= \frac{1}{L}\sum_2\int dy\ \ \left. f^{0*}_{\mathcal{G}_j L/R}\right|_{\Omega_F}(y)\ \left. f^{0}_{\mathcal{G}_i L/R}\right|_{\Omega_F}(y) \\ &= \delta_{ij}\,, \end{aligned} \tag{6.47}$$

implying that $\left| p^2_{4,0\mathcal{G}_i}(\Omega_F)\right\rangle_{L/R}$ $(i=1,2)$ are ortho-normalized via arbitrary angle $\Omega_F\in[0,2\pi)$. So, we can consider $\left| p^2_{4,0\mathcal{G}_i}(0)\right\rangle_{L/R}$ $(i=1,2)$ as a set of ortho-normalized basis

23. Note that $\Omega_F=\pi/2, 3\pi/2$ will make generic solutions $\left. f^0_{\mathcal{G}_{1(2)} L}\right|_{\Omega_F}(y)$ (6.44)-(6.45) localized in the second(first) sub-region. So, $\left. f^0_{\mathcal{G}_i L}\right|_{\Omega_F}(y)$ $(i=1,2)$ will have wrong names according to their localization. However, in the realistic case, this weird phenomenon will not happen since we can realize all reasonable configurations without $\Omega_F=\pi/2,3\pi/2$ and $\Omega_F$ can be fixed by experimental quantities (see Section 6.3).

states from the generic solutions (6.44)-(6.45),

$$\begin{cases} f^0_{\mathcal{G}_1 L/R}\Big|_0 (y) &= \left\langle y \middle| p^2_{4,0\mathcal{G}_1}(0) \right\rangle_{L/R} = \theta\left(y\right)\theta\left(L_F - y\right)\sqrt{\dfrac{L}{\Delta L_F^1}}\,, \\ f^0_{\mathcal{G}_2 L/R}\Big|_0 (y) &= \left\langle y \middle| p^2_{4,0\mathcal{G}_2}(0) \right\rangle_{L/R} = \theta\left(y - L_F^+\right)\theta\left(L - y\right)\sqrt{\dfrac{L}{\Delta L_F^2}}\,, \end{cases} \tag{6.48}$$

which span a two-dimensional 0-mode Hilbert space. Indeed, the first generation profile $\left\langle y \middle| p^2_{4,0\mathcal{G}_1}(0) \right\rangle_{L/R}$ vanishes on the second sub-region $(L_F, L]$, while the second generation profile $\left\langle y \middle| p^2_{4,0\mathcal{G}_2}(0) \right\rangle_{L/R}$ vanishes on the first sub-region $[0, L_F]$, as illustrated in Figure 6.3.

Then, Eq. (6.44)-(6.45) can re-written as developments on these basis elements (6.48), thanks to Eq. (6.37),

$$\begin{aligned} \left\langle y \middle| p^2_{4,0\mathcal{G}_1}(\Omega_F) \right\rangle_{L/R} &= \cos\Omega_F e^{i\alpha_F^{10}} \left\langle y \middle| p^2_{4,0\mathcal{G}_1}(0) \right\rangle_{L/R} \\ &\quad + \sin\Omega_F e^{i\alpha_F^{20}} \left\langle y \middle| p^2_{4,0\mathcal{G}_2}(0) \right\rangle_{L/R}, \\ \left\langle y \middle| p^2_{4,0\mathcal{G}_2}(\Omega_F) \right\rangle_{L/R} &= -\sin\Omega_F e^{i\left(\alpha_F^{10}+\delta_F\right)} \left\langle y \middle| p^2_{4,0\mathcal{G}_1}(0) \right\rangle_{L/R} \\ &\quad + \cos\Omega_F e^{i\left(\alpha_F^{20}+\delta_F\right)} \left\langle y \middle| p^2_{4,0\mathcal{G}_2}(0) \right\rangle_{L/R}, \end{aligned} \tag{6.49}$$

which can be illustrated geometrically in the case of vanishing complex phases, i.e. $\alpha_F^{10} = \alpha_F^{20} = \delta_F = 0$. This is what we perform in Figure 6.2, using $\left| p^2_{4,0\mathcal{G}_{1(2)}}(0) \right\rangle_{L/R}$ as two ortho-normal basis vector representations with coordinates (1, 0) and (0, 1) respectively. Then, the angle $\Omega_F$ can be interpreted as the projection angle made by $\left| p^2_{4,0\mathcal{G}_{1(2)}}(\Omega_F) \right\rangle_{L/R}$ with respect to these basis elements.

In the absence of complex phases, we can also see the decomposition equation (6.49) as a basis rotation from the initial basis states $\left| p^2_{4,0\mathcal{G}_i}(0) \right\rangle_{L/R}$ $(i = 1, 2)$ into another pair of ortho-normal states $\left| p^2_{4,0\mathcal{G}_i}(\Omega_F) \right\rangle_{L/R}$ $(i = 1, 2)$ with respect to the rotation angle $\Omega_F$:

$$\begin{bmatrix} \left| p^2_{4,0\mathcal{G}_1}(\Omega_F) \right\rangle_{L/R} \\ \left| p^2_{4,0\mathcal{G}_2}(\Omega_F) \right\rangle_{L/R} \end{bmatrix} = \begin{bmatrix} \cos\Omega_F & \sin\Omega_F \\ -\sin\Omega_F & \cos\Omega_F \end{bmatrix} \begin{bmatrix} \left| p^2_{4,0\mathcal{G}_1}(0) \right\rangle_{L/R} \\ \left| p^2_{4,0\mathcal{G}_2}(0) \right\rangle_{L/R} \end{bmatrix}. \tag{6.50}$$

In Figure 6.3, we represent the fermion profiles corresponding to the quantum states of the Hilbert space shown in Figure 6.2. Clearly, the angle $\Omega_F$ controls the location of the two fermion profiles with respect to the intermediate BBT brane at $y = L_F$. In this sense, the partition mechanism arranges the location of the various flavors along the extra dimension. Similarly, we can directly extend our partition mechanism to the realistic case of 3 flavors, by introducing two intermediate branes via certain BBT (see Section 6.2.1.1). Then, a similar formalism will induce a 3-dimensional Hilbert space for the zero modes.

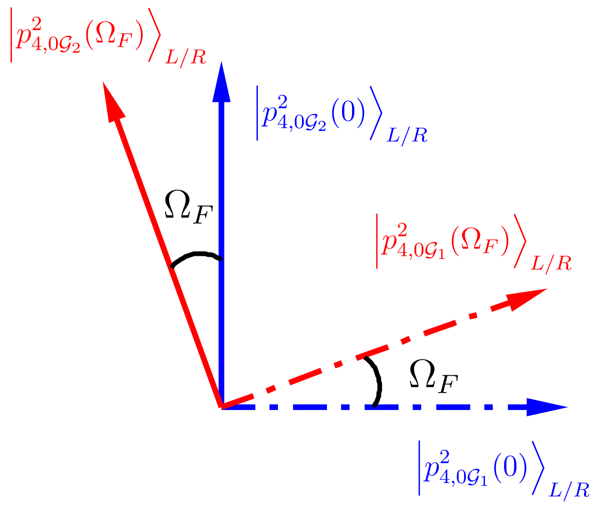


Figure 6.2 – Basis rotation (6.48) in the 0-mode Hilbert space with respect to a rotation angle $\Omega_F$ (6.50).

In particular, according to the localization, we can span the 0-mode Hilbert space by two kinds of basis states,

$$\begin{cases} f^0_{I(II)L/R} \;\hat{=}\; \left\langle y \middle| p^2_{4,0\mathcal{G}_{1(2)}}(0)\right\rangle_{L/R}, \quad \Omega_F = 0\,, \\ f^0_{a(b)L/R} \;\hat{=}\; \left\langle y \middle| p^2_{4,0a(b)}\right\rangle_{L/R}, \quad \Omega_F \neq \dfrac{k\pi}{2}\,(k\in\mathbb{N})\,, \end{cases} \tag{6.51}$$

which refers to existing definitions in Eq. (6.37)-(6.48). $f^0_{\mathcal{G}_i L/R}(y)$ $[i=1,2,\ \mathcal{G}_{1(2)} = I(II)]$ include two generations with the zero mode localized in the two regions respectively with the corresponding 4D fields $F^0_{\mathcal{G}_i L/R}(x^\mu)$ $[i=1,2,\ \mathcal{G}_{1(2)} = I(II)]$. In contrast, $f^0_{\mathcal{G}_i L/R}(y)$ $[i=1,2,\ \mathcal{G}_{1(2)} = a(b)]$ include two generations with the zero mode propagating in the whole region $[0,L_F]\cup(L_F,L]$ with the corresponding 4D fields $F^0_{\mathcal{G}_i L/R}(x^\mu)$ $[i=1,2,\ \mathcal{G}_{1(2)} = a(b)]$. Then, we present explicit non-zero modes in Table 6.1.

Finally, in Figure 6.4, we draw the two localized SM-like zero modes of $F = Q, D$ [i.e. $\Omega_Q = \Omega_D = 0$ (6.51)] in two SM-like BBT configurations (6.7) respectively ($\sigma_1^Q$ and $\sigma_2^D$) presented in in Eq. (6.44)-(6.45) with a shifted intermediate brane ($L_Q \neq L_D$), in the simple real case, $\alpha_F^{10} = \alpha_F^{20} = \delta_F = 0$. In contrast, in Figure 6.5, we draw SM flavors with a unique intermediate brane $L_Q = L_D$ and $\Omega_{Q,D} \neq \dfrac{k\pi}{2}$ $(k\in\mathbb{N})$ (6.51). Therefore, both of these two scenarios will allow to realize a flavor mixing in two different approaches as will be discussed when we will introduce the bulk Yukawa couplings.

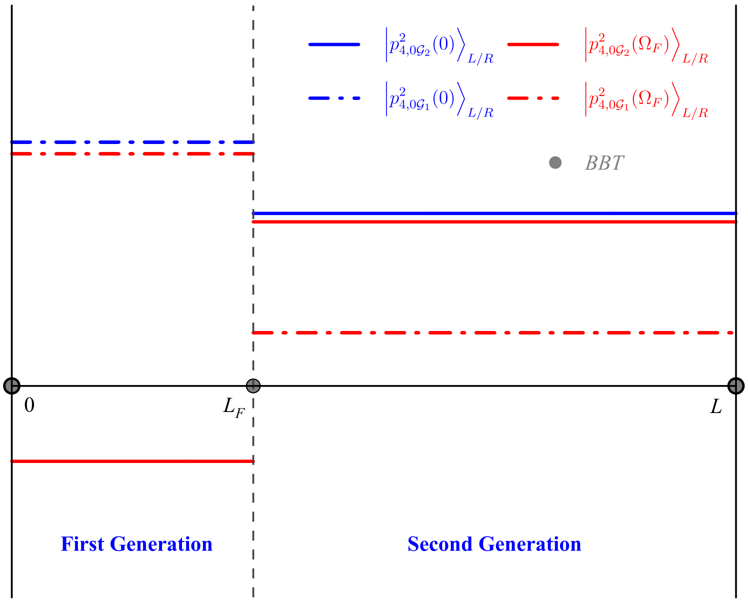


Figure 6.3 – Two pairs of profiles corresponding to $\left\langle y\middle|p^2_{4,0\mathcal{G}_i}(0)\right\rangle_{L/R}$ $(i = 1,2)$ (6.48) and $\left\langle y\middle|p^2_{4,0\mathcal{G}_i}(\Omega_F)\right\rangle_{L/R}$ $(i = 1,2)$ respectively, before and after a rotation of angle $\Omega_F$ in Eq. (6.50), which exactly matches the rotation in the Hilbert space visualized in Figure 6.2 (same solid/dash-dot and color codes are used).

### 6.2.4 Introducing the Fermion Currents

Let us now discuss the probability currents. The global $\mathrm{U}(1)_\mathrm{F}$ symmetry of the field, letting $\mathcal{L}^F_{\text{bulk}}$ invariant, acts as,

$$F \mapsto \mathrm{e}^{i\alpha} F\,, \quad \bar{F} \mapsto \mathrm{e}^{-i\alpha} \bar{F}\,, \tag{6.52}$$

where $\alpha \in \mathbb{R}$ is a continuous constant phase. Based on the global gauge symmetry and the bulk EOM (6.11), the Noether's theorem predicts the local conservation relation holding over the the entire domain $\mathcal{I}_1$,

$$\partial_M j^M_F = 0\,, \tag{6.53}$$

for the conserved probability current,

$$j^M_F = -\alpha \bar{F}\,\Gamma^M F\,, \tag{6.54}$$

as derived in details within the Appendix F. By the way, the addition of the $S_B$ (6.5) part, which is a $\mathrm{U}(1)_\mathrm{F}$ invariant form, to $S^F_{\text{bulk}}$ is not affecting the current equations (6.53)-(6.54). Nevertheless, we can check that $j^4_F|_{0,L^1_F,L^+_F,L}$ is well vanishing due to the Dirichlet BC for $F_{L/R}$ (6.13) [24], which forces $j^4_F$ to be continuous on the entire domain $\mathcal{I}_1$ including intermediate branes. Therefore, the BBT play the role of making the probability current $j^4_F$ vanish at the associated intermediate branes.

24. The vanishing can be derived clearly from the chiral form of $j^4_F$ (6.54) in Eq. (3.23).

| $\sigma^F$ | Fields | Generations | |
|---|---|---|---|
| | | $f^n_{\mathcal{G}_1 L/R}(y)\, F^n_{\mathcal{G}_1 L/R}(x^\mu)$ , $n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 L/R}(y)\, F^n_{\mathcal{G}_2 L/R}(x^\mu)$ , $n \in \mathbb{N}^*$ |
| $\sigma^F_1$ | $F_L$ | $f^n_{\mathcal{G}_1 L}(y)/e^{i\alpha_F^{1n}}$ , $n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 L}(y)/e^{i\alpha_F^{2n}}$ , $n \in \mathbb{N}^*$ |
| | | $\theta(y)\,\theta(L_F - y)\,\sqrt{2}\cos\left(m^F_{\mathcal{G}_1 n}\, y\right)$ | $\theta\left(y - L_F^+\right)\theta(L - y)\,\sqrt{2}\cos\left[m^F_{\mathcal{G}_2 n}\,(y - L_F)\right]$ |
| | $F_R$ | $f^n_{\mathcal{G}_1 R}(y)/e^{i\alpha_F^{1n}}$ , $n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 R}(y)/e^{i\alpha_F^{2n}}$ , $n \in \mathbb{N}^*$ |
| | | $-\theta(y)\,\theta(L_F - y)\,\sqrt{2}\sin\left(m^F_{\mathcal{G}_1 n}\, y\right)$ | $-\theta\left(y - L_F^+\right)\theta(L - y)\,\sqrt{2}\sin\left[m^F_{\mathcal{G}_2 n}\,(y - L_F)\right]$ |
| $\sigma^F_2$ | $F_L$ | $f^n_{\mathcal{G}_1 L}(y)/e^{i\alpha_F^{1n}}$ , $n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 L}(y)/e^{i\alpha_F^{2n}}$ , $n \in \mathbb{N}^*$ |
| | | $\theta(y)\,\theta(L_F - y)\,\sqrt{2}\sin\left(m^F_{\mathcal{G}_1 n}\, y\right)$ | $\theta\left(y - L_F^+\right)\theta(L - y)\,\sqrt{2}\sin\left[m^F_{\mathcal{G}_2 n}\,(y - L_F)\right]$ |
| | $F_R$ | $f^n_{\mathcal{G}_1 R}(y)/e^{i\alpha_F^{1n}}$ , $n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 R}(y)/e^{i\alpha_F^{2n}}$ , $n \in \mathbb{N}^*$ |
| | | $\theta(y)\,\theta(L_F - y)\,\sqrt{2}\cos\left(m^F_{\mathcal{G}_1 n}\, y\right)$ | $\theta\left(y - L_F^+\right)\theta(L - y)\,\sqrt{2}\cos\left[m^F_{\mathcal{G}_2 n}\,(y - L_F)\right]$ |
| KK Masses | | $\left\lvert m^F_{\mathcal{G}_1 n}\right\rvert$ , $n \in \mathbb{N}^*$ | $\left\lvert m^F_{\mathcal{G}_2 n}\right\rvert$ , $n \in \mathbb{N}^*$ |
| | | $\dfrac{n\pi}{\Delta L^1_F}$ , $n \in \mathbb{N}^*$ | $\dfrac{n\pi}{\Delta L^2_F}$ , $n \in \mathbb{N}^*$ |

Table 6.1 – Two sets of SM-like BBT configurations (6.7) induce respectively two generations of SM-like free fermionic $f^n_{\mathcal{G}_{1(2)} L/R}(y)$ ($n \in \mathbb{N}^*$) profiles – ortho-normalized (6.38) up to the indicated complex phases – on the entire domain, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$, corresponding to the solution in Eq. (6.30). The associated mass spectrum (6.32) is included as well for completeness. The non-zero mode profiles are given for the two generations localized in the first and the second region respectively.

### 6.2.5 Interpretation of the Partition Mechanism

Theoretically, the partition mechanism must be interpreted along the following words. At low energies, the three flavors of particles of the SM (quarks and leptons) appear to be three distinct particles. It is an illusion in the sense that at high energies – where one can 'feel' and test the extra dimension – the three flavors represent in fact three different position states along extra dimension(s) of a unique higher-dimensional field. This mechanism thus provides an explanation for the identical quantum numbers of the three flavor fields, or in other words, for the existence itself of flavors as coming from a replication of fermions with the same origin.

The different masses of the three flavors of particles allow to distinguish them at low energies and originate in this model from the various flavor wave function overlaps with the Higgs boson profile. The SM fermion mass hierarchy is easily implemented from an exponential Higgs profile.

Now, the experimental signature of the present scenario would be to detect three KK fermion towers – with three different KK spectra – associated to the three position states (three flavors). There is also a possible kind of measurement distinguishing this scenario with a model explaining the fermion mass hierarchies but not the three flavor appearance, like the framework with a higher-dimensional field introduced for each generation together with a bulk mass to control the fermion profile overlaps with a brane-localized Higgs boson (as inspired by the now standard mechanism studied within the context of a warped extra dimension): it is the gap of separation between two consecutive KK fermion masses.

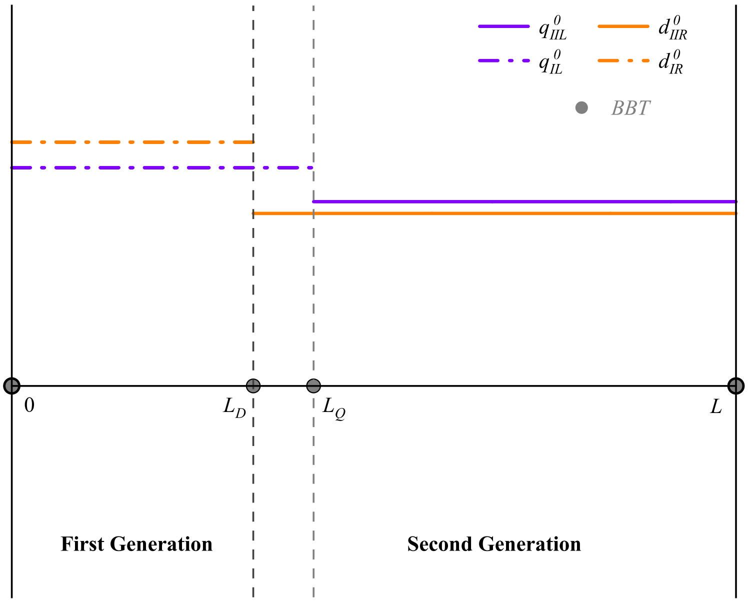


Figure 6.4 – Two ortho-normalized (6.38) generations of SM-like zero-mode wave functions, $f^0_{I(II)L/R}$ (6.51) ($f = q, d$), obtained by two sets of SM-like BBT configurations $\sigma_1^Q$, $\sigma_2^D$ (6.7) respectively along the entire domain, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$ ($F = Q,\, D$), corresponding to localized free solutions of Eq. (6.44)-(6.45) ($\Omega_{Q,D} = 0$) with a shifted intermediate BBT brane $L_D < L_Q$, in the simplified case, $\alpha_F^{10} = \alpha_F^{20} = \delta_F = 0$.

Indeed, this gap is constant at any $n^{\text{th}}$ ($n \in \mathbb{N}$) KK level:

$$\Delta \left|m_n^F\right| = \left|m_{n+1}^F\right| - \left|m_n^F\right| = \frac{\pi}{\Delta L_F}\,, \tag{6.55}$$

in our model [see the spectra (6.32)-(6.33) of both types of chirality], while it is not regular when affected by a bulk fermion mass [see the spectrum in Eq. (6.64)]. In our model, the three towers of KK fermion excitations, at different locations along the extra dimension, also possess exponentially different 4D effective couplings with the Higgs boson being located towards a boundary (or with the KK gauge bosons close to the TeV-brane in the mentioned warped model), which might also help to discriminate with other higher-dimensional mass models.

### 6.2.6 Introduce Bulk Masses

After the bulk massless fields investigation in the last section, we can add another bulk mass terms to the bulk kinetic terms (6.3),

$$\mathcal{L}^F_{\text{bulk}} = \mathcal{L}^F_{\text{kin}} + \mathcal{L}^F_{\text{mass}}\,, \quad \text{with} \quad \mathcal{L}^F_{\text{mass}} = -\widetilde{m}_F \bar{F} F\,, \tag{6.56}$$

where $\widetilde{m}_F$ is the bulk mass of the fermion $F$, which is a constant on each segment as

$$\widetilde{m}_F(y) = \begin{cases} \widetilde{m}_F^1 & , \quad y \in [0, L_F]\,, \\ \widetilde{m}_F^2 & , \quad y \in (L_F, L]\,. \end{cases} \tag{6.57}$$

such that $\partial_4 \widetilde{m}_F = 0$ on the whole physical domain, $y \in \mathcal{I}_1 = [0, L_F] \cup (L_F, L]$. The mass terms $\mathcal{L}^F_{\text{mass}}$ (6.56), can also be rewritten via the chiral decomposition (1.17) as,

$$\mathcal{L}^F_{\text{mass}} = -\widetilde{m}_F \left(F_L^\dagger F_R + F_R^\dagger F_L\right) = -\widetilde{m}_F \left(\bar{\mathcal{F}}_L \mathcal{F}_R + \bar{\mathcal{F}}_R \mathcal{F}_L\right)\,. \tag{6.58}$$

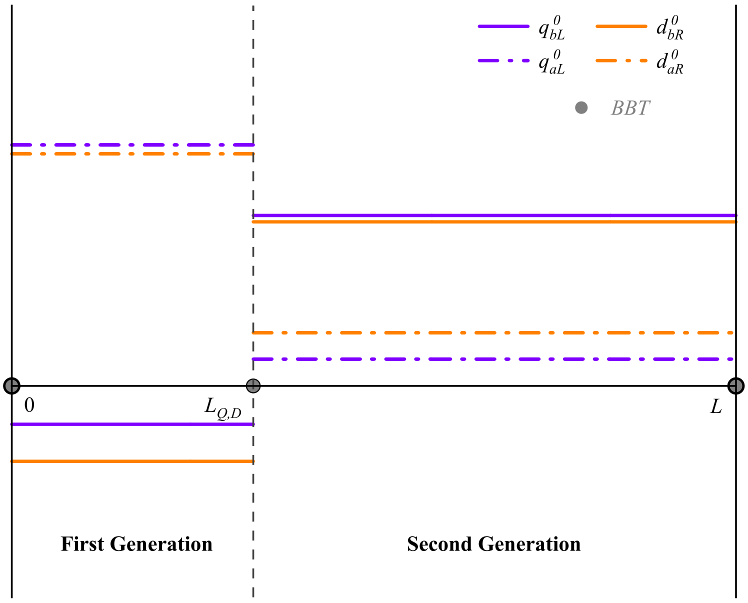


Figure 6.5 – Two ortho-normalized (6.38) generations of SM-like zero-mode wave functions, $f^0_{a(b)L/R}$ (6.51) ($f = q, d$), obtained by two sets of SM-like BBT configurations $\sigma^Q_1$, $\sigma^D_2$ (6.7) respectively along the entire domain, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$ ($F = Q$, $D$), corresponding to non-localized free solutions of Eq. (6.44)-(6.45) $[\Omega_{Q,D} \neq \dfrac{k\pi}{2}$ ($k \in \mathbb{N}$)] with a unique intermediate BBT brane $L_Q = L_D$, in the simplified case, $\alpha^{10}_F = \alpha^{20}_F = \delta_F = 0$.

The least action principle is applied to

$$S^F_{\text{bulk}} + S^F_B \,,$$

where $S^F_{\text{bulk}}$ is constituted of $\mathcal{L}^F_{\text{bulk}}$ (6.56) via Eq. (6.2), and - after the similar process in Eq. (6.9) - leads to the bulk EOM with the piece-wise constant bulk mass $\widetilde{m}_F$ (6.57),

$$\forall\, x^\mu,\, y \in \mathcal{I}_1 = [0, L^1_F] \cup (L_F, L]\,, \quad \Big(i\,\Gamma^M \partial_M - \widetilde{m}_F\Big)\, F = 0\,, \tag{6.59}$$

and the NBC remain identical to Eq. (6.13). Then, following the same procedure in the bulk massless case (see Section 6.2.2), inserting the 5D factorization (6.14) and the 4D Dirac equations (1.25) into the 5D EOM (6.59) and NBC (6.13), one directly obtains the EOM for the profiles on $[0, L_F] \cup (L_F, L]$:

$$\forall\, n \in \mathbb{N}, \left\{ \begin{array}{rcl} (\partial_4 + \widetilde{m}^i_F)\, f^n_L(y) - m^F_n\, f^n_R(y) & = & 0\,, \\ (\partial_4 - \widetilde{m}^i_F)\, f^n_R(y) + m^F_n\, f^n_L(y) & = & 0\,, \end{array} \right. \tag{6.60}$$

which can be combined into the decoupled second order equations,

$$\forall\, n \in \mathbb{N},\ \partial^2_4 f^n_{L/R}(y) + \left[ -\left(\widetilde{m}^i_F\right)^2 + \left(m^F_n\right)^2 \right] f^n_{L/R}(y) = 0\,, \tag{6.61}$$

which are the regular Sturm-Liouville equations on the two segments $[0, L_F] \cup (L_F, L]$ for $i = 1, 2$ independently. Firstly, the continuous SM-like profiles and the mass spectrum on $y \in [0, L_F]$ can be derived via two SM-like BBT configurations (6.7) (cf. Appendix J) as the following two possible sets of chiral profiles together with the associated KK mass spectrum [25],

$$1)\left[\begin{array}{lcl}(++): f_L^0(y) &=& \mathcal{N}_1^{F10} e^{-\widetilde{m}_F^1 y},\ f_L^n(y)=\mathcal{N}_1^{F1n}\cos\left[\sqrt{\left(m_n^F\right)^2-\left(\widetilde{m}_F^1\right)^2}\,y+\zeta_n^{F1}\right],\ n\in\mathbb{N}^*,\\ (--): f_R^0(y) &=& 0,\ f_R^n(y)=-\mathcal{N}_1^{F1n}\sin\left[\sqrt{\left(m_n^F\right)^2-\left(\widetilde{m}_F^1\right)^2}\,y\right],\ n\in\mathbb{N}^*,\end{array}\right. \tag{6.62}$$

$$2)\left[\begin{array}{lcl}(--): f_L^0(y) &=& 0,\ f_L^n(y)=\mathcal{N}_2^{F1n}\sin\left[\sqrt{\left(m_n^F\right)^2-\left(\widetilde{m}_F^1\right)^2}\,y\right],\ n\in\mathbb{N}^*,\\ (++): f_R^0(y) &=& \mathcal{N}_2^{F10} e^{\widetilde{m}_F^1 y},\ f_R^n(y)=\mathcal{N}_2^{F1n}\cos\left[\sqrt{\left(m_n^F\right)^2-\left(\widetilde{m}_F^1\right)^2}\,y-\zeta_n^{F1}\right],\ n\in\mathbb{N}^*,\end{array}\right. \tag{6.63}$$

where the argument $\zeta_n^{F1}$ $(n\in\mathbb{N}^*)$ is defined as,

$$\sin\left(\zeta_n^{F1}\right) \hat{=} \frac{\widetilde{m}_F^1}{m_n^F}\,,\quad \cos\left(\zeta_n^{F1}\right) \hat{=} \frac{\sqrt{\left(m_n^F\right)^2-\left(\widetilde{m}_F^1\right)^2}}{m_n^F}\,,$$

and the associated mass spectrum reads,

$$\left|m_n^F\right| = \left\{\begin{array}{ll} 0 & ,\quad n=0\,,\\ \sqrt{\left[\dfrac{n\pi}{\Delta L_F^1}\right]^2+\left(\widetilde{m}_F^1\right)^2} & ,\quad n\in\mathbb{N}^*\,,\end{array}\right. \tag{6.64}$$

which is parameterized by the *bulk mass* and the associated length and recovers the results of profiles and the mass spectrum in Eq. (6.32) in the bulk massless limit ($\widetilde{m}_F\to 0$).

