

Quantum corrections to quasi-periodic solution of Sine-Gordon model and periodic solution of ϕ^4 model

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Abstract. Analytical form of quantum corrections to quasi-periodic solution of Sine-Gordon model and periodic solution of ϕ^4 model is obtained through zeta function regularisation with account of all rest variables of a d-dimensional theory. Qualitative dependence of quantum corrections on parameters of the classical systems is also evaluated for a much broader class of potentials $u(x) = b^2 f(bx) + C$ with b and C as arbitrary real constants.

1. Introduction

Since the seminal work of Dashen et al. [1], where the determinant approximation of path integrals proposed by Maslov [2] was first used for quantisation of a ϕ^4 kink, there is a constant interest in semiclassical quantisation of nonlinear field theories. Over the years many different methods of obtaining quantum corrections to energy basing either on the Feynman form of propagation operator (see [3]), or alternative form of path integral formulation of quantum mechanics proposed by Garrod [4] were developed. A year after publications of Dashen et al. Korepin and Faddeev calculated energy corrections for the static Sine-Gordon soliton [5] in $1 + 1$ dimensions. In the late seventies generalised zeta function was introduced as a powerful tool for regularisation procedure. This development allowed Konoplich to quantise the Sine-Gordon soliton in a general d-dimensional space [6] as well as account for non-zero temperature [7]. At the same time Hawking used generalized zeta function to define field quantisation on curved manifolds [8]. Many other methods and variations of existing ones were developed in subsequent years including direct mode summation [9] and use of contour integrals for zeta function construction [10, 11], which allowed Pawellek to quantise periodic solutions of Sine-Gordon and ϕ^4 models [12, 13] without inclusion of any rest variables. It is of note, that the final integrals obtained by Pawellek have a very similar form to those derived by Bordag in [9].

Our aim is to quantise the quasi-periodic solution of Sine-Gordon system as well as the periodic solution of ϕ^4 model using zeta function regularisation in form used by Konoplich [6] in order to examine the influence of rest variables on the energy corrections in those cases, since results for the kink solutions of respective models show a very strong dependence on the overall number of dimensions [6, 14, 15]. Additionally we provide general qualitative analysis of semiclassical quantisation for a large class of potentials, which provides an interesting new insight into effects of scaling of the classical system onto its quantum counterpart as well as the meaning of the mass scale and its dependence on physical parameters.



2. Semiclassical corrections to energy

For the purpose of this publication we use quantisation procedure presented by Maslov in [2] with regularisation procedure proposed by Konoplich in [6]. The scheme is based on the expansion of the action (with unitless variables $x_n = \frac{x'_n}{a}$ and $t = \frac{t'}{T}$)

$$S(\psi, T) = Ta^d \int_0^1 \int_{\mathbb{R} \times [0, l]^{d-1}} \left(\frac{M}{2T^2} \left(\frac{\partial \psi}{\partial t} \right)^2 - \frac{G}{2a^2} \sum_{n=1}^d \left(\frac{\partial \psi}{\partial x_n} \right)^2 - V(\psi) \right) \prod_{n=1}^d dx_n dt \quad (1)$$

in path integral formulation of the propagator

$$\langle \phi | e^{-\frac{i}{\hbar} TH} | \phi \rangle = \int_{C_{\phi, \phi}^{0, T}} D\psi e^{\frac{i}{\hbar} S(\psi, T)} \quad (2)$$

in a Taylor series around the classical solution φ and cutting it at the first non-trivial term with T as an arbitrary time period. The series is cut at the first nontrivial element (second derivative) and energy corrections can be formally represented as

$$\Delta E = -\frac{\hbar}{iT} \ln(\det[L]), \quad (3)$$

with L as the second functional derivative of the Lagrangian taken at the classical solution

$$L = -\frac{iTa^d}{2\pi\hbar r^2} \left(-\frac{M}{T^2} \frac{\partial^2}{\partial t^2} + \frac{G}{a^2} \sum_{n=1}^d \frac{\partial^2}{\partial x_n^2} - V''(\varphi) \right) \quad (4)$$