In analogy to the bulk mass case in Section 6.2.2, based on the localization and the quantum superposition principle, the complete 5D field $F_{L/R}\left(x^\mu,y\right)$ should still be written as Eq. (6.35)-(6.36), including two generations $F_{\mathcal{G}_iL/R}\left(x^\mu,y\right)$ $(i=1,2)$. Note that the EOM with bulk masses (6.59) remains the linearity as the bulk massless one in Eq. (6.11), while homogeneous boundary conditions exactly remain identical to Eq. (6.13). Similarly, zero mode profile $f^0_{\mathcal{G}_iL/R}\big|_{\Omega_F}(y)$ $(i=1,2)$ is still labeled by a rotation angle $\Omega_F\in[0,2\pi)$, which will be described later in the precise solutions (6.74)-(6.75). In analogy to Eq. (6.64), the KK mass spectra for two SM-like generations are explicitly determined by,

$$\left|m^F_{\mathcal{G}_{1(2)}n}\right| = \left\{\begin{array}{ll} 0 & ,\quad n=0\,,\\ \sqrt{\left[\dfrac{n\pi}{\Delta L_F^{1(2)}}\right]^2+\left[\widetilde{m}_F^{1(2)}\right]^2} & ,\quad n\in\mathbb{N}^*\,.\end{array}\right. \tag{6.65}$$

Following the same comments in Section 6.2.2, one can not build more than 2 (families of) orthogonal fields satisfying the ortho-normalization conditions of Eq. (6.38), which can be justified by injecting firstly to the kinetic terms $\mathcal{L}^F_{\text{kin}}$ (6.4). However, the kinetic terms would no longer split as in Eq. (6.39) but with non-diagonal terms,

$$\begin{aligned}&\sum_{2}\int dy\ \mathcal{L}^F_{\text{kin}} \to \sum_{i=1,2}\sum_{2}\int dy\ \mathcal{L}^F_{\mathcal{G}_i\text{kin}} + \text{non-diagonal terms}\,,\\ &\text{with}\quad \mathcal{L}^F_{\mathcal{G}_i\text{kin}} = \frac{1}{2}\left(iF^\dagger_{\mathcal{G}_iR}\sigma^\mu\overleftrightarrow{\partial_\mu}F_{\mathcal{G}_iR}+iF^\dagger_{\mathcal{G}_iL}\bar{\sigma}^\mu\overleftrightarrow{\partial_\mu}F_{\mathcal{G}_iL}-F^\dagger_{\mathcal{G}_iR}\overleftrightarrow{\partial_4}F_{\mathcal{G}_iL}+F^\dagger_{\mathcal{G}_iL}\overleftrightarrow{\partial_4}F_{\mathcal{G}_iR}\right)\,,\end{aligned} \tag{6.66}$$

25. To be clear, normalization factors $\mathcal{N}_a^{Fin}$ and argument $\zeta_n^{Fi}$, $i=1,2$ – region label, $n\in\mathbb{N}$ – KK mode, $a=1,2$ – type of solutions.

while non-diagonal kinetic terms read as,

$$
\begin{aligned}
&\sum_{i\neq j\in\{1,2\}}\sum_{2}\int dy\ \frac{1}{2}\left(iF^{\dagger}_{\mathcal{G}_iR}\sigma^{\mu}\overleftrightarrow{\partial_\mu}F_{\mathcal{G}_jR}+iF^{\dagger}_{\mathcal{G}_iL}\bar{\sigma}^{\mu}\overleftrightarrow{\partial_\mu}F_{\mathcal{G}_jL}-F^{\dagger}_{\mathcal{G}_iR}\overleftrightarrow{\partial_4}F_{\mathcal{G}_jL}+F^{\dagger}_{\mathcal{G}_jL}\overleftrightarrow{\partial_4}F_{\mathcal{G}_iR}\right)\\
&=\sum_{i\neq j\in\{1,2\}}\left[\frac{i}{2L}\sum_{n,m=0}^{+\infty}\left(\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iR}\,f^{m}_{\mathcal{G}_jR}\right)F^{n\dagger}_{\mathcal{G}_iR}\left(x^{\mu}\right)\sigma^{\mu}\overleftrightarrow{\partial_\mu}F^{m}_{\mathcal{G}_jR}\left(x^{\mu}\right)\right.\\
&\qquad\qquad\left.+\frac{i}{2L}\sum_{n,m=0}^{+\infty}\left(\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jL}\right)F^{n\dagger}_{\mathcal{G}_iL}\left(x^{\mu}\right)\bar{\sigma}^{\mu}\overleftrightarrow{\partial_\mu}F^{m}_{\mathcal{G}_jL}\left(x^{\mu}\right)\right]\\
&+\sum_{i\neq j\in\{1,2\}}\left[\frac{1}{2L}\sum_{n,m=0}^{+\infty}\left(\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\overleftrightarrow{\partial_4}f^{m}_{\mathcal{G}_jR}\right)F^{n\dagger}_{\mathcal{G}_iL}\left(x^{\mu}\right)F^{m}_{\mathcal{G}_jR}\left(x^{\mu}\right)+\text{H.c.}\right]\\
&=\sum_{i\neq j\in\{1,2\}}\left[\sum_{n,m=0}^{+\infty}\left(\frac{1}{2L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\overleftrightarrow{\partial_4}f^{m}_{\mathcal{G}_jR}\right)F^{n\dagger}_{\mathcal{G}_iL}\left(x^{\mu}\right)F^{m}_{\mathcal{G}_jR}\left(x^{\mu}\right)+\text{H.c.}\right]\\
&=\sum_{i\neq j\in\{1,2\}}\left[\sum_{n,m=0}^{+\infty}\left(\frac{1}{L}\sum_{2}\int dy\ \widetilde{m}_F\,f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jR}\right)F^{n\dagger}_{\mathcal{G}_iL}\left(x^{\mu}\right)\ F^{m}_{\mathcal{G}_jR}\left(x^{\mu}\right)+\text{H.c.}\right]\\
&=\sum_{i\neq j\in\{1,2\}}\sum_{2}\int dy\ \widetilde{m}_F\left(F^{\dagger}_{\mathcal{G}_iL}F_{\mathcal{G}_jR}+\text{H.c.}\right)\,,
\end{aligned}\tag{6.67}
$$

where the orthogonal relation (6.38) is inserted and

$$
\begin{aligned}
&\frac{1}{2L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\overleftrightarrow{\partial_4}f^{m}_{\mathcal{G}_jR}=\frac{1}{L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\,\partial_4 f^{m}_{\mathcal{G}_jR}-\frac{1}{2L}\sum_{2}\int dy\ \partial_4\left(f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jR}\right)\\
&=\frac{1}{L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\,\partial_4 f^{m}_{\mathcal{G}_jR}-\frac{1}{2L}\left(f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jR}\Big|_{0}^{L_F}+f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jR}\Big|_{L_F^{+}}^{L}\right)\\
&=\frac{1}{L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\,\partial_4 f^{m}_{\mathcal{G}_jR}\\
&=\frac{1}{L}\sum_{2}\int dy\ f^{n*}_{\mathcal{G}_iL}\left(\widetilde{m}_F\,f^{m}_{\mathcal{G}_jR}-m^{F}_{\mathcal{G}_in}\,f^{m}_{\mathcal{G}_jL}\right)\\
&=-m^{F}_{\mathcal{G}_in}\,\delta_{\mathcal{G}_i\mathcal{G}_j}\,\delta_{nm}+\frac{1}{L}\sum_{2}\int dy\ \widetilde{m}_F\,f^{n*}_{\mathcal{G}_iL}\,f^{m}_{\mathcal{G}_jR}\,,
\end{aligned}\tag{6.68}
$$

with the Dirichlet BC (6.13) for $f^{n}_{\mathcal{G}_iL}$ or $f^{m}_{\mathcal{G}_jR}$ inserted at $y=0,L_F,L_F^{+},L$.

The similar generation splitting procedure exists in the mass terms, $\mathcal{L}^{F}_{\text{mass}}$ (6.58),

$$
\begin{aligned}
&\sum_{2}\int dy\ \mathcal{L}^{F}_{\text{mass}}\to\sum_{i=1,2}\sum_{2}\int dy\ \mathcal{L}^{F}_{\mathcal{G}_i\text{mass}}+\text{non-diagonal terms}\,,\\
&\text{with}\ \ \mathcal{L}^{F}_{\mathcal{G}_i\text{mass}}=-\widetilde{m}_F\left(F^{\dagger}_{\mathcal{G}_iL}F_{\mathcal{G}_iR}+F^{\dagger}_{\mathcal{G}_iR}F_{\mathcal{G}_iL}\right)\,,
\end{aligned}\tag{6.69}
$$

with non-diagonal mass terms,

$$
\sum_{i\neq j\in\{1,2\}}\sum_{2}\int dy\ \left(-\widetilde{m}_F\right)\left(F^{\dagger}_{\mathcal{G}_iL}F_{\mathcal{G}_jR}+\text{H.c.}\right)\,.\tag{6.70}
$$

Finally, the bulk terms, $\mathcal{L}_{\text{bulk}}$ (6.3), split into two generations again,

$$
\begin{aligned}
&\sum_{2}\int dy\ \mathcal{L}_{\text{bulk}}\to\sum_{i=1,2}\sum_{2}\int dy\ \mathcal{L}^{F}_{\mathcal{G}_i\text{bulk}}\,,\\
&\text{with}\ \ \mathcal{L}^{F}_{\mathcal{G}_i\text{bulk}}=\mathcal{L}^{F}_{\mathcal{G}_i\text{kin}}+\mathcal{L}^{F}_{\mathcal{G}_i\text{mass}}\,,
\end{aligned}\tag{6.71}
$$

since non-diagonal kinetic (6.67) and mass (6.70) terms cancel each other fortunately. This cancellation is interestingly non-trivial especially for the generation splitting form of $F_{\mathcal{G}_1 L/R}(x^\mu, y)$ and $F_{\mathcal{G}_2 L/R}(x^\mu, y)$ whose $\Omega_F \neq \dfrac{k\pi}{2}$ $(k \in \mathbb{N})$ [i.e. $\mathcal{G}_{1(2)} = a(b)$ (6.51)] in Eq. (6.74)-(6.75) and in turn non-diagonal kinetic (6.67) and mass (6.70) terms do not vanish to zero.

Analogy to the calculation in the bulk massless case (6.43), $\sum\limits_{2} \int dy\ \mathcal{L}^F_{\mathcal{G}_i \text{kin}}$ is modified due to the additional bulk masses in the EOM (6.60) for each generation,

$$
\begin{aligned}
\sum_{2} \int dy\ \mathcal{L}^F_{\mathcal{G}_i \text{kin}} &= \sum_{2} \int dy\ \frac{1}{2} \left( i F^\dagger_{\mathcal{G}_i R} \sigma^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_i R} + i F^\dagger_{\mathcal{G}_i L} \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F_{\mathcal{G}_i L} - F^\dagger_{\mathcal{G}_i R} \overleftrightarrow{\partial_4} F_{\mathcal{G}_i L} + F^\dagger_{\mathcal{G}_i L} \overleftrightarrow{\partial_4} F_{\mathcal{G}_i R} \right) \\
&= \sum_{n=n^F_L}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i L} \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i L} + \sum_{n=n^F_R}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i R} \sigma^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i R} \\
&\qquad + \sum_{2} \int dy\ \frac{1}{2L} \sum_{n,m=0}^{+\infty} \left( f^{n*}_{\mathcal{G}_i L} \overleftrightarrow{\partial_4} f^m_{\mathcal{G}_i R} F^{n\dagger}_{\mathcal{G}_i L} F^m_{\mathcal{G}_i R} + \text{H.c.} \right) \\
&= \sum_{n=n^F_L}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i L} \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i L} + \sum_{n=n^F_R}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i R} \sigma^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i R} \\
&\qquad - \sum_{n=1}^{+\infty} m^F_{\mathcal{G}_i n} \left( F^{n\dagger}_{\mathcal{G}_i L} F^n_{\mathcal{G}_i R} + \text{H.c.} \right) + \sum_{2} \int dy\ \widetilde{m}_F \left( F^\dagger_{\mathcal{G}_i L} F_{\mathcal{G}_i R} + \text{H.c.} \right), \\
&\text{with} \quad n^F_{L\langle R \rangle} = 0\langle 1 \rangle \text{ or } 1\langle 0 \rangle\,, 
\end{aligned}
\tag{6.72}
$$

where the relation (6.68) is inserted. Combining with the mass terms in Eq. (6.69), one can obtain the 4D Lagrangian,

$$
\begin{aligned}
\sum_{2} \int dy\ \mathcal{L}^F_{\mathcal{G}_i \text{bulk}} &= \sum_{2} \int dy\ \left( \mathcal{L}^F_{\mathcal{G}_i \text{kin}} + \mathcal{L}^F_{\mathcal{G}_i \text{mass}} \right) \\
&= \sum_{n=n^F_L}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i L} \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i L} + \sum_{n=n^F_R}^{+\infty} \frac{i}{2} F^{n\dagger}_{\mathcal{G}_i R} \sigma^\mu \overleftrightarrow{\partial_\mu} F^n_{\mathcal{G}_i R} \\
&\qquad - \sum_{n=1}^{+\infty} m^F_{\mathcal{G}_i n} \left( F^{n\dagger}_{\mathcal{G}_i L} F^n_{\mathcal{G}_i R} + F^{n\dagger}_{\mathcal{G}_i R} F^n_{\mathcal{G}_i L} \right),
\end{aligned}
\tag{6.73}
$$

which maintains the same formalism as the bulk massless case (6.42)-(6.43) with the bulk masse $\widetilde{m}_F$ involved in KK masses $m^F_{\mathcal{G}_i n}$.

As the end of this section, we study explicit profile solutions, satisfying the ortho-normalization conditions (6.38). First, zero modes over the whole domain for the SM-like profile $f^0_{\mathcal{G}_i L/R}\Big|_{\Omega_F}(y)$ $(i = 1, 2)$ are taken from Eq. (6.62)-(6.63) via the Heaviside step function (6.17) with an additional ingredient from different bulk masses $\widetilde{m}_F(y)$ (6.57) depending on the region, leading to non-vanishing zero modes in the $\sigma^F_1$ BBT configuration (6.7),

$$\left\{\begin{array}{lcl} f^0_{\mathcal{G}_1 L}\Big|_{\Omega_F}(y) &=& \theta(y)\,\theta(L_F-y)\sqrt{\dfrac{2\,\widetilde{m}^1_F\,L}{1-e^{-2\widetilde{m}^1_F\,\Delta L^1_F}}}\cos\Omega_F\,e^{i\alpha^{10}_F}\,e^{-\widetilde{m}^1_F\,y} \\ && +\theta\left(y-L^+_F\right)\theta(L-y)\sqrt{\dfrac{2\,\widetilde{m}^2_F\,L}{1-e^{-2\widetilde{m}^2_F\,\Delta L^2_F}}}\sin\Omega_F\,e^{i\alpha^{20}_F}\,e^{-\widetilde{m}^2_F\,(y-L_F)}\,, \\ f^0_{\mathcal{G}_2 L}\Big|_{\Omega_F}(y) &=& -\theta(y)\,\theta(L_F-y)\sqrt{\dfrac{2\,\widetilde{m}^1_F\,L}{1-e^{-2\widetilde{m}^1_F\,\Delta L^1_F}}}\sin\Omega_F\,e^{i\left(\alpha^{10}_F+\delta_F\right)}\,e^{-\widetilde{m}^1_F\,y} \\ && +\theta\left(y-L^+_F\right)\theta(L-y)\sqrt{\dfrac{2\,\widetilde{m}^2_F\,L}{1-e^{-2\widetilde{m}^2_F\,\Delta L^2_F}}}\cos\Omega_F\,e^{i\left(\alpha^{20}_F+\delta_F\right)}\,e^{-\widetilde{m}^2_F\,(y-L_F)}\,, \end{array}\right. \tag{6.74}$$

and that in the $\sigma^F_2$ BBT configuration (6.7),

$$\left\{\begin{array}{lcl} f^0_{\mathcal{G}_1 R}\Big|_{\Omega_F}(y) &=& \theta(y)\,\theta(L_F-y)\sqrt{\dfrac{2\,\widetilde{m}^1_F\,L}{e^{2\widetilde{m}^1_F\,\Delta L^1_F}-1}}\cos\Omega_F\,e^{i\alpha^{10}_F}\,e^{\widetilde{m}^1_F\,y} \\ && +\theta\left(y-L^+_F\right)\theta(L-y)\sqrt{\dfrac{2\,\widetilde{m}^2_F\,L}{e^{2\widetilde{m}^2_F\,\Delta L^2_F}-1}}\sin\Omega_F\,e^{i\alpha^{20}_F}\,e^{\widetilde{m}^2_F\,(y-L_F)}\,, \\ f^0_{\mathcal{G}_2 R}\Big|_{\Omega_F}(y) &=& -\theta(y)\,\theta(L_F-y)\sqrt{\dfrac{2\,\widetilde{m}^1_F\,L}{e^{2\widetilde{m}^1_F\,\Delta L^1_F}-1}}\sin\Omega_F\,e^{i\left(\alpha^{10}_F+\delta_F\right)}\,e^{\widetilde{m}^1_F\,y} \\ && +\theta\left(y-L^+_F\right)\theta(L-y)\sqrt{\dfrac{2\,\widetilde{m}^2_F\,L}{e^{2\widetilde{m}^2_F\,\Delta L^2_F}-1}}\cos\Omega_F\,e^{i\left(\alpha^{20}_F+\delta_F\right)}\,e^{\widetilde{m}^2_F\,(y-L_F)}\,, \end{array}\right. \tag{6.75}$$

with arbitrary phases $\alpha^{1(2)0}_F$, $\delta_F$ and the relative phase angle $\Omega_F \in [0, 2\pi)$, which can generate a rotation in zero mode Hilbert space as the bulk massless case in Eq. (6.50). Moreover, $f^0_{\mathcal{G}_i L/R}$ $(i = 1, 2)$ recovers the bulk massless results in Eq. (6.44)-(6.45) in the zero bulk mass limit. Zero mode profiles in Eq. (6.44)-(6.45) would be tuned from piecewise constants to exponential curves by $\widetilde{m}_F$ (6.57) to achieve one more step closer to our final goal to realize the flavor mass hierarchy. In Table 6.2, we present the explicit non-zero modes over the whole domain for the SM-like profile $f^n_{\mathcal{G}_i L/R}(y)$ $(n \in \mathbb{N}^*,\ i = 1, 2)$ taken from Eq. (6.62)-(6.63) analogy to Table 6.1, with arguments $\zeta^{Fi}_n$ $(n \in \mathbb{N}^*,\ i = 1, 2)$,

$$\sin\left[\zeta^{F1(2)}_n\right] \mathrel{\hat{=}} \frac{\widetilde{m}^{1(2)}_F}{m^F_{\mathcal{G}_{1(2)}n}}\,, \quad \cos\left[\zeta^{F1(2)}_n\right] \mathrel{\hat{=}} \frac{\sqrt{\left[m^F_{\mathcal{G}_{1(2)}n}\right]^2-\left[\widetilde{m}^{1(2)}_F\right]^2}}{m^F_{\mathcal{G}_{1(2)}n}}\,,$$

where $\mathcal{G}_i$ $(i = 1, 2)$ is the generation label.

Finally, in Figure 6.6, we draw the two SM-like zero modes of $F = Q$, $D$ in two SM-like BBT configurations (6.7) respectively presented in Eq. (6.74)-(6.75) within the simple real case, $\alpha^{10}_F = \alpha^{20}_F = \delta_F = 0$.

## 6.3 Mass Hierarchy and Mixing Model

Once the free flavor model is addressed via the BBT in Section 6.2.2, the fermion generation splitting mechanism is revealed together with the associated profiles and mass

| $\sigma^F$ | Fields | Generations | |
|---|---|---|---|
| | | $f^n_{\mathcal{G}_1 L/R}(y)\, F^n_{\mathcal{G}_1 L/R}(x^\mu)\,, \quad n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 L/R}(y)\, F^n_{\mathcal{G}_2 L/R}(x^\mu)\,, \quad n \in \mathbb{N}^*$ |
| $\sigma_1^F$ | $F_L$ | $f^n_{\mathcal{G}_1 R}(y)/e^{i\alpha_F^{1n}}\,, \quad n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 R}(y)/e^{i\alpha_F^{2n}}\,, \quad n \in \mathbb{N}^*$ |
| | | $\theta\,(y)\,\theta\,(L_F^1 - y)\,\sqrt{2}\cos\left[\sqrt{\left(m^F_{\mathcal{G}_1 n}\right)^2 - (\widetilde{m}_F^1)^2}\, y + \zeta_n^{F1}\right]$ | $\theta\left(y - L_F^+\right)\theta\,(L - y)\,\sqrt{2}\cos\left[\sqrt{\left(m^F_{\mathcal{G}_2 n}\right)^2 - (\widetilde{m}_F^2)^2}\,(y - L_F) + \zeta_n^{F2}\right]$ |
| | $F_R$ | $f^n_{\mathcal{G}_1 R}(y)/e^{i\alpha_F^{1n}}\,, \quad n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 R}(y)/e^{i\alpha_F^{2n}}\,, \quad n \in \mathbb{N}^*$ |
| | | $-\theta\,(y)\,\theta\,(L_F^1 - y)\,\sqrt{2}\sin\left[\sqrt{\left(m^F_{\mathcal{G}_1 n}\right)^2 - (\widetilde{m}_F^1)^2}\, y\right]$ | $-\theta\left(y - L_F^+\right)\theta\,(L - y)\,\sqrt{2}\sin\left[\sqrt{\left(m^F_{\mathcal{G}_2 n}\right)^2 - (\widetilde{m}_F^2)^2}\,(y - L_F)\right]$ |
| $\sigma_2^F$ | $F_L$ | $f^n_{\mathcal{G}_1 L}(y)/e^{i\alpha_F^{1n}}\,, \quad n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 L}(y)/e^{i\alpha_F^{2n}}\,, \quad n \in \mathbb{N}^*$ |
| | | $\theta\,(y)\,\theta\,(L_F^1 - y)\,\sqrt{2}\sin\left[\sqrt{\left(m^F_{\mathcal{G}_1 n}\right)^2 - (\widetilde{m}_F^1)^2}\, y\right]$ | $\theta\left(y - L_F^+\right)\theta\,(L - y)\,\sqrt{2}\sin\left[\sqrt{\left(m^F_{\mathcal{G}_2 n}\right)^2 - (\widetilde{m}_F^2)^2}\,(y - L_F)\right]$ |
| | $F_R$ | $f^n_{\mathcal{G}_1 R}(y)/e^{i\alpha_F^{1n}}\,, \quad n \in \mathbb{N}^*$ | $f^n_{\mathcal{G}_2 R}(y)/e^{i\alpha_F^{2n}}\,, \quad n \in \mathbb{N}^*$ |
| | | $\theta\,(y)\,\theta\,(L_F^1 - y)\,\sqrt{2}\cos\left[\sqrt{\left(m^F_{\mathcal{G}_1 n}\right)^2 - (\widetilde{m}_F^1)^2}\, y - \zeta_n^{F1}\right]$ | $\theta\left(y - L_F^+\right)\theta\,(L - y)\,\sqrt{2}\cos\left[\sqrt{\left(m^F_{\mathcal{G}_2 n}\right)^2 - (\widetilde{m}_F^2)^2}\,(y - L_F) - \zeta_n^{F2}\right]$ |
| KK Masses | | $\left\|m^F_{\mathcal{G}_1 n}\right\|\,, \quad n \in \mathbb{N}^*$ | $\left\|m^F_{\mathcal{G}_2 n}\right\|\,, \quad n \in \mathbb{N}^*$ |
| | | $\sqrt{\left(\frac{n\pi}{\Delta L_F^1}\right)^2 + (\widetilde{m}_F^1)^2}\,, \quad n \in \mathbb{N}^*$ | $\sqrt{\left(\frac{n\pi}{\Delta L_F^2}\right)^2 + (\widetilde{m}_F^2)^2}\,, \quad n \in \mathbb{N}^*$ |

Table 6.2 – Two sets of SM-like BBT configurations (6.7) induce respectively two generations of SM-like free fermionic $f^n_{\mathcal{G}_i L/R}(y)$ ($n \in \mathbb{N}^*$, $i = 1, 2$) profiles – orthonormalized (6.38) up to the indicated complex phases – on the entire domain, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$, corresponding to the solution in Eq. (6.62)-(6.63). The associated mass spectrum (6.32) is included as well for completeness. The non-zero mode profiles are given for the two generations localized in the first and the second region respectively.

spectra. Then, let us now introduce the bulk Higgs scalar field $H$ (mass dimension 3/2) (cf. Section 1.2) to study the presence of the bulk Yukawa couplings, which would further induce the fermion mass hierarchy.

### 6.3.1 SM-like Fermion Content

In analogy to the interval case in Section 3.2.2, let us introduce the minimal spin-3/2 fermion field content (cf. Section 6.2.1.2) which allows to write down a bulk SM Yukawa-like coupling between zero mode fermions (of different chiralities) and the bulk Higgs scalar – spin-0 – field (cf. Section 1.2). It is constituted by a pair of fermion fields called $Q$ and $D$, potentially splitting into two generations. Those particles propagate along the interval $\mathcal{I}_1$ (see Section 6.2.1.1) but with individual intermediate branes at $y = L_Q, L_D$ respectively, i.e.

$$\mathcal{I}_1 = [0, L_Q] \cup (L_Q, L] = [0, L_D] \cup (L_D, L]\,,$$

as we have in mind an extension of this toy model to a realistic SM scenario with three generations where $Q, D$ will represent respectively the $\mathrm{SU}(2)_\mathrm{L}$ gauge doublet down-component quark and the singlet down-quark. The lengths of the two segments for $Q$ and $D$ respectively are denoted respectively as in Eq. (6.1) with $F = Q, D$.