Necessary regularisation consists of two steps: subtraction of analogous expression for classical vacuum solution

$$\Delta E = -\frac{\hbar}{iT} \ln \left(\frac{\det[L]}{\det[L_0]} \right), \quad (5)$$

and a choice of a multiplicative constant r^2 in L and L_0 operators (the so called mass scale) connected to the norm of base functions used in (2), which cuts logarithmic divergences. On purely mathematical level r^2 can be viewed as a free parameter of the theory (which is a reason for keeping it unspecified in works of Konoplich [6]). Yet, to obtain physically relevant results one has to find a way of fitting its value. This problem will be further discussed in section 3. Similar issues might arise with the L_0 for fields spanning over a finite domain, where any choice of the constant potential leads to finite results. However, known results for fields spanning over infinite domains suggest, that the lowest eigenvalue of L_0 should coincide with start of the unbound states band. The above regularisation is realised by means of generalised zeta function

$$\Delta E = -\frac{\hbar}{iT} \lim_{s \rightarrow 0+} \frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \int_{[0,1] \times \mathbb{R} \times [0, l]^{d-1}} (g_L(\tau, \vec{x}, \vec{x}) - g_{L_0}(\tau, \vec{x}, \vec{x})) d\vec{x} d\tau, \quad (6)$$

with \vec{x} covering all variables of the classical system including time and g_L as a Green function of the following equation

$$\left(\frac{\partial}{\partial \tau} + L \right) g_L(\tau, \vec{x}, \vec{x}_0) = \delta(\tau) \delta(\vec{x} - \vec{x}_0) \quad (7)$$

For convenience we will also define

$$\gamma_L(\tau) = \int_{[0,1] \times \mathbb{R} \times [0, l]^{d-1}} g_L(\tau, \vec{x}, \vec{x}) d\vec{x}, \quad (8)$$

$$\gamma(\tau) = \int_{[0,1] \times \mathbb{R} \times [0,l]^{d-1}} (g_L(\tau, \vec{x}, \vec{x}) - g_{L_0}(\tau, \vec{x}, \vec{x})) d\vec{x} \quad (9)$$

and

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \gamma(\tau) d\tau. \quad (10)$$

Since the classical fields considered in this publication depend on a single spatial variable, we will use the fact, that for an operator L expressible as a sum of operators working on independent variables ($L = \sum_i L_n$), heat equation Green function can be constructed as a product of Green functions for L_n (see [16]). For this purpose we will define

$$L_1 = A \left(\frac{\partial^2}{\partial x_1^2} - \frac{a^2 V''(\varphi(x_1))}{G} \right) \quad (11)$$

$$L_2 = -\frac{A}{c^2} \frac{\partial^2}{\partial t^2} \quad (12)$$

$$L_3 = A \sum_{n=2}^d \frac{\partial^2}{\partial x_n^2} \quad (13)$$

with

$$A = -\frac{iTGa^{d-2}}{2\pi\hbar r^2} \quad (14)$$

$$c^2 = \frac{GT^2}{Ma^2} \quad (15)$$

and can readily write

$$\gamma_{L_2} = \sqrt{\frac{c^2}{4\pi A\tau}} \quad (16)$$

$$\gamma_{L_3} = \left(-\frac{l}{4\pi A\tau} \right)^{\frac{d-1}{2}} \quad (17)$$