To be simple, we consider the bulk massless 5D fermion fields with the bulk Lagrangian density,

$$\mathcal{L}^F_{\text{bulk}} = \mathcal{L}^F_{\text{kin}}\,, \quad F = Q, D\,,$$

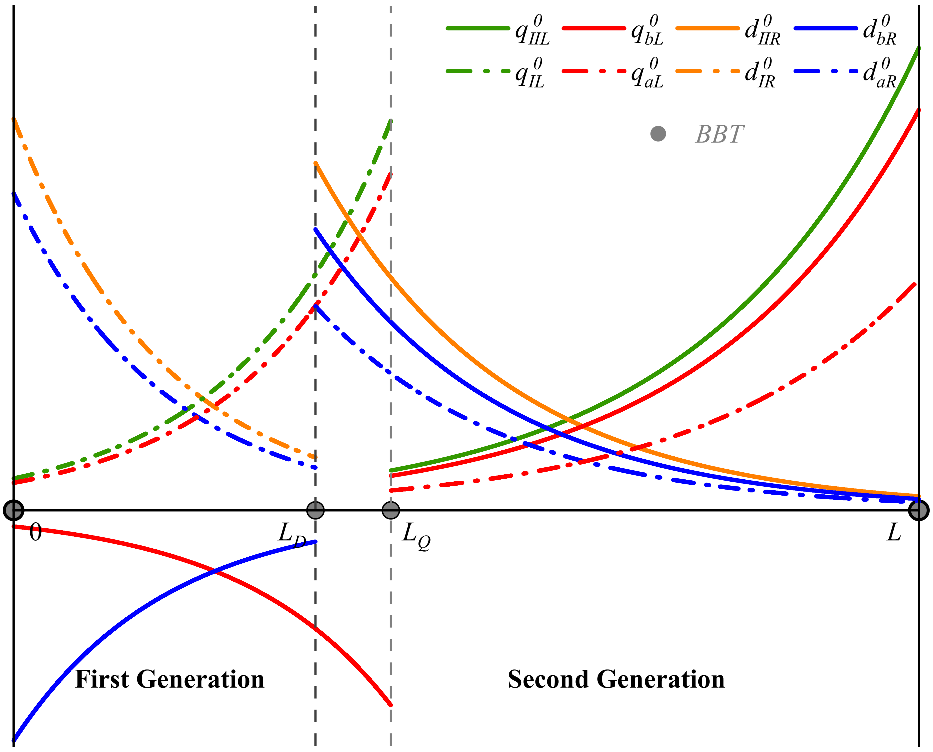


Figure 6.6 – Two ortho-normalized (6.38) generations of SM-like zero-mode wave functions, $f^0_{\mathcal{G}_{1(2)}L/R}(y)$ [$\mathcal{G}_{1(2)} = I(II), a(b)$, $f = q, d$], obtained by two sets of SM-like BBT configurations $\sigma^Q_1$, $\sigma^D_2$ (6.7) respectively along the entire domain, $\mathcal{I}_1 = [0, L_F] \cup (L_F, L]$ ($F = Q, D$), corresponding to the free solutions of Eq. (6.74)-(6.75) in the simplified case, $\alpha^{10}_F = \alpha^{20}_F = \delta_F = 0$, bulk masses $\widetilde{m}^i_F < 0$ ($i = 1, 2$).

which constitute the bulk action via Eq. (6.2) as

$$S_{\text{bulk}} = \sum_{F=Q,D} S^F_{\text{bulk}} \,. \tag{6.76}$$

The individual kinetic Lagrangian density $\mathcal{L}^F_{\text{kin}}$ is presented in Eq. (6.3).

The BBT – inducing the intermediate branes at $y = L_Q, L_D$ (see Figure 6.7) would also be a double replica for $Q$ and $D$ respectively as

$$S_B = \sum_{F=Q,D} S^F_B \,, \tag{6.77}$$

where $S^F_B$ is the individual BBT for $F = Q, D$ and the dimensionless BBT signs are selected as the chiral SM configuration $\sigma^Q_1$ and $\sigma^D_2$ in Eq. (6.7) such that only $Q_L$ and $D_R$ would be non-vanishing for the zero mode (see Section 6.2.2).

### 6.3.2 Yukawa Interactions

We consider the following bulk Yukawa interactions up to dimension 11/2 allowing to study the phenomenology induced by the coupling of the bulk Higgs scalar field (cf.

Section 1.2) and bulk fermions regarding to the distribution of intermediate branes at $y = L_Q, L_D$ in Figure 6.7 ($L_Q \neq L_D$) and Figure 6.8 ($L_Q = L_D$) [26],

$$S_Y = \int d^4x \sum_{D,Q} \int dy \; \mathcal{L}_Y(x^\mu, y)\,,$$
$$\text{with} \;\; \mathcal{L}_Y = -Y_5\, Q_L^\dagger\, H\, D_R - Y_5\, Q_R^\dagger\, H\, D_L + \text{H.c.}\,. \tag{6.78}$$

When calculating the fermion mass spectrum, we restrict our considerations to the VEV function $v(y)$ (1.11) of the bulk Higgs scalar field $H$. Based on the spontaneous $\mathbb{Z}_2$ symmetry breaking in Eq. (1.7), the complete Yukawa sector reads as,

$$S_Y = S_X + S_{hQD}\,.$$

We concentrate our attention on $S_X$ potentially generating mass terms,

$$S_X = \int d^4x \sum_{D,Q} \int dy \; \mathcal{L}_X(x^\mu, y)\,,$$
$$\text{with} \;\; \mathcal{L}_X = -X\, Q_L^\dagger\, D_R - X\, Q_R^\dagger\, D_L + \text{H.c.}\,, \tag{6.79}$$

with the compact Yukawa coupling notations

$$X(y) \mathrel{\hat{=}} \frac{v(y) Y_5}{\sqrt{2}}\,. \tag{6.80}$$

The complete toy model studied for the fermion mass hierarchy is characterized by the action

$$S_{5\text{D}} = S_{\text{bulk}} + S_B + S_X\,, \tag{6.81}$$

where the bulk action terms $S_{\text{bulk}}$ (6.76) consist of the kinetic terms for both of the 5D fields $Q$ and $D$ since we only consider the bulk massless case. Moreover, $S_B$ involves the BBT for $F = Q, D$, allowing to deduce the free profile solutions of $Q$ and $D$ respectively as in Section 6.2. Then, we take into account of the Yukawa action part $S_X$ (6.79) and present a perturbation method to obtain the physical mass including the effects of the Yukawa terms for two generation fermions in Section 6.3.3.

### 6.3.3 Mass Matrix & Hierarchy

In the 5D approach, the EOM is coupled to the bulk Yukawa coupling, which would cause great difficulty for an analytical solution. On the other hand, we have in mind that the interactions of the bulk Higgs scalar field with the bulk fermions can be treated in perturbation theory, as usual in the quantum field theory. Since the zero modes provide the main contribution to the physical flavor basis, we approximately take the zero modes as the SM flavor basis for the $\text{SU}(2)_\text{L}$ gauge doublet down-component quark $Q$ and the singlet down-quark $D$ respectively. Thus, the 4D effective kinetic Lagrangian for generation $\mathcal{G}_i$

26. To be compact,

$$\sum_{D,Q} \int dy \mathrel{\hat{=}} \begin{cases} \left( \int_0^{L_D} + \int_{L_D^+}^{L_Q} + \int_{L_Q^+}^{L} \right) dy\,, & L_Q \neq L_D\,, \\ \left( \int_0^{L_{Q,D}} + \int_{L_{Q,D}^+}^{L} \right) dy & , \;\; L_Q = L_D\,. \end{cases}$$

($i=1,2$) of the fermion field $F=Q,D$ in Eq. (6.43) can be simplified as [via the chiral SM configuration $\sigma_1^Q$ and $\sigma_2^D$ in Eq. (6.7)],

$$\sum_2 \int dy\ \mathcal{L}^F_{\mathcal{G}_i\text{kin}} \sim \begin{cases} \dfrac{i}{2} Q^{0\dagger}_{\mathcal{G}_iL}\bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} Q^0_{\mathcal{G}_iL} & , \quad F=Q\,, \\ \dfrac{i}{2} D^{0\dagger}_{\mathcal{G}_iR}\sigma^\mu \overleftrightarrow{\partial_\mu} D^0_{\mathcal{G}_iR} & , \quad F=D\,, \end{cases} \tag{6.82}$$

so all physical masses in the complete action $S_{\text{5D}}$ (6.81) must come from the Yukawa terms (6.79). We need to emphasize that the $S_B$ (6.77) will not contribute to the mass eigenvalues in the perturbation approach since it leads to the Dirichlet BC for free KK wave functions in one chirality at intermediate branes at $y=0, L_F, L_F^+, L$ ($F=Q,D$) [see Eq. (6.13)], which can be seen clearly in its chiral form of Eq. (6.5) and has also been highlighted in the simple interval scenario (see Section 3.2.3).

The Yukawa mass terms $S_X$ (6.79) – inserting the generation splitting relation (6.35) and the KK decomposition (6.36) – can be rewritten with zero modes as,

$$\begin{aligned} \sum_{D,Q}\int dy\ \mathcal{L}_X &\sim -\sum_{D,Q}\int dy \left[\sum_{i,j\in\{1,2\}} \frac{X(y)}{L}\ q^{0*}_{\mathcal{G}_iL} d^0_{\mathcal{G}_jR} Q^{0\dagger}_{\mathcal{G}_iL} D^0_{\mathcal{G}_jR} + \text{H.c.}\right] \\ &= -\begin{bmatrix} Q^{0\dagger}_{\mathcal{G}_1L} & Q^{0\dagger}_{\mathcal{G}_2L}\end{bmatrix} \mathcal{M} \begin{bmatrix} D^0_{\mathcal{G}_1R} \\ D^0_{\mathcal{G}_2R}\end{bmatrix} + \text{H.c.}\,, \end{aligned} \tag{6.83}$$

where the physical mass matrix $\mathcal{M}$ reads,

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{bmatrix}\,, \tag{6.84}$$

with $\mathcal{M}_{ij}$ ($i,j\in\{1,2\}$) defined as the overlap of $q^0_{\mathcal{G}_iL}$ (6.44), $d^0_{\mathcal{G}_jR}$ (6.45) and the Higgs VEV $v(y)$ under the configuration presented in Figure 6.7 ($L_Q \neq L_D$) and Figure 6.8 ($L_Q = L_D$) schematically,

$$\mathcal{M}_{ij} \hat{=} \sum_{D,Q}\int dy\ \frac{X(y)}{L} q^{0*}_{\mathcal{G}_iL} d^0_{\mathcal{G}_jR}\,, \quad i,j\in\{1,2\}\,. \tag{6.85}$$

with explicit formalism of $\mathcal{M}_{ij}$ (6.84) in Appendix K.

Finally, the mass eigenvalues can be determined by bidiagonalizing the mass matrix, $\mathcal{M}$ (6.84) with the associated characteristic equation,

$$\det\left(\mathcal{M}^\dagger\mathcal{M} - \mathcal{M}^2_{\mathcal{G}_i} I\right) = 0\,.$$

where $\mathcal{M}_{\mathcal{G}_i}$ ($i=1,2$) is the physical mass eigenvalue for the $i^{\text{th}}$ ($i=1,2$) fermion generation with the analytic formula,

$$\begin{aligned} |\mathcal{M}_{\mathcal{G}_1}| &= \sqrt{\frac{|\mathcal{M}|^2 - \sqrt{\left(|\mathcal{M}_{11}|^2+|\mathcal{M}_{21}|^2-|\mathcal{M}_{12}|^2-|\mathcal{M}_{22}|^2\right)^2 + 4\left|\mathcal{M}^*_{11}\mathcal{M}_{12}+\mathcal{M}^*_{21}\mathcal{M}_{22}\right|^2}}{2}}\,, \\ |\mathcal{M}_{\mathcal{G}_2}| &= \sqrt{\frac{|\mathcal{M}|^2 + \sqrt{\left(|\mathcal{M}_{11}|^2+|\mathcal{M}_{21}|^2-|\mathcal{M}_{12}|^2-|\mathcal{M}_{22}|^2\right)^2 + 4\left|\mathcal{M}^*_{11}\mathcal{M}_{12}+\mathcal{M}^*_{21}\mathcal{M}_{22}\right|^2}}{2}}\,. \end{aligned} \tag{6.86}$$

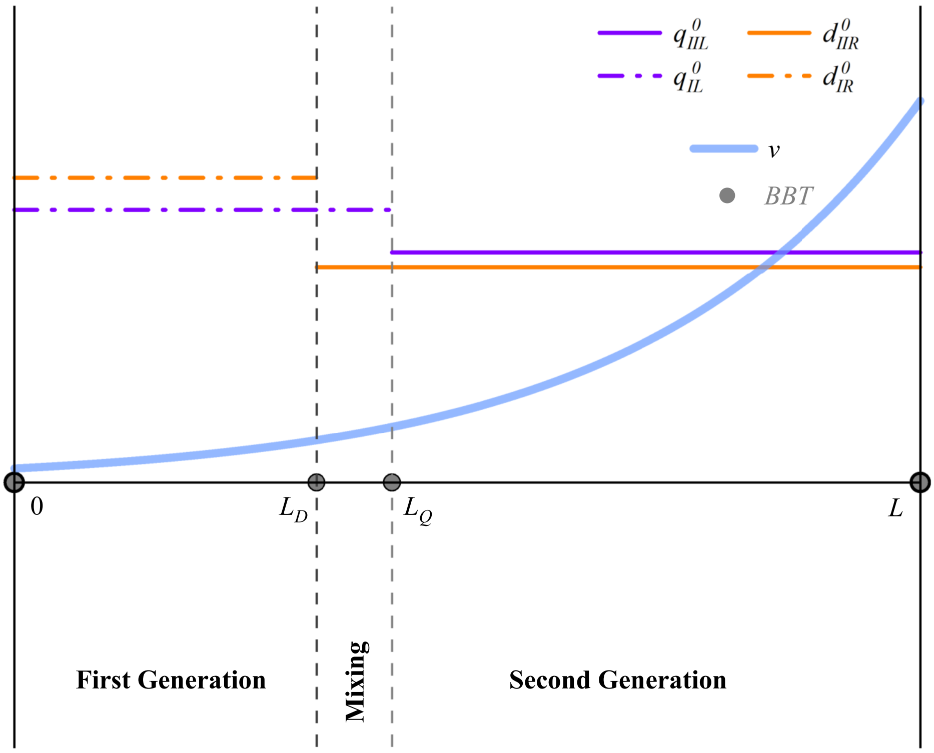


Figure 6.7 – Two generations of otho-normalized zero-mode wave functions, $q^0_{I(II)L}(y)$, $d^0_{I(II)R}(y)$ (6.51), along the entire domain, corresponding to the free solutions in the simplified real case and the BBT as well as the bulk Higgs VEV function, $v(y)$ (1.11), are indicated on the graph. The positions of the intermediate branes are chosen as $L_D < L_Q$.

with

$$|\mathcal{M}|^2 = |\mathcal{M}_{11}|^2 + |\mathcal{M}_{12}|^2 + |\mathcal{M}_{21}|^2 + |\mathcal{M}_{22}|^2 \ .$$

#### 6.3.3.1 Localized SM Flavors

For localized solutions of $q^0_{I(II)L}(y)$, $d^0_{I(II)R}(y)$ (6.51) presented in Figure 6.7 ($L_D < L_Q$), elements of mass matrix $\mathcal{M}$ (6.84) can be derived as (cf. Appendix K),

$$\begin{cases} \mathcal{M}_{11} &= \mathcal{N}_{11}\, \mathrm{e}^{i\left(\alpha_D^{10}-\alpha_Q^{10}\right)}\,, \\ \mathcal{M}_{22} &= \mathrm{e}^{i\left(\delta_D-\delta_Q\right)}\, \mathcal{N}_{22}\, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{20}\right)}\,, \\ \mathcal{M}_{12} &= \mathrm{e}^{i\delta_D}\, \mathcal{N}_{12}\, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{10}\right)}\,, \\ \mathcal{M}_{21} &= 0\,, \end{cases} \tag{6.87}$$

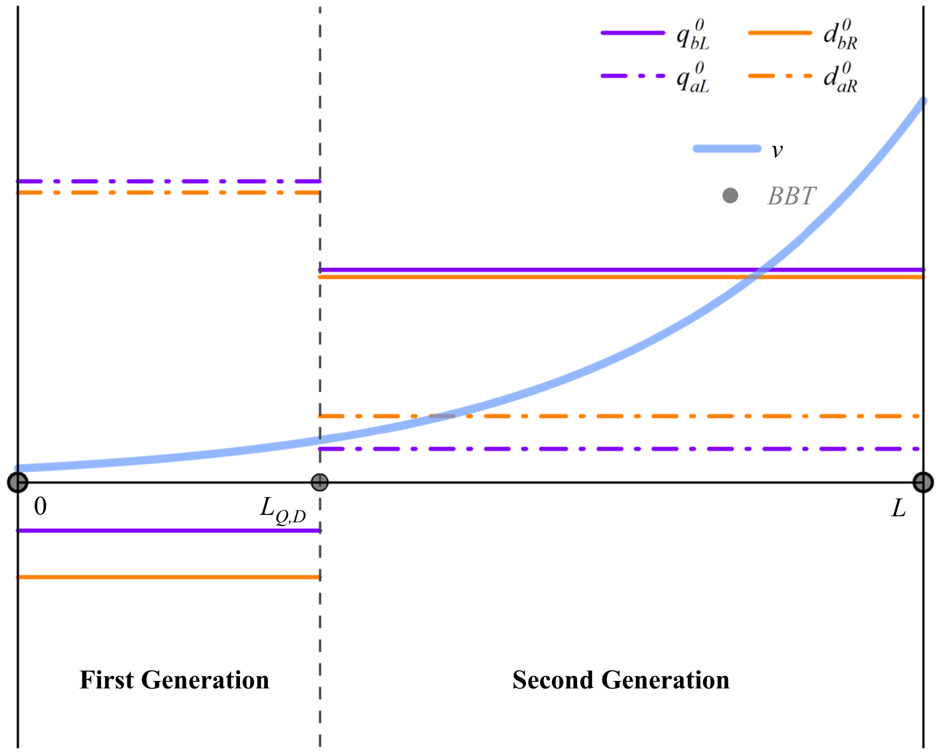


Figure 6.8 – Two generations of otho-normalized zero-mode wave functions, $q^0_{a(b)L}(y)$, $d^0_{a(b)R}(y)$ (6.51), along the entire domain, corresponding to the free solutions in the simplified real case and the BBT as well as the bulk Higgs VEV function, $v(y)$ (1.11), are indicated on the graph. The positions of the intermediate branes are chosen as $L_D = L_Q$.

and the mass eigenvalues can be derived via Eq. (6.86),

$$\left\{\begin{array}{lcl} |\mathcal{M}_{\mathcal{G}_1}| & = & \sqrt{\dfrac{|\mathcal{N}_{11}|^2+|\mathcal{N}_{12}|^2+|\mathcal{N}_{22}|^2-\sqrt{\left(|\mathcal{N}_{11}|^2-|\mathcal{N}_{12}|^2-|\mathcal{N}_{22}|^2\right)^2+4\,|\mathcal{N}_{11}|^2\,|\mathcal{N}_{12}|^2}}{2}}\,, \\ |\mathcal{M}_{\mathcal{G}_2}| & = & \sqrt{\dfrac{|\mathcal{N}_{11}|^2+|\mathcal{N}_{12}|^2+|\mathcal{N}_{22}|^2+\sqrt{\left(|\mathcal{N}_{11}|^2-|\mathcal{N}_{12}|^2-|\mathcal{N}_{22}|^2\right)^2+4\,|\mathcal{N}_{11}|^2\,|\mathcal{N}_{12}|^2}}{2}}\,. \end{array}\right. \tag{6.88}$$

where we have used coefficients $\mathcal{N}_{ij}$ defined in Eq. (K.2). To test the hierarchy in numerical method, we choose parameters in Table 6.3, which produce physical masses of two generations,

$$|\mathcal{M}_{\mathcal{G}_1}| = 98.80\,\text{MeV}\,,\quad |\mathcal{M}_{\mathcal{G}_2}| = 4.19\,\text{GeV}\,, \tag{6.89}$$

which recover the latest experimental values in Ref. [262] precisely, where the masses of strange and bottom quarks are $90-100$ MeV and $4.15-4.21$ GeV respectively. It's possible to make the hierarchy even better by the use of bulk masses and a warp factor in *AdS* geometry where each generation profile would be localized at the corresponding brane. We hope to make this upgrade in the future.

Another crucial phenomenological implement of the partition model is to realize the lepton universality. Three (normalized) fermion profiles, associated to three lepton flavors, are distributed in three sub-regions via the partition mechanism. The SM gauge boson zero modes have flat profiles propagating on the extra dimension, which play a similar role of the Higgs VEV to realize the fermion mass hierarchy in this section. Thus, three lepton flavors would have the identical overlap with the gauge boson, leading to the universality of the effective 4D couplings.

| $\Delta L_D^1$ | $\Delta L_D^2$ | $\Delta L_Q^1$ | $\Delta L_Q^2$ | $M_H$ | $\mathcal{N}_v \frac{Y_5}{\sqrt{2}} \mathrm{e}^{M_H L}$ |
|---|---|---|---|---|---|
| $0.604 \cdot L$ | $0.396 \cdot L$ | $0.705 \cdot L$ | $0.295 \cdot L$ | $8.056 \cdot L^{-1}$ | 12.718 GeV |

Table 6.3 – Numerical parameters to realize SM Fermion mass hierarchy for localized flavors.

#### 6.3.3.2 Non-Localized SM Flavors

For non-localized solutions of $q^0_{a(b)L}(y)$, $d^0_{a(b)R}(y)$ (6.51) presented in Figure 6.8 ($L_D = L_Q$), elements of mass matrix $\mathcal{M}$ (6.84) can be derived in the real case (i.e. $\delta_{D,Q} = \alpha^{10}_{D,Q} = \alpha^{20}_{D,Q} = 0$) as (cf. Appendix K),

$$\begin{cases} \mathcal{M}_{11} &= \mathcal{N}_{11} \cos\Omega_Q \cos\Omega_D + \mathcal{N}_{22} \sin\Omega_Q \sin\Omega_D \,, \\ \mathcal{M}_{22} &= \mathcal{N}_{11} \sin\Omega_Q \sin\Omega_D + \mathcal{N}_{22} \cos\Omega_Q \cos\Omega_D \,, \\ \mathcal{M}_{12} &= -\mathcal{N}_{11} \cos\Omega_Q \sin\Omega_D + \mathcal{N}_{22} \sin\Omega_Q \cos\Omega_D \,, \\ \mathcal{M}_{21} &= -\mathcal{N}_{11} \sin\Omega_Q \cos\Omega_D + \mathcal{N}_{22} \cos\Omega_Q \sin\Omega_D \,, \end{cases} \tag{6.90}$$

and the mass eigenvalues can be derived via Eq. (6.86),

$$\begin{cases} |\mathcal{M}_{\mathcal{G}_1}| &= |\mathcal{N}_{11}| \,, \\ |\mathcal{M}_{\mathcal{G}_2}| &= |\mathcal{N}_{22}| \,, \end{cases} \tag{6.91}$$

where we have used coefficients $\mathcal{N}_{ij}$ defined in Eq. (K.2) ($L_Q = L_D$). To recover the numerical hierarchy in Eq. (6.89), we choose parameters in Table 6.4.

| $\Delta L^1_{Q,D}$ | $\Delta L^2_{Q,D}$ | $M_H$ | $\mathcal{N}_v \frac{Y_5}{\sqrt{2}} \mathrm{e}^{M_H L}$ |
|---|---|---|---|
| $0.5 \cdot L$ | $0.5 \cdot L$ | $7.497 \cdot L^{-1}$ | 16.096 GeV |

Table 6.4 – Numerical parameters to realize SM Fermion mass hierarchy for non-localized flavors.

### 6.3.4 Localization Model

In the previous section, we have built a partition mechanism using the BBT. In this section, we use the same BBT terms to build a universal (for all flavors) localization mechanism, by introducing particularly a 5D field for each fermion flavor together with certain BBT (for each flavor as well). Then, if all the BBT are located at the same point along an extra dimension, it follows that all the fermion flavors can be localized in the same interval (in the case of a model defined by the absence of BBT on the complementary interval).

The space time geometry is similar to the partition scenario and defined as $\mathcal{E}^5 = \mathcal{M}^4 \times \widetilde{\mathcal{I}_1}$. Distinguishing from $\mathcal{I}_1$ in the partition model, the interval $\widetilde{\mathcal{I}_1} = [0, L_F] \cup (L_F, L]$ is constructed by the BBT only existing at $y = 0, L_F$ for the fermion $F$, as illustrated symbolically in Figure 6.9. This toy model is developed to localize a fermion field in a certain region, with the possible limit of the brane-localization of a fermion field. For example, if we take the limit of length (6.1): $\Delta L_F^1 \to 0$, the fermion field will be localized at the brane $y = 0$ and vanish outside.

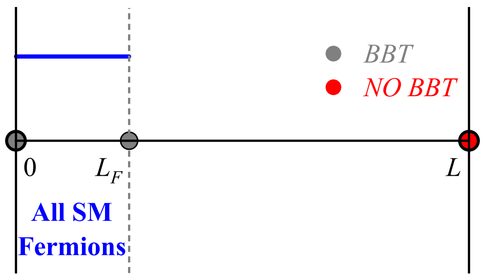


Figure 6.9 – A picture of the interval with one intermediate brane. Two 3-branes (two solid lines) on two boundaries (two black points) at $y = 0, L$. Unique intermediate brane (dashed line) at $y = L_F$ are induced by the BBT (two gray points).

The initial action preserves the identical formula as that (6.2)-(6.8) in the partition model and fermionic particles propagate along $\widetilde{\mathcal{I}_1}$. Meanwhile, the BBT (6.5) will change to new configurations with $\sigma^F_{L_F^+, L} = 0$, leading to a set of natural boundary conditions localized at $y = 0, L_F$ in Eq. (6.12). For example, the two sets of SM configurations in Eq. (6.7) are replaced by

$$\begin{cases} \sigma_1^F : \sigma^F_{0,L_F} & = & -1\,, \\ \sigma_2^F : \sigma^F_{0,L_F} & = & +1\,, \end{cases} \quad \text{and} \quad \sigma^F_{L_F^+, L} = 0\,. \tag{6.92}$$

Notice that the lack of BBT at $y = L_F^+, L$ will lead to vanishing profiles on $(L_F, L]$ via the Dirichlet BC at $y = L_F^+, L$,

$$F_L|_{L_F^+, L} = F_R|_{L_F^+, L} = 0\,, \tag{6.93}$$

as demonstrated in Section 3.3.1. Finally, one can obtain normalizable profiles localized on $[0, L_F]$, clarified as four series in Eq. (6.35) with two sets of KK mass spectra in Eq. (6.32)-(6.33).

Note that such a localization mechanism could be extended to the limit of an infinitesimal size for the sub-interval $[0, L_F]$ ($L_F \to 0$), by taking the BBT position as close as appreciated to the brane at the origin, which could have the effect of localizing all the SM fermions on the 3-brane at $y = 0$. This process could realize for instance the ADD [52–54] configuration, by invoking other mechanisms to brane-localize as well all the bosons of the SM.

## 6.4 Conclusions

Based on an extra spatial dimension scenario, we have proposed a new simple geometrical mechanism, where a higher-dimensional field can be written as a sum of individual flavor fields lying in distinct intervals connected by some intermediate branes, generating the fermion replications needed to realize the three SM families, without introducing new fields. We have presented different versions of this scenario: different realizations of quark/lepton mixing, possible intermediate region with vanishing profile,...