3. General results

All potentials considered in following sections have many similarities, so it is useful to study general properties of a family of potentials of form

$$u(x) = b^2 f(bx) + C \quad (18)$$

where b and C are some real constants and f is an arbitrary integrable function. Our goal will be to extract as much information from (6) as possible without explicitly solving the Green function problem. This will be most useful for quasi-periodic solution of Sine-Gordon and periodic solution of ϕ^4 model, where exact analytic solutions are difficult to obtain. Let us begin with the Green function equation for L_1 with (18) as the potential

$$\left(\frac{\partial}{\partial \tau} + A \left(\frac{\partial^2}{\partial x^2} - b^2 f(bx) - C \right) \right) g_{L_1}(\tau, x, x_0) = \delta(\tau) \delta(x - x_0) \quad (19)$$

We will now try to remove b from the equation by a series of substitutions

$$x_b = bx \quad (20)$$

will give us

$$\left(\frac{\partial}{\partial \tau} + Ab^2 \left(\frac{\partial^2}{\partial x_b^2} - f(x_b) - \frac{C}{b^2} \right) \right) g_{L_1}(\tau, x_b, x_{b,0}) = b\delta(\tau)\delta(x_b - x_{b,0}) \quad (21)$$

Next we rescale τ

$$\tau = \frac{\tau_b}{|A|b^2} \quad (22)$$

obtaining

$$\left(\frac{\partial}{\partial \tau_b} - i \left(\frac{\partial^2}{\partial x_b^2} - f(x_b) - \frac{C}{b^2} \right) \right) g_{L_1}(\tau_b, x_b, x_{b,0}) = b\delta(\tau_b)\delta(x_b - x_{b,0}) \quad (23)$$

From here we can extract both b and C by rescaling the Green function

$$g_{L_1}(\tau_b, x_b, x_{b,0}) = be^{\frac{C}{b^2}\tau_b} g_{L_{1,b}}(\tau_b, x_b, x_{b,0}) \quad (24)$$

after inserting that form into (23), we find that $g_{L_{1,b}}(\tau_b, x_b, x_{b,0})$ solves the equation

$$\left(\frac{\partial}{\partial \tau_b} - i \left(\frac{\partial^2}{\partial x_b^2} - f(x_b) \right) \right) g_{L_{1,b}}(\tau_b, x_b, x_{b,0}) = \delta(\tau_b)\delta(x_b - x_{b,0}) \quad (25)$$

The same procedure has to be done for the Green function of operator L_0 - both rescaling of the x and τ variables as well as potential shift by $\frac{C}{b^2}$ (if done after rescaling of τ) even if the constant potential in L_0 has a different value than $\frac{C}{b^2}$. As for the Green functions (or γ functions (8)) of L_2 and L_3 operators, it is most convenient to express them through τ_b as well

$$\gamma_{L_2}(\tau_b) = \sqrt{\frac{ic^2b^2}{4\pi\tau_b}} \quad (26)$$

$$\gamma_{L_3}(\tau_b) = \left(\frac{l^2b^2}{i4\pi\tau_b} \right)^{\frac{d-1}{2}} \quad (27)$$

With this information we can reproduce the formula for energy corrections remembering to change the integration over x to integration over x_b .

$$\Delta E = \Re \left(\lim_{s \rightarrow 0} \frac{\partial}{\partial s} \frac{\hbar cb^d l^{d-1}}{2^{d+1} \pi^{\frac{d}{2}} T(|A|b^2)^{-s} \Gamma(s)} \int_0^\infty i^{1-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}} e^{\frac{C}{b^2}\tau_b} \int_{\mathbb{R}} (g_{L_{1,b}}(\tau_b, x_b, x_b) - g_{L_{0,b}}(\tau_b, x_b, x_b)) dx_b d\tau_b \right) \quad (28)$$

Interestingly, in all cases considered in this work C is proportional to b^2 , so we can rightfully substitute

$$C = C_b b^2 \quad (29)$$

and obtain

$$\Delta E = \Re \left(\lim_{s \rightarrow 0} \frac{\partial}{\partial s} \frac{\hbar cb^d l^{d-1}}{2^{d+1} \pi^{\frac{d}{2}} T(|A|b^2)^{-s} \Gamma(s)} \int_0^\infty i^{1-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}} e^{C_b \tau_b} \int_{\mathbb{R}} (g_{L_{1,b}}(\tau_b, x_b, x_b) - g_{L_{0,b}}(\tau_b, x_b, x_b)) dx_b d\tau_b \right) \quad (30)$$

Unfortunately, extracting C_b out of the Mellin transform is not a trivial task. Nevertheless at this point we have most of physical parameters of the classical system extracted out of the Green function. Apart from qualitative estimations of quantum corrections, it helps us in choosing the renormalisation parameter r^2 , since the logarithmic divergences will arise from differentiation of exponential components of the zeta function. This does not concern divergence in T parameter

only - it is also reasonable to assume, that if the classical solution vanishes, corrections should vanish as well. This means, that r^2 should contain b^2 . The same argument can be used to cut divergences in all physically relevant parameters, thus it seems valid to propose for the renormalisation factor to cancel the whole $(|A|b^2)^{-s}$ term. Yet, there is no strict way of choosing the value of r^2 , so in practice we will fit its value to recover well known results in the cases obtainable by different methods.

At this point it is also worth noting, that in our chosen dimensionless variables c is linearly dependent on T , so it cancels out the T component in the denominator. This would not be the case, if we didn't include the kinetic energy component in the action integral and later on in calculation of energy corrections. Combined with the strong dependence of the result on the number of spatial dimensions included it indicates that one should never omit any components of the action integral even if it would be valid for the classical solution.

One of the most important findings going beyond the chosen class of potentials, if we consider a $\tau_A = |A|\tau$ scaling, is that quantum corrections to energy don't depend on the scale of the classical system. If we were to amplify the action integral by a constant factor, it would have no impact on the quantum corrections.

4. Cnoidal waves

We will consider periodic solutions of two well known models (Sine-Gordon and ϕ^4). Considering the findings of the previous section, we will present them in a simplified form:

$$S(\psi, T) = \int_0^1 \int_{\mathbb{R} \times [0, l]^{d-1}} \left(\frac{1}{2c^2} \left(\frac{\partial \psi}{\partial t} \right)^2 - \sum_{n=1}^d \left(\frac{\partial \psi}{\partial x_n} \right)^2 - V(\psi) \right) \prod_{n=1}^d dx_n dt \quad (31)$$

with

$$V_{SG}(\psi) = m^2(1 - \cos(\psi)) \quad (32)$$

for Sine-Gordon and

$$V_{\phi^4} = \frac{m^2}{2V^2} \psi^4 - \frac{m^2}{2} \psi^2 + \frac{m^2 V^2}{8} \quad (33)$$

for ϕ^4 model. Static, periodic solution of respective systems:

$$\psi_{SG}(x) = 2 \arcsin(k \operatorname{sn}(mx; k)) + \pi \quad (34)$$

$$\psi_{\phi^4}(x) = \sqrt{\frac{k^2}{1+k^2}} V \operatorname{sn} \left(\frac{m}{\sqrt{1+k^2}} x; k \right) \quad (35)$$

They will give following potential for the L_1 operator:

$$V_{SG}''(\psi_{SG}(x)) = u_{SG}(x) = m^2(2k^2 - 1 - 2k^2 \operatorname{cn}^2(mx; k)) \quad (36)$$

$$u_{\phi^4}(x) = m^2 \left(5k^2 - 1 - 6k^2 \operatorname{cn}^2 \left(\frac{m}{\sqrt{1+k^2}} x; k \right) \right) \quad (37)$$