We have shown that distinct flavor fermion wave functions split by BBT points, would possibly have different (non-vanishing) amplitudes on each BBT side, which in turn addresses the fermion mass hierarchy via a 5D Higgs boson profile exponential along the flat extra dimension (obtained from a bulk scalar mass). Indeed, overlaps between different families of profiles at their respective locations and the Higgs boson profile will then differ exponentially, which generate the observed strong fermion mass hierarchy through hierarchical effective 4D Yukawa couplings. This mass hierarchy is realized without hierarchical 5D coupling configuration in the model. A numerical analysis has also been presented to illustrate that the realistic numerical values can be easily reached for the fermion masses.

Hence, we have proposed a simple geometrical scenario addressing two puzzles of the SM flavor sector – the origin of fermion families and the mass scale hierarchies. In addition, a warped version of the present model would bring a common solution to relate the three flavor appearance, the quark/lepton mass hierarchies and the gauge hierarchy, via the curved Higgs profile peaked at the TeV-brane.

Some extensions of the present model are possible. First, some of the SM fermions can be brane-localized in order to realize the models which have interpreted the flavor anomalies in the $B$ meson decays. Another possibility is to localize the Yukawa coupling at the intermediate 3-branes connecting two flavor intervals in order to generate the quark/lepton mixing differently (without relying on the overlap of different flavor profiles). Furthermore, a bulk mass of 5D fermion fields would modify the shape of profiles and in turn bring even more degrees of freedom in the parameter space.

We have also proposed a new type of spin-1/2 fermion localization mechanism along an extra dimension which possesses the following features:

(i) Intermediate partitioning branes are induced by the BBT, and do not rely on some bulk fermion interactions with other fields (like solitons) or a specific gravitational background. Thus, this point-like mechanism is suitable for free fermion fields as well.

(ii) The mechanism can localize a (chiral or vectorial) fermion [any KK mode] to a sub-interval strictly, so that the profile exactly vanishes outside this interval.

(iii) This interval width, determined by the BBT points, can be easily controlled and selected as small as wanted down to the minimum limit of a point (representing then a brane-localization).

# Summary & Outlook

In the modern theory of particle physics, the two hierarchy problems – gauge and flavor hierarchies – motivate model building beyond the SM, introducing new fields to stabilize radiative corrections (superpartners, KK modes, composite states) and generate mass hierarchies (like scalar fields charged under flavor symmetries). The paradigm of higher-dimensional models, that we have studied under some formal aspects, has allowed us to follow an alternative geometrical approach – extendable to a warped version addressing the gauge hierarchy – reducing the fermionic field content of the SM by a factor 3 (corresponding to the flavor number: $12 \mapsto 4$). This reduction can be seen as an 'economy' from the theoretical point of view. Let us summarise the Ph.D. thesis results and their impacts, below, with more precisions.

In the RS1 model, spacetime is a slice of $AdS_5$, and the warp factor redshifts the scale at which gravity becomes strongly coupled, from the 4D Planck scale to the TeV scale, on the IR-brane where the SM Higgs boson is localized, an appealing solution to the gauge hierarchy problem. In the attractive picture with SM fields in the bulk, it is essential to understand deeply the theoretical treatment of 5D fermion fields coupled to a Higgs field localized on a 3-brane, which has been treated in the literature through a physically unnecessary and mathematically incorrect brane regularization procedure. In the first part of this thesis, we have explicitly derived [Chapter 3] the proper analytical treatment for the flat interval, leading to new physical results at the Lagrangian level for the bulk fermions (KK mass spectrum and 4D effective Yukawa couplings). This method relies on additional bilinear fermion terms (BBT) in the action for the 5D fermions – or on certain essential boundary conditions derived from geometrical model definitions through fermion probability currents (if one accepts a UV philosophy where the model is not only defined through the action itself) – allowing to deduce the natural boundary conditions without over-constraining the system. The BBT can be used for both free and brane-coupled bulk fermions, as confirmed by the exact converging results of the 4D and 5D approaches, and show how elaborating a UV origin of the chiral nature of the SM as well as its chirality distribution among quarks/leptons.

The new calculations presented, implying the independence of excited fermion masses and 4D effective Yukawa couplings on the 'wrong-chirality' Yukawa terms, have impacts on phenomenological results like the relaxing of previously obtained strong bounds on Kaluza-Klein masses induced by flavor-changing reactions generated by the tree-level exchanges of the Higgs field.

In Chapter 4, based on the techniques developed for the interval scenario, we have studied the famous $\mathcal{S}^1/\mathbb{Z}_2$ orbifold, a circle equipped with an additional discrete $\mathbb{Z}_2$-parity, and shed a new light on the duality between the $\mathcal{S}^1/\mathbb{Z}_2$ orbifold and the basic interval scenario. The obtained fermion profiles along the extra dimension turn out to undergo some discontinuities, particularly at the Higgs brane, which we have shown to be possibly mathematically consistent, if the 5D action is well written with improper integrals. We have also shown that the $\mathbb{Z}_2$ parity transformations in the bulk do not affect the fermion chiralities, masses and couplings, in contrast with the essential BC or BBT. Besides when the parity is extended to the orbifold fixed points, it represents a UV interpretation for the chiral nature of the low-energy theory and selects the SM chirality setup.

Besides, we have suggested a formalism which is appropriate to field theories in extra dimensions: it is based on an extension of fields as functions to fields as distributions (sometimes called generalised functions). The initial action, written as an applied Lagrangian-distribution, possesses the interest that, after its explicit development, the Lagrangian in terms of fields as functions is recovered including automatically the BBT necessary to define the fermion behavior at the orbifold fixed points (dual to the interval boundaries). More precisely, considering all the possible $\mathbb{Z}_2$ symmetry configurations, the so-called weak derivatives of the fermion field-distributions entering the kinetic terms induce naturally the presence of the needed BBT (except, so far, for vector-like 0-mode solutions associated for instance to custodians in custodially protected warped scenarios). We are still working on the additional possibility to give the test functions, for the field-distributions, the role to implement the definition information about the extra compact space.

As a further application of the BBT prescription, we have built a geometrical spin-1/2 fermion partition mechanism allowing to explain the origin of the three SM fermion families: a higher-dimensional field can be written as a sum of individual flavor fields lying in distinct intervals of a compact space connected by some intermediate BBT branes. This construction is thus realizing the needed SM fermion replication: the reason why the three families would have identical quantum numbers would be that they originate from the same unique higher-dimensional field. This scenario with fermion flavors split along an extra dimension, where the Higgs boson would have an exponential profile, further permits to produce the huge fermion mass hierarchies of the SM. We propose different buildings to create generically the quark/lepton mixings imposed by the experimental results. Moreover, after introducing the SM gauge interactions, our construction allows to realize the higher-dimensional models proposed recently in the literature to explain the deviations from lepton flavour universality – observed through neutral/charged-current semi-leptonic $B$ meson decays.

In addition, the mechanism principle presented right above and based on BBT-like terms can be applied, independently of the flavor puzzles, to *strictly localize* [vanishing wave functions outside] all/some SM fermions on a (thick) 3-brane, or even on well-controlled domains (squares, disks,...) of extra spaces with $N = 2, 3, \ldots$ dimensions. In other words, we propose a new mechanism of fermion localization which represents an alternative to the standard localization procedure based on a generic fermion coupling to a solitonic background (like a domain wall).

As a general opening remark ending up this global conclusion, we state that in this manuscript, we have scrutinized fundamental aspects of bulk fermions undergoing some brane-localized phenomenon (like from brane-localized interactions or BBT). Such an analysis could maybe have implications in other higher-dimensional contexts with similar geometrical field configurations. We have in mind, for instance, the attractive warped version of the Minimal Supersymmetric SM (MSSM) where supersymmetry is broken at an extra dimension boundary (TeV-brane) [263]. Such a scenario can push the supersymmetry breaking scale $\Lambda_{SUSY}$ around the electroweak symmetry breaking scale, $\Lambda_{EW} \simeq 100$ GeV, as required by the gauge hierarchy. There, the gauge boson and graviton superfields propagate in the bulk while the matter and Higgs superfields are assumed to be confined on the Planck-brane, which sufficiently suppresses all higher-dimension operators associated with dangerous proton decay and flavor changing neutral current processes. The hidden sector of supersymmetry breaking would then lie at the TeV-brane where the graviton and its superpartner – the gravitino – would feel different boundary conditions: Neumann and Dirichlet BC respectively (as for the gauge bosons and their associated gauginos), so that the gaugino zero mode would no longer be massless. On the other hand, scalars at the Planck-brane would obtain a soft supersymmetry breaking mass generated at the one-loop level from interactions with the bulk gauge vector multiplets (negligible gravitational processes): this is how the satisfactory breaking scale, $\Lambda_{SUSY} \sim 100$ GeV, would be associated to the Planck-brane superpartners (squarks and sleptons). It would thus maybe be interesting to revisit for example the treatment of the brane-phenomenon giving mass to bulk gauginos, through our approaches.

# Appendices

# Appendix A

# Notations & Conventions

Throughout the manuscript, we use the natural units where we have the reduced Planck constant $\hbar = 1$ and the speed of light $c = 1$ and the conventions of Ref. [112]. The 4D Minkowski metric is,

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,, \tag{A.1}$$

where $\mu, \nu = 0, 1, 2, 3$. The 5D Minkowski metric is,

$$\eta_{MN} = \mathrm{diag}(+1, -1, -1, -1, -1)\,, \tag{A.2}$$

where $M, N = 0, 1...4$.

The 4D Dirac matrices are taken in the Weyl representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{with} \qquad \left\{ \begin{array}{lcl} \sigma^\mu & = & \left(\mathbb{1}_{2\times 2}, \sigma^i\right)\,, \\ \bar{\sigma}^\mu & = & \left(\mathbb{1}_{2\times 2}, -\sigma^i\right)\,, \end{array} \right. \tag{A.3}$$

where $\mu = 0, 1, 2, 3$ and $\sigma^i$ $(i = 1, 2, 3)$ are the three Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.4}$$

One has also the 4D chirality operator,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}\,. \tag{A.5}$$

In our conventions, the 5D Dirac matrices $\Gamma^M$ $(M = 0, 1...4)$ obey $\left\{\Gamma^A, \Gamma^B\right\} = 2\eta^{AB}$ $(A, B = 0, 1...4)$ and read as,

$$\Gamma^M = \left(\gamma^\mu, -i\gamma^5\right)\,. \tag{A.6}$$

# Appendix B

# 5D EOM & BC Splitting for the VEV

In this section, we would present a splitting mechanism for the 5D EOM (1.5) and BC (1.6) due to the existence of the VEV (1.7), which is induced from the spontaneous $\mathbb{Z}_2$ symmetry breaking as described in Section 1.2. Thus, we can obtain two sets of EOM and BC for the VEV $v(y)$ and the 5D scalar field $h(x^\mu, y)$ respectively. Insert the VEV decomposition (1.7) into the EOM (1.5) and the BC (1.6), one would directly obtain,

$$\begin{aligned} \left(\partial_M \partial^M + M_H^2\right) [v(y) + h(x^\mu, y)] = 0 \\ \left(-\partial_4^2 + M_H^2\right) v(y) + \left(\partial_M \partial^M + M_H^2\right) h(x^\mu, y) = 0 \,, \end{aligned} \tag{B.1}$$

which should be valid for any 4D position $x^\mu$, such that

$$\begin{cases} (-\partial_4^2 + M_H^2)\, v(y) &= C \,, \\ \left(\partial_M \partial^M + M_H^2\right) h(x^\mu, y) &= -C \,, \end{cases}$$

where $C$ is a constant to be determined. Considering the 4D asymptotic condition for the scalar field $h(x^\mu, y)$, it should vanish to zero at 4D infinity boundaries due to the localization comments. Hence, we can split the EOM (1.5) for $v(y)$ and $h(x^\mu, y)$ respectively as,

$$\begin{cases} (-\partial_4^2 + M_H^2)\, v(y) &= 0 \,, \\ \left(\partial_M \partial^M + M_H^2\right) h(x^\mu, y) &= 0 \,. \end{cases} \tag{B.2}$$

Then, we can apply similar treatments to BC (1.6). For the BC at $y = 0$,

$$\begin{aligned} (\partial_4 - M_0) \left[v(y) + h(x^\mu, y)\right]\big|_0 = 0 \\ (\partial_4 - M_0)\, v(y)\big|_0 + (\partial_4 - M_0)\, h(x^\mu, y)\big|_0 = 0 \,, \end{aligned} \tag{B.3}$$

which can be split by the 4D $x^\mu$ independence as,

$$\begin{cases} (\partial_4 - M_0)\, v(y)\big|_0 &= C_1 \,, \\ (\partial_4 - M_0)\, h(x^\mu, y)\big|_0 &= -C_1 \,, \end{cases}$$

where $C_1$ is a constant, which is suspended by the 4D localization comments for $h(x^\mu, y)$ again,

$$\begin{cases} (\partial_4 - M_0)\, v(y)\big|_0 &= 0 \,, \\ (\partial_4 - M_0)\, h(x^\mu, y)\big|_0 &= 0 \,. \end{cases} \tag{B.4}$$

Turn to the BC at $y = L$,

$$\begin{aligned}(\partial_4 - M_L)\,[v(y) + h(x^\mu, y)]|_L &= -\frac{\lambda_H/2}{3!}\,[v(y) + h(x^\mu, y)]^3\Big|_L \\ \left[(\partial_4 - M_L)\,v + \frac{\lambda_H/2}{3!}\,v^3\right]\Big|_L &= \left[-(\partial_4 - M_L)\,h - \frac{\lambda_H/2}{3!}\left(3v^2h + 3vh^2 + h^3\right)\right]\Big|_L\,,\end{aligned} \tag{B.5}$$

where the right-handed should be independent of $x^\mu$. Considering $h(x^\mu, y)$ must vanish at $x^\mu \to \infty$, we can finally obtain the BC for $v(y)$ and $h(x^\mu, y)$ respectively at $y = L$,

$$\left\{\begin{aligned}\left[(\partial_4 - M_L)\,v + \frac{\lambda_H/2}{3!}\,v^3\right]\Big|_L &= 0\,, \\ \left[(\partial_4 - M_L)\,h + \frac{\lambda_H/2}{3!}\,(3v^2h + 3vh^2 + h^3)\right]\Big|_L &= 0\,,\end{aligned}\right. \tag{B.6}$$

which can be further simplified if we impose the hypothesis that $v(y) \gg h(x^\mu, y)$,

$$\left\{\begin{aligned}\left[(\partial_4 - M_L)\,v + \frac{\lambda_H}{12}\,v^3\right]\Big|_L &= 0\,, \\ \left[(\partial_4 - M_L)\,h + \frac{\lambda_H}{4}\,v^2h\right]\Big|_L &= 0\,.\end{aligned}\right. \tag{B.7}$$

# Appendix C

# From Spinor Components to Compact Notations

## C.1 Spinor Components and Their Variations

The generic spinor field $F$ ($F = Q, D$) introduced via Eq. (1.17) can be written in terms of its four explicit components $F_\alpha$ [$\alpha = 1, 2, 3, 4$]:

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} , \tag{C.1}$$

and similarly, $\bar{F}$ can be expressed in terms of its own four components $\bar{F}_\alpha$:

$$\bar{F} = \begin{pmatrix} \bar{F}_1, & \bar{F}_2, & \bar{F}_3, & \bar{F}_4 \end{pmatrix} \hat{=} F^\dagger \gamma^0 = \begin{pmatrix} F_3^*, & F_4^*, & F_1^*, & F_2^* \end{pmatrix} . \tag{C.2}$$

These 8 components constitute the fundamental variables of the bulk kinetic Lagrangian (4.8). Hence, the variation of the associated action $S_{\text{bulk}}$ [see Eq. (4.5)], involves the following 8 elementary variations, that we can group into new 4-component (transposed) vectorial objects defined as:

$$\delta F \hat{=} \begin{pmatrix} \delta F_1 \\ \delta F_2 \\ \delta F_3 \\ \delta F_4 \end{pmatrix} , \quad \delta \bar{F} \hat{=} \begin{pmatrix} \delta \bar{F}_1, & \delta \bar{F}_2, & \delta \bar{F}_3, & \delta \bar{F}_4 \end{pmatrix} , \tag{C.3}$$

introducing the 8 components $\delta F_\alpha$ and $\delta \bar{F}_\alpha$. We then define,

$$\delta F = \begin{pmatrix} \delta F_L \\ \delta F_R \end{pmatrix} , \ \delta \bar{F} \hat{=} \begin{pmatrix} \delta F_R^\dagger, & \delta F_L^\dagger \end{pmatrix} , \ \text{with for instance, } \delta \mathrm{F}_{\mathrm{R}}^\dagger \hat{=} \begin{pmatrix} \delta \bar{F}_1, & \delta \bar{F}_2 \end{pmatrix} , \tag{C.4}$$

inspired by the following generic relations, based on Eq. (1.17),

$$F = \begin{pmatrix} F_L \\ F_R \end{pmatrix} , \quad \bar{F} \hat{=} F^\dagger \gamma^0 = (F_R^\dagger, \, F_L^\dagger) \, . \tag{C.5}$$

## C.2 A Typical Compact Form Calculation

Using the Lagrangian $\mathcal{L}_{\rm kin}$ of Eq. (4.8), let us work out explicitly the following quantity entering Eq. (4.22) in a compact form (no explicit spinor index of type $\alpha$),

$$\begin{aligned}
\delta\bar{F}\,\frac{\partial\mathcal{L}_{\rm kin}}{\partial\bar{F}} &\hat{=} \sum_{\alpha=1}^{4}\delta\bar{F}_\alpha\frac{\partial\mathcal{L}_{\rm kin}}{\partial\bar{F}_\alpha} = \sum_{\alpha=1}^{4}\delta\bar{F}_\alpha\frac{\partial}{\partial\bar{F}_\alpha}\left(\frac{i}{2}\,\bar{F}\Gamma^M\partial_M F\right) \\
&= \sum_{\alpha=1}^{4}\delta\bar{F}_\alpha\frac{\partial}{\partial\bar{F}_\alpha}\left(\frac{i}{2}\sum_{\beta=1}^{4}\bar{F}_\beta[\Gamma^M\partial_M F]_\beta\right) = \sum_{\alpha=1}^{4}\delta\bar{F}_\alpha\frac{i}{2}\,[\Gamma^M\partial_M F]_\alpha \\
&= \frac{i}{2}\,\delta\bar{F}\Gamma^M\partial_M F\,, \qquad\qquad (C.6)
\end{aligned}$$

where the spinor components of Eq. (C.1) and (C.2) have appeared, as well as the variations of Eq. (C.3).

## C.3 $\mathbb{Z}_2$ Transformations of Field Variations

Finally, we can derive the $\mathbb{Z}_2$ transformation for the compact form $\delta\bar{F}$ of Eq. (C.3). Accordingly to the $\mathbb{Z}_2$ transformations (4.9)-(4.12), we have,

$$\bar{F}|_{-y} = F^\dagger|_{-y}\gamma^0 = \left(\pm\gamma^5 F\right)^\dagger|_y\gamma^0 = \pm F^\dagger|_y\gamma^5\gamma^0 = \mp F^\dagger|_y\gamma^0\gamma^5 = \mp\bar{F}|_y\gamma^5\,,$$

due to the anti-commutator relation $\{\gamma^5,\gamma^\mu\} = 0$. Then one must rewrite this relation by making the spinor components of Eq. (C.2) appear explicitly:

$$\bar{F}_\alpha|_{-y} = \mp\sum_{\beta=1}^{4}\bar{F}_\beta|_y\gamma^5_{\beta\alpha}\,,$$

in order to deduce the relation on the variations of these components:

$$\delta\bar{F}_\alpha|_{-y} = \mp\sum_{\beta=1}^{4}\delta\bar{F}_\beta|_y\gamma^5_{\beta\alpha}\,.$$

Thanks to Eq. (C.3), this equation can be contracted back to the compact notation as,

$$\delta\bar{F}|_{-y} = \mp\delta\bar{F}|_y\gamma^5\,. \qquad (C.7)$$

# Appendix D

# The Spin Connection on $AdS_5$

Recall the original definition of the spin connection $\omega_M$ in the covariant derivative $D_M$ (2.22) [1],

$$\omega_M = \frac{i}{2}\,\omega_{MAB}\,\mathcal{J}^{AB}\,, \tag{D.1}$$

where

$$\mathcal{J}^{AB} = -\frac{i}{4}\left[\Gamma^A,\,\Gamma^B\right]\,,\quad \omega_{MAB} = \eta_{AC}\,e_N{}^C\left(\partial_M\,e^N{}_B + e^S{}_B\,\Gamma^N_{SM}\right)\,, \tag{D.2}$$

and $\Gamma^N_{SM}$ are Christoffel symbols,

$$\Gamma^K_{MN} = \frac{1}{2}\,g^{KL}\left(\partial_N\,g_{LM} + \partial_M\,g_{LN} - \partial_L\,g_{MN}\right)\,, \tag{D.3}$$

containing non-vanishing components in the $AdS_5$ as,

$$\begin{cases} \Gamma^\nu_{\mu4} &= \quad \Gamma^\nu_{4\mu} = \dfrac{1}{2}\,e^{2ky}\partial_4 e^{-2ky}\delta^\nu_\mu = -k\delta^\nu_\mu\,, \\ \Gamma^4_{\mu\nu} &= \quad \Gamma^4_{\nu\mu} = \dfrac{1}{2}\,\eta_{\mu\nu}\partial_4 e^{-2ky} = -k\eta_{\mu\nu}\,e^{-2ky}\,. \end{cases} \tag{D.4}$$

Insert $\Gamma^N_{SM}$ (D.4) into $\omega_{MAB}$ (D.2), one would obtain non-vanishing components from,

$$\begin{aligned} \omega_{\mu AB} &= \eta_{AC}\,e_N{}^C\,e^S{}_B\,\Gamma^N_{S\mu} \\ &= \eta_{AC}\,e_4{}^C\,e^\nu{}_B\,\Gamma^4_{\nu\mu} + \eta_{AC}\,e_\nu{}^C\,e^4{}_B\,\Gamma^\nu_{4\mu} \\ &= -k\,e^{-ky}\,\eta_{A4}\,\eta_{B\mu} - k\,e^{-ky}\,\eta_{A\mu}\,\delta^4_B\,, \end{aligned}$$

which satisfy an anti-symmetric relation,

$$\omega_{\mu4\nu} = -\omega_{\mu\nu4} = k\,e^{-ky}\,\eta_{\mu\nu}\,. \tag{D.5}$$

Combining with the commutator of $\Gamma^M$ matrices,

$$\left[\Gamma^4,\,\Gamma^\mu\right] = 2\Gamma^4\Gamma^\mu = 2i\gamma^\mu\gamma^5\,,$$

1. One can also use the RS index alternatively,

$$\omega_M = \frac{i}{2}\,\omega_{MPQ}\,\mathcal{J}^{PQ}\,,$$

where $\omega_{MPQ} = e_P{}^A\,e_Q{}^B\,\omega_{MAB}$ and $\mathcal{J}^{PQ} = e^P{}_A\,e^Q{}_B\,\mathcal{J}^{PQ} = -\frac{i}{4}\left[e^P{}_A\,\Gamma^A,\,e^Q{}_B\,\Gamma^B\right]$.

one can finally obtain the spin connection $\omega_M$ (D.1),

$$\begin{aligned}\omega_\mu &= \frac{1}{8}\,\omega_{\mu AB}\left[\Gamma^A,\,\Gamma^B\right]\\ &= \frac{1}{4}\,\omega_{\mu 4\nu}\left[\Gamma^4,\,\Gamma^\nu\right]\\ &= i\,\frac{k}{2}\,e^{-ky}\gamma_\mu\gamma^5\,,\end{aligned} \tag{D.6}$$

while the other components are zero.

# Appendix E

# Profile Solutions of Bulk Fermions on $AdS_5$

## E.1 Bulk Massless Solutions

In this chapter, we would present general solutions of the second order differential equation in Eq. (2.36) with $m_n^F \neq 0$, which are regular Sturm-Liouville equations with a weight function,

$$w(y) = e^{-3y}\,,$$

and it's consistent with the ortho-normalization conditions in Eq. (2.27).

Rewrite the Sturm-Liouville equation of Eq. (2.36) in an explicit form,

$$\left(\partial_4^2 - 5k\partial_4\right) f_{L/R}^n + \left[6k^2 + \left(m_n^F\right)^2 e^{2ky}\right] f_{L/R}^n = 0\,. \tag{E.1}$$

In order to obtain a specific form of particular equations, we do a variable transformation,

$$\xi \mathrel{\hat{=}} \frac{m_n^F}{k}\, e^{ky}\,, \tag{E.2}$$

such that

$$\begin{aligned} \partial_4 &= k\varepsilon\partial_\xi\,, \\ \partial_4^2 &= k^2\left(\xi^2\partial_\xi^2 + \xi\partial_\xi\right)\,. \end{aligned} \tag{E.3}$$

Insert Eq. (E.2)-(E.3) to Eq. (E.1), we would obtain a better formula,

$$\left(\xi^2\partial_\xi^2 - 4\xi\partial_\xi\right) f_{L/R}^n + \left(6 + \xi^2\right) f_{L/R}^n = 0\,. \tag{E.4}$$

In order to realize a Bessel equation, we need to introduce an auxiliary function $g(y)$ and factors $\nu_{L/R}$, such that

$$\left(\xi^2\partial_\xi^2 - 4\xi\partial_\xi\right)\left(g f_{L/R}^n\right) + \left(\xi^2 - \nu_{L/R}^2\right) g f_{L/R}^n = 0\,, \tag{E.5}$$

which is a typical formula of the Bessel equation. The next step is to find out the auxiliary function $g(\xi)$ and factors $\nu_{L/R}$, which must exist if the Bessel equation is truly suitable to our case. Expand Eq. (E.5) and manage it to the following form [1],

$$\xi^2 g\partial_\xi^2 f_{L/R}^n + \left(2\xi^2 g' + \xi g\right)\partial_\xi f_{L/R}^n + \left[\xi^2 g'' + \xi g' + \left(\xi^2 - \nu_{L/R}^2\right) g\right] f_{L/R}^n = 0\,. \tag{E.6}$$

1. $g' \mathrel{\hat{=}} \partial_\xi g$ and $g'' \mathrel{\hat{=}} \partial_\xi^2 g$.