After the scaling procedure from previous section (with $b = m$ for Sine-Gordon and $b = \frac{m}{\sqrt{1+k^2}}$ for ϕ^4 model) we calculated respective Laplace transforms of Green function diagonals using the method described in [17]:

$$G_{SG}(p, z) = i \frac{ip - k^2 z}{2\sqrt{ip(ip - (k^2 - 1))(k^2 - ip)}} \quad (38)$$

$$G_{\phi^4}(p, z) = i \frac{-p^2 + 9k^2(1 + k^2(-1 + z))z - 3(ip + k^2pz)}{2\sqrt{(3 - ip)ip(ip - 3k^2)((ip - 1 - k^2)^2 - 4(1 - k^2 + k^4))}} \quad (39)$$

with $z = \text{cn}^2(x_b; k)$. At this point we perform the vacuum cut-off with

$$G_0(p) = \frac{i}{2\sqrt{C - ip}} \quad (40)$$

with the constant C chosen separately for both systems to coincide with the only root of the polynomial in the denominator, which does not become degenerate in the $k \rightarrow 1$ limit, which corresponds to the boundary of the unbound state band of our potential. Next we integrate the function over half-period of the classical solution ($2\mathcal{K}(k)$) in both cases with \mathcal{K} as complete elliptic integral of the first kind to obtain Laplace transform of the γ function

$$\hat{\gamma}(p) = \int_{-\mathcal{K}(k)}^{\mathcal{K}(k)} (iG(ip, x_b) - iG_0(ip)) dx_b \quad (41)$$

$$\hat{\gamma}_{SG}(p) = i \frac{-\mathcal{E}(k) + \left(-1 + k^2 + ip - ip\sqrt{\frac{1-k^2}{ip} + 1}\right) \mathcal{K}(k)}{\sqrt{(ip + 1 - k^2)(k^2 - ip)ip}} \quad (42)$$

$$\hat{\gamma}_{\phi^4}(p) = i \frac{3(1+k^2-ip)\mathcal{E}(k) + (-3+k^2(3-3ip)-p^2)\mathcal{K}(k)}{\sqrt{(3k^2-ip)p(3i+p)((ip-1-k^2)^2-4(1-k^2+k^4))}} + \frac{i \mathcal{K}(k)}{\sqrt{(-1-k^2-2\sqrt{1-k^2+k^4}+ip)}} \quad (43)$$

where \mathcal{E} is the complete elliptic integral of the second kind. We now have to perform an inverse Laplace transform of obtained functions, which unfortunately proves to be beyond our reach at the moment. However, we managed to obtain approximate results by expanding problematic parts of $\hat{\gamma}$ functions in a power series around $k = 1$ treating elliptic integrals as independent parameters. We will showcase the procedure on the Sine-Gordon case.

$$\begin{aligned} \hat{\gamma}_{SG}(p) \approx & -i \frac{\mathcal{E}(k)}{ip\sqrt{1-ip}} + i(k-1) \frac{\mathcal{E}(k)(1-2ip)+3\mathcal{K}(k)(p^2+ip)}{(-1+ip)p^2\sqrt{1-ip}} \\ & -i(k-1)^2 \frac{2\mathcal{E}(k)(-3+7ip+5p^2+2ip^3)+\mathcal{K}(k)(7ip+17p^2-7ip^3+3p^4)}{(-1+ip)^2p^3\sqrt{ip-1}} + \dots \end{aligned} \quad (44)$$

Inverse Laplace transform of this function is easily obtained, since any function of form $\frac{\prod_{j=1}^N(p-a_j)}{\prod_{j=1}^N(p-b_j)\sqrt{b_0-p}}$ can be rewritten as a sum of $\frac{1}{(p-b_j)^n\sqrt{b_0-p}}$ for which inverse Laplace transforms are well known. We will obtain