Then, compare Eq. (E.6) with Eq. (E.4) multiplied by the function $g$,

$$\xi^2 g\partial_\xi^2 f^n_{L/R} - 4\xi g\partial_\xi f^n_{L/R} + \left(6+\xi^2\right) g f^n_{L/R} = 0\,, \tag{E.7}$$

and now we can write the characteristic equation of the function $g$ and factors $\nu_{L/R}$,

$$\left\{\begin{array}{rcl} 2\xi^2 g' + \xi g &=& -4\xi g\,, \\ \xi^2 g'' + \xi g' + \left(\xi^2 - \nu^2_{L/R}\right) g &=& \left(6+\xi^2\right) g\,, \end{array}\right.$$

which induces the appreciate formula of the function $g$ and the factor $\nu_{L/R}$,

$$\left\{\begin{array}{rcl} g &=& \xi^{-\frac{5}{2}}\,, \\ \nu^2_{L/R} &=& \dfrac{1}{4}\,. \end{array}\right. \tag{E.8}$$

Since the function $gf^n_{L/R}(y)$ satisfies the Bessel equation (E.5) with $\nu_{L/R} = -/+\frac{1}{2}$, we can now recover the general solutions of Eq. (2.36),

$$f^n_{L/R}(y) = e^{\frac{5}{2}ky}\left[A^n_{L/R}\,J_{-/+\frac{1}{2}}\left(\frac{m^F_n}{k}e^{ky}\right) + B^n_{L/R}\,Y_{-/+\frac{1}{2}}\left(\frac{m^F_n}{k}e^{ky}\right)\right]\,, \tag{E.9}$$

where $J_\nu$ and $Y_\nu$ are the Bessel functions of the first and the second kind respectively and $A^n_{L/R}$, $B^n_{L/R}$ are complex coefficients related by the coupled equations in Eq. (2.33). Here, we choose $\nu_{L/R} = -/+\frac{1}{2}$ corresponding to the left/right chirality respectively as the set of two independent fundamental solution bases. This assignment is configured to obtain a compact relation among $A^n_{L/R}$, $B^n_{L/R}$,

$$\begin{aligned} \partial_4 f^n_L(y) = &\frac{5}{2}kf^n_L(y) \\ &+ e^{\frac{5}{2}ky}k\xi\left\{A^n_L\left[J_{-\frac{3}{2}}(\xi) + \frac{1}{2\xi}J_{-\frac{1}{2}}(\xi)\right] + B^n_L\left[Y_{-\frac{3}{2}}(\xi) + \frac{1}{2\xi}Y_{-\frac{1}{2}}(\xi)\right]\right\}\,, \end{aligned} \tag{E.10}$$

where the Bessel function relations have been injected,

$$Z'_\nu(\xi) = Z_{\nu-1}(\xi) - \frac{\nu}{\xi}Z_\nu(\xi)\,,\; Z_\nu = J_\nu,\, Y_\nu Z_\nu\,, \tag{E.11}$$

and consider another Bessel function relation,

$$Z_{\nu-1}(\xi) + Z_{\nu+1}(\xi) = \frac{2\nu}{\xi}Z_\nu(\xi)\,,\; Z_\nu = J_\nu,\, Y_\nu Z_\nu\,, \tag{E.12}$$

we can do a further step,

$$\partial_4 f^n_L(y) = e^{\frac{5}{2}ky}\left\{A^n_L\left[-k\xi J_{\frac{1}{2}}(\xi) + 2kJ_{-\frac{1}{2}}(\xi)\right] + B^n_L\left[-k\xi Y_{\frac{1}{2}}(\xi) + 2kY_{-\frac{1}{2}}(\xi)\right]\right\}\,, \tag{E.13}$$

so that according to EOM in Eq. (2.36),

$$\left(\partial_4 - 2k\right) f^n_L(y) = -k\xi e^{\frac{5}{2}ky}\left[A^n_L J_{\frac{1}{2}}(\xi) + B^n_L Y_{\frac{1}{2}}(\xi)\right]\,, \tag{E.14}$$

must be equal to

$$-k\xi f^n_R(y) = -k\xi e^{\frac{5}{2}ky}\left[A^n_R J_{\frac{1}{2}}(\xi) + B^n_R Y_{\frac{1}{2}}(\xi)\right]\,, \tag{E.15}$$

which would finally derive the relation among $A^n_{L/R}$, $B^n_{L/R}$,

$$\left\{\begin{array}{rcl} A^n_L &=& A^n_R\,, \\ B^n_L &=& B^n_R\,. \end{array}\right. \tag{E.16}$$

## E.2 Bulk Massive Solutions

In this chapter, we would present general solutions of the second order differential equation in Eq. (2.48) with $m_n^F \neq 0$, which are regular Sturm-Liouville equations with a weight function,

$$w(y) = e^{-3y},$$

and it's consistent with the ortho-normalization conditions in Eq. (2.27).

Rewrite the Sturm-Liouville equation of Eq. (2.48) in an explicit form,

$$\left(\partial_4^2 - 5k\partial_4\right) f_{L/R}^n + \left[\left(6 - / + c - c^2\right) k^2 + \left(m_n^F\right)^2 e^{2ky}\right] f_{L/R}^n = 0\,. \tag{E.17}$$

Then, a variable transformation of $\xi$ in Eq. (E.2) would be repeated, such that the differential equation (E.17) would become a better formula,

$$\left(\xi^2\partial_\xi^2 - 4\xi\partial_\xi\right) f_{L/R}^n + \left[\left(6 - / + c - c^2\right) + \xi^2\right] f_{L/R}^n = 0\,. \tag{E.18}$$

A Bessel equation form in Eq. (E.5) would be expected again, following the treatment in the bulk massless case in Section E.1, we need to introduce an auxiliary function $g(y)$ and factors $\nu_{L/R}$. The next step is to find out the auxiliary function $g(\xi)$ and factors $\nu_{L/R}$, which must exist if the Bessel equation is truly suitable to our case. Then, compare Eq. (E.6) with Eq. (E.18) multiplied by the function $g$,

$$\xi^2 g\partial_\xi^2 f_{L/R}^n - 4\xi g\partial_\xi f_{L/R}^n + \left[\left(6 - / + c - c^2\right) + \xi^2\right] g f_{L/R}^n = 0\,, \tag{E.19}$$

and now we can write the characteristic equation of the function $g$ and factors $\nu_{L/R}$,

$$\begin{cases} 2\xi^2 g' + \xi g &= -4\xi g\,, \\ \xi^2 g'' + \xi g' + \left(\xi^2 - \nu_{L/R}^2\right) g &= \left[(6 - / + c - c^2) + \xi^2\right] g\,, \end{cases}$$

which induces the appreciate formula of the function $g$ and the factor $\nu$,

$$\begin{cases} g &= \xi^{-\frac{5}{2}}\,, \\ \nu_{L/R}^2 &= \left(c - / + \frac{1}{2}\right)^2\,. \end{cases} \tag{E.20}$$

Since the function $g f_{L/R}^n(y)$ satisfies the Bessel equation (E.5) with $\nu_{L/R} = c - / + \frac{1}{2}$, we can now recover the general solutions of Eq. (2.48),

$$f_{L/R}^n(y) = e^{\frac{5}{2}ky}\left[A_{L/R}^n J_{c-/+\frac{1}{2}}\left(\frac{m_n^F}{k}e^{ky}\right) + B_{L/R}^n Y_{c-/+\frac{1}{2}}\left(\frac{m_n^F}{k}e^{ky}\right)\right]\,, \tag{E.21}$$

where $J_\nu$ and $Y_\nu$ are the Bessel functions of the first and the second kind respectively and $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients related by the coupled equations in Eq. (2.45). Here, we choose $\nu_{L/R} = c - / + \frac{1}{2}$ corresponding to the left/right chirality respectively as the set of two independent fundamental solution bases. This assignment is configured to obtain a simple and compact relation among $A_{L/R}^n$, $B_{L/R}^n$,

$$\begin{aligned} \partial_4 f_L^n(y) = &\frac{5}{2}k f_L^n(y) \\ &+ e^{\frac{5}{2}ky}k\xi\left\{A_L^n\left[J_{c-\frac{3}{2}}(\xi) + \frac{1}{2\xi}J_{c-\frac{1}{2}}(\xi)\right] + B_L^n\left[Y_{c-\frac{3}{2}}(\xi) + \frac{1}{2\xi}Y_{c-\frac{1}{2}}(\xi)\right]\right\}\,, \end{aligned} \tag{E.22}$$

where the Bessel function relations in Eq. (E.11) have been injected. We can do a further step with another Bessel function relation in Eq. (E.12),

$$\partial_4 f_L^n(y) = e^{\frac{5}{2}ky}\left\{A_L^n\left[-k\xi J_{c+\frac{1}{2}}(\xi) + 2kJ_{c-\frac{1}{2}}(\xi)\right] + B_L^n\left[-k\xi Y_{c+\frac{1}{2}}(\xi) + 2kY_{c-\frac{1}{2}}(\xi)\right]\right\}, \tag{E.23}$$

so that according to EOM in Eq. (2.48),

$$(\partial_4 - 2k)\, f_L^n(y) = -k\xi e^{\frac{5}{2}ky}\left[A_L^n J_{c+\frac{1}{2}}(\xi) + B_L^n Y_{c+\frac{1}{2}}(\xi)\right], \tag{E.24}$$

must be equal to

$$-k\xi f_R^n(y) = -k\xi e^{\frac{5}{2}ky}\left[A_R^n J_{c+\frac{1}{2}}(\xi) + B_R^n Y_{c+\frac{1}{2}}(\xi)\right], \tag{E.25}$$

which would finally derive the relation among $A_{L/R}^n$, $B_{L/R}^n$,

$$\begin{cases} A_L^n &= A_R^n\,, \\ B_L^n &= B_R^n\,. \end{cases} \tag{E.26}$$

# Appendix F

# Noether's Theorem including Brane-localized Terms

Here we demonstrate the Noether's theorem in the presence of boundary-localized Yukawa couplings and the BBT. We first consider the free bulk action constituted via kinetic terms (3.1) together with the BBT (3.4) [or the custodian BBT (3.7)] being invariant under the transformations (3.21) affecting the fields but not the coordinates $x^M$. The infinitesimal action variation under such a transformation on the field $F$ reads generically as [1],

$$\underline{\delta}(S_{\text{bulk}} + S_B) = \int d^4x \; dy \left\{ \underline{\delta}F \frac{\partial \mathcal{L}_{kin}}{\partial F} + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial \bar{F}} + \underline{\delta}(\partial_M F) \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta}(\partial_M \bar{F}) \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})} \right\}$$
$$+ \int d^4x \; \left\{ - \underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_0 - \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_0 + \underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_L + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_L \right\} . \quad \text{(F.1)}$$

Now we invoke the generic version of the EOM,

$$\frac{\partial \mathcal{L}_{kin}}{\partial F} = \partial_M \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} , \quad \text{(F.2)}$$

as found in Eq. (1.30) [2], which isn't affected by the BBT or the EBC. Using these EOM to rewrite the bulk terms of Eq. (F.1) and then grouping those with the last two terms to make global derivatives appear, we find:

$$\underline{\delta}(S_{\text{bulk}} + S_B) = \int d^4x \; \left\{ - \underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_0 - \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_0 + \underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_L + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_L \right\}$$
$$+ \int d^4x \; dy \left\{ \partial_M \left[ \underline{\delta}F \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})} \right] \right\} , \quad \text{(F.3)}$$

where the four brane BBT terms vanish since the infinitesimal field variations (3.22) lead for instance to,

$$- \underline{\delta}Q \frac{\partial \mathcal{L}_B}{\partial Q}\bigg|_0 - \underline{\delta}\bar{Q} \frac{\partial \mathcal{L}_B}{\partial \bar{Q}}\bigg|_0 = \frac{1}{2}\bar{Q}(i\alpha Q)\bigg|_0 + \frac{1}{2}(-i\alpha\bar{Q})Q\bigg|_0 = 0 . \quad \text{(F.4)}$$

A similar cancellation, due to the symmetry of the model, arises for the last two terms at $y = L$ and the $D$ field contributions [relying on the $\alpha'$ parameter in Eq. (3.21)].

1. Using the compact notations defined in Appendix C.
2. Of course similar EOM hold for the complex conjugate fields.

The infinitesimal variation of the bulk (kinetic) terms (3.1) and the BBT (3.4) vanishes when integrated over the whole space [$\underline{\delta}(S_{\text{bulk}} + S_B) = 0$] and even over any 5D domain $\Omega$, since the Lagrangian density keeps invariant everywhere. Note that the brane terms of Eq. (F.3) vanishes independently for any integration volume $\Omega$ with or without [absence of $S_B$ (3.4)] the boundaries $y = 0, L$.

Therefore, mathematically, Eq. (F.3) implies the vanishing of its bulk terms for any integration region $\Omega$ and in turn the local conservation relation for the 5D probability current of the field $F$,

$$\partial_M j_F^M = 0\,,\ \forall\, x^M, \quad \text{with } j_F^M \;=\; \underline{\delta}F \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})}\,. \tag{F.5}$$

Notice that an alternative reading, based on the global derivatives of the bulk terms in Eq. (F.3), with integration over a generic 5D domain $\Omega$, is that the bulk terms lead to the equality between the net fluxes [3] and the change of charge (time component of the 5D currents). It's nothing but an equivalent form of the local current conservation (F.5). As a consistency check, let's consider the entire 5D domain, i.e. $\Omega$ represents the whole 5D bulk. The integration of fluxes at boundaries along the standard axes [4-coordinates $x^\mu$] $j_F^\mu(\pm\infty, y)$ tend to zero – due to the vanishing of fields at 4D infinity boundaries, which in turn suspend the difference of current components along the extra dimension,

$$\int d^4x \, \left[ j_F^4(x^\mu, L) - j_F^4(x^\mu, 0) \right] = 0\,.$$

It is compatible with the finite geometrical model, $j_F^4(x^\mu, L) = j_F^4(x^\mu, 0) = 0$, $\forall\, x^\mu$. By the way, this global current continuity condition (F.6) could be realized as well in a periodic setup like an orbifold scenario (identification of the boundary points $y = 0, L$ and hence of the currents there), which is precisely described in Chapter 4.

Let us now extend the demonstration of the Noether's theorem to the presence of the BBT and boundary-localized Yukawa couplings by considering the bulk (kinetic) terms (3.1) together with the BBT (3.4) and the Yukawa terms (3.11). This whole action $S_{\text{bulk}} + S_B + S_Y$ is invariant under the transformation (3.77). The infinitesimal action variation under this transformation reads as,

$$\begin{aligned}
\underline{\delta}(S_{\text{bulk}} + S_B + S_Y) = &\sum_{F_{\mathcal{C}} = Q_{L/R}, D_{L/R}} \int d^4x \, \left\{ \underline{\delta}F_{\mathcal{C}} \frac{\partial \mathcal{L}_Y}{\partial F_{\mathcal{C}}}\bigg|_L + \underline{\delta}F_{\mathcal{C}}^\dagger \frac{\partial \mathcal{L}_Y}{\partial F_{\mathcal{C}}^\dagger}\bigg|_L \right\} \\
+ &\sum_{F=Q,D} \int d^4x \, \left\{ -\,\underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_0 - \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_0 + \underline{\delta}F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_L + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_L \right\} \\
+ &\sum_{F=Q,D} \int d^4x \, dy \, \left\{ \underline{\delta}F \frac{\partial \mathcal{L}_{kin}}{\partial F} + \underline{\delta}\bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial \bar{F}} + \underline{\delta}(\partial_M F) \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta}(\partial_M \bar{F}) \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})} \right\}.
\end{aligned} \tag{F.6}$$

Invoking once more the EOM (F.2), including neither the possible BBT contributions nor the Yukawa terms (both rather entering the boundary conditions), we can rewrite the first two terms in the bulk terms of Eq. (F.6) and then grouping those with the last two terms

3. The difference between the ingoing and the out going currents on spacial dimensions.

to make global derivatives arise:

$$\underline{\delta}(S_{\text{bulk}} + S_B + S_Y) = \sum_{F_{\mathcal{C}}=Q_{L/R},D_{L/R}} \int d^4x \, \left\{ \underline{\delta} F_{\mathcal{C}} \frac{\partial \mathcal{L}_Y}{\partial F_{\mathcal{C}}}\bigg|_L + \underline{\delta} F_{\mathcal{C}}^\dagger \frac{\partial \mathcal{L}_Y}{\partial F_{\mathcal{C}}^\dagger}\bigg|_L \right\}$$
$$+ \sum_{F=Q,D} \int d^4x \, \left\{ - \underline{\delta} F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_0 - \underline{\delta} \bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_0 + \underline{\delta} F \frac{\partial \mathcal{L}_B}{\partial F}\bigg|_L + \underline{\delta} \bar{F} \frac{\partial \mathcal{L}_B}{\partial \bar{F}}\bigg|_L \right\}$$
$$+ \sum_{F=Q,D} \int d^4x \, dy \, \left\{ \partial_M \left[ \underline{\delta} F \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta} \bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})} \right] \right\} . \tag{F.7}$$

Here, the four terms of BBT cancel each other since for example the infinitesimal field variations (3.78) lead to,

$$- \underline{\delta} Q \frac{\partial \mathcal{L}_B}{\partial Q}\bigg|_0 - \underline{\delta} \bar{Q} \frac{\partial \mathcal{L}_B}{\partial \bar{Q}}\bigg|_0 = \frac{1}{2} \bar{Q}(i\alpha Q)\bigg|_0 + \frac{1}{2}(-i\alpha \bar{Q})Q\bigg|_0 = 0 \, , \tag{F.8}$$

and the Yukawa terms vanishes as for instance the infinitesimal field variations of type (3.78) lead to,

$$\sum_{F_{\mathcal{C}}=Q_{L/R},D_{L/R}} \left[ - \underline{\delta} F_{\mathcal{C}} \frac{\partial (Y_5 \, Q_L^\dagger H D_R)}{\partial F_{\mathcal{C}}}\bigg|_L - \underline{\delta} F_{\mathcal{C}}^\dagger \frac{\partial (Y_5 \, Q_L^\dagger H D_R)}{\partial F_{\mathcal{C}}^\dagger}\bigg|_L \right]$$
$$= - Y_5 \, Q_L^\dagger H \, (i\alpha D_R)\Big|_L - Y_5 \left( -i\alpha Q_L^\dagger \right) H D_R \Big|_L = 0 \, . \tag{F.9}$$

Therefore, the vanishing infinitesimal variation (F.7) over a generic 5D domain $\Omega$, similar as in the free case, leads to the local conservation relation for the 5D probability current,

$$\partial_M j^M = 0 \, , \; \forall \, x^M, \quad \text{with } j^M = \sum_{F=Q,D} \underline{\delta} F \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M F)} + \underline{\delta} \bar{F} \frac{\partial \mathcal{L}_{kin}}{\partial(\partial_M \bar{F})} \, . \tag{F.10}$$

# Appendix G

# Boundary Variations

In this Appendix, we write down explicitly the global boundary variations derived from the initial variation of the action $S_{5\mathrm{D}}^{\mathrm{m}}$ in Eq. (3.64):

$$\sum_{F=Q,D}\left(\delta_{F_L^\dagger}S|_{\mathrm{brane}}+\delta_{F_R^\dagger}S|_{\mathrm{brane}}+\delta_{F_L}S|_{\mathrm{brane}}+\delta_{F_R}S|_{\mathrm{brane}}\right)=0\,,$$

where

$$\begin{aligned}
\delta_{Q_L^\dagger}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{Q_L^\dagger}S|_{\mathrm{brane}}&\triangleq\int d^4x\left\{\left[\delta Q_L^\dagger\left(-XD_R-\frac{1}{2}Q_R\right)\right]\Big|_L+\left(\delta Q_L^\dagger Q_R\right)\Big|_0\right\},\\
\delta_{Q_R^\dagger}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{Q_R^\dagger}S|_{\mathrm{brane}}&\triangleq\int d^4x\left[\delta Q_R^\dagger\left(-X'D_L+\frac{1}{2}Q_L\right)\right]\Big|_L,\\
\delta_{D_L^\dagger}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{D_L^\dagger}S|_{\mathrm{brane}}&\triangleq\int d^4x\left[\delta D_L^\dagger\left(-X'^*Q_R-\frac{1}{2}D_R\right)\right]\Big|_L,\\
\delta_{D_R^\dagger}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{D_R^\dagger}S|_{\mathrm{brane}}&\triangleq\int d^4x\left\{\left[\delta D_R^\dagger\left(-X^*Q_L+\frac{1}{2}D_L\right)\right]\Big|_L-\left(\delta D_R^\dagger D_L\right)\Big|_0\right\},\\
\delta_{Q_L}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{Q_L}S|_{\mathrm{brane}}&\triangleq\int d^4x\left\{\left[\left(-X^*D_R^\dagger-\frac{1}{2}Q_R^\dagger\right)\delta Q_L\right]\Big|_L+\left(Q_R^\dagger\delta Q_L\right)\Big|_0\right\},\\
\delta_{Q_R}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{Q_R}S|_{\mathrm{brane}}&\triangleq\int d^4x\left[\left(-X'^*D_L^\dagger+\frac{1}{2}Q_L^\dagger\right)\delta Q_R\right]\Big|_L,\\
\delta_{D_L}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{D_L}S|_{\mathrm{brane}}&\triangleq\int d^4x\left[\left(-X'Q_R^\dagger-\frac{1}{2}D_R^\dagger\right)\delta D_L\right]\Big|_L,\\
\delta_{D_R}S_{5\mathrm{D}}^{\mathrm{m}}\ni\delta_{D_R}S|_{\mathrm{brane}}&\triangleq\int d^4x\left\{\left[\left(-XQ_L^\dagger+\frac{1}{2}D_L^\dagger\right)\delta D_R\right]\Big|_L-\left(D_L^\dagger\delta D_R\right)\Big|_0\right\}.
\end{aligned}\tag{G.1}$$

# Appendix H

# Distribution Theory on $\mathcal{S}^1$

## H.1 Schwartz's Distribution Theory on an Interval

In this appendix, we adapt Schwartz's distribution theory [115, 116] to the case of test functions defined on the interval.

### H.1.1 Basics

**Test function on the interval** – The vector space $D(\mathcal{I}, \mathbb{C})$ of test S-functions on the interval $\mathcal{I} = y \in [0, \pi R]$, is the set of all functions $\varphi \in \mathcal{C}^\infty(\mathcal{I}, \mathbb{C})$, i.e.

$$D(\mathcal{I}, \mathbb{C}) \triangleq \mathcal{C}^\infty(\mathcal{I}, \mathbb{C}) \,, \tag{H.1}$$

such that the sequence of functions $\left\{\partial_y^k \varphi \,,\; k \in \mathbb{N}\right\}$ is uniformly bounded on $\mathcal{I}$.

**Convergence in $D(\mathcal{I}, \mathbb{C})$** – A test S-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ $[\forall\, \varphi_n \in D(\mathcal{I}, \mathbb{C})]$ converges to a test S-function $\varphi \in D(\mathcal{I}, \mathbb{C})$, i.e. convergent on $D(\mathcal{I}, \mathbb{C})$, if and only if the sequence $\left\{\partial_y^k \varphi_n\right\}_{n\in\mathbb{N}}$ converges uniformly to $\partial_y^k \varphi$ on $\mathcal{I}$ for each $k \in \mathbb{N}$.

**Continuity of a linear functional on $D(\mathcal{I}, \mathbb{C})$** – A linear functional

$$T \;:\; D(\mathcal{I}, \mathbb{C}) \to \mathbb{C} \,,$$
$$\forall\, \varphi_1, \varphi_2 \in D(\mathcal{I}, \mathbb{C}) \,, \quad T[\lambda_1 \varphi_1 + \lambda_2 \varphi_2] = \lambda_1 T[\varphi_1] + \lambda_2 T[\varphi_2] \,, \quad \lambda_1, \lambda_2 \in \mathbb{C} \,,$$

is continuous if and only if the sequence $\{T[\varphi_n]\}_{n\in\mathbb{N}}$ converges to $T[\varphi] \in \mathbb{C}$, for any test S-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ on $D(\mathcal{I}, \mathbb{C})$.

**S-distribution** – A S-distribution [1] $T$ on $\mathcal{I}$ is a continuous linear functional on $D(\mathcal{I}, \mathbb{C})$,

$$T \;:\; D(\mathcal{I}, \mathbb{C}) \to \mathbb{C} \,.$$

The set containing all the S-distributions,

$$D'(\mathcal{I}, \mathbb{C}) \triangleq \{T \mid T \text{ is an S-distribution on } D(\mathcal{I}, \mathbb{C}).\} \,,$$

forms a vector space over $\mathbb{C}$ by

(i) $(T_1 + T_2)[\varphi] = T_1[\varphi] + T_2[\varphi]\,;$

(ii) $(\lambda T)[\varphi] = \lambda\,(T[\varphi])\,,\; \lambda \in \mathbb{C}\,,$

which are valid for $\forall\, T_1, T_2 \in D'(\mathcal{I}, \mathbb{C})$, $\varphi \in D(\mathcal{I}, \mathbb{C})$.

1. "S" stands for Schwartz.

**Regular S-distribution** – For any integrable function $f \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$ [2], one can define a regular S-distribution $\widetilde{f} \in D'(\mathcal{I}, \mathbb{C})$ $[\mathcal{I} = y \in [0, \pi R]]$ such that

$$\forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ \widetilde{f}[\varphi] \hat{=} \int_0^{\pi R} dy\, f(y)\, \varphi(y)\,, \tag{H.2}$$

which clearly follows the commutative law,

$$\forall\, f, g \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})\,,\ \widetilde{f * g} = \widetilde{g * f}\,. \tag{H.3}$$

Note that an S-distribution which is not regular is singular.

**Product S-distribution** – If $T \in D'(\mathcal{I}, \mathbb{C})$ and $f \in D(\mathcal{I}, \mathbb{C})$, one can define their product as an S-distribution denoted as $f\,T$ via

$$\forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ (f\,T)[\varphi] \hat{=} T[f * \varphi]\,, \tag{H.4}$$

satisfying the bi-linear conditions,

(i) $f\,(\lambda_1\, T_1 + \lambda_2\, T_2) = \lambda_1\,(f\,T_1) + \lambda_2\,(f\,T_2)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$
(ii) $(\lambda_1\, f + \lambda_2\, g)\, T = \lambda_1\,(f\,T) + \lambda_2\,(g\,T)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\, T, T_1, T_2 \in D'(\mathcal{I}, \mathbb{C})$, $f, g \in D(\mathcal{I}, \mathbb{C})$.

It can also be proved to preserve two following calculation laws:

(i) The associative composition law

$$\forall\, f, g \in D(\mathcal{I}, \mathbb{C})\,,\ f(g\,T) = (f * g)T\,, \tag{H.5}$$

can be proved as,

$$\begin{aligned} \forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ f(g\,T)[\varphi] &= (g\,T)[f * \varphi] \\ &= T[g * f * \varphi] \\ &= T[(f * g) * \varphi] \\ &= (f * g)T[\varphi]\,. \end{aligned}$$

(ii) The commutative law

$$\forall\, f, g \in D(\mathcal{I}, \mathbb{C})\,,\ f(g\,T) = g(f\,T)\,, \tag{H.6}$$

can also be guaranteed as

$$f(g\,T) = (f * g)T = (g * f)T = g(f\,T)\,,$$

where we inserted the associative composition law (H.5).

In particular, we can derive some specific results,

(i) $T \in D'(\mathcal{I}, \mathbb{C})$ is a regular S-distribution associated to $\tau \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$, we should have

$$f\,T = \widetilde{f * \tau}\,, \tag{H.7}$$

which should be demonstrated as

$$\begin{aligned} \forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ f\,T[\varphi] &= T[f * \varphi] \\ &= \int_0^{\pi R} dy\, \tau(y)\, f(y)\, \varphi(y) \\ &= \int_0^{\pi R} dy\, [f(y)\, \tau(y)]\, \varphi(y) = \widetilde{f * \tau}\,[\varphi]\,. \end{aligned}$$

2. $D(\mathcal{I}, \mathbb{C})$ is a subset of $\mathcal{C}^0(\mathcal{I}, \mathbb{C})$, i.e. $D(\mathcal{I}, \mathbb{C}) \subset \mathcal{C}^0(\mathcal{I}, \mathbb{C})$.

(ii) $T \in D'(\mathcal{I}, \mathbb{C})$ is a regular S-distribution associated to $\tau \in D(\mathcal{I}, \mathbb{C})$, we should have

$$f\, T = \tau \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.8}$$

where the first equality should be demonstrated as

$$\begin{aligned}\forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ f\, T[\varphi] &= T[f * \varphi] \\ &= \int_0^{\pi R} dy\, \tau(y)\, f(y)\, \varphi(y) \\ &= \int_0^{\pi R} dy\, f(y)\, [\tau(y)\, \varphi(y)] = \widetilde{f}[\tau * \varphi] = (\tau \widetilde{f})\, [\varphi]\,,\end{aligned}$$

and the second one could be obtained from Eq. (H.7).

If $T \in D'(\mathcal{I}, \mathbb{C})$ and $\widetilde{f} \in D'(\mathcal{I}, \mathbb{C})$ is the regular S-distribution associated to $f \in D(\mathcal{I}, \mathbb{C})$, the product S-distribution $\widetilde{f}T$ with respect to $T$ and $\widetilde{f}$ is defined as

$$\widetilde{f}\, T \mathrel{\hat{=}} f\, T\,, \tag{H.9}$$

satisfying the bi-linear conditions,

(i) $\widetilde{f}\, (\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1\, (\widetilde{f}\, T_1) + \lambda_2\, (\widetilde{f}\, T_2)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1 \widetilde{f} + \lambda_2 \widetilde{g})\, T = \lambda_1\, (\widetilde{f}\, T) + \lambda_2\, (\widetilde{g}\, T)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\, T, T_1, T_2 \in D'(\mathcal{I}, \mathbb{C})$, $f, g \in D(\mathcal{I}, \mathbb{C})$ and we invoke the definition of $f\, T \in D'(\mathcal{I}, \mathbb{C})$ in Eq. (H.4). It can also be proved to follow

(i) The associative composition law

$$\forall\, f, g \in D(\mathcal{I}, \mathbb{C})\,,\ \widetilde{f}(\widetilde{g}\, T) = (\widetilde{f}\, \widetilde{g})T\,, \tag{H.10}$$

can be proved as,

$$\widetilde{f}(\widetilde{g}\, T) = f(g\, T) = (f * g)T = \widetilde{f * g}\, T = (f\, \widetilde{g})T = (\widetilde{f}\, \widetilde{g})T\,,$$

where $\widetilde{f}, \widetilde{g} \in D'(\mathcal{I}, \mathbb{C})$ are the regular S-distributions associated to $f, g \in D(\mathcal{I}, \mathbb{C})$ respectively via Eq. (H.2) and we invoke the Eq. (H.8).

(ii) The commutative law

$$\forall\, f, g \in D(\mathcal{I}, \mathbb{C})\,,\ \widetilde{f}(\widetilde{g}\, T) = \widetilde{g}(\widetilde{f}\, T)\,, \tag{H.11}$$

can also be derived as

$$\widetilde{f}(\widetilde{g}\, T) = f(g\, T) = g(f\, T) = \widetilde{g}(\widetilde{f}\, T)\,,$$

where $\widetilde{f}, \widetilde{g} \in D'(\mathcal{I}, \mathbb{C})$ are the regular S-distributions associated to $f, g \in D(\mathcal{I}, \mathbb{C})$ respectively via Eq. (H.2) and we inserted the commutative law (H.6).

In particular, if $T \in D'(\mathcal{I}, \mathbb{C})$ is the regular S-distribution associated to $\tau \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$, we can rewrite Eq. (H.7) via Eq. (H.9) as

$$\widetilde{f}\, T = \widetilde{f * \tau}\,. \tag{H.12}$$

In contrast, when $\tau \in D(\mathcal{I}, \mathbb{C})$, we can rewrite Eq. (H.8) via Eq. (H.9) as

$$\widetilde{f}\, T = T\, \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.13}$$

which indicates the commutative law of the product of two regular S-distributions in Eq. (H.9). So, we can conclude that no matter $T$ is a regular S-distribution or not, $\widetilde{f}\, T$ and $T\, \widetilde{f}$ would not induce an ambiguity so that we would not distinguish them afterwards.

**Dirac S-distribution** – The Dirac S-distribution $\delta_{y_0}$ , $y_0 \in \mathcal{I}$ is a singular S-distribution, which is defined as

$$\forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ \delta_{y_0}[\varphi] \,\hat{=}\, \varphi(y_0)\,. \tag{H.14}$$

### H.1.2 Weak Derivative

In contrast to the derivative of a function, we expect to define a weak derivative for regular S-distributions. Following the methodology of its definition in Schwartz's distribution theory on $\mathbb{R}^1$ (see Ref. [115, 116]), let us consider an integrable function

$$f \in \mathcal{C}^1(\mathcal{I}, \mathbb{C})\,,$$

whose first order derivative $\partial_y f \in \mathcal{C}^0(\mathcal{I}, \mathbb{C})$ is also integrable on $\mathcal{I}$. Thus, one can obtain the regular S-distribution associated to $f$ as $\widetilde{f} \in D'(\mathcal{I}, \mathbb{C})$ and $\{\partial_y f\} \in D'(\mathcal{I}, \mathbb{C})$ [3] is the regular S-distribution induced by $\partial_y f$ via Eq. (H.2). The weak derivative $\partial_y \widetilde{f}$ of $\widetilde{f}$ is defined as

$$\partial_y \widetilde{f} \,\hat{=}\, \{\partial_y f\}\ ,$$

so that we can rewrite it with the Dirac S-distribution $\delta_{y_0}$ (H.14) as

$$\begin{aligned}\forall\, \varphi \in D(\mathcal{I}, \mathbb{C})\,,\ \partial_y \widetilde{f}[\varphi] &= \int_0^{\pi R} dy\, \partial_y f(y)\, \varphi(y) \\ &= -\int_0^{\pi R} dy\, f(y)\, \partial_y \varphi(y) + [f(y)\varphi(y)]|_0^{\pi R} \\ &= -\widetilde{f}[\partial_y \varphi] - \delta_0[f\varphi] + \delta_{\pi R}[f\varphi]\,,\end{aligned} \tag{H.15}$$

where the first equality is just the definition of the regular S-distribution $\{\partial_y f\}$ in Eq. (H.2). The boundary terms in the second equality are generated from the partial integration. Finally, the definition of $\widetilde{f}$ in Eq. (H.2) is invoked to obtain $\widetilde{f}[\partial_y \varphi]$ in the last equality.

## H.2 Schwartz's Distribution Theory on $\mathcal{S}^1$

In this appendix, we adapt Schwartz's distribution theory [115, 116] to the case of test functions defined on the circle $\mathcal{S}^1$ labeled by $y \in [-\pi R^+, 0^-] \cup [0, \pi R \equiv -\pi R]$ [4].

### H.2.1 Basics

**Test function on $\mathcal{S}^1$** – The vector space $D(\mathcal{S}^1, \mathbb{C})$ of test S-functions on $\mathcal{S}^1$ is the set of all smooth functions $\varphi \in \mathcal{C}^\infty(\mathcal{S}^1, \mathbb{C})$, i.e.

$$D(\mathcal{S}^1, \mathbb{C}) \,\hat{=}\, \mathcal{C}^\infty(\mathcal{S}^1, \mathbb{C})\,, \tag{H.16}$$

such that the sequence of functions $\left\{\partial_y^k \varphi\,,\ k \in \mathbb{N}\right\}$ is uniformly bounded on $\mathcal{S}^1$.

**Convergence in $D(\mathcal{S}^1, \mathbb{C})$** – A test S-function sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ $[\forall\, \varphi_n \in D(\mathcal{S}^1, \mathbb{C})]$ converges to a test S-function $\varphi \in D(\mathcal{S}^1, \mathbb{C})$, i.e. convergent on $D(\mathcal{S}^1, \mathbb{C})$, if and only if the sequence $\left\{\partial_y^k \varphi_n\right\}_{n \in \mathbb{N}}$ converges uniformly to $\partial_y^k \varphi$ on $\mathcal{S}^1$ for each $k \in \mathbb{N}$.

3. We want to emphasize the specific regular S-distribution induced from a function derivative $\partial_y f$ so that a new notation $\{\partial_y f\}$ is introduced instead of the usual one $\widetilde{\partial_y f}$ via Eq. (H.2).

4. We use the notation $[-\pi R^+, 0^-] \;\Leftrightarrow\; (-\pi R, 0)$ defined in Section 4.2.1.

**Continuity of a linear functional on** $D(\mathcal{S}^1,\mathbb{C})$ **–** A linear functional

$$T \;:\; D(\mathcal{S}^1,\mathbb{C}) \to \mathbb{C}\,,$$
$$\forall\,\varphi_1,\varphi_2 \in D(\mathcal{S}^1,\mathbb{C})\,,\quad T[\lambda_1\,\varphi_1 + \lambda_2\,\varphi_2] = \lambda_1\,T[\varphi_1] + \lambda_2\,T[\varphi_2]\,,\quad \lambda_1,\lambda_2 \in \mathbb{C}\,,$$

is continuous if and only if the sequence $\{T[\varphi_n]\}_{n\in\mathbb{N}}$ converges to $T[\varphi] \in \mathbb{C}$, for any test S-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ on $D(\mathcal{S}^1,\mathbb{C})$.

**S-distribution –** A S-distribution $T$ on $\mathcal{S}^1$ is a continuous linear functional on $D(\mathcal{S}^1,\mathbb{C})$,

$$T \;:\; D(\mathcal{S}^1,\mathbb{C}) \to \mathbb{C}\,.$$

The set containing all the S-distributions,

$$D'(\mathcal{S}^1,\mathbb{C}) \;\hat{=}\; \Big\{T \mid T \text{ is an S-distribution on } D(\mathcal{S}^1,\mathbb{C}).\Big\}\,,$$

forms a vector space over $\mathbb{C}$ by

(i) $(T_1 + T_2)[\varphi] = T_1[\varphi] + T_2[\varphi]\,;$

(ii) $(\lambda T)[\varphi] = \lambda\,(T[\varphi])\,,\ \lambda \in \mathbb{C}\,,$

which are valid for $\forall\,T_1,T_2 \in D'(\mathcal{S}^1,\mathbb{C})$, $\varphi \in D(\mathcal{S}^1,\mathbb{C})$.

**Regular S-distribution –** For any integrable piece-wise continuous function $f \in \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$ [5], one can define a regular S-distribution $\widetilde{f} \in D'(\mathcal{S}^1,\mathbb{C})$ such that

$$\forall\,\varphi \in D(\mathcal{S}^1,\mathbb{C})\,,\ \widetilde{f}[\varphi] \;\hat{=}\; \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\,\varphi(y)\,, \tag{H.17}$$

which clearly follows the commutative law,

$$\forall\,f,g \in \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})\,,\ \widetilde{f*g} = \widetilde{g*f}\,. \tag{H.18}$$

Note that an S-distribution which is not regular is singular.

**Product S-distribution –** If $T \in D'(\mathcal{S}^1,\mathbb{C})$ and $f \in D(\mathcal{S}^1,\mathbb{C})$, one can define their product as an S-distribution denoted as $f\,T$ via

$$\forall\,\varphi \in D(\mathcal{S}^1,\mathbb{C})\,,\ (f\,T)[\varphi] \;\hat{=}\; T[f*\varphi]\,, \tag{H.19}$$

satisfying the bi-linear conditions,

(i) $f\,(\lambda_1\,T_1 + \lambda_2\,T_2) = \lambda_1\,(f\,T_1) + \lambda_2\,(f\,T_2)\,,\quad \lambda_1,\lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1\,f + \lambda_2\,g)\,T = \lambda_1\,(f\,T) + \lambda_2\,(g\,T)\,,\quad \lambda_1,\lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\,T,T_1,T_2 \in D'(\mathcal{S}^1,\mathbb{C})$, $f,g \in D(\mathcal{S}^1,\mathbb{C})$.

It also follows two other calculation laws:

(i) The associative composition law

$$\forall\,f,g \in D(\mathcal{S}^1,\mathbb{C})\,,\ f(g\,T) = (f*g)T\,, \tag{H.20}$$

can be proved as,

$$\begin{aligned}\forall\,\varphi \in D(\mathcal{S}^1,\mathbb{C})\,,\ f(g\,T)[\varphi] &= (g\,T)[f*\varphi]\\ &= T[g*f*\varphi]\\ &= T[(f*g)*\varphi]\\ &= (f*g)T[\varphi]\,.\end{aligned}$$

5. $D(\mathcal{S}^1,\mathbb{C})$ is a subset of $\mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$, i.e. $D(\mathcal{S}^1,\mathbb{C}) \subset \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$.

(ii) The commutative law

$$\forall\, f, g \in D(\mathcal{S}^1, \mathbb{C})\,,\ \ f(g\,T) = g(f\,T)\,, \tag{H.21}$$

can also be guaranteed as

$$f(g\,T) = (f * g)T = (g * f)T = g(f\,T)\,,$$

where we inserted the associative composition law (H.20).

In particular, we can derive some specific results,

(i) $T \in D'(\mathcal{S}^1, \mathbb{C})$ is a regular S-distribution associated to $\tau \in \mathcal{C}^0([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$, we should have

$$f\,T = \widetilde{f * \tau}\,, \tag{H.22}$$

which should be demonstrated as

$$\begin{aligned}\forall\, \varphi \in D(\mathcal{S}^1, \mathbb{C})\,,\ \ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\, [f(y)\, \tau(y)]\, \varphi(y) = \widetilde{f * \tau}\, [\varphi]\,.\end{aligned}$$

(ii) $T \in D'(\mathcal{S}^1, \mathbb{C})$ is a regular S-distribution associated to $\tau \in D(\mathcal{S}^1, \mathbb{C})$, we should have

$$f\,T = \tau \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.23}$$

where the first equality should be demonstrated as

$$\begin{aligned}\forall\, \varphi \in D(\mathcal{S}^1, \mathbb{C})\,,\ \ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\, [\tau(y)\, \varphi(y)] = \widetilde{f}[\tau * \varphi] = (\tau \widetilde{f})\, [\varphi]\,,\end{aligned}$$

and the second one could be obtained from Eq. (H.22).

If $T \in D'(\mathcal{S}^1, \mathbb{C})$ and $\widetilde{f} \in D'(\mathcal{S}^1, \mathbb{C})$ is the regular S-distribution associated to $f \in D(\mathcal{S}^1, \mathbb{C})$, the product S-distribution $\widetilde{f}\,T$ with respect to $T$ and $\widetilde{f}$ is defined by

$$\widetilde{f}T \mathrel{\hat{=}} f\,T\,, \tag{H.24}$$

satisfying the bi-linear conditions,

(i) $\widetilde{f}\,(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1\,(\widetilde{f}\,T_1) + \lambda_2\,(\widetilde{f}\,T_2)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1 \widetilde{f} + \lambda_2 \widetilde{g})\,T = \lambda_1\,(\widetilde{f}\,T) + \lambda_2\,(\widetilde{g}\,T)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\, T, T_1, T_2 \in D'(\mathcal{S}^1, \mathbb{C})$, $f, g \in D(\mathcal{S}^1, \mathbb{C})$ and we invoke the definition of $f\,T \in D'(\mathcal{S}^1, \mathbb{C})$ in Eq. (H.19). Besides, it respects two following laws as well,

(i) The associative composition law

$$\forall\, f, g \in D(\mathcal{S}^1, \mathbb{C})\,,\ \ \widetilde{f}(\widetilde{g}\,T) = (\widetilde{f}\,\widetilde{g})T\,, \tag{H.25}$$

can be proved as,

$$\widetilde{f}(\widetilde{g}\,T) = f(g\,T) = (f * g)T = \widetilde{f * g}\,T = (f\,\widetilde{g})T = (\widetilde{f}\,\widetilde{g})T\,,$$

where $\widetilde{f}, \widetilde{g} \in D'(\mathcal{S}^1, \mathbb{C})$ are the regular S-distributions associated to $f, g \in D(\mathcal{S}^1, \mathbb{C})$ respectively via Eq. (H.17) and we invoke the Eq. (H.23).

(ii) The commutative law

$$\forall\, f,g \in D(\mathcal{S}^1,\mathbb{C})\,,\ \widetilde{f}(\widetilde{g}\,T) = \widetilde{g}(\widetilde{f}\,T)\,, \tag{H.26}$$

can also be derived as

$$\widetilde{f}(\widetilde{g}\,T) = f(g\,T) = g(f\,T) = \widetilde{g}(\widetilde{f}\,T)\,,$$

where $\widetilde{f},\widetilde{g} \in D'(\mathcal{S}^1,\mathbb{C})$ are the regular S-distributions associated to $f,g \in D(\mathcal{S}^1,\mathbb{C})$ respectively via Eq. (H.17) and we inserted the commutative law (H.21).

In particular, if $T \in D'(\mathcal{S}^1,\mathbb{C})$ is the regular S-distribution associated to $\tau \in \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$, we can rewrite Eq. (H.22) via Eq. (H.24) as

$$\widetilde{f}\,T = \widetilde{f*\tau}\,. \tag{H.27}$$

In contrast, when $\tau \in D(\mathcal{S}^1,\mathbb{C})$, we can rewrite Eq. (H.23) via Eq. (H.24) as

$$\widetilde{f}\,T = T\,\widetilde{f} = \widetilde{f*\tau}\,, \tag{H.28}$$

which indicates the commutative law of the product of two regular S-distributions in Eq. (H.24). So, we can conclude that no matter $T$ is a regular S-distribution or not, $\widetilde{f}\,T$ and $T\,\widetilde{f}$ would not induce an ambiguity so that we would not distinguish them afterwards.

**Dirac S-distribution –** The Dirac S-distribution $\delta_{y_0}$ , $y_0 \in \mathcal{S}^1$ is a singular S-distribution, which is defined as

$$\forall\,\varphi \in D(\mathcal{S}^1,\mathbb{C})\,,\ \delta_{y_0}[\varphi] \,\hat{=}\, \varphi(y_0)\,. \tag{H.29}$$

### H.2.2 Weak Derivative

Analogy to the interval case in Appendix H.1, we consider an integrable function

$$f \in \mathcal{C}^0(\mathcal{S}^1,\mathbb{C}) \cap \mathcal{C}^1([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})\,, \tag{H.30}$$

which is continuous at $y = 0,\pi R$ [forming a subset of $D(\mathcal{S}^1,\mathbb{C})$] and its corresponding derivative $\partial_y f \in \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$ is also integrable on $\mathcal{S}^1$. Let us introduce the regular S-distribution $\widetilde{f} \in D'(\mathcal{S}^1,\mathbb{C})$ associated to $f$ (H.30) and $\{\partial_y f\}$ is the regular S-distribution associated to $\partial_y f$. The weak derivative $\partial_y\widetilde{f}$ of $\widetilde{f}$ is thus defined as

$$\partial_y\widetilde{f} \,\hat{=}\, \{\partial_y f\}\,,$$

which can rewritten as

$$\begin{aligned}
\forall\,\varphi \in D(\mathcal{S}^1,\mathbb{C})\,,\ \partial_y\widetilde{f}[\varphi] &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \partial_y f(y)\,\varphi(y) \\
&= -\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\,\partial_y\varphi(y) \\
&\quad + [f(y)\,\varphi(y)]|_{-\pi R^+}^{0^-} + [f(y)\,\varphi(y)]|_0^{\pi R} \\
&= -\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\,\partial_y\varphi(y) \\
&\quad + \{[f(y)\,\varphi(y)]|_{0^-} - [f(y)\,\varphi(y)]|_0\} \\
&\quad + \left\{[f(y)\,\varphi(y)]|_{\pi R} - [f(y)\,\varphi(y)]|_{-\pi R^+}\right\} \\
&= -\left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\,\partial_y\varphi(y) \\
&= -\widetilde{f}[\partial_y\varphi]\,,
\end{aligned} \tag{H.31}$$

where the first equality is just the definition of the regular S-distribution $\{\partial_y f\}$ in Eq. (H.17). To go from the first to the second equality, we have performed integrations by parts. Note that the two pairs of obtained boundary terms at $y = 0, 0^-$, $y = -\pi R^+, \pi R$ cancel each other respectively because $f(y)$ and $\varphi(y)$ are continuous at $y = 0, \pi R$. In the last step, we have used the definition of $\widetilde{f}$ in Eq. (H.17).

Now, as in Schwartz's distribution theory on $\mathbb{R}^1$, in contrast to Eq. (H.30), we generalize the weak derivative definition to any regular S-distribution $\widetilde{f} \in D'(\mathcal{S}^1, \mathbb{C})$ associated to

$$f \in \mathcal{C}^1([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C}) \,, \tag{H.32}$$

which can be discontinuous at $y = 0, \pi R$ and its derivative $\partial_y f \in \mathcal{C}^0([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$ is integrable on $\mathcal{S}^1$ so that

$$\forall \, \varphi \in D(\mathcal{S}^1, \mathbb{C}) \,, \; \partial_y \widetilde{f}[\varphi] \; \widehat{=} \; - \, \widetilde{f}[\partial_y \varphi] \,. \tag{H.33}$$

Since $f$ (H.32) can be discontinuous at $y = 0, \pi R$, we define the jumps as

$$\begin{aligned} \beta_0[f] \; &\widehat{=} \; \left. f(y) \right|_{0^-}^{0} = f(0) - f(0^-) \,, \\ \beta_{\pi R}[f] \; &\widehat{=} \; \left. f(y) \right|_{\pi R}^{-\pi R^+} = f(-\pi R^+) - f(\pi R) \,. \end{aligned} \tag{H.34}$$

In contrast to the Jump K-distribution defined in Eq. (H.70), note that $\beta_{0, \pi R}$ (H.34) isn't an S-distribution but a conventional notation. Using $\{\partial_y f\} \in D'(\mathcal{S}^1, \mathbb{C})$ as the regular S-distribution associated to $\partial_y f \in \mathcal{C}^0([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$, we can have

$$\begin{aligned} \forall \, \varphi \in D(\mathcal{S}^1, \mathbb{C}) \,, \; \partial_y \widetilde{f}[\varphi] &= - \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \; f(y) \, \partial_y \varphi(y) \\ &= \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \; \varphi(y) \, \partial_y f(y) \\ &\quad - \left. [f(y)\varphi(y)] \right|_{-\pi R^+}^{0^-} - \left. [f(y)\varphi(y)] \right|_0^{\pi R} \\ &= \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy \; \varphi(y) \, \partial_y f(y) \\ &\quad + \varphi(0) \; \left. f(y) \right|_{0^-}^{0} + \varphi(\pi R) \; \left. f(y) \right|_{\pi R}^{-\pi R^+} \\ &= \{\partial_y f\} \, [\varphi] + \beta_0[f] \, \delta_0[\varphi] + \beta_{\pi R}[f] \, \delta_{\pi R}[\varphi] \,, \end{aligned}$$

where the first equality is just the definition of the weak derivative (H.33). Then, we have performed integrations by parts. In turn, the continuity of $\varphi$ at $y = 0, \pi R$ is involed. Finally, we use the definitions of $\{\partial_y f\}$ (H.17), $\delta_{0/\pi R}$ (H.29), and $\beta_{0/\pi R}[f]$ (H.34). We have thus shown that

$$\partial_y \widetilde{f} = \{\partial_y f\} + \sum_{y_0 = 0, \pi R} \beta_{y_0}[f] \delta_{y_0} \,. \tag{H.35}$$

This definition of the weak derivative on $\mathcal{S}^1$ is similar to what we have on $\mathbb{R}^1$.

## H.3 Kurasov's Distribution Theory on an Interval

In this appendix, we adapt Kurasov's distribution theory [264] to the case of test functions defined on the interval labeled by $y \in [0, L'] \cup [L'^+, L]$ ($L' < L$).

### H.3.1 Basics

**Test function on the interval –** The set $K(\mathcal{I}_1, \mathbb{C})$ of test K-functions on $\mathcal{I}$ is the set of all piece-wise smooth functions $\varphi \in \mathcal{C}^\infty([0, L'] \cup [L'^+, L], \mathbb{C})$, i.e.

$$K(\mathcal{I}_1, \mathbb{C}) \triangleq \mathcal{C}^\infty([0, L'] \cup [L'^+, L], \mathbb{C})\,, \tag{H.36}$$

such that the sequence of functions $\left\{\partial_y^k \varphi\,,\ k \in \mathbb{N}\right\}$ is uniformly bounded on $[0, L'] \cup [L'^+, L]$. In contrast to $D(\mathcal{I}, \mathbb{C})$ in Eq. (H.16), we have following two remarks:

(i) Test K-functions in $K(\mathcal{I}_1, \mathbb{C})$ can be discontinuous at $y = L'$ ($L' < L$), but the (left/right) limits of the functions $\partial_y^k \varphi$, $k \in \mathbb{N}$, on both sides of $y = L'$ ($L' < L$) exist and are finite.

(ii) The set of test S-functions $D(\mathcal{I}, \mathbb{C})$ is a subspace of $K(\mathcal{I}_1, \mathbb{C})$, i.e. the test K-functions are smooth at $y = L'$ ($L' < L$) particularly.

**Convergence in $K(\mathcal{I}_1, \mathbb{C})$ –** A test K-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ $[\forall\, \varphi_n \in K(\mathcal{I}_1, \mathbb{C})]$ converges to a test K-function $\varphi \in K(\mathcal{I}_1, \mathbb{C})$, i.e. convergent on $K(\mathcal{I}_1, \mathbb{C})$, if and only if the sequence $\left\{\partial_y^k \varphi_n\right\}_{n\in\mathbb{N}}$ converges uniformly to $\partial_y^k \varphi$ on $[0, L'] \cup [L'^+, L]$ for each $k \in \mathbb{N}$.

**Continuity of a linear functional on $K(\mathcal{I}_1, \mathbb{C})$ –** A linear functional

$$T \ :\ K(\mathcal{I}_1, \mathbb{C}) \to \mathbb{C}\,,$$
$$\forall\, \varphi_1, \varphi_2 \in K(\mathcal{I}_1, \mathbb{C})\,,\quad T[\lambda_1\, \varphi_1 + \lambda_2\, \varphi_2] = \lambda_1\, T[\varphi_1] + \lambda_2\, T[\varphi_2]\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$$

is continuous if and only if the sequence $\{T[\varphi_n]\}_{n\in\mathbb{N}}$ converges to $T[\varphi] \in \mathbb{C}$, for any test K-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ on $K(\mathcal{I}_1, \mathbb{C})$.

**K-distribution –** A K-distribution [6] $T$ on $\mathcal{I}$ is a continuous linear functional on $K(\mathcal{I}_1, \mathbb{C})$,

$$T \ :\ K(\mathcal{I}_1, \mathbb{C}) \to \mathbb{C}\,.$$

The set containing all the K-distributions,

$$K'(\mathcal{I}_1, \mathbb{C}) \triangleq \{T \mid T \text{ is a K-distribution on } K(\mathcal{I}_1, \mathbb{C}).\}\,,$$

forms a vector space over $\mathbb{C}$ by

(i) $(T_1 + T_2)[\varphi] = T_1[\varphi] + T_2[\varphi]\,;$

(ii) $(\lambda T)[\varphi] = \lambda\,(T[\varphi])\,,\ \lambda \in \mathbb{C}\,,$

which are valid for $\forall\, T_1, T_2 \in K'(\mathcal{I}_1, \mathbb{C})$, $\varphi \in K(\mathcal{I}_1, \mathbb{C})$.

**Regular K-distribution –** For any integrable piece-wise continuous function $f \in \mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})$ [7], one can define a regular K-distribution $\widetilde{f} \in K'(\mathcal{I}_1, \mathbb{C})$ such that

$$\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ \ \widetilde{f}[\varphi] \triangleq \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ f(y)\, \varphi(y)\,, \tag{H.37}$$

which clearly follows the commutative law,

$$\forall\, f, g \in \mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})\,,\ \ \widetilde{f * g} = \widetilde{g * f}\,. \tag{H.38}$$

Note that a K-distribution which is not regular is singular.

6. "K" stands for Kurasov.
7. $K(\mathcal{I}_1, \mathbb{C})$ is a subset of $\mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})$, i.e. $D(\mathcal{I}, \mathbb{C}) \subset K(\mathcal{I}_1, \mathbb{C}) \subset \mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})$.