$$\begin{aligned} \gamma_{SG}(\tau_b) = & -\mathcal{E}(k)\text{Erf}(\sqrt{i\tau_b}) + (k-1) \left(\frac{1}{\sqrt{\pi}} e^{-i\tau_b} \mathcal{E}(k) \sqrt{i\tau_b} + \frac{1}{2} (6\mathcal{K}(k) - \mathcal{E}(k)(1+2i\tau_b)) \text{Erf}(\sqrt{i\tau_b}) \right) \\ & - \frac{1}{8} (k-1)^2 \left(\frac{2}{\sqrt{\pi}} e^{-i\tau_b} \sqrt{i\tau_b} (8\mathcal{K}(k)(1+i\tau_b) + 5\mathcal{E}(k)(1+2i\tau_b)) + \right. \\ & \left. \text{Erf}(\sqrt{i\tau_b}) (4\mathcal{K}(k)(1-14i\tau_b) + \mathcal{E}(k)(-5+4i\tau_b+12\tau_b^2)) \right) + \dots \end{aligned} \quad (45)$$

With this result we can readily use formula (30) to obtain energy corrections assuming value of r^2 as noted in the previous section

$$\Delta E_1(k) = \frac{\hbar cm}{4T\pi} \left(-4\mathcal{E}(k) + 12(k-1)\mathcal{K}(k) - \frac{(k-1)^2}{8} (80\mathcal{K}(k) - 10\mathcal{E}(k)) + \dots \right) \quad (46)$$

$$\begin{aligned} \Delta E_2(k) = & \frac{\hbar cm^2 l}{8T\pi} (2\mathcal{E}(k) + (k-1)(-6\mathcal{K}(k) + \mathcal{E}(k)(-\ln(4))) - \\ & \frac{(k-1)^2}{8} (\mathcal{K}(k)(-20+28(-\ln(4))) + \mathcal{E}(k)(8-2(-\ln(4)))) + \dots) \end{aligned} \quad (47)$$

$$\Delta E_3(k) = \frac{\hbar c m^3 l^2}{16 T \pi^2} \left(\frac{10}{9} \mathcal{E}(k) + (k-1) \left(-\frac{15}{3} \mathcal{K}(k) + \mathcal{E}(k) \left(-5 + \frac{5}{9} \right) \right) - \frac{(k-1)^2}{8} \left(\mathcal{K}(k) \left(-\frac{20}{9} - 128 \right) + \mathcal{E}(k) \left(10 + \frac{25}{9} \right) \right) + \dots \right) \quad (48)$$

The $d = 1$ case in the $k \rightarrow 1$ limit coincides with other quantisation methods [5, 12]. Using the same method we also obtained energy corrections for the ϕ^4 cnoidal wave with r^2 chosen in a way, that ensures $4|A|b^2 = 1$ in order to reproduce the results of Daschen et al. [1] for the $k \rightarrow 1$ limit in the $d = 1$ case.

$$\Delta E_1(k) = -\frac{\hbar m c}{4 T \pi \sqrt{1+k^2}} \left(2\mathcal{E}(k) \left(6 - \frac{\pi}{\sqrt{3}} \right) + (k-1) \left(-\frac{2\pi}{\sqrt{3}} \mathcal{K}(k) - 6(\mathcal{E}(k) - 2\mathcal{K}(k)) \right) + (k-1)^2 \left(-15\mathcal{K}(k) + \frac{3}{2}\mathcal{E}(k) + \frac{\pi}{\sqrt{3}}(\mathcal{K}(k) - 2\mathcal{E}(k)) \right) + \dots \right) \quad (49)$$

$$\Delta E_2(k) = -\frac{\hbar l c m^2}{8 T \pi (1+k^2)} \left(-\mathcal{E}(k)(12 + 3 \ln(3)) - (k-1) \frac{3}{2} (\mathcal{E}(k) + 2\mathcal{K}(k))(4 + \ln(3)) + (k-1)^2 \frac{3}{8} (\mathcal{E}(k)(6 - 33 \operatorname{arccoth}(2)) + \mathcal{K}(k)(-40 + 6 \ln(3))) + \dots \right) \quad (50)$$

$$\Delta E_3(k) = -\frac{\hbar l^2 m^3 c}{48 \pi^2 T (1+k^2)^{\frac{3}{2}}} \left(4\mathcal{E}(k)(\sqrt{3}\pi - 18) + (k-1)4(\sqrt{3}(\mathcal{K}(k) + \mathcal{E}(k))\pi - 9(2\mathcal{K}(k) + \mathcal{E}(k))) - (k-1)^2(2\mathcal{K}(k)(7 + 2\sqrt{3}\pi) - \mathcal{E}(k)(14\sqrt{3}\pi - 27)) + \dots \right) \quad (51)$$