**Product K-distribution** – If $T \in K'(\mathcal{I}_1, \mathbb{C})$ and $f \in K(\mathcal{I}_1, \mathbb{C})$, one can define their product $f\,T$ as

$$\forall\,\varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ (f\,T)[\varphi] \mathrel{\hat{=}} T[f * \varphi]\,. \tag{H.39}$$

satisfying the bi-linear conditions,

(i) $f\,(\lambda_1\,T_1 + \lambda_2\,T_2) = \lambda_1\,(f\,T_1) + \lambda_2\,(f\,T_2)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1\,f + \lambda_2\,g)\,T = \lambda_1\,(f\,T) + \lambda_2\,(g\,T)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\,T, T_1, T_2 \in K'(\mathcal{I}_1, \mathbb{C})$, $f, g \in K(\mathcal{I}_1, \mathbb{C})$.

It also follows two other calculation laws:

(i) The associative composition law

$$\forall\,f, g \in K(\mathcal{I}_1, \mathbb{C})\,,\ f(g\,T) = (f * g)T\,, \tag{H.40}$$

can be proved as,

$$\begin{aligned}\forall\,\varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ f(g\,T)[\varphi] &= (g\,T)[f * \varphi] \\ &= T[g * f * \varphi] \\ &= T[(f * g) * \varphi] \\ &= (f * g)T[\varphi]\,.\end{aligned}$$

(ii) The commutative law

$$\forall\,f, g \in K(\mathcal{I}_1, \mathbb{C})\,,\ f(g\,T) = g(f\,T)\,, \tag{H.41}$$

can also be guaranteed as

$$f(g\,T) = (f * g)T = (g * f)T = g(f\,T)\,,$$

where we inserted the associative composition law (H.40).

In particular, we can derive some specific results,

(i) $T \in K'(\mathcal{I}_1, \mathbb{C})$ is a regular K-distribution associated to $\tau \in \mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})$, we should have

$$f\,T = \widetilde{f * \tau}\,, \tag{H.42}$$

which should be demonstrated as

$$\begin{aligned}\forall\,\varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\, [f(y)\, \tau(y)]\, \varphi(y) = \widetilde{f * \tau}\, [\varphi]\,.\end{aligned}$$

(ii) $T \in K'(\mathcal{I}_1, \mathbb{C})$ is a regular K-distribution associated to $\tau \in K(\mathcal{I}_1, \mathbb{C})$, we should have

$$f\,T = \tau \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.43}$$

where the first equality should be demonstrated as

$$\begin{aligned}\forall\,\varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ f(y)\, [\tau(y)\, \varphi(y)] = \widetilde{f}[\tau * \varphi] = (\tau \widetilde{f})\, [\varphi]\,,\end{aligned}$$

and the second one could be obtained from Eq. (H.42).

If $T \in K'(\mathcal{I}_1, \mathbb{C})$ and $\widetilde{f} \in K'(\mathcal{I}_1, \mathbb{C})$ is the regular K-distribution associated to $f \in K(\mathcal{I}_1, \mathbb{C})$, the product K-distribution $\widetilde{f}\, T$ with respect to $T$ and $\widetilde{f}$ is defined by

$$\widetilde{f}\, T \mathrel{\hat{=}} f\, T\,, \tag{H.44}$$

satisfying the bi-linear conditions,

(i) $\widetilde{f}\,(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1\,(\widetilde{f}\, T_1) + \lambda_2\,(\widetilde{f}\, T_2)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1 \widetilde{f} + \lambda_2\, \widetilde{g})\, T = \lambda_1\,(\widetilde{f}\, T) + \lambda_2\,(\widetilde{g}\, T)\,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\, T, T_1, T_2 \in K'(\mathcal{I}_1, \mathbb{C})$, $f, g \in K(\mathcal{I}_1, \mathbb{C})$ and we invoke the definition of $f\, T \in K'(\mathcal{I}_1, \mathbb{C})$ in Eq. (H.39). Besides, it respects two following laws as well,

(i) The associative composition law

$$\forall\, f, g \in K(\mathcal{I}_1, \mathbb{C})\,,\ \widetilde{f}(\widetilde{g}\, T) = (\widetilde{f}\, \widetilde{g})T\,, \tag{H.45}$$

can be proved as,

$$\widetilde{f}(\widetilde{g}\, T) = f(g\, T) = (f * g)T = \widetilde{f * g}\, T = (f\, \widetilde{g})T = (\widetilde{f}\, \widetilde{g})T\,,$$

where $\widetilde{f}, \widetilde{g} \in K'(\mathcal{I}_1, \mathbb{C})$ are the regular K-distributions associated to $f, g \in K(\mathcal{I}_1, \mathbb{C})$ respectively via Eq. (H.37) and we invoke the Eq. (H.43).

(ii) The commutative law

$$\forall\, f, g \in K(\mathcal{I}_1, \mathbb{C})\,,\ \widetilde{f}(\widetilde{g}\, T) = \widetilde{g}(\widetilde{f}\, T)\,, \tag{H.46}$$

can also be derived as

$$\widetilde{f}(\widetilde{g}\, T) = f(g\, T) = g(f\, T) = \widetilde{g}(\widetilde{f}\, T)\,,$$

where $\widetilde{f}, \widetilde{g} \in K'(\mathcal{I}_1, \mathbb{C})$ are the regular K-distributions associated to $f, g \in K(\mathcal{I}_1, \mathbb{C})$ respectively via Eq. (H.37) and we inserted the commutative law (H.41).

In particular, if $T \in K'(\mathcal{I}_1, \mathbb{C})$ is the regular K-distribution associated to $\tau \in \mathcal{C}^0([0, L'] \cup [L'^+, L], \mathbb{C})$, we can rewrite Eq. (H.42) via Eq. (H.44) as

$$\widetilde{f}\, T = \widetilde{f * \tau}\,. \tag{H.47}$$

In contrast, when $\tau \in K(\mathcal{I}_1, \mathbb{C})$, we can rewrite Eq. (H.43) via Eq. (H.44) as

$$\widetilde{f}\, T = T\, \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.48}$$

which indicates the commutative law of the product of two regular K-distributions in Eq. (H.44). So, we can conclude that no matter $T$ is a regular K-distribution or not, $\widetilde{f}\, T$ and $T\, \widetilde{f}$ would not induce an ambiguity so that we would not distinguish them afterwards.

**Jump K-distribution –** The Jump K-distribution $\beta_{y_0}$, $y_0 \in \mathcal{I}$ is a singular K-distribution, which is defined as

$$\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ \beta_{y_0}[\varphi] \mathrel{\hat{=}} \begin{cases} 0 & \text{for} \quad y_0 \neq L'\,, \\ \varphi(y)|_{L'}^{L'^+} = \varphi(L'^+) - \varphi(L') & \text{for} \quad y_0 = L'\,, \end{cases} \tag{H.49}$$

so that $\beta_{y_0}[\varphi]$ gives the jump of the test K-function $\varphi(y) \in K(\mathcal{I}_1, \mathbb{C})$ at $y_0$. Thus, we have the following remark:

$$\beta_{y_0}[\varphi] \neq 0 \text{ if and only if } y_0 = L' \text{ and } \varphi(y) \in K(\mathcal{I}_1, \mathbb{C}) \text{ is discontinuous there.} \tag{H.50}$$

**Dirac K-distribution –** The Dirac K-distribution $\delta_{y_0}$, $y_0 \in \mathcal{I}$ is a singular K-distribution, which is defined as

$$\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\;\; \delta_{y_0}[\varphi] \,\hat{=}\, \varphi(y_0)\,. \tag{H.51}$$

### H.3.2 Weak Derivative

We want to define a weak derivative for regular K-distributions. However, we will not define it as the distributional derivative in Kurasov's original article (see Ref. [264]) since his definition is not appropriate for practical reason [8]. Here, we follow the way of the weak derivative defined in Schwartz's distribution theory (see Appendix H.2).

First , let us consider an integrable function

$$f \in \mathcal{C}^0(\mathcal{I}, \mathbb{C}) \cap K(\mathcal{I}_1, \mathbb{C})\,, \tag{H.52}$$

which is continuous at $y = L'$ $(L' < L)$ [forming a subset of $K(\mathcal{I}_1, \mathbb{C})$] and its corresponding derivative $\partial_y f \in K(\mathcal{I}_1, \mathbb{C})$ is also integrable on $\mathcal{I}$. Then, introduce the regular K-distribution $\widetilde{f} \in K'(\mathcal{I}_1, \mathbb{C})$ associated to $f$ (H.52) and $\{\partial_y f\}$ is the regular K-distribution associated to $\partial_y f$. The weak derivative $\partial_y \widetilde{f}$ of $\widetilde{f}$ is thus defined as

$$\partial_y \widetilde{f} \,\hat{=}\, \{\partial_y f\}\;,$$

which can be rewritten via $\beta_{y_0}$ (H.49) and $\delta_{y_0}$ (H.51),

$$\begin{aligned}
\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\;\; \partial_y \widetilde{f}[\varphi] &= \left(\int_0^{L'} + \int_{L'+}^{L}\right) dy\; \partial_y f(y)\, \varphi(y) \\
&= -\left(\int_0^{L'} + \int_{L'+}^{L}\right) dy\; f(y)\, \partial_y \varphi(y) \\
&\quad + [f(y)\, \varphi(y)]|_0^{L'} + [f(y)\, \varphi(y)]|_{L'+}^{L} \\
&= -\left(\int_0^{L'} + \int_{L'+}^{L}\right) dy\; f(y)\, \partial_y \varphi(y) \\
&\quad + \{[f(y)\, \varphi(y)]|_{L'} - [f(y)\, \varphi(y)]|_{L'+}\} \\
&\quad + \{[f(y)\, \varphi(y)]|_{L} - [f(y)\, \varphi(y)]|_{0}\} \\
&= -\widetilde{f}[\partial_y \varphi] - \delta_{L'}[f]\, \beta_{L'}[\varphi] + \widetilde{f}\,(\delta_L - \delta_0)\,[\varphi]\,,
\end{aligned} \tag{H.53}$$

where we have used the continuity of $f$ at $y = L'$ $(L' < L)$ in Eq. (H.52) and the definition of $\widetilde{f}$ in Eq. (H.37). The non-vanishing boundary terms which defer from what we have for a S-distribution on $\mathcal{I}$ in Eq. (H.31), come from possible jumps at $y = L'$ $(L' < L)$ of the test K-functions.

Now, as in Schwartz's distribution theory on $\mathcal{I}$ in Appendix H.2, we generalize this result to define the weak derivative of any regular K-distribution $\widetilde{f} \in K'(\mathcal{I}_1, \mathbb{C})$ associated to a test K-function

$$f \in K(\mathcal{I}_1, \mathbb{C})\,, \tag{H.54}$$

which can be discontinuous at $y = L'$ $(L' < L)$ and its derivative $\partial_y f \in K(\mathcal{I}_1, \mathbb{C})$ is also integrable on $\mathcal{I}$ so that

$$\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\;\; \partial_y \widetilde{f}[\varphi] \,\hat{=}\, -\,\widetilde{f}[\partial_y \varphi] - \delta_{L'}[f]\, \beta_{L'}[\varphi] + \widetilde{f}\,(\delta_L - \delta_0)\,[\varphi]\,. \tag{H.55}$$

8. Kurasov discusses in Ref. [264] that his distributional derivative does not match with the usual derivative for differentiable functions, which is however essential for a weak derivative in physical applications [115].

Using $\{\partial_y f\} \in K'(\mathcal{I}_1, \mathbb{C})$ as the regular K-distribution associated to $\partial_y f \in K(\mathcal{I}_1, \mathbb{C})$, we can have

$$\forall\, \varphi \in K(\mathcal{I}_1, \mathbb{C})\,,\ \partial_y \widetilde{f}[\varphi] = \{\partial_y f\}[\varphi] + \beta_{L'}[f]\left\{\delta_{L'}[\varphi] + \beta_{L'}[\varphi]\right\}\,,$$

where we inserted

$$\begin{aligned}
\widetilde{f}[\partial_y \varphi] &= \left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ f(y)\,\partial_y \varphi(y) \\
&= -\left[\left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ \varphi(y)\,\partial_y f(y)\right] + [f(y)\varphi(y)]|_0^{L'} + [f(y)\varphi(y)]|_{L'^+}^{L} \\
&= -\left[\left(\int_0^{L'} + \int_{L'^+}^{L}\right) dy\ \varphi(y)\,\partial_y f(y)\right] + \{\,[f(y)\,\varphi(y)]|_{L'} - [f(y)\,\varphi(y)]|_{L'^+}\} \\
&\qquad + \{\,[f(y)\,\varphi(y)]|_{L} - [f(y)\,\varphi(y)]|_{0}\} \\
&= -\{\partial_y f\}[\varphi] - \{\beta_{L'}[f]\delta_{L'}[\varphi] + \delta_{L'}[f]\beta_{L'}[\varphi]\} - \beta_{L'}[f]\beta_{L'}[\varphi] + \widetilde{f}\,(\delta_L - \delta_0)\,[\varphi]\,,
\end{aligned}$$

with

$$\begin{aligned}
[f(y)\,\varphi(y)]|_{L'} - [f(y)\,\varphi(y)]|_{L'^+} &= f(L')\varphi(L') - f(L')\varphi(L'^+) + f(L')\varphi(L'^+) - f(L'^+)\varphi(L'^+) \\
&= -f(L')\beta_{L'}[\varphi] - \varphi(L'^+)\beta_{L'}[f] \\
&= -\{\beta_{L'}[f]\delta_{L'}[\varphi] + \delta_{L'}[f]\beta_{L'}[\varphi]\} - \beta_{L'}[f]\beta_{L'}[\varphi]\,.
\end{aligned}$$

We have thus shown that

$$\partial_y \widetilde{f} = \{\partial_y f\} + \beta_{L'}[f]\,(\delta_{L'} + \beta_{L'})\ . \tag{H.56}$$

## H.4 Kurasov's Distribution Theory on $\mathcal{S}^1$

In this appendix, we adapt Kurasov's distribution theory [264] to the case of test functions defined on $\mathcal{S}^1$ labeled by $y \in [-\pi R^+, 0^-] \cup [0, \pi R \equiv -\pi R]$.

### H.4.1 Basics

**Test function on $\mathcal{S}^1$ –** The set $K(\mathcal{S}^1, \mathbb{C})$ of test K-functions on $\mathcal{S}^1$ is the set of all piece-wise smooth functions $\varphi \in \mathcal{C}^\infty([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$, i.e.

$$K(\mathcal{S}^1, \mathbb{C}) \,\hat{=}\, \mathcal{C}^\infty([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})\,, \tag{H.57}$$

such that the sequence of functions $\left\{\partial_y^k \varphi\,,\ k \in \mathbb{N}\right\}$ is uniformly bounded on $[-\pi R^+, 0^-] \cup [0, \pi R]$. In contrast to $D(\mathcal{S}^1, \mathbb{C})$ in Eq. (H.16), we have following two remarks:

(i) Test K-functions in $K(\mathcal{S}^1, \mathbb{C})$ can be discontinuous at $y = 0, \pi R$, but the (left/right) limits of the functions $\partial_y^k \varphi$, $k \in \mathbb{N}$, on both sides of $y = 0, \pi R$ exist and are finite.

(ii) The set of test S-functions $D(\mathcal{S}^1, \mathbb{C})$ is a subspace of $K(\mathcal{S}^1, \mathbb{C})$, i.e. the test K-functions are smooth at $y = 0, \pi R$ particularly.

**Convergence in $K(\mathcal{S}^1, \mathbb{C})$ –** A test K-function sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ $[\forall\, \varphi_n \in K(\mathcal{S}^1, \mathbb{C})]$ converges to a test K-function $\varphi \in K(\mathcal{S}^1, \mathbb{C})$, i.e. convergent on $K(\mathcal{S}^1, \mathbb{C})$, if and only if the sequence $\left\{\partial_y^k \varphi_n\right\}_{n \in \mathbb{N}}$ converges uniformly to $\partial_y^k \varphi$ on $[-\pi R^+, 0^-] \cup [0, \pi R]$ for each $k \in \mathbb{N}$.

**Continuity of a linear functional on $K(\mathcal{S}^1,\mathbb{C})$ –** A linear functional

$$T \;:\; K(\mathcal{S}^1,\mathbb{C}) \to \mathbb{C}\,,$$
$$\forall\,\varphi_1,\varphi_2 \in K(\mathcal{S}^1,\mathbb{C})\,,\quad T[\lambda_1\,\varphi_1+\lambda_2\,\varphi_2] = \lambda_1\,T[\varphi_1]+\lambda_2\,T[\varphi_2]\,,\quad \lambda_1,\lambda_2\in\mathbb{C}\,,$$

is continuous if and only if the sequence $\{T[\varphi_n]\}_{n\in\mathbb{N}}$ converges to $T[\varphi]\in\mathbb{C}$, for any test K-function sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ on $K(\mathcal{S}^1,\mathbb{C})$.

**K-distribution –** A K-distribution $T$ on $\mathcal{S}^1$ is a continuous linear functional on $K(\mathcal{S}^1,\mathbb{C})$,

$$T \;:\; K(\mathcal{S}^1,\mathbb{C}) \to \mathbb{C}\,.$$

The set containing all the K-distributions,

$$K'(\mathcal{S}^1,\mathbb{C}) \mathrel{\hat{=}} \Big\{T \mid T \text{ is a K-distribution on } K(\mathcal{S}^1,\mathbb{C}).\Big\}\,,$$

forms a vector space over $\mathbb{C}$ by

(i) $(T_1+T_2)[\varphi] = T_1[\varphi]+T_2[\varphi]\,;$

(ii) $(\lambda T)[\varphi] = \lambda\,(T[\varphi])\,,\ \lambda\in\mathbb{C}\,,$

which are valid for $\forall\,T_1,T_2\in K'(\mathcal{S}^1,\mathbb{C})$, $\varphi\in K(\mathcal{S}^1,\mathbb{C})$.

**Regular K-distribution –** For any integrable piece-wise continuous function $f \in \mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$ [9], one can define a regular K-distribution $\widetilde{f}\in K'(\mathcal{S}^1,\mathbb{C})$ such that

$$\forall\,\varphi\in K(\mathcal{S}^1,\mathbb{C})\,,\ \widetilde{f}[\varphi] \mathrel{\hat{=}} \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\,\varphi(y)\,, \tag{H.58}$$

which clearly follows the commutative law,

$$\forall\,f,g\in\mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})\,,\ \widetilde{f*g} = \widetilde{g*f}\,. \tag{H.59}$$

Note that a K-distribution which is not regular is singular.

**Product K-distribution –** If $T\in K'(\mathcal{S}^1,\mathbb{C})$ and $f\in K(\mathcal{S}^1,\mathbb{C})$, one can define their product $f\,T$ as

$$\forall\,\varphi\in K(\mathcal{S}^1,\mathbb{C})\,,\ (f\,T)[\varphi] \mathrel{\hat{=}} T[f*\varphi]\,. \tag{H.60}$$

satisfying the bi-linear conditions,

(i) $f\,(\lambda_1\,T_1+\lambda_2\,T_2) = \lambda_1\,(f\,T_1)+\lambda_2\,(f\,T_2)\,,\quad \lambda_1,\lambda_2\in\mathbb{C}\,;$

(ii) $(\lambda_1\,f+\lambda_2\,g)\,T = \lambda_1\,(f\,T)+\lambda_2\,(g\,T)\,,\quad \lambda_1,\lambda_2\in\mathbb{C}\,,$

which are valid for $\forall\,T,T_1,T_2\in K'(\mathcal{S}^1,\mathbb{C})$, $f,g\in K(\mathcal{S}^1,\mathbb{C})$.

It also follows two other calculation laws:

(i) The associative composition law

$$\forall\,f,g\in K(\mathcal{S}^1,\mathbb{C})\,,\ f(g\,T) = (f*g)T\,, \tag{H.61}$$

can be proved as,

$$\begin{aligned}\forall\,\varphi\in K(\mathcal{S}^1,\mathbb{C})\,,\ f(g\,T)[\varphi] &= (g\,T)[f*\varphi]\\ &= T[g*f*\varphi]\\ &= T[(f*g)*\varphi]\\ &= (f*g)T[\varphi]\,.\end{aligned}$$

9. $K(\mathcal{S}^1,\mathbb{C})$ is a subset of $\mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$, i.e. $D(\mathcal{S}^1,\mathbb{C})\subset K(\mathcal{S}^1,\mathbb{C})\subset\mathcal{C}^0([-\pi R^+,0^-]\cup[0,\pi R],\mathbb{C})$.

(ii) The commutative law

$$\forall\, f, g \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ f(g\,T) = g(f\,T)\,, \tag{H.62}$$

can also be guaranteed as

$$f(g\,T) = (f * g)T = (g * f)T = g(f\,T)\,,$$

where we inserted the associative composition law (H.61).

In particular, we can derive some specific results,

(i) $T \in K'(\mathcal{S}^1, \mathbb{C})$ is a regular K-distribution associated to $\tau \in \mathcal{C}^0([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$, we should have

$$f\,T = \widetilde{f * \tau}\,, \tag{H.63}$$

which should be demonstrated as

$$\begin{aligned}\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\, [f(y)\, \tau(y)]\, \varphi(y) = \widetilde{f * \tau}\, [\varphi]\,.\end{aligned}$$

(ii) $T \in K'(\mathcal{S}^1, \mathbb{C})$ is a regular K-distribution associated to $\tau \in K(\mathcal{S}^1, \mathbb{C})$, we should have

$$f\,T = \tau \widetilde{f} = \widetilde{f * \tau}\,, \tag{H.64}$$

where the first equality should be demonstrated as

$$\begin{aligned}\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ f\,T[\varphi] &= T[f * \varphi] \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ \tau(y)\, f(y)\, \varphi(y) \\ &= \left(\int_{-\pi R^+}^{0^-} + \int_0^{\pi R}\right) dy\ f(y)\, [\tau(y)\, \varphi(y)] = \widetilde{f}[\tau * \varphi] = (\tau \widetilde{f})\, [\varphi]\,,\end{aligned}$$

and the second one could be obtained from Eq. (H.63).

If $T \in K'(\mathcal{S}^1, \mathbb{C})$ and $\widetilde{f} \in K'(\mathcal{S}^1, \mathbb{C})$ is the regular K-distribution associated to $f \in K(\mathcal{S}^1, \mathbb{C})$, the product K-distribution $\widetilde{f}\,T$ with respect to $T$ and $\widetilde{f}$ is defined by

$$\widetilde{f}\,T \mathrel{\hat{=}} f\,T\,, \tag{H.65}$$

satisfying the bi-linear conditions,

(i) $\widetilde{f}\,(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1\,(\widetilde{f}\,T_1) + \lambda_2\,(\widetilde{f}\,T_2)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,;$

(ii) $(\lambda_1 \widetilde{f} + \lambda_2 \widetilde{g})\,T = \lambda_1\,(\widetilde{f}\,T) + \lambda_2\,(\widetilde{g}\,T)\,,\quad \lambda_1, \lambda_2 \in \mathbb{C}\,,$

which are valid for $\forall\, T, T_1, T_2 \in K'(\mathcal{S}^1, \mathbb{C})$, $f, g \in K(\mathcal{S}^1, \mathbb{C})$ and we invoke the definition of $f\,T \in K'(\mathcal{S}^1, \mathbb{C})$ in Eq. (H.60). Besides, it respects two following laws as well,

(i) The associative composition law

$$\forall\, f, g \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ \widetilde{f}(\widetilde{g}\,T) = (\widetilde{f}\,\widetilde{g})T\,, \tag{H.66}$$

can be proved as,

$$\widetilde{f}(\widetilde{g}\,T) = f(g\,T) = (f * g)T = \widetilde{f * g}\,T = (f\,\widetilde{g})T = (\widetilde{f}\,\widetilde{g})T\,,$$

where $\widetilde{f}, \widetilde{g} \in K'(\mathcal{S}^1, \mathbb{C})$ are the regular K-distributions associated to $f, g \in K(\mathcal{S}^1, \mathbb{C})$ respectively via Eq. (H.58) and we invoke the Eq. (H.64).

(ii) The commutative law

$$\forall\, f, g \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ \widetilde{f}(\widetilde{g}\,T) = \widetilde{g}(\widetilde{f}\,T)\,, \tag{H.67}$$

can also be derived as

$$\widetilde{f}(\widetilde{g}\,T) = f(g\,T) = g(f\,T) = \widetilde{g}(\widetilde{f}\,T)\,,$$

where $\widetilde{f}, \widetilde{g} \in K'(\mathcal{S}^1, \mathbb{C})$ are the regular K-distributions associated to $f, g \in K(\mathcal{S}^1, \mathbb{C})$ respectively via Eq. (H.58) and we inserted the commutative law (H.62).

In particular, if $T \in K'(\mathcal{S}^1, \mathbb{C})$ is the regular K-distribution associated to $\tau \in \mathcal{C}^0([-\pi R^+, 0^-] \cup [0, \pi R], \mathbb{C})$, we can rewrite Eq. (H.63) via Eq. (H.65) as

$$\widetilde{f}\,T = \widetilde{f * \tau}\,. \tag{H.68}$$

In contrast, when $\tau \in K(\mathcal{S}^1, \mathbb{C})$, we can rewrite Eq. (H.64) via Eq. (H.65) as

$$\widetilde{f}\,T = T\,\widetilde{f} = \widetilde{f * \tau}\,, \tag{H.69}$$

which indicates the commutative law of the product of two regular K-distributions in Eq. (H.65). So, we can conclude that no matter $T$ is a regular K-distribution or not, $\widetilde{f}\,T$ and $T\,\widetilde{f}$ would not induce an ambiguity so that we would not distinguish them afterwards.

**Jump K-distribution –** The Jump K-distribution $\beta_{y_0}$, $y_0 \in \mathcal{S}^1$ is a singular K-distribution, which is defined as

$$\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ \beta_{y_0}[\varphi] \mathrel{\widehat{=}} \begin{cases} \varphi(y)|_{y_0^-}^{y_0} = \varphi(y_0) - \varphi(y_0^-) & \text{for} \quad y_0 \neq \pi R\,, \\ \\ \varphi(y)|_{\pi R}^{-\pi R^+} = \varphi(-\pi R^+) - \varphi(\pi R) & \text{for} \quad y_0 = \pi R\,, \end{cases} \tag{H.70}$$

so that $\beta_{y_0}[\varphi]$ gives the jump of the test K-function $\varphi(y) \in K(\mathcal{S}^1, \mathbb{C})$ at $y_0$. Thus, we have the following remark:

$$\beta_{y_0}[\varphi] \neq 0 \text{ if and only if } y_0 = 0, \pi R \text{ and } \varphi(y) \in K(\mathcal{S}^1, \mathbb{C}) \text{ is discontinuous there.} \tag{H.71}$$

**Dirac K-distribution –** The Dirac K-distribution $\delta_{y_0}$, $y_0 \in \mathcal{S}^1$ is a singular K-distribution, which is defined as

$$\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\ \ \delta_{y_0}[\varphi] \mathrel{\widehat{=}} \varphi(y_0)\,. \tag{H.72}$$

### H.4.2 Weak Derivative

We want to define a weak derivative for regular K-distributions. However, we will not define it as the distributional derivative in Kurasov's original article (see Ref. [264]) since his definition is not appropriate for practical reason [10]. Here, we follow the way of the weak derivative defined in Schwartz's distribution theory (see Appendix H.2).

First , let us consider an integrable function

$$f \in \mathcal{C}^0(\mathcal{S}^1, \mathbb{C})\, \cap\, K(\mathcal{S}^1, \mathbb{C})\,, \tag{H.73}$$

10. Kurasov discusses in Ref. [264] that his distributional derivative does not match with the usual derivative for differentiable functions, which is however essential for a weak derivative in physical applications [115].

which is continuous at $y = 0, \pi R$ [forming a subset of $K(\mathcal{S}^1, \mathbb{C})$] and its corresponding derivative $\partial_y f \in K(\mathcal{S}^1, \mathbb{C})$ is also integrable on $\mathcal{S}^1$. Then, introduce the regular K-distribution $\widetilde{f} \in K'(\mathcal{S}^1, \mathbb{C})$ associated to $f$ (H.73) and $\{\partial_y f\}$ is the regular K-distribution associated to $\partial_y f$. The weak derivative $\partial_y \widetilde{f}$ of $\widetilde{f}$ is thus defined as

$$\partial_y \widetilde{f} \;\hat{=}\; \{\partial_y f\} \;,$$

which can be rewritten via $\beta_{y_0}$ (H.70) and $\delta_{y_0}$ (H.72),

$$\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\;\; \partial_y \widetilde{f}[\varphi] = -\widetilde{f}[\partial_y \varphi] - \sum_{y_0 = 0, \pi R} \delta_{y_0}[f]\, \beta_{y_0}[\varphi]\,, \tag{H.74}$$

where we perform the same calculation in Eq. (H.31) except the last step. We have used the continuity of $f$ at $y = 0, \pi R$ in Eq. (H.73) and the definition of $\widetilde{f}$ in Eq. (H.58). The non-vanishing boundary terms which defer from what we have for a S-distribution on $\mathcal{S}^1$ in Eq. (H.31), come from possible jumps at $y = 0, \pi R$ of the test K-functions.

Now, as in Schwartz's distribution theory on $\mathcal{S}^1$ in Appendix H.2, we generalize this result to define the weak derivative of any regular K-distribution $\widetilde{f} \in K'(\mathcal{S}^1, \mathbb{C})$ associated to a test K-function

$$f \in K(\mathcal{S}^1, \mathbb{C})\,, \tag{H.75}$$

which can be discontinuous at $y = 0, \pi R$ and its derivative $\partial_y f \in K(\mathcal{S}^1, \mathbb{C})$ is also integrable on $\mathcal{S}^1$ so that

$$\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\;\; \partial_y \widetilde{f}[\varphi] \;\hat{=}\; -\, \widetilde{f}[\partial_y \varphi] - \sum_{y_0 = 0, \pi R} \delta_{y_0}[f]\, \beta_{y_0}[\varphi]\,. \tag{H.76}$$

Using $\{\partial_y f\} \in K'(\mathcal{S}^1, \mathbb{C})$ as the regular K-distribution associated to $\partial_y f \in K(\mathcal{S}^1, \mathbb{C})$, we can have

$$\forall\, \varphi \in K(\mathcal{S}^1, \mathbb{C})\,,\;\; \partial_y \widetilde{f}[\varphi] = \{\partial_y f\}\,[\varphi] + \{\beta_{\pi R}[f] \beta_{\pi R}[\varphi] - \beta_0[f] \beta_0[\varphi]\} + \sum_{y_0 = 0, \pi R} \beta_{y_0}[f] \delta_{y_0}[\varphi]\,,$$

where we inserted

$$\begin{aligned}
\widetilde{f}[\partial_y \varphi] &= \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\;\, f(y)\, \partial_y \varphi(y) \\
&= -\left[ \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\;\, \varphi(y)\, \partial_y f(y) \right] + [f(y) \varphi(y)]|_{-\pi R^+}^{0^-} + [f(y) \varphi(y)]|_0^{\pi R} \\
&= -\left[ \left( \int_{-\pi R^+}^{0^-} + \int_0^{\pi R} \right) dy\;\, \varphi(y)\, \partial_y f(y) \right] + \{\, [f(y)\, \varphi(y)]|_{0^-} - [f(y)\, \varphi(y)]|_0 \} \\
&\qquad + \left\{ [f(y)\, \varphi(y)]|_{\pi R} - [f(y)\, \varphi(y)]|_{-\pi R^+} \right\} \\
&= -\, \{\partial_y f\}\, [\varphi] + \{\beta_0[f] \beta_0[\varphi] - \beta_{\pi R}[f] \beta_{\pi R}[\varphi]\} - \sum_{y_0 = 0, \pi R} \{\beta_{y_0}[f] \delta_{y_0}[\varphi] + \delta_{y_0}[f]\, \beta_{y_0}[\varphi]\}\;,
\end{aligned}$$

with

$$\begin{aligned}
[f(y)\,\varphi(y)]|_{0^-} - [f(y)\,\varphi(y)]|_{0} &= f(0^-)\varphi(0^-) - f(0^-)\varphi(0) + f(0^-)\varphi(0) - f(0)\varphi(0) \\
&= -f(0^-)\beta_0[\varphi] - \varphi(0)\beta_0[f] \\
&= \{\beta_0[f] - \delta_0[f]\}\,\beta_0[\varphi] - \delta_0[\varphi]\beta_0[f] \\
&= -\{\beta_0[f]\delta_0[\varphi] + \delta_0[f]\beta_0[\varphi]\} + \beta_0[f]\beta_0[\varphi]\,, \\
[f(y)\,\varphi(y)]|_{\pi R} - [f(y)\,\varphi(y)]|_{-\pi R^+} &= f(\pi R)\varphi(\pi R) - f(\pi R)\varphi(-\pi R^+) \\
&\quad + f(\pi R)\varphi(-\pi R^+) - f(-\pi R^+)\varphi(-\pi R^+) \\
&= -f(\pi R)\beta_{\pi R}[\varphi] - \varphi(-\pi R^+)\beta_{\pi R}[f] \\
&= -\{\beta_{\pi R}[f]\delta_{\pi R}[\varphi] + \delta_{\pi R}[f]\beta_{\pi R}[\varphi]\} - \beta_{\pi R}[f]\beta_{\pi R}[\varphi]\,.
\end{aligned}$$

We have thus shown that

$$\partial_y \widetilde{f} = \{\partial_y f\} + (\beta_{\pi R}[f]\beta_{\pi R} - \beta_0[f]\beta_0) + \sum_{y_0=0,\pi R} \beta_{y_0}[f]\delta_{y_0}\,, \tag{H.77}$$

which keeps a similar formalism of the weak derivative for an S-distribution on $\mathcal{S}^1$ in Eq. (H.35).

# Appendix I

# Dimensional analysis

The MKS unit system is recovered by the conventional formalism in Ref. [265].

## I.1 4D Analysis

The generic action is contructed by the 4D Lagrangian as,

$$
\begin{aligned}
S &= \frac{1}{c}\int d^4x \; \mathcal{L}^{4D} \,, \; x^\mu = (ct, \mathbf{x}) \;, \\
[S] &= [E] \times [T] = [J] = [P] \times [L] \;, \\
\left[\mathcal{L}^{4D}\right] &= [E] \times [L]^{-3} \;.
\end{aligned} \tag{I.1}
$$

— 4D scalar field $\phi$

$$
\begin{aligned}
\mathcal{L}^{4D} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2 \,, \\
\left[\mathcal{L}^{4D}\right] &= [L]^{-2} \times [\phi]^2 \;, \\
[\phi]^2 &= [E] \times [L]^{-1} = [m] \times [T]^{-2} \times [L] \;, \\
[\phi] &= [m]^{\frac{1}{2}} \times [T]^{-1} \times [L]^{\frac{1}{2}} = [E]^{\frac{1}{2}} \times [L]^{-\frac{1}{2}} \;.
\end{aligned} \tag{I.2}
$$

— 4D spinor field $\psi$

$$
\begin{aligned}
\mathcal{L}^{4D} &= i\,(\hbar c)\,\bar{\psi}\gamma^\mu\partial_\mu\psi - \left(mc^2\right)\bar{\psi}\psi \,, \\
[\psi] &= [L]^{-\frac{3}{2}} \;.
\end{aligned} \tag{I.3}
$$

## I.2 5D Analysis

The generic action is contructed by the 5D bulk and brane-localized Lagrangian (4D) as,

$$
\begin{aligned}
S &= \frac{1}{c}\int d^4x \left(\int dy \; \mathcal{L}^{5D} + \mathcal{L}^{4D}\right) \,, \; x^M = (ct, \mathbf{x}, y) \;, \\
\left[\mathcal{L}^{5D}\right] &= \left[\mathcal{L}^{4D}\right] \times [L]^{-1} = [E] \times [L]^{-4} \;.
\end{aligned} \tag{I.4}
$$

— 5D bulk scalar field $\Phi$

$$\begin{aligned}
\mathcal{L}^{5D} &= \frac{1}{2}\partial_M \Phi \partial^M \Phi - \frac{1}{2}\left(\frac{M_\Phi c}{\hbar}\right)^2 \Phi^2 \,, \\
\left[\mathcal{L}^{5D}\right] &= [L]^{-2} \times [\Phi]^2 \ , \\
[\Phi]^2 &= [E] \times [L]^{-2} = [m] \times [T]^{-2} \ , \\
[\Phi] &= [m]^{\frac{1}{2}} \times [T]^{-1} = [E]^{\frac{1}{2}} \times [L]^{-1} \ .
\end{aligned} \tag{I.5}$$

— 5D bulk spinor field $\Psi$

$$\begin{aligned}
\mathcal{L}^{5D} &= i\,(\hbar c)\,\bar{\Psi}\Gamma^M \partial_M \Psi - \left(M_\Psi c^2\right)\bar{\Psi}\Psi \,, \\
[\Psi] &= [L]^{-2} \ .
\end{aligned} \tag{I.6}$$

— BBT terms (3.4) as a brane-localized Lagrangian

$$\begin{aligned}
S_B &\ni \frac{1}{c}\int d^4x \ \frac{1}{2}\,(\hbar c)\,\sigma_P^\Psi \ \bar{\Psi}\Psi\Big|_P \,, \ P = 0, L \,, \\
[\hbar c] &= [E] \times [L] \ .
\end{aligned} \tag{I.7}$$

# Appendix J

# Bulk Massive KK Modes on an Interval

Here, we give an example demonstration on continuous solutions of the EOM (6.60) with the NBC (6.12) on the first region, $[0, L_F]$.

For $\left(m_n^F\right)^2 - (\widetilde{m}_F^1)^2 < 0$ in Eq. (6.61), the solutions have the general form on $y \in [0, L_F]$,

$$f_{L/R}^n(y) = A_{L/R}^n \, e^{\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\, y} + B_{L/R}^n \, e^{-\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\, y}\,, \tag{J.1}$$

where $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients. For the Dirichlet BC for $f_R^n(y)$ (6.12), i.e. $f_R^n|_0 = f_R^n|_{L_F} = 0$, we have the following set of equations to determine $A_R^n$, $B_R^n$,

$$\begin{cases} A_R^n \, e^{\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot 0} + B_R^n \, e^{-\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot 0} &= 0\,, \\ A_R^n \, e^{\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot L_F} + B_R^n \, e^{-\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot L_F} &= 0\,, \end{cases}$$

and the associated characteristic determinant,

$$\Delta = e^{\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot (0-L_F)} - e^{\sqrt{\left(\widetilde{m}_F^1\right)^2 - (m_n^F)^2}\cdot (L_F-0)} \neq 0\,,$$

which leads to the absence of $A_R^n$, $B_R^n$, i.e. $A_R^n = B_R^n = 0$, such that,

$$f_R^n(y) = 0\,, \tag{J.2}$$

which is valid on $y \in [0, L_F]$ and after injecting to Eq. (6.60), leads to,

$$\begin{cases} f_L^n(y) &= 0,\ m_n^F \neq 0\,, \\ \left(\partial_4 + \widetilde{m}_F^1\right) f_L^n(y) &= 0,\ m_n^F = 0\,, \end{cases}$$

which means $f_L^n(y)$ has non-zero solutions in this case if and only if $m_n^F = 0$, i.e. the zero mode $f_L^0(y)$, and directly leads to the form of $f_L^0(y)$ with the normalization factor $\mathcal{N}_L^{F10}$, derived from the ortho-normalisation conditions (6.38) [1],

$$f_L^0(y) = \mathcal{N}_L^{F10} \, e^{-\widetilde{m}_F^1 \, y}\,, \quad \mathcal{N}_L^{F10} = \sqrt{\frac{2\, \widetilde{m}_F^1 \, L}{1 - e^{-2\widetilde{m}_F^1 \, \Delta L_F^1}}}\, e^{i\alpha_F^{10}}\,, \tag{J.3}$$

1. Here, we just consider the normalization on $[0, L_F]$,

$$\forall\, n, m \in \mathbb{N},\ \frac{1}{L}\int_0^{L_F} dy\ f_{L/R}^{n*}(y)\, f_{L/R}^m(y) = \delta_{nm}\,.$$

where $\Delta L_F^1 \mathrel{\hat{=}} L_F - 0$ is the length of the segment, $[0, L_F]$, and $e^{i\alpha_F^{10}}$ ($\alpha_F^{10} \in \mathbb{R}$) is the global phase.

For $\left(m_n^F\right)^2 - (\widetilde{m}_F^1)^2 \geq 0$ in Eq. (6.61), the solutions have the general form on $y \in [0, L_F]$,

$$\forall\, n \in \mathbb{N}^*,\ f_{L/R}^n(y) = A_{L/R}^n \cos\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y\right] + B_{L/R}^n \sin\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y\right] , \tag{J.4}$$

where $A_{L/R}^n$, $B_{L/R}^n$ are complex coefficients. Then, the Dirichlet BC for $f_R^n(y)$ would constrain solutions as the following form with the normalization factors $\mathcal{N}_R^{F1n}$ ($n \in \mathbb{N}^*$),

$$\forall\, n \in \mathbb{N}^*,\ f_R^n(y) = \mathcal{N}_R^{F1n} \sin\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y\right] ,\quad \mathcal{N}_R^{F1n} = \sqrt{2}\, e^{i\alpha_F^{1n}} , \tag{J.5}$$

which generates the mass spectrum,

$$\left|m_n^F\right| = \sqrt{\left[\frac{n\pi}{\Delta L_F^1}\right]^2 + (\widetilde{m}_F^1)^2} ,\ n \in \mathbb{N}^* . \tag{J.6}$$

Combining with the zero mode, one can obtain the complete mass spectrum. Inserting the solutions of $f_R^n(y)$ (J.5) into the coupled EOM (6.60), one can obtain the solutions of $f_L^n(y)$,

$$\begin{aligned} &\forall\, n \in \mathbb{N}^*,\ f_L^n(y) = \mathcal{N}_L^{F1n} \cos\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y + \zeta_n^{F1}\right] , \\ &\text{with},\ \mathcal{N}_L^{F1n} = -\mathcal{N}_R^{F1n} ,\ \sin\left(\zeta_n^{F1}\right) \mathrel{\hat{=}} \frac{\widetilde{m}_F^1}{m_n^F} ,\ \cos\left(\zeta_n^{F1}\right) \mathrel{\hat{=}} \frac{\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}}{m_n^F} . \end{aligned} \tag{J.7}$$

Note that the KK mass $\left|m_n^F\right|$ (J.6) can't be equal to $|\widetilde{m}_F^1|$. Otherwise, both of the left and the right profiles are suspended to vanish due to the coupled EOM (6.60).

The similar analysis can be applied on the Dirichlet BC for $f_L^n(y)$ (6.12), i.e. $f_L^n|_0 = f_L^n|_{L_F} = 0$, which would lead to the vanishing $f_L^0(y)$ on $y \in [0, L_F]$. The normalized non-vanishing zero mode $f_R^0(y)$ reads,

$$f_R^0(y) = \mathcal{N}_R^{F10}\, e^{\widetilde{m}_F^1\, y} ,\quad \mathcal{N}_R^{F10} = \sqrt{\frac{2\, \widetilde{m}_F^1\, L}{e^{2\widetilde{m}_F^1\, \Delta L_F^1} - 1}}\, e^{i\alpha_F^{10}} , \tag{J.8}$$

and the higher order KK modes ($n \in \mathbb{N}^*$),

$$\forall\, n \in \mathbb{N}^*,\ \begin{cases} f_L^n(y) \;=\; \mathcal{N}_L^{F1n} \sin\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y\right] ,\ \mathcal{N}_L^{F1n} = \sqrt{2}\, e^{i\alpha_F^{1n}} , \\ f_R^n(y) \;=\; \mathcal{N}_R^{F1n} \cos\left[\sqrt{(m_n^F)^2 - (\widetilde{m}_F^1)^2}\, y - \zeta_n^{F1}\right] ,\ \mathcal{N}_R^{F1n} = \mathcal{N}_L^{F1n} , \end{cases} \tag{J.9}$$

which generates the identical mass spectrum of Eq. (J.6).

# Appendix K

# Explicit Formula of the Mass Matrix

## K.1 Generic Elements of Mass Matrix

Using the explicit profile solution of $q^0_{\mathcal{G}_i L}$ (6.44) and $d^0_{\mathcal{G}_j R}$ (6.45) $(i,j=1,2)$, let us work out following explicit results of the mass matrix $\mathcal{M}_{ij}$ in Eq. (6.85) and the bulk Yukawa coupling $X(y)$ (6.80),

$$
\begin{aligned}
\mathcal{M}_{11} &= \int_0^{L_D} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_1 R} + \int_{L_D^+}^{L_Q} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_1 R} + \int_{L_Q^+}^{L} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_1 R} \\
&= \mathcal{N}_{11} \cos\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{10}-\alpha_Q^{10}\right)} + \mathcal{N}_{12} \cos\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{10}\right)} \\
&\qquad + \mathcal{N}_{22} \sin\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{20}\right)} ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{22} &= \int_0^{L_D} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_2 R} + \int_{L_D^+}^{L_Q} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_2 R} + \int_{L_Q^+}^{L} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_2 R} \\
&= \mathrm{e}^{i(\delta_D-\delta_Q)} \Big[ \mathcal{N}_{11} \sin\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{10}-\alpha_Q^{10}\right)} - \mathcal{N}_{12} \sin\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{10}\right)} \\
&\qquad + \mathcal{N}_{22} \cos\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{20}\right)} \Big] ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{12} &= \int_0^{L_D} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_2 R} + \int_{L_D^+}^{L_Q} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_2 R} + \int_{L_Q^+}^{L} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_1 L} d^0_{\mathcal{G}_2 R} \\
&= \mathrm{e}^{i(\delta_D-\pi)} \Big[ \mathcal{N}_{11} \cos\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{10}-\alpha_Q^{10}\right)} - \mathcal{N}_{12} \cos\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{10}\right)} \\
&\qquad - \mathcal{N}_{22} \sin\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{20}\right)} \Big] ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{21} &= \int_0^{L_D} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_1 R} + \int_{L_D^+}^{L_Q} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_1 R} + \int_{L_Q^+}^{L} dy\, \frac{X(y)}{L} q^{0*}_{\mathcal{G}_2 L} d^0_{\mathcal{G}_1 R} \\
&= \mathrm{e}^{i(\pi-\delta_Q)} \Big[ \mathcal{N}_{11} \sin\Omega_Q \cos\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{10}-\alpha_Q^{10}\right)} + \mathcal{N}_{12} \sin\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{10}\right)} \\
&\qquad - \mathcal{N}_{22} \cos\Omega_Q \sin\Omega_D \, \mathrm{e}^{i\left(\alpha_D^{20}-\alpha_Q^{20}\right)} \Big] , \qquad \text{(K.1)}
\end{aligned}
$$

with,

$$\mathcal{N}_{11} \hat{=} \mathcal{N}_v \frac{Y_5}{\sqrt{2}} \sqrt{\frac{L^2}{\Delta L_Q^1 \Delta L_D^1} \frac{\mathrm{e}^{M_H L_D} - 1}{M_H L}} = \widetilde{\mathcal{N}} \sqrt{\frac{L^2}{\Delta L_Q^1 \Delta L_D^1} \frac{\mathrm{e}^{-M_H \Delta L_D^2} - \mathrm{e}^{-M_H L}}{M_H L}} \,,$$

$$\mathcal{N}_{12} \hat{=} \mathcal{N}_v \frac{Y_5}{\sqrt{2}} \sqrt{\frac{L^2}{\Delta L_Q^1 \Delta L_D^2} \frac{\mathrm{e}^{M_H L_Q} - \mathrm{e}^{M_H L_D}}{M_H L}} = \widetilde{\mathcal{N}} \sqrt{\frac{L^2}{\Delta L_Q^1 \Delta L_D^2} \frac{\mathrm{e}^{-M_H \Delta L_Q^2} - \mathrm{e}^{-M_H \Delta L_D^2}}{M_H L}} \,,$$

$$\mathcal{N}_{22} \hat{=} \mathcal{N}_v \frac{Y_5}{\sqrt{2}} \sqrt{\frac{L^2}{\Delta L_Q^2 \Delta L_D^2} \frac{\mathrm{e}^{M_H L} - \mathrm{e}^{M_H L_Q}}{M_H L}} = \widetilde{\mathcal{N}} \sqrt{\frac{L^2}{\Delta L_Q^2 \Delta L_D^2} \frac{1 - \mathrm{e}^{-M_H \Delta L_Q^2}}{M_H L}} \,, \tag{K.2}$$

where

$$\widetilde{\mathcal{N}} \hat{=} \mathcal{N}_v \frac{Y_5}{\sqrt{2}} \mathrm{e}^{M_H L} \,,$$

and $\mathcal{N}_v$ denotes the Higgs VEV amplitude in Eq. (1.11)-(1.12).

## K.2 Matrix Elements of Localized Profiles

For the localization solutions of $q^0_{a(b)L}(y)$, $d^0_{a(b)R}(y)$ (6.51) presented in Figure 6.8 ($L_D = L_Q$), elements of mass matrix $\mathcal{M}$ (6.84) can be derived in the real case (i.e. $\delta_{D,Q} = \alpha^{10}_{D,Q} = \alpha^{20}_{D,Q} = 0$) as (cf. Appendix K),

$$\begin{cases} \mathcal{M}_{11} &= \mathcal{N}_{11} \cos\Omega_Q \cos\Omega_D + \mathcal{N}_{22} \sin\Omega_Q \sin\Omega_D \,, \\ \mathcal{M}_{22} &= \mathcal{N}_{11} \sin\Omega_Q \sin\Omega_D + \mathcal{N}_{22} \cos\Omega_Q \cos\Omega_D \,, \\ \mathcal{M}_{12} &= -\mathcal{N}_{11} \cos\Omega_Q \sin\Omega_D + \mathcal{N}_{22} \sin\Omega_Q \cos\Omega_D \,, \\ \mathcal{M}_{21} &= -\mathcal{N}_{11} \sin\Omega_Q \cos\Omega_D + \mathcal{N}_{22} \cos\Omega_Q \sin\Omega_D \,, \end{cases} \tag{K.3}$$

and in turn related crucial terms for mass eigenvalues of Eq. (6.86) can be written as,

$$\begin{cases} |\mathcal{M}|^2 &= \mathcal{N}_{11}^2 + \mathcal{N}_{22}^2 \,, \\ |\mathcal{M}_{11}|^2 + |\mathcal{M}_{21}|^2 - |\mathcal{M}_{12}|^2 - |\mathcal{M}_{22}|^2 &= (\mathcal{N}_{11}^2 - \mathcal{N}_{22}^2) \cos(2\Omega_D) \,, \\ \mathcal{M}_{11}^* \mathcal{M}_{12} + \mathcal{M}_{21}^* \mathcal{M}_{22} &= (\mathcal{N}_{22}^2 - \mathcal{N}_{11}^2) \sin\Omega_D \cos\Omega_D \,, \end{cases} \tag{K.4}$$

so that

$$\sqrt{\left(|\mathcal{M}_{11}|^2 + |\mathcal{M}_{21}|^2 - |\mathcal{M}_{12}|^2 - |\mathcal{M}_{22}|^2\right)^2 + 4\,|\mathcal{M}_{11}^* \mathcal{M}_{12} + \mathcal{M}_{21}^* \mathcal{M}_{22}|^2} = \mathcal{N}_{22}^2 - \mathcal{N}_{11}^2 \,. \tag{K.5}$$

# Glossary

**ADD** : **A**rkani-Hamed, **D**imopoulos and **D**vali

**AdS** : **A**nti-**d**e **S**itter

**BBN** : **B**ig **B**ang **N**ucleosynthesis

**BBT** : **B**ilinear **B**oundary **T**erms

**BC** : **B**oundary **C**onditions

**BSM** : **B**eyond the **S**tandard **M**odel

**CDM** : **C**old **D**ark **M**atter

**CFT** : **C**onformal **F**ield **T**heory

**CMB** : **C**osmic **M**icrowave **B**ackground

**DGP** : **D**vali, **G**abadadze and **P**orrati

**DLS** : **D**iscrete **L**agrangian and **S**ymmetries

**EBC** : **E**ssential **B**oundary **C**ondition

**EFT** : **E**ffective **F**ield **T**heory

**EOM** : **E**quations **O**f **M**otion

**EWSB** : **E**lectro**W**eak **S**ymmetry **B**reaking

**FCC pp** : **F**uture **C**ircular **C**ollider **p**roton-**p**roton

**FCNC** : **F**lavor **C**hanging **N**eutral **C**urrent

**GBBT** : **G**eneric **B**ilinear **B**oundary **T**erms

**GUT** : **G**rand **U**nified **T**heory

**GSW** : **G**lashow-**S**alam-**W**einberg

**IR** : **I**nfra**R**ed

**KK** : **K**aluza-**K**lein

**LED** : **L**arge **E**xtra **D**imension

**LHC** : **L**arge **H**adron **C**ollider

**LHS** : **L**eft-**H**and **S**ide

**LKP** : **L**ightest **K**aluza-Klein **P**article

**LQG** : **L**oop **Q**uantum **G**ravity

**nD** : **n-D**imensional space

**NBC** : **N**atural **B**oundary **C**ondition

**NPGO** : **N**on-**P**erturbative **G**ravitationnal **O**bject

**PGO** : **P**erturbative **G**ravitational **O**bject

**QBH** : **Q**uantum **B**lack **H**ole

**QCD** : **Q**uantum **C**hromo**D**ynamics

**QFT** : **Q**uantum **F**ield **T**heory

**RHS** : **R**ight-**H**and **S**ide

**RS** : **R**andall and **S**undrum

**SM** : **S**tandard **M**odel

**SUGRA** : **SU**per**GRA**vity

**SUSY** : **SU**per**SY**mmetry

**UED** : **U**niversal **E**xtra **D**imension

**UV** : **U**ltra**V**iolet

**VEV** : **V**acuum **E**xpectation **V**alue

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breaking, must be either stuck exactly on the so-called TeV-brane (boundary of the finite extra dimension) or located in the bulk with a Kaluza-Klein (KK) wave function peaking at the TeV- brane. In particular, I consider the bulk Higgs with the bulk mass term as well as the brane localized quartic self-interaction, which is necessary to build new flavor model contributing to our final goal.

Then, I continued to work on the Fermion fields with and without Yukawa sector. In this step, I worked on both the Natural Boundary Conditions (NBCs) generated from Bilinear Boundary Terms (BBTs) with integer factor to realize the Neumann Boundary Conditions and the Dirichlet Boundary Conditions at two boundary branes and the Essential Boundary Conditions (EBCs) added by hand, which is also crucial for our new flavor model.

As the starting of the next period, I have already started to work on the problem of the unicity of the BBTs, which is not studied in the literatures on this field. After that, I'll focus more on the spacetime geometry in 2 types of orbifolds, i.e. circle and the periodic translation like crystal. Finally, I'll work on the multi-brane extra spatial dimension to build new flavor model to realize the fermion hierarchy and flavor mixing. Therefore, the work plan described in our initial program is followed as expected.

**Date de soutenance envisagée :** .......... /.......... / ..........

**Orientation envisagée** *(pour les doctorants en dernière année de thèse)*..........

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