It is important to note, that in the $k \rightarrow 1$ limit the results converge to those obtained by Konoplich in [6]. As yet we were unable to obtain convergence radius for the power series approximation and in this sense the results are incomplete, we are however able to calculate as many terms in this series as needed. It is evident, that the dependence of the results on the elliptic parameter (which is connected to the period of classical solutions) is heavily influenced by the number of spatial dimensions included in the quantisation procedure.

5. Conclusions

If we carry all physical constants through the calculations, it becomes evident, that quantum corrections to energy are independent of the energy scale of the classical system. If we were to multiply the classical action integral by any constant, it would not affect the corrections due to the way parameter A is cut by regularisation. Even if our choice of the regularisation coefficient r^2 was incorrect, scaling of the classical Hamiltonian would at most result in a logarithmic change in quantum corrections. In a way it coincides with the intuition, that quantum effects should only be noticeable in small scale systems.

The zeta-function regularisation scheme is incomplete in the sense, that it doesn't give clear method of choosing the regularisation coefficient r^2 . The choice of vacuum cutoff in the case of fields over a finite interval is not necessarily straightforward as well. We can solve the problem by comparing the energy corrections for cases solved by other methods and extending the results to those otherwise unattainable as we did for the ϕ^4 and Sine-Gordon models.

Energy corrections show strong dependence on the overall number of dimensions of the classical system. It would be of interest to calculate energy corrections without the continuum approximation to research the system's geometry effect on energy in semiclassical regime.

References

- [1] R F Dashen, B Hasslacher and A Neveu, Phys. Rev. D 10, p. 4130 (1974).
- [2] V P Maslov, Zhurnal Vychislitel'noy Matematiki i Matematicheskoy Fiziki 1, p. 638 (1961).
V P Maslov, Teor. i Mat. Fiz. 2, p. 30 (1970).
- [3] R P Feynman, Reviews of Modern Physics 20, 2, p. 367 (1948).
- [4] C Garrod, Reviews of Modern Physics 38, 3, p. 483 (1966).
- [5] W E Korepin and L D Faddeev, Teor. i Mat. Fiz. 25, 2, p. 147 (1975).
- [6] R V Konoplich, Teoreticheskaya i Matematicheskaya Fizika 73, 3, p. 379 (1987).
- [7] R V Konoplich, Teor. Mat. Fiz. 78, 3, p. 444 (1989).

- [8] S W Hawking, Commun. Math. Phys. 55, 2, p. 133 (1977).
- [9] M Bordag, J. Phys. A Math. Gen. 28, 3, p. 755 (1995).
- [10] K Kirsten and A J McKane, Annals. Phys. 308, 2, p. 502 (2003).
- [11] K Kirsten and P Loya, Am. J. Phys. 76, 1 p. 60 (2008).
- [12] M Pawellek, Nuclear Physics B 810, 3, p. 527 (2009).
- [13] M Pawellek, arXiv:0802.0710v1 [hep-th] (6 Feb 2008).
- [14] G Kwiatkowski and S Leble, Physics Letters A 376, 1-2, p. 991 (2012).
- [15] G Kwiatkowski and S Leble, Central European Journal of Physics (Jul 2013).
- [16] S Leble, Theoretical and Mathematical Physics 160, 1, p. 976 (2009).
- [17] G Kwiatkowski and S Leble, Applied Mathematics and Computation 219, 11, p. 6084 (2013